

Exercitiul 3.3.10

• Fie $f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$f[x_1, x_2, x_3] = [x_2 - x_1]$$

si $v = [1, 1, 0], [0, 1, 1], [1, 0, 1]^T$

si $w = [1, 1], [1, -2]^T$

a) Să se arate că $f \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$

b) Să se arate că v și w sunt baze în \mathbb{R}^3 , respectiv \mathbb{R}^2 și să se determine matricele $[f]_{v,w}$, ești si $[f]_{v,v}$, unde v este baza canonică din \mathbb{R}^3 .

c) Să se determine dimensiunea și căte o bază în $\text{Ker}(f)$ și $\text{im}(f)$.

a) $\text{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2) = \{ f: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \mid f \text{ este liniară} \}$

f este liniară (z)

$$f(x+y) = f(x) + f(y)$$

$$f(\alpha x) = \alpha f(x)$$

Deci

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

$$a) f(x+y) = f(x) + f(y)$$

$$\begin{aligned}x &= [x_1, x_2, x_3] \\y &= [y_1, y_2, y_3]\end{aligned}\quad \Rightarrow f(x+y) = f([x_1, x_2, x_3] + [y_1, y_2, y_3])$$
$$= f([x_1+y_1, x_2+y_2, x_3+y_3]) \in [x_2+y_2, -x_1-y_1] \quad (1)$$

$$\begin{aligned}f(\cancel{x}) &= f(x) + f(y) = f([x_1, x_2, x_3]) + f([y_1, y_2, y_3]) \\&= [x_2, -x_1] + [y_2, -y_1] \in [x_2+y_2, -x_1-y_1] \quad (2)\end{aligned}$$

$$\dim(1) + (2) \Rightarrow f(x+y) = f(x) + f(y) \quad (1)$$

$$f(\alpha x) = \alpha f(x)$$

$$f(\alpha [x_1, x_2, x_3]) = \alpha \cdot f([x_1, x_2, x_3])$$

$$f([\alpha x_1, \alpha x_2, \alpha x_3]) = \alpha \cdot [x_2, -x_1]$$

$$[\alpha x_2, -\alpha x_1] = [\alpha x_2, -\alpha x_1] \quad (\textcircled{A} \Rightarrow)$$
$$\Rightarrow f(\alpha x) = \alpha f(x) \quad (\textcircled{2})$$

$$\text{Dim } (\textcircled{1}) + (\textcircled{2}) \Rightarrow f(x+y) = f(x) + f(y) \quad \begin{matrix} \text{S} \\ \text{w} \end{matrix} \quad (\textcircled{1})$$
$$f(\alpha x) = \alpha f(x) \quad (\textcircled{2})$$

\Rightarrow f este liniară

$$\text{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2) = \left\{ f: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \mid f \text{ este liniară} \right\} \supset$$
$$\Rightarrow f \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2) \quad (\textcircled{A})$$

b) v și w sunt baze ale lui \mathbb{R}^3 , respectiv \mathbb{R}_+^2
 $\Rightarrow v$ și w sunt linear independente

Pentru v : \mathbb{R}_+

$$v = \left[[1, 1, 0], [0, 1, 1], [1, 0, 1] \right]^T = [v_1, v_2, v_3]^T$$

v linear independentă $\Leftrightarrow v_1 \cdot v_1 + v_2 \cdot v_2 + v_3 \cdot v_3 \neq 0$

$$\alpha_1(1, 1, 0) + \alpha_2(0, 1, 1) + \alpha_3(1, 0, 1) \neq 0$$

$$(\alpha_1, \alpha_2, 0) + (\alpha_2, \alpha_3, 1) + (\alpha_3, 0, \alpha_3) \neq 0$$

$$(\alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3) \neq 0$$

$$\begin{cases} \alpha_1 + \alpha_3 \neq 0 \\ \alpha_1 + \alpha_2 \neq 0 \\ \alpha_2 + \alpha_3 \neq 0 \end{cases} \quad \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2 \neq 0$$

$$= 1^3 + 0 + 1^3 - (0 + 0 + 0) = 1 + 1 = 2 \neq 0$$

$\Rightarrow v$ este linear independentă $\Rightarrow v$ este baza

Pentru w :

$$w = \left[[1, 1], [1, -2] \right]^T = [w_1, w_2]$$

w linear independentă $\Leftrightarrow w_1 \cdot w_1 + w_2 \cdot w_2 \neq 0$

$$\alpha_1 \cdot (1, 1) + \alpha_2 \cdot (1, -2) \neq 0$$

$$(\alpha_1, \alpha_1) + (\alpha_2, -2\alpha_2) \neq 0$$

$$(\alpha_1 + \alpha_2, \alpha_1 - 2\alpha_2) \neq 0$$

$$\begin{cases} \alpha_1 + \alpha_2 \neq 0 \\ \alpha_1 - 2\alpha_2 \neq 0 \end{cases}$$

$$\alpha_1 - 2\alpha_2 \neq 0$$

$$\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = 1 \cdot (-2) - 1 \cdot 1 = -2 - 1 = -3 \neq 0 \Rightarrow$$

\Rightarrow v este linia independentă $\Rightarrow v$ este bază

e bază canonica dim $\mathbb{R}^2 \Rightarrow \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right]$

$$\Rightarrow e = [e_1, e_2]^T$$

$$e_1 = [1, 0]^T$$

$$e_2 = [0, 1]^T$$

$$\rightarrow [f]_{v,e} = [f(v)]_e$$

$$\begin{aligned} [f]_{v,e} &= \begin{bmatrix} [f(v_1)]_e \\ [f(v_2)]_e \\ [f(v_3)]_e \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} f(v_1) &= a_{11} \cdot e_1 + a_{21} \cdot e_2 = a_{11} \cdot [1, 0]^T + a_{21} [0, 1]^T \\ &= [a_{11}, 0]^T + [a_{21}, 0]^T = \\ &= [a_{11}, a_{21}] = \underline{\underline{[1, -1]}} \end{aligned}$$

$$\begin{aligned} f(v_2) &= a_{21} \cdot e_1 + a_{22} \cdot e_2 = a_{21} \cdot [1, 0]^T + a_{22} [0, 1]^T \\ &= [a_{21}, 0]^T + [0, a_{22}]^T = [a_{21}, a_{22}] = \underline{\underline{[1, 0]}} \end{aligned}$$

$$\begin{aligned} f(v_3) &= a_{31} \cdot e_1 + a_{32} \cdot e_2 = a_{31} [1, 0]^T + a_{32} [0, 1]^T \\ &= [a_{31}, 0]^T + [0, a_{32}]^T = [a_{31}, a_{32}] = \underline{\underline{[0, -1]}} \end{aligned}$$

$$\rightarrow [f]_{v,w} \neq [f(v)]_w = \begin{bmatrix} f(v_1)|_w \\ f(v_2)|_w \\ f(v_3)|_w \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\begin{aligned} f(v_1) &= a_{11} \cdot w_1 + a_{12} \cdot w_2 = a_{11} \cdot [1, 1]^T + a_{12} \cdot [1, -2]^T = \\ &= [a_{11}, a_{11}] + [a_{12}, -2a_{12}] = [a_{11} + a_{12}, a_{11} - 2a_{12}] \end{aligned}$$

$$\begin{aligned} f(v_2) &= a_{21} \cdot w_1 + a_{22} \cdot w_2 = a_{21} \cdot [1, 1]^T + a_{22} \cdot [1, -2]^T = \\ &= [a_{21}, a_{21}] + [a_{22}, -2a_{22}] = [a_{21} + a_{22}, a_{21} - 2a_{22}] \end{aligned}$$

$$f(v_3) = a_{31} \cdot w_1 + a_{32} \cdot w_2 = a_{31} \cdot [1, 1] + a_{32} \cdot [1, -2] = \\ = [a_{31}, a_{31}] + [a_{32}, -2a_{32}] = \\ = [a_{31} + a_{32}, a_{31} - 2a_{32}]$$

$$f(v_1) = [1, -1]$$

$$f(v_2) = [1, 0]$$

$$f(v_3) = [0, -1]$$

$$\begin{aligned} &= 1 \left\{ \begin{array}{l} a_{11} + a_{12} = 1 \\ a_{11} - 2a_{12} = -1 \end{array} \right. \quad \left(\begin{array}{l} 2 \\ 1 \end{array} \right) \left\{ \begin{array}{l} 2a_{11} + 2a_{12} = 2 \\ a_{11} - 2a_{12} = -1 \end{array} \right. \quad (+) \\ &\qquad \qquad \qquad 3a_{11} = 1 \Rightarrow a_{11} = \frac{1}{3} \\ &\qquad \qquad \qquad a_{12} = 1 - \frac{1}{3} = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} &\left\{ \begin{array}{l} a_{21} + a_{22} = 1 \\ a_{21} - 2a_{22} = 0 \end{array} \right. \quad \left(\begin{array}{l} 1 \\ 2 \end{array} \right) \left\{ \begin{array}{l} 2a_{21} + 2a_{22} = 2 \\ a_{21} - 2a_{22} = 0 \end{array} \right. \quad (-) \\ &\qquad \qquad \qquad 3a_{21} = 2 \Rightarrow a_{21} = \frac{2}{3} \\ &\qquad \qquad \qquad a_{22} = 1 - \frac{2}{3} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} &\left\{ \begin{array}{l} a_{31} + a_{32} = 0 \\ a_{31} - 2a_{32} = -1 \end{array} \right. \quad \left(\begin{array}{l} 1 \\ 2 \end{array} \right) \left\{ \begin{array}{l} 2a_{31} + 2a_{32} = 0 \\ a_{31} - 2a_{32} = -1 \end{array} \right. \quad (-) \\ &\qquad \qquad \qquad 3a_{31} = -1 \Rightarrow a_{31} = -\frac{1}{3} \\ &\qquad \qquad \qquad a_{32} = 0 + \frac{1}{3} = \frac{1}{3} \end{aligned}$$

$$(f) \quad v_1, w = \left[\begin{array}{cc} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{array} \right]$$

$$c) \text{Ker } f = \left\{ x \in \mathbb{R}^3 \mid f(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\text{im } f = \left\{ f(x) \mid x \in \mathbb{R}^3 \right\}$$

$$f(x) = 0 \Leftrightarrow f[x_1, x_2, x_3] = [0, 0]$$

$$\Rightarrow \begin{cases} x_2 = 0 \\ -x_1 = 0 \Rightarrow x_1 = 0 \end{cases} \quad \begin{matrix} x_1 \in \mathbb{R}^3 \\ x_1 = 0 \end{matrix}$$

$$\Rightarrow \text{Ker } f = \left\{ \begin{pmatrix} x_1 & 0 & 0 \\ 0 & 0 & x_3 \end{pmatrix} \mid x_3 \in \mathbb{R}^3 \right\} \quad \dim \text{Ker } f = \mathbb{R}^3$$

$$\text{im } f = (a, b) \in \mathbb{R}^2. \quad (\exists) (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ ai. } f[x_1, x_2, x_3] = [a, b]$$

$$\text{im } f : (\forall)(a, b) \in \mathbb{R}^2, \quad (\exists) (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ ai.}$$

$$f[x_1, x_2, x_3] = [a, b]$$

$$[x_2, -x_3] = [a, b]$$

$$\begin{cases} x_2 = a \in \mathbb{R} \\ -x_3 = b \Rightarrow x_3 = -b \in \mathbb{R} \end{cases} \quad \begin{matrix} x_1, x_2, x_3 \in \mathbb{R}^3 \\ \text{im } f = \mathbb{R}^2 \end{matrix}$$