

Exercițiu 3.3.10

• Fie $f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$f[x_1, x_2, x_3] = [x_2 - x_1]$$

$$v = [1, 1, 0], [0, 1, 1], [1, 0, 1]$$

$$w = [1, 1], [1, -2]$$

a) Să se arate că $f \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$

b) Să se arate că v și w sunt baze în \mathbb{R}^3 , respectiv \mathbb{R}^2 și să se determine matricele $[f]_v, e$ și $[f]_v, w$, unde e este baza canonică din \mathbb{R}^2 .

c) Să se determine dimensiunea și câte o bază în $\text{Ker}(f)$ și $\text{Im}(f)$.

a) $\text{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2) = \{ f: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \mid f \text{ este liniară} \}$

f este liniară \Leftrightarrow

$$f(x+y) = f(x) + f(y)$$

$$f(\alpha x) = \alpha f(x)$$

Sau

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

$$a) f(x+y) = f(x) + f(y)$$

$$x = [x_1, x_2, x_3] \quad y = [y_1, y_2, y_3] \quad | \quad (1) f(x+y) = f([x_1, x_2, x_3] + [y_1, y_2, y_3])$$

$$= f([x_1+y_1, x_2+y_2, x_3+y_3]) = [x_2+y_2, -x_1-y_1] \quad (1)$$

$$\cancel{f(x+y)} \quad f(x) + f(y) = f([x_1, x_2, x_3]) + f([y_1, y_2, y_3]) \\ = [x_2, -x_1] + [y_2, -y_1] = [x_2+y_2, -x_1-y_1] \quad (2)$$

$$\text{Sim (1) + (2) } \Rightarrow f(x+y) = f(x) + f(y) \quad (1)$$

$$f(\alpha x) = \alpha f(x)$$

$$f(\alpha [x_1, x_2, x_3]) = \alpha \cdot f([x_1, x_2, x_3])$$

$$f([\alpha x_1, \alpha x_2, \alpha x_3]) = \alpha \cdot [x_2, -x_1]$$

$$[\alpha x_2, -\alpha x_1] = [\alpha x_2, -\alpha x_1] \quad \textcircled{A} \Rightarrow$$

$$\Rightarrow f(\alpha x) = \alpha f(x) \quad (2)$$

$$\text{Din } (1) + (2) \Rightarrow f(x+y) = f(x) + f(y) \\ \text{si} \\ f(\alpha x) = \alpha f(x) \quad \Rightarrow$$

$\Rightarrow f$ este liniară

$$\text{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2) = \{f: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \mid f \text{ este liniară}\} \Rightarrow$$

$$\Rightarrow f \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2) \quad \textcircled{A}$$

b) v_i w sunt baze ale lui \mathbb{R}^3 , respectiv \mathbb{R}^2
 $\Rightarrow v_i$ w sunt linear independente

• Pentru v : \mathbb{R}_2

$$v = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \stackrel{+}{=} \underset{\text{rot.}}{[v_1, v_2, v_3]}$$

v linear independentă $\Rightarrow \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 \neq 0$

$$\alpha_1 (1, 1, 0) + \alpha_2 (0, 1, 1) + \alpha_3 (1, 0, 1) \neq 0$$

$$(\alpha_1, \alpha_1, 0) + (0, \alpha_2, \alpha_2) + (\alpha_3, 0, \alpha_3) \neq 0$$

$$(\alpha_1 + \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3) \neq 0$$

$$\begin{cases} \alpha_1 + \alpha_3 \neq 0 \\ \alpha_1 + \alpha_2 \neq 0 \\ \alpha_2 + \alpha_3 \neq 0 \end{cases} \quad \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} =$$

$$= 1^3 + 0 + 1^3 - (0 + 0 + 0) = 1 + 1 = 2 \neq 0$$

$\Rightarrow v$ este linear independentă $\Rightarrow v$ este bază

• Pentru w :

$$w = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \stackrel{+}{=} [w_1, w_2]$$

w linear independentă $\Rightarrow \alpha_1 \cdot w_1 + \alpha_2 \cdot w_2 \neq 0$

$$\Rightarrow \alpha_1 (1, 1) + \alpha_2 (1, -2) \neq 0$$

$$(\alpha_1, \alpha_1) + (\alpha_2, -2\alpha_2) \neq 0$$

$$(\alpha_1 + \alpha_2, \alpha_1 - 2\alpha_2) \neq 0$$

$$\begin{cases} \alpha_1 + \alpha_2 \neq 0 \\ \alpha_1 - 2\alpha_2 \neq 0 \end{cases}$$

$$\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 1 \cdot (-2) - 1 = -2 - 1 = -3 \neq 0 \Rightarrow$$

\Rightarrow w este liniear independente $\Rightarrow w$ este bază

e bază canonică din \mathbb{R}^2 , ~~$e = [0, 1]$~~

$$\Rightarrow e = [e_1, e_2]^T$$

$$e_1 = [1, 0]$$

$$e_2 = [0, 1]$$

$$\rightarrow [f]_{v,e} = [f(v_i)]_e$$

$$[f]_{v,e} = \begin{bmatrix} [f(v_1)]_e \\ [f(v_2)]_e \\ [f(v_3)]_e \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} f(v_1) &= a_{11} \cdot e_1 + a_{12} \cdot e_2 = a_{11} \cdot [1, 0] + a_{12} \cdot [0, 1] \\ &= [a_{11}, 0] + [0, a_{12}] = \\ &= [a_{11}, a_{12}] = \underline{\underline{(1, -1)}} \end{aligned}$$

$$\begin{aligned} f(v_2) &= a_{21} \cdot e_1 + a_{22} \cdot e_2 = a_{21} \cdot [1, 0] + a_{22} \cdot [0, 1] \\ &= [a_{21}, 0] + [0, a_{22}] = [a_{21}, a_{22}] = \underline{\underline{(1, 0)}} \end{aligned}$$

$$\begin{aligned} f(v_3) &= a_{31} \cdot e_1 + a_{32} \cdot e_2 = a_{31} \cdot [1, 0] + a_{32} \cdot [0, 1] \\ &= [a_{31}, 0] + [0, a_{32}] = [a_{31}, a_{32}] = \underline{\underline{(0, -1)}} \end{aligned}$$

$$- [f]_{v,w} = [f(v_i)]_w = \begin{bmatrix} [f(v_1)]_w \\ [f(v_2)]_w \\ [f(v_3)]_w \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\begin{aligned} f(v_1) &= a_{11} \cdot w_1 + a_{12} \cdot w_2 = a_{11} \cdot [1, 1] + a_{12} \cdot [1, -2] = \\ &= [a_{11}, a_{11}] + [a_{12}, -2a_{12}] = [a_{11} + a_{12}, a_{11} - 2a_{12}] \end{aligned}$$

$$\begin{aligned} f(v_2) &= a_{21} \cdot w_1 + a_{22} \cdot w_2 = a_{21} \cdot [1, 1] + a_{22} \cdot [1, -2] = \\ &= [a_{21}, a_{21}] + [a_{22}, -2a_{22}] = [a_{21} + a_{22}, a_{21} - 2a_{22}] \end{aligned}$$

$$f(v_3) = a_{31} \cdot w_1 + a_{32} \cdot w_2 = a_{31} \cdot [1, 1] + a_{32} \cdot [1, -2] =$$

$$= [a_{31}, a_{31}] + [a_{32}, -2a_{32}] =$$

$$= [a_{31} + a_{32}, a_{31} - 2a_{32}] =$$

$$f(v_1) = [1, -1]$$

$$f(v_2) = [1, 0]$$

$$f(v_3) = [0, -1]$$

$$\Rightarrow \begin{cases} a_{11} + a_{12} \geq 1 & (1) \\ a_{11} - 2a_{12} \geq -1 & (2) \end{cases} \Rightarrow \begin{cases} 2a_{11} + 2a_{12} \geq 2 \\ a_{11} - 2a_{12} \geq -1 & (+) \end{cases}$$

$$\underline{3a_{11} \geq 1 \Rightarrow a_{11} \geq \frac{1}{3}}$$

$$a_{12} \geq 1 - \frac{1}{3} = \frac{2}{3}$$

$$\begin{cases} a_{21} + a_{22} \geq 1 & (1) \\ a_{21} - 2a_{22} \geq 0 & (2) \end{cases} \Rightarrow \begin{cases} 2a_{21} + 2a_{22} \geq 2 \\ a_{21} - 2a_{22} \geq 0 & (+) \end{cases}$$

$$\underline{3a_{21} \geq 2 \Rightarrow a_{21} \geq \frac{2}{3}}$$

$$a_{22} \geq 1 - \frac{2}{3} = \frac{1}{3}$$

$$\begin{cases} a_{31} + a_{32} \geq 0 & (1) \\ a_{31} - 2a_{32} \geq -1 & (2) \end{cases} \Rightarrow \begin{cases} 2a_{31} + 2a_{32} \geq 0 \\ a_{31} - 2a_{32} \geq -1 & (+) \end{cases}$$

$$\underline{3a_{31} \geq -1 \Rightarrow a_{31} \geq -\frac{1}{3}}$$

$$a_{32} \geq 0 + \frac{1}{3} = \frac{1}{3}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} v, w = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$c) \text{Ker } f = \{x \in \mathbb{R}^3 \mid f(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

$$\text{im } f = \{f(x) \mid x \in \mathbb{R}^3\}$$

$$f(x) = 0 \Rightarrow f[x_1, x_2, x_3] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1, x_2 = 0 \\ -x_1 = 0 \Rightarrow x_1 = 0 \end{cases} \quad x_3 \in \mathbb{R}$$

$$\Rightarrow \text{Ker } f = \left\{ \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} \mid x_3 \in \mathbb{R} \right\} \Rightarrow \dim \text{Ker } f = 1$$

$$\text{im } f = \{(a, b) \in \mathbb{R}^2 \mid \exists (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ s.t. } f[x_1, x_2, x_3] = \begin{bmatrix} a \\ b \end{bmatrix}\}$$

$$\text{im } f = \{(a, b) \in \mathbb{R}^2 \mid \exists (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ s.t.}$$

$$f[x_1, x_2, x_3] = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ -x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{cases} x_2 = a \in \mathbb{R} \\ -x_3 = b \Rightarrow x_3 = -b \in \mathbb{R} \end{cases}$$

$$\Rightarrow \text{im } f = \mathbb{R}^2$$

$$x_1, x_2, x_3 \in \mathbb{R}$$