Exerciții Seminar 4 Sisteme dinamice

1. a) $\begin{cases} y(1+e^x)\, y' - e^x = 0 \\ y(0) = 1 \end{cases}$

$(1+e^x)\, y \cdot y' - e^x = 0$

$y y' = \dfrac{e^x}{1+e^x}$

$y' = \dfrac{e^x}{1+e^x} \cdot \dfrac{1}{y}$

$y' = \dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{e^x}{1+e^x} \cdot \dfrac{1}{y} \Rightarrow y\,dy = \dfrac{e^x}{1+e^x}\,dx \,\Big|\int$

$\int y\,dy = \int \dfrac{e^x}{1+e^x}\,dx$

...ii este:

$c_1 \in \mathbb{R}$

$\dfrac{y^2}{2} = \ln|1+e^x| + c_1 \quad \Big|\cdot 2$

$2c_1 = c$

$y^2 = 2\ln|1+e^x| + 2c_1$

de forma:

$y^2 = 2\ln|1+e^x| + c \,\Big|\sqrt{}$

...$c_2 \cdot y_2(x) \quad c_1, c_2 \in \mathbb{R}$

$y = \pm\sqrt{2\ln|1+e^x| + c}$

...$c_2 \cdot \sin(3x) \quad c_1, c_2 \in \mathbb{R}$

I $\; y(x) = \sqrt{2\ln|1+e^x| + c}$

... (*)

$y(0) = 1$ Condiția pt. pb. Cauchy

$$\Rightarrow y(0) = \sqrt{2\ln 2 + c} \Rightarrow \sqrt{2\ln 2 + c} = 1 \quad |\ ^2$$

$$2\ln 2 + c = 1$$

$$\boxed{c = 1 - 2\ln 2}$$

$-3c_2\cos 3x)$

$-\frac{3}{5}\cos 3x)$

II Analog $\boxed{c = 1 - 2\ln 2}$

$$\Rightarrow y(x) = \pm\sqrt{2\ln(1 + e^x) + 1 - 2\ln 2}$$

b) $\begin{cases} y_1' = y_1 + y_2 \\ y_2' = -2y_1 + 4y_2 \\ y_1(0) = 0 \\ y_2(0) = -1 \end{cases}$

$$y_1' = y_1 + y_2 \,\Big|' \Rightarrow y_1'' = y_1' + y_2'$$

$$y_1'' = y_1 + y_2 - 2y_1 + 4y_2$$

$$y_1'' = -y_1 + 5y_2$$

Avem sist. $\begin{cases} y_1'' = -5y_1 + 5y_2 \\ y_1' = y_1 + y_2 \end{cases} \Rightarrow y_2 = y_1' - y_1$

$$y_1'' = -y_1 + 5y_1' - 5y_1 = -6y_1 + 5y_1'$$

$$y_1'' - 5y_1' + 6y_1 = 0$$

$$r^2 - 5r + 6 = 0$$

$$\Delta = 1 > 0 \Rightarrow r_1 = 3, \quad r_2 = 2 \;\Big|\Rightarrow \begin{array}{l} \phi_1(x) = e^{3x} \\ \phi_2(x) = e^{2x} \end{array}$$

$$y_1(x) = c_1 e^{3x} + c_2 e^{2x}$$

$$y_2(x) = y_1' - y_1 = 3c_1 e^{3x} + 2c_2 e^{2x} - c_1 e^{3x} - c_2 e^{2x}$$

$$y_2(x) = 2c_1 e^{3x} + c_2 e^{2x}$$

$c_1, c_2 \in \mathbb{R}$

$$\begin{cases} y_1(x) = c_1 e^{3x} + c_2 e^{2x} \\ y_2(x) = 2c_1 e^{3x} + c_2 e^{2x} \end{cases}$$

Pb. Cauchy

$$y_1(0) = 0 \Rightarrow y_1(0) = c_1 + c_2 \Rightarrow c_1 + c_2 = 0$$

$$y_2(0) = -1 \Rightarrow y_2(0) = 2c_1 + c_2 \Rightarrow \frac{2c_1 + c_2 = -1 \quad (-)}{c_1 = -1 \Rightarrow c_2 = 1}$$

$$\begin{cases} y_1(x) = -e^{3x} + e^{2x} \\ y_2(x) = -2e^{3x} + e^{2x} \end{cases}$$

2.

a) $\begin{cases} y'' - 5y' + 4y = 0 \\ y(0) = 5 \\ y'(0) = 8 \end{cases}$

$$y'' - 5y' + 4y = 0$$

$$r^2 - 5r + 4 = 0$$

$$\left.\begin{array}{l} \Delta = 9 \Rightarrow r_1 = 4 \\ \qquad\quad r_2 = 1 \end{array}\right) \Rightarrow \begin{array}{l} \phi_1(x) = e^{4x} \\ \phi_2(x) = e^{x} \end{array}$$

$$y(x) = c_1 e^{4x} + c_2 e^{x} \qquad c_1, c_2 \in \mathbb{R}$$

$$y(0) = 5 \Rightarrow y(0) = c_1 + c_2 = 5$$

$$y'(x) = 4c_1 e^{4x} + c_2 e^{x}$$

$$y'(0) = 8 \Rightarrow 4c_1 + c_2 = 8$$

$$\begin{cases} c_1 + c_2 = 5 \\ 4c_1 + c_2 = 8 \end{cases} \quad (-)$$

$$3c_1 = 3 \Rightarrow c_1 = 3$$

$$c_2 = 2$$

$y(x) = e^{4x} + 4e^{x}$

b) $\begin{cases} y'' + 4y = 4x \\ y(\pi) = 0 \\ y'(\pi) = 1 \end{cases}$

$y'' + 4y = 4x$ ec. neomog.

$\underline{\text{I}}$ Ec. omogenă

$y'' + 4y = 0$

$r^2 + 4 = 0$

$r^2 = 4i^2 \Rightarrow r_{1,2} = \pm 2i$

$r_1 = 2i \Rightarrow y_1(x) = \cos 2x$

$r_2 = -2i \Rightarrow y_2(x) = \sin 2x$

$y_0 = c_1 \cos 2x + c_2 \sin 2x \qquad c_1, c_2 \in \mathbb{R}$

$\underline{\text{II}}$ Ec. particulară

$y'' + 4y = 4x$

$f(x) = 4x$

Funcția f este de forma unui polinom

$f(x) = P_1$, adică cazul I, b)

$y_p = x(ax + b) = ax^2 + bx$

$y_p' = 2ax + b$

$y_p'' = 2a$

$$2a + 4ax^2 + 4bx = 4x$$

identificare coeff.

$$\begin{cases} 4b = 4 \Rightarrow b = 1 \\ 2a = 0 \\ 4a = 0 \end{cases} \Big/ \Rightarrow a = 0$$

$$\Rightarrow y_r = x$$

$$y(x) = y_0 + y_r = C_1 \cos 2x + C_2 \sin 2x + x$$

$$y(\pi) = 0 \Rightarrow y(\pi) = C_1 \cos 2\pi + C_2 \sin 2\pi + \pi$$

$$y(\pi) = C_1 + \pi$$

$$\Rightarrow C_1 + \pi = 0 \Rightarrow C_1 = -\pi$$

$$y'(\pi) = 1 \Leftrightarrow y'(x) = -2C_1 \sin 2x + 2C_2 \cos 2x + 1$$

$$y'(\pi) = -2C_1 \sin 2\pi + 2C_2 \cos 2\pi + 1$$

$$y'(\pi) = 2C_2 + 1$$

$$2C_2 + 1 = 1 \Rightarrow C_2 = 0$$

$$y(x) = -\pi \cos 2x + x$$

3. a) $\begin{cases} y'' + \pi^2 y = 0 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$

$y'' + \pi^2 y = 0$

$r^2 + \pi^2 = 0$

$\Delta = b^2 - 4ac$

$\Delta = -4\pi^2$

$r_{1,2} = \pm \pi i$

$r_1 = \pi i \Rightarrow \phi_1(x) = \cos \pi x$

$\phi_2(x) = \sin \pi x$

$y(x) = c_1 \cdot \cos \pi x + c_2 \sin \pi x$, $c_1, c_2 \in \mathbb{R}$

$y(0) = 0 \Rightarrow y(0)' = C_1 \Rightarrow C_1 = 0$

$y(1) = 0 \Rightarrow y(1) = C_1 \cdot (-1) + C_2 \cdot 0$

$-C_1 = 0 \Rightarrow C_1 = 0$

$\Rightarrow \begin{cases} C_1 = 0 \\ C_2 = 0 \end{cases} \Rightarrow y(x) = 0$

b) $\begin{cases} y'' + y = x \\ y(0) = 1 \\ y(\frac{\pi}{2}) = \frac{\pi}{2} \end{cases}$

I Ec. omog.

$y'' + y = 0$

$r^2 + 1 = 0$

$r^2 = 1 i^2 \Rightarrow r_{1,2} = \pm i$

$r_1 = i \Rightarrow \phi_1(x) = \cos x$

$r_2 = -i \Rightarrow \phi_2(x) = \sin x$

$y_0 = C_1 \cos x + C_2 \sin x$, $C_1, C_2 \in \mathbb{R}$

II Ec. particulară

II Ec. particulară

$$y'' + y = x$$

$$f(x) = x$$

Suntem în cazul $\overline{1}$, b) cu $f(x) = P_1(x)$

$$y_p(x) = x(ax + b) = ax^2 + bx$$

$$y_p' = 2ax + b$$

$$y_p'' = 2a$$

$$2a + ax^2 + bx = x \Rightarrow \begin{cases} a = 0 \\ b = 1 \end{cases}$$

$$\Rightarrow y_p = 1$$

$$y = y_0 + y_p = C_1 \cos x + C_2 \sin x + x$$

Bilocală

$$y(0) = 1 \Rightarrow y(0) = C_1 \Leftrightarrow C_1 = 1$$

$$y\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \Rightarrow y\left(\frac{\pi}{2}\right) = C_2 + \frac{\pi}{2} \Rightarrow C_2 = 0$$

$$\begin{cases} C_1 = 1 \\ C_2 = 0 \end{cases} \Rightarrow y(x) = \cos x + x$$

5\. a) $\begin{cases} x^2 y' \cos\left(\frac{1}{x}\right) - y \sin\left(\frac{1}{x}\right) = -1 \\ \lim\limits_{x \to \infty} y(x) = 0 \end{cases}$ $(*)$

$$x^2 y' \cos\left(\frac{1}{x}\right) - y \sin\left(\frac{1}{x}\right) = -1 \Big/ : x^2 \cos\left(\frac{1}{x}\right)$$

$$y'(x) - \frac{y}{x^2} \, tg\left(\frac{1}{x}\right) = -\frac{1}{x^2 \cos \frac{1}{x}}$$

I Ec. omogenă

$$y' - \frac{y}{x^2} \, tg \frac{1}{x} = 0$$

$$y' = \frac{y \cdot tg\left(\frac{1}{x}\right)}{x^2}$$

$$\frac{dy}{dx} = y \frac{tg \frac{1}{x}}{x^2} \Rightarrow \frac{dy}{y} = \frac{tg \frac{1}{x}}{x^2} dx \Big/ \int$$

$$\int \frac{dy}{y} = \int \frac{tg \frac{1}{x}}{x^2} dx$$

$$\ln|y| = \ln\left|\cos\left(\frac{1}{x}\right)\right| + C_1$$

$$y_0 = c \cdot \cos \frac{1}{x}$$

II Ec. particulară (met. var. const.)

$\overline{\text{II}}$ Ec. particulară (met. var. const.)

$$y_p = f(x)\cos\frac{1}{x}$$

$$y_p' = f'(x)\cos\frac{1}{x} + \frac{1}{x^2} f(x)\sin\frac{1}{x}$$

Ne întoarcem la * și înlocuim y_p, y_p'

← ec. inițială neomog.

$\Rightarrow f'(x) = \ldots$

$f(x) = \ldots$

$$\Rightarrow y(x) = \sin\frac{1}{x} + c\cdot\cos\frac{1}{x}$$

- $\lim_{x\to\infty} y(x) = 0$

$$\lim_{x\to\infty} \sin\frac{1}{x} + c\cos\frac{1}{x} = c$$

$$\Rightarrow c = 0 \Rightarrow y(x) = \sin\frac{1}{x}$$

c) I Ec. omogenă

$y'' - 4y' + 5y = 0$

$r^2 - 4r + 5 = 0$

Cazul c)

$r_{1,2} \in \mathbb{C}$

$\Delta < 0$

$r_{1,2} = \alpha \pm \beta i$ $\Rightarrow$ $\phi_2(x) = e^{\alpha x} \sin \beta x$

$\phi_1(x) = e^{\alpha x} \cos \beta x$

$y = c_1 \cdot \phi_1 + c_2 \cdot \phi_2$

$\Delta = b^2 - 4ac = 16 - 4 \cdot 1 \cdot 5 = 16 - 20 = -4$

$y_o = c_1 \cdot e^{2x} \cos x + c_2 e^{2x} \sin x$

$r_{1,2} = \frac{-b \pm i\sqrt{-\Delta}}{2a} = \frac{4 \pm 2i}{2} = \underset{\alpha}{2} \pm \underset{\beta}{i}$

II Particulară

$f(x) = \sin x = e^{\alpha x} \cdot P_m(x) \cdot \sin \beta x$

$\alpha = 0, \beta = 1, P_0(x)$

$\alpha + \beta \cdot i$

$y_p(x) = e^{\alpha x} [Q_m(x) \cos \beta x + R_m(x) \sin \beta x]$

$y_p(x) = a \cos x + b \sin x \Rightarrow y_p = \frac{1}{8} \cos x + \frac{1}{8} \sin x$

$y_p'(x) = -a \sin x + b \cos x$

$y_p''(x) = -a \cos x - b \sin x$

$-a \cos x - b \sin x + 4a \sin x - 4b \cos x + 5a \cos x + 5b \sin x = \sin x$

$\sin x(\underbrace{-b + 4a + 5b}) + \cos x(-a - 4b + 5a) = 1 \sin x$

$$-a\cos x - b\sin x + 4a\sin x - 4b\cos x + 5a\cos x + 5b\sin x = \sin x$$

$$\sin x\underbrace{(-b + 4a + 5b)} + \cos x(-a - 4b + 5a) = 1\sin x$$

$$4b + 4a = 1$$

$$4a - 4b = 0 \quad (+)$$

$$8a = 1 \Rightarrow a = \frac{1}{8} \Rightarrow \frac{1}{2} - 4b = 0$$

$$4b = \frac{1}{2}$$

$$b = \frac{1}{8}$$

Exerciții temă Seminar 4

1\. b) $\begin{cases} y_1' = 3y_1 - y_2 \\ y_2' = 10y_1 - 4y_2 \\ y_1(0) = 1 \\ y_2(0) = 5 \end{cases}$

$y_1'' = 3y_1' - y_2'$

$y_1'' = 3(3y_1 - y_2) - (10y_1 - 4y_2)$

$y_1'' = 9y_1 - 3y_2 - 10y_1 + 4y_2$

$y_1'' = -y_1 + y_2$

$y_1' = 3y_1 - y_2$

$y_1' + y_2 = 3y_1$

$y_2 = 3y_1 - y_1'$

$y_1'' = -y_1 + 3y_1 - y_1'$

$y_1'' = 2y_1 - y_1'$

$y_1'' + y_1' - 2y_1 = 0$

$r^2 + r - 2 = 0$

$-1 + 2$

$(r-1)(r+2) = 0 \Rightarrow r_1 = 1, r_2 = -2$

Suntem în cazul I, a)

$\left.\begin{matrix} r_1 \neq r_2 \\ r_1, r_2 \in \mathbb{R} \\ \Delta > 0 \end{matrix}\right| \Rightarrow \begin{matrix} \Phi_1(x) = e^{r_1 x} \\ \Phi_2(x) = e^{r_2 x} \end{matrix}$

$y_1(x) = c_1 \cdot \Phi_1 + c_2 \cdot \Phi_2 \quad c_1, c_2 \in \mathbb{R}$

$y_1(x) = c_1 \cdot e^x + c_2 \cdot e^{-2x}$

$y_1'(x) = c_1 \cdot e^x - 2c_2 \cdot e^{-2x} \quad c_1, c_2 \in \mathbb{R}$

$y_2(x) = 3y_1 - y_1' \Rightarrow y_2(x) = 3c_1 \cdot e^x + 3c_2 \cdot e^{-2x} - c_1 \cdot e^x + 2c_2 \cdot e^{-2x}$

$y_2(x) = 2c_1 \cdot e^x + 5c_2 \cdot e^{-2x} \quad c_1, c_2 \in \mathbb{R}$

$$\begin{cases} y_1(x) = c_1 \cdot e^x + c_2 \cdot e^{-2x} \\ y_2(x) = 2c_1 \cdot e^x + 5c_2 \cdot e^{-2x} \end{cases} \quad c_1, c_2 \in \mathbb{R}$$

Problema Cauchy

$y_1(0) = 1 \Rightarrow c_1 + c_2 = 1 \quad | \cdot (-2) \Leftrightarrow \begin{cases} -2c_1 - 2c_2 = -2 \\ 2c_1 + 5c_2 = 5 \quad (+) \end{cases}$

$y_2(0) = 5 \Rightarrow 2c_1 + 5c_2 = 5$

$/ \quad 3c_2 = 3 \Rightarrow c_2 = 1 \Rightarrow c_1 = 0$

$$\begin{cases} y_1(x) = e^{-2x} \\ y_2(x) = 5e^{-2x} \end{cases} \quad c_1, c_2 \in \mathbb{R}$$

(*)

(*)

c) $\begin{cases} xy' + y = e^x \\ y(a) = b \end{cases}$ $a, b \in \mathbb{R}$

I) Ec. omogenă

$xy' + y = 0$

$xy' = -y$

$y' = -\frac{y}{x}$

$\frac{dy}{dx} = -\frac{y}{x} \Rightarrow \frac{dy}{y} = -\frac{dx}{x} \quad \Big| \int$

$\int \frac{dy}{y} = -\int \frac{dx}{x}$

$c_1 = \ln c$

$\ln|y| = -\ln|x| + c_1$

$\ln|y| = \ln \frac{c}{x}$

$y_0 = \frac{c}{x}, \quad c \in \mathbb{R}$

II Ec. particulară (met. variației constantelor)

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$$y_p = \frac{\varphi}{x}$$

$$y_p' = \frac{\varphi' \cdot x - \varphi \cdot x'}{x^2} = \frac{\varphi' \cdot x - \varphi}{x^2}$$

- Ne întoarcem la * și înlocuim:

$$x \cdot \frac{\varphi' x - \varphi}{x^2} + \frac{\varphi}{x} = x e^x$$

$$\varphi' x - \varphi + \varphi = x \cdot e^x$$

$$\varphi' \cdot x = x \cdot e^x$$

$$\varphi' = e^x \quad \Big| \int$$

$= \ln c_2$

$$\varphi = e^x \Rightarrow y_p = \frac{e^x}{x}$$

$= c$

$$y = y_0 + y_p = \frac{c}{x} + \frac{e^x}{x} \qquad c \in \mathbb{R}$$

Problema Cauchy

$$y(a) = b \Rightarrow \frac{c}{a} + \frac{e^a}{a} = b$$

$$c + e^a = ab$$

$$c = ab - e^a$$

$$y(x) = \frac{e^x + ab - e^a}{x} \qquad a, b \in \mathbb{R}$$