CURS 7

Sisteme dinamice generate de ecuații diferențiale autonome

$x = x(t)$ fct. nec.

$x' = f(t, x)$ ec. dif. neautonomă (ex. $x' = t^2 x + x^3$)

$\boxed{x' = f(x)}$ ec. dif. autonomă (ex.: $x' = x - x^3$)

1. Flux. Portret fazic

(1) $\quad x' = f(x)$.

Teoremă. Dacă $f \in C^1(\mathbb{R})$ atunci probl. Cauchy:

$$(2) \quad \begin{cases} x' = f(x) \\ x(0) = \eta \end{cases}$$

are o unică soluție saturată pt. $\forall \eta \in \mathbb{R}$.

soluție saturată $\Leftrightarrow$ soluția definită pe cel mai mare interval posibil.

I_η – interval maximal $\Rightarrow$ interval deschis.

$x(t,\eta)$ sol. probl. Cauchy (2)

$\eta \in \mathbb{R}$, $x(\cdot,\eta) : I_\eta \to \mathbb{R}$

$$I_\eta = (\alpha_\eta, \beta_\eta)$$

$$0 \in I_\eta \Rightarrow \alpha_\eta < 0 < \beta_\eta$$

$W = \{ I_\eta \times \{\eta\} \mid \eta \in \mathbb{R} \}$.

$\varphi : W \to \mathbb{R}$

$\varphi(t,\eta) = x(t,\eta)$ fluxul generat de ec. dif. (1).

operatorul $\eta \mapsto \varphi(t,\eta)$ se numește sistemul dinamic generat de ec. dif. (1).

Proprietăți

1. $\varphi(0,\eta)=\eta$, $\forall \eta\in\mathbb{R}$
2. $\varphi(t+s,\eta)=\varphi(t,\varphi(s,\eta))$, $\forall t,s\in I_\eta$, $\forall \eta\in\mathbb{R}$.
3. φ este continuă.

Def. $\eta\in\mathbb{R}$.

$$\gamma^+(\eta)=\bigcup_{t\in[0,\beta_\eta)}\varphi(t,\eta)$$ orbita pozitivă a lui η

$$\gamma^-(\eta)=\bigcup_{t\in(\alpha_\eta,0]}\varphi(t,\eta)$$ orbita negativă a lui η.

$$\gamma(\eta)=\gamma^+(\eta)\cup\gamma^-(\eta)$$ orbita lui η.

Def. Reuniunea tuturor orbitelor împreună cu sensul de parcurgere al acestora se numește portret fazic.

Exemple.

1) $x' = -x \implies \boxed{x(t) = c\cdot e^{-t},\ c \in \mathbb{R}}$ sol. gen. a ec.

fluxul: $\begin{cases} x' = -x \\ x(0) = \eta,\ \eta \in \mathbb{R}. \end{cases}$

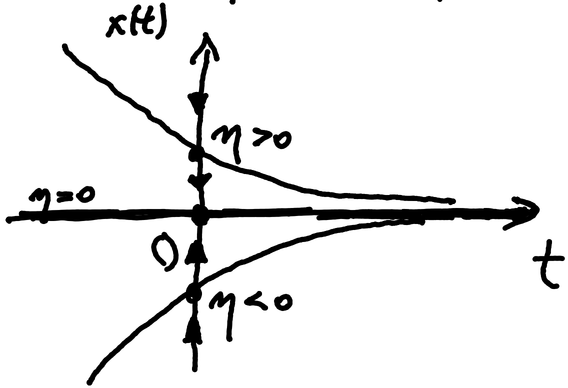
$x(0) = \eta \implies c = \eta \implies x(t,\eta) = \eta\cdot e^{-t}$

$I_\eta = \mathbb{R},\ \forall \eta \in \mathbb{R}.$

$W = \{ I_\eta \times \{\eta\} \mid \eta \in \mathbb{R} \} = \mathbb{R}\times\mathbb{R}$

$\varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$

$\varphi(t,\eta) = x(t,\eta) = \eta\cdot e^{-t}$ fluxul generat de ecuație.



1. $\eta = 0 \Rightarrow \varphi(t,0) \equiv 0$

$$\gamma(0) = \bigcup_{t\in\mathbb{R}} \varphi(t,0) = \bigcup_{t\in\mathbb{R}} \{0\} = \{0\}$$

2. $\eta > 0 \Rightarrow \varphi(t,\eta) = \eta\cdot e^{-t}$

$$\gamma^+(\eta) = \bigcup_{t\in[0,+\infty)} \varphi(t,\eta) = \bigcup_{t\in[0,+\infty)} (\eta\cdot e^{-t}) = (0,\eta]$$

$$\gamma^-(\eta) = \bigcup_{t\in(-\infty,0]} \varphi(t,\eta) = [\eta,+\infty)$$

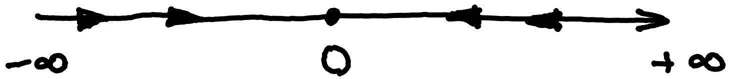
$$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta) = (0,+\infty)$$

3. $\eta < 0 \Rightarrow \varphi(t,\eta) = \eta e^{-t}$

$$\gamma^+(\eta) = \bigcup_{t\in[0,+\infty)} \varphi(t,\eta) = [\eta,0)$$

$$\gamma^-(\eta) = \bigcup_{t\in(-\infty,0]} \varphi(t,\eta) = [-\infty,\eta]$$

$$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta) = (-\infty,0)$$



portretul fazic

2) $x' = x \Rightarrow \boxed{x(t) = c \cdot e^t,\ c \in \mathbb{R}}$ sol. gen.

fluxul : $\begin{cases} x' = x \\ x(0) = \eta \end{cases}, \eta \in \mathbb{R}$

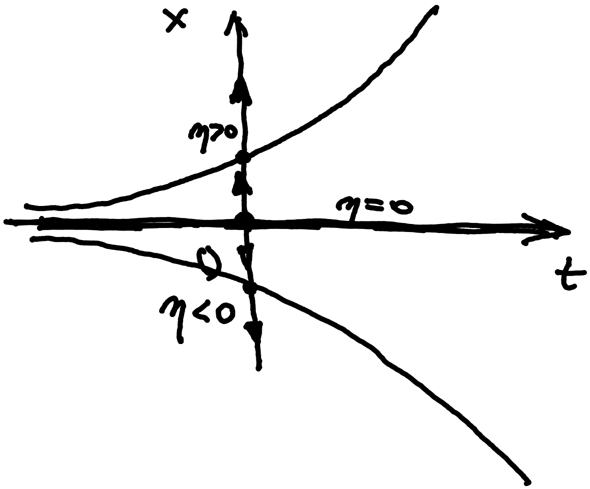
$x(0) = \eta \Rightarrow c = \eta \Rightarrow x(t, \eta) = \eta \cdot e^t$

$I_\eta = \mathbb{R},\ \forall \eta \in \mathbb{R} \Rightarrow W = \mathbb{R} \times \mathbb{R}$

$\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$\varphi(t, \eta) = x(t, \eta) = \eta \cdot e^t$

fluxul generat de ecuație



1. $\eta = 0 \Rightarrow \varphi(t,0) \equiv 0$

$$\gamma(0) = \bigcup_{t\in\mathbb{R}} \varphi(t,0) = \bigcup_{t\in\mathbb{R}} \{0\} = \{0\}.$$

2. $\eta > 0 \Rightarrow \varphi(t,\eta) = \eta \cdot e^t$

$$\gamma^+(\eta) = \bigcup_{t\in[0,+\infty)} \varphi(t,\eta) = \underrightarrow{[\eta,+\infty)}$$

$$\gamma^-(\eta) = \bigcup_{t\in[-\infty,0]} \varphi(t,\eta) = \underrightarrow{(0,\eta]}$$

$$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta) = \underrightarrow{(0,+\infty)}$$

3. $\eta < 0 \Rightarrow \varphi(t,\eta) = \underline{\eta}\, e^t$

$$\gamma^+(\eta) = \bigcup_{t\in[0,+\infty)} \varphi(t,\eta) = \underleftarrow{(-\infty,\eta]}$$

$$\gamma^-(\eta) = \bigcup_{t\in(-\infty,0]} \varphi(t,\eta) = \underleftarrow{[\eta,0)}$$

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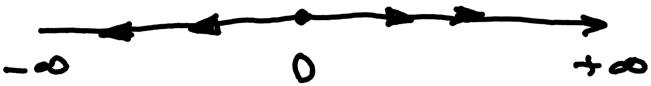


portretul fazic

$x' = f(x)$ portretul fazic poate fi determinat pe baza semnului funcției f.

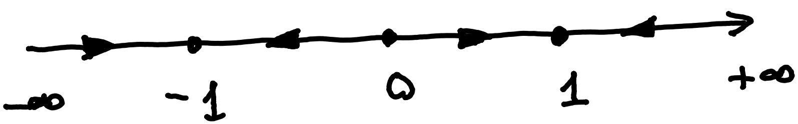
$f(x) = 0 \Rightarrow x_1, x_2, \dots, x_n \in \mathbb{R}$ rădăcini

x	$-\infty$		x_1		x_2		x_3	$\dots$	x_n	$+\infty$
$f(x)$		$+$	0	$-$	0	$+$	0	$\dots$	0	
		$\rightarrow$		$\leftarrow$		$\rightarrow$		$\dots$		

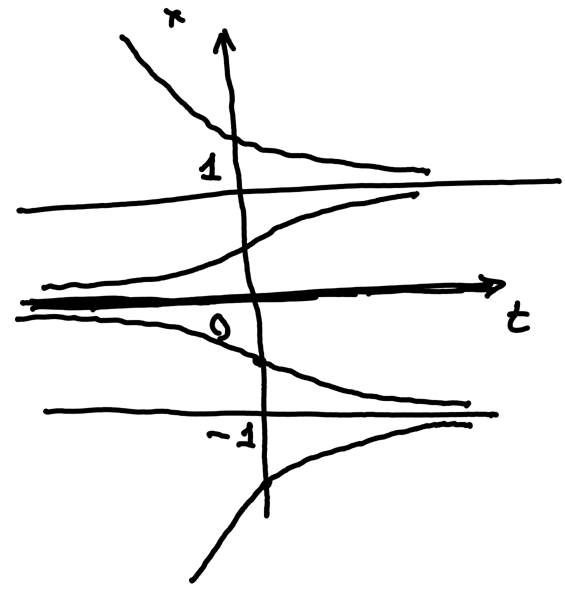
3) $x' = x - x^3$ $\qquad f(x) = x - x^3$

$$f(x) = 0 \Rightarrow x - x^3 = 0$$
$$x(1 - x^2) = 0$$
$$x_1 = 0, \quad x_{2,3} = \pm 1$$

x	$-\infty$		-1		0		1		$+\infty$
$f(x)$		$+$	0	$-$	0	$+$	0	$-$	
		$\rightarrow$		$\leftarrow$		$\rightarrow$		$\leftarrow$	



portretul fazic



2. Puncte de echilibru. Stabilitate

(1) $\quad x' = f(x)$

Def. Soluțiile constante ale ec. (1) se numesc soluții de echilibru (staționare).

$x(t) \equiv x^*$ sol. de echil.

$x^* \in \mathbb{R}$. — punct de echilibru.

Punctele de echilibru sunt soluțiile reale ale ec.

$$\boxed{f(x) = 0}$$

Def. Fie $x^* \in \mathbb{R}$ un pct de echil. pt (1).

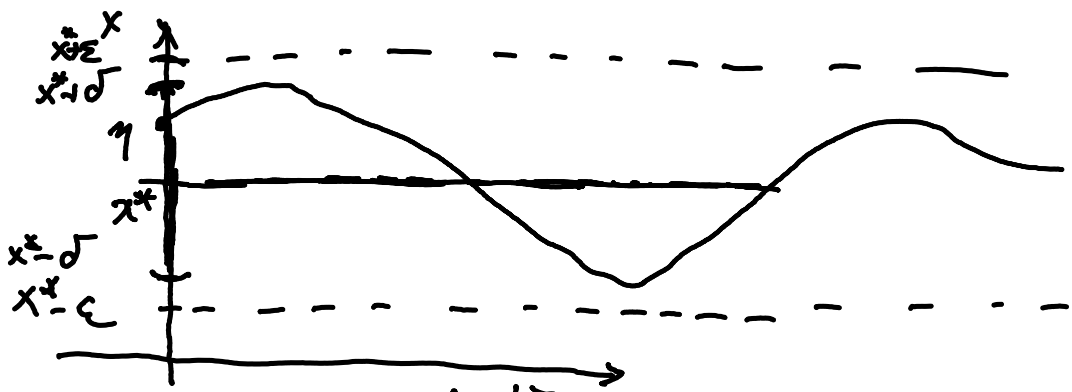
Spunem că x^* este:

a) local stabil dacă $\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0$ a.î. dacă $|\eta - x^*| < \delta$ atunci $|\varphi(t,\eta) - x^*| < \varepsilon, \forall t \geq 0$, unde $\varphi(t,\eta)$ fluxul generat de (1).

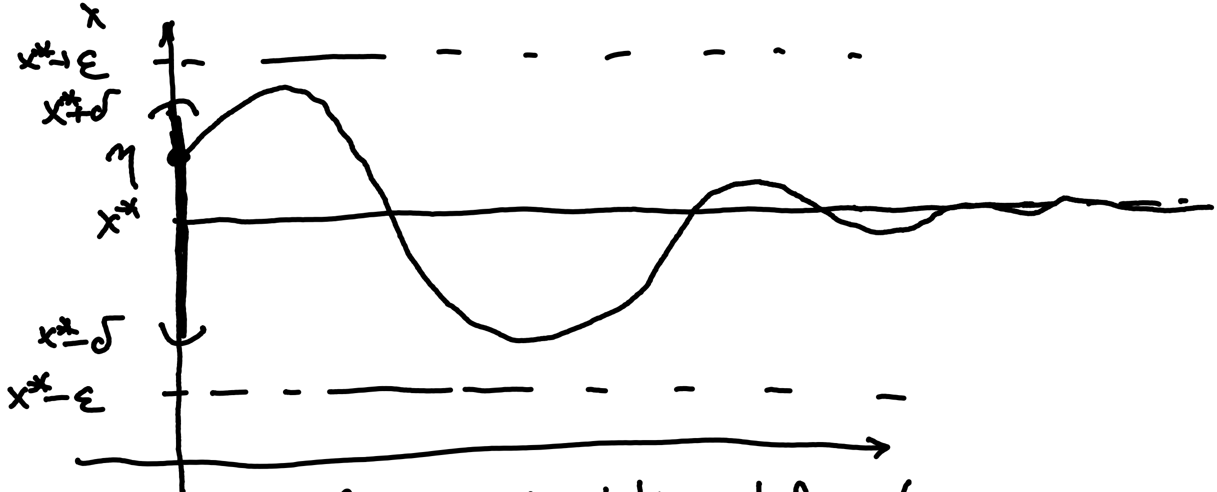
b) local asimptotic stabil dacă este local stabil și în plus.

$$|\varphi(t,\eta) - x^*| \xrightarrow[t \to +\infty]{} 0$$

c) instabil dacă nu este local stabil.



stabilitate locală



local asimptotic stab.

Exemple:

1) $x' = -x$ $\quad$ $f(x) = -x$

$f(x) = 0 \Rightarrow x^* = 0$ pct de echil.

$$\varphi(t,\eta) = \eta \cdot e^{-t}$$

$$|\varphi(t,\eta) - \underbrace{x^*}_{=0}| = |\eta \cdot e^{-t}| = |\eta| \cdot e^{-t} = |\eta - x^*| \cdot \underbrace{e^{-t}}_{\leq 1,\, t \geq 0} \leq$$

$$\leq |\eta - x^*|$$

fie $\varepsilon>0$ alegem $\delta=\varepsilon$ $\Rightarrow$

$$|\varphi(t,\eta)-x^*| \leq |\eta - x^*| < \delta = \varepsilon$$

$\Rightarrow x^*=0$ este local stabil.

mai mult $\quad |\varphi(t,\eta)-x^*| = |\eta|\cdot \underbrace{e^{-t}}_{\substack{\downarrow t\to+\infty \\ 0}} \xrightarrow[t\to\infty]{} 0$

$\Rightarrow x^*=0$ este asimptotic stabil.

asimpt. stab.

portretul fazic

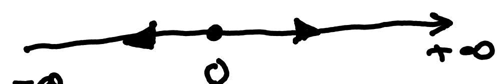
2) $x'=x \qquad f(x)=x$

$f(x)=0 \Rightarrow x^*=0$ pct. de echil.

$$\varphi(t,\eta)=\eta\cdot e^t$$

$$|\varphi(t,\eta)-x^*| = |\eta\cdot e^t| = |\eta|\cdot e^t = |\eta - x^*|\cdot \underbrace{e^t}_{\substack{\downarrow t\to+\infty \\ +\infty}}$$

$\to x^*=0$ instabil.



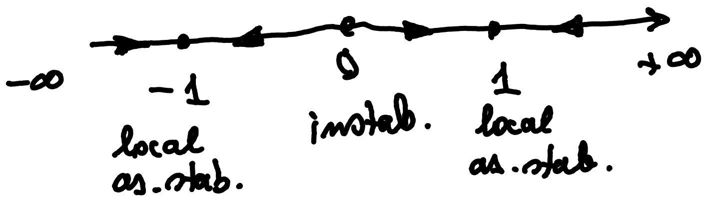
instabil.

3) $x' = x - x^3$

$f(x) = x - x^3$

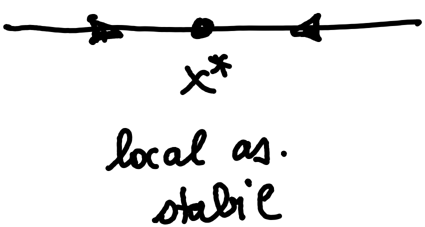
$f(x) = 0 \Rightarrow x - x^3 = 0 \rightarrow x(1 - x^2) = 0$

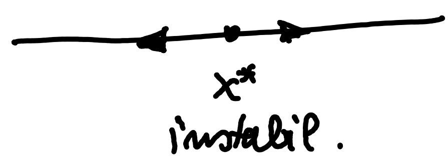
$x_1^* = 0$, $x_{2,3}^* = \pm 1$ pct. de echil.



x		-1		0		1	
$f(x)$	$+$	0	$-$	0	$+$	0	$-$
	$\rightarrow$		$\leftarrow$		$\rightarrow$		$\leftarrow$

In general: $x^* \in \mathbb{R}$ pct. de echil. pt $x' = f(x)$:





Teorema stabilității în prima aproximație

$x^* \in \mathbb{R}$ un pct de echilibru pt ec. (1) și $f \in C^1$

a) Dacă $f'(x^*) < 0 \Rightarrow x^*$ este local as. stabil

b) Dacă $f'(x^*) > 0 \Rightarrow x^*$ este instabil.

Exemplu

$$x' = x - x^3$$

$$f(x) = x - x^3, \quad f'(x) = 1 - 3x^2$$

$x_1^* = 0$, $x_{2,3}^* = \pm 1$ pct. de echil.

$x_1^* = 0$: $\quad f'(0) = 1 > 0 \Rightarrow x_1^* = 0$ instab.

$x_{2,3}^* = \pm 1$: $\quad f'(\pm 1) = 1 - 3 = -2 < 0 \Rightarrow x_{2,3}^* = \pm 1$ local as. stabile.