Temă Sisteme dinamice

Seminar 1

1. Ecuații cu variabile separate

4. $xy + (2x-1)y' = 0$

$xy = -(2x-1)y'$

$y' = -\frac{xy}{2x-1}$ $\qquad 2x-1 \neq 0 \Rightarrow x \neq \frac{1}{2}$

$f(x) = -\frac{x}{2x-1}$, $f: \mathbb{R} \setminus \{\frac{1}{2}\} \to \mathbb{R}$

$g(y) = y$, $f: \mathbb{R} \to \mathbb{R}$

- Soluția singulară este $y = 0$
- $y \neq 0 \Rightarrow y' = -\frac{xy}{2x-1}$

Știm $y' = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{xy}{2x-1} \Big/ \cdot \frac{dx}{y}$

$\Rightarrow \frac{dy}{y} = -\frac{x}{2x-1} \cdot dx \Big/ \int$

$\int \frac{dy}{y} = -\int \frac{x}{2x-1} dx$

$\ln|y| = -\frac{1}{2} \int \frac{2x-1+1}{2x-1} dx$

$\ln|y| = -\frac{1}{2} \int \cancel{\frac{2x-1}{2x-1}} dx - \frac{1}{2} \int \frac{1}{2x-1} dx$

sol. în formulă implicită $\quad \ln|y| = -\frac{1}{2}x - \frac{1}{4}\ln|2x-1| + c$

$\Rightarrow \cancel{y = e^{-\frac{1}{2}x - \frac{1}{4}\ln|2x-1| + c}}$, $c \in \mathbb{R}$

Înlocuim $c \to \ln c$ $\qquad$ soluție explicită

$\ln|y| = \ln e^{-\frac{1}{2}x} - \frac{1}{4}\ln|2x-1| + \ln c \Rightarrow y = e^{-\frac{1}{2}x} \cdot |2x-1|^{-\frac{1}{4}} \cdot c$

(5.) $y' = K \cdot \frac{y}{x}$, $K \in \mathbb{R}^*$

$f(x) = \frac{K}{x}$, $f : \mathbb{R}^* \longrightarrow \mathbb{R}$, $K \in \mathbb{R}^*$

$g(y) = y$, $f : \mathbb{R} \longrightarrow \mathbb{R}$

- Soluția singulară este $y = 0$
- $y \neq 0 \Rightarrow y' = K \cdot \frac{y}{x}$

Știm $y' = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = K \cdot \frac{y}{x} \;\Big|\; \cdot \frac{dx}{y}$

$\frac{dy}{y} = K \cdot \frac{dx}{x} \;\Big|\; \int$

$\int \frac{dy}{y} = \int \frac{K\,dx}{x}$

$\ln|y| = K \cdot \ln|x| + C$ soluție implicită

$y = e^{K \cdot \ln|x| + C}$, $C \in \mathbb{R}$

Sau $y = e \cdot x^{K}$ soluție explicită

(6.) $y - xy' = a(1 + x^2 y')$, $a \in \mathbb{R}^*$

$y - xy' = a + ax^2 y'$

$y - a = ax^2 y' + xy'$

$y - a = y'(ax^2 + x)$

$\frac{y-a}{ax^2+x} = y'$

$ax^2 + x \neq 0$

$x(ax+1) \neq 0$

$x \neq 0$

$x \neq -\frac{1}{a}$

$f(x) = \frac{1}{ax^2+x}$

$f : \mathbb{R} \setminus \{0, -\frac{1}{a}\} \longrightarrow \mathbb{R}$

$g(y) = y - a$

$g : \mathbb{R} \longrightarrow \mathbb{R}$

- Soluția singulară este $y = \{a\}$
- $y \neq \;\Rightarrow y' = \frac{y-a}{ax^2+x}$

Știm că $y' = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y-a}{ax^2+x} \;\Big|\; \cdot \frac{dx}{y-a}$

$$\frac{dy}{y-a} = \frac{dx}{ax^2+x} \quad \Big| \int$$

$$\int \frac{dy}{y-a} = \int \frac{dx}{ax^2+x}$$

$$\ln|y-a| = \int \frac{dx}{x(ax+1)}$$

$$\ln|y-a| = \int \frac{1+ax-ax}{x(ax+1)}\,dx$$

$$\ln|y-a| = \int \frac{ax+1}{x\cancel{(ax+1)}}\,dx - \int \frac{ax}{\cancel{x}(ax+1)}\,dx$$

$$\ln|y-a| = \int \frac{1}{x}\,dx - \int \frac{a}{ax+1}\,dx$$

$$\ln|y-a| = \ln|x| - \ln|ax+1| + c \quad \text{soluție implicită}$$

$$\ln|y-a| = \ln|x| - \ln|ax+1| + \ln c_1 \qquad c = \ln c_1,\ c_1 > 0$$

$$\ln|y-a| = \ln\left|\frac{c_1 x}{ax+1}\right| \Rightarrow y = \frac{c_1 x}{ax+1} \Rightarrow y = \frac{cx}{ax+1};\ a, c \in \mathbb{R}$$

$a \in \mathbb{R}^*$ soluție explicită

(2) Ecuații omogene în sens Euler

(4) $xy' = \sqrt{x^2-y^2} + y$

$$y' = \frac{\sqrt{x^2-y^2}}{x} + \frac{y}{x}$$

$$y' = \sqrt{\frac{x^2-y^2}{x^2}} + \frac{y}{x}$$

$$y' = \sqrt{1-\frac{y^2}{x^2}} + \frac{y}{x}$$

$$y' = \sqrt{1-\left(\frac{y}{x}\right)^2} + \frac{y}{x}$$

$z = \frac{y}{x} \Rightarrow$ ~~$y' = \sqrt{1-z^2} + z$~~

$y = z \cdot x$

$y' = z'x + z$

$\boxed{z = \frac{y}{x}}$

⊙ $z \neq \pm 1$:

$$z'x + \cancel{z} = \sqrt{1-z^2} + \cancel{z}$$

$$z' = \frac{\sqrt{1-z^2}}{x}$$

$$\frac{dz}{dx} = \frac{\sqrt{1-z^2}}{x}$$

$$\frac{dz}{\sqrt{1-z^2}} = \frac{dx}{x} \quad \Big| \int$$

$$\int \frac{dz}{\sqrt{1-z^2}} = \int \frac{dx}{x}$$

$\arcsin z = \ln|x| + c$ sol generală

⊙ ~~$z = \pm 1$~~ Soluția singulară este $z = \pm 1$

$$z = \sin(\ln|x| + c)$$

$$y = x \cdot \sin(\ln|x| + c) \qquad c \in \mathbb{R}$$

Scanned with CamScanner

(5) $y' = \frac{y}{x} + tg\frac{y}{x}$

$z = \frac{y}{x} \Rightarrow y = zx$

$y' = z'x + z$

$z'x + \cancel{z} = \cancel{z} + tg\, z$

$z' = \frac{tg\, z}{x}$

$\frac{dz}{dx} = \frac{tg\, z}{x}$

$\frac{dz}{tg\, z} = \frac{dx}{x} \Big| \int$

$\int \frac{dz}{tg\, z} = \int \frac{dx}{x}$

$\int ctg\, z\, dz = \int \frac{dx}{x}$

$\ln|\sin z| = \ln|x| + c, \; c \in \mathbb{R}$

$\ln|\sin z| = \ln|x| + \ln c_1 \quad c = \ln c_1, \; c_1 > 0$

$\ln|\sin z| = \ln x c_1$

$\sin z = x c_1$ ~~sol generala~~

$\sin z = x c, \quad c \in \mathbb{R}$

$z = \arcsin(x \cdot c)$

$\boxed{z = \frac{y}{x}}$

$y = x \cdot \arcsin(x \cdot c)$ sol. gen.

(6.) $x - y\cos\frac{y}{x} + x\cos\left(\frac{y}{x}\right) y' = 0$

$x - y\cos\frac{y}{x} = -x\cos\left(\frac{y}{x}\right) \cdot y' \Big| \cdot \left(-\frac{1}{x}\right)$

$-1 + \frac{y}{x}\cos\frac{y}{x} = \cos\left(\frac{y}{x}\right) \cdot y'$

$-\frac{1}{\cos\frac{y}{x}} + \frac{y}{x} = y' \Rightarrow -\frac{1}{\cos z} + \cancel{z} = z'x + \cancel{z}$

$\frac{y}{x} = z \Rightarrow y = zx$

$y' = z'x + z$

$z' = -\frac{1}{\cos z \cdot x}$

$\frac{dz}{dx} = -\frac{1}{x\cos z} \Big| \cdot \cos z \cdot dx$

$\cos z\, dz = -\frac{dx}{x} \Big| \int$

$\int \cos z\, dz = -\int \frac{dx}{x}$

$\sin z = -\ln|x| + c, \; c \in \mathbb{R}$

$\sin z = -\ln|x| + \ln c_1 \quad c = \ln c_1$

sol. generală

$\sin z = \ln\frac{c_1}{x}$

6. $y' + \frac{x}{1-x^2}\, y = x + \arcsin x \quad (*)$

I Omogenă

$$y' + y\cdot\frac{x}{1-x^2} = 0$$

$$y' = -y\cdot\frac{x}{1-x^2}$$

$$\frac{dy}{dx} = -y\cdot\frac{x}{1-x^2} \Rightarrow \frac{dy}{y} = -\frac{x}{1-x^2}\cdot dx \;\Big|\int$$

$$\int\frac{dy}{y} = \int\frac{-x}{1-x^2}\,dx$$

$c = \ln c_1$
$c_1 > 0$

$$\ln|y| = \frac{1}{2}\ln|1-x^2| + c$$

$$\ln y = \frac{1}{2}\ln|1-x^2| + \ln c_1$$

$$\ln|y| = \ln|\sqrt{1-x^2}\cdot c_1|$$

$$y_0 = c\cdot\sqrt{1-x^2}$$

II Particulară

$$y_p = f\cdot\sqrt{1-x^2}$$

$$y_p' = f'\cdot\sqrt{1-x^2} + f\cdot\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} = f'\cdot\sqrt{1-x^2} - f\frac{x}{\sqrt{1-x^2}}$$

Ne întoarcem la (*) și înlocuim:

$$\cancel{f'\sqrt{1-x^2} - f\frac{x}{\sqrt{1-x^2}}\cdot\frac{x}{1-x^2} = x+\arcsin x}$$

$$f'\sqrt{1-x^2} - \cancel{f\frac{x}{\sqrt{1-x^2}}} + \cancel{f\cdot\sqrt{1-x^2}\cdot\frac{x}{1-x^2}} = x + \arcsin x$$

$$f'\sqrt{1-x^2} = x + \arcsin x$$

$$f' = \frac{x}{\sqrt{1-x^2}} + \frac{\arcsin x}{\sqrt{1-x^2}} \;\Big|\int$$

$$f = -\sqrt{1-x^2} + \arcsin x$$

$$y_p = \left(-\sqrt{1-x^2} + \arcsin x\right) \cdot \sqrt{1-x^2}$$

$$y_p = (x^2-1) + (\arcsin x)\sqrt{1-x^2}$$

$$\boxed{y = y_0 + y_p} = c \cdot \sqrt{1-x^2} + (x^2-1) + (\arcsin x) \cdot \sqrt{1-x^2}$$

Temă Seminar 2

10.03.2021

Sisteme dinamice

1. Ecuații de forma $y''(x)=f(x)$

3. $y''=\frac{1}{x} \quad \Big| \int$

$y'=\int \frac{1}{x}\,dx$

$y'=\ln x + c_1 \quad \Big| \int \qquad c_1 \in \mathbb{R}$

$y=\int(\ln x + c_1)\,dx$

$y=\int (x)'\ln x\,dx + c_1 x$

$y=x\ln x-\int x\cdot\frac{1}{x}\,dx + c_1 x$

$y=x\ln x - x + c_1 x + c_2$

$c_1, c_2 \in \mathbb{R}$

4. $y''=\ln x \quad \Big| \int$

$y'=\int \ln x\,dx$

$y' \overset{(3)}{=} x\ln x - x + c_1 \quad \Big| \int$

$y=\int(x\ln x - x + c_1)\,dx$

$y=\int\left(\frac{x^2}{2}\right)'\ln x\,dx - \frac{x^2}{2} + c_1 x$

$y=\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx - \frac{x^2}{2} + c_1 x$

$y=\frac{x^2}{2}\ln x - \frac{x^2}{4} - \frac{x^2}{2} + c_1 x + c_2$

$y=\frac{x^2}{2}\ln x - \frac{3x^2}{4} + c_1 x + c_2, \quad c_1, c_2 \in \mathbb{R}$

5. $y'' = x \cdot e^x \quad \Big|\Big| \int$

$y' = \int x \cdot e^x \, dx$

$y' = \int x \cdot (e^x)' \, dx$

$y' = xe^x - \int e^x \, dx$

$y' = xe^x - e^x + C_1 \quad \Big|\Big| \int \quad C_1 \in \mathbb{R}$

$y = \int (xe^x - e^x + C_1) \, dx$

$y = xe^x - e^x - e^x + C_1 x + C_2$

$y = xe^x - 2e^x + C_1 x + C_2, \quad C_1, C_2 \in \mathbb{R}$

6. $y'' = \frac{2x^2}{(1+x^2)^2} \quad \Big|\Big| \int$

$y' = 2\int \frac{x^2+1-1}{(1+x^2)^2} \, dx$

$y' = 2\int \frac{\cancel{1+x^2}}{(1+x^2)^{\cancel{2}}} \, dx - 2\int \frac{1}{(1+x^2)^2} \, dx$

$\cancel{y' = 2 \operatorname{arctg} x - 2\int \frac{1}{1+2x^2+x^4} \, dx}$

$y' = 2\operatorname{arctg} x - \frac{x}{x^2+1} - \int \frac{1}{x^2+1} \, dx$

$y' = 2\operatorname{arctg} x - \frac{x}{x^2+1} - \operatorname{arctg} x + C_1$

$y' = \operatorname{arctg} x - \frac{x}{x^2+1} + C_1 \quad \Big|\Big| \int \quad C_1 \in \mathbb{R}$

$y = \int \left(\operatorname{arctg} x - \frac{x}{x^2+1} + C_1\right) dx$

$y = \int \operatorname{arctg} x \, dx - \frac{1}{2}\int \frac{2x}{x^2+1} \, dx + C_1 x$

$$y = x \operatorname{arctg} x - \int \frac{x}{x^2+1}\,dx - \int \frac{x}{x^2+1}\,dx + C_1 x$$

$$y = x \operatorname{arctg} x - \int \frac{2x}{x^2+1}\,dx + C_1 x$$

$$y = x \operatorname{arctg} x - \ln|x^2+1| + C_1 x + C_2 \qquad C_1, C_2 \in \mathbb{R}$$

2. Ecuatii de forma $y''(x) = f(x, y')$

3. $y'' - 2y' = -x^2 \Rightarrow z + x\cdot z' - 2z = -x^2$

$y' = z$ $\qquad x\cdot z' = \cancel{2}z - x^2$

$y'' = z + x\cdot z'$ $\qquad x\cdot z' - z = -x^2 \,\Big|\, \frac{1}{x^2}$

$$\frac{z'}{x} - \frac{z}{x^2} = -1 \qquad \text{Ec. neomogenă}$$

I Ec. omogenă

$$\frac{z'}{x} - \frac{z}{x^2} = 0$$

$$\frac{z'}{x} = \frac{z}{x^2} \;\Big|\; \cdot x$$

$$z' = \frac{z}{x} \Rightarrow \frac{dz}{dx} = \frac{z}{x} \Rightarrow \frac{dz}{z} = \frac{dx}{x} \;\Big|\; \int$$

$$z' = \frac{dz}{dx}$$

$$\int \frac{dz}{z} = \int \frac{dx}{x} \qquad C_1 \in \mathbb{R}$$

$$\ln|z| = \ln|x| + \ln|C_1|$$

$$\ln|z| = \ln|x\cdot C_1|$$

$$z_0 = x\cdot C_2, \qquad C_2 \in \mathbb{R}$$

Scanned with CamScanner

II Ec. particulară

$z_p = \frac{f}{\cancel{x}} \cdot x$ (met. variației const.)

$z_p' = f' \cdot x + f$

$$\frac{f'x + f}{x} - \frac{f \cdot \cancel{x}}{x^{\cancel{2}}} = -1$$

$$\frac{f'x + f - f}{x} = -1$$

$f' = -1 \Rightarrow f = -x \Rightarrow z_p = -x^2$

Sol. generală a ec. liniare neomogene în x este:

$z = z_0 + z_p = x \cdot c - x^2$

Revin la substituția $y' = z$

$y' = x \cdot c - x^2 \; / \int$

$y = \int (x \cdot c - x^2)\, dx$

$y = c \cdot \frac{x^2}{2} \cdot \cancel{\ } - \frac{x^3}{3} + C$

4. $(1 + x^2)\, y'' + y' \cdot x^2 + 1 = 0 \Rightarrow (1 + x^2)\,(x \cdot z' + z) + z \cdot x^2 + 1 = 0$

$y' = z$ $\qquad x \cdot z' + z + x^3 \cdot z' + x^2 \cdot z + x^2 \cdot z + 1 = 0$

$y'' = x \cdot z' + z$ $\qquad z'(x + x^3) + 2x^2 \cdot z + z + 1 = 0$

$z'(x + x^3) + z(2x^2 + 1) = -1$ Ec. neomogenă

Scanned with CamScanner