Seminar 1: Ecuații diferențiale de ordinul întâi

În acest seminar vom studia ec. dif. de ordinul întâi în formă normală rezolvabile efectiv.

$$(1) \quad y'(x) = f(x, y(x))$$

1. Ecuații cu variabile separabile

Au următoarea formă:

$$(2) \quad \boxed{y'(x) = f(x)\cdot g(y(x))}$$

, $f \in C(I)$
$g \in C(J, \mathbb{R}^*)$, $J \subset \mathbb{R}$
sunt continue.

Fie y o sol. a ec. (2) și

$$f : (x_1; x_2) \to \mathbb{R}$$
$$g : (y_1; y_2) \to \mathbb{R}^*$$

Din (2) obținem:

$$\frac{y'}{g(y)} = f(x)$$

Știm că $\boxed{y' = \frac{dy}{dx}}$ deducem $\frac{dy}{g(y)} = f(x)\,dx$

Fie $x_0 \in (x_1; x_2)$ și $y_0 = y(x_0)$

Integrăm rel. de mai sus

$$(3) \quad \int_{y_0}^{y} \frac{dt}{g(t)} = \int_{x_0}^{x} f(s)\,ds$$

Considerăm $G(\gamma) = \int_{y_0}^{\gamma} \frac{dt}{g(t)}$ derivabilă și strict monotonă dată de semnul funcției g

$\Rightarrow \exists$ inversa G^{-1} $\overset{\text{Dim}}{\Rightarrow}$ (3) obținem

$$G(y) = \int_{x_0}^{x} f(s)\,ds \Rightarrow y(x) = G^{-1}\left(\int_{x_0}^{x} f(s)\,ds\right)$$

Ex 1: 1. $y' = 2x(1+y^2)$

$f(x) = 2x$, $f, g : \mathbb{R} \to \mathbb{R}$

$g(y) = 1+y^2 > 0$

$$\frac{y'}{1+y^2} = 2x \qquad \frac{dy}{dx}\cdot\frac{1}{1+y^2} = 2x \,\Big|\, dx$$

$$\frac{dy}{1+y^2} = 2x\,dx$$

$$\frac{y'}{1+y^2} = 2x$$

Știm $\boxed{y' = \frac{dy}{dx}}$ $\Rightarrow \frac{dy}{1+y^2} = 2x\,dx \;\Big|\int$

$$\Rightarrow \int \frac{dy}{1+y^2} = \int 2x\,dx$$

$$\Rightarrow \operatorname{arctg}(y) = x^2 + c \quad \Big|\, tg \quad , c \in \mathbb{R}$$

$$\Rightarrow \quad y = tg\,(x^2 + c) \quad , c \in \mathbb{R}$$

Obs! Orice funcție de forma de mai sus definită pe un interval este soluție a ecuației.

2. $(x^2-1)\,y' + 2xy^2 = 0$

$$y' = -\frac{2x}{x^2-1}\cdot y^2$$

$f(x) = -\frac{2x}{x^2-1}$, $f : \mathbb{R}\setminus\{-1,1\} \to \mathbb{R}$

$g(y) = y^2$, $g : \mathbb{R} \to \mathbb{R}$

- Soluția singulară este $y = 0$

[$y_0 = 0 \Rightarrow g(y_0) = 0$ astfel $\Uparrow$]

$\odot$ $y \neq 0 \Rightarrow y' = -\frac{2x}{x^2-1} \cdot y^2$

Știm $y' = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{2x}{x^2-1} \cdot y^2$

$$\Rightarrow \frac{dy}{y^2} = -\frac{2x}{x^2-1}\, dx \quad \Big| \int$$

$$\Rightarrow \int \frac{dy}{y^2} = \int -\frac{2x}{x^2-1}\, dx$$

$$-\frac{1}{y} = -\ln|x^2-1| - c$$

$$y = \frac{1}{\ln|x^2-1| + c}, \quad c \in \mathbb{R}$$

3. $xy' = y^3 + y$

$$y' = \frac{y^3+y}{x} = \frac{1}{x}(y^3+y)$$

$f(x) = \frac{1}{x}$, $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$g(y) = y^3 + y$, $g : \mathbb{R} \to \mathbb{R}$

$\odot$ $y_0 = 0 \Rightarrow g(y_0) = 0 \Rightarrow y = 0$ sol. singulară

$\odot$ $y \neq 0$ $\frac{y'}{y^3+y} = \frac{1}{x}$

$$y' = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} \cdot \frac{1}{y^3+y} = \frac{1}{x}$$

$$\frac{dy}{y^3+y} = \frac{dx}{x} \quad \Big| \int$$

$$\Rightarrow \int \frac{dy}{y^3+y} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{1}{y^3+y}\, dy = \int \frac{1}{y(y^2+1)}\, dy = \int \frac{1+y^2-y^2}{y(y^2+1)}\, dy =$$

$$= \int \frac{\cancel{1+y^2}}{y(\cancel{1+y^2})}\,dy - \int \frac{y^{\cancel{2}}}{\cancel{y}(y^2+1)}\,dy =$$

$$= \int \frac{1}{y}\,dy - \int \frac{y}{y^2+1}\,dy =$$

$$= \ln|y| - \frac{1}{2}\ln|y^2+1| + c$$

$$\Rightarrow \ln y - \frac{1}{2}\ln(y^2+1) = \ln x + c_0 \qquad c_0 := \ln c_1 \,,\ c_1 > 0$$

$$\ln y - \ln\sqrt{y^2+1} = \ln x + \ln c_1 \qquad \text{sol. în formă implicită}$$

$$\ln \frac{y}{\sqrt{y^2+1}} = \ln(c_1 \cdot x) \quad \Big|\, e$$

$$\frac{y}{\sqrt{y^2+1}} = c_1 \cdot x \Rightarrow y = c_1 x\sqrt{y^2+1} \quad \Big|^2$$

$$y^2 = c_1^2 x^2 (y^2+1)$$

$$y^2 = c_1^2 x^2 y^2 + c_1^2 x^2$$

$$y^2(1 - c_1^2 x^2) = c_1^2 x^2$$

$$y^2 = \frac{c_1^2 x^2}{1 - c_1^2 x^2} \quad \Big|\sqrt{\ } \qquad c_1 := c \in \mathbb{R}$$

$$\Rightarrow y = \pm\sqrt{\frac{c^2 x^2}{1 - c^2 x^2}}$$

soluție explicită

Temă:

4. $xy + (2x-1)y' = 0$

5. $y' = k \cdot \frac{y}{x} \quad , k \in \mathbb{R}^*$

6. $y - xy' = a(1 + x^2 y') \quad , a \in \mathbb{R}^*$

2. Ecuații omogene în sens Euler

– forma gen. $y'(x) = g(x,y)$

îl scriem pe y' astfel

$$y' = f\left(\frac{y}{x}\right)$$

Subs. $\boxed{z(x) = \frac{y(x)}{x}}$

Ex 2: 1. $2x^2 y' = x^2 + y^2$, $x > 0$

$$y' = \frac{x^2+y^2}{2x^2}$$

$$y' = \frac{x^2+y^2}{2x^2} = \frac{\cancel{x^2}}{2\cancel{x^2}} + \frac{y^2}{2x^2} = \frac{1}{2} + \frac{y^2}{2x^2} =$$

$$y' = \frac{1}{2}\left(1 + \frac{y^2}{x^2}\right) = \frac{1}{2}\left(1 + \left(\frac{y}{x}\right)^2\right)$$

Facem substituția

$$\boxed{z = \frac{y}{x} \Rightarrow y = z \cdot x, \quad y' = x \cdot z' + z}$$

$$x \cdot z' + z = \frac{1}{2}(1 + z^2)$$

$$z' = \frac{1}{2x}(1 + z^2 - 2z)$$

$$z' = \frac{1}{2x}(1-z)^2$$

ⓘ $z = 1$ sol. singulară

ⓘ $z \neq 1$ $\quad z' = \frac{dz}{dx}$

$$\frac{dz}{dx} = \frac{1}{2x}(1-z)^2$$

$$\frac{dz}{(1-z)^2} = \frac{1}{2x}\,dx \quad \Big| \int$$

$$\int \frac{dz}{(1-z)^2} = \int \frac{1}{2x}\,dx$$

$$\Rightarrow \frac{1}{1-z} = \frac{1}{2}\ln x + c_0 \quad , \quad c_0 = \frac{c}{2}$$

$$\frac{1}{1-z} = \frac{\ln x + c}{2}$$

$$1 - z = \frac{2}{\ln x + c} \quad , \quad x > 0$$

$$z = 1 - \frac{2}{\ln x + c}$$

$$\Rightarrow \begin{cases} y = x \cdot z = x\left(1 - \dfrac{2}{\ln x + c}\right), & c \in \mathbb{R},\ x \in (0, e^{-c}) \text{ SAU } x \in (e^{-c}, \infty) \\ y = x, & x > 0 \end{cases}$$

2. $$y' = -\frac{x+y}{y}$$

$$y' = -\frac{x}{y} - 1$$

$$\frac{1}{z} = \frac{x}{y} \Rightarrow z = \frac{y}{x} \Rightarrow y = z \cdot x$$

$$y' = z'x + z$$

$$z'x + z = -\frac{1}{z} - 1$$

$$z' = -\frac{1}{zx} - {}^{z)}\frac{1}{x} - {}^{z)}\frac{z}{x} = \frac{-1 - z - z^2}{zx}$$

$$z' = \frac{dz}{dx} \Rightarrow \frac{dz}{dx} = \frac{-z^2 - z - 1}{zx}$$

$$\frac{dz}{dx} = \frac{-z - 1 - \frac{1}{z}}{x}$$

$$\frac{1}{-{}^{z)}z - 1 - \frac{1}{z}}\, dz = \frac{1}{x}\, dx \quad \Big| \cdot (-1)$$

$$\frac{z}{z^2+z+1}\,dz = -\frac{1}{x}\,dx \quad \Big|\int$$

$$\int \frac{z}{z^2+z+1}\,dz = -\ln|x| + c$$

$$I = \frac{1}{2}\int \frac{2z+1-1}{z^2+z+1}\,dz = \frac{1}{2}\int \frac{2z+1}{z^2+z+1}\,dz - \frac{1}{2}\int \frac{1}{z^2+z+1}\,dz =$$

$$= \frac{1}{2}\ln(z^2+z+1) - \frac{1}{2}\int \frac{1}{\left(z+\frac{1}{2}\right)^2+\frac{3}{4}}\,dz =$$

f. canonică $= a\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a}$

$$= \frac{1}{2}\ln(z^2+z+1) - \frac{1}{2}\int \frac{1}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dz =$$

$$= \frac{1}{2}\ln(z^2+z+1) - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\,\mathrm{arctg}\,\frac{z+\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$\Rightarrow \frac{1}{2}\ln(z^2+z+1) - \frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2z+1}{\sqrt{3}} = -\ln|x| + c$$

$\Rightarrow$ sol. generală $\frac{1}{2}\ln\left(\frac{y^2+xy+x^2}{x^2}\right) - \frac{1}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{\sqrt{3}}{3}\left(\frac{2y+x}{x}\right)\right) = -\ln|x| + c$

3. $y' = e^{\frac{y}{x}} + \frac{y}{x}$

$$z = \frac{y}{x} \Rightarrow y = zx$$

$$y' = z'x + z$$

$$z'x + \cancel{z} = e^z + \cancel{z}$$

$$xz' = e^z$$

$$z' = \frac{e^z}{x}$$

$$\frac{dz}{dx} = \frac{e^z}{x} \Rightarrow \frac{dz}{e^z} = \frac{dx}{x} \quad \Big|\int$$

Temă:

4. $x y' = \sqrt{x^2 - y^2} + y$

5. $y' = \frac{y}{x} + tg\, \frac{y}{x}$

6. $x - y \cos\left(\frac{y}{x}\right) + x \cos\left(\frac{y}{x}\right) y' = 0$

3. Ec. liniare

Ex 3: 1. $y' + y \cdot tg(x) = \frac{1}{\cos(x)}$ (*)

I Omogenă

$$y' + y \cdot tg(x) = 0$$

$$\frac{y'}{y} = -tg(x) \quad , \quad y' = \frac{dy}{dx}$$

$$\frac{dy}{y} = -tg(x)\, dx \quad \Big| \int$$

$$\Rightarrow \int \frac{dy}{y} = -\int tg(x)\, dx$$

$$\Rightarrow \ln|y| = \ln|\cos x| + c$$

$$\ln y = \ln|\cos x| + \ln c$$

$$\ln y = \ln(c \cdot \cos x)$$

$$\Rightarrow y_o = c \cdot \cos x$$

II Particulară (det. sol. partic. prin met. variației constantelor)

$$y_p = \varphi \cdot \cos x$$

$$y_p' = \varphi' \cdot \cos(x) - \varphi \sin(x)$$

[Ne întoarcem la (*) și înlocuim]

$$\varphi' \cdot \cos x - \cancel{\varphi \sin x} + \cancel{\varphi \cdot \cos x \cdot \underset{\frac{\sin x}{\cos x}}{\underset{\shortparallel}{\operatorname{tg} x}}} = \frac{1}{\cos x}$$

$$\varphi' = \frac{1}{\cos^2 x} \quad \Big| \int \quad \Rightarrow \varphi = \operatorname{tg} x$$

$$y_p = \operatorname{tg} x \cdot \cos x = \sin x$$

$$\Rightarrow \boxed{y = y_0 + y_p} = c \cdot \cos x + \sin x$$

Temă: 5. $y' - y = \sin(x)$

6. $y' + \frac{x}{1-x^2} y = x + \arcsin(x)$