26.01.2021

Mihăiță Andra
gr. 214

Lucrare scrisă la analiză matematică

- Sesiunea iarnă 2021 -

1. Determinați mulțimea de convergență a seriei de puteri:

$$\sum_{n=0}^{\infty} \left(\frac{n+1}{n+2}\right)^n \cdot (x-2)^n \quad , \quad x \in \mathbb{R}$$

2. Determinați punctele critice și punctele de extrem local, (specificând tipul acestora), pentru funcția:

$$f: \mathbb{R}^3 \to \mathbb{R} \quad , \quad f(x,y,z) = x^3 - x + y^2 + z^2$$

3. Se dă funcția

$$f(x,y) = \arcsin \frac{y}{\sqrt{x^2+y^2}}$$. Determinați constanta $\alpha \in \mathbb{R}$

astfel încât: ~~derivata~~

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \alpha \cdot \frac{\partial^2 f}{\partial y^2}(x,y) \ , \ (\forall)\ (x,y) \in (0,+\infty)^2$$

4. Studiați natura integralei improprii în funcție de valorile parametrului $\alpha \in \mathbb{R}$,

$$I(\alpha) = \int_0^1 \frac{x^\alpha}{(e^{\sqrt{x}}-1)^2}\,dx$$, și calculați apoi $I(-\frac{1}{2})$.

1. $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

$a_n = \left(\frac{n+1}{n+2}\right)^n, \quad x_0 = 2$

$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\left(\frac{n+1}{n+2}\right)^n \cdot \left(\frac{n+2}{n+3}\right)^{n+1}\right| =$

$= \lim_{n\to\infty} \frac{(n+1)^n}{\cancel{(n+2)^n}} \cdot \frac{(n+2)^{\cancel{n+1}}}{(n+3)^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^n}{(n+3)^n} \cdot \frac{n+2}{n+3} =$

$= \lim_{n\to\infty} \left(\frac{n+1}{n+3}\right)^n \cdot \frac{n+2}{n+3} = \lim_{n\to\infty} \left(\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(1+\frac{3}{n})}\right)^n \cdot \frac{\cancel{n}(1+\frac{2}{n})}{\cancel{n}(1+\frac{3}{n})} =$

$= 1^{\infty} \cdot 1 = 1 \Rightarrow r = 1$

$(x_0 - r, x_0 + r) \subseteq I \subseteq [x_0 - r, x_0 + r]$

$(2-1, 2+1) \subseteq I \subseteq [2-1, 2+1]$

$(1, 3) \subseteq I \subseteq [1, 3]$

pt. $x = 1 \Rightarrow \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n \cdot (1-2)^n = \sum_{n=1}^{\infty} (-1)^n \cdot \left(\frac{n+1}{n+2}\right)^n$

Verificăm absolut convergența

$\cancel{\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\left(\frac{n+1}{n+2}\right)^n \cdot (-1)^n}{\left(\frac{n+2}{n+3}\right)^n \cdot (-1)^{n+1}} = \lim_{n\to\infty}}$

$\left|(-1)^n \cdot \left(\frac{n+1}{n+2}\right)^n\right| = \left|\underbrace{\left(\frac{n+1}{n+2}\right)^n}_{\leq 1}\right| \leq 1$

Scanned with CamScanner

Pentru $(-1)^n = 1$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n = \lim_{n\to\infty} \left(\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(1+\frac{2}{n})}\right)^n = 1^\infty = 1$$

~~Pentru $(-1)^n = -1$~~

$$\cancel{\Rightarrow \lim_{n\to\infty} -1 a_n = \lim_{n\to\infty} -\left(\frac{n+1}{n+2}\right)^n = \lim_{n\to\infty} -\left(\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(1+\frac{2}{n})}\right)^n = -1 = -1}$$

$$\Rightarrow \cancel{\nexists \lim_{n\to\infty} (-1)^n a_n \Rightarrow \sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{n+1}{n+2}\right)^n \text{ divergentă}}$$

$$\Rightarrow \lim_{n\to\infty} (-1)^n \cdot a_n \neq 0 \overset{C.L.}{\Longrightarrow} \text{seria nu este convergentă} \Rightarrow$$

$\Rightarrow$ seria ~~diver~~ este divergentă

pt. $x = 3 \Rightarrow \sum_{n=0}^{\infty} \left(\frac{n+1}{n+2}\right)^n \cdot (3-2)^n = \sum_{n=0}^{\infty} \left(\frac{n+1}{n+2}\right)^n$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n = 1 \neq 0 \Rightarrow$ seria este divergentă

$\left|\left(\frac{n+1}{n+2}\right)^n\right| \leq 1$

$\Rightarrow j = (1, 3)$

2. $f: \mathbb{R}^3 \to \mathbb{R}$

$f(x,y,z) = x^3 - x + y^2 + z^2$

I Puncte critice

$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z) \right) =$

$= (3x^2 - 1, 2y, 2z)$

$\nabla f(x,y,z) = 0_3 \Rightarrow \begin{cases} 3x^2 - 1 = 0 \\ 2y = 0 \Rightarrow y = 0 \\ 2z = 0 \Rightarrow z = 0 \end{cases}$

$3x^2 - 1 = 0 \Rightarrow (\sqrt{3}x - 1)(\sqrt{3}x + 1) = 0$

$\sqrt{3}x - 1 = 0 \Rightarrow x\sqrt{3} = 1 \Rightarrow x = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

sau

$\sqrt{3}x + 1 = 0 \Rightarrow x\sqrt{3} = -1 \Rightarrow x = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$\left.\begin{array}{l} \left(\frac{\sqrt{3}}{3}, 0, 0\right) \\ \left(-\frac{\sqrt{3}}{3}, 0, 0\right) \end{array}\right\}$ puncte critice

II Natura punctelor critice

$H f(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$

$$= \begin{pmatrix} 6x & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Pentru $\left(\frac{\sqrt{3}}{3}, 0, 0\right) = a$

$$\left.\begin{array}{l} \Delta_1 = 6 \cdot \frac{\sqrt{3}}{3} = 2\sqrt{3} > 0 \\ \Delta_2 = 6 \cdot \frac{\sqrt{3}}{3} \cdot 2 - 0 \cdot 0 = 4\sqrt{3} > 0 \\ \Delta_3 = 6 \cdot \frac{\sqrt{3}}{3} \cdot 2 \cdot 2 = 8\sqrt{3} > 0 \end{array}\right\} \Rightarrow d^2 f(a) \text{ pozitiv definită}$$

$\Rightarrow a = \left(\frac{\sqrt{3}}{3}, 0, 0\right)$ punct de minim local

Pentru $\left(-\frac{\sqrt{3}}{3}, 0, 0\right) = b$

$$\left.\begin{array}{l} \Delta_1 = 6 \cdot \left(-\frac{\sqrt{3}}{3}\right) = -2\sqrt{3} < 0 \\ \Delta_2 = 6 \cdot \left(-\frac{\sqrt{3}}{3}\right) \cdot 2 = -4\sqrt{3} < 0 \\ \Delta_3 = 6 \left(-\frac{\sqrt{3}}{3}\right) \cdot 2 \cdot 2 = -8\sqrt{3} < 0 \end{array}\right\} \text{nu respectă niciun criteriu Sylvester}$$

$\Rightarrow d^2 f(b)(u_1, u_2, u_3) = -6 \frac{\sqrt{3}}{3} \cdot u_1^2 + 2u_2^2 + 2u_3^2 =$

$= -2\sqrt{3}\, u_1^2 + 2u_2^2 + 2u_3^2$

$d^2 f(b)(1, 0, 0) = -2\sqrt{3} < 0$

$d^2 f(b)(0, 1, 1) = 2 + 2 = 4 > 0$

$\Rightarrow$ nu este ~~negativ definită~~

~~nu este indefinită~~ $\Rightarrow$

$\Rightarrow$ ~~teorema nu se aplică~~

$d^2 f(b)$ indefinită, $b = \left(-\frac{\sqrt{3}}{3}, 0, 0\right)$ punct șa

3. $f(x,y) = \arcsin \frac{y}{\sqrt{x^2+y^2}}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{\sqrt{1-\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}} \cdot \frac{0 - y \cdot \frac{2x}{2\sqrt{x^2+y^2}}}{\left(\sqrt{x^2+y^2}\right)^2} =$$

$$= \frac{1}{\sqrt{1-\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}} \cdot \left(- \frac{xy}{\sqrt{x^2+y^2}} \cdot \frac{1}{x^2+y^2} \right)$$

$$= \frac{1}{\sqrt{1-\frac{y^2}{x^2+y^2}}} \cdot \left(- \frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}} \right)$$

$$= \frac{1}{\sqrt{\frac{x^2+y^2-y^2}{x^2+y^2}}} \cdot \left(- \frac{xy}{(x^2+y^2)(\sqrt{x^2+y^2}} \right)$$

$$= \frac{\cancel{\sqrt{x^2+y^2}}}{x} \cdot \left(- \frac{xy}{(x^2+y^2)(\cancel{\sqrt{x^2+y^2}})} \right)$$

$$= - \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = \left(-\frac{y}{x^2+y^2}\right)'_x = + \cdot \frac{0 + y \cdot 2x}{(x^2+y^2)^2} =$$

$$= \frac{2xy}{(x^2+y^2)^2} \quad (1)$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{1}{\sqrt{1-\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}}\cdot\frac{1\left(\sqrt{x^2+y^2}\right)-y\cdot\cancel{2}y\cdot\frac{1}{\cancel{2}\sqrt{x^2+y^2}}}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$=\frac{1}{\sqrt{1-\frac{y^2}{x^2+y^2}}}\cdot\frac{\sqrt{x^2+y^2}-\frac{y^2}{\sqrt{x^2+y^2}}}{(x^2+y^2)}$$

$$=\frac{1}{\sqrt{\frac{x^2+\cancel{y^2}-\cancel{y^2}}{x^2+y^2}}}\cdot\frac{(x^2+y^2)\cancel{\sqrt{x^2+y^2}}-y^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$=\frac{\cancel{\sqrt{x^2+y^2}}}{x}\cdot\frac{(x^2+y^2)\cancel{\sqrt{x^2+y^2}}-y^2}{(x^2+y^2)\cancel{\sqrt{x^2+y^2}}}=\frac{x^2+\cancel{y^2}-\cancel{y^2}}{x(x^2+y^2)}=\frac{x^{\cancel{2}}}{\cancel{x}(x^2+y^2)}=$$

$$=\frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y)=\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y)=\left(\frac{x}{x^2+y^2}\right)'_y=$$

$$=\frac{0-x\cdot 2y}{(x^2+y^2)^2}=-\frac{2xy}{(x^2+y^2)^2}\quad(2)$$

Dar (1) și (2)

$$\left.\begin{cases}\frac{\partial^2 f}{\partial x^2}(x,y)=\alpha\cdot\frac{\partial^2 f}{\partial y^2}(x,y)\\ \frac{\partial^2 f}{\partial x^2}(x,y)=(-1)\cdot\frac{\partial^2 f}{\partial y^2}(x,y)\end{cases}\right|\Rightarrow\boxed{\alpha=-1\in\mathbb{R}}$$

Scanned with CamScanner

(4.) $i(\alpha)=\int_0^1 \frac{x^{\alpha}}{(e^{\sqrt{x}}-1)^2}\,dx$

I Pentru $\alpha<0 \Rightarrow i(\alpha)=\int_0^1 \frac{1}{x^{\alpha}(e^{\sqrt{x}}-1)^2}\,dx$

$$\lambda=\lim_{x\searrow 0} x^{p}\cdot\frac{1}{x^{\alpha}(e^{\sqrt{x}}-1)^2}=\lim_{x\searrow 0}\frac{x^{p-\alpha}}{(e^{\sqrt{x}}-1)^2}\overset{\frac{0}{0}}{\underset{L'H}{=}}$$

$$\lim_{x\searrow 0}\frac{(p-\alpha)x^{p-\alpha-1}}{\cancel{2}(e^{\sqrt{x}}-1)\cdot e^{\sqrt{x}}\cdot\frac{1}{\cancel{2}\sqrt{x}}}=\lim_{x\searrow 0}\frac{(p-\alpha)x^{p-\alpha-1}\cdot x^{\frac{1}{2}}}{e^{\sqrt{x}}(e^{\sqrt{x}}-1)}$$

$$=\lim_{x\searrow 0}\frac{(p-\alpha)x^{p-\alpha-\overset{2}{1}+\frac{1}{2}}}{e^{\sqrt{x}}(e^{\sqrt{x}}-1)}=\lim_{x\searrow 0}\frac{(p-\alpha)x^{p-\alpha-\frac{1}{2}}}{e^{\sqrt{x}}(e^{\sqrt{x}}-1)}\overset{\frac{0}{0}}{\underset{L'H}{=}}$$

$$=\lim_{x\searrow 0}\frac{(p-\alpha)(p-\alpha-\frac{1}{2})x^{p-\alpha-\frac{1}{2}-\overset{2}{1}}}{\frac{e^{\sqrt{x}}}{2\sqrt{x}}(e^{\sqrt{x}}-1)+e^{\sqrt{x}}\cdot\frac{e^{\sqrt{x}}}{2\sqrt{x}}}$$

$$=\lim_{x\searrow 0}\frac{(p-\alpha)(p-\alpha-\frac{1}{2})\cdot x^{p-\alpha-\frac{3}{2}}\cdot 2\sqrt{x}}{e^{\sqrt{x}}(e^{\sqrt{x}}-1+e^{\sqrt{x}})}$$

$$=\lim_{x\searrow 0}\frac{2(p-\alpha)(p-\alpha-\frac{1}{2})\cdot x^{p-\alpha-\frac{3}{2}+\frac{1}{2}}}{e^{\sqrt{x}}(2e^{\sqrt{x}}-1)}=$$

$$=\frac{2(p-\alpha)(p-\alpha-\frac{1}{2})\,0^{p-\alpha-1}}{e^{\sqrt{0}}(2e^{\sqrt{0}}-1)}=\frac{0}{1\cdot(2-1)}=\frac{0}{1}=0$$

$\Rightarrow \lambda = 0$

Pentru $p < 1$, $\alpha < 0$ $\Big| \Rightarrow I(\alpha)$ convergentă

II $\alpha = 0 \Rightarrow \int_{0,\searrow 0}^{1} \frac{1}{(e^{\sqrt{x}}-1)^2} dx$

$$\lambda = \lim_{x \searrow 0} (x-0)^p \cdot \frac{1}{(e^{\sqrt{x}}-1)^2} = \lim_{x\searrow 0} \frac{x^p}{(e^{\sqrt{x}}-1)^2} \overset{\frac{0}{0}}{\underset{L'H}{=}}$$

$$= \lim_{x \searrow 0} \frac{p \cdot x^{p-1}}{\cancel{2}(e^{\sqrt{x}}-1) \cdot e^{\sqrt{x}} \cdot \frac{1}{\cancel{2}\sqrt{x}}} = \lim_{x\searrow 0} \frac{p \cdot x^{p-1} \cdot x^{\frac{1}{2}}}{(e^{\sqrt{x}}-1)e^{\sqrt{x}}}$$

$$= \lim_{x\searrow 0} \frac{p \cdot x^{p-1+\frac{1}{2}}}{(e^{\sqrt{x}}-1)e^{\sqrt{x}}} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x \searrow 0} \frac{p \cdot (p-\frac{1}{2}) \cdot x^{p-\frac{1}{2}-1}}{\frac{e^{\sqrt{x}} \cdot e^{\sqrt{x}}}{2\sqrt{x}} + \frac{(e^{\sqrt{x}}-1)e^{\sqrt{x}}}{2\sqrt{x}}}$$

$$= \lim_{x\searrow 0} \frac{p(p-\frac{1}{2}) \cdot x^{p-\frac{3}{2}} \cdot 2x^{\frac{1}{2}}}{e^{\sqrt{x}}(e^{\sqrt{x}}+e^{\sqrt{x}}-1)} = \lim_{x\searrow 0} \frac{2p(p-\frac{1}{2}) \cdot x^{p-1}}{e^{\sqrt{x}}(2e^{\sqrt{x}}-1)}$$

$$= \frac{2p(p-\frac{1}{2}) \cdot 0^{p-1}}{e^{\sqrt{0}}(2e^{\sqrt{0}}-1)} = \frac{0}{1(2-1)} = \frac{0}{1} = 0$$

$\Rightarrow \lambda = 0$

Pentru $p < 1$, $\alpha = 0$ $\Big| \Rightarrow I(\alpha)$ convergentă

III $\alpha > 0 \Rightarrow \int_{0+0}^{1} \frac{x^\alpha}{(e^{\sqrt{x}}-1)^2}\,dx$

$$\lambda = \lim_{x \searrow 0} \frac{x^p \cdot x^\alpha}{(e^{\sqrt{x}}-1)^2} = \lim_{x\searrow 0} \frac{x^{p+\alpha}}{(e^{\sqrt{x}}-1)^2} \underset{L'H}{\overset{0/0}{=}}$$

$$= \lim_{x\searrow 0} \frac{(p+\alpha)\, x^{p+\alpha-1}}{\cancel{2}(e^{\sqrt{x}}-1)\cdot e^{\sqrt{x}} \cdot \frac{1}{\cancel{2}\sqrt{x}}} = \lim_{x\searrow 0} \frac{(p+\alpha)\, x^{p+\alpha-1+\frac{1}{2}}}{e^{\sqrt{x}}(e^{\sqrt{x}}-1)} \underset{L'H}{\overset{0/0}{=}}$$

$$= \lim_{x\searrow 0} \frac{(p+\alpha)(p+\alpha-\frac{1}{2})\, x^{p+\alpha-\frac{1}{2}-1}}{\frac{e^{\sqrt{x}}(e^{\sqrt{x}}-1)}{2\sqrt{x}} + \frac{e^{\sqrt{x}}\cdot e^{\sqrt{x}}}{2\sqrt{x}}} =$$

$$= \lim_{x\searrow 0} \frac{2(p+\alpha)(p+\alpha-\frac{1}{2})\cdot x^{p+\alpha-\cancel{\frac{1}{2}}-1+\cancel{\frac{1}{2}}}}{e^{\sqrt{x}}(e^{\sqrt{x}}-1+e^{\sqrt{x}})}$$

$$= \lim_{x\searrow 0} \frac{2(p+\alpha)(p+\alpha-\frac{1}{2})\cdot x^{p+\alpha-1}}{e^{\sqrt{x}}(2e^{\sqrt{x}}-1)} =$$

$$= \frac{2(p+\alpha)(p+\alpha-\frac{1}{2})\cdot 0^{p+\alpha-1}}{e^{\sqrt{0}}(2e^{\sqrt{0}}-1)} = \frac{0}{1(2-1)} = 0$$

$\Rightarrow \lambda = 0$

$\left.\text{Pentru } p < 1, \alpha > 0\right\} \Rightarrow I(\alpha)$ convergentă

$$I\left(-\frac{1}{2}\right) = \int_0^1 \frac{x^{-\frac{1}{2}}}{(e^{\sqrt{x}}-1)^2}\,dx = \int_0^1 \frac{1}{\sqrt{x}\,(e^{\sqrt{x}}-1)^2}\,dx$$

$$= \int_0^1 \frac{2}{2\sqrt{x}\,(e^{\sqrt{x}}-1)^2}\,dx$$

$\sqrt{x} = t$

$\frac{1}{2\sqrt{x}}\,dx = dt$

$x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$$= \int_0^1 \frac{2}{(e^t-1)^2}\,dt = 2\int_0^1 \frac{1}{(e^t-1)^2}\,dt$$

$$= 2\int_0^1 \frac{e^t}{(e^t-1)^2 \cdot e^t}\,dt = 2\int_1^e \frac{du}{(u-1)^2 \cdot u}$$

$e^t = u$

$e^t\,dt = du$

$t = 0 \Rightarrow u = 1$

$t = 1 \Rightarrow u = e$

$$= 2\int_1^e \frac{-(u-1)+u}{(u-1)^2 \cdot u}\,du$$

$$= -2\int_1^e \frac{\cancel{u-1}}{(u-1)^2 \cdot u}\,du + 2\int_1^e \frac{\cancel{u}}{(u-1)^2 \cdot \cancel{u}}\,du$$

$$= -2\int_1^e \frac{1}{(u-1)u}\,du + 2\cdot\int_1^e \frac{1}{(u-1)^2}\,du$$

$\cancel{u-1 = z}$

$\cancel{du = dz}$

$\cancel{u = 1 \Rightarrow z = 0}$

$\cancel{u = e \Rightarrow z = e-1}$

$$= -2\int_1^e \frac{-(u-1)+u}{(u-1)u}\,du + 2\int_1^e (u-1)^{-2}\,du$$

$$= 2\int_1^e \frac{\cancel{u-1}}{(\cancel{u-1})u}\,du - 2\int_1^e \frac{\cancel{u}}{(u-1)\cancel{u}}\,du + 2\cdot\frac{(u-1)^{-2+1}}{-2+1}\Bigg|_1^e$$

$$= 2\ln u\Big|_1^e - 2\ln(u-1)\Big|_1^e + 2\,\frac{(e-1)^{-1}}{-1}$$

$$= 2\ln e - 2\ln 1 - 2\ln(e-1) + \cancel{2\ln} - 2\cdot\frac{1}{e-1}$$

$$= 2 - 2\ln(e-1) - \frac{2}{e-1}$$