Root finding
Steffensen's method
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THE PROBLEM:

We want to compute the zeros, or the roots of a nonlinear equation f(x) = 0, in a way to avoid the use of derivatives, without affecting the order of convergence.

THE SOLUTION:

Steffensen's method was first published in 1933, in a paper called "Remarks on iteration", Scandinavian Actuarial Journal. It is one very popular root finding method, named after the Danish mathematician, Johan Frederik Steffensen.

DESCRIPTION:

There are different methods for computing the roots of a nonlinear equation f(x) = 0; the most well known of these methods is the classical Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots,$$

Newton's method has been modified in a number of ways to avoid the use of derivatives without affecting the order of convergence. For example, on replacing in the formula above the derivative by the forward approximation

$$f'(x_n) \approx \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}.$$

Newton's method becomes:

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)},$$

which is called Steffensen's method. This method still has quadratic convergence, in spite of being derivative free and using only two functional evaluations per step.

ADVANTAGES:

Steffensen's method has quadratic convergence like Newton's method, moreover in particular cases it converges faster than Newton method. This means that this method finds roots to an equation f 'quickly' . In this case, quickly means that the number of correct digits in the answer doubles with each step.

The main difference from Newton's method: the formula for Newton's method requires evaluation of the function's derivative ***f'*** as well as the function ***f***, while

Steffensen’s method only requires ***f*** itself. This is important when the derivative is not easily or efficiently available.

Steffensen’s method is of high interest because it could be generalized for solving nonlinear equations in abstract spaces with preserving good characteristics mentioned above.

TWO VERSIONS:

There are two versions of the Steffensen’s method - but they are the same thing:

$$x_n - \frac{(x_{n+1} - x_n)^2}{x_n - 2x_{n+1} + x_{n+2}} \quad vs \quad x_n - \frac{\left(f(x_n)\right)^2}{f\left(x_n + f(x_n)\right) - f(x_n)}$$

they are both derived from Aitken's delta-squared method.

You can see that both versions are the same by observing that the numerators for both these are the same as well as the denominators.

The proof for numerators equality:

$$x_{n+1} = g(x_n)$$
$$x_{n+1} - x_n = g(x_n) - x_n$$
$$\lim_{n\to\infty} \{x_n\} = r$$
$$\text{When } n \text{ is large}$$
$$g(x_n) - x_n = 0$$
$$f(x_n) = 0$$
$$g(x_n) - x_n = f(x_n)$$
$$x_{n+1} - x_n = f(x_n)$$

The proof for denominators equality:

$$f(x_n) = g(x_n) - x_n$$
$$x_{n+1} = g(x_n)$$
$$x_{n+2} = g(x_{n+1}) = g\left(g(x_n)\right)$$
$$f\left(x_n + f(x_n)\right) - f(x_n)$$
$$f(x_n + g(x_n) - x_n) - (g(x_n) - x_n)$$
$$f\left(g(x_n)\right) - g(x_n) + x_n$$
$$g\left(g(x_n)\right) - g(x_n) - g(x_n) + x_n$$
$$g\left(g(x_n)\right) - 2g(x_n) + x_n$$
$$x_{n+2} - 2x_{n+1} + x_n$$
$$x_n - 2x_{n+1} + x_{n+2}$$

DRAWBACKS:

The main drawback is the double function evaluation. For comparison, the secant method needs only one function evaluation per step. (and also the secant method converges faster than Steffensen's method).

Another crucial weakness in Steffensen's method is the choice of the starting value. If the starting value is not ‘close enough’ to the actual solution, the method may fail.

ALGORITHM:

Input:

g = function in the non-accelerated method

ε = the precision

nr_max = maximum number of iterations allowed

x = the initial approximation

Output:

x = the approximate solution satisfying ε or no_max

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k $\leftarrow 0$

$x_0 \leftarrow x$

$x_1 \leftarrow g(x_0)$

$x_2 \leftarrow g(x_1)$

$x \leftarrow x_0 - (x_1 - x_0)^2/(x_2 - 2x_1 + x_0)$

while $|x - x_0| \geq \epsilon$ *and* $k \leq nr_max$

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k$\leftarrow k + 1$

$x_0 \leftarrow x$

$x_1 \leftarrow g(x_0)$

$x_2 \leftarrow g(x_1)$

$x \leftarrow x_0 - (x_1 - x_0)^2/(x_2 - 2x_1 + x_0)$

}

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