Logică pentru Informatică - Săptămâna 11
Deducția naturală
Exerciții pentru Seminar

1 Regulile deducției naturale

$$\wedge i\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi'),} \qquad \wedge e_1\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi,} \qquad \wedge e_2\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi',}$$

$$\to e\ \frac{\Gamma \vdash (\varphi \to \varphi') \qquad \Gamma \vdash \varphi}{\Gamma \vdash \varphi',} \qquad \to i\ \frac{\Gamma, \varphi \vdash \varphi'}{\Gamma \vdash (\varphi \to \varphi'),} \qquad \vee i_1\ \frac{\Gamma \vdash \varphi_1}{\Gamma \vdash (\varphi_1 \vee \varphi_2),}$$

$$\vee i_2\ \frac{\Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \vee \varphi_2),} \qquad \vee e\ \frac{\Gamma \vdash (\varphi_1 \vee \varphi_2) \qquad \Gamma, \varphi_1 \vdash \varphi' \qquad \Gamma, \varphi_2 \vdash \varphi'}{\Gamma \vdash \varphi',}$$

$$\neg e\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \neg\varphi}{\Gamma \vdash \bot,} \qquad \neg i\ \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi,} \qquad \bot e\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi,}$$

$$\textsc{Ipoteză}\ \frac{}{\Gamma \vdash \varphi}\ \varphi \in \Gamma, \qquad \textsc{Extindere}\ \frac{\Gamma \vdash \varphi}{\Gamma, \varphi' \vdash \varphi,} \qquad \neg\neg e\ \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi.} \qquad \forall e\ \frac{\Gamma \vdash \forall x.\varphi}{\Gamma \vdash \varphi[x \mapsto t]}$$

$$\exists e\ \frac{\Gamma \vdash \exists x.\varphi \qquad \Gamma \cup \{\varphi[x \mapsto x_0]\} \vdash \psi}{\Gamma \vdash \psi}\ x_0 \notin vars(\Gamma, \varphi, \psi) \qquad \forall i\ \frac{\Gamma \vdash \varphi[x \mapsto x_0]}{\Gamma \vdash \forall x.\varphi}\ x_0 \notin vars(\Gamma, \varphi)$$

$$\exists i\ \frac{\Gamma \vdash \varphi[x \mapsto t]}{\Gamma \vdash \exists x.\varphi}$$

2 Exerciții rezolvate

1. Arătați că secvența $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(b)}$ este validă.

Rezolvare:

1. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(a)}$; (Ipoteză)
2. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \neg\mathtt{P(a)}$; (Ipoteză)
3. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \bot$; ($\neg e$, 1, 2)
4. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(b)}$. ($\bot e$, 3)

2. Arătați că secvența $\{(\mathtt{P(a)} \vee \mathtt{Q(a)})\} \vdash (\mathtt{q} \vee \mathtt{p})$ este validă.

Rezolvare:

1. $\{(\mathtt{P}(\mathtt{a}) \vee \mathtt{Q}(\mathtt{a})), \mathtt{P}(\mathtt{a})\} \vdash \mathtt{P}(\mathtt{a})$; (Ipoteză)
2. $\{(\mathtt{P}(\mathtt{a}) \vee \mathtt{Q}(\mathtt{a})), \mathtt{P}(\mathtt{a})\} \vdash (\mathtt{Q}(\mathtt{a}) \vee \mathtt{P}(\mathtt{a}))$; $(\vee i_2,\ 1)$
3. $\{(\mathtt{P}(\mathtt{a}) \vee \mathtt{Q}(\mathtt{a})), \mathtt{Q}(\mathtt{a})\} \vdash \mathtt{Q}(\mathtt{a})$; (Ipoteză)
4. $\{(\mathtt{P}(\mathtt{a}) \vee \mathtt{Q}(\mathtt{a})), \mathtt{Q}(\mathtt{a})\} \vdash (\mathtt{Q}(\mathtt{a}) \vee \mathtt{P}(\mathtt{a}))$; $(\vee i_1,\ 1)$
5. $\{(\mathtt{P}(\mathtt{a}) \vee \mathtt{Q}(\mathtt{a}))\} \vdash (\mathtt{P}(\mathtt{a}) \vee \mathtt{Q}(\mathtt{a}))$; (Ipoteză)
6. $\{(\mathtt{P}(\mathtt{a}) \vee \mathtt{Q}(\mathtt{a}))\} \vdash (\mathtt{Q}(\mathtt{a}) \vee \mathtt{P}(\mathtt{a}))$. $(\vee e,\ 5,\ 2,\ 4)$

3. Arătați că secvența $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \mathtt{P}(\mathtt{a})\} \vdash \exists \mathtt{x}.\mathtt{Q}(\mathtt{x})$ este validă.

Rezolvare:

1. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \mathtt{P}(\mathtt{a})\} \vdash \forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x}))$ (Ipoteză)
2. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \mathtt{P}(\mathtt{a})\} \vdash \mathtt{P}(\mathtt{a})$ (Ipoteză)
3. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \mathtt{P}(\mathtt{a})\} \vdash (\mathtt{P}(\mathtt{a}) \to \mathtt{Q}(\mathtt{a}))$ $(\forall e, 1, \mathtt{a})$
4. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \mathtt{P}(\mathtt{a})\} \vdash \mathtt{Q}(\mathtt{a})$ $(\to e, 3, 2)$
5. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \mathtt{P}(\mathtt{a})\} \vdash \exists \mathtt{x}.\mathtt{Q}(\mathtt{x})$ $(\exists i, 4)$

4. Arătați că secvența $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash \exists \mathtt{x}.\mathtt{Q}(\mathtt{x})$ este validă.

Rezolvare:

1. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash \exists \mathtt{x}.\mathtt{P}(\mathtt{x})$ (Ipoteză)
2. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x}), \mathtt{P}(\mathtt{x}_0)\} \vdash \mathtt{P}(\mathtt{x}_0)$ (Ipoteză)
3. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x}), \mathtt{P}(\mathtt{x}_0)\} \vdash \forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x}))$ (Ipoteză)
4. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x}), \mathtt{P}(\mathtt{x}_0)\} \vdash (\mathtt{P}(\mathtt{x}_0) \to \mathtt{Q}(\mathtt{x}_0))$ $(\forall e, 3, \mathtt{x}_0)$
5. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x}), \mathtt{P}(\mathtt{x}_0)\} \vdash \mathtt{Q}(\mathtt{x}_0)$ $(\to e, 4, 2)$
6. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x}), \mathtt{P}(\mathtt{x}_0)\} \vdash \exists \mathtt{x}.\mathtt{Q}(\mathtt{x})$ $(\exists i, 5)$
7. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash \exists \mathtt{x}.\mathtt{Q}(\mathtt{x})$ $(\exists e, 1, 6)$

5. Arătați că secvența $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \mathtt{P}(\mathtt{x})\} \vdash \forall \mathtt{x}.\mathtt{Q}(\mathtt{x})$ este validă

Rezolvare:

1. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \forall \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash \forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x}))$ (Ipoteză)
2. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \forall \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash \forall \mathtt{x}.\mathtt{P}(\mathtt{x})$ (Ipoteză)
3. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \forall \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash (\mathtt{P}(\mathtt{x}_0) \to \mathtt{Q}(\mathtt{x}_0))$ $(\forall e, 1, \mathtt{x}_0)$
4. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \forall \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash \mathtt{P}(\mathtt{x}_0)$ $(\forall e, 2, \mathtt{x}_0)$
5. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \forall \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash \mathtt{Q}(\mathtt{x}_0)$ $(\to e, 3, 4)$
6. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \forall \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash \forall \mathtt{x}.\mathtt{Q}(\mathtt{x})$ $(\forall i, 5)$

3 Exerciții propuse

Secvențele de mai jos sunt valide?

1. $\{((\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a})) \wedge \forall \mathtt{x}.\mathtt{P}(\mathtt{x}))\} \vdash (\mathtt{Q}(\mathtt{a}) \wedge \forall \mathtt{x}.\mathtt{P}(\mathtt{x}))$;
2. $\{((\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a})) \wedge \forall \mathtt{x}.\mathtt{P}(\mathtt{x})), \forall \mathtt{x}.\mathtt{Q}(\mathtt{x})\} \vdash (\forall \mathtt{x}.\mathtt{Q}(\mathtt{x}) \wedge \mathtt{Q}(\mathtt{a}))$;
3. $\{((\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a})) \wedge \forall \mathtt{x}.\mathtt{P}(\mathtt{x}))\} \vdash (\forall \mathtt{x}.\mathtt{P}(\mathtt{x}) \wedge (\mathtt{Q}(\mathtt{a}) \wedge \mathtt{P}(\mathtt{a})))$;
4. $\{((\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a})) \rightarrow \forall \mathtt{x}.\mathtt{P}(\mathtt{x})), \mathtt{P}(\mathtt{a}), \mathtt{Q}(\mathtt{a})\} \vdash \forall \mathtt{x}.\mathtt{P}(\mathtt{x})$;
5. $\{(\mathtt{P}(\mathtt{a}) \rightarrow \forall \mathtt{x}.\mathtt{P}(\mathtt{x})), \mathtt{P}(\mathtt{a}), \mathtt{Q}(\mathtt{a})\} \vdash (\mathtt{Q}(\mathtt{a}) \wedge \forall \mathtt{x}.\mathtt{P}(\mathtt{x}))$;
6. $\{(\mathtt{P}(\mathtt{a}) \rightarrow \mathtt{P}(\mathtt{b})), (\mathtt{Q}(\mathtt{a}) \rightarrow \mathtt{P}(\mathtt{b}))\} \vdash ((\mathtt{P}(\mathtt{a}) \vee \mathtt{Q}(\mathtt{a})) \rightarrow \mathtt{P}(\mathtt{b}))$;
7. $\{\neg(\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a}))\} \vdash (\neg\mathtt{P}(\mathtt{a}) \vee \neg\mathtt{Q}(\mathtt{a}))$;
8. $\{\neg(\neg\mathtt{P}(\mathtt{a}) \vee \neg\mathtt{Q}(\mathtt{a}))\} \vdash (\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a}))$;
9. $\{\neg(\neg\mathtt{P}(\mathtt{a}) \wedge \neg\mathtt{Q}(\mathtt{a}))\} \vdash (\mathtt{P}(\mathtt{a}) \vee \mathtt{Q}(\mathtt{a}))$;
10. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \wedge \mathtt{Q}(\mathtt{x}))\} \vdash \forall \mathtt{x}.\mathtt{P}(\mathtt{x})$;
11. $\{\forall \mathtt{x}.\mathtt{Q}(\mathtt{x}), \mathtt{P}(\mathtt{a})\} \vdash \mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a})$;
12. $\{\forall \mathtt{x}.\mathtt{P}(\mathtt{x}), \forall \mathtt{x}.\mathtt{Q}(\mathtt{x}))\} \vdash \forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \wedge \mathtt{Q}(\mathtt{x}))$;
13. $\{\exists \mathtt{x}.\exists \mathtt{y}.\mathtt{P}(\mathtt{x}, \mathtt{y})\} \vdash \exists \mathtt{y}.\exists \mathtt{x}.\mathtt{P}(\mathtt{x}, \mathtt{y})$;
14. $\{\exists \mathtt{x}.\forall \mathtt{y}.\mathtt{P}(\mathtt{x}, \mathtt{y})\} \vdash \forall \mathtt{y}.\exists \mathtt{x}.\mathtt{P}(\mathtt{x}, \mathtt{y})$; Dar invers: $\{\forall \mathtt{y}.\exists \mathtt{x}.\mathtt{P}(\mathtt{x}, \mathtt{y})\} \vdash \exists \mathtt{x}.\forall \mathtt{y}.\mathtt{P}(\mathtt{x}, \mathtt{y})$?
15. $\{\neg(\exists \mathtt{x}.\mathtt{P}(\mathtt{x}))\} \vdash \forall \mathtt{x}.\neg\mathtt{P}(\mathtt{x})$;
16. $\{\forall \mathtt{x}.\neg\mathtt{P}(\mathtt{x})\} \vdash \neg(\exists \mathtt{x}.\mathtt{P}(\mathtt{x}))$;