Please enter the weight of two items.
For each item give its weight in pounds and ounces, separated by a space:

Item #1: 3 12

Item #2: 5 7

The combined weight is 9 pounds and 3 ounces

I

$$\begin{array}{r} \overset{1}{2}8 \\ +\ 34 \\ \hline 6\overset{\cancel{12}}{2} \end{array}$$

$$\begin{array}{rr} \overset{1}{3} & 12 \\ +\ 5 & 7 \\ \hline 9 & \overset{\cancel{19}}{3} \end{array}$$

$19 \div 16 = 1R$

II

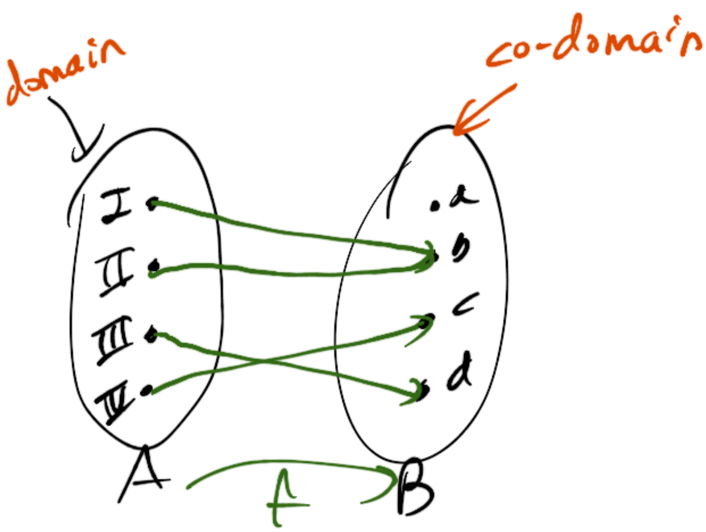
$3\ 12 \rightarrow 60$

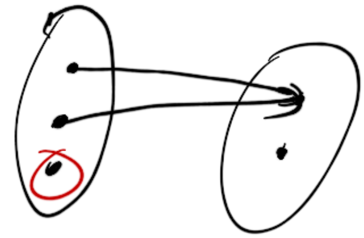
$5\ 7 \rightarrow 87$

$9\ 3 \leftarrow 147$

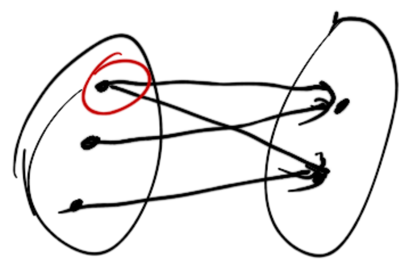
Functions:

Definition: Let A, B be two sets. We say that $f: A \to B$ is a function from A to B if for every element in A it assigns exactly one element from B.





not a function

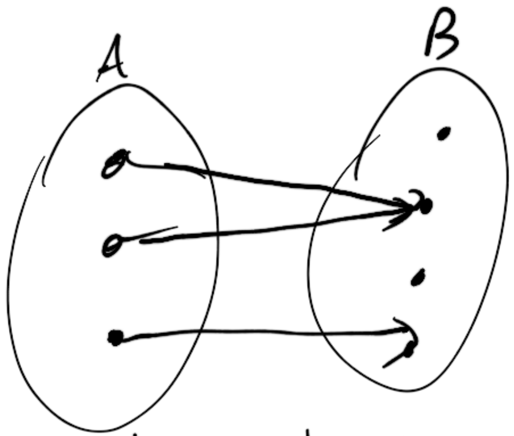


not a function

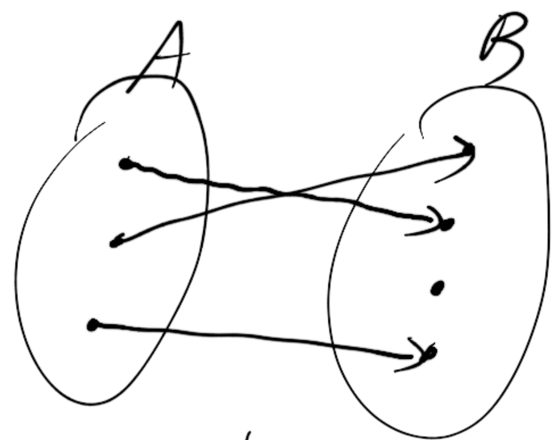
Definitions:

I) Let $f: A \to B$ be a function. We say that f is one-to-one if $\forall_{a_1, a_2 \in A} (a_1 \neq a_2) \to f(a_1) \neq f(a_2)$

$$\equiv \forall_{a_1, a_2 \in A} [f(a_1) = f(a_2)] \to (a_1 = a_2)$$

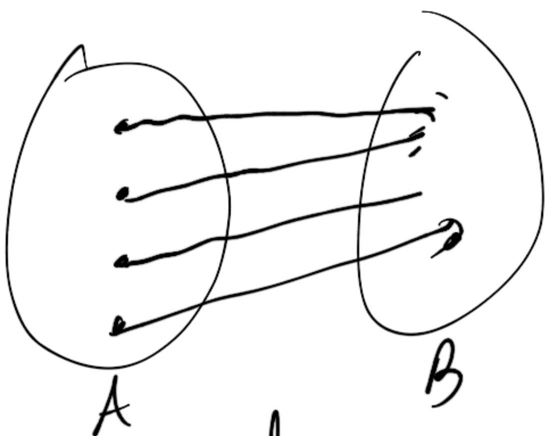


not one-to-one

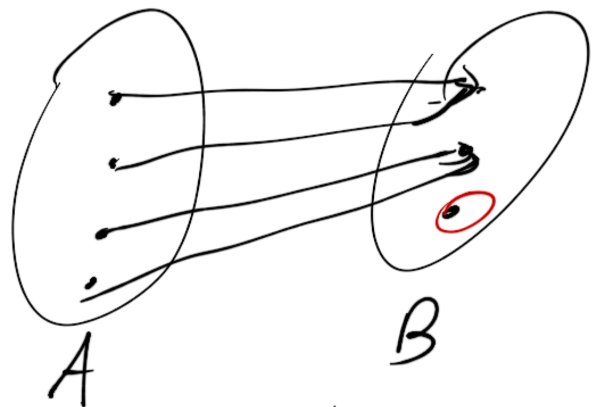


one-to-one

II) Let $f: A \to B$ be a function. We say that f is onto if $\forall_{b \in B} \exists_{a \in A}\ f(a) = b$



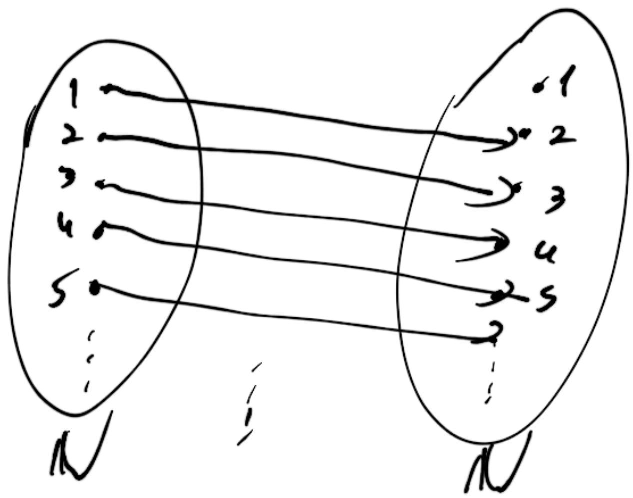
onto



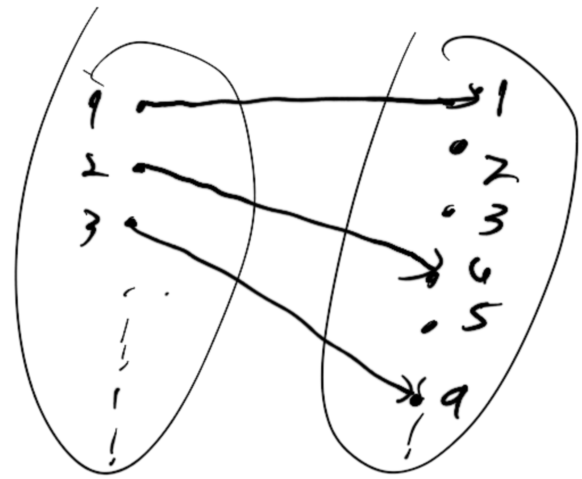
not onto

Ex1: Give an example of $f:\mathbb{N}\to\mathbb{N}$, that is

a) one-to-one but not onto

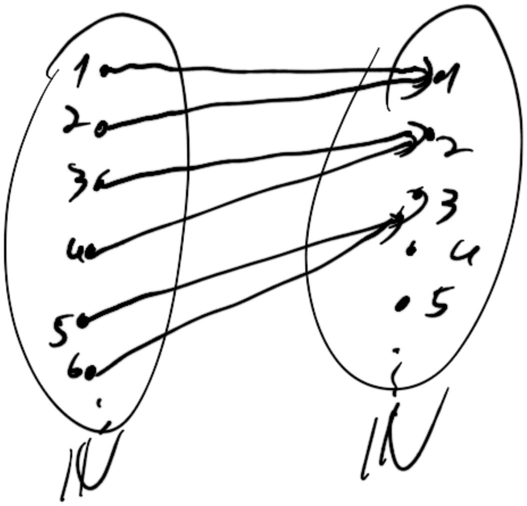


$f(n)=n+1$

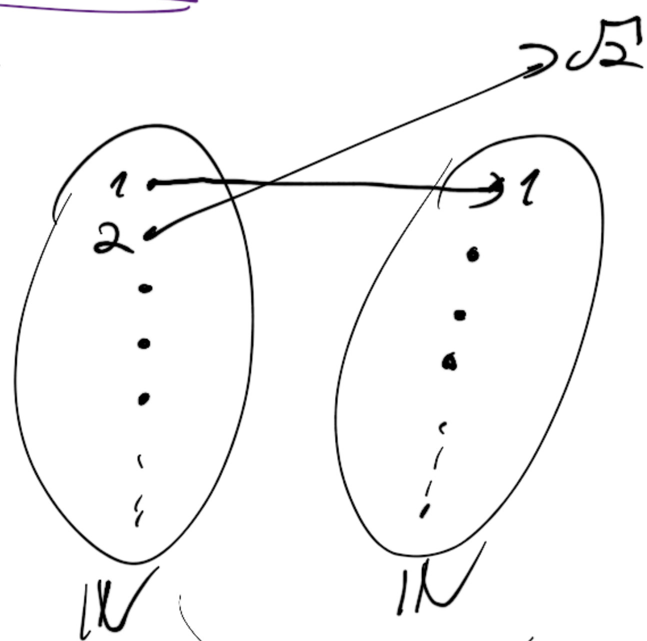


$f(n)=n^2$

b) onto but not one-to-one



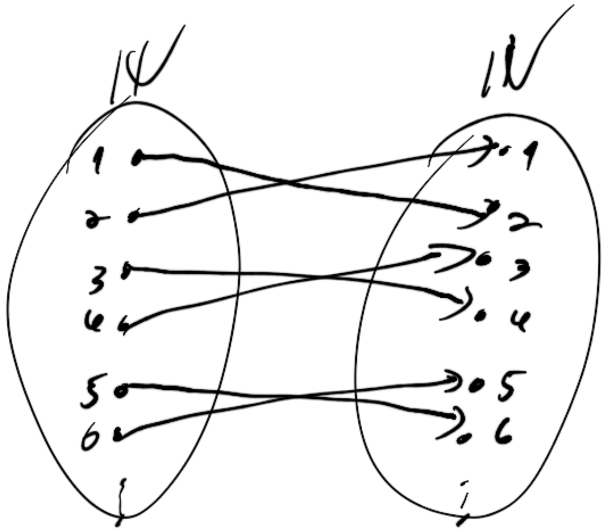
$f(n) = \left\lceil \frac{n}{2} \right\rceil$



$\cancel{f(n) = \sqrt{n}}$

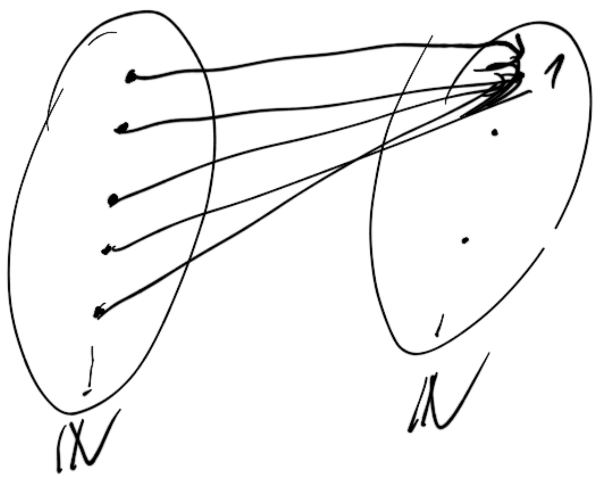
not a function from $\mathbb{N}$ to $\mathbb{N}$

c) both one-to-one and onto (but not the identity function)



$$f(n) = \begin{cases} n+1 & n \text{ is odd} \\ n-1 & n \text{ is even} \end{cases}$$

d) neither one-to-one nor onto



$f(n) = 1$

Ex 2: Determine if the following functions are one-to-one and/or onto

a) $\begin{cases} f:\mathbb{Z}\to\mathbb{Z} \\ f(n)=n^2+1 \end{cases}$

not one-to-one

$-1 \to 2$
$1 \to 2$

take $n_1=1$
$n_2=-1$

we notice that

$n_1 \neq n_2$ and $\underbrace{f(n_1)}_{1^2+1 \,=\, 2} = \underbrace{f(n_2)}_{(-1)^2+1 \,=\, 2}$

not onto

if we take

$y=(-1)$. since n^2+1 is always positive there is no $n \in \mathbb{Z}$ $f(n)=(-1)$

b $\begin{cases} f: \mathbb{N} \to \mathbb{N} \\ f(n) = \text{sum of digits in } n \end{cases}$

$f(17) = 8$

$f(273) = 12$

not one-to-one

take $n_1 = 2$

$n_2 = 11$

we have

$n_1 \neq n_2$ and $\underbrace{f(n_1)}_{\substack{\shortparallel \\ 2}} = \underbrace{f(n_2)}_{\substack{\shortparallel \\ 1+1 \\ \shortparallel \\ 2}}$

onto $\forall_{b \in B} \exists_{a \in A} f(a) = b$

Given an arbitrary $n_0 \in \mathbb{N}$

If we take $a_0 = \underbrace{111\ldots1}_{n_0 \text{ times}}$

$f(a_0) = f(\underbrace{11\ldots1}_{n_0 \text{ times}}) = \underbrace{1+1+\ldots+1}_{n_0 \text{ times}} = n_0$

Since n_0 was arbitrary, we get $\forall_{n \in \mathbb{N}} \exists_{a \in \mathbb{N}} f(a) = n$

c) $\begin{cases} f:(0,\infty)\to(0,1) \\ f(x)=\dfrac{x}{1+x} \end{cases}$

one-to-one $\left[\forall_{a_1,a_2\in A}\; f(a_1)=f(a_2)\to a_1=a_2\right]$:

Given $\boxed{a_1,a_2\in(0,\infty)}$ two arbitrary elements

Assume $\boxed{f(a_1)=f(a_2)}$, we now show that $\boxed{a_1=a_2}$

Since $f(a_1)=f(a_2)$, we have $\dfrac{a_1}{1+a_1}=\dfrac{a_2}{1+a_2}$

$$\Downarrow$$

$$a_1(1+a_2)=a_2(1+a_1)$$

$$\Downarrow$$

$$a_1+\cancel{a_1a_2}=a_2+\cancel{a_1a_2}$$

$$\Downarrow$$

$$a_1=a_2$$

onto $\left[\forall_{b\in B}\ \exists_{a\in A}\ f(a)=b\right]$

Given $b_0\in(0,1)$ an arbitrary element, we show that $\exists_{x\in(0,\infty)}\ f(x)=b_0$

take $x=\dfrac{b_0}{1-b_0}$

note that:

1. x is positive (since numerator and denominator are positive)

2. $$f(x)=\frac{\frac{b_0}{1-b_0}}{1+\frac{b_0}{1-b_0}}=\frac{\frac{b_0}{\cancel{1-b_0}}}{\frac{1-b_0+b_0}{\cancel{1-b_0}}}=\frac{b_0}{1\cancel{-b_0}\cancel{+b_0}}=\frac{b_0}{1}=b_0$$

$$\Downarrow$$

$$f(x)=b_0$$

Draft

$$\begin{cases} f : (0,\infty) \to (0,1) \\ f(x) = \dfrac{x}{1+x} \end{cases}$$

$$\frac{x}{1+x} = b_0$$

$$\Downarrow$$

$$x = b_0 + b_0 \cdot x$$

$$x - b_0 x = b_0$$

$$(1-b_0)x = b_0$$

$$x = \frac{b_0}{1-b_0}$$

$$\frac{x}{1+x} = \frac{1}{2}$$

$$\Downarrow$$

$$2x = 1 + x$$

$$x = 1$$

$$\frac{x}{1+x} = \frac{1}{4}$$

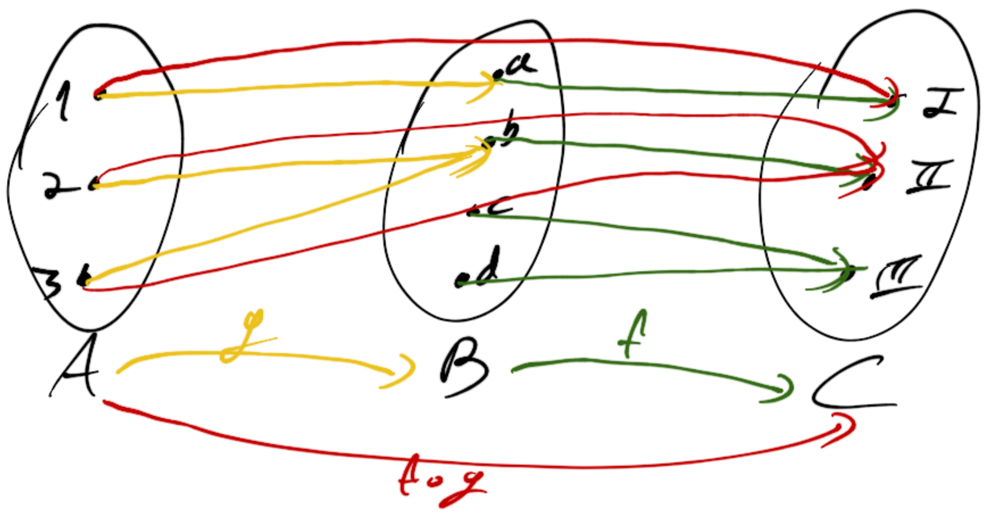
$$\Downarrow$$

$$4x = 1 + x$$

$$3x = 1$$

$$x = \frac{1}{3}$$

Definition: Let $g:A\to B$, $f:B\to$ be two functions. We define the composition function: $f\circ g: A\to C$ as a function from A to C with $f\circ g(a)=f(g(a))$



$f\circ g(1)=\mathrm{I}$

$f\circ g(2)=\mathrm{II}$

$f\circ g(3)=\mathrm{II}$

$\underbrace{f\left(\underbrace{g(3)}_{b}\right)}_{\mathrm{II}}$

given the following functions:

$$\begin{cases} f : \mathbb{R} \to \mathbb{R} \\ f(x) = x^2 + 1 \end{cases} \qquad \begin{cases} g : \mathbb{R} \to \mathbb{R} \\ g(x) = x + 2 \end{cases}$$

$$\begin{cases} f \circ g : \mathbb{R} \to \mathbb{R} \\ f \circ g(x) = (x+2)^2 + 1 \end{cases} \qquad \begin{cases} g \circ f : \mathbb{R} \to \mathbb{R} \\ g \circ f(x) = x^2 + 3 \end{cases}$$

$f \circ g(x) = f(g(x)) = f(x+2) =$
$= (x+2)^2 + 1$

$g \circ f(x) = g(f(x)) =$
$= g(x^2+1) = x^2 + 1 + 2$

Ex 3: Let $g: A \to B$, $f: B \to C$ be functions. If $f \circ g: A \to C$ is one-to-one, does it follow that both f and g are one-to-one? Justify your answer.

Solution: No!

$g(1) = a$

$g(2) = b$

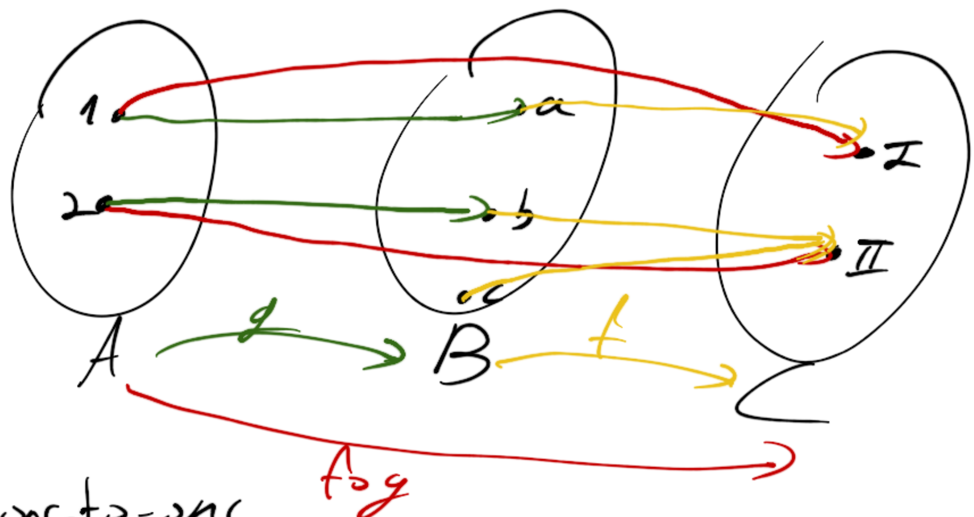
$f(a) = I$

$f(b) = II$

$f(c) = II$

$\Downarrow$

f is not one-to-one



$f \circ g(1) = I$

$f \circ g(2) = II$

$\Downarrow$

$f \circ g$ is one-to-one