Advanced Signal Processing

Mathematical Background

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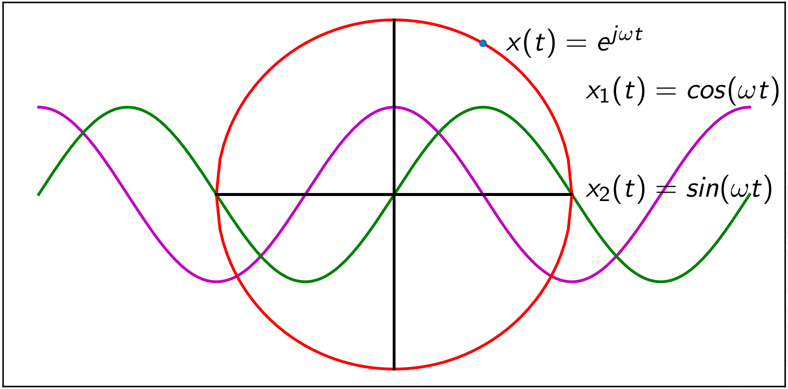
PROGRAMA DE MAESTRÍA EN TELECOMUNICACIONES

Outline

Oscillations Everywhere



Discrete-time exponential

- a frequency ω (units: radians)
- an initial phase ϕ (units: radians)
- an amplitude A

$$x[n] = Ae^{j(\omega n+\phi)}$$

$$= A[cos(\omega n + \phi) + jsin(\omega n + \phi)]$$

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&= A[cos(\omega n + \phi) + jsin(\omega n + \phi)]
\end{aligned}
$$

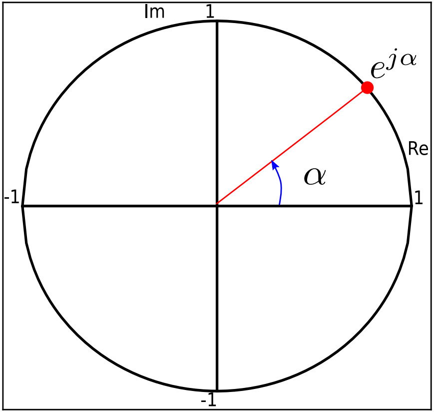
Why complex exponentials

Advantages

- Every sinusoid can always be written as a sum of sine and cosine
- math is simpler: trigonometry becomes algebra
- phase shifts is simple multiplication
- notation is simpler

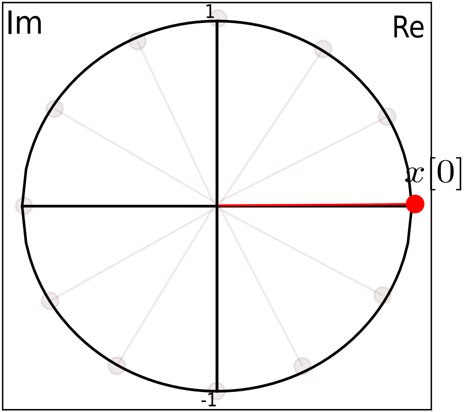
Complex Exponentials I

$$e^{j\alpha} = cos\alpha + jsen\alpha \qquad \left|e^{j\alpha}\right| = 1$$



Complex Exponentials II

$$x[n] = e^{j\omega n};\ \ x[n+1] = e^{j\omega}x[n]$$



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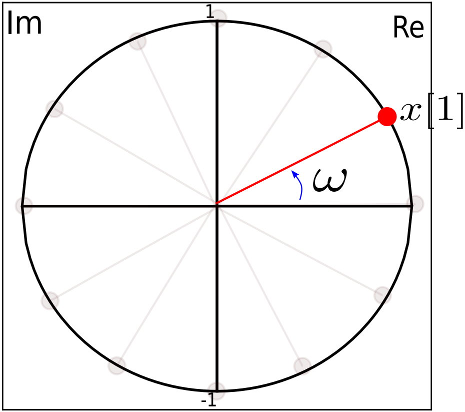

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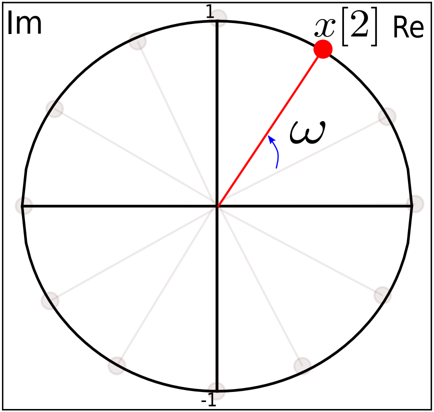

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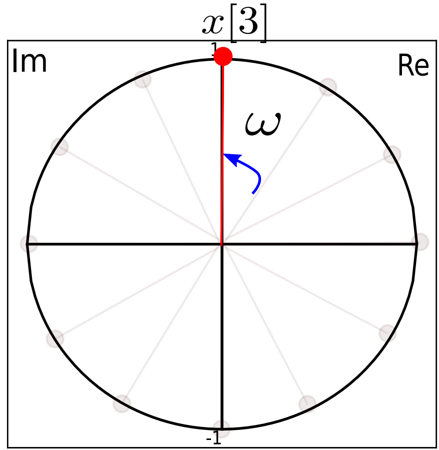

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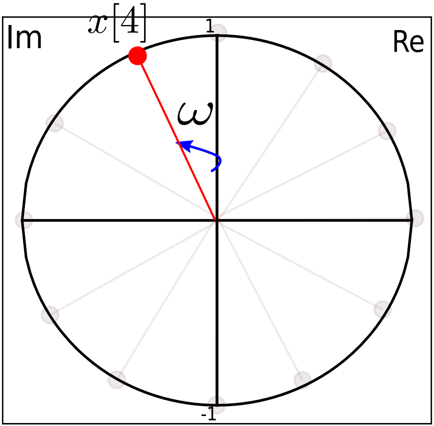

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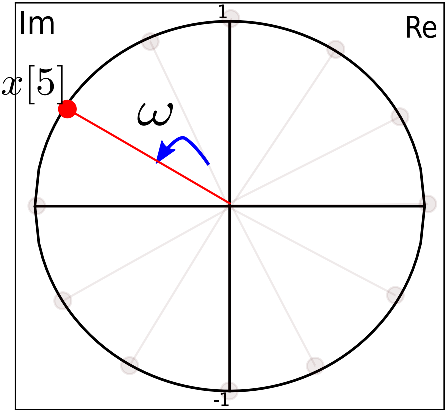

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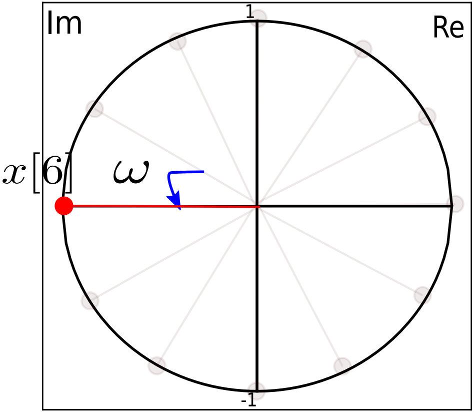

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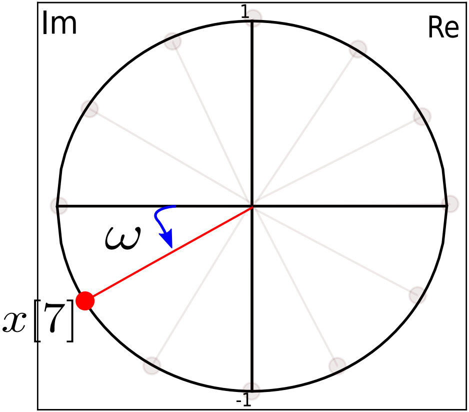

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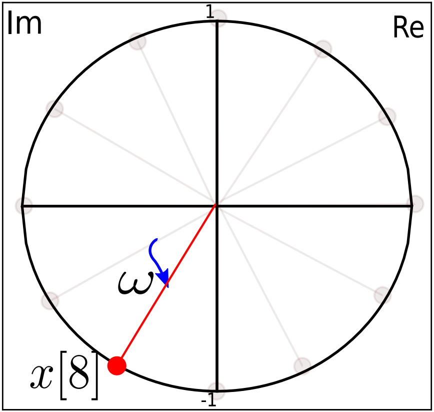

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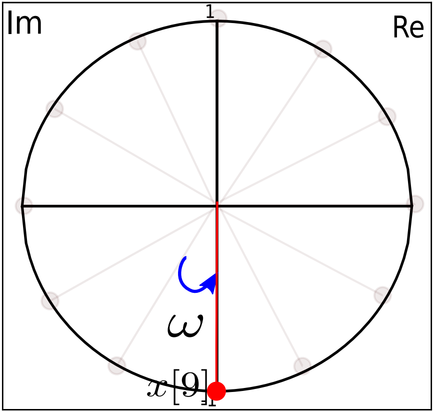

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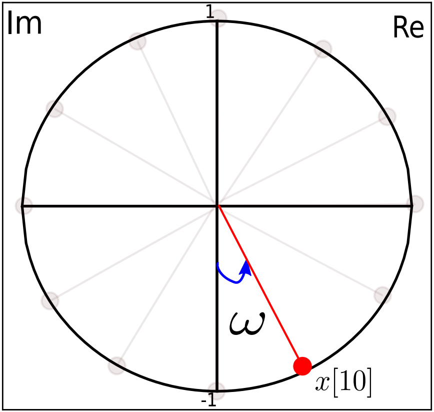

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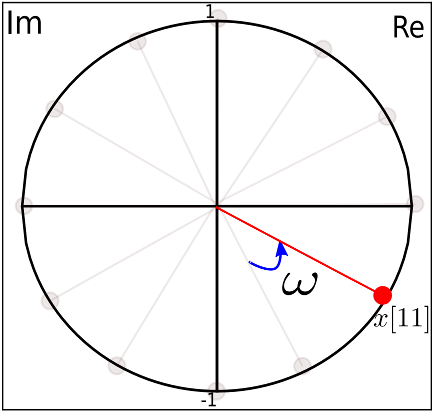

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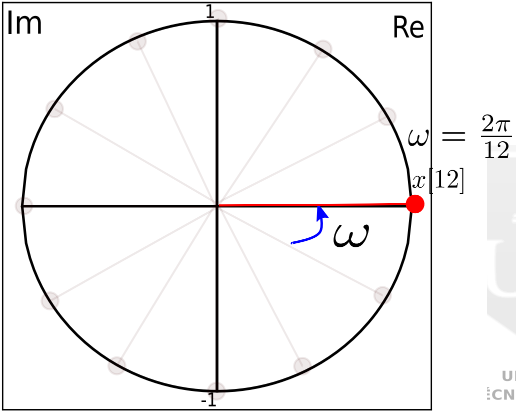
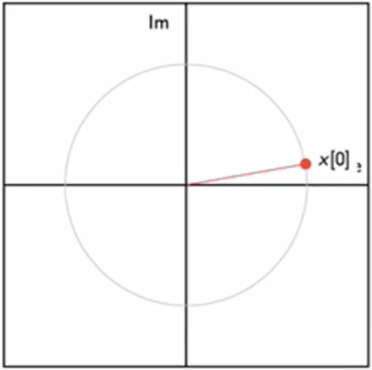


Complex Exponentials - Initial Phase

$$x[n] = e^{j(\omega n + \phi)};\ \ x[n+1] = e^{j\omega} x[n],\ \ x[0] = e^{j\phi}$$



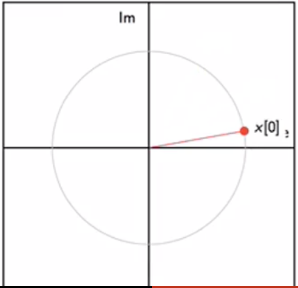
Complex Exponentials - Discrete signasl

$$x[n] = e^{j\omega n};\ x[n+1] = e^{j\omega} x[n]$$

WARNING

Not every sinusoid is periodic in discrete time, it is only periodic if:

$$\omega = \frac{M}{N} 2\pi, M, N, \in \mathbb{N}$$



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$$x[n] = x[n+N]$$

$$e^{(j\omega n+\phi)} = e^{j(\omega(n+N)+\phi)}$$

$$e^{j\omega n}e^{j\phi} = e^{j\omega n}e^{j\omega N}e^{j\phi}$$

$$e^{j\omega N} = 1$$

$$\omega N = 2M\pi, \;\; M \in \mathbb{Z}$$

$$\omega = \frac{M}{N}2\pi$$

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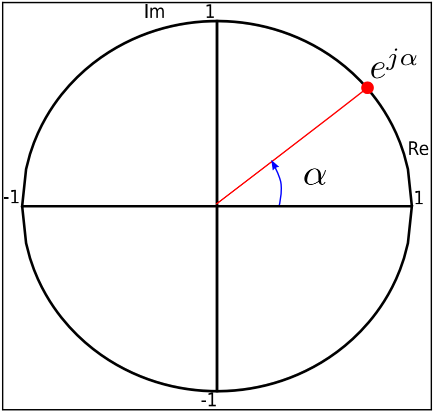
Aliasing 2π-periodicity

One point many names



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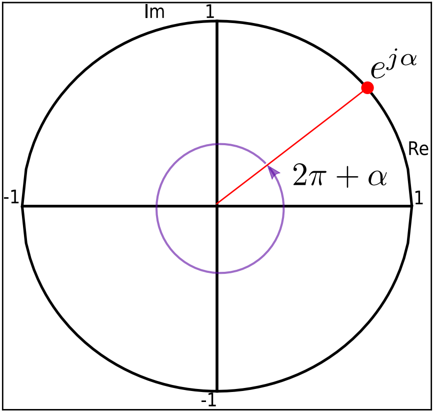

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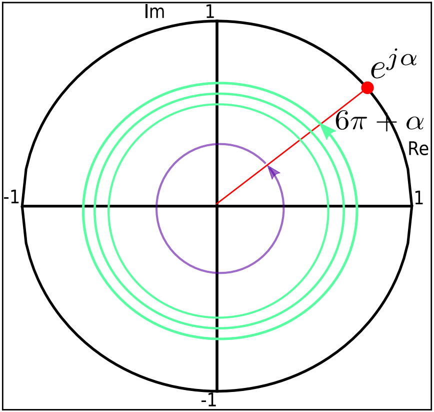


Outline

Why Vector Spaces

- same framework for different classes of signals
- same framework for continuos-time signals
- easy explanation of the Fourier transform
- easy explanation of sampling and interpolation
- useful in approximation and compression
- fundamental in communication system design
- vector spaces are very general objects
- vector spaces are defined by their properties
- if the properties of a vector space are satified, you can use all the tools for the space

Vector Spaces

- $\mathbb{R}^2$, $\mathbb{R}^3$: Euclidean space, geometry
- $\mathbb{R}^N$, $\mathbb{C}^N$: Linear algebra

Others less know spaces

- $\ell_2(\mathbb{Z})$: space of squarable-summable infinite sequences
- $L_2([a, b])$: space of square-integrable functions over an interval

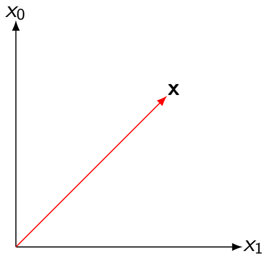
IMPORTANT

Vectors can be Functions

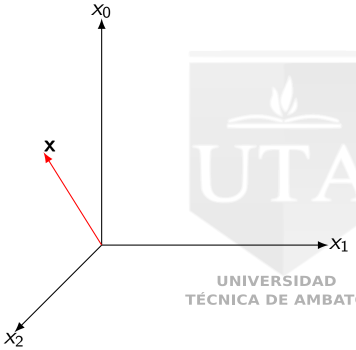
Vector Spaces representation

Some vectors space can be represented graphically

$\mathbb{R}^2$: $\mathbf{x} = [x_0 \ x_1]^T$



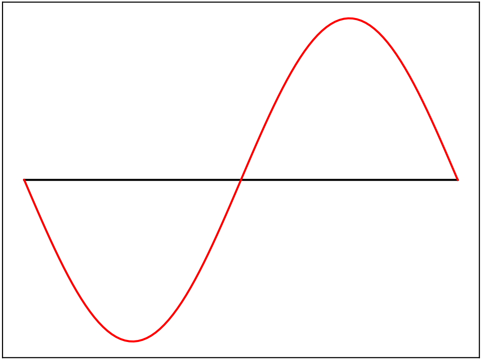
$\mathbb{R}^3$: $\mathbf{x} = [x_0 \ x_1 \ x_2]^T$



Vector Spaces representation

$$L_2([-1,1]) : \mathbf{x} = x(t), \;\; t \in [-1,1]$$

$x = sin(\pi t)$



Formal properties of a vector Vector Spaces:

For $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in V$ and , $\beta \in \mathbb{C}$

- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
- $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$
- $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$
- $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$
- $\exists\ 0 \in V \mid \mathbf{x} + 0 = 0 + \mathbf{x} = \mathbf{x}$
- $\forall\mathbf{x} \in V\ \exists(-\mathbf{x}) \mid \mathbf{x} + (-\mathbf{x})$

Additional properties of a vector Vector Spaces:

Something More

Measure and compare vectors: **inner product (aka dot product)**

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

- measure of similarity between vectors
- inner product is zero? vectors are orthogonal (maximally different)

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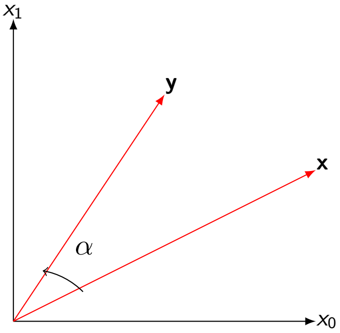
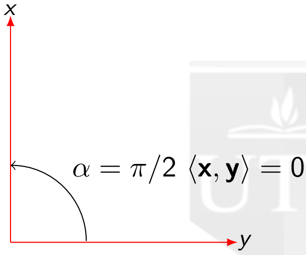
Inner Product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x_0 y_0} + \mathbf{x_1 y_1} = \|\mathbf{x}\| \|\mathbf{y}\| cos\alpha$$

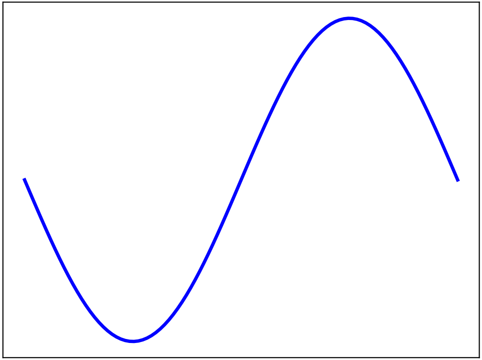




Inner Product in $L2[-1,1]$: the norm

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^{1} x(t)y(t) \qquad x = \sin(\pi t)$$

$$\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2 = \int_{-1}^{1} x(t)x(t) = \int_{-1}^{1} sin^2(\pi t)dt = 1$$



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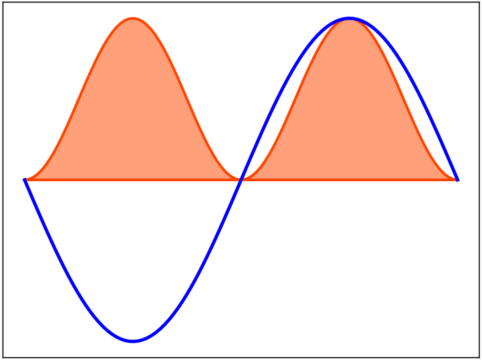
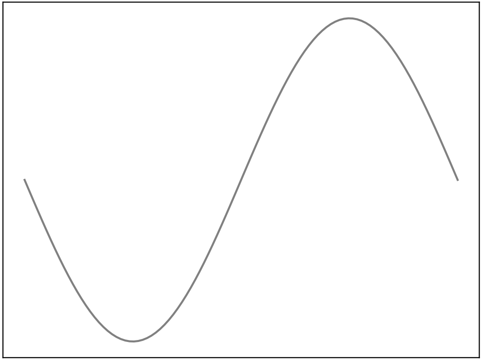


Inner Product in $L2[-1,1]$: the norm

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^{1} \sqrt{3/2}t \; sin(\pi t) dt = (2/\pi)\sqrt{3/2} \approx 0.78$$



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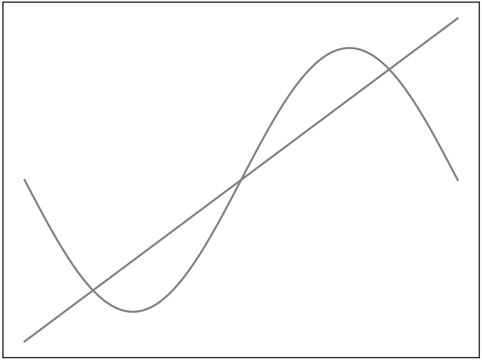

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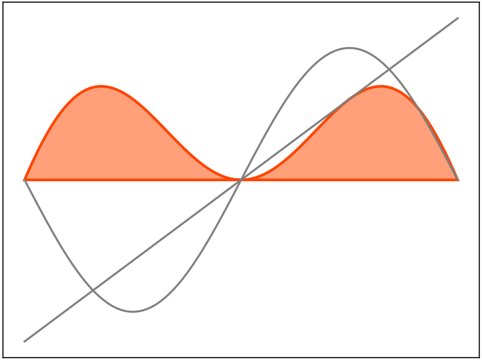


Norm and Distance

- inner product defines a norm: $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$
- $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$
- in $L_2[-1, 1]$ the distance is the mean square error

$$\|\mathbf{x} - \mathbf{y}\| = \int_{-1}^{1} |x(t) - y(t)|^2 dt$$

- inner product for finite-length vectors

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=0}^{N-1} x^*[n] y[n]$$

Infinite-Length Signals

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{n=-\infty}^{\infty} x^*[n]y[n]$$

- it requires sequences to be square-summable $\sum |x[n]|^2 < \infty$
- $\sum |x[n]|^2 < \infty$ means finite energy
- space of square-summable sequences $\ell_2(\mathbb{Z})$

Hilbert Space

1. a vector space: $H(V, \mathbb{C})$
2. an inner product: $\langle ., . \rangle : V \times V \to \mathbb{C}$
3. completeness

Outline

Complex Exponentials

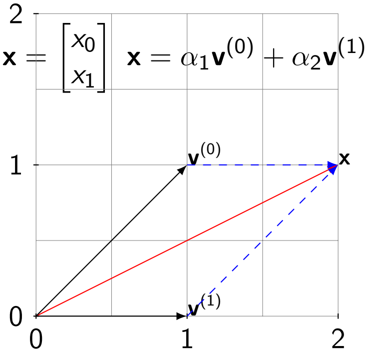
Signal Processing and Vector Spaces

Bases

Subspace-based approximations

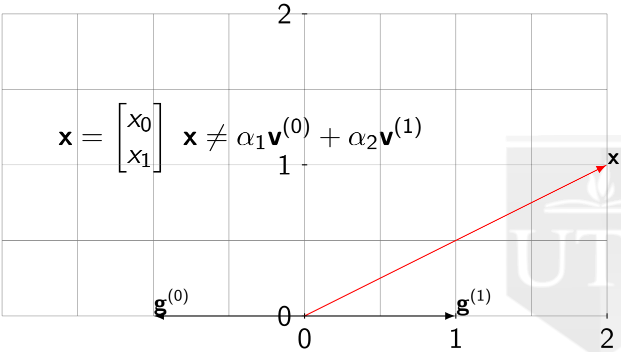
Bases

Find a set of vectors $\{\mathbf{w}^{(k)}\}$ so that we can write any vector as a a linear combination of the $\{\mathbf{w}^{(k)}\}$



Bases

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \quad \mathbf{x} \neq \alpha_1 \mathbf{v}^{(0)} + \alpha_2 \mathbf{v}^{(1)}$$



Bases - Definition

Given:

- a vector space H
- a set of K vectors from H: $W = \mathbf{w}^{(k)}{}_{k=0,1,\ldots..,K-1}$

W is a basis for H if:

1. we can write for all $\mathbf{x} \in H$:

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}, \ \alpha_k \in \mathbb{C}$$

2. the coefficients α_k are unique

Infinite-dimensional spaces

- infinite-dimensional

$$\mathbf{x} = \sum_{k=0}^{\infty} \alpha_k \mathbf{w}^{(k)}$$

- For instance: basis for $\ell_2(\mathbb{Z})$

$$\mathbf{e}^k = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad k\text{-th } \textit{position}, \;\; k \;\in \mathbb{Z}$$

Functions vector spaces

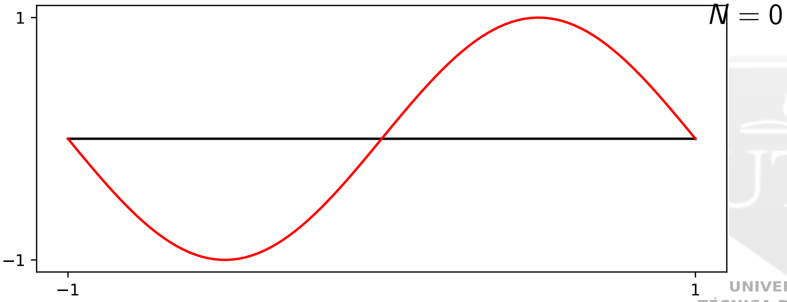
- functions for vector spaces

$$f(t) = \sum_k \alpha_k h^{(t)}(t)$$

- the Fourier basis for [-1,1], is one of the most famous function basis:
  $\frac{1}{\sqrt{2}}$, $\cos(\pi t)$, $\sin(\pi t)$, $\cos(2\pi t)$, $\sin(2\pi t)$, $\cos(3\pi t)$, $\sin(3\pi t)$....
- can we represent any square-integrable function over $[-1, 1]$ as a linear combination of the fourier basis functions?

Square Wave

$$\sum_{k=0}^{N} \frac{\sin(2k+1)\pi t}{2k+1} = \sum_{k=0}^{N} \frac{\mathbf{w}^{4k+2}}{2k+1}$$



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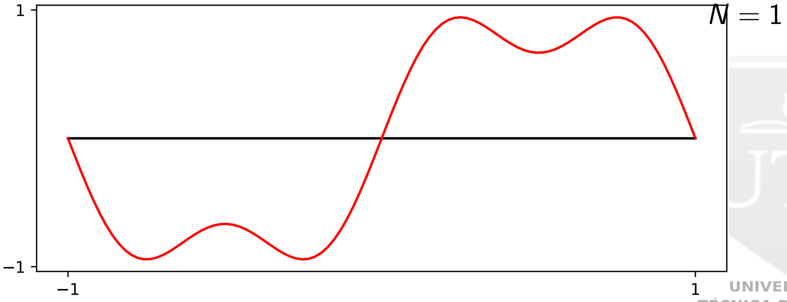

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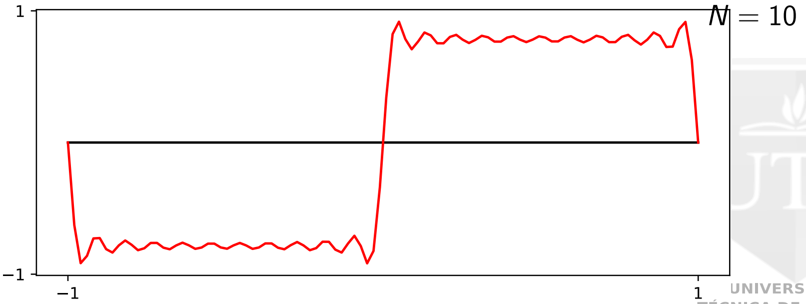

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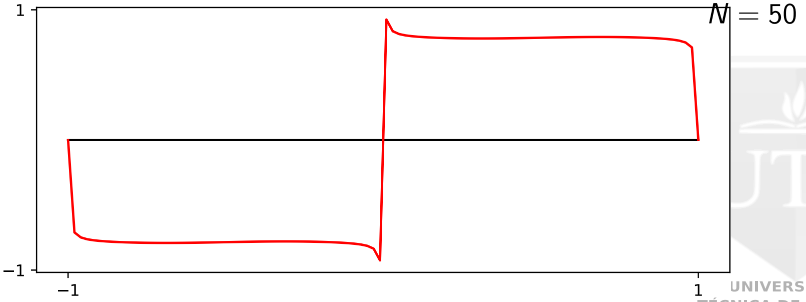
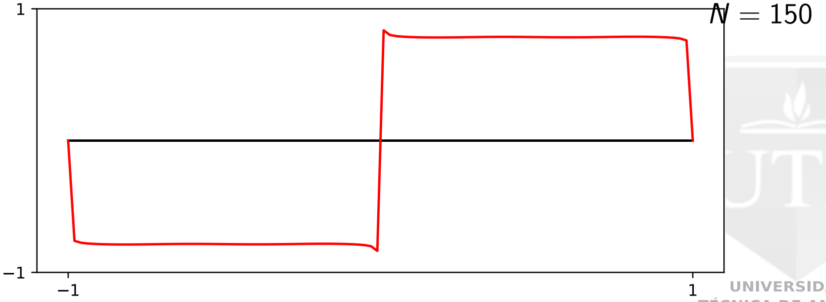


Square Wave

$$\sum_{k=0}^{N} \frac{\sin(2k+1)\pi t}{2k+1} = \sum_{k=0}^{N} \frac{\mathbf{w}^{4k+2}}{2k+1}$$



Special Bases

- Orthogonal basis

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = 0 \text{ for } k \neq n$$

- Orthonormal basis

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = \delta[n-k]$$

if the vectors are not the same vector, the inner product will be equal to zero

- Gram-Schmidt algorithm (Orthonormalize)

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Basis expansion

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}$$

- how do we find the α's?

  Orthonormal bases are the best

$$\alpha_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle$$

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Change of Basis

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} = \sum_{k=0}^{K-1} \beta_k \mathbf{v}^{(k)}$$

- if $\{\mathbf{v}^{(k)}\}$ is orthonormal:

$$\beta_h = \langle \mathbf{v}^{(h)}, \mathbf{x} \rangle$$

$$= \langle \mathbf{v}^{(h)}, \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} \rangle$$

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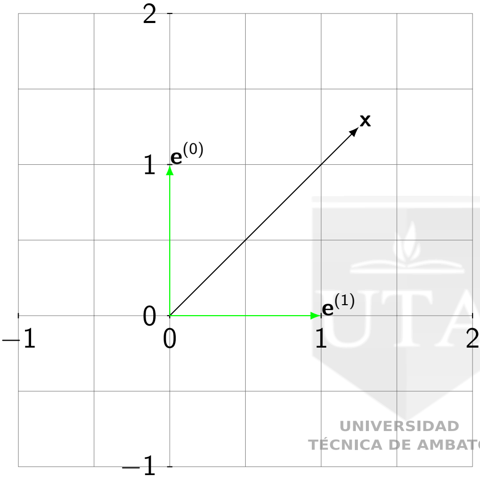
Change of Basis

$$
\begin{aligned}
\beta_h &= \sum_{k=0}^{K-1} \alpha_k \langle \mathbf{v}^{(h)}, \mathbf{w}^{(k)} \rangle \\
&= \sum_{k=0}^{K-1} \alpha_k c_{hk}
\end{aligned}
$$

$$
= \begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0(K-1)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{(K-1)0} & c_{(K-1)1} & \cdots & c_{(K-1)(K-1)} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{(K-1)} \end{bmatrix}
$$

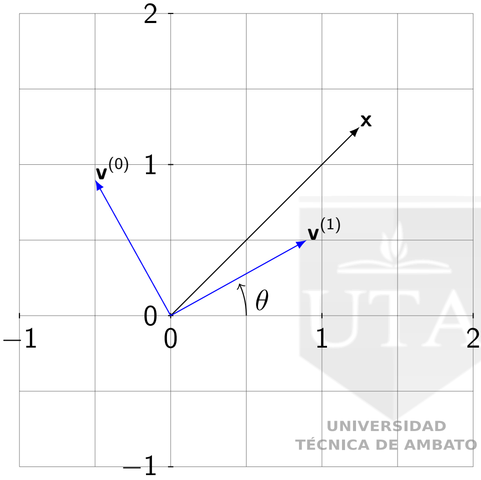
Example: change of basis

- canonical basis
  $E = \{\mathbf{e}^{(0)}, \mathbf{e}^{(1)}\}$
- $\mathbf{x} = \alpha_0 \mathbf{e}^{(0)} + \alpha_1 \mathbf{e}^{(1)}$



Example: change of basis

- canonical basis
  $E = \{\mathbf{e}^{(0)}, \mathbf{e}^{(1)}\}$
- $\mathbf{x} = \alpha_0 \mathbf{e}^{(0)} + \alpha_1 \mathbf{e}^{(1)}$
- new basis
  $V = \{\mathbf{v}^{(0)}, \mathbf{v}^{(1)}\}$
  $\mathbf{v}^{(0)} = [\cos\theta \;\; \sin\theta]^T$
  $\mathbf{v}^{(1)} = [-\sin\theta \;\; \cos\theta]^T$
- $\mathbf{x} = \beta_0 \mathbf{v}^{(0)} + \beta_1 \mathbf{v}^{(1)}$



Example: change of basis

- new basis is orthonormal:

$$c_{hk} = \langle \mathbf{v}^{(h)}, \mathbf{e}^{(k)} \rangle$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha_0 \\ 1 \end{bmatrix} = \mathbf{R}_\alpha$$

- $\mathbf{R}$ rotation matrix (rotate coordinates in computer vision)
- $\mathbf{R}^T\mathbf{R} = I$

Outline

Complex Exponentials

Signal Processing and Vector Spaces

Bases

Subspace-based approximations

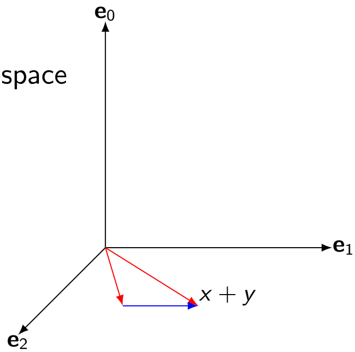
Vector subspace

Definition

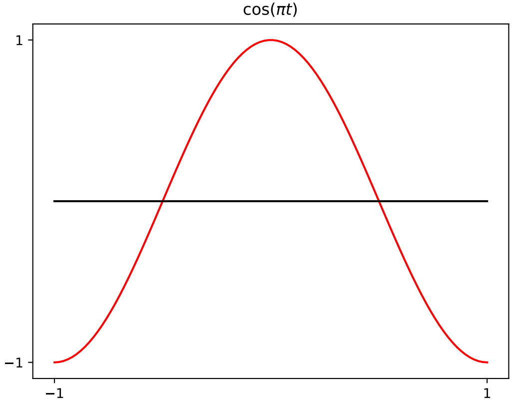
A subset of vectors closed under addition and scalar multiplication

$\mathbb{R}^2 \subset \mathbb{R}^3$

a plane is a subspace



Subspace of symmetric functions over $L_2[-1, 1]$



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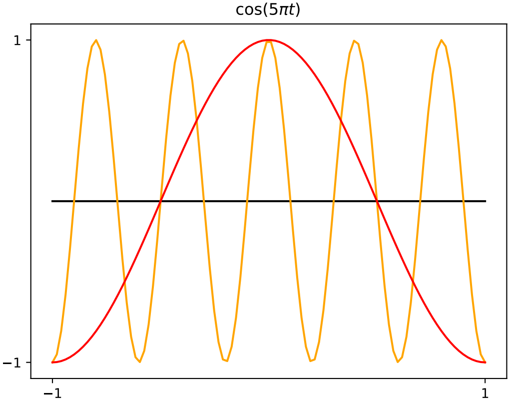

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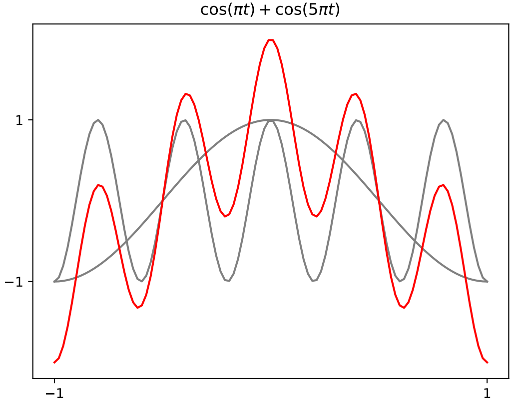
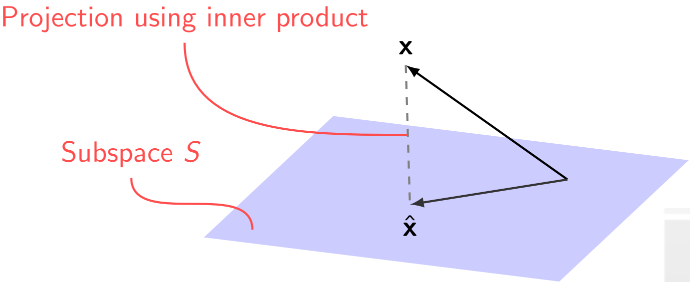


Approximation



Problem

- vector $\mathbf{x} \in V$
- subspace $S \supseteq V$
- vector approximate $\mathbf{x}$ *with* $\hat{\mathbf{x}} \ \in \ S$

Least-Squares Approximation

- $\{\mathbf{s}^{(k)}\}_{k=0,1,\ldots\ldots,K-1}$ orthonormal basis for S
- orthogonal projection:

$$\hat{\mathbf{x}} = \sum_{k=0}^{K-1} \langle \mathbf{s^{(k)}}, \mathbf{x} \rangle \mathbf{s^{(k)}}$$

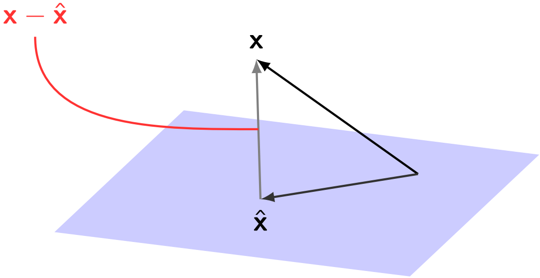
- orthogonal projection is the best approximation over S:
  - orthogonal projection has minimum norm error

$$\arg\min_{y \in S} \|\mathbf{x} - \mathbf{y}\| = \hat{\mathbf{x}}$$

  - error is orthogonal to approximation:

$$\langle \mathbf{x} - \hat{\mathbf{x}}, \hat{\mathbf{x}} \rangle = 0$$

Least-Squares Approximation



Example: polynomial approximation

- vector space $P_N[-1,1] \subset L_2[-1,1]$
- $\mathbf{p} = a_0 + a_1 t + \ldots + a_{N-1} t^{N-1}$
- a self-evident, naive basis: $\mathbf{s}^{(k)} = t^{(k)}$, $k = 0, 1, 2, \ldots, N-1$
- but naive basis is not orthonormal

Example: polynomial approximation

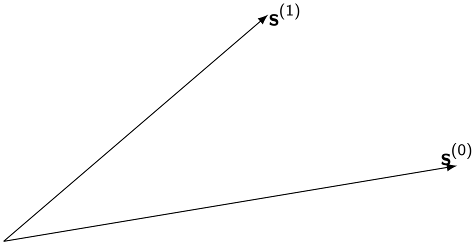
approximate $\mathbf{x} = \sin t \ \in \ L_2[-1, 1]$ *over* $P_3[-1, 1]$

- build orthonormal basis from naive basis
- Gram- Schmidt orthonormalization

$$\{\mathbf{s}^{(k)}\} \rightarrow \{\mathbf{u}^{(k)}\}$$

- project $\mathbf{x}$ over the orthonormal basis
- compute approximation error
- compare error to Taylor approximation (well known but not optimal over the interval)

Build orthonormal basis



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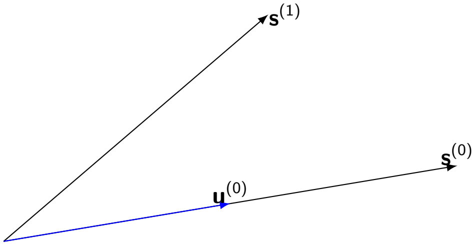

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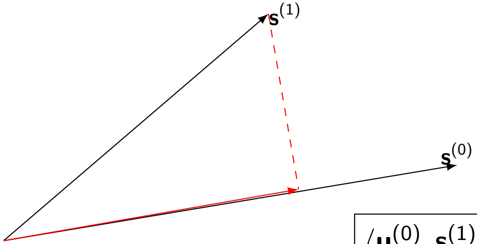
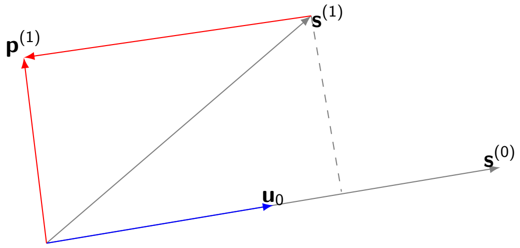


$$\boxed{\langle \mathbf{u}^{(0)}, \mathbf{s}^{(1)} \rangle \mathbf{u}^{(0)}}$$

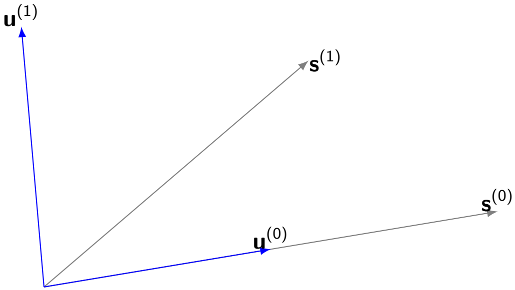
Build orthonormal basis



$$\boxed{p^{(k)} = s^{(k)} - \sum_{n=0}^{K-1} \langle \mathbf{u}^{(n)}, \mathbf{s}^{(k)} \rangle \mathbf{u}^{(n)}}$$

$$\boxed{p^{(1)} = s^{(1)} - \langle \mathbf{u}^{(0)}, \mathbf{s}^{(1)} \rangle \mathbf{u}^{(0)}}$$

Build orthonormal basis



$$\boxed{u^{(1)} = \frac{p^{(1)}}{||p^{(1)}||}}$$

Example - PYTHON

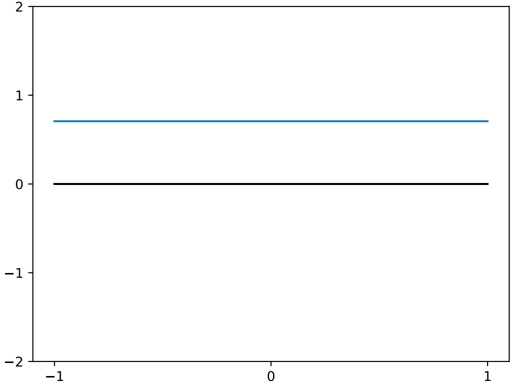
Polynomial Approximation

Approximate $\mathbf{x} = \sin t \ \in \ L_2[-1,1]$ *over* $P_3[-1,1]$

$$\mathbf{s}^{(k)} = t^{(k)}$$

$$1, t, t^2, t^3, \ldots, t^n \text{ for } k = 0, 1, 2, \ldots, N-1$$

Example - PYTHON



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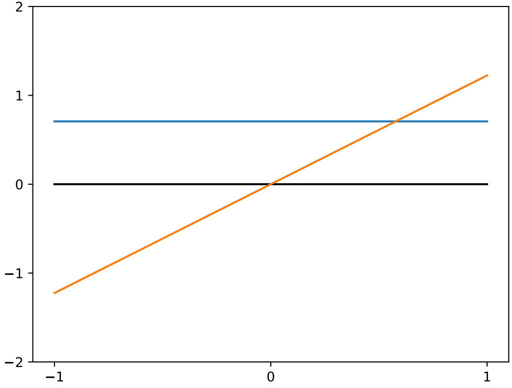

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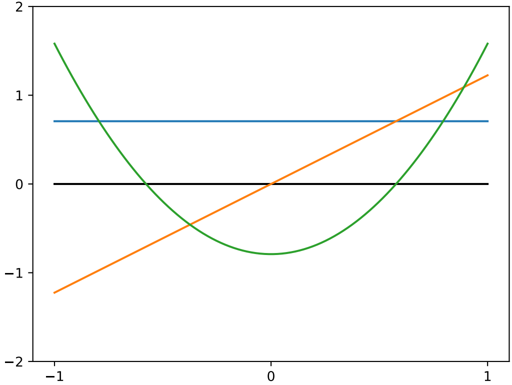

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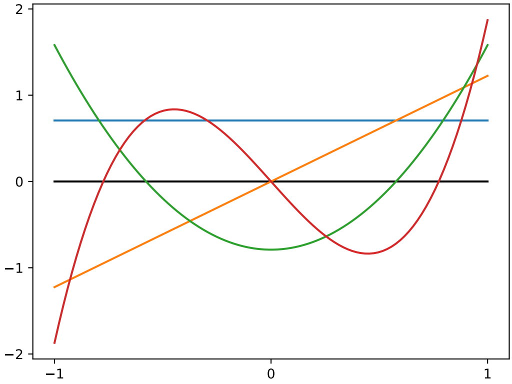

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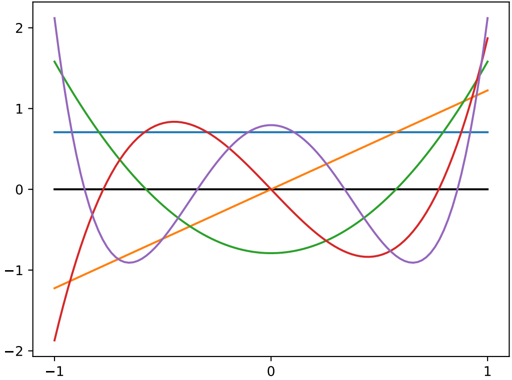


Example - PYTHON

$$\alpha_k = \langle \mathbf{u}^{(k)}, \mathbf{x}^{(k)} \rangle = \int_{-1}^{1} \mathbf{u}_k(t) \sin t dt$$

- $\alpha_0 = \langle \sqrt{1/2}, \sin t \rangle = 0$
- $\alpha_1 = \langle \sqrt{3/2}t, \sin t \rangle \approx 0.7377$
- $\alpha_2 = \langle \sqrt{5/8}(3t^2 - 1), \sin t \rangle = 0$

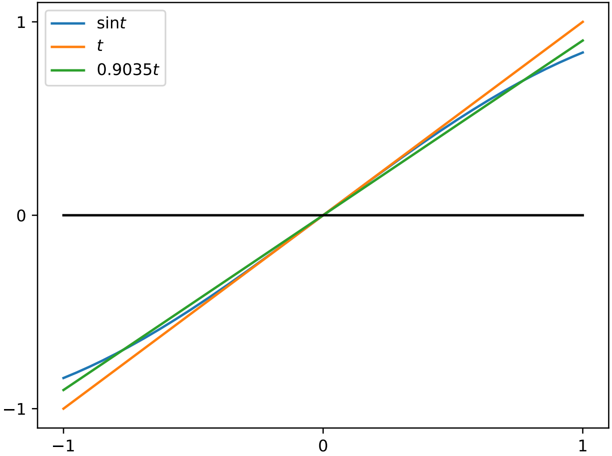
Using the orthogonal projection over $P_3[-1, 1]$:

$$\sin t \to \alpha_1 \mathbf{u}^{(1)} \approx 0.9035t$$

Using Taylor series:

$$\sin t \approx t$$

Example - PYTHON



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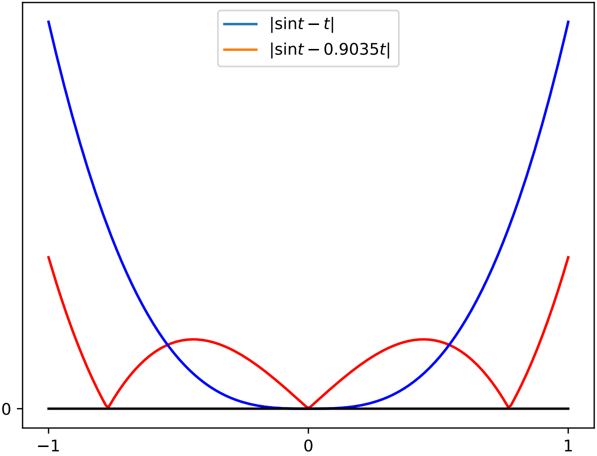

Example - PYTHON

- Taylor approximation is a local approximation
- The orthogonal projection minimizes the global distance

PYTHON - Practice