# Advanced Signal Processing

## Fourier Analysis and Wavelet Transform

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PROGRAMA DE MAESTRÍA EN TELECOMUNICACIONES

# Outline

# Oscillations Everywhere

- sustainable dynamic systems exhibit oscillatory behaviour
- intuitively things that don't move in circles can't last:
    - bombs
    - rockets
    - human beings...

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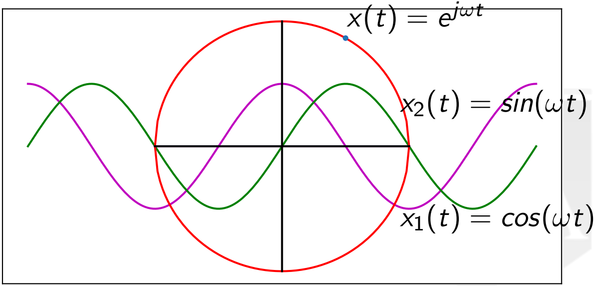
# Circular motion

- period $P$
- frequency $f = \frac{1}{P}$



# Circular Motion Cosine

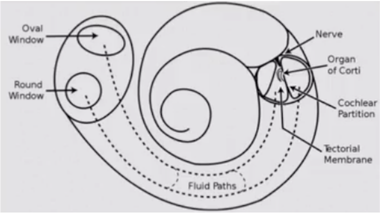
# Circular Motion Cosine-Sine

# Circular Motion Cosine-Sine

# Detect sinusoids

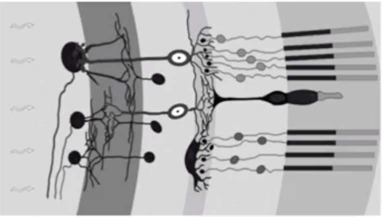
The human body has two receptors for sinusoidal

Cochlea (inner ear)




- air pressure sinusoids
- frequencies from 20 Hz to 20KHz

rods and cones (retina)



- electromagnetic sinusoids
- frequencies from 430 THz to 790 THz

# Frequency Domain

- humans analyze complex signals in terms of their sinusoidal components
- we can build instruments that “resonate” at one or multiple frequencies
- the “frequency domain” seems to be as important as the time domain

# Fourier Analysis

Can we decompose any signal into sinusoidal elements??
Yes, FOURIER ANALYSIS

analysis

- from time to frequency domain
- find the contribution of different frequencies
- discover "hidden" signal properties

synthesis

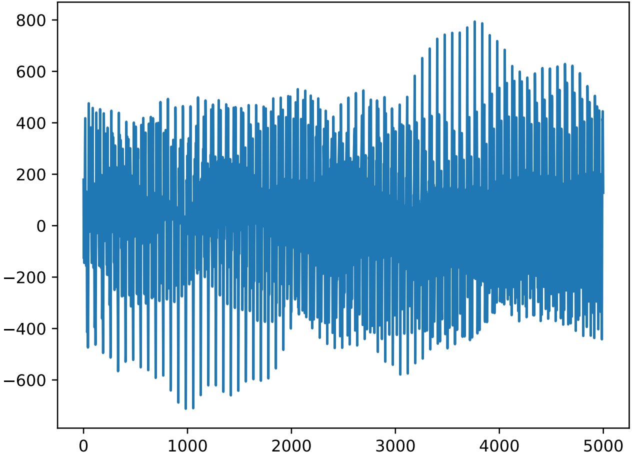
- from frequency to time domain
- create signals with known frequency content
- fit signals to specific frequency regions

# Outline

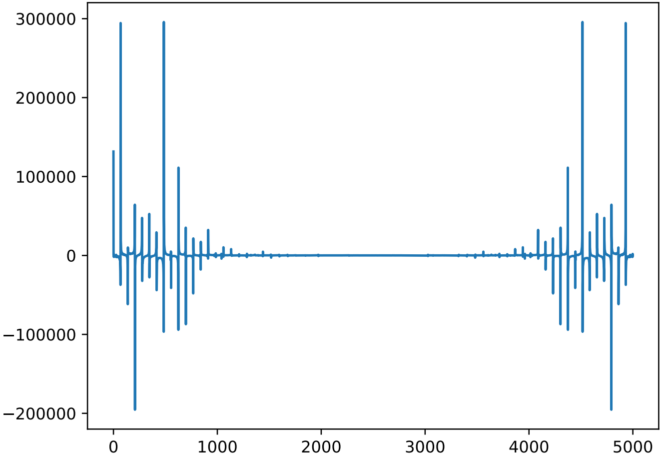
# Mathematical Setup

- Fourier analysis is a simple change of basis
- a change of basis is a change of perspective
- a good basis can reveal information

# Unknown Signal



# Unknown Signal



# Fourier Basis for $\mathbb{C}^N$

Claim: the set of $N$ signals in $\mathbb{C}^N$

$$w_k[n] = e^{\frac{2\pi}{N} nk}, \;\; n, k \;=\; 0, 1, 2, ..., N-1$$

is a orthogonal basis in $\mathbb{C}^N$

- k is the index of the signal
- n is the index of the element

$$\omega = \frac{2\pi k}{N}$$

# Fourier Basis - Vector Notation

In vector notation

$\{\mathbf{w}^{(k)}\}_{k=0,1,2,...N-1}$

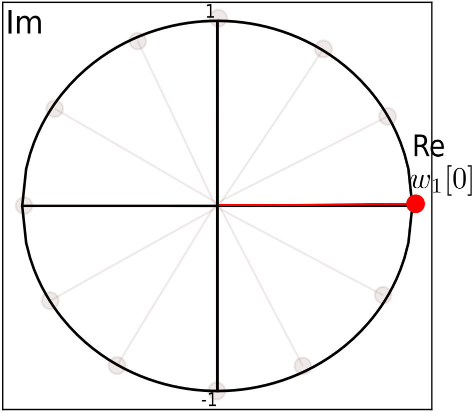
with

$$w_k[n] = e^{\frac{2\pi}{N} nk}, \; n, k \; = \; 0, 1, 2, ..., N-1$$

is a orthogonal basis in $\mathbb{C}^N$

$$\omega = \frac{2\pi k}{N}$$

# Complex Generating Machine



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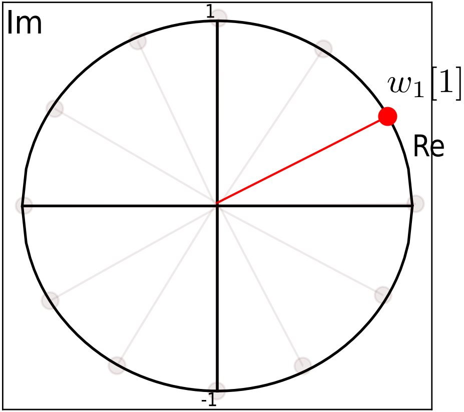

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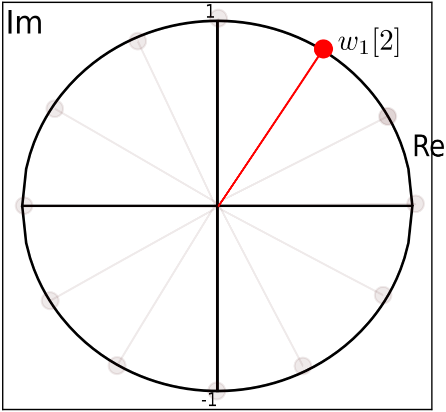

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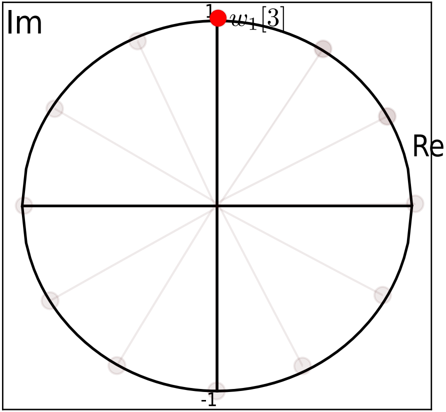

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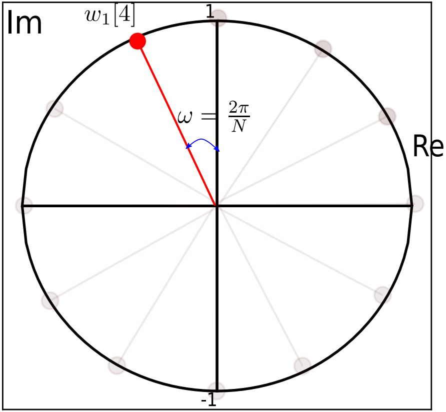

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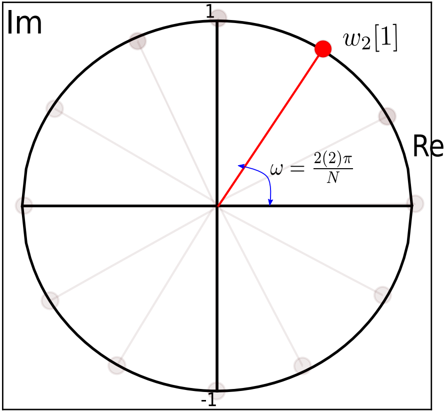
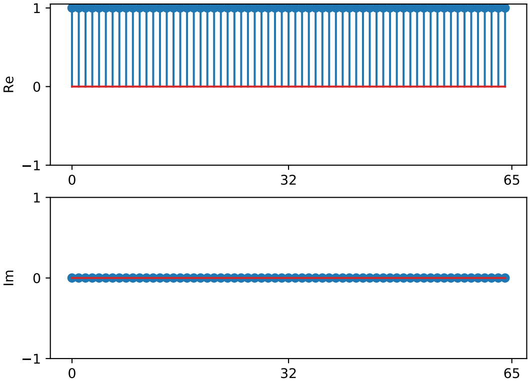


# Basis vector



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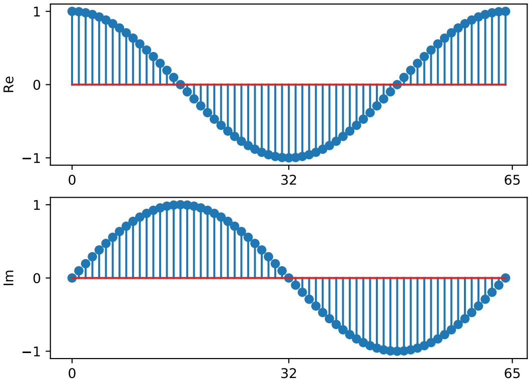

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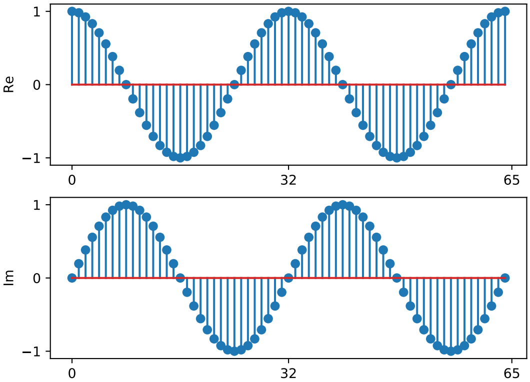

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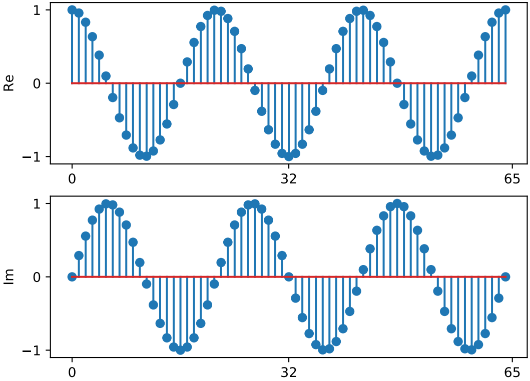

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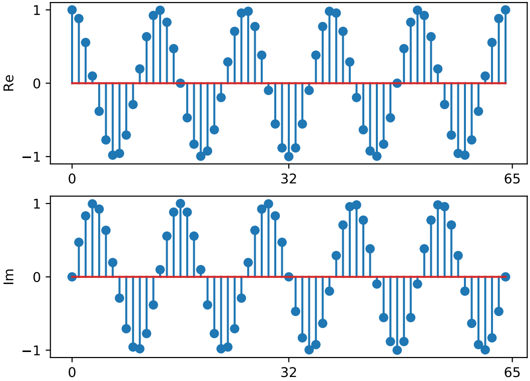

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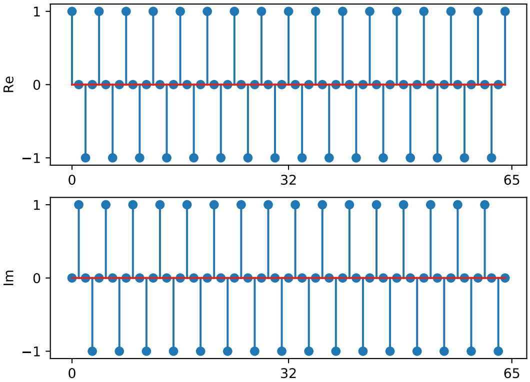

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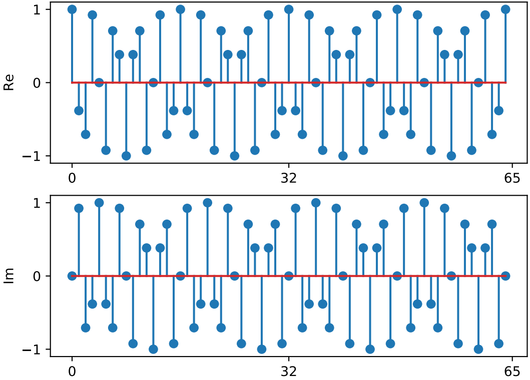

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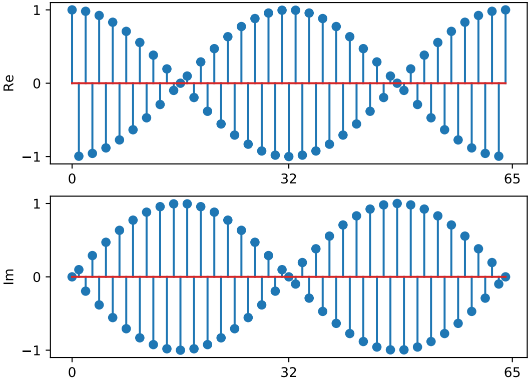

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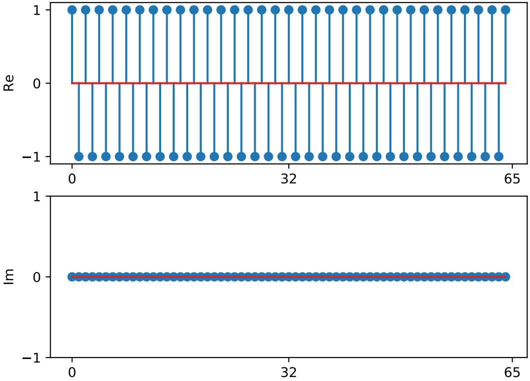

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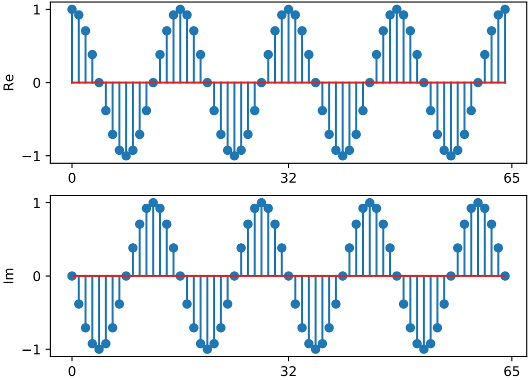

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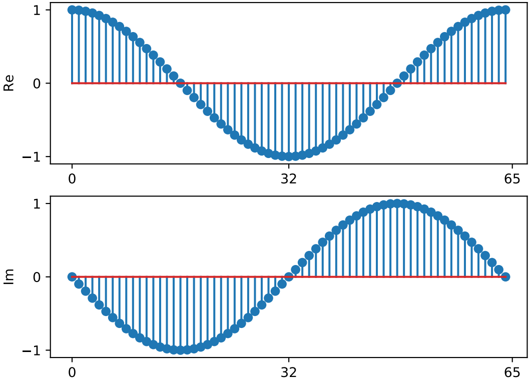
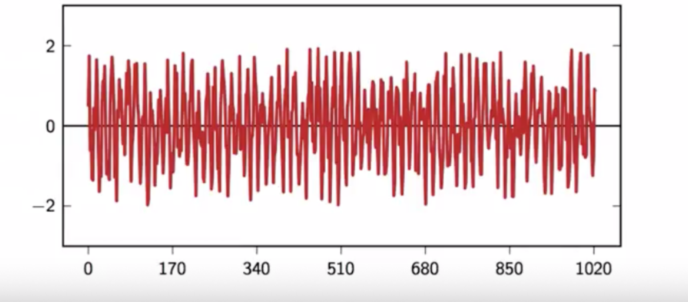


## Remarks

- N orthogonal vectors $\rightarrow$ basis for $\mathbb{C}^N$
- vectors are not orthonormal. Normalization swould be $1/\sqrt{N}$

# Mistery signal



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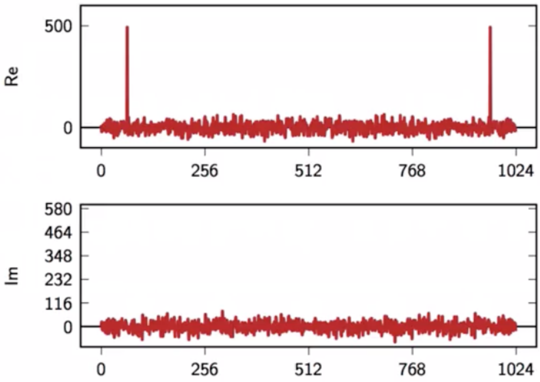


## Mistery signal

$x[n] = \cos(\omega n + \phi) + noise$

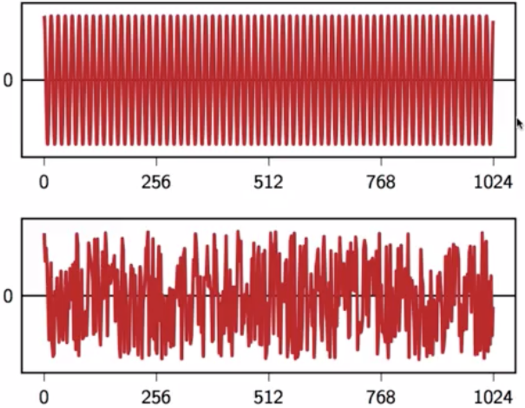
with

$\phi = 0$

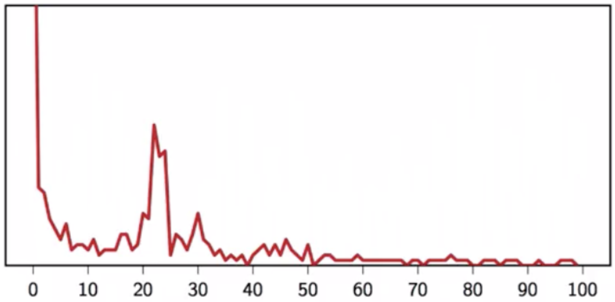
$$\omega = \frac{2\pi}{1024} 64$$

Peak occurs in 64

# Mistery signal revelead



# Example



# Mistery signal

- DFT main peak k=22
- 22 cycles over 2904 months

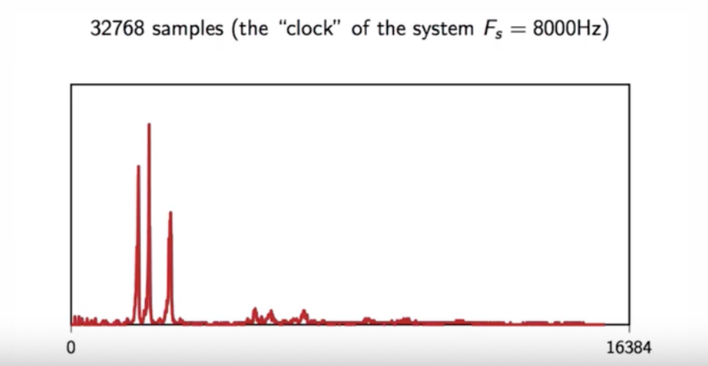
$$period = \frac{2904}{22} \approx 11 \text{ years}$$

# Labelling the frequency axis

if we know the clock of the system

- fastest (positive) frequency is $\omega = \pi$
- sinusoids at $\omega = \pi$ need two sample to do full revolution
- time between samples $T_s = 1/F_s$
- real-world period for fastest sinusoid $2T_s$
- real-world frequency for fastest sinusoid $F_s/2$

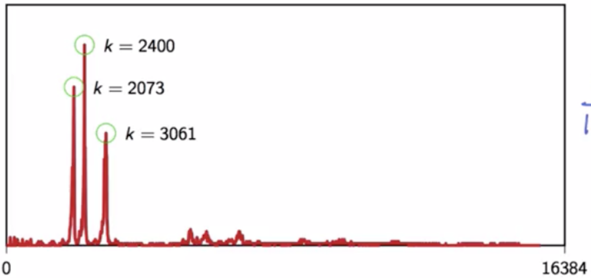
# Train Whistle




$F_s/2 \to k = N/2$
$16384 \to 4KHz$

# Train Whistle

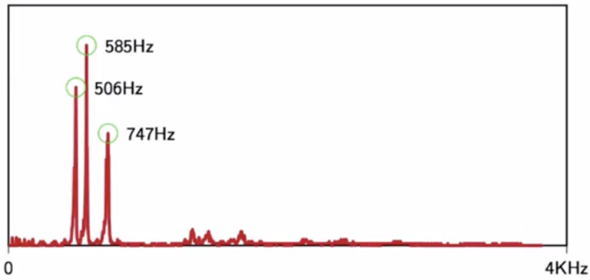
32768 samples (the "clock" of the system $F_s = 8000\text{Hz}$)




$\frac{4\text{kHz}}{16384} \cdot k$

# Train Whistle

the “clock” of the system $F_s = 8000\text{Hz}$

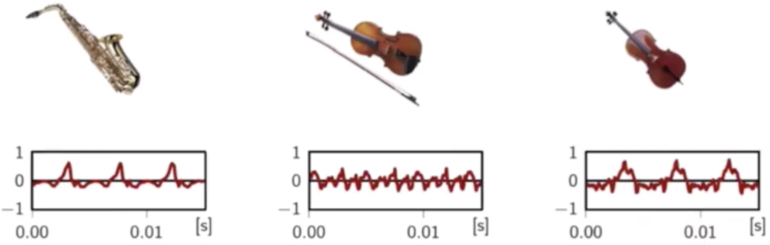


# Outline

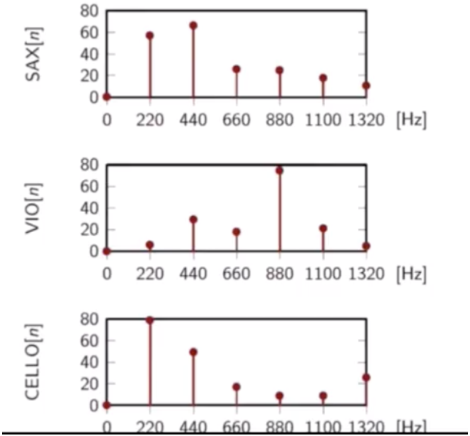
# Analysis of musical isntrument

- Pitch - Frequency
- Harmonics??
- What give the timbre of an instrument??

# Musical instrument



# Musical instrument



# Outline

Introduction to Fourier analysis

DFT - Change of basis

Musical Instruments

Short Time Fourier Transform

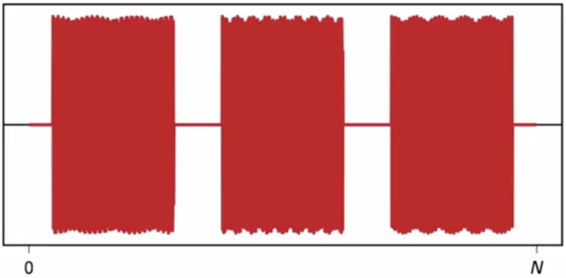
Spectogram

Advanced concepts in Fourier

# Dual-tone Multi Frequency Dialing

| | 1209Hz | 1336Hz | 1477Hz |
|---|---|---|---|
| 697Hz | 1 | 2 | 3 |
| 770Hz | 4 | 5 | 6 |
| 852Hz | 7 | 8 | 9 |
| 941Hz | * | 0 | # |

# Dialing 1-5-9



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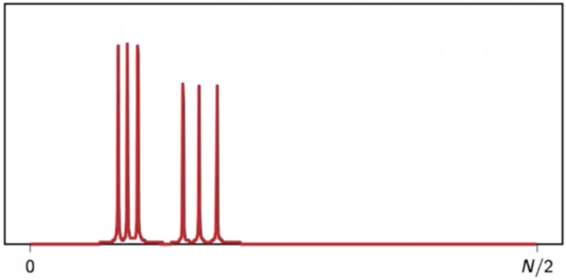


# Problem

- time representation hide the frequency
- frequency representation hide the time, We know frequency but why don't know when it happen

# Solution

Idea:

- take small signal pieces of length L
- compute DFT of each piece

$$X[m;k] = \sum_{n=0}^{L-1} x[m+n] e^{-j\frac{2\pi}{L}nk}$$

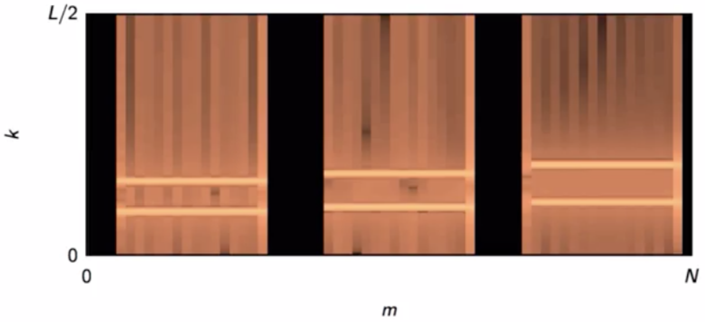
# Outline

# Spectogram

Idea:

- color code the magnitude: dark is small, white is large
- use $10log_{10}(|X[m;k]|)$
- plot spectra slices one after other

# Dialing 1-5-9 -spectogram



## Labelling the Spectogram

- width of the analysis windows??
- position of the windows (overlapping??)
- shape of the window(weighting samples??)

# wideband vs narrowband

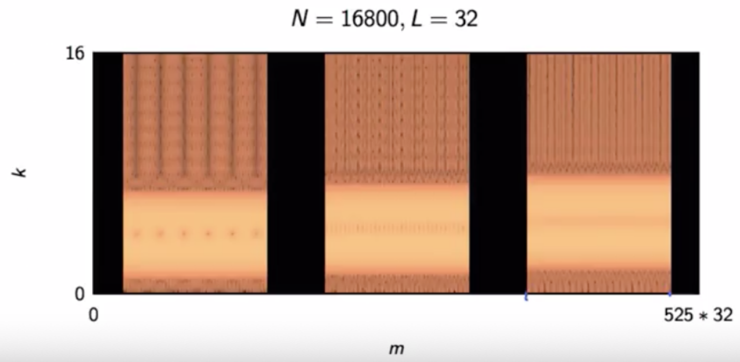
Long window: narrowband spectrogram

- long window $\Rightarrow$ more DFT points $\Rightarrow$ more frequency resolution
- long window $\Rightarrow$ more "things can happen" $\Rightarrow$ less precision in time

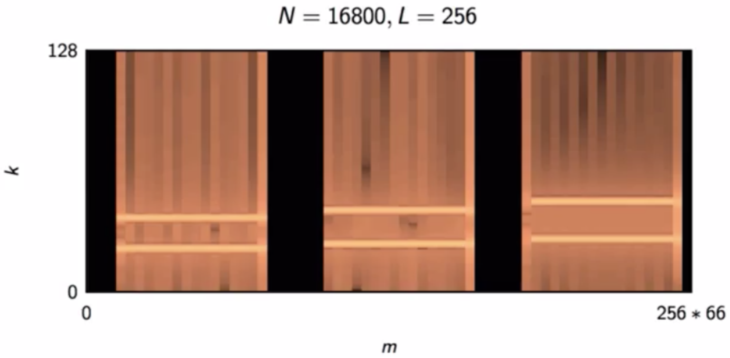
Short window: wideband spectrogram

- short window $\Rightarrow$ many time slices $\Rightarrow$ precise location of transitions
- short window $\Rightarrow$ fewer DFT points $\Rightarrow$ poor frequency resolution

# wideband vs narrowband



# wideband vs narrowband



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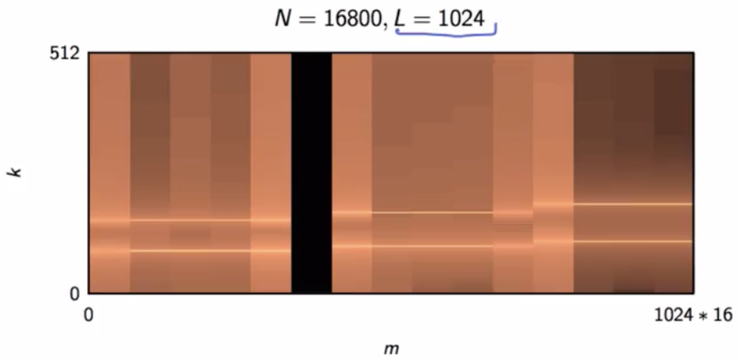
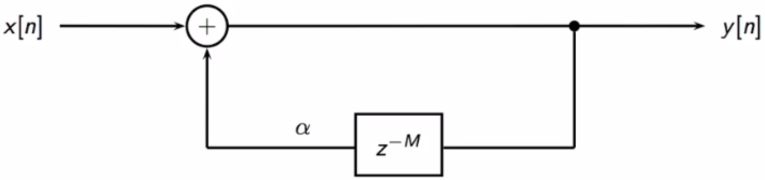


# Outline

# Discrete Fourier Series (DFS)

- DFS = DFT with periodicity explicit
- DFS maps an N-periodic signal onto an N-periodic sequence of Fourier coefficients
- the DFS of an N-periodic signal is mathematically equivalent to the DFT of one period
- DFS: only N Fourier coefficients capture all the information

# Karplus-Strong revisited and DFS




$$y[n] = \alpha\, y[n-M] + x[n]$$

# Karplus-Strong revisited and DFS

- choose a signal $\bar{x}[n]$ that is nonzero only for $0 \leq n < M$
- $\alpha = 1$ (for now)

$$y[n] = \underbrace{\bar{x}[0], \bar{x}[1], \ldots, \bar{x}[M-1]}_{1^{\text{st}}\text{ period}}, \underbrace{\bar{x}[0], \bar{x}[1], \ldots, \bar{x}[M-1]}_{2^{\text{nd}}\text{ period}}, \underbrace{\bar{x}[0], \bar{x}[1], \ldots}_{\ldots}$$

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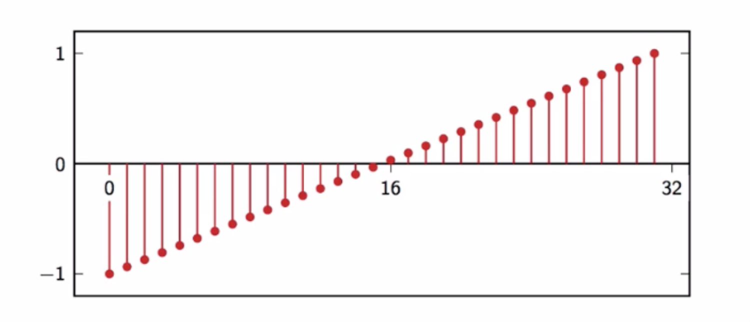

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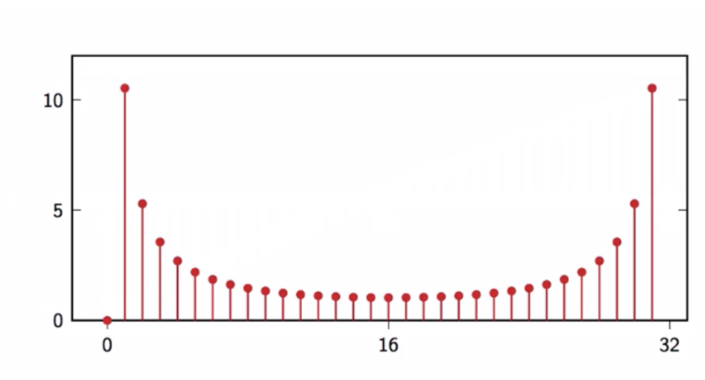

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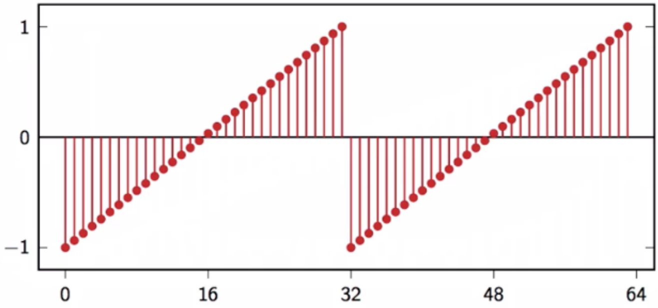

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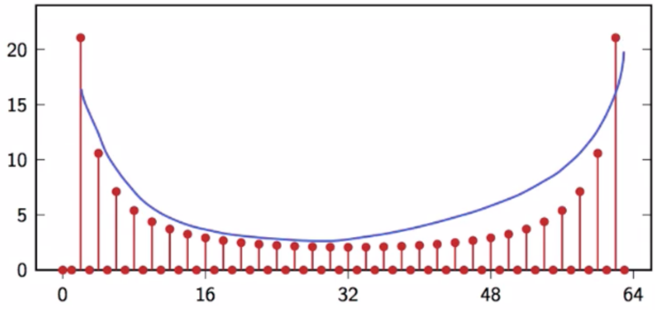

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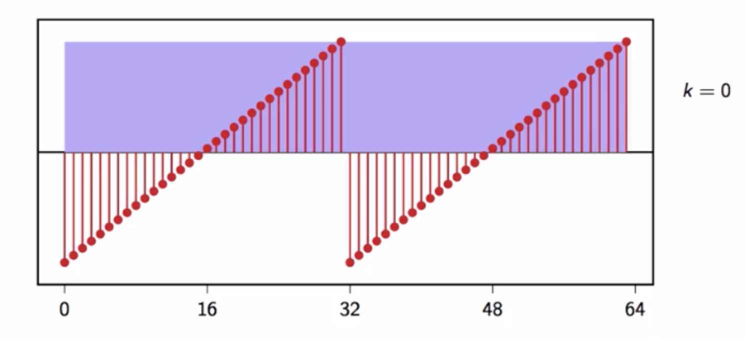

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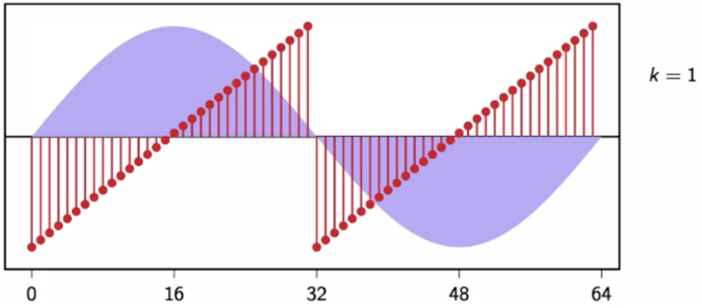

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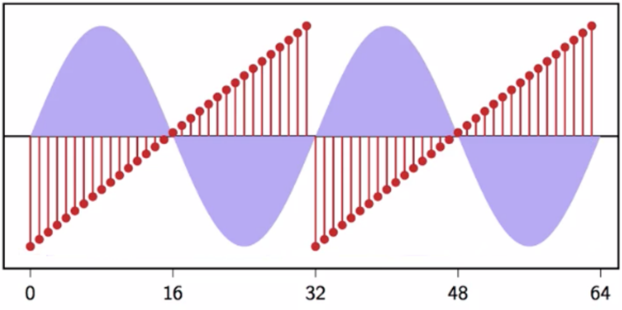
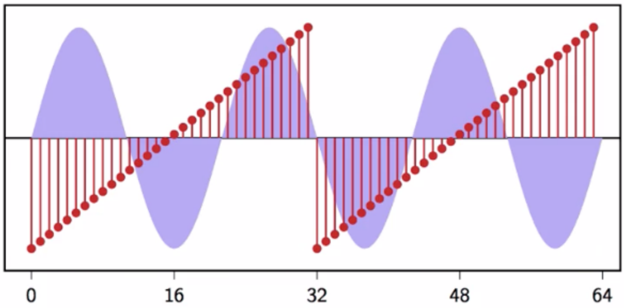



$k = 2$

# Karplus-Strong revisited and DFS




$k = 3$

# DFT of L periods

$$X_L[k] = \sum_{n=0}^{LM-1} y[n]e^{-j\frac{2\pi}{LM}nk} \qquad k = 0, 1, 2 \ldots, LM-1$$

$$= \sum_{p=0}^{L-1}\sum_{n=0}^{M-1} y[n+pM]e^{-j\frac{2\pi}{LM}(n+pM)k}$$

$$= \sum_{p=0}^{L-1}\sum_{n=0}^{M-1} y[n]e^{-j\frac{2\pi}{LM}nk}e^{-j\frac{2\pi}{L}pk}$$

$$= \left(\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk}\right)\sum_{n=0}^{M-1} \bar{x}[n]\, e^{-j\frac{2\pi}{LM}nk}$$

# DFT of L periods

$$\sum_{p=0}^{L-1} e^{-j\frac{2\pi}{L}pk} = \begin{cases} L & \text{if } k \text{ multiple of } L \\ 0 & \text{otherwise} \end{cases}$$

(remember the orthogonality proof for the DFT basis)

# DFT of L periods

$$X_L[k] = \begin{cases} L\bar{X}[k/L] & \text{if } k = 0, L, 2L, 3L, \ldots \\ 0 & \text{otherwise} \end{cases}$$

# DFT of L periods

- again, all the spectral information for a periodic signal is contained in the DFT coefficients of a single period
- to stress the periodicity of the underlying signal, we use the term DFS

# Discrete-time Fourier Transform (DTFT)

- Start with the DFT. What happens when $N \to \infty$?
- $(2\pi/N)k$ becomes denser in $[0, 2\pi]$
- in the limit $(2\pi/N)k \to \omega$

$$\sum_n x[n] \exp^{-j\omega n} \quad \omega \in \mathbb{R}$$

# Discrete-time Fourier Transform (DTFT)

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$$\sum_n x[n] \exp^{-j\omega n} \quad \omega \in \mathbb{R}$$

# Discrete-time Fourier Transform (DTFT)

Formal definition:

- $x[n] \in \ell_2(\mathbb{Z})$
- define the *function* of $\omega \in \mathbb{R}$

$$\underbrace{F(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}}$$

- inversion (when $F(\omega)$ exists):

$$\underbrace{x[n]} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega, \qquad n \in \mathbb{Z}$$

# Discrete-time Fourier Transform (DTFT)

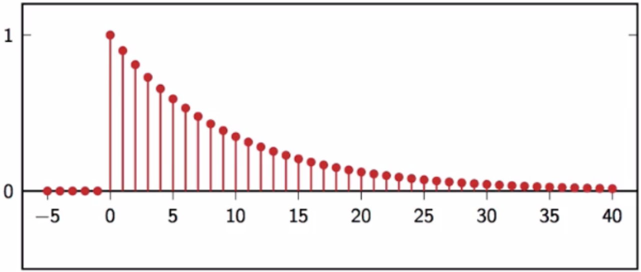
- $F(\omega)$ is $2\pi$-periodic
- to stress periodicity (and for other reasons) we will write

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- by convention, $X(e^{j\omega})$ is represented over $[-\pi, \pi]$

# Discrete-time Fourier Transform (DTFT)

$$x[n] = \alpha^n u[n], \quad |\alpha| < 1$$



# Discrete-time Fourier Transform (DTFT)

## DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} \alpha^n\, e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} (\alpha\, e^{-j\omega})^n \\
&= \frac{1}{1 - \alpha e^{-j\omega}}
\end{aligned}
$$

# Discrete-time Fourier Transform (DTFT)

Formal definition:

- $x[n] \in \ell_2(\mathbb{Z})$
- define the *function* of $\omega \in \mathbb{R}$

$$\underbrace{F(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}}$$

- inversion (when $F(\omega)$ exists):

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega, \qquad n \in \mathbb{Z}$$

# Discrete-time Fourier Transform (DTFT)

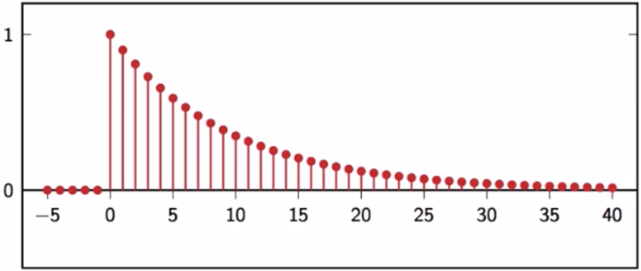
- $F(\omega)$ is $2\pi$-periodic
- to stress periodicity (and for other reasons) we will write

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- by convention, $X(e^{j\omega})$ is represented over $[-\pi, \pi]$

# Discrete-time Fourier Transform (DTFT)

$x[n] = \alpha^n u[n], \quad |\alpha| < 1$

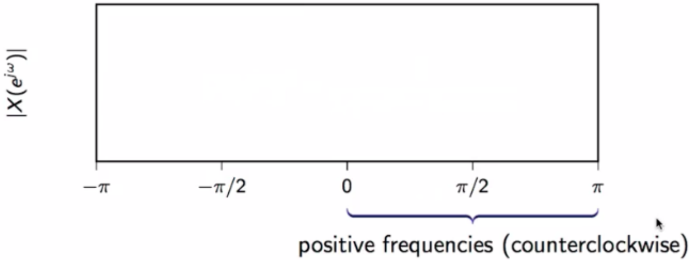


# Discrete-time Fourier Transform (DTFT)

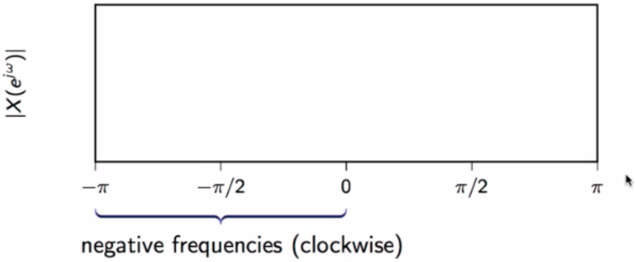
DTFT of $x[n] = \alpha^n u[n], \quad |\alpha| < 1$

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n\, e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} (\alpha\, e^{-j\omega})^n \\ &= \frac{1}{1 - \alpha e^{-j\omega}} \end{aligned}$$

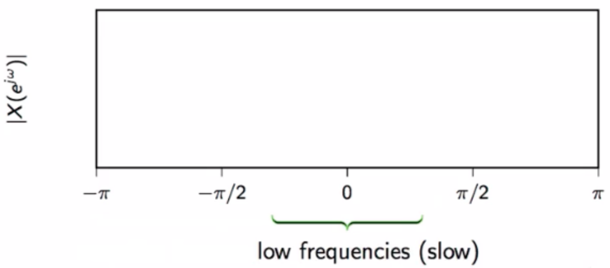
# Plotting the DTFT



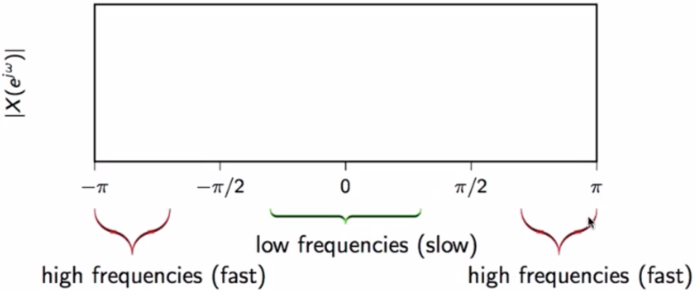
# Plotting the DTFT



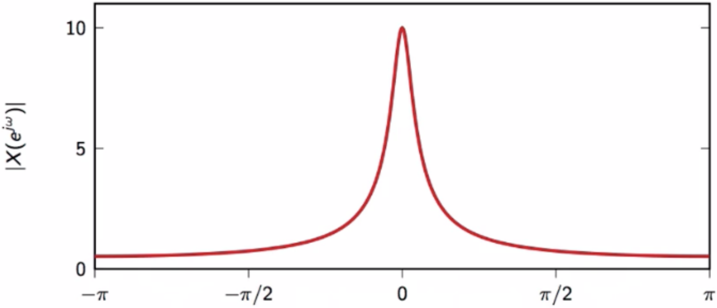
# Plotting the DTFT



# Plotting the DTFT

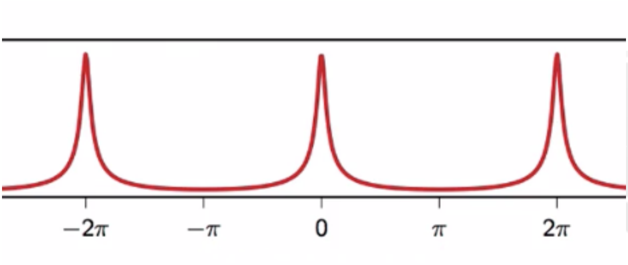


# Plotting the DTFT

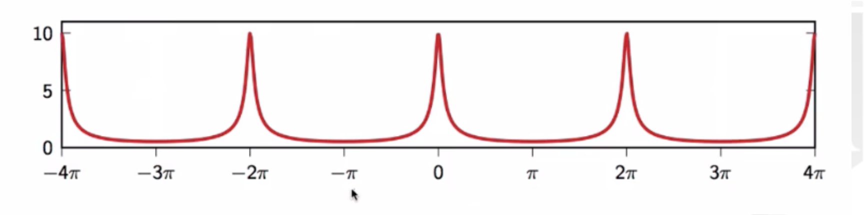



$\alpha = 0.9$

# Plotting the DTFT



# Plotting the DTFT



## Remarks

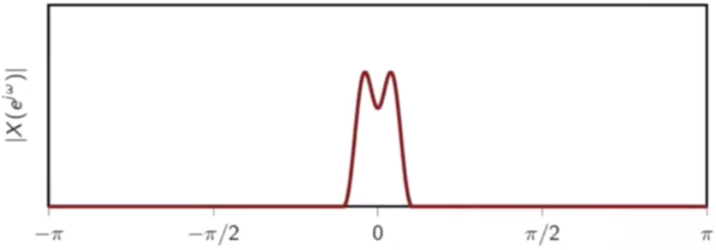
- The DTFT thus maps an infinite length sequence $x[n]$ onto a function of real variable $\omega$
- the DTFT is $2\pi$-periodic and we write $X(\exp^{j\omega})$
- for convention DTFT is defined on the $[-\pi, \pi]$

# Classifying signals in frequency

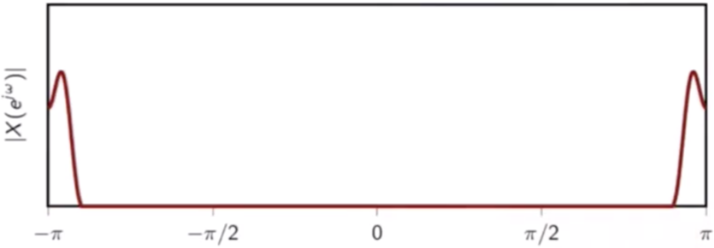
Three broad categories according to where most of the spectral energy resides:

- lowpass signals (also known as “baseband” signals)
- highpass signals
- bandpass signals

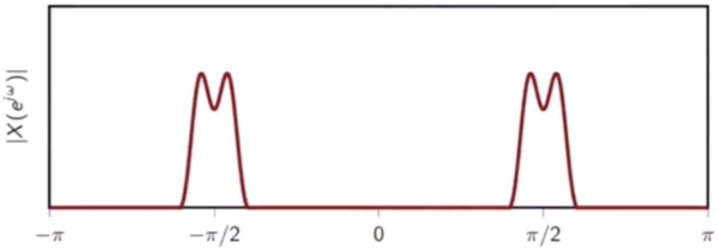
# Lowpass Example



# Highpass Example



# Bandpass Example

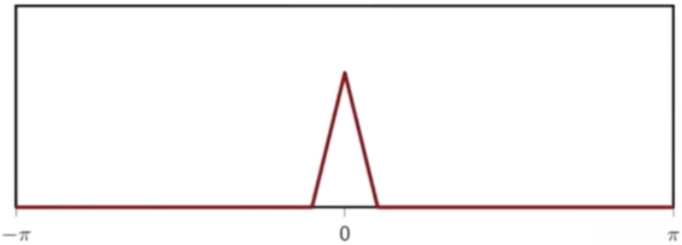


# Sinosoidal Modulation

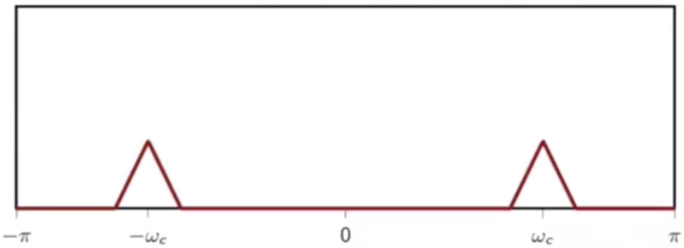
$$\begin{aligned}\text{DTFT}\left\{x[n]\cos(\omega_c n)\right\} &= \text{DTFT}\left\{\frac{1}{2}e^{j\omega_c n}x[n] + \frac{1}{2}e^{-j\omega_c n}x[n]\right\} \\ &= \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]\end{aligned}$$

- usually $x[n]$ baseband
- $\omega_c$ is the *carrier* frequency

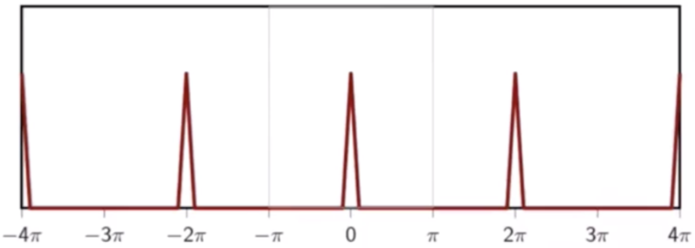
# Example



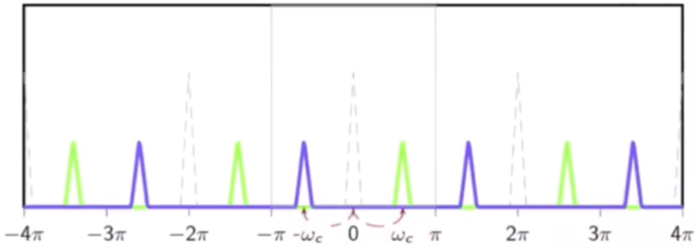
# Example



# Example



# Example



# Example

- voice and music are lowpass signals
- radio channels are bandpass, in much higher frequencies
- modulation brings the baseband signal in the transmission band
- demodulation at the receiver brings it back

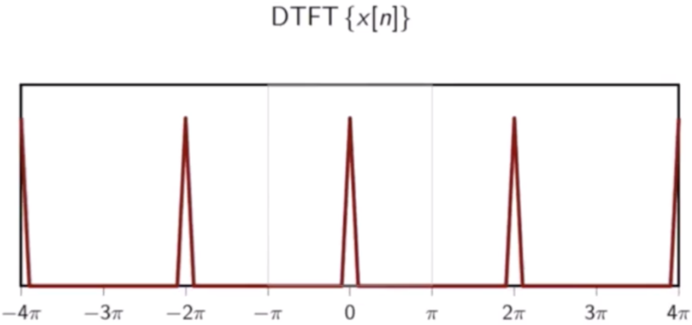
# Demodulation

just multiply the received signal by the carrier again

$$y[n] = x[n]\cos(\omega_c n) \qquad Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]$$

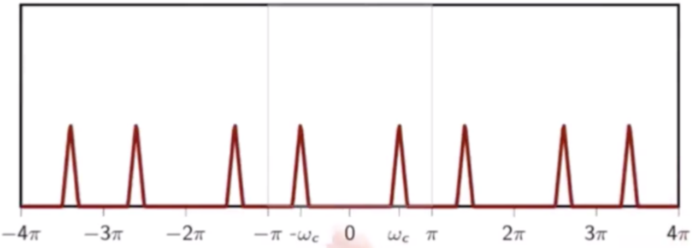
$$\begin{aligned}\mathrm{DTFT}\left\{y[n]\cdot 2\cos(\omega_c n)\right\} &= Y(e^{j(\omega-\omega_c)}) + Y(e^{j(\omega+\omega_c)}) \\ &= \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega)}) + X(e^{j(\omega)}) + X(e^{j(\omega+2\omega_c)})\right] \\ &= X(e^{j(\omega)}) + \frac{1}{2}\left[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega+2\omega_c)})\right]\end{aligned}$$

# Example-Demodulation



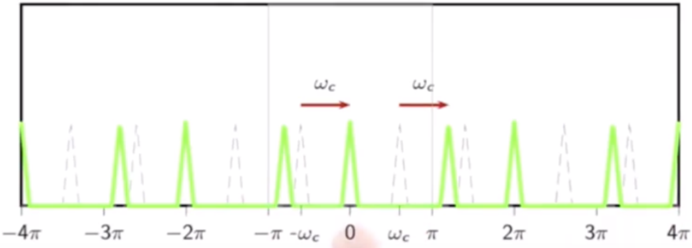
# Example-Demodulation

$$\mathrm{DTFT}\{y[n]\} = \mathrm{DTFT}\{x[n]\cos\omega_c n\}$$



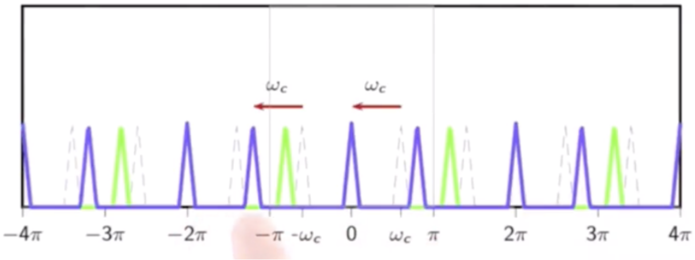
# Example-Demodulation

$$\mathrm{DTFT}\{y[n]\cos\omega_c n\}$$



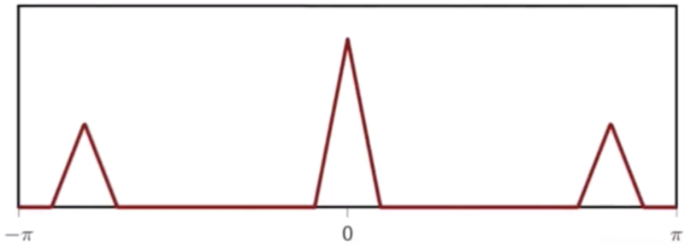
# Example-Demodulation

$$\text{DTFT}\{y[n]\cos\omega_c n\}$$



# Example-Demodulation

$$\text{DTFT}\{y[n]\cos\omega_c n\}$$



# FFT