Cuadratura gaussiana

Problema: Hallar α_0 y x_0 tales que la fórmula $\int_{-1}^{1} f(x)\,dx \cong \alpha_0 f(x_0)$ sea exacta para todo polinomio de grado n, $n \leq 1$.

$$f(x) = 1 \Rightarrow \int_{-1}^{1} 1\,dx = \alpha_0 \Longrightarrow \boxed{\alpha_0 = 2}$$

$$f(x) = x \Rightarrow \int_{-1}^{1} x\,dx = \alpha_0 x_0 \Longrightarrow \alpha_0 x_0 = 0$$

$$2x_0 = 0$$

$$\boxed{x_0 = 0}$$

$$\boxed{\int_{-1}^{1} f(x) \approx 2 f(0)}$$

Problema: Hallar α_0, x_0 y α_1, x_1 tales que la fórmula

$$\int_{-1}^{1} f(x)\,dx \approx \alpha_0 f(x_0) + \alpha_1 f(x_1)$$

sea exacta para todo polinomio de grado menor o igual que $n=3$.

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$$\int_{-1}^{1} 1\,dx = \alpha_0 + \alpha_1$$

$$\int_{-1}^{1} x\,dx = \alpha_0 x_0 + \alpha_1 x_1$$

$$\int_{-1}^{1} x^2\,dx = \alpha_0 x_0^2 + \alpha_0 x_1^2$$

$$\int_{-1}^{1} x^3\,dx = \alpha_0 x_0^3 + \alpha_0 x_1^3$$

Sistema no Lineal

$$\alpha_0 + \alpha_1 = 2$$

$$\alpha_0 x_0 + \alpha_1 x_1 = 0$$

$$\alpha_0 x_0^2 + \alpha_0 x_1^2 = \frac{2}{3}$$

$$\alpha_0 x_0^3 + \alpha_0 x_1^3 = 0$$

Solución: $\alpha_0 = \alpha_1 = 1$ $\quad x_0 = \frac{-\sqrt{3}}{3}$

$x_1 = \frac{\sqrt{3}}{3}$

$$\boxed{\int_{-1}^{1} f(x)\,dx \simeq f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)}$$

Referencia 1

$$\int_{-1}^{1} f(x)\,dx \simeq \alpha_0 f(x_0) + \alpha_1 f(x_1) + \cdots + \alpha_m f(x_m)$$

Se calculan α_j, x_j de tal forma que la fórmula sea exacta para todo polinomio de grado menor o igual que $\boxed{2m+1}$

③ cuadratura de Gauss-Legendre

Raíces y coeficientes

$n=2$	$\pm\sqrt{\frac{1}{3}}$	1
$n=3$	$\pm\sqrt{\frac{3}{5}}\,,\ 0$	$\frac{5}{9}\,,\ \frac{8}{9}$
$n=4$	$\pm\frac{1}{35}\sqrt{525+70\sqrt{30}}$	$\frac{1}{36}\left(18-\sqrt{30}\right)$
	$\pm\frac{1}{35}\sqrt{525-70\sqrt{30}}$	$\frac{1}{36}\left(18+\sqrt{30}\right)$
$n=5$	0	$\frac{128}{225}$
	$\pm\frac{1}{21}\sqrt{245-14\sqrt{70}}$	$\frac{1}{900}\left(322+13\sqrt{70}\right)$
	$\pm\frac{1}{21}\sqrt{245+14\sqrt{70}}$	$\frac{1}{900}\left(322-13\sqrt{70}\right)$

nodos: X_i — coeficientes: α_i

Ejemplo: $\displaystyle\int_0^2 e^{-x}\,\text{sen}(5x)\,dx$, $u = x-1$

$$= \int_{-1}^{1} e^{-(u+1)}\,\text{sen}(5u+5)\,du$$

$$= e^{-1}\int_{-1}^{1} e^{-u}\,\text{sen}(5u+5)\,du$$

$$\simeq e^{-1}\left[\sum_{j=0}^{4} c_j\, f(x_j)\right]$$

5 nodos, cuadrat gauss-Legendre

$$\simeq e^{-1}\Big[0.236926885\, f(-0.906179846) + 0.47862867\, f(-0.53846931) + 0.56888889\, f(0) + 0.47862867\, f(0.53846931) + 0.236926885\, f(0.906179846)\Big]$$

$$\simeq 0.21795444$$

- Cuadratura de Gauss-Legendre. $\circledast$

$$\int_{-1}^{1} f(z)\,dz$$

Los nodos son las raíces de los polinomios de legendre.

Producto interno : $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$

$w(x) = 1$

$$\int_{0}^{+\infty} e^{-z} f(z)\,dz$$

polinomios de Laguerre.

producto interno $\langle P, Q\rangle = \int_{0}^{+\infty} e^{-x} p(x)q(x)\,dx$.

Cuadratura de Gauss-Laguerre $\circledast$

$w(x) = e^{-x}$

$$\int_{-\infty}^{+\infty} e^{-z^2} f(z)\,dz$$

polinomios de Hermite

producto interno $\langle p, q\rangle = \int_{-\infty}^{\infty} e^{-x^2} p(x)\,q(x)\,dx$

$w(x) = e^{-x^2}$

cuadratura de gauss-hermite

Cuadratura de Gauss-Chebyshev

$$\int_{-1}^{1} \frac{f(z)}{\sqrt{1-z^2}}\, dz \quad \Big\} \text{ polinomios de Chebyshev}$$

$$\langle p, q \rangle = \int_{-1}^{1} \frac{p(x)q(x)}{\sqrt{1-x^2}}\, dx$$

$$W(x) = \frac{1}{\sqrt{1-x^2}}$$

En cada caso: Error en la integral: E_r

$$E_r = \frac{f^{(2n)}(\xi)}{(2n)!} \int_a^b \left[W(x) \prod_{i=1}^{n} (x - x_i) \right] dx \qquad \xi \in (a,b)$$

con "n-nodos" $\rightarrow x_1, x_2, \ldots, x_n$

Ejemplo

$$\int_0^{+\infty} e^{-x^2/2}\,dx = \sqrt{\frac{\pi}{2}} \simeq 1.253314137$$

(5-nodos)

$\boxed{\text{Método 1:}}$ usando cuadratura de Gauss-Laguerre

$$\int_0^{+\infty} e^{x^2/2}\,dx = \int_0^{+\infty} e^{-x} f(x)\,dx \quad , \quad f(x) = e^{x-\frac{x^2}{2}}$$

$$\simeq c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) + c_4 f(x_4)$$

$$+ c_5 f(x_5) \simeq 1.263673858$$

$c_1 = 0.521756$

$c_2 = 0.398667$

$c_3 = 0.0759424$

$c_4 = 0.00361176$

$c_5 = 0.00002337$

$x_1 = 0.26356$, $x_2 = 1.4134$

$x_3 = 3.59643$

$x_4 = 7.08581$

$x_5 = 12.6408$

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$\boxed{\text{Método 2}}$

$$\int_0^{+\infty} e^{-x^2/2}\,dx = \int_0^{2} e^{-x^2/2}\,dx + \int_2^{+\infty} e^{-x^2/2}\,dx$$

$$= \mathrm{I} + \mathrm{II}$$

$\mathrm{I} \simeq 1.19628013$ (gauss-Legendre 20 nodos)

para II (aplicamos un cambio de variable) $\boxed{a>0}$

$$\left\{ \begin{array}{l} z = \dfrac{1}{x} \\[2ex] \displaystyle\int_a^{+\infty} f(x)\,dx = \int_0^{1/a} \left[\frac{f\left(\frac{1}{z}\right)}{z^2}\right] dz \end{array} \right.$$

$$\mathrm{II} = \int_0^{1/2} \frac{1}{z^2} e^{-\frac{1}{2z^2}}\,dz \simeq 0.57026419$$

gauss-Legendre 20 nodos

$\mathrm{I} + \mathrm{II} \simeq 1.253314432$

Ejemplo.

$$\int_0^1 \frac{e^x}{\sqrt{x}}\,dx = \int_0^1 \left(\frac{e^x - P_4(x)}{\sqrt{x}} + \frac{P_4(x)}{\sqrt{x}} \right) dx$$

$P_4(x)$ = polinomio de Maclaurin de e^x, de grado 4.

$$= \int_0^1 \frac{e^x - P_4(x)}{\sqrt{x}}\,dx + \int_0^1 \frac{P_4(x)}{\sqrt{x}}\,dx$$

(gauss-Legendre, 20 nodos)

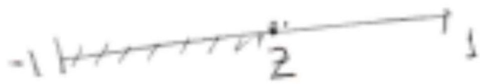
$$\cong 0.001758518 + 2.881082472$$

$$\simeq \boxed{2.882840990}$$

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f(\varphi(z))\,\varphi'(z)\,dz = \frac{b-a}{2}\int_{-1}^{1} H(z)\,dz$$

$$\frac{x-a}{b-a} = \frac{z+1}{2}, \quad x = \frac{b-a}{2}(z+1) + a$$

$$x = \varphi(z)$$



Esto es para [illegible],

$$\frac{b-a}{2}\int_{-1}^{1} H(z)\,dz \simeq \frac{b-a}{2}\left(\sum_{i=1}^{n} C_i\, H(z_i)\right)$$

$$\simeq \frac{b-a}{2}\sum_{i=1}^{n} C_i\, f\left(\frac{b-a}{2}(z_i+1)+a\right)$$