Integración numérica

Teorema fundamental del cálculo.

Si $f \in C[a,b]$ y $g'(x) = f(x)$ $\forall x \in [a,b]$

entonces $$\int_a^b f(x)\,dx = g(b) - g(a)$$

Dificultad usual: Hallar $g(x) = \int f(x)\,dx$

Fórmulas de Newton-Cotes cerradas

Consideremos una discretización uniforme del intervalo $[a,b]$

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$$

n+1 puntos

La fórmula cerrada de newton cotes, de n+1 puntos, consiste en aproximar

$$\int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx$$ donde $P_n(x)$

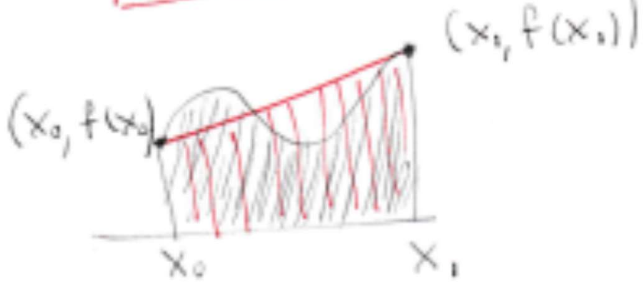
es el Polinomio interpolante de Lagrange de grado menor o igual que n, para los nodos $(x_j, y_j) = (x_j, f(x_j))$

$0 \le j \le n$

Caso $n=1$ Regla trapezoidal

$$\int_a^b f(x)\,dx = \boxed{\frac{h}{2}\left[f(x_0) + f(x_1)\right]} - \frac{h^3}{12} f''(\xi)$$

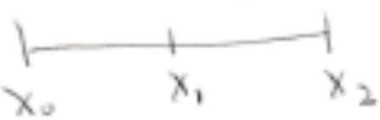
$\xi \in (x_0, x_1)$



Caso $n=2$ Regla de Simpson

$$\int_a^b f(x)\,dx = \int_a^b P_2(x)\,dx + \text{Error}$$

$$= \boxed{\frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]} - \frac{h^5}{90} f^{(4)}(\xi)$$



$h = \Delta x = \dfrac{x_2 - x_0}{2}$

caso $n=3$ Regla de Simpson $\frac{3}{8}$

$a=x_0 \quad x_1 \quad x_2 \quad x_3=b \qquad h=\frac{b-a}{3}$

$$\int_a^b f(x)\,dx = \boxed{\frac{3}{8}h\Big(f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\Big)} - \frac{3}{80}h^5 f^{(4)}(\xi)$$

caso $n=4$ $\qquad a=x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4=b \qquad h=\frac{b-a}{4}$

$$\int_a^b f(x)\,dx = \frac{2h}{45}\Big(7f(x_0)+32f(x_1)+12f(x_2) + 32f(x_3)+7f(x_4)\Big) - \frac{8}{945}h^7 f^{(6)}(\xi)$$

Ejemplo: $\displaystyle\int_0^{\pi/4} \operatorname{Sen}(x)\,dx = I = 1-\frac{\sqrt{2}}{2} = 0.29289322$

R. trapecio: $0 \quad \pi/2$

$I \approx 0.27768018$

R. simpson $0 \quad \frac{\pi}{4} \quad \frac{\pi}{2}$

$I \approx 0.29293264$

R simpson $\frac{3}{8}$ $\quad 0 = x_0 \quad \frac{\pi}{6} \quad \frac{\pi}{3} \quad x_3 = \frac{\pi}{2}$, $I \approx 0.2929107$

R. con 5 nodos $\quad x_0 \quad x_1 = \pi/8 \quad x_2 = \pi/4 \quad x_3 = \frac{3\pi}{8} \quad x_4 = \frac{\pi}{2}$

$I \approx 0.\underline{292893}82$

Demostración Regla de Simpson

$x_0 \quad x_1 \quad x_2$

Polinomio interpolación de Lagrange P_2

taylor

$$f(x) = f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2}(x - x_1)^2$$

$$+ \frac{f'''(x_1)}{6}(x - x_1)^3 + \frac{f^{(4)}(\xi_1)}{24}(x - x_1)^4$$

ξ_1 pertenece a (x_0, x_2)

(5)

$$\int_a^b f(x)\,dx = \int_a^b \left(f(x_1) + f'(x_1)(x-x_1) + \frac{f''(x_1)}{2}(x-x_1)^2 + \frac{f'''(x_1)}{6}(x-x_1)^3 \right) dx$$

$$+ \left(\frac{f^{(4)}(\xi_1)}{5(24)} (x-x_1)^5 \right]_a^b$$

$$\int_a^b f(x)\,dx = 2h\, f(x_1) + \left(\frac{f'(x_1)\,(x-x_1)^2}{2} \right]_a^b + \left(\frac{f''(x_1)}{6} (x-x_1)^3 \right]_a^b$$

$$+ \left(\frac{f'''(x_1)}{(6)(4)} (x-x_1)^4 \right]_a^b + \frac{f^{(4)}(\xi_1)}{120}\, 2\left(h^5\right)$$

$$\int_a^b f(x)\,dx = 2h\, f(x_1) + \frac{f'(x_1)}{2}\left(h^2 - (-h)^2 \right)$$

$$+ \frac{f''(x_1)}{6}\left(h^3 + h^3 \right) + \frac{f'''(x_1)}{24}\left(h^4 - (-h)^4 \right) + \frac{f^{(4)}(\xi_1)}{60} h^5$$

$$\int_a^b f(x)\,dx = 2h\,f(x_1) + \frac{2f''(x_1)}{6}h^3 + \frac{f^{(4)}(\varepsilon_1)}{60}h^5$$

De otra parte tenemos

$$f''(x_1) = \frac{f(x_0) - 2f(x_1) + f(x_2)}{h^2} - \frac{h^2}{12}f^{(4)}(\varepsilon_2)$$

$$\frac{f''(x_1)h^3}{3} = \frac{h}{3}\left(f(x_0) - 2f(x_1) + f(x_2)\right) - \frac{h^5}{36}f^{(4)}(\varepsilon_2)$$

al reemplazar se obtiene.

$$\int_a^b f(x)\,dx = 2h\,f(x_1) + \frac{h}{3}\left(f(x_0) - 2f(x_1) + f(x_2)\right) - \frac{h^5}{36}f^{(4)}(\varepsilon_2) + \frac{h^5}{60}f^{(4)}(\varepsilon_1)$$

$$= \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) + \boxed{\text{ERROR}}$$

Lema auxiliar

Existe $K \in \mathbb{R}$ tal que

$\forall f \in C^{(4)}[a,b]$ existe ξ tal $\xi \in (a,b)$

que $\quad -\dfrac{h^5}{36} f^{(4)}(\xi_2) + \dfrac{h^5}{60} f^{(4)}(\xi_1) = K f^{(4)}(\xi)$

$\forall \xi_1, \xi_2$

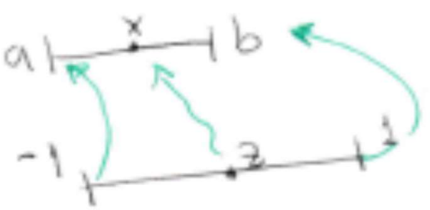
¿ cómo calcular K ?

$-1 = x_0 \quad 0 \quad 1 = x_2$

consideremos $\displaystyle\int_{-1}^{1} x^4\, dx$ $\qquad f(x) = x^4$

$$\int_{-1}^{1} x^4\, dx = \frac{1}{3}\left((-1)^4 + 4\,0^4 + 1^4\right) + \hat{k} f^{(4)}(\xi)$$

$$\frac{2}{5} = \frac{1}{3} + 24k \longrightarrow \hat{k} = -\frac{1}{90}$$



$$x = \varphi(z) = \frac{b-a}{2}(z+1) + a$$

$$\int_a^b f(x)\, dx = \int_{-1}^{1} (f \circ \varphi)(z)\, \varphi'(z)\, dz$$

Como sale este integral cambio de variable.

bajo la misma funcion,

Que mapea los extremos de la integral en -1 — 1.

⑧

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left[\int_{-1}^{1} H(z)\,dz\right] \qquad H(z) = (f\circ\Phi)(z)$$

$$= \frac{b-a}{2}\left[\frac{1}{3}\Big(H(-1) + 4H(0) + H(1)\Big) - \frac{1}{90} H^{(4)}(\eta)\right]$$

$$= \frac{b-a}{2}\left[\frac{1}{3}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right]$$

$$- \boxed{\frac{b-a}{2(90)}\left(\frac{b-a}{2}\right)^4 f^{(4)}(\xi)} = -\frac{h^5}{90} f^{(4)}(\xi)$$

$$\frac{d^4}{dz^4}\big(f\circ\Phi(z)\big) = \left(\frac{b-a}{2}\right)^4 f^{(4)}(\xi)$$

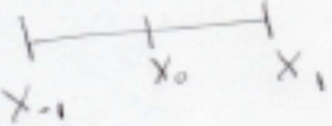
$$\frac{d}{dz}(f\circ\varphi)(z) = f'(\varphi(z))\,\Phi'(z) = \frac{b-a}{2} f'(\xi)$$

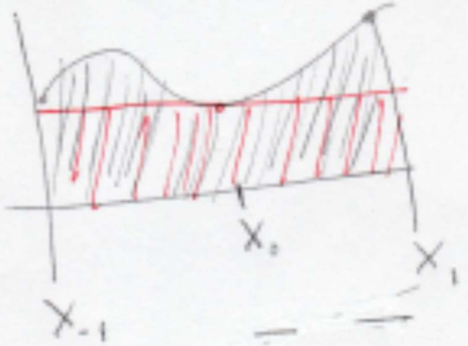
$$= (f'\circ\varphi)(z)\left(\frac{b-a}{2}\right)$$

fórmulas de newton-cotes abiertas

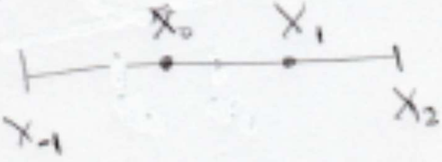
$n=0$

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2h\,f(x_0) + \frac{h^3}{3} f''(\xi)$$



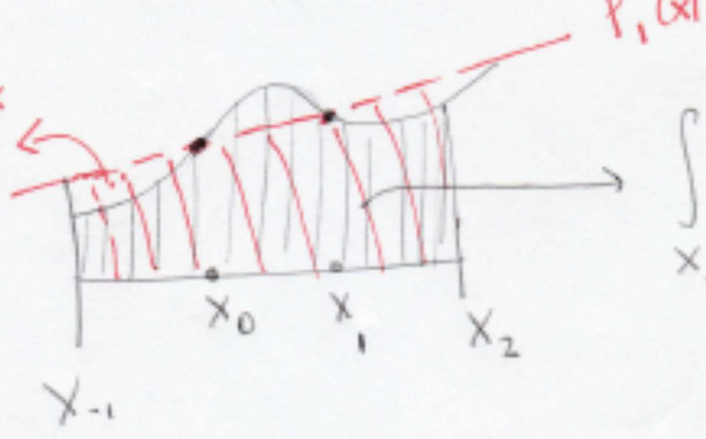


$n=1$



$$\int_{x_{-1}}^{x_2} f(x)\,dx = \frac{3}{2}h\,(f(x_0) + f(x_1)) + \frac{3h^3}{4} f''(\xi)$$

$$\int_{x_{-1}}^{x_2} P_1(x)\,dx$$



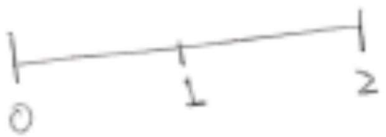
$$\int_{x_{-1}}^{x_2} f(x)\,dx$$

ejemplo

$$\int_0^2 \left(1+e^x \operatorname{sen}(2x-1)\right) dx =$$

$$= 2+\frac{1}{5}\left(\operatorname{Sen}(1)+2\cos(1)-2e^2\cos(3)+e^2\operatorname{sen}(3)\right)$$

$$\simeq \underline{5.51900789}$$



Regla de simpson

$$\int_0^2 f(x)\,dx \simeq \frac{h}{3}\left(f(x_0)+4f(x_1)+f(x_2)\right)$$

$$= \frac{1}{3}\left(f(0)+4f(1)+f(2)\right)$$

$$= \frac{1}{3}\left(15.35069382\right)$$

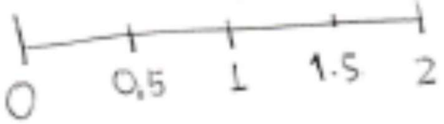
$$\simeq \underline{5.11689794}$$

Regla del punto medio

$$\int_0^2 f(x)dx \simeq 2hf(1) = 2f(1)$$

$$= \underline{6.57471057}$$

fórmula de newton, cerrada con $n=4$



$h = 0.5$

$$\int_0^2 \left(1 + e^x \operatorname{sen}(2x-1)\right) dx \approx$$

$$\frac{2h}{45}\Big(7f(0) + 32f(0.5) + 12f(1) + \cdots$$
$$32f(1.5) + 7f(2)\Big) \simeq \underline{5.53918220}$$

Integración numérica compuesta

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} f\,dx + \int_{x_1}^{x_2} f\,dx + \cdots + \int_{x_{n-1}}^{x_n} f\,dx$$

$$\boxed{h = \frac{b-a}{n}} \simeq \frac{h}{2}\big(f(x_0) + f(x_1)\big) + \frac{h}{2}\big(f(x_1) + f(x_2)\big) + \cdots$$
$$\cdots + \frac{h}{2}\big(f(x_{n-1}) + f(x_n)\big)$$

Regla trapezoidal compuesta

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)\right]$$

$$\underbrace{- \frac{h^3}{12}}_{\text{Error}}\left(f''(\varepsilon_1) + f''(\varepsilon_2) + \dots + f''(\varepsilon_n)\right)$$

$$\varepsilon_i \in (x_{i-1}, x_i)$$

$$\text{Error} = -\frac{h^3 n}{12}\left(\frac{f''(\varepsilon_1) + \dots + f''(\varepsilon_n)}{n}\right)$$

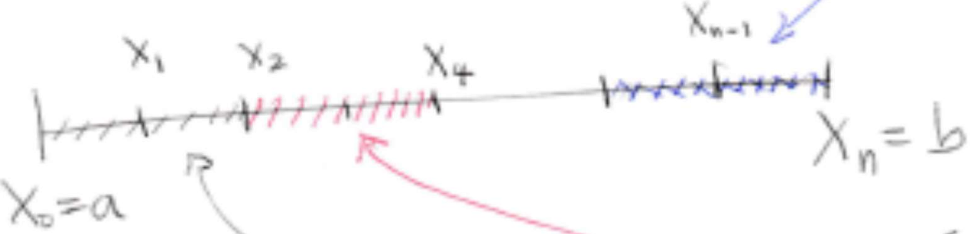
aplicando el teorema del valor intermedio existe $\xi \in (a,b)$ tal que

$$f''(\xi) = \frac{f''(\varepsilon_1) + \dots + f''(\varepsilon_n)}{n}$$

$$\frac{h^3 n}{12} = \frac{h^2 h n}{12} = \frac{h^2}{12}\left(\frac{b-a}{n}\right)n = \frac{b-a}{12}h^2$$

$$\text{Error} = -\frac{(b-a)}{12}h^2 f''(\xi)$$

Regla de simpson compuesta



n: número par

$$\int_a^b f(x)\,dx \simeq \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) + \frac{h}{3}\left(f(x_2) + 4f(x_3) + f(x_4)\right) + \cdots + \frac{h}{3}\left(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

$$\int_a^b f(x)\,dx = \frac{h}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big) - \frac{h^5}{90}\left(f^{(4)}(\theta_1) + f^{(4)}(\theta_2) + \cdots + f^{(4)}(\theta_{\frac{n}{2}})\right)$$

$\theta_1 \in (x_0, x_2)$

$\theta_2 \in (x_2, x_4)$

$\longrightarrow \theta_{\frac{n}{2}} \in (x_{n-2}, x_n)$

$$\mathcal{E}rror = -\frac{h^5}{90}\frac{n}{2}\left(\frac{f^{(4)}(\theta_1) + \dots + f^{(4)}(\theta_{\frac{n}{2}})}{\frac{n}{2}}\right)$$

$$= -\frac{h^5}{90}\frac{n}{2}\, f^{(4)}(\hat{\xi}) \qquad \hat{\xi} \in (x_0, x_n)$$

$$= -\frac{h^4}{90}\frac{n}{2}\left(\frac{b-a}{n}\right) f^{(4)}(\hat{\xi})$$

$$= -\frac{h^4(b-a)}{180} f^{(4)}(\hat{\xi})$$

$$I = \int_0^2 \left(1 + e^x \operatorname{sen}(2x-1)\right) dx .$$

usando 4 subintervalos (5 nodos)

$h = 0.5$

$I \simeq 5.23158998$ Trap.- compuesta

$I \simeq 5.51278943$ Simpson - compuesta