Ejemplo

$$\int_0^2 e^{-x} \operatorname{Sen}(5x)\, dx = I$$

$$\frac{d^2}{dx^2}\left(e^{-x} \operatorname{Sen}(5x)\right) = \left[-24 \sin(5x) - 10 \cos(5x)\right] e^{-x}$$

$$\frac{d^4}{dx^4}\left(e^{-x} \operatorname{Sen}(5x)\right) = \left[476 \sin(5x) + 480 \cos(5x)\right] e^{-x}$$

ⓐ Calcule mediante la regla compuesta del trapecio una aproximación de I, con un error menor o igual que $2(10^{-5})$

ⓑ Calcule mediante la regla compuesta de Simpson una aproximación de I con un error menor o igual que $2(10^{-5})$

usaremos las siguientes cotas para f'' y $f^{(4)}$

$$\forall x \in [0,2] \qquad |f''(x)| \leq 22$$

$$|f^{(4)}(x)| \leq 600$$

(a) $$\text{error} = \frac{b-a}{12} h^2 |f^{(2)}(\mu)|$$

$$\leq \frac{1}{6} h^2 (22) = \frac{11}{3} h^2$$

se resuelve $\frac{11}{3} h^2 \leq 2(10^{-5})$

$$h^2 \leq \frac{6}{11} \cdot 10^{-5} = 0.545454 \, (10^{-5})$$

$$h = 0.002335$$

$$\longrightarrow \frac{2}{n} = 0.002335$$

$$\boxed{n = 857}$$

(b) $$\text{error} = \frac{b-a}{180} h^4 |f^{(4)}(\mu)| \leq \frac{1}{90} h^4 (600)$$

Se resuelve:

$$\frac{h^4}{9\cancel{0}} 6\cancel{0}\cancel{0} \leq 2\,(10^{-5})$$

$$h^4 \leq \frac{18}{60}\, 10^{-5} = \frac{3}{10}\,(10^{-5})$$

$$h^4 \leq 3\,(10^{-6})$$

$$h \leq 0.0416179 1$$

$$\frac{2}{n} \leq 0.04161791 \quad , \quad n \geqslant \frac{2}{0.0416179}$$

$$\boxed{n = 50}$$

$$\int_0^2 e^{-x}\operatorname{Sen}(5x)\,dx = \frac{-1}{26}\left(\frac{5\cos(10) + \operatorname{sen}(10)}{e^2}\right) + \frac{5}{26}$$

$$\simeq \underline{0.216977122}$$

$S = \text{trapr1}(f, 0, 2, 857) = 0.\underline{21697}4628$

$|\text{error}| = 0.00000249$

$S = \text{Simpr1}(f, 0, 2, 50) = 0.\underline{21697}8792$

$|\text{error}| \simeq 0.00000167$

Integración de Romberg

Regla Compuesta $\underbrace{\text{del trapecio}}$ $\rightarrow R_{n,1}$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{m-1} f(x_j) + f(b)\right] + K_1 h^2 + K_2 h^4$$

$$+ \cdots + K_j h^{2j} + \cdots$$

K_j depende de $f^{(2j-1)}(a)$ y $f^{(2j-1)}(b)$.

$$\int_a^b f(x)\,dx = R_{n,1}$$

$\rightarrow$ Regla del trapecio, compuesta.
subintervalos : $2^{n-1} = m$

$R_{1,1} \rightarrow$ un intervalo, $h = b - a$

$R_{2,1} \rightarrow$ 2 intervalos $h = \frac{b-a}{2}$

$R_{3,1} \rightarrow$ 4 intervalos $h = \frac{b-a}{2^2}$

$\vdots$

$R_{n,1} \rightarrow 2^{n-1}$ intervalos $h = \frac{b-a}{2^{n-1}}$

(1) $$\int_a^b f(x)dx = R_{n,1} + K_1 h^2 + K_2 h^4 + \cdots K_j h^{2j} + \cdots$$

(2) $$\int_a^b f(x)dx = R_{n+1,1} + K_1 \frac{h^2}{4} + \frac{K_2 h^4}{2^4} + \cdots + K_j \frac{h^{2j}}{2^{2j}}$$

$$\downarrow$$
$$= I$$

de (1) y (2)

$$4I - I = 4R_{n+1,1} - R_n + \left(\frac{2^2}{2^4} - 1\right) K_2 h^4 + \cdots + \left(\frac{2^2}{2^{2j}} - 1\right) K_j h^{2j}$$

$$3I = 3R_{n+1,1} + \left(R_{n+1,} - R_n\right) + \cdots C_j \left(\frac{h}{2}\right)^{2j} + \cdots$$

$$I = R_{n+1,1} + \frac{\left[R_{n+1} - R_n\right]}{3} + \cdots D_j \left(h\right)^{2j}$$

$$I = R_{n+1,1} + \frac{\left[R_{n+1,1} - R_{n,1}\right]}{3} + D_1 h^4 + D_2 h^6 + \cdots + D_j h^{2(j+1)}$$

tenemos

$$I = R_{n,1} + O(h^2)$$

$$I = R_{n+1,1} + O\left(\frac{h^2}{4}\right)$$

$R_{n+1,2}$

$$I = R_{n+1,2} + O(h^4) = \left[R_{n+1,1} + \frac{[R_{n+1,1} - R_{n,1}]}{3}\right] + O(h^4)$$

- - - duplicamos el número de subintervalos

(3) $I = R_{m,2} + D_1 h^4 + D_2 h^6 + \cdots + D_j h^{2j+2} + \cdots$

(4) $I = R_{m+1,2} + \frac{D_1}{2^4} h^4 + D_2 \left(\frac{h}{2}\right)^6 + \cdots + D_j \left(\frac{h}{2}\right)^{2j+2} + \cdots$

$$2^4 I - I = 2^4 R_{m+1,2} - R_{m,2} + E_2 h^6 + \cdots + E_j h^{2j+2}$$

$$(2^4 - 1) I = R_{m+1,2} + (2^4 - 1) R_{m+1,2} - R_{m,2} + \cdots + O(h^6)$$

$$I = \left[R_{m+1,2} + \frac{R_{m+1,2} - R_{m,2}}{2^4 - 1}\right] + \cdots + F_j h^{2j+4}$$

$$I = R_{m+1,3} + O(h^6)$$

Fórmula de avance.

$$(5) \quad I = R_{L,3} + g_1 h^6 + g_2 h^8 + \cdots$$

$$(6) \quad I = R_{L+1,3} + g_1 \frac{h^6}{4^3} + g_2 \frac{h^8}{4^4} + \cdots$$

$$4^3 I - I = 4^3 R_{L+1,3} - R_{L,3} + \hat{g}_1 h^8 + \hat{g}_2 h^{10} + \cdots$$

$$= (4^3 - 1) R_{L+1,3} + R_{L+1,3} - R_{L,3} + O(h^8)$$

$$I = R_{L+1,3} + \boxed{\frac{R_{L+1,3} - R_{L,3}}{4^3 - 1}} + O(h^8)$$

$$R_{n+1,2} = R_{n+1,1} + \frac{[R_{n+1,1} - R_{n,1}]}{4^1 - 1}$$

$$R_{n+1,3} = R_{n+1,2} + \frac{[R_{n+1,2} - R_{n,2}]}{4^2 - 1}$$

$$R_{n+1,4} = R_{n+1,3} + \frac{[R_{n+1,3} - R_{n,3}]}{4^3 - 1}$$