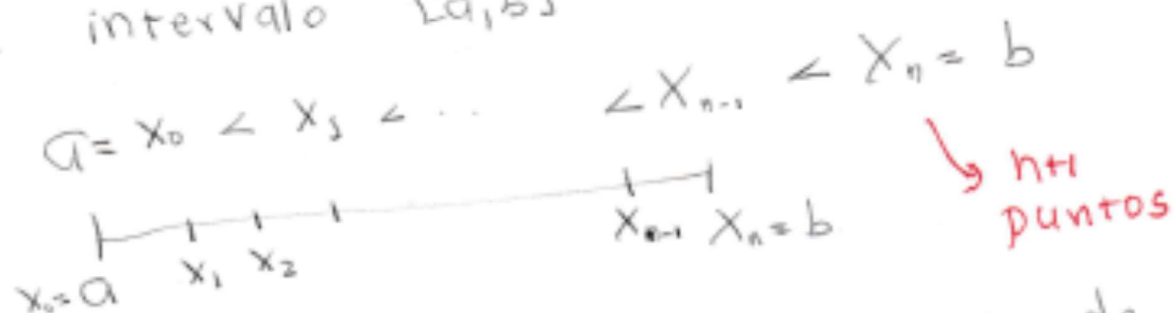


①

Integración numérica

Teorema fundamental del cálculo.

Si $f \in C[a, b]$ y $g'(x) = f(x) \quad \forall x \in [a, b]$ entonces $\int_a^b f(x) dx = g(b) - g(a)$ Dificultad usual: Hallar $g(x) = \int f(x) dx$ Fórmulas de Newton-Cotes cerradasConsideremos una discretización uniforme del intervalo $[a, b]$ La fórmula cerrada de Newton-Cotes, de $n+1$ puntos, consiste en aproximar

$$\int_a^b f(x) dx \approx \int_a^b P_n(x) dx \quad \text{donde } P_n(x)$$

es el polinomio interpolante de Lagrange

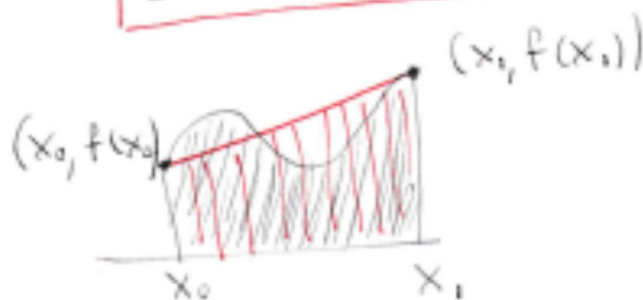
(2)

de grado menor o igual que n , para
los nodos $(X_j, Y_j) = (X_j, f(X_j))$

$$0 \leq j \leq n$$

Caso $n=1$ Regla trapezoidal

$$\int_a^b f(x) dx = \frac{h}{2} [f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi)$$



Caso $n=2$ Regla de Simpson

$$\int_a^b f(x) dx = \int_a^b P_2(x) dx + \text{Error}$$

$$= \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi)$$

$$h = \Delta x = \frac{x_2 - x_0}{2}$$

caso $n=3$ Regla de Simpson $\frac{3}{8}$ ③

$$a = x_0 \quad x_1 \quad x_2 \quad x_3 = b \quad h = \frac{b-a}{3}$$

$$\int_a^b f(x) dx = \frac{3}{8} h \left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right) - \frac{3}{80} h^5 f^{(4)}(\xi)$$

caso $n=4$

$$a = x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 = b \quad h = \frac{b-a}{4}$$

$$\int_a^b f(x) dx = \frac{2h}{45} \left(7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4) \right) - \frac{8}{945} h^7 f^{(4)}(\xi)$$

Ejemplo: $\int_0^{\pi/4} \text{Sen}(x) dx = I = 1 - \frac{\sqrt{2}}{2} = \underline{0.29289322}$

R trapecio: diagram

$$I \approx 0.27768018$$

R. Simpson diagram

$$I \approx 0.29293264$$

(4)

R Simpson $\frac{3}{8}$ $0=x_0$ $\frac{\pi}{6}$ $\frac{\pi}{3}$ $x_3 = \frac{\pi}{2}$, $I \approx 0.2929107$

R. con 5 nodos $\frac{\pi}{8}$ $\frac{\pi}{4}$ $\frac{3\pi}{8}$
 x_0 x_1 x_2 x_3 $x_4 = \frac{\pi}{2}$

$$I \approx 0.29289382$$

Demostración Regla de Simpson

Polinomio interpolación de Lagrange P_2

x_0 x_1 x_2
 Taylor

$$f(x) = f(x_1) + f'(x_1)(x-x_1) + \frac{f''(x_1)}{2}(x-x_1)^2$$

$$+ \frac{f'''(x_1)}{6}(x-x_1)^3 + \frac{f^{(4)}(\xi_1)}{24}(x-x_1)^4$$

ξ_1 pertenece a (x_0, x_2)

(5)

$$\int_a^b f(x) dx = \int_a^b \left(f(x_1) + f'(x_1)(x-x_1) + \frac{f''(x_1)}{2}(x-x_1)^2 + \frac{f'''(x_1)}{6}(x-x_1)^3 + \left(\frac{f^{(4)}(\xi_1)}{5(24)}(x-x_1)^5 \right) \right) dx$$

$$\int_a^b f(x) dx = 2h f(x_1) + \left(\frac{f'(x_1)}{2}(x-x_1)^2 \right) \Big|_a^b + \left(\frac{f''(x_1)}{6}(x-x_1)^3 \right) \Big|_a^b + \left(\frac{f'''(x_1)}{(6)(4)}(x-x_1)^4 \right) \Big|_a^b + \frac{f^{(4)}(\xi_1)}{120} 2(h^5)$$

$$\int_a^b f(x) dx = 2h f(x_1) + \frac{f'(x_1)}{2} (h^2 - (-h)^2) + \frac{f''(x_1)}{6} (h^3 + h^3) + \frac{f'''(x_1)}{24} (h^4 - (-h)^4) + \frac{f^{(4)}(\xi_1)}{60} h^5$$

$$\int_a^b f(x) dx = 2h f(x_1) + \frac{2f''(x_1)}{6} h^3 + \frac{f^{(4)}(\xi_1)}{60} h^5 \quad (6)$$

De otra parte tenemos

$$f''(x_1) = \frac{f(x_0) - 2f(x_1) + f(x_2)}{h^2} - \frac{h^2}{12} f^{(4)}(\xi_2)$$

$$\left(\frac{f''(x_1) h^3}{3} = \frac{h}{3} (f(x_0) - 2f(x_1) + f(x_2)) - \frac{h^5}{36} f^{(4)}(\xi_2) \right)$$

al reemplazar se obtiene.

$$\int_a^b f(x) dx = 2h f(x_1) + \frac{h}{3} (f(x_0) - 2f(x_1) + f(x_2))$$

$$- \frac{h^5}{36} f^{(4)}(\xi_2) + \frac{h^5}{60} f^{(4)}(\xi_1)$$

$$= \frac{h}{3} (f(x_0) + 4f(x_1) + f(x_2)) + \boxed{\text{ERROR}}$$

Lema auxiliar

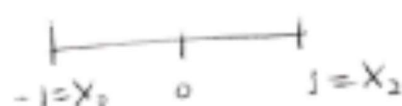
Existe $K \in \mathbb{R}$ tal que

$\forall f \in C^{(4)}[a,b]$ existe ξ tal

que $\forall \xi_1, \xi_2$ $-\frac{h^5}{36} f^{(4)}(\xi_2) + \frac{h^5}{60} f^{(4)}(\xi_1) = K f^{(4)}(\xi)$

¿Cómo calcular K ?

Consideremos $\int_{-1}^1 x^4 dx$



$$f(x) = x^4$$

$$\int_{-1}^1 x^4 dx = \frac{1}{5} \left((-1)^4 + 4 \cdot 0^4 + 1^4 \right) + \hat{K} f^{(4)}(\xi)$$

$$\frac{2}{5} = \frac{1}{3} + 24K \longrightarrow \hat{K} = -\frac{1}{90}$$



$$x = \varphi(z) = \frac{b-a}{2}(z+1) + a$$

$$\int_a^b f(x) dx = \int_{-1}^1 (f \circ \varphi)(z) \varphi'(z) dz$$

$$\int_a^b f(x) dx = \frac{b-a}{2} \left[\int_{-1}^1 H(z) dz \right]$$

(8) $H(z) = (f \circ \Phi)(z)$

$$= \frac{b-a}{2} \left[\frac{1}{3} (H(-1) + 4H(0) + H(1)) - \frac{1}{90} H^{(4)}(\eta) \right]$$

$$= \frac{b-a}{2} \left[\frac{1}{3} (f(a) + 4f(\frac{a+b}{2}) + f(b)) \right]$$

$$- \frac{b-a}{2(90)} \left(\frac{b-a}{2} \right)^4 f^{(4)}(\xi)$$

$$= -\frac{h^5}{90} f^{(4)}(\xi)$$

$$\frac{d^4}{dz^4} (f \circ \Phi(z)) = \left(\frac{b-a}{2} \right)^4 f^{(4)}(\xi)$$

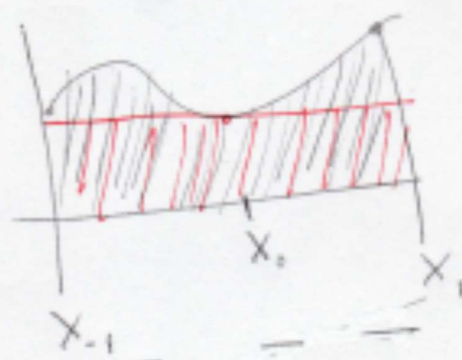
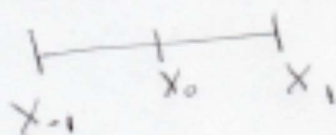
$$\begin{aligned} \frac{d}{dz} (f \circ \Phi)(z) &= \overbrace{f'(\Phi(z))}^{\text{chain rule}} \overbrace{\Phi'(z)}^{\text{chain rule}} = \frac{b-a}{2} f'(\xi) \\ &= (f' \circ \Phi)(z) \left(\frac{b-a}{2} \right) \end{aligned}$$

(9)

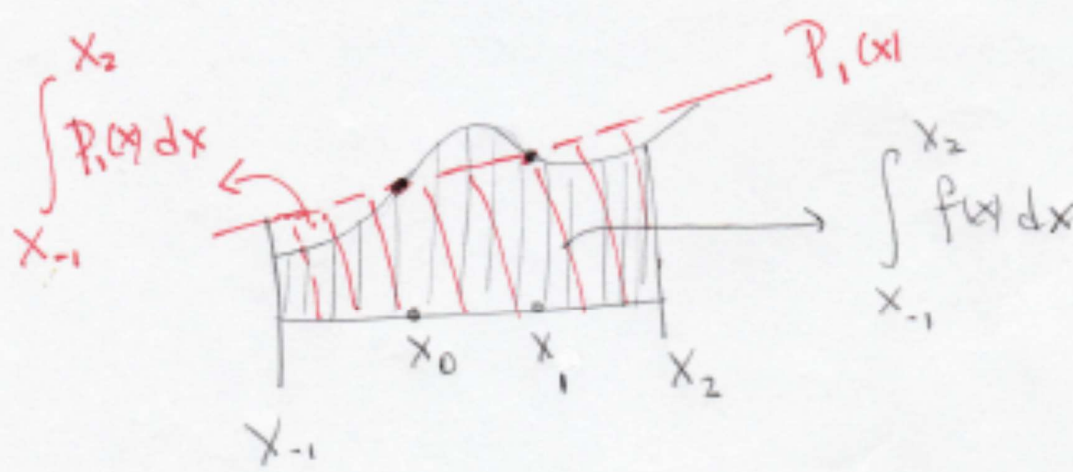
fórmulas de newton-cotes abiertas

 $n=0$

$$\int_{x_{-1}}^{x_1} f(x) dx = 2h f(x_0) + \frac{h^3}{3} f''(\xi)$$


 $n=1$

$$\int_{x_{-1}}^{x_2} f(x) dx = \frac{3}{2}h (f(x_0) + f(x_1)) + \frac{3h^3}{4} f''(\xi)$$

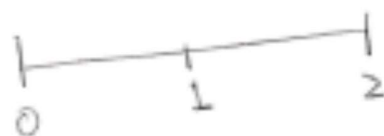


ejemplo

$$\int_0^2 (1 + e^x \sin(2x-1)) dx =$$

$$= 2 + \frac{1}{5} (\sin(1) + 2\cos(1) - 2e^2 \cos(3) + e^2 \sin(3))$$

$$\approx \underline{5.51900789}$$



Regla de Simpson

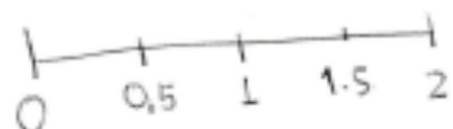
$$\begin{aligned} \int_0^2 f(x) dx &\approx \frac{h}{3} (f(x_0) + 4f(x_1) + f(x_2)) \\ &= \frac{1}{3} (f(0) + 4f(1) + f(2)) \\ &= \frac{1}{3} (15.35069382) \end{aligned}$$

$$\approx \underline{5.11689794}$$

Regla del punto medio

$$\begin{aligned} \int_0^2 f(x) dx &\approx 2h f(1) = 2f(1) \\ &= \underline{6.57471057} \end{aligned}$$

(11)

fórmula de newton, cerrada con $n=4$ 

$$h=0.5$$

$$\int_0^2 (1 + e^x \sin(2x-1)) dx \approx$$

$$\frac{2h}{45} (7f(0) + 32f(0.5) + 12f(1) + \dots + 32f(1.5) + 7f(2)) \approx \underline{5.53918220}$$

Integración numérica compuesta



$$\int_a^b f(x) dx = \int_{x_0}^{x_1} f dx + \int_{x_1}^{x_2} f dx + \dots + \int_{x_{n-1}}^{x_n} f dx$$

$$\left| h = \frac{b-a}{n} \right| \approx \frac{h}{2} (f(x_0) + f(x_1)) + \frac{h}{2} (f(x_1) + f(x_2)) + \dots + \frac{h}{2} (f(x_{n-1}) + f(x_n))$$

(12)

Regla trapezoidal compuesta

$$\int_a^b f(x) dx = \frac{h}{2} \left[f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n) \right] - \frac{h^3}{12} \left(f''(\xi_1) + f''(\xi_2) + \dots + f''(\xi_n) \right)$$

$\xi_i \in (X_{i-1}, X_i)$

Error \swarrow

$$\text{Error} = - \frac{h^3 n}{12} \left(\frac{f''(\xi_1) + \dots + f''(\xi_n)}{n} \right)$$

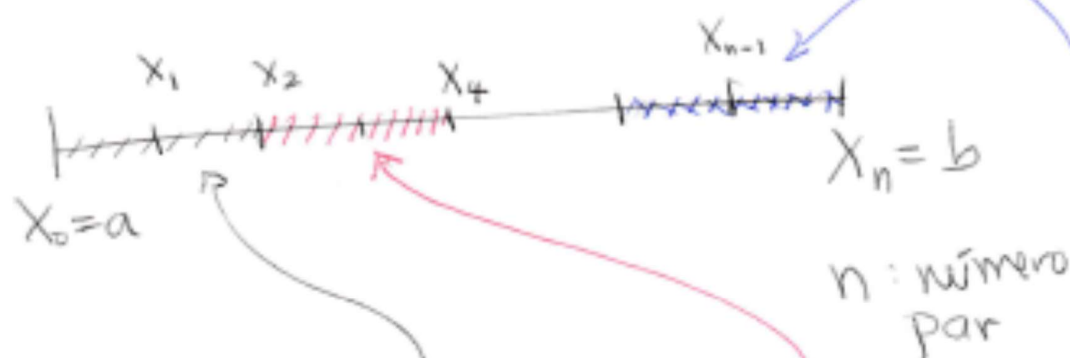
Aplicando el teorema del valor intermedio existe $\xi \in (a, b)$

tal que $f''(\xi) = \frac{f''(\xi_1) + \dots + f''(\xi_n)}{n}$

$$\frac{h^3 n}{12} = \frac{h^2 h n}{12} = \frac{h^2}{12} \left(\frac{b-a}{n} \right) n = \frac{b-a}{12} h^2$$

$$\text{Error} = - \frac{(b-a) h^2}{12} f''(\xi)$$

Regla de Simpson compuesta



$$\int_a^b f(x) dx \approx \frac{h}{3} (f(x_0) + 4f(x_1) + f(x_2)) + \frac{h}{3} (f(x_2) + 4f(x_3) + f(x_4)) + \dots + \frac{h}{3} (f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

$$\int_a^b f(x) dx = \frac{h}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)) - \frac{h^5}{90} (f^{(4)}(\theta_1) + f^{(4)}(\theta_2) + \dots + f^{(4)}(\theta_{\frac{n}{2}}))$$

$$\theta_1 \in (x_0, x_2) \quad \theta_2 \in (x_2, x_4) \quad \dots \quad \theta_{\frac{n}{2}} \in (x_{n-2}, x_n)$$

(14)

$$\epsilon_{\text{error}} = -\frac{h^5}{90} \frac{n}{2} \left(\frac{f^{(4)}(\theta_1) + \dots + f^{(4)}(\theta_{\frac{n}{2}})}{\frac{n}{2}} \right)$$

$$= -\frac{h^5}{90} \frac{n}{2} f^{(4)}(\xi) \quad \xi \in (x_0, x_n)$$

$$= -\frac{h^4}{90} \frac{n}{2} \left(\frac{b-a}{n} \right) f^{(4)}(\xi)$$

$$= -\frac{h^4 (b-a)}{180} f^{(4)}(\xi)$$

$$I = \int_0^2 (1 + e^x \sin(2x-1)) dx$$

usando 4 subintervalos (5 nodos)
 $h = 0.5$

$$I \approx 5.23158998$$

Trap.-compuesta

$$I \approx 5.51278943$$

Simpson-compuesta