COERCION IN FCOM

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1. Preliminaries

The typing rules for coercion, homogeneous composition and heterogeneous composition are:

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \qquad \Gamma \vdash u : A(r/i)}{\Gamma \vdash \mathsf{coe}_i^{r \to s}\, A\, u : A(s/i)[r = s \mapsto u]}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \qquad \Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \\ \Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash u_0 : A[\varphi \mapsto u(r/i)]\end{array}}{\Gamma \vdash \mathsf{hcom}_i^{r \to s}\, A\, [\varphi \mapsto u]\, u_0 : A[\varphi \mapsto u(s/i), r = s \mapsto u_0]}$$

$$\frac{\begin{array}{c}\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \\ \Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash u_0 : A(r/i)[\varphi \mapsto u(r/i)]\end{array}}{\Gamma \vdash \mathsf{com}_i^{r \to s}\, A\, [\varphi \mapsto u]\, u_0 : A(s/i)[\varphi \mapsto u(s/i), r = s \mapsto u_0]}$$

The rules for fcom types:

$$\frac{\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \qquad \Gamma, \varphi, i : \mathbb{I} \vdash B \qquad \Gamma \vdash A[\varphi \mapsto B(r/i)]}{\Gamma \vdash \mathsf{fcom}_i^{r \to s}\, [\varphi \mapsto B]\, A\, [r = s \mapsto A, \varphi \mapsto B(s/i)]}$$

$$\frac{\begin{array}{c}\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \qquad \Gamma, \varphi, i : \mathbb{I} \vdash B \\ \Gamma \vdash A[\varphi \mapsto B(r/i)] \qquad \Gamma, \varphi \vdash u : B(s/i) \qquad \Gamma \vdash u_0 : A[\varphi \mapsto \mathsf{coe}_i^{s \to r}\, B\, u]\end{array}}{\Gamma \vdash \mathsf{box}^{r \to s}\, [\varphi \mapsto u]\, u_0 : \mathsf{fcom}_i^{r \to s}\, [\varphi \mapsto B]\, A[r = s \mapsto u_0, \varphi \mapsto u]}$$

$$\frac{\begin{array}{c}\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \\ \Gamma, \varphi, i : \mathbb{I} \vdash B \qquad \Gamma \vdash A[\varphi \mapsto B(r/i)] \qquad \Gamma \vdash u : \mathsf{fcom}_i^{r \to s}\, [\varphi \mapsto B]\, A\end{array}}{\Gamma \vdash \mathsf{cap}_i^{r \leftarrow s}\, [\varphi \mapsto B]\, u : A[r = s \mapsto u, \varphi \mapsto \mathsf{coe}_i^{s \to r}\, B\, u]}$$

$$\frac{\begin{array}{c}\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \qquad \Gamma, \varphi, i : \mathbb{I} \vdash B \\ \Gamma \vdash A[\varphi \mapsto B(r/i)] \qquad \Gamma, \varphi \vdash u : B(s/i) \qquad \Gamma \vdash u_0 : A[\varphi \mapsto \mathsf{coe}_i^{s \to r}\, B\, u]\end{array}}{\Gamma \vdash \mathsf{cap}_i^{r \leftarrow s}\, [\varphi \mapsto B]\, (\mathsf{box}^{r \to s}\, [\varphi \mapsto u]\, u_0) = u_0 : A}$$

$$\frac{\begin{array}{c}\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \\ \Gamma, \varphi, i : \mathbb{I} \vdash B \qquad \Gamma \vdash A[\varphi \mapsto B(r/i)] \qquad \Gamma \vdash u : \mathsf{fcom}_i^{r \to s}\, [\varphi \mapsto B]\, A\end{array}}{\Gamma \vdash \mathsf{box}^{r \to s}\, [\varphi \mapsto u]\, (\mathsf{cap}_i^{r \leftarrow s}\, [\varphi \mapsto 1B]\, u) = u : \mathsf{fcom}_i^{r \to s}\, [\varphi \mapsto B]\, A}$$

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The fiber of a map $e : A \to B$ over $x : B$ is defined as:

$$\mathsf{Fiber}\, e\, x := (y : A) \times \mathsf{Path}\ B\ (e\ y)\ x$$

Lemma ("Dagstuhl lemma")**.** *Given*

$$i : \mathbb{I} \vdash A \qquad \vdash r, s : \mathbb{I} \qquad \vdash y : A(s/i) \qquad \vdash \varphi : \mathbb{F}$$

we can extend

$$\varphi \vdash (x, \langle\_\rangle\, \mathsf{coe}_i^{r \to s}\, A\, x) : \mathsf{Fiber}\, (\mathsf{coe}_i^{r \to s}\, A)\, y$$

to a total element

$$\vdash (a, p) : (\mathsf{Fiber}\, (\mathsf{coe}_i^{r \to s}\, A)\, y)\ [\varphi \mapsto (x, \langle\_\rangle\, \mathsf{coe}_i^{r \to s}\, A\, x)]$$

Proof. Note that $\varphi \vdash x : A(r/i)$ and $\varphi \vdash \mathsf{coe}_i^{r \to s}\, A\, x = y$ by assumption so that we can first let

$$\vdash a := \mathsf{com}_i^{s \to r}\, A\, [\varphi \mapsto \mathsf{coe}_i^{r \to i}\, A\, x]\, y : A(r/i)$$

We then let

$$i : \mathbb{I} \vdash \alpha_0 := \mathsf{coe}_i^{r \to i}\, A\, a : A$$
$$i : \mathbb{I} \vdash \alpha_1 := \mathsf{com}_i^{s \to i}\, A\, [\varphi \vdash \mathsf{coe}_i^{r \to i}\, A\, x]\, y : A$$

so that we can take

$$\vdash p := \langle j \rangle\, \mathsf{com}_i^{r \to s}\, A\, [(j = 0) \mapsto \alpha_0, (j = 1) \mapsto \alpha_1, \varphi \mapsto \mathsf{coe}_i^{r \to i}\, A\, x]\, a$$

for a fresh dimension $j : \mathbb{I}$. Note that

$$\vdash p\, 0 = \alpha_0(s/i) = \mathsf{coe}_i^{r \to s}\, A\, a$$
$$\vdash p\, 1 = \alpha_1(s/i) = y$$

as desired. □

2. Coercion in fcom

The goal of this section is to explain how to compute:

$$\mathsf{coe}_i^{r \to r'}\, (\mathsf{fcom}_j^{s \to s'}\, [\varphi \mapsto B]\, A)\, u_0 : (\mathsf{fcom}_j^{s \to s'}\, [\varphi \mapsto B]\, A)(r'/i)$$

We omit the ambient context Γ and are given:

$$i : \mathbb{I} \vdash \varphi : \mathbb{F}$$
$$i : \mathbb{I} \vdash s, s' : \mathbb{I}$$
$$i : \mathbb{I}, \varphi, j : \mathbb{I} \vdash B$$
$$i : \mathbb{I} \vdash A[\varphi \mapsto B(s/j)]$$

so that

$$i : \mathbb{I} \vdash F := \mathsf{fcom}_j^{s \to s'}\, [\varphi \mapsto B]\, A$$

is a well-formed fcom type such that:

$$i : \mathbb{I}, \varphi \vdash F = B(s'/j)$$
$$i : \mathbb{I}, (s = s') \vdash F = A$$

Given $\vdash r, r' : \mathbb{I}$ and $\vdash u_0 : F(r/i)$ the goal is to define

$$\vdash w := \mathsf{coe}_i^{r \to r'}\, F\, u_0 : F(r'/i)$$

such that

$$\begin{aligned}
\forall i.\varphi &\vdash w = \mathsf{coe}_i^{r\to r'}\, B(s'/j)\, u_0 : B(s'/j)(r'/i)\\
(r=r') &\vdash w = u_0 : F(r/i)\\
\forall i.(s=s') &\vdash w = \mathsf{coe}_i^{r\to r'}\, A\, u_0 : A(r'/i)
\end{aligned}$$

Step 1. Compute the cap of u_0:

$$\begin{aligned}
\vdash a_0 :=\ & \mathsf{cap}_j^{s(r/i)\leftarrow s'(r/i)}\, [\varphi(r/i) \mapsto B(r/i)]\, u_0\\
&: A(r/i)[\,(s(r/i) = s'(r/i)) \mapsto u_0\\
&\qquad ,\varphi(r/i) \mapsto \mathsf{coe}_j^{s'(r/i)\to s(r/i)}\, B(r/i)\, u_0]
\end{aligned}$$

Note that if i does not occur in s and s' then the first restriction reads like $(s = s') \mapsto u_0$.

Step 2. Note that $\varphi \vdash u_0 : B(s'/j)(r/i)$ and we can coerce in $B(s'/j)$ to get a line along i:

$$\forall i.\varphi, i : \mathbb{I} \vdash \tilde{b} := \mathsf{coe}_i^{r\to i}\, B(s'/j)\, u_0 : B(s'/j)$$

so that

$$\begin{aligned}
\forall i.\varphi &\vdash \tilde{b}(r/i) = u_0 : B(s'/j)(r/i)\\
\forall i.\varphi &\vdash b_1 := \tilde{b}(r'/i) = \mathsf{coe}_i^{r\to r'}\, B(s'/j)\, u_0 : B(s'/j)(r'/i)
\end{aligned}$$

Note that $(r = r') \vdash b_1 = u_0$.

Step 3. We now compute a composition of a_0 with the tubes given by $\tilde{b}$ and a suitable correction:

$$\vdash a_1 := \mathsf{com}_i^{r\to r'}\, A\, [\forall i.\varphi \mapsto \mathsf{coe}_j^{s'\to s}\, B\, \tilde{b}, \forall i.(s=s') \mapsto \mathsf{coe}_i^{r\to i}\, A\, u_0]\, a_0 : A(r'/i)$$

which is well-formed as $\forall i.\varphi \leqslant \varphi(r/i)$ so that

$$\forall i.\varphi \vdash (\mathsf{coe}_j^{s'\to s}\, B\, \tilde{b})(r/i) = \mathsf{coe}_j^{s'(r/i)\to s(r/i)}\, B(r/i)\, u_0 = a_0$$

and

$$\forall i.(s=s') \vdash a_1 = \mathsf{coe}_i^{r\to r}\, A\, u_0 = u_0 = a_0$$

by the remark at the end of Step 1.

Furthermore this satisfies:

$$\begin{aligned}
\forall i.\varphi &\vdash a_1 = \mathsf{coe}_j^{s'(r'/i)\to s(r'/i)}\, B(r'/i)\, (\mathsf{coe}_i^{r\to r'}\, B(s'/j)\, u_0) : B(r'/i)(s(r'/i)/j)\\
(r=r') &\vdash a_1 = a_0 : A(r/i)\\
\forall i.(s=s') &\vdash a_1 = \mathsf{coe}_i^{r\to r'}\, A\, u_0 : A(r'/i)
\end{aligned}$$

where the first equation makes sense as $\varphi(r'/i) \vdash A(r'/i) = B(s/j)(r'/i) = B(r'/i)(s(r'/i)/j)$.

Step 4. Note that we have an element in the fiber of

$$e := \mathsf{coe}_j^{s'(r'/i)\to s(r'/i)}\, B(r'/i)\ : B(r'/i)(s'(r'/i)/j) \to B(r'/i)(s(r'/i)/j)$$

over a_1 given by:

$$\varphi(r'/i), \forall i.\varphi \vdash (b_1, \langle \_ \rangle\, a_1) : \mathsf{Fiber}\, e\, a_1$$

where

$$\mathsf{Fiber}\, e\, a_1 := (x : B(r'/i)(s'(r'/i)/j)) \times \mathsf{Path}\ B(r'/i)(s(r'/i)/j)\ (e\ x)\ a_1$$

which makes sense by the remark in the end of Step 3 above.

The fact that the pair $(b_1, \langle \_ \rangle\, a_1)$ has type $\mathsf{Fiber}\, e\, a_1$ on this restriction is justified by:

$$\varphi(r'/i), \forall i.\varphi \vdash e\, b_1 = \mathsf{coe}_j^{s'(r'/i)\to s(r'/i)}\, B(r'/i)\, (\mathsf{coe}_i^{r\to r'}\, B(s'/j)\, u_0) = a_1$$

Furthermore, we have:

$$\varphi(r'/i), (r = r') \vdash e\, b_1 = \mathsf{coe}_j^{s'(r/i)\to s(r/i)}\, B(r/i)\, u_0 = a_0 = a_1$$

The second equality holds as $\varphi(r/i) \vdash \mathsf{coe}_j^{s'(r/i)\to s(r/i)}\, B(r/i)\, u_0 = a_0$, and the other equalities hold by the remarks about what happens on $(r = r')$ in Step 2 and 3. We hence get

$$\varphi(r'/i), \forall i.\varphi \vee (r = r') \vdash (b_1, \langle \_ \rangle\, a_1) : \mathsf{Fiber}\, e\, a_1$$

Step 5. As $\varphi(r'/i), \forall i.\varphi \vee (r = r') \vdash a_1 = e\, b_1 = \mathsf{coe}_j^{s'(r'/i)\to s(r'/i)}\, B(r'/i)\, b_1$ we can use Lemma 1 to extend

$$\varphi(r'/i), \forall i.\varphi \vee (r = r') \vdash (b_1, \langle \_ \rangle\, e\, b_1) : \mathsf{Fiber}\, e\, a_1$$

to an element $\varphi(r'/i) \vdash (a, p) : (\mathsf{Fiber}\, e\, a_1)[\forall i.\varphi \vee (r = r') \mapsto (b_1, \langle \_ \rangle\, e\, b_1]$.

Step 6. We can then use p to extend this to a total element in $A(r'/i)$:

$$\begin{aligned} \vdash a_1' := \mathsf{hcom}_k^{1\to 0}\, A(r'/i)\, [\, & \varphi(r'/i) \mapsto p\, k \\ , & (r = r') \mapsto a_0 \\ , & \forall i.(s = s') \mapsto a_1]\, a_1 : A(r'/i) \end{aligned}$$

Note that on $\varphi(r'/i)$ we have $p\, 1 = a_1$ (as $(a, p) : \mathsf{Fiber}\, e\, a_1$) and on $(r = r')$ we have $a_1 = a_0$, making the homogeneous composition well-formed.

Step 7. Finally we combine all the parts we have computed:

$$\vdash w := \mathsf{box}^{s(r'/i)\to s'(r'/i)}\, [\varphi(r'/i) \mapsto a]\, a_1' : F(r'/i)$$

To see that w is well-formed note that:

$$\varphi(r'/i) \vdash a_1' = p\, 0 = e\, a = \mathsf{coe}_j^{s'(r'/i)\to s(r'/i)}\, B(r'/i)\, a$$

where the second equality holds as $(a, p) : \mathsf{Fiber}\, e\, a_1$.

To see that w satisfies the three necessary properties note that $\forall i.\varphi \leqslant \varphi(r'/i)$ so that

$$\forall i.\varphi \vdash w = a = b_1 = \mathsf{coe}_i^{r\to r'}\, B(s'/j)\, u_0$$

as desired. Furthermore,

$$\begin{aligned}\varphi(r'/i), (r = r') &\vdash a = b_1 = u_0 \\ (r = r') &\vdash a'_1 = a_0 = \mathsf{cap}_j^{s(r/i) \leftarrow s'(r/i)}\,[\varphi(r/i) \mapsto B(r/i)]\, u_0\end{aligned}$$

so that

$$\begin{aligned}(r = r') \vdash w &= \mathsf{box}^{s(r/i) \to s'(r/i)}\,[\varphi(r/i) \mapsto u_0] \\ &\qquad (\mathsf{cap}_j^{s(r/i) \leftarrow s'(r/i)}\,[\varphi(r/i) \mapsto B(r/i)]\, u_0) \\ &= u_0\end{aligned}$$

by the η-rule for box and cap.

Finally, we have:

$$\begin{aligned}\forall i.(s = s') \vdash w &= \mathsf{box}^{s(r'/i) \to s'(r'/i)}\,[\varphi(r/i) \mapsto a]\, a'_1 \\ &= a'_1 \\ &= a_1 \\ &= \mathsf{coe}_i^{r \to r'}\, A\, u_0\end{aligned}$$

3. Summary of the algorithm

Using the variable names above the goal is to define:

$$\vdash w := \mathsf{coe}_i^{r \to r'}\, F\, u_0 : F(r'/i)$$

The algorithm to compute w is then:

$$
\begin{aligned}
a_0 &= \mathsf{cap}_j^{s(r/i) \leftarrow s'(r/i)}\, [\varphi(r/i) \mapsto B(r/i)]\, u_0 \\
a_1 &= \mathsf{com}_i^{r \to r'}\, A\, [\forall i.\varphi \mapsto \mathsf{coe}_j^{s' \to s}\, B\, (\mathsf{coe}_i^{r \to i}\, B(s'/j)\, u_0), \forall i.(s = s') \mapsto \mathsf{coe}_i^{r \to i}\, A\, u_0]\, a_0 \\
b_1 &= \mathsf{coe}_i^{r \to r'}\, B(s'/j)\, u_0 \\
\bar{a} &= \mathsf{com}_j^{s(r'/i) \to j}\, B(r'/i)\, [\forall i.\varphi \vee (r = r') \mapsto \mathsf{coe}_j^{s'(r'/i) \to j}\, B(r'/i)\, b_1]\, a_1 \\
a &= \bar{a}(s'(r'/i)/j) \\
p &= \mathsf{com}_j^{s'(r'/i) \to s(r'/i)}\, B(r'/i)\, [\,(k = 0) \mapsto \mathsf{coe}_j^{s'(r'/i) \to j}\, B(r'/i)\, a \\
&\qquad , (k = 1) \mapsto \bar{a} \\
&\qquad , \forall i.\varphi \vee (r = r') \mapsto \mathsf{coe}_j^{s'(r'/i) \to j}\, B(r'/i)\, b_1]\, a \\
a_1' &= \mathsf{hcom}_k^{1 \to 0}\, A(r'/i)\, [\,\varphi(r'/i) \mapsto p, (r = r') \mapsto a_0, \forall i.(s = s') \mapsto a_1]\, a_1 \\
w &= \mathsf{box}^{s(r'/i) \to s'(r'/i)}\, [\,\varphi(r'/i) \mapsto a]\, a_1'
\end{aligned}
$$

Here is the algorithm from Part III:

$$
\begin{aligned}
a_0 &= \mathsf{cap}_j^{s(r/i) \leftarrow s'(r/i)}\, [\varphi(r/i) \mapsto B(r/i)]\, u_0 \\
a_0' &= \mathsf{hcom}_j^{s'(r/i) \to j}\, A(r/i)\, [\varphi(r/i) \mapsto \mathsf{coe}_j^{j \to s(r/i)}\, B(r/i)\, (\mathsf{coe}_j^{s'(r/i) \to j}\, B(r/i)\, u_0)]\, a_0 \\
a_1 &= \mathsf{com}_i^{r \to r'}\, A\, [\forall i.\varphi \mapsto \mathsf{coe}_j^{s' \to s}\, B\, (\mathsf{coe}_i^{r \to i}\, B(s'/j)\, u_0), \forall i.(s = s') \mapsto \mathsf{coe}_i^{r \to i}\, A\, u_0]\, a_0'(s(r/i)/j) \\
b_1 &= \mathsf{coe}_i^{r \to r'}\, B(s'/j)\, u_0 \\
\bar{a} &= \mathsf{com}_j^{s(r'/i) \to j}\, B(r'/i)\, [\forall i.\varphi \vee (r = r') \mapsto \mathsf{coe}_j^{s'(r'/i) \to j}\, B(r'/i)\, b_1]\, a_1 \\
p &= \mathsf{coe}_j^{j \to s(r'/i)}\, B(r'/i)\, \bar{a} \\
a_1' &= \mathsf{hcom}_j^{s(r'/i) \to s'(r'/i)}\, A(r'/i)\, [\,\varphi(r'/i) \mapsto p, (r = r') \mapsto a_0']\, a_1 \\
w &= \mathsf{box}^{s(r'/i) \to s'(r'/i)}\, [\,\varphi(r'/i) \mapsto \bar{a}(s'(r'/i)/j)]\, a_1'
\end{aligned}
$$

The major difference is the simplification in p, which in turn leads to an extra hcom in a_0'.