Formalizing $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ and Computing a Brunerie Number in Cubical Agda

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***Abstract*—Brunerie's 2016 PhD thesis contains the first synthetic proof in Homotopy Type Theory (HoTT) of the classical result that the fourth homotopy group of the 3-sphere is $\mathbb{Z}/2\mathbb{Z}$. The proof is one of the most impressive pieces of synthetic homotopy theory to date and uses a lot of advanced classical algebraic topology rephrased synthetically. Furthermore, the proof is fully constructive and the main result can be reduced to the question of whether a particular "Brunerie number" β can be normalized to ± 2. The question of whether Brunerie's proof could be formalized in a proof assistant, either by computing this number or by formalizing the pen-and-paper proof, has since remained open. In this paper, we present a complete formalization in `Cubical Agda`. We do this by modifying Brunerie's proof so that a key technical result, whose proof Brunerie only sketched in his thesis, can be avoided. We also present a formalization of a new and much simpler proof that β is ± 2. This formalization provides us with a sequence of simpler Brunerie numbers, one of which normalizes very quickly to -2 in `Cubical Agda`, resulting in a fully formalized computer-assisted proof that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.**

I. Introduction

Homotopy theory originated in algebraic topology, but is by now a central tool in many branches of modern mathematics, such as algebraic geometry and category theory. One of the central notions in homotopy theory is that of the *homotopy groups* of a space X, denoted $\pi_n(X)$. These groups constitute a topological invariant, making them a powerful tool for establishing if two given spaces are homotopy equivalent. The first two such groups of a space are easy to understand: $\pi_0(X)$ characterizes the connected components of X and $\pi_1(X)$ is the fundamental group, i.e. the group of equivalence classes of loops in X up to homotopy. This idea generalizes to higher values of n, for which $\pi_n(X)$ consists of n-dimensional loops up to homotopy. For many spaces, these groups are increasingly esoteric and difficult to compute for large n. This is true also for seemingly tame spaces like spheres where $\pi_n(\mathbb{S}^m)$ in general is highly irregular when $n > m \geq 2$.

This paper concerns the first computer formalization of the classical result that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$, a result which is particularly interesting because it gives the whole first stable stem of homotopy groups of spheres, i.e. $\pi_{n+1}(\mathbb{S}^n)$ for $n \geq 3$. The fact that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ was proved already in the 1930's by Pontryagin using cobordism theory, but we instead follow the synthetic approach to homotopy theory developed in Homotopy Type Theory (HoTT) and popularized by the HoTT Book [1]. In this new approach, spaces up to homotopy are represented directly as higher inductive types (HITs) and homotopy groups are computed using Voevodsky's univalence axiom [2]. This gives a logical approach suitable for computer formalization, where the results are interpretable in any suitably structured $(\infty, 1)$-topos [3].

The basis for our formalization is the 2016 PhD thesis of Brunerie [4] which contains the first synthetic proof in HoTT that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$. The proof is one of the most impressive pieces of synthetic homotopy theory to date and uses advanced machinery from classical algebraic topology developed synthetically, including the symmetric monoidal structure of smash products, (integral) cohomology rings, the Mayer-Vietoris and Gysin sequences, the Hopf invariant, Whitehead products, etc. The formalization of Brunerie's proof has since remained open, primarily due to the highly technical nature of some of the details. In this paper, we present such a formalization in `Cubical Agda` [5], a *cubical* extension of `Agda` [6] with computational support for univalence and HITs.

In addition to being a very impressive proof in synthetic homotopy theory, Brunerie's proof is particularly interesting as it is fully constructive. The proof consists of two parts, with the first one culminating at the end of Chapter 3 with the definition of a number $\beta : \mathbb{Z}$ such that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$. Since then, this β has been commonly referred to as the *Brunerie number*. Brunerie writes the following about it:

> *This result is quite remarkable in that even though it is a constructive proof, it is not at all obvious how to actually compute this [β]. At the time of writing, we still haven't managed to extract its value from its definition. [4, Page 85]*

In fact, [4, Appendix B] contains a complete and concise definition of β as the image of 1 under a sequence of 12 maps:

$$
\begin{aligned}
&\mathbb{Z} \longrightarrow \Omega(\mathbb{S}^1) \longrightarrow \Omega^2(\mathbb{S}^2) \longrightarrow \Omega^3(\mathbb{S}^3) \\
&\longrightarrow \Omega^3(\mathbb{S}^1 * \mathbb{S}^1) \longrightarrow \Omega^3(\mathbb{S}^2) \longrightarrow \Omega^3(\mathbb{S}^1 * \mathbb{S}^1) \to \Omega^3(\mathbb{S}^3) \\
&\longrightarrow \Omega^2\|\mathbb{S}^2\|_2 \longrightarrow \Omega\|\Omega(\mathbb{S}^2)\|_1 \to \|\Omega^2(\mathbb{S}^2)\|_0 \longrightarrow \Omega(\mathbb{S}^1) \to \mathbb{Z}
\end{aligned}
$$

By implementing this number in a proof assistant with computational support for univalence and HITs, one should be able to normalize it using a computer to establish that $\beta = \pm 2$ and hence that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$. In 2016, by the time Brunerie

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was finishing his thesis, there were some experimental proof assistants based on the cubical type theory of [7], but these were too slow to perform such a complex computation. So, instead of relying on normalization, Brunerie spends the second part of the thesis (Chapters 4–6) to prove, using a lot of the advanced machinery mentioned above, that $|\beta|$ is propositionally equal to 2. However, if one were instead able to compute the number automatically in a proof assistant, this equality would hold definitionally—effectively reducing the complexity and length of the proof by an order of magnitude.

The intriguing possibility of a computer-assisted formal proof made many people interested and countless attempts to normalize it has been made using increasingly powerful computers. However, to date, no one has succeeded and it is still unclear if β is normalizable in a reasonable amount of time. In light of this, it is natural to wonder if it is possible to simplify it to be able to compute it. For example, the original definition only involves 1-HITs, as the status of higher HITs was still quite understudied at the time. With a better understanding of higher HITs [8], [9], [10], one quickly sees that the first 3 maps can be combined into one sending 1 to the 3-cell of $\mathbb{S}^3$ defined as a 3-HIT instead of as an iterated suspension as in [4]. Unfortunately, simple optimizations like this do not seem to be sufficient and, after several unsuccessful attempts, we decided to formalize the second half of Brunerie's thesis instead. However, this is by no means straightforward. The first issue appears already in Section 4.1 about smash products. The main result, Proposition 4.1.2, says that the smash product is a 1-coherent symmetric monoidal product on pointed types. However, its proof is just a sketch and Brunerie writes the following about it:

> *The following result is the main result of this section even though we essentially admit it. [4, Page 90]*

Unfortunately, this result is then used to construct integral cohomology rings, $H^*(X)$, whose cup product, $\smile$, appears in the definition of the so-called Hopf invariant which is crucially used to prove that $|\beta|$ is 2. While one might be convinced that Brunerie's informal proof sketch is correct, it is not obvious how one convinces a proof assistant of this. A complete formalization would either have to fill in the holes in the proof sketch or find an alternative construction which avoids Proposition 4.1.2. In fact, Brunerie tried very hard to fill these holes using Agda metaprogramming [11]. However, he never managed to typecheck his computer generated proof of the pentagon identity. Hence, this approach also seems infeasible with current proof assistant technology. Luckily, Brunerie, Ljungström and Mörtberg [12] recently gave an alternative synthetic definition of the cup product on $H^*(X)$ which completely avoids smash products. This has allowed us to completely skip the problematic Chapter 4 and, in particular, Proposition 4.1.2, while still following the proofs in Chapters 5 and 6. Having a strategy for a formal proof, we were then able to embark on the ambitious project of formalizing Brunerie's proof. Even though we do not need smash products, there was still a lot left to formalize and our formalization closely follows Brunerie's proofs, except for various smaller simplifications and adjustments which we discuss in the paper.

In addition to this, we have also formalized a new proof by Ljungström [13] which completely circumvents Chapters 4–6. This major simplification builds on manually calculating the image of the element $\eta : \pi_3(\mathbb{S}^2)$, corresponding to β under the isomorphism $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, by dividing this isomorphism into several maps, tracing η in each step. In particular, the new proof is completely elementary and does not rely on advanced tools such as cohomology. The elements that one obtains while tracing η are all new "Brunerie numbers" that should normalize to ± 2. In fact, one of these normalizes to -2 in `Cubical Agda` in just under 4 seconds on a regular laptop at the time of writing. So, despite still not being able to compute the original β, this work can be seen as an alternative solution to Brunerie's conjecture about obtaining a computational proof that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ which relies on simplifying the Brunerie number until it becomes effectively computable.

Outline. The paper closely follows the structure of Brunerie's proof. In Section II, we discuss key results from HoTT that we will need and their formalization in `Cubical Agda`. Section III, which roughly corresponds to Chapter 2 of Brunerie's thesis, contains some first results on homotopy groups of spheres—e.g. the computation of $\pi_n(\mathbb{S}^m)$ for $n \leq m$. We then give Brunerie's definition of β and prove that $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$, the formalization of which involves the James construction and Whitehead products. The remainder of the paper is then devoted to the formalization of the different proofs that $\beta = \pm 2$. We first discuss the formalization of Chapters 4–6 of Brunerie's proof in Section V. This involves a lot of technical machinery like cohomology, the Hopf invariant, etc. We then, in Section VI, turn our attention to the new elementary proof that $\beta = \pm 2$ and the new Brunerie number which quickly normalizes to -2 in `Cubical Agda`. We conclude in Section VII with a discussion and comparison of the different formal proofs, as well as some directions for future work.

Formalization. All results in the paper have been formalized in `Cubical Agda` and is part of the `agda/cubical` library (https://github.com/agda/cubical/). The code in the paper is mainly literal `Agda` code taken verbatim from the library, but we have taken some liberties when typesetting, e.g. shortening notations and omitting some universe levels. A `Cubical Agda` summary file linking the formalization and paper can be found here. The development typechecks with `Agda`'s `--safe` flag, which ensures that there are no admitted goals or postulates.

II. Homotopy type theory in `Cubical Agda`

In this section, we concisely summarize the key HoTT concepts needed for the proofs and their formalization in `Cubical Agda`. This roughly corresponds to [4, Chap. 1]. For a more in depth introduction see [1], which also serves as a reference for the formal language "Book HoTT". In this paper, we mainly use cubical notations, but almost all of the results hold with minor changes in Book HoTT where paths are represented using Martin-Löf's inductive Id-types [14] instead of cubical path types. In Section VII we discuss in more detail which proofs crucially rely on cubical features.

All of the results presented in this section were already part of the `agda/cubical` library before we began our formalization and, while useful as a resource for our notations, experts on HoTT and `Cubical Agda` can safely skim this section.

A. Elementary HoTT notions and `Cubical Agda` notations

We write $(x : A) \to B\,x$ for dependent function types and denote the identity function by $\mathsf{id}_A : A \to A$. We write $\Sigma_{x:A}(B\,x)$ for dependent pair types and $\mathsf{fst}/\mathsf{snd}$ for its projections. We write Bool for the type of booleans and $\mathbb{1}$ for the unit/singleton type with a single point $\star_1$.

Cubically, paths correspond to functions out of the unit interval, just like in traditional topology. In `Cubical Agda`, there is a primitive interval (pre-)type I with endpoints $\mathsf{i0}$ and $\mathsf{i1}$. A path of type $x \equiv y$ between two points $x, y : A$ is a function $p : \mathsf{I} \to A$ such that $p\,\mathsf{i0} = x$ and $p\,\mathsf{i1} = y$. As an example, the constant path at x is defined by:

```
refl : (x : A) → x ≡ x
refl x = λ i → x
```

Note that we use "$=$" for definitional/judgmental equality and "$\equiv$" for `Cubical Agda`'s path-equality. This can be contrasted with [1] which uses the opposite convention where "$=$" is propositional/typal equality. This type of notational conventions is not the only difference between `Cubical Agda` and Book HoTT. Many proofs by path induction in Book HoTT can instead be replaced by direct proofs involving cubical path types. For example, cong (ap in Book HoTT), which applies a function to a path, can be proved as:

```
cong : (f : A → B) (p : x ≡ y) → f x ≡ f y
cong f p i = f (p i)
```

Two other interesting applications of cubical path types are the direct proofs of function extensionality and its inverse funExt^- which are both one-liners [5, Sect. 2.1]. Although the treatment of paths in `Cubical Agda` differs somewhat from Book HoTT, we may still prove *path induction*: for any dependent type $B : (y : A)\,(p : x \equiv y) \to \mathsf{Type}$, all dependent functions $f : (y : A)\,(p : x \equiv y) \to B\,x\,p$ are uniquely determined by $f\,x\,(\mathsf{refl}\,x)$. In Book HoTT, this can be used, among other things, to define the notion of a *dependent* path, which formalizes the situation when two points $a : A$ and $b : B$ are equal up to a path $p : A \equiv B$. In `Cubical Agda`, however, the type of dependent paths is primitive:

```
PathP : (A : I → Type) → A i0 → A i1 → Type
```

In fact, $\_\equiv\_$ is just the special case of PathP where the line of paths $A : \mathsf{I} \to \mathsf{Type}$ is constant.

Following [1], a *pointed type* is a dependent pair $(A, \star_A)$ consisting of a type A and a fixed basepoint $\star_A : A$. We often omit the basepoint and simply write A for the pointed type $(A, \star_A)$. Given pointed types A and B, the type of *pointed functions* $A \to_\star B$ consists of pairs $(f, \star_f)$ where $f : A \to B$ and $\star_f : f\,\star_A \equiv \star_B$. Again, we often simply write $f : A \to_\star B$ and take $\star_f$ implicit.

A defining feature of HoTT, as opposed to Martin-Löf type theory [15], is the existence of *higher inductive types* (HITs). This is a generalization of inductive types where we are not only allowed to specify the generating points of the type in question, but also identifications between these points (and possibly identifications of these identifications, and so on). This is useful for defining quotient types, but also for defining spaces when working in the *types-as-spaces* interpretation of HoTT [1, Table 1]. `Cubical Agda` natively supports HITs and a type representing the circle can be defined as follows:

```
data S¹ : Type where
  base : S¹
  loop : base ≡ base
```

This captures precisely the representation of the circle as a cell complex with one 0-cell (base) and one 1-cell (loop). We always take $\mathbb{S}^1$ to be pointed by base. A dependent function $f : (x : \mathbb{S}^1) \to B\,x$ is determined by a point $b : B\,\mathsf{base}$ and a loop $\ell : \mathsf{PathP}(\lambda i \to B(\mathsf{loop}\,i))\,b\,b$. In `Cubical Agda`, this would be written using pattern matching, as in the left-most definition below, which is introduced side-by-side with the way it would commonly be written in informal HoTT (e.g. as in Brunerie's thesis):

f base = b	$f(\mathsf{base}) = b$
f (loop i) = $\ell\ i$	$\mathsf{ap}_f(\mathsf{loop}) = \ell$

B. More higher inductive types

Let us now introduce the remaining HITs used in [4]. These come equipped with induction principles analogous to that of $\mathbb{S}^1$. To define higher spheres, we need suspensions:

```
data Susp (A : Type) : Type where
  north : Susp A
  south : Susp A
  merid : A → north ≡ south
```

We always take suspensions to be pointed by north. We may now define the n-sphere, for $n \geq 1$, by $\mathbb{S}^n = \mathsf{Susp}^{n-1}\,\mathbb{S}^1$ where Susp^{n-1} denotes $(n-1)$-fold suspension. We also define $\mathbb{S}^{-1} = \perp$ (the empty type) and $\mathbb{S}^0 = \mathsf{Bool}$. We remark that we could equivalently have defined $\mathbb{S}^1$ as the suspension of $\mathbb{S}^0$ as is done in [4]. Our reason for not doing so is that certain functions using $\mathbb{S}^1$ appear to compute better with the $\mathsf{base}/\mathsf{loop}$ definition. Furthermore, this is the definition used in already existing code in the `agda/cubical` library.

We may also capture the (homotopy) pushout of a span $B \xleftarrow{f} A \xrightarrow{g} C$ by the following HIT:

```
data Pushout (f : A → B) (g : A → C) : Type where
  inl : B → Pushout f g
  inr : C → Pushout f g
  push : (a : A) → inl (f a) ≡ inr (g a)
```

We use pushouts to define the wedge sum of two pointed types, denoted $A \vee B$, the join of two types, denoted $A \star B$, and the cofiber of a map $f : A \to B$, denoted $\mathsf{cofib}\, f$:

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A \vee B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} A \times B \arrow[r, "\mathsf{snd}"] \arrow[d, "\mathsf{fst}"'] & B \arrow[d] \\ A \arrow[r] & A \star B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ \mathbb{1} \arrow[r] & \mathsf{cofib}\, f \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Two particularly important functions out of wedge sums are:

$$\begin{aligned} &\nabla : A \vee A \to A & \qquad & i^{\vee} : A \vee B \to A \times B \\ &\nabla\ (\mathsf{inl}\ x) = x & & i^{\vee}\ (\mathsf{inl}\ a) = (a\ ,\ \star_B) \\ &\nabla\ (\mathsf{inr}\ x) = x & & i^{\vee}\ (\mathsf{inr}\ b) = (\star_A\ ,\ b) \\ &\nabla\ (\mathsf{push}\ \star_1\ i) = \star_A & & i^{\vee}\ (\mathsf{push}\ \star_1\ i) = (\star_A\ ,\ \star_B) \end{aligned}$$

C. Truncation levels and n-truncations

An important concept in HoTT are Voevodsky's *h-levels* [16], which gives rise to the notion of an *n-type*. Since types in HoTT are interpreted as spaces (or rather, as homotopy types), they are not only determined by their points but also by which higher paths they may contain. We say that a type A is an n-type if all $(n+1)$-dimensional structure of A is trivial. Formally, this is captured by an inductive definition. We say that A is a (-2)-type if it is contractible, i.e. consisting of a single point, as captured by $\mathsf{isContr}\, A = \Sigma_{a_0 : A}((a : A) \to a_0 \equiv a)$. We inductively say that A is an $(n+1)$-type if for any $x, y : A$, the type $x \equiv y$ is an n-type. We call (-1)-types *propositions* and 0-types *sets*.

We can turn any type A into an n-type by *n-truncation*, denoted $\| A \|_n$. We often use the direct definitions of (-1)- and 0-truncation in our formalization, and similar constructions work for any fixed value of n, but not when n is arbitrary. For higher n we rely on the hub-and-spoke construction [1, Sect. 7.3]. One caveat with truncations is that a map $f : A \to B$ does *not*, in general, induce a map $f : \| A \|_n \to B$. This is, however, the case when B is an n-type. In particular, f always induces a function $\| f \|_n : \| A \|_n \to \| B \|_n$.

D. Univalence, loop spaces, and h-spaces

In order to introduce Voevodsky's univalence principle [2], we need to define the (homotopy) fiber of a function. Given a function $f : A \to B$ and a point $b : B$, we define the fiber of f over b by $\mathsf{fib}\, f\, b = \Sigma_{x:A}(f\, a \equiv b)$. We say that $f : A \to B$ is an equivalence, written $f : A \simeq B$, if $\mathsf{fib}\, f\, b$ is contractible for all $b : B$. In order to prove that a function $f : A \to B$ is an equivalence, it suffices to provide an inverse $f^{-} : B \to A$ and two paths $f \circ f^{-} \equiv \mathsf{id}_B$ and $f^{-} \circ f \equiv \mathsf{id}_A$. If f is also pointed, we write $f : A \simeq_\star B$.

Univalence states that the canonical map $A \equiv B \to A \simeq B$, defined by path induction, is an equivalence. In particular, we get a map $\mathsf{ua} : A \simeq B \to A \equiv B$ promoting equivalences to paths. This provides us with a useful method for transferring proofs between equivalent types.

Transferring proofs is, however, not the only use case of univalence in HoTT. It can also be used to characterize *loop spaces* of HITs. This is often done using the *encode-decode method* [1, Sect. 8.1.4], a type theoretic analogue of proofs by contractibilty of total spaces of fibrations. In HoTT, we define the loop space of a pointed type A, by $\Omega\, A = (\star_A \equiv \star_A)$. This is again pointed by $\mathsf{refl}\, \star_A$, so we may iterate this definition to get the nth loop space of A, denoted $\Omega^n\, A$. Loop spaces belong to a particularly important class of types called h-spaces. These consist of a pointed type B equipped with a magma structure

$$\begin{aligned} &\mu : B \times B \to B \\ &\mu_l : (b : B) \to \mu(\star_B, b) \equiv b \\ &\mu_r : (b : B) \to \mu(b, \star_B) \equiv b \end{aligned}$$

satisfying $\mu_l\, \star_B \equiv \mu_r\, \star_B$. Another particularly important h-space for our purposes is $\mathbb{S}^1$, for which we will use $+$ to denote its binary operation. $\mathbb{S}^1$ also comes equipped with a notion of inversion which we will denote by $-$. In fact, $\mathbb{S}^1$ is a commutative and associative h-space.

III. First results on homotopy groups of spheres

In this section, we cover [4, Chap. 2], which introduces some elementary results on the homotopy groups of spheres. All of these results can also be found in [1]. Before even stating them, we need homotopy groups:

Definition 1 (Homotopy groups)**.** *For $n : \mathbb{N}$, we define the nth* homotopy group *of a pointed type A by:*

$$\pi_n(A) = \| \mathbb{S}^n \to_\star A \|_0$$

The name homotopy *group* should be taken with a grain of salt: it, in general, only has a group structure when $n \geq 1$ (abelian when $n \geq 2$). The structure may be defined, much like in [17, Section 5], by considering the equivalence $(\mathbb{S}^n \to_\star A) \simeq (\mathbb{S}^{n-1} \to_\star \Omega\, A)$, where the latter type has a multiplication given by pointwise path composition. An alternative definition of $\pi_n(A)$ is via loop spaces. There is an equivalence $\omega_n : \Omega^n\, A \simeq (\mathbb{S}^n \to_\star A)$ and, hence, we could equivalently have defined $\pi_n(A)$ by setting $\pi_n(A) = \| \Omega^n\, A \|_0$. This makes the group structure on $\pi_n(A)$ more transparent: it is simply path composition. This is the definition used in [1]. Brunerie uses both definitions and often passes between the two without comment.

An elementary but crucial result for the computation of homotopy groups is the existence of the *long exact sequence of homotopy groups*. Its proof is usually phrased using the loop space definition of homotopy groups [1, Theorem 8.4.6]. For ease of notation, let us simply write $\mathsf{fib}\, f$ for the fiber of a pointed function $f : A \to_\star B$ over the basepoint of B.

Proposition 2 (LES of homotopy groups)**.** *For any pointed map $f : A \to_\star B$, there is a long exact sequence*

$$\begin{tikzcd} & \ldots \arrow[r] & \pi_{n+1}(B) \arrow[dll, bend left=10] \\ \pi_n(\mathsf{fib}\, f) \arrow[r] & \pi_n(A) \arrow[r] & \pi_n(B) \arrow[dll, bend left=10] \\ \pi_{n-1}(\mathsf{fib}\, f) \arrow[r] & \ldots & \end{tikzcd}$$

Above, we have implicitly taken the kernel and image of a group homomorphism $\phi : G \to H$ to be defined by

$$\begin{aligned} \ker \phi &= \mathsf{fib}\, \phi\, 0_H \\ \mathsf{im}\, \phi &= \Sigma_{h:H} \| \Sigma_{g:G} (\phi(g) \equiv h) \|_{-1} \end{aligned}$$

When analyzing loop spaces and homotopy groups of suspensions, the following function is of great importance. It will be used in many constructions to come.

Definition 3 (The suspension map)**.** *Given a pointed type A, there is a canonical map $\sigma : A \to \Omega\,(\mathsf{Susp}\, A)$ given by*

$$\sigma\, x = \mathsf{merid}\ x \cdot (\mathsf{merid}\ \star_A)^{-1}$$

This induces a homomorphism on homotopy groups by post-composition:

$$\pi_n(A) \xrightarrow{\sigma_*} \pi_n(\Omega\,(\mathsf{Susp}\, A)) \xrightarrow{\cong} \pi_{n+1}(\mathsf{Susp}\, A)$$

We will often, with some abuse of notation, simply write σ_* for this composition. We also define $\sigma_n : \| A \|_n \to \Omega\,(\| \mathsf{Susp}\, A \|_{n+1})$ by

$$\sigma_n | \, x \, | = \mathsf{cong}\ |\_|\ (\sigma\, x)$$

We will soon see the suspension map in action, but first we need the following elementary result.

Proposition 4 (Join of spheres)**.** $\mathbb{S}^n * \mathbb{S}^m \simeq \mathbb{S}^{n+m+1}$.

Proof: The statement is easily proved by induction, using $\mathbb{S}^{n+1} \simeq \mathsf{Susp}\, \mathbb{S}^n$ and $(\mathsf{Susp}\, A) * B \simeq \mathsf{Susp}\,(A * B)$. ■

In particular, Proposition 4 gives us an equivalence $\mathbb{S}^1 * \mathbb{S}^1 \simeq \mathbb{S}^3$. Using this fact, we define the following map, which will play a crucial role in the analysis of $\pi_4(\mathbb{S}^3)$.

Definition 5 (Hopf map)**.** *We define $\mathsf{hopf} : \mathbb{S}^3 \to \mathbb{S}^2$ by the composition $\mathbb{S}^3 \xrightarrow{\sim} \mathbb{S}^1 * \mathbb{S}^1 \xrightarrow{\mathsf{h}} \mathbb{S}^2$ where h is given by*

```
h : S¹ * S¹ → S²
h (inl x) = north
h (inr y) = north
h (push (x , y) i) = σ (y - x) i
```

where y - x is defined using the h-space and inversion structure on $\mathbb{S}^1$.

It turns out that the following is true.

Proposition 6 ([1, Theorem 8.5.1])**.** *The fiber of hopf is equivalent to $\mathbb{S}^1$, i.e. $\mathsf{fib}\, \mathsf{hopf} \simeq \mathbb{S}^1$.*

This gives us a fibration sequence $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$ which, in particular, will allow us to connect homotopy groups of $\mathbb{S}^2$ with those of $\mathbb{S}^3$ and $\mathbb{S}^1$. For this, we need to introduce the notion of *connectedness*. We say that a type A is n-connected if $\| A \|_n$ is contractible. Similarly, we say that a function $f : A \to B$ is n-connected if all of its fibers are n-connected. This means, in particular, that the induced function $\| f \|_n : \| A \|_n \to \| B \|_n$ is an equivalence. The following is an immediate consequence of the definition of n-truncations.

Lemma 7 (Connectedness of spheres)**.** *For $n \geq -1$, $\mathbb{S}^n$ is $(n-1)$-connected.*

Using Lemma 7, we can easily prove the following:

Proposition 8 ([4, Prop. 2.4.1])**.** *For $n < m$, the group $\pi_n(\mathbb{S}^m)$ is trivial.*

For the sake of completeness, let us take the liberty of mentioning some results from [4, Chap. 3] already here, since they also concern low-dimensional homotopy groups of spheres. A crucial result is [1, Theorem 8.6.4]:

Theorem 9 (Freudenthal suspension theorem)**.** *Given an n-connected and pointed type A, the map $\sigma : A \to \Omega\,(\mathsf{Susp}\, A)$ is $2n$-connected.*

On can easily deduce from Theorem 9 that, in particular, $\sigma_n : \| A \|_n \to \| \Omega\,(\mathsf{Susp}\, A) \|_n$ is an equivalence. This allows us to prove the following result:

Corollary 10. *For $n \geq 1$, we have $\pi_n(\mathbb{S}^n) \cong \mathbb{Z}$. Furthermore, $\pi_n(\mathbb{S}^n)$ is generated by $i_n = |\, \mathsf{id}_{\mathbb{S}^n} |$.*

Proof: The synthetic proof of the classical result that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ is due to Licata and Shulman [18]. The fact that $\pi_2(\mathbb{S}^2) \cong \pi_1(\mathbb{S}^1)$ is given by the LES associated to the Hopf fibration combined with Proposition 8. The fact that $\pi_{n+1}(\mathbb{S}^{n+1}) \cong \pi_n(\mathbb{S}^n)$ is an immediate consequence of Theorem 9. The second statement follows by induction on n. ■

We have now analyzed all homotopy groups $\pi_n(\mathbb{S}^m)$ with $n \leq m$. This yields the following:

Proposition 11. *Post-composition by hopf induces an isomorphism $\pi_3(\mathbb{S}^3) \cong \pi_3(\mathbb{S}^2)$.*

Proof: By Propositions 2 and 6, we get an exact sequence

$$\pi_3(\mathbb{S}^1) \to \pi_3(\mathbb{S}^3) \xrightarrow{\mathsf{hopf}_*} \pi_3(\mathbb{S}^2) \to \pi_2(\mathbb{S}^1)$$

as $\pi_n(\mathbb{S}^1)$ vanishes for $n > 1$, hopf_* is an isomorphism. ■

Corollary 12. *There is an isomorphism $\psi : \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$. Furthermore, $\pi_3(\mathbb{S}^2)$ is generated by hopf.*

Proof: By Corollary 10 we know that $\pi_3(\mathbb{S}^3)$ is generated by the identity function on $\mathbb{S}^3$. We know that the isomorphism $\pi_3(\mathbb{S}^3) \cong \pi_3(\mathbb{S}^2)$ is given by post-composition by hopf and thus the generator of $\pi_3(\mathbb{S}^3)$ is mapped to hopf. ■

A. Formalization of Brunerie's Chapter 2

Most of these results have already been added to `agda/cubical` by Mörtberg & Pujet [19], Ljungström [20], and Brunerie, Ljungström & Mörtberg [12]. The Freudenthal suspension theorem was formalized in `Cubical Agda` by Cavallo [21], using a direct cubical proof following [1, Thm. 8.6.4]. Corollary 10 was given a direct proof, following the computation of cohomology groups of spheres in [12].

There were some technical difficulties related to the equivalence $\omega_n : \Omega^n\, A \simeq (\mathbb{S}^n \to_\star A)$, which is used to show that the two different definitions of homotopy groups are equivalent. In several proofs, it is more natural to work on the left-hand-side of ω_n. At the same time, working on the right-hand-side often makes constructing elements easier (compare, for instance, an explicit description of the generator of $i_3 : \pi_3(\mathbb{S}^3)$ described as a 3-loop in $\mathbb{S}^3$ to the very compact definition $i_3 = |\,\mathsf{id}_{\mathbb{S}^3}\,|$). This means that we often have to translate between the two definitions. One particularly important example is the LES of homotopy groups associated to a function $A \to_\star B$. On each level, the maps are given as follows:

$$\Omega^n\,(\mathsf{fib}\,f) \xrightarrow{\Omega^n\,\mathsf{fst}} \Omega^n\,A \xrightarrow{\Omega^n\,f} \Omega^n\,B$$

This is then transported to the definition of homotopy groups as maps from spheres via ω_n. For the proof of e.g. Corollary 12, we need to know that the maps in the sequence are given as follows:

$$\pi_n(\mathsf{fib}\,f) \xrightarrow{\mathsf{fst}_*} \pi_n(A) \xrightarrow{f_*} \pi_n(B)$$

What we need is then more than just an equivalence $\omega_n : \Omega^n\,A \simeq (\mathbb{S}^n \to_\star A)$ – we need to show that this equivalence is functorial. This is implicitly assumed in Brunerie's thesis, but, in `Cubical Agda`, we need to make it precise. Formalizing this fact is not entirely trivial. First, we need a tractable definition of the equivalence in question. It can be described inductively with base case $\omega_1 : \Omega\,A \to (\mathbb{S}^1 \to_\star A)$ given by:

$$\begin{aligned}\omega_1\,p\,\mathsf{base} &= \star_A\\ \omega_1\,p\,(\mathsf{loop}\,i) &= p\,i\end{aligned}$$

which we take to be pointed by refl. It is easy to verify that this is an equivalence. We define ω_{n+1} by the composition:

$$\begin{aligned}\Omega^{n+1}\,A = \Omega\,(\Omega^n\,A) &\xrightarrow{\Omega\,\omega_n} \Omega\,(\mathbb{S}^n \to_\star A)\\ &\xrightarrow{\mathsf{funExt}_\star^-} (\mathbb{S}^n \to_\star \Omega\,A)\\ &\xrightarrow{\qquad} (\mathbb{S}^{n+1} \to_\star A)\end{aligned}$$

where the last arrow comes from the adjunction $\mathsf{Susp} \dashv \Omega$. This is a composition of equivalences, and hence an equivalence. We then need to verify that the following commutes

$$\begin{tikzcd} \Omega^n\,A \arrow[r, "\omega_n"] \arrow[d, "\Omega^n f"'] & (\mathbb{S}^n \to_\star A) \arrow[d, "f_*"] \\ \Omega^n\,B \arrow[r, "\omega_n"'] & (\mathbb{S}^n \to_\star B) \end{tikzcd}$$

This can be proved inductively. The base case is easy and the inductive step is given by the following diagram

$$\begin{tikzcd} & & \Omega\,(\mathbb{S}^n \to_\star A) \arrow[dl, "\simeq"] \arrow[ddd, "\Omega\,f_*"] \\ \Omega^{n+1}\,A \arrow[urr, "\Omega\,\omega_n"] \arrow[r, "\omega_{n+1}"'] \arrow[d, "\Omega^{n+1}\,f"'] & (\mathbb{S}^{n+1} \to_\star A) \arrow[d, "f_*"] & \\ \Omega^{n+1}\,B \arrow[r, "\omega_{n+1}"] \arrow[drr, "\Omega\,\omega_n"'] & (\mathbb{S}^{n+1} \to_\star B) & \\ & & \Omega\,(\mathbb{S}^n \to_\star B) \arrow[ul, "\simeq"'] \end{tikzcd}$$

where the commutativity of the outer square comes from the base case paired with the inductive hypothesis, the triangles from the definition of ω_{n+1} and the right-most square from a straightforward argument.

IV. The Brunerie number

Here we give an overview of the first half of Brunerie's proof. This corresponds to [4, Chap. 3] and culminates in the isomorphism $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$ for an at this point unknown "Brunerie number" $\beta : \mathbb{Z}$. We also discuss the formalization of this part of the proof and various simplifications found during the formalization.

A. The James construction

To define β, Brunerie uses the *James construction* [22], which he introduced in HoTT and partially formalized in [23].

Proposition 13 (James construction)**.** *Let A be a pointed $(k \geq 0)$-connected type. There are types $\mathsf{J}_n\,A$ with inclusions*

$$\mathsf{J}_0\,A \xhookrightarrow{j_0} \mathsf{J}_1\,A \xhookrightarrow{j_1} \mathsf{J}_2\,A \xhookrightarrow{j_2} \cdots$$

such that its sequential colimit $\mathsf{J}_\infty\,A \simeq \Omega\,(\mathsf{Susp}\,A)$. Furthermore, $j_n : \mathsf{J}_n\,A \hookrightarrow \mathsf{J}_{n+1}\,A$ is $(n(k+1)+(k-1))$-connected.

A consequence of Proposition 13 is the following fact

Proposition 14. *Given a $(k \geq 0)$-connected type A, there is a $(3k+1)$-connected map $\mathsf{J}_2\,A \to \Omega\,(\mathsf{Susp}\,A)$.*

The proof of Proposition 14 uses that $\mathsf{J}_\infty\,A$, the sequential colimit of the sequence in Proposition 13, can be shown to be equivalent to $\Omega\,(\mathsf{Susp}\,A)$. This, paired with some results on the connectivity of sequential colimits, gives the statement.

Theorem 15. $\pi_4(\mathbb{S}^3) \cong \pi_3(\mathsf{J}_2\,\mathbb{S}^2)$

Proof: Because $\mathbb{S}^2$ is 1-connected, Proposition 14 tells us that there is a 4-connected map

$$\mathsf{J}_2\,\mathbb{S}^2 \to \Omega\,(\mathsf{Susp}\,\mathbb{S}^2) = \Omega\,(\mathbb{S}^3)$$

In particular, it is 3-connected and induces an equivalence $\|\,\mathsf{J}_2\,\mathbb{S}^2\,\|_3 \simeq \|\,\Omega\,\mathbb{S}^3\,\|_3$. We get

$$\pi_4(\mathbb{S}^3) \cong \pi_3(\Omega\,\mathbb{S}^3) \cong \pi_3(\mathsf{J}_2\,\mathbb{S}^2)$$

as desired. ∎

B. Formalization of the James construction

This is a particularly technical part of Brunerie's thesis, primarily due to the high number of higher coherences which need to be verified in the proof of Proposition 13. While this has, subsequent to our efforts, been formalized in its entirety by Kang [24], we have taken a shortcut by giving a direct proof of Theorem 15, which means we do not in fact need the full James construction. Consequently, we instead give direct definitions of $\mathsf{J}_n\,A$ for $n \leq 2$ for a pointed type A.

Definition 16 (Low dimensional James construction)**.** *We define $\mathsf{J}_0\,A = \mathbb{1}$ and $\mathsf{J}_1\,A = A$. The type $\mathsf{J}_2\,A$ is defined as the pushout of the span $A \times A \xleftarrow{i^\vee} A \vee A \xrightarrow{\nabla} A$.*

We remark that the construction in Definition 16 is not definitionally the same as Brunerie's; in his thesis, these constructions are theorems rather than definitions. Here we take them as definitions. With $\mathsf{J}_n\, A$ defined this way, the map $j_0 : \mathsf{J}_0\, A \to \mathsf{J}_1\, A$ is just the constant pointed map and $j_1 : \mathsf{J}_1\, A \to \mathsf{J}_2\, A$ is inr.

Before we continue, let us temporarily redefine $\mathbb{S}^2$ to be the following equivalent HIT. This will make some of the following constructions more compact.

```
data S² : Type where
  base : S²
  surf : refl base ≡ refl base
```

The next lemma will be crucial. It is a special case of the *Wedge Connectivity Lemma* [1, Lemma 8.6.2], of which we have formalized a version of the proof of the sphere case in [12, Lemma 8]. From the point of view of formalization, this proof is easier to work with since it gives more useful definitional equalities.

Lemma 17 (Wedge connectivity for $\mathbb{S}^2$)**.** *Let $P : \mathbb{S}^2 \times \mathbb{S}^2 \to$ 2-Type. Any function $f : (x : \mathbb{S}^2 \times \mathbb{S}^2) \to P\, x$ is induced by the following data:*

$$f_l : (x : \mathbb{S}^2) \to P(x\, , \mathsf{base})$$
$$f_r : (y : \mathbb{S}^2) \to P(\mathsf{base}\, , y)$$
$$f_{lr} : f_l\, \mathsf{base} \equiv f_r\, \mathsf{base}$$

Before we discuss the formalization of Theorem 15 stated with the low dimensional James construction, we first construct a family of equivalences $f_x : \| \mathsf{J}_2\, \mathbb{S}^2 \|_3 \simeq \| \mathsf{J}_2\, \mathbb{S}^2 \|_3$ over $x : \mathbb{S}^2$. We do this by truncation elimination and pattern matching on x, starting with the base case:

$$f_{\mathsf{base}}\, |\, \mathsf{inl}\, (x, y)\, | = |\, \mathsf{inl}\, (x\, , y)\, |$$
$$f_{\mathsf{base}}\, |\, \mathsf{inr}\, z\, | = |\, \mathsf{inl}\, (\mathsf{base}\, , z)\, |$$

We omit the path constructors, which are all easy coherences which are omitted due to space constraints. It is an easy lemma that f_{base} is equal to the identity on $\| \mathsf{J}_2\, \mathbb{S}^2 \|_3$. To complete the definition of f_x, we need to consider the case when $x = \mathsf{surf}\, i\, j$. This amounts to providing a dependent function:

$$f_{\mathsf{surf}} : (x : \| \mathsf{J}_2\, \mathbb{S}^2 \|_3) \to \Omega^2\, (\| \mathsf{J}_2\, \mathbb{S}^2 \|_3\, , f_{\mathsf{base}}\, x)$$

To do this, we will, in particular, need to provide a family of fillers $Q_{(x\, , y)} : \mathsf{refl}\, |\, \mathsf{inl}\, (x\, , y)\, | \equiv \mathsf{refl}\, |\, \mathsf{inl}\, (x\, , y)\, |$. This is a 1-type, and thus Lemma 17 applies. We define:

$$Q_{(\mathsf{base}\, , y)}\, i\, j = |\, \mathsf{inl}\, (\mathsf{surf}\, i\, j\, , y)\, |$$
$$Q_{(x\, , \mathsf{base})}\, i\, j = |\, \mathsf{inl}\, (x\, , \mathsf{surf}\, i\, j)\, |$$

The fact that these two constructions agree when both x and y are base is a technical, but relatively straightforward lemma. Thereby, $Q_{(x\, , y)}$ is defined. We may now define f_{surf}:

$$f_{\mathsf{surf}}\, |\, \mathsf{inl}\, (x\, , y)\, | = Q_{(x\, , y)}$$
$$f_{\mathsf{surf}}\, |\, \mathsf{inr}\, z\, | = Q_{(\mathsf{base}\, , z)}$$

The higher cases are easy due to the fact that the goal becomes 0-truncated, making it sufficient to define them for $\mathsf{base} : \mathbb{S}^2$. Thus, f_x is defined for all $x : \mathbb{S}^2$.

Lemma 18. *For $x : \mathbb{S}^2$, f_x is an automorphism on $\| \mathsf{J}_2\, \mathbb{S}^2 \|_3$.*

This can be proved by explicitly constructing an inverse to f_x which makes later proofs easier. As the construction is completely analogous to that of f_x we have omitted it due to space constraints.

We are now ready to prove the following statement, which is a rephrasing of Theorem 15.

Proposition 19. $\Omega\, \| \mathbb{S}^3 \|_4 \simeq \| \mathsf{J}_2\, \mathbb{S}^2 \|_3$

Proof: We take $\mathbb{S}^3 = \mathsf{Susp}\, \mathbb{S}^2$, where $\mathbb{S}^2$ is defined using $\mathsf{base}/\mathsf{surf}$ as above. We employ the encode-decode method and define $\mathsf{Code} : \| \mathbb{S}^3 \|_4 \to$ 3-Type. Since the universe of 3-types is a 4-type, we may do so by truncation elimination, letting $\mathsf{Code}\, |\, \mathsf{north}\, | = \mathsf{Code}\, |\, \mathsf{south}\, | = \| \mathsf{J}_2\, \mathbb{S}^2 \|_3$ and $\mathsf{Code}\, |\, \mathsf{merid}\ x\, i\, | = \mathsf{ua}\, f_x\, i$. We now need to define a family of functions $\mathsf{decode}_x : \mathsf{Code}\, x \to |\, \mathsf{north}\, | \equiv x$ over $x : \| \mathbb{S}^3 \|_4$. The key step is defining $\mathsf{decode}_{|\, \mathsf{north}\, |} : \| \mathsf{J}_2\, \mathbb{S}^2 \|_3 \to \Omega\, \| \mathbb{S}^3 \|_4$. On point constructors, it is given by:

$$\mathsf{decode}_{|\, \mathsf{north}\, |}\, (\mathsf{inl}\, (x\, , y)) = \sigma\, x \cdot \sigma\, y$$
$$\mathsf{decode}_{|\, \mathsf{north}\, |}\, (\mathsf{inr}\, z) = \sigma\, z$$

which is easily verified to be coherent with the higher constructors. At this point, we may follow the usual encode-decode heuristic [1, Section 8.9] to prove that $\mathsf{decode}_{|\, \mathsf{north}\, |}$ is an equivalence in a technical, but direct manner. ∎

We get Theorem 15 as an immediate corollary of Proposition 19 via the same sequence of isomorphisms as in the proof of Theorem 15.

C. Definition of the Brunerie number

Brunerie's goal is now to analyze $\pi_3(\mathsf{J}_2\, \mathbb{S}^2)$. The first result needed is the following:

Definition 20 (Whitehead map)**.** *Given two pointed types A and B, there is a map:*

```
W : A ⋆ B → Susp A ∨ Susp B
W (inl a) = inr north
W (inr b) = inl north
W (push (a , b) i) =
  (cong inr (σ b) ∙ push ⋆₁ ⁻¹ ∙ cong inl (σ a)) i
```

For our purposes, we only need the case when $A = B = \mathbb{S}^1$ (although all of the following results appear in full generality in Brunerie's thesis). We get a composite map:

$$\mathbb{S}^3 \xrightarrow{\simeq} \mathbb{S}^1 \star \mathbb{S}^1 \xrightarrow{\mathsf{W}} \mathbb{S}^2 \vee \mathbb{S}^2$$

This induces, via pre-composition, a map

$$\| \mathbb{S}^2 \vee \mathbb{S}^2 \to_\star \mathbb{S}^2 \|_0 \to \pi_3(\mathbb{S}^2)$$

which via the obvious map

$$\pi_2(\mathbb{S}^2) \times \pi_2(\mathbb{S}^2) \to \| \mathbb{S}^2 \vee \mathbb{S}^2 \to_\star \mathbb{S}^2 \|_0$$

defines a *Whitehead product*:

$$[-,-] : \pi_2(\mathbb{S}^2) \times \pi_2(\mathbb{S}^2) \to \pi_3(\mathbb{S}^2)$$

Recall that we denote by i_2 the generator of $\pi_2(\mathbb{S}^2)$. Brunerie shows, in particular, the following about its relation to the Whitehead product.

Theorem 21. *The kernel of the suspension map $\sigma_* : \pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$ is generated by $[i_2, i_2]$.*

The key technical component in the proof is the *Blakers-Massey Theorem*, first formalized in HoTT by Favonia, Finster, Licata & Lumsdaine in [25] and later generalized by Anel, Biedermann, Finster & Joyal [26]:

Theorem 22 (Blakers-Massey)**.** *Consider the diagram*

$$\begin{tikzcd}
A \arrow[dr, dashed, "f \sqcup g"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & C \arrow[d, "\mathsf{inr}"] \\
& B \arrow[r, "\mathsf{inl}"'] & \mathsf{Pushout}\; f\; g \arrow[ul, phantom, "\ulcorner" very near start]
\end{tikzcd}$$

where P is the pullback along inl *and* inr*, i.e.* $P = \Sigma_{(b,c):B\times C}(\mathsf{inl}\, b \equiv \mathsf{inr}\, c)$*, and $f \sqcup g$ is defined by*

$$(f \sqcup g)\, a = (f\, a \,, g\, a \,, \mathsf{push}\, a)$$

If f and g are n- respectively m-connected, then $f \sqcup g$ is $(n+m)$-connected.

Theorem 21 is proved by considering the following diagram

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[dr] \arrow[drr, bend left, "\nabla \circ \mathsf{W}"] \arrow[ddr, bend right] & & \\
& P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathbb{S}^2 \arrow[d] \\
& \mathbb{1} \arrow[r] & J_2\mathbb{S}^2 \arrow[ul, phantom, "\ulcorner" very near start]
\end{tikzcd}$$

Verifying that the outer square is a pushout square is technical and we omit the proof here. Above, P is simply the fiber of $\mathsf{inr} : \mathbb{S}^2 \to \mathsf{J}_2\,\mathbb{S}^2$. The leftmost map is 2-connected since $\mathbb{S}^3$ is 2-connected and the top map is 0-connected since $\mathbb{S}^3$ and $\mathbb{S}^2$ are both 1-connected. Consequently, by Theorem 22, we get that the map $\mathbb{S}^3 \to P$ is 2-connected and thus induces a surjection after application of π_3. This gives the diagram:

$$\begin{tikzcd}
\pi_3(P) \arrow[r] & \pi_3(\mathbb{S}^2) \arrow[r] \arrow[dr, "\sigma_*"'] & \pi_3(\mathsf{J}_2\,\mathbb{S}^2) \arrow[d, "\cong"] \\
\pi_3(\mathbb{S}^3) \arrow[u] \arrow[ur, dashed] & & \pi_4(\mathbb{S}^3)
\end{tikzcd}$$

where the sequence on the top comes from the LES of homotopy groups associated to P. The dashed map sends the generator $i_3 : \pi_3(\mathbb{S}^3)$ to $[i_2, i_2] : \pi_3(\mathbb{S}^2)$ by definition.

Theorem 21 motivates the following definition. Recall that we denote by ψ the isomorphism $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$.

Definition 23 (Brunerie number)**.** *We define the Brunerie number $\beta : \mathbb{Z}$ by $\beta = \psi\,[i_2, i_2]$.*

We may now state the main result of [4, Chap. 3].

Corollary 24. $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/\beta\mathbb{Z}$*.*

Proof: The statement follows immediately from Theorem 21 and the fact that $\sigma_* : \pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$ is surjective, which is a consequence of Theorem 9. ∎

D. Formalization of the Brunerie number

The formalization of this part was straightforward. Arguably the most technical result, the Blakers-Massey theorem, was already available in the library thanks to Kang [27]. Most of the remaining results were essentially just diagram chases which, in a proof assistant, can be somewhat technical. Most work went into verifying that $\mathsf{J}_2\,\mathbb{S}^2$ is the cofiber of $\nabla \circ \mathsf{W}$, the proof of which followed Brunerie's closely.

In this section we found the only obvious mistake in Brunerie's thesis on page 82. In his definition of the push-case for W, the path component in the middle was not inverted, making the term ill-typed. Naturally, this was of no mathematical significance and something Brunerie immediately would have noticed if he would have attempted to provide a computer formalization of this construction.

V. Brunerie's proof that $|\beta| \equiv 2$

This section concerns the final three Chapters (4–6) of Brunerie's thesis. The main goal here is proving that $|\beta| \equiv 2$.

We will not discuss Chapter 4 in much detail. Chapter 4 is devoted to smash products and, in particular, their symmetric monoidal structure. Brunerie used this in subsequent chapters to define and prove properties about the *cup product*, a graded multiplicative operation on cohomology groups which will be used to show that $|\beta| \equiv 2$. This chapter has turned out to be incredibly difficult to formalize due to the large number of higher coherences involved in the proofs [11].

Luckily, it turns out that Chapter 4 can be avoided altogether and that this in fact makes some difficult proofs later on very direct. For this reason, the results in Chapter 4 were omitted completely from our formalization. The reason for this is that all results regarding smash products in Brunerie's thesis concern, in some way, pointed maps out of smash products. In this case, we may exploit the adjunction of maps out of smash products and bi-pointed maps:

$$(A \wedge B \to_\star C) \simeq (A \to_\star (B \to_\star C))$$

Here, $B \to_\star C$ is taken to be pointed by the constant map. As shown in [12], it is arguably easier to define the cup product on the right-hand side of the adjunction, which effectively means that we never have to work with smash products when formalizing cohomology theory.

A. Cohomology and the Hopf invariant

[4, Chap. 5] introduces integral cohomology groups and rings, and gives a construction of the Mayer-Vietoris sequence. In more detail, Brunerie defines the integral Eilenberg-MacLane spaces by $\mathsf{K}_0 = \mathbb{Z}$ and $\mathsf{K}_n = \|\, \mathbb{S}^n \,\|_n$ for $n \geq 1$. This allows for a definition of the (integral) cohomology of X:

$$\mathsf{H}^n(X) = \|\, X \to \mathsf{K}_n \,\|_0$$

The fact that $\Omega\,\mathsf{K}_{n+1} \simeq \mathsf{K}_n$ follows by a proof completely analogous to that of Corollary 10. Brunerie uses this equivalence to carry over the (commutative) h-space structure on $\Omega\,\mathsf{K}_{n+1}$ to that of K_n. This provides a notion of addition $+_k$: $\mathsf{K}_n \times \mathsf{K}_n \to \mathsf{K}_n$ which lifts to $\mathsf{H}^n(X)$ by post-composition, thereby endowing $\mathsf{H}^n(X)$ with a group structure. Brunerie also defines a notion of multiplication $\smile_k$: $\mathsf{K}_n \to \mathsf{K}_m \to \mathsf{K}_{n+m}$ which induces the graded-commutative multiplication $\smile : \mathsf{H}^n(X) \to \mathsf{H}^m(X) \to \mathsf{H}^{n+m}(X)$ called the cup product. This construction is far less direct in Brunerie's thesis, and due to space constraints we cannot discuss it in more detail here. For a more direct construction (which is also the one we have used in our formalization), we refer to [12, Section 4.1].

The synthetic construction of the Mayer-Vietoris sequence concerns the long exact sequence

$$\begin{tikzcd} \mathsf{H}^0(D) \arrow[r] & \mathsf{H}^0(B) \times \mathsf{H}^0(C) \arrow[r] & \mathsf{H}^0(A) \arrow[dll] \\ \mathsf{H}^1(D) \arrow[r] & \dots & \end{tikzcd}$$

where D denotes the pushout of a span $B \xleftarrow{f} A \xrightarrow{g} C$. A direct application gives us, for $n \geq 1$, that $\mathsf{H}^n(\mathbb{S}^m) \cong \mathbb{Z}$ if $n = m$ and $\mathsf{H}^n(\mathbb{S}^m) \cong \mathbb{1}$ otherwise. This gives, by another application of the sequence, the following result:

Lemma 25. *For any $f : \mathbb{S}^3 \to \mathbb{S}^2$ we have*

$$\mathsf{H}^n(\mathsf{cofib}\, f) \;\cong\; \begin{cases} \mathbb{Z} & n \in \{0, 2, 4\} \\ \mathbb{1} & \text{otherwise} \end{cases}$$

Let us briefly fix $f : \mathbb{S}^3 \to \mathbb{S}^2$. Denote by γ_2 and γ_4 the generators of $\mathsf{H}^2(\mathsf{cofib}\, f)$ and $\mathsf{H}^4(\mathsf{cofib}\, f)$ respectively given by the image of $1 : \mathbb{Z}$ under the isomorphism in Lemma 25. These generators may be used to define an invariant on $\mathbb{S}^3 \to \mathbb{S}^2$ called the *Hopf Invariant*. This is done as follows:

Definition 26 (Hopf invariant)**.** *The* Hopf invariant *of f is the unique integer $\mathsf{HI}\, f : \mathbb{Z}$ such that $\gamma_2 \smile \gamma_2 \equiv \mathsf{HI}\, f \cdot \gamma_4$.*

We remark that the above definition is given for the more general class of maps $\mathbb{S}^{2n-1} \to \mathbb{S}^n$ in Brunerie's thesis. For our purposes, the above special case suffices. In particular, we may see HI as a function $\pi_3(\mathbb{S}^2) \to \mathbb{Z}$. The following turns out to be true:

Proposition 27. *HI is a homomorphism $\pi_3(\mathbb{S}^2) \to \mathbb{Z}$.*

Proof: We first rephrase $f + g : \pi_3(\mathbb{S}^2)$ as a composition

$$\mathbb{S}^3 \to \mathbb{S}^3 \vee \mathbb{S}^3 \xrightarrow{f \vee g} \mathbb{S}^2$$

By analyzing the cohomology of $\mathsf{cofib}\,(f \vee g)$ and the action on generators of the obvious maps from $\mathsf{cofib}\,(f \vee g)$, $\mathsf{cofib}\, f$ and $\mathsf{cofib}\, g$ into $\mathsf{cofib}\,(f + g)$, one arrives at the result with some elementary algebra. ∎

Finally, the Hopf invariant of our element of interest $[i_2, i_2]$ is computed (up to a sign), using an argument similar to that of the proof of Proposition 27.

Proposition 28. $|\mathsf{HI}\,[i_2, i_2]| \equiv 2$

We are now almost done: if there is an element $f : \pi_3(\mathbb{S}^2)$ such that $\mathsf{HI}\, f \equiv 1$, then HI is an isomorphism $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$. Since isomorphisms of this type are unique up to a sign, Proposition 28 tells us that also for the standard isomorphism $\psi : \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, we must have $|\psi[i_2, i_2]| \equiv 2$, i.e. $|\beta| \equiv 2$. Hence, we have so far shown the following:

Lemma 29. *If there is $f : \pi_3(\mathbb{S}^2)$ with $\mathsf{HI}\, f \equiv 1$, then $|\beta| \equiv 2$.*

The final chapter of Brunerie's thesis is devoted to proving the antecedent of Lemma 29.

B. Formalization of cohomology and the Hopf invariant

This section was largely covered by Brunerie, Ljungström and Mörtberg in [12] and thus also available in `agda/cubical`. Hence, what remained to be formalized in Chapter 5 was the Hopf invariant and Propositions 27 and 28. The formalization of these propositions was straightforward and we were able to translate Brunerie's proofs in a direct manner. This is not surprising as the proofs are very algebraic.

For simplicity, we only formalized these propositions as they stand here and not their generalizations to higher spheres (i.e. as in [4, Prop. 5.4.3 & 5.4.4]). We remark, however, that the formalized proofs easily should be rephrasable for the general Hopf invariant of maps $\mathbb{S}^{2n-1} \to \mathbb{S}^n$.

C. The Gysin sequence

This section corresponds to [4, Chap. 6]. In order to be able to apply Lemma 29, this chapter is devoted to proving that $|\mathsf{HI}\ \mathsf{hopf}| \equiv 1$, where, recall, $\mathsf{hopf} : \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf map—the generator of $\pi_3(\mathbb{S}^2)$ from Definition 5. This amounts to analyzing the cup product on the cohomology of $\mathsf{cofib}\ \mathsf{hopf}$. It is well-known that $\mathsf{cofib}\ \mathsf{hopf}$ gives a model of the complex projective plane $\mathbb{C}P^2$ (see e.g. [28, Example 4.45]), so let us simply write $\mathbb{C}P^2$ from now on. In order to show that $|\mathsf{HI}\ \mathsf{hopf}| \equiv 1$, it suffices to show that $- \smile \gamma_2 : \mathsf{H}^2(\mathbb{C}P^2) \to \mathsf{H}^4(\mathbb{C}P^2)$ is an isomorphism for $\gamma_2 : \mathsf{H}^2(\mathbb{C}P^2)$ a generator. Brunerie does this by constructing the Gysin sequence synthetically.

Proposition 30 (Gysin sequence)**.** *Let B be a pointed and 0-connected type and $P : B \to \mathsf{Type}$ be a fibration with $P \star_B \simeq_\star \mathbb{S}^{n-1}$. Let $E = \Sigma_{b:B}(P\, b)$ be the total space of P. If there is a family of maps $c : (b : B) \to (\mathsf{Susp}\,(P\, b) \to_\star \mathsf{K}_n)$ with $c_{\star_B}$ a generator of $\mathsf{H}^n(\mathbb{S}^m)$, then there is an element $e_n : \mathsf{H}^n(B)$ and a long exact sequence*

$$\begin{tikzcd} & \dots \arrow[r] & \mathsf{H}^{i-1}(B) \arrow[dll] \\ \mathsf{H}^{i-1}(E) \arrow[r] & \mathsf{H}^{i-n}(B) \arrow[r, "- \smile e_n"] & \mathsf{H}^i(B) \arrow[dll] \\ \mathsf{H}^i(E) \arrow[r] & \dots & \end{tikzcd}$$

Moreover, c (and also e_n) exists when B is 1-connected.

In order to make use of this, we need the following result.

Proposition 31. *There is a fibration* $P : \mathbb{C}P^2 \to \mathsf{Type}$ *with* $P \star_{\mathbb{C}P^2} \simeq_\star \mathbb{S}^1$ *and total space* $\mathbb{S}^5$.

Proposition 31 is a special case of the following result.

Proposition 32 (Iterated Hopf construction)**.** *Given an associative h-space* A*, let* $h_A : A \star A \to \mathsf{Susp}\, A$ *denote the associated Hopf map. There is a fibration* $\mathsf{cofib}\, h_A \to \mathsf{Type}$ with fiber A and total space $A \star A \star A$.

We consider the particular case when $A = \mathbb{S}^1$ in Proposition 32. In this case, the map $h_{\mathbb{S}^1} : \mathbb{S}^1 \star \mathbb{S}^1 \to \mathbb{S}^2$ corresponds to the usual Hopf map under the equivalence $\mathbb{S}^1 \star \mathbb{S}^1 \simeq \mathbb{S}^3$ and hence $\mathsf{cofib}\, h_{\mathbb{S}^1} \simeq \mathbb{C}P^2$. The total space of this is $\mathbb{S}^1 \star \mathbb{S}^1 \star \mathbb{S}^1$ which is equivalent to $\mathbb{S}^5$ by Proposition 4 and thus we have proved Proposition 31. The associated Gysin sequence gives us the main result of this section:

Proposition 33. $|\mathsf{HI}\, h| \equiv 1$

Proof: Since $\mathbb{C}P^2$ is 1-connected, Proposition 30 combined with Proposition 31 gives us an element $e_2 : \mathsf{H}^2(\mathbb{C}P^2)$ and a sequence

$$\mathsf{H}^{i-1}(\mathbb{S}^5) \to \mathsf{H}^{i-2}(\mathbb{C}P^2) \xrightarrow{- \smile e_2} \mathsf{H}^i(\mathbb{C}P^2) \to \mathsf{H}^i(\mathbb{S}^5)$$

When $1 \leq i \leq 4$, $\mathsf{H}^i(\mathbb{S}^5)$ vanishes. Setting $i = 2$, we get that e_2 must be a generator of $\mathsf{H}^2(\mathbb{C}P^2)$, and thus equal to the generator $\gamma_2 : \mathsf{H}^2(\mathbb{C}P^2)$ up to a sign. Setting $i = 4$, we get that $- \smile e_2$ must be an isomorphism of groups $\mathsf{H}^2(\mathbb{C}P^2) \cong \mathsf{H}^4(\mathbb{C}P^2)$ and hence $e_2 \smile e_2$ is a generator. Consequently, so is $\gamma_2 \smile \gamma_2$, and thus $|\mathsf{HI}\ \mathsf{hopf}| \equiv 1$. ■

Proposition 33 combined with Lemma 29 gives the desired path: $|\beta| \equiv 2$. This completes Brunerie's proof and Corollary 24 gives us the main result:

Theorem 34. $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$

D. Formalization of the Gysin sequence

Formalizing the results from Chapter 6 was more challenging, but was greatly aided by the alternative construction of the cup product discussed above. The first technical lemma, which is crucial for the construction of the Gysin sequence is:

Lemma 35. *Given* $x : \mathsf{K}_n$ *and* $y : \mathsf{K}_m$*, we have*

$$\mathsf{cong}\ (\lambda\, a \to a \smile_k y)\ (\sigma_n\, x) \equiv \sigma_{n+m}(x \smile_k y)$$

In Brunerie's thesis, this lemma relies on a result which in turn requires the symmetric monoidal structure of the smash product (in particular, it uses the *pentagon identity*). With the alternative construction of the cup product, however, this result follows immediately from the definition of the cup product.

Lemma 35 is used to show that the map

$$\begin{aligned} g^i &: \mathsf{K}_i \to (\mathbb{S}^n \to_\star \mathsf{K}_{i+n}) \\ g^i\, x &= \lambda y \to x \smile_k \iota y \end{aligned}$$

is an equivalence, which is crucially used in the construction of the Gysin sequence. Above, $\iota : \mathbb{S}^n \to \mathsf{K}_n$ is a generator of $\mathsf{H}^n(\mathbb{S}^n)$. For reference, g^i is the map $g^i_{\star_B}$ in the proof of [4, Prop. 6.1.2]. While the general idea of Brunerie's proof of this statement is correct, it was difficult to formalize directly. The primary reason for this is that Brunerie does not pay much attention to the fact that the objects of interest are not just functions, but *pointed* functions. Fortunately for us, the whole proof is very direct with the alternative definition of the cup product. Formalizing Brunerie's proof with pointedness of functions respected would have been hard, especially without machinery external to [4] (e.g. [12, A.2, Lemma 27]).

After these subtleties were dealt with, the formalization of the Gysin sequence could proceed following Brunerie's proof closely. In our initial formalization, we made a slight adjustment to the indexing of the Gysin sequence. This removed some bureaucracy but happened at the cost of generality.[1] This made verifying that Proposition 33 slightly less direct, because we no longer had access to the case

$$\mathsf{H}^1(\mathbb{S}^5) \to \mathsf{H}^0(\mathbb{C}P^2) \xrightarrow{- \smile e_2} \mathsf{H}^2(\mathbb{C}P^2) \to \mathsf{H}^2(\mathbb{S}^5)$$

which is used by Brunerie to show that the element $e_2 : \mathsf{H}^2(\mathbb{C}P^2)$, for which $- \smile e_2 : \mathsf{H}^2(\mathbb{C}P^2) \to \mathsf{H}^4(\mathbb{C}P^2)$ is an isomorphism, is indeed a generator. However, in practice, this is not a big problem. In fact, it provides a nice example of a proof by computation. It is very direct to manually show that the map $i : \mathbb{C}P^2 \to \mathsf{K}_2$ induced by $i(\mathsf{inl}\, x) = |\, x\,|$ is equal to the underlying map of e_2. The fact that i generates $\mathsf{H}^2(\mathbb{C}P^2)$ can then be verified by computation: applying the isomorphism $\mathsf{H}^2(\mathbb{C}P^2) \cong \mathbb{Z}$ to $|\, i\,|$ returns 1 by normalization in `Cubical Agda`. We stress, for those skeptical of this method, that it also is very direct to provide a "manual" formalization of this fact.

The final step of the formalization was Proposition 32, i.e. the iterated Hopf construction. Although technical, the formalization could be carried out following Brunerie closely.

VI. The simplified new proof and normalization of a Brunerie number

It turns out that not only Chapter 4, but also Chapters 5–6 can be avoided. As conjectured by Brunerie, it would be possible to do this by simply normalizing the Brunerie number. While we still cannot normalize his original definition of it, we can at least provide a computation of a substantially simplified Brunerie number. This is defined via a more tractable description of the isomorphism $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ as a composition of simpler isomorphisms, relying on an alternative definition of π_3 in terms of $\mathbb{S}^1 \star \mathbb{S}^1$. The idea is then to trace $[i_2, i_2] : \pi_3(\mathbb{S}^2)$ step by step through these isomorphisms. This gives a sequence of new Brunerie numbers and one of these quite surprisingly normalizes to -2 in `Cubical Agda` in a matter of seconds.

The trick to give a more tractable definition of $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ is to redefine the third homotopy group of a type A as $\pi_3^*(A) = \|\, \mathbb{S}^1 \star \mathbb{S}^1 \to_\star A \,\|_0$. This reformulation of π_3 can be given an explicit group structure, such that pre-composition by $\mathbb{S}^1 \star \mathbb{S}^1 \simeq \mathbb{S}^3$ induces an isomorphism $\pi_3(A) \cong \pi_3^*(A)$. We briefly outline the proof by first defining the following product:

[1] A more general form of the Gysin sequence using Brunerie's indexing has later been added to `agda/cubical`.

$$
\begin{aligned}
&\_\smile_1\_ : \mathbb{S}^1 \to \mathbb{S}^1 \to \mathbb{S}^2\\
&\mathsf{base} \smile_1 y = \mathsf{north}\\
&\mathsf{loop}\ i \smile_1 y = \sigma\ y\ i
\end{aligned}
$$

In fact, $\smile_1$ behaves like a "cup product" on $\mathbb{S}^1$:

Proposition 36. *For $x, y : \mathbb{S}^1$, we have*

$$
\begin{aligned}
x \smile_1 y &\equiv -(y \smile_1 x)\\
x \smile_1 \mathsf{base} &\equiv \mathsf{base}\\
x \smile_1 (x + y) &\equiv x \smile_1 y
\end{aligned}
$$

where $-$ denotes inversion on $\mathbb{S}^2$.

Proof: All three identities are direct by pattern matching on x and, for the first one, also on y. The first one uses, in the case where x and y are both loops, an easy lemma which states for any 2-loop $p : \Omega^2\ A$ we have that $p^{-1} \equiv (\lambda\, i\, j \to p\, j\, i)$. ∎

This operation plays an important role in the definition of the equivalence $\mathbb{S}^1 \star \mathbb{S}^1 \simeq \mathbb{S}^3$.

Proposition 37. *The following map is an equivalence:*

$$
\begin{aligned}
&\mathsf{F} : \mathbb{S}^1 \star \mathbb{S}^1 \to \mathbb{S}^3\\
&\mathsf{F}\ (\mathsf{inl}\ x) = \mathsf{north}\\
&\mathsf{F}\ (\mathsf{inr}\ y) = \mathsf{north}\\
&\mathsf{F}\ (\mathsf{push}\ (x\ ,\ y)\ i) = \sigma\ (x \smile_1 y)\ i
\end{aligned}
$$

We omit the proof which is essentially just technical path-algebra. The fact that F uses $\smile_1$, which satisfies the laws in Proposition 36, lets us analyze $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ in a more algebraic manner. We now redefine $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ via the following decomposition, primarily defined in terms of post- and pre-composition with F and its inverse. We remind the reader of the map $\mathsf{h} : \mathbb{S}^1 \star \mathbb{S}^1 \to \mathbb{S}^2$ from Definition 5 for which $\mathsf{h}_* : \pi_3^*(\mathbb{S}^1 \star \mathbb{S}^1) \to \pi_3^*(\mathbb{S}^2)$ is an isomorphism—this follows from Proposition 11.

Definition 38. *Let $\theta : \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ be defined by the following sequence of isomorphisms*

$$
\begin{tikzcd}
\pi_3(\mathbb{S}^2) \arrow[r, "\mathsf{F}^*"] & \pi_3^*(\mathbb{S}^2) \arrow[r, "(\mathsf{h}_*)^{-1}"] & \pi_3^*(\mathbb{S}^1 \star \mathbb{S}^1) \arrow[dll, bend left, "\mathsf{F}_*"] \\
\pi_3^*(\mathbb{S}^3) \arrow[r, "(\mathsf{F}^{-1})^*"] & \pi_3(\mathbb{S}^3) \arrow[r, "\xi"] & \mathbb{Z}
\end{tikzcd}
$$

where ξ can be chosen to be any reasonable description of the isomorphism $\pi_3(\mathbb{S}^3) \cong \mathbb{Z}$ sending i_3 to 1.

The goal is to trace the image of $[i_2, i_2] : \pi_3(\mathbb{S}^2)$ under θ. Let us define the following three underlying functions of elements $\eta_1 : \pi_3^*(\mathbb{S}^2)$, $\eta_2 : \pi_3^*(\mathbb{S}^1 \star \mathbb{S}^1)$ and $\eta_3 : \pi_3^*(\mathbb{S}^3)$:

$$
\begin{aligned}
&\eta_1\text{-}\mathsf{fun} : \mathbb{S}^1 \star \mathbb{S}^1 \to \mathbb{S}^2\\
&\eta_1\text{-}\mathsf{fun}\ (\mathsf{inl}\ x) = \mathsf{north}\\
&\eta_1\text{-}\mathsf{fun}\ (\mathsf{inr}\ y) = \mathsf{north}\\
&\eta_1\text{-}\mathsf{fun}\ (\mathsf{push}\ (x\ ,\ y)\ i) = (\sigma\ y \cdot \sigma\ x)\ i
\end{aligned}
$$

$$
\begin{aligned}
&\eta_2\text{-}\mathsf{fun} : \mathbb{S}^1 \star \mathbb{S}^1 \to \mathbb{S}^1 \star \mathbb{S}^1\\
&\eta_2\text{-}\mathsf{fun}\ (\mathsf{inl}\ x) = \mathsf{inr}\ (\text{-}\ x)\\
&\eta_2\text{-}\mathsf{fun}\ (\mathsf{inr}\ y) = \mathsf{inr}\ y\\
&\eta_2\text{-}\mathsf{fun}\ (\mathsf{push}\ (x\ ,\ y)\ i) =\\
&\quad (\mathsf{push}\ (y\ \text{-}\ x\ ,\ \text{-}\ x)\ ^{-1} \cdot \mathsf{push}\ (y\ \text{-}\ x\ ,\ y))\ i
\end{aligned}
$$

$$
\begin{aligned}
&\eta_3\text{-}\mathsf{fun} : \mathbb{S}^1 \star \mathbb{S}^1 \to \mathbb{S}^3\\
&\eta_3\text{-}\mathsf{fun}\ (\mathsf{inl}\ x) = \mathsf{north}\\
&\eta_3\text{-}\mathsf{fun}\ (\mathsf{inr}\ y) = \mathsf{north}\\
&\eta_3\text{-}\mathsf{fun}\ (\mathsf{push}\ (x\ ,\ y)\ i) =\\
&\quad (\sigma\ (x \smile_1 y)\ ^{-1} \cdot \sigma\ (x \smile_1 y)\ ^{-1})\ i
\end{aligned}
$$

The claim is now that the image of $[i_2, i_2]$ under the chain of isomorphisms can be described as follows:

$$
\begin{tikzcd}
{[i_2, i_2]} \arrow[r, mapsto, "\mathsf{F}^*"] & \eta_1 \arrow[r, mapsto, "(\mathsf{h}_*)^{-1}"] & \eta_2 \arrow[dll, bend left, "\mathsf{F}_*"] \\
\eta_3 \arrow[r, mapsto, "(\mathsf{F}^{-1})^*"] & (-2) i_3 \arrow[r, mapsto, "\xi"] & \pm 2
\end{tikzcd}
$$

Lemma 39. $\mathsf{F}^*\, [i_2, i_2] \equiv \eta_1$

Proof: The definition of η_1 matches that of $|\nabla \circ \mathsf{W}| : \pi_3^*(\mathbb{S}^2)$, and so the statement holds by construction of the Whitehead product. ∎

Lemma 40. $(\mathsf{h}_*)^{-1}\, \eta_1 \equiv \eta_2$

Proof: Applying h_* on both sides gives the equation $\eta_1 \equiv \mathsf{h}_*\, \eta_2$. The underlying functions of these elements agree definitionally on inl and inr, and the push-case reduces to a simple application of the laws described in Proposition 36. ∎

Lemma 41. $\mathsf{F}_*\, \eta_2 \equiv \eta_3$

Proof: The proof is similar to that of Lemma 40. ∎

Theorem 42. $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$

Proof: By uniqueness (up to a sign) of isomorphisms $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, it suffices, according to Corollary 24, to show that the image of $[i_2, i_2]$ under θ is ± 2. That is:

$$(\xi \circ (\mathsf{F}^{-1})^* \circ \mathsf{F}_* \circ (\mathsf{h}_*)^{-1} \circ \mathsf{F}^*)[i_2, i_2] \equiv \pm 2$$

By Lemmas 39 to 41, it suffices to show that

$$(\xi \circ (\mathsf{F}^{-1})^*)\, \eta_3 \equiv \pm 2$$

One can easily show that $(\mathsf{F}^{-1})^*\, \eta_3 \equiv (-2)\, i_3$, and hence

$$(\xi \circ (\mathsf{F}^{-1})^*)\, \eta_3 \equiv (-2)\, (\xi\, i_3) \equiv -2$$

∎

In addition to providing a new and much shorter proof of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$, this gives us a sequence of new Brunerie numbers, $\beta_1, \beta_2, \beta_3 : \mathbb{Z}$, of decreasing complexity:

$$
\begin{aligned}
\beta_1 &= (\xi \circ (\mathsf{F}^{-1})^* \circ \mathsf{F}_* \circ (\mathsf{h}_*)^{-1})\, \eta_1\\
\beta_2 &= (\xi \circ (\mathsf{F}^{-1})^* \circ \mathsf{F}_*)\, \eta_2\\
\beta_3 &= (\xi \circ (\mathsf{F}^{-1})^*)\, \eta_3
\end{aligned}
$$

This gives new hope for Brunerie's conjecture about a proof by normalization. This may be captured as follows:

Theorem 43 (New Brunerie numbers)**.** *If either of $\beta_1, \beta_2, \beta_3 : \mathbb{Z}$ normalizes to ± 2, then $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.*

Ideally, we could normalize β_1. This, however, turns out to be difficult, as it does not bypass the main hurdle of computing the inverse of the isomorphism $\pi_3^*(\mathbb{S}^2) \cong \pi_3^*(\mathbb{S}^1 \star \mathbb{S}^1)$ induced by the Hopf map, which has a rather indirect construction coming from the LES of homotopy groups associated to the Hopf fibration. This problem does not apply to β_2, for which the computation does not rely on the problematic inverse. Unfortunately, also β_2 fails to normalize in reasonable time in `Cubical Agda`. This is surprising, as the only maps playing a fundamental role here are two applications of the equivalence $\mathbb{S}^1 \star \mathbb{S}^1 \simeq \mathbb{S}^3$, which is not too involved, and one application of ξ which may be compactly described via

$$\pi_3(\mathbb{S}^3) \xrightarrow{|\_|_*} \mathsf{H}^3(\mathbb{S}^3) \xrightarrow{\cong} \mathbb{Z}$$

and computes relatively well if the last isomorphism is constructed as in [12].[2] We have hence, at the time of writing, not been able to normalize even β_2, despite many optimizations of the functions involved. We are, however, able to normalize β_3 after some minor modifications to η_3 and the map $\pi_3^*(\mathbb{S}^3) \to \mathbb{Z}$. This optimized version of β_3, normalizes to -2 in `Cubical Agda` in just under 4 seconds, thereby giving us an at least partially computer-assisted proof of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.

We emphasize again that β_2 is a vastly simplified version of β since the isomorphism $\pi_3(\mathbb{S}^2) \cong \pi_3(\mathbb{S}^3)$ never has to be computed. Hence, it is rather surprising that computations break down already at this stage. This tells us that `Cubical Agda` has a long way to go before any direct computation of the original β is feasible. We hope that this could be useful for benchmarking in future optimizations of `Cubical Agda` and related systems.

Finally, we address the elephant in the room: why is there a minus sign popping up? In other words, have we really chosen the, in some way, canonical isomorphism? The isomorphism $\pi_3(\mathbb{S}^3) \cong \mathbb{Z}$ maps, as expected, i_3 to 1, so it can hardly be the culprit. Neither can the equivalence $\mathsf{F} : \mathbb{S}^1 \star \mathbb{S}^1 \simeq \mathbb{S}^3$, since it is applied equally in the constructions of hopf and of $[i_2, i_2]$. We could, however, have defined the push-case for h by

```
h (push (x , y) i) = σ (x - y) i
```

in which case θ would have sent $[i_2, i_2]$ to 2 and hopf to 1 (note that this is only possible since altering h would alter the definition of θ). The construction of h that we have given is, however, precisely the one which fell out by unfolding our formalization Brunerie's construction of the corresponding map. If this indeed is what Brunerie intended, we may also conclude that the original Brunerie number β is equal to -2. We stress that this merely is a fun fact and of no mathematical importance to Brunerie's proof or our formalization.

VII. Conclusion

In this paper, we have presented three formalizations of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ in the `Cubical Agda` system:

[2]As noted in [20], the Freudenthal suspension theorem should be avoided here as it has a tendency to lead to very slow computations. This is another way in which we deviate from Brunerie's β.

1) Brunerie's original proof [$\sim 17,000$ LOC]
2) A direct calculation of β [~ 600 LOC]
3) A computer-assisted reformulation of 2) [~ 400 LOC]

As always, the number of lines of code (LOC) should be taken with a grain of salt: on the one hand, the $17,000$ LOC in the first formalization exclude over 8000 LOC from [27], [21], [12] which we have imported as libraries. In addition, these numbers also exclude many elementary results used in the formalization, including ~ 9000 LOC for Chapters 1-3. We also stress that the line count for formalizations 2) and 3) only concern the part of the proof discussed in Section VI.

Formalization 1), which constituted the bulk of this paper, was a formalization of Brunerie's pen-and-paper proof, taking some convenient shortcuts when possible. The problem of formalizing Brunerie's proof has been a widely discussed open problem in HoTT/UF, and we hope that our efforts here provide a satisfactory solution to it. Formalizations 2) and 3) were of a simplified calculation of the Brunerie number, β. The very similar proofs 2) and 3) differ in that 3) uses `Cubical Agda` to carry out part of the computation of the new Brunerie number automatically. Perhaps equally important, we have seen that 3) provides us with new Brunerie numbers $\beta_1, \beta_2 : \mathbb{Z}$ which are far simpler than the original one, but still do not normalize in a reasonable amount of time. Our hope is that these can prove useful in future optimizations of `Cubical Agda` and related systems, as they could help shed some light on where the normalization of the original Brunerie number break down.

We remark that proofs 1) and 2) could be done in Book HoTT and do not use any cubical machinery in a fundamental way, making them interpretable in any suitably structured $(\infty, 1)$-topos [3]. We hence claim that, in our formalizations, we do not crucially rely on computations using univalence and HITs to prove anything that we could not have proved by hand in Book HoTT. Nevertheless, the `Cubical Agda` system has been very helpful in the formalization, primarily due to its native support for HITs and definitional computation rules for higher constructors. Formalization 3), however, is only valid in a system with computational support for univalence as it crucially relies on normalization of proof terms involving univalence. It would be interesting to run this in other cubical systems, like `cubicaltt` [29], `redtt` [30], `cooltt` [31], etc.

We also remark that our formalization of Brunerie's proof does not cover all results of Brunerie's thesis in full generality. For instance, we have not developed his proof concerning Whitehead products in full generality. We leave this generalization for future work. This would tie in nicely with another possible direction of future research, namely that of investigating whether the approach outlined in Section VI can be used to compute other Whitehead products. In addition, describing their graded quasi-Lie algebra structure is work in progress. Another related project is the proof of the symmetric monoidal structure on smash products, i.e. the main result of [4, Chap. 4]. While this would not make the proof of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ easier, it would be interesting to see whether Brunerie's proof could actually be formalized in the way that he intended it. Naturally, this question is also interesting in its own right. In fact, this problem was recently solved by Ljungström [32].

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