Towards a better implementation of cubical type theory

Thierry Coquand

The Poset Model

The Poset model of type theory is the presheaf model $\mathsf{Psh}(\Box)$ over the base category $\Box$ of finite non empty posets. This category contains as a full subcategory the category Δ of finite linear posets. We write $[n]$ the finite linear poset with $n+1$ elements.

The interval is defined to be the presheaf $\mathbf{I}$ represented by $[1]$. The cofibration classifier Φ is defined by taking $\Phi(X)$ to be the set of sieves that are finite union of principal sieves determined by embedding of codomain X (i.e. maps $i : Y \to X$ such that $i(y_0) \leqslant i(y_1)$ iff $y_0 \leqslant y_1$).

Dually any poset X corresponds to a f.p. distributive lattice $D(X) = [1]^X$. The elements of $\Phi(X)$ can then be seen as finite disjunction of sieves given by a finite set of constraints on the generators of this lattice. Since classical logic is conservative over the logic of distributive lattices, we can use a SAT solver to decide equality in $\Phi(X)$.

We can define $\Gamma^{\mathbf{I}}(X)$ to be $\Gamma(X \times [1])$. Since the category $\Box$ is cartesian closed, the right adjoint of exponentiation with $\mathbf{I}$ also has an explicit description: $\Delta_{\mathbf{I}}(X)$ can be taken to be $\Delta(X^{[1]})$. We have the canonical evaluation map $\mathsf{ev} : X^{[1]} \times [1] \to X$ and the *dependent* right adjoint is defined by

$$A_{\mathbf{I}}(X, \rho) = A(X^{[1]}, \rho\mathsf{ev})$$

so that $A_{\mathbf{I}}$ is in $\mathsf{Type}(\Gamma)$ if A is in $\mathsf{Type}(\Gamma^{\mathbf{I}})$.

The evaluation map $\mathsf{ev} : X^{[1]} \times [1] \to X$ has the following description in term of lattices. First, the lattice $X^{[1]}$ is described as follows. If $X \to [1]$ is given by some generators $z_1, z_2, \ldots$ and relations, we take two copies of the generators $u_0(z_1), u_0(z_2), \ldots, u_1(z_1), u_1(z_2), \ldots$ and a corresponding copies of the relations, with the constraints $u_0(z_1) \leqslant u_1(z_1), u_0(z_2) \leqslant u_1(z_2), \ldots$. We then have two maps $u_0 \leqslant u_1 : D(X) \to D(X^{[1]})$. For the evaluation map ev, we add a new generator z and we consider the map $d \mapsto u_0(d) \vee (z \wedge u_1(d))$, and this is the lattice version of the map ev.

Fibrant types

For A in $\mathsf{Type}(\Gamma)$ we can define the type of composition $T(A)$ in $\mathsf{Type}(\Gamma^{\mathbf{I}})$ as

$$\Pi_{\psi:\Phi}\Pi_{u:\Pi_{z:\mathbf{I}}T(z=0\vee\psi)\to A(\gamma z)}A(\gamma 1)[\psi, u1]$$

and we define $\mathsf{Fib}(A) = T(A)_{\mathbf{I}}$ which is in $\mathsf{Type}(\Gamma)$, so that

$$\mathsf{Fib}(A)(X, \rho) = T(A)(X^{[1]}, \rho\mathsf{ev})$$

There is a problem of notations for the restriction operation: if c in $\mathsf{Fib}(A)(X, \rho)$ and $f : Y \to X$ we want to define the restriction cf in $\mathsf{Fib}(A)(Y, \rho f)$ but it will be defined by $cf = c(f^{[1]} \times [1])$ with restriction operation for $T(A)$ in $\mathsf{Type}(\Gamma^{\mathbf{I}})$. I am not sure what notations we should have for the restriction operation.

Closure operations on fibrant types

We have the canonical map $\epsilon : X \to (X \times [1])^{[1]}$ defined by $\epsilon\ x\ z = (x, z)$

We have an isomorphism between $\mathsf{Elem}(\Gamma, \mathsf{Fib}(A))$ and $\mathsf{Elem}(\Gamma^{\mathbf{I}}, T(A))$ which is natural in Γ. If c in $\mathsf{Elem}(\Gamma, \mathsf{Fib}(A))$ we have $\mathsf{tf}(c)$ in $\mathsf{Elem}(\Gamma^{\mathbf{I}}, T(A))$ by defining $\mathsf{tf}(c)(X, \nu)$ in $T(A)(X, \nu)$ with ν in $\Gamma^{\mathbf{I}}(X) = \Gamma(X \times [1])$ to be $c(X \times [1], \nu)(\epsilon \times [1])$.

Conversely, if d in $\mathsf{Elem}(\Gamma^{\mathbf{I}}, T(A))$ we define $\mathsf{ft}(d)$ in $\mathsf{Elem}(\Gamma, \mathsf{Fib}(A))$ by taking $\mathsf{ft}(d)(X, \rho)$ for ρ in $\Gamma(X)$ to be $d(X^{[1]}, \rho\mathsf{ev})$.

Given this isomorphism, we can now define closure operations on types with a Fib structure by transport of structure.

For instance, we have an operation $d_\pi\ d_A\ d_B$ in $\mathsf{Elem}(\Gamma^{\mathbf{I}}, \Pi\ A\ B)$ for d_A in $\mathsf{Elem}(\Gamma^{\mathbf{I}}, A)$ and d_B in $\mathsf{Elem}((\Gamma.A)^{\mathbf{I}}, B)$. We can then define the corresponding operation $c_\pi\ c_A\ c_B$ by

$$c_\pi\ c_A\ c_B = \mathsf{ft}(d_\pi\ \mathsf{tf}(c_A)\ \mathsf{tf}(c_B))$$

Towards an implementation

We want to structure the implementation as a *relativization* of the presheaf model. A type over a presheaf Γ should be interpreted as a pair A, c_A where A is in $\mathsf{Type}(\Gamma)$ and c_A an element of $\mathsf{Elem}(\Gamma, \mathsf{Fib}(A))$. An element of A, c_A is an element of $\mathsf{Elem}(\Gamma, A)$.

Since we have a denotational description of $\mathsf{Fib}(A)$ it should be possible to have an implementation which relies on the usual notion of closure, without having to do evaluation under a binder as it is done now.