

Efficient Evaluation for Cubical Type Theories

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Many more definitions to go!

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- ① Normalization-by-evaluation for MLTT
- ② NbE for CTT
- ③ Implementation & benchmarks
- ④ Conclusions

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Solution

- Separate *syntax* (program code) from *semantic values* (runtime objects).
- The syntax only supports *evaluation* into values.
- Values support efficient β -reduction, without using recursive substitution.

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I focus on a *practical flavor* of NbE which has several differences to the nicest *formal* NbE.

Informal NbE (1)

We omit types of things for brevity.

Syntax & values

$\Gamma, \Delta : \text{Con}$	$\sigma, \delta : \text{Env } \Gamma \Delta$
$t, u : \text{Tm } \Gamma$	$v : \text{Val } \Gamma$
$\sigma, \delta : \text{Sub } \Gamma \Delta$	

Operations

$\text{eval} : \text{Env } \Gamma \Delta \rightarrow \text{Tm } \Delta \rightarrow \text{Val } \Gamma$
$\text{quote} : \text{Val } \Gamma \rightarrow \text{Tm } \Gamma$
$\text{conv} : \text{Val } \Gamma \rightarrow \text{Val } \Gamma \rightarrow \text{Bool}$

$\text{Val } \Gamma$ has the same structure as $\text{Tm } \Gamma$, except each binder is replaced with a **closure**. A closure stores a variable name x , an environment $\sigma : \text{Env } \Gamma \Delta$ and a $t : \text{Tm } (\Delta, x)$.

Informal NbE (2)

$\text{eval} : \text{Env } \Gamma \Delta \rightarrow \text{Tm } \Delta \rightarrow \text{Val } \Gamma$

$\text{eval } \sigma x \quad : \equiv \sigma x$

$\text{eval } \sigma (\lambda x. t) : \equiv \lambda_{\text{Val}} (x, \sigma, t)$

$\text{eval } \sigma (t u) \quad : \equiv \text{case eval } \sigma t \text{ of}$

$\lambda_{\text{Val}} (x, \delta, t) \rightarrow \text{eval } (\delta, x \mapsto \text{eval } \sigma u) t$

$v \quad \rightarrow v (\text{eval } \sigma u)$

$\text{quote} : \text{Val } \Gamma \rightarrow \text{Tm } \Gamma$

$\text{quote } x \quad : \equiv x$

$\text{quote } (\lambda_{\text{Val}} (x, \delta, t)) : \equiv \lambda x'. \text{quote } (\text{eval } (\delta, x \mapsto x') t)$

where x' is fresh in Γ

$\text{quote } (t u) \quad : \equiv (\text{quote } t) (\text{quote } u)$

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Cubical NbE

In the following we consider Cartesian a CTT with coe , hcom , HITs and Glue.

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In analogy to MLTT NbE, cubical NbE should take a “semantic interpretation” of the context as input.

- An interval substitution $\sigma : \text{Sub}^I \Psi_0 \Psi_1$.
- A cofibration implication $f : \alpha_0 \Rightarrow \alpha_1[\sigma]$.
- A value environment $\delta : \text{Env } \Gamma_0 (\Gamma_1[\sigma, f])$.

$$\begin{aligned}
 \text{eval} : & \forall \Psi_0 \alpha_0 \Gamma_0 \Psi_1 \alpha_1 \Gamma_1 \\
 & (\sigma : \text{Sub}^I \Psi_0 \Psi_1) \\
 & (f : \alpha_0 \Rightarrow \alpha_1[\sigma]) \\
 & (\delta : \text{Env } \Gamma_0 (\Gamma_1[\sigma, f])) \\
 & \rightarrow \text{Tm } (\Psi_1; \alpha_1; \Gamma_1) \rightarrow \text{Val } (\Psi_0; \alpha_0; \Gamma_0)
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6 out of 10 inputs are computationally relevant in implementation:

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- Ψ_0 marks the next fresh interval variable.
- α_0 is used for “forcing” (see later).
- Γ_0 is passed to detect when there are no fibrant free variables.
- σ , δ and t are evidently required.

Trouble with interval substitution

MLTT NbE: Val substitution is inefficient.

$$-[-] : \text{Val } \Delta \rightarrow \text{Env } \Gamma \Delta \rightarrow \text{Val } \Gamma$$

Evaluation creates **shared structure**. Recursive substitution destroys all such sharing by creating fresh copies of values.

Example for sharing:

$\text{let } x := f \ y \text{ in } (x, x, x, x, x)$

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Likewise: recursive **interval substitution** destroys all structure sharing.

- MLTT NbE: no need for value substitution.
- CTT NbE: **must** support interval substitution on values.

Two extra operations.

1. Interval substitution

$$-[-] : \text{Val}(\Psi_0; \alpha; \Gamma) \rightarrow (\sigma : \text{Sub}^I \Psi_1 \Psi_0) \rightarrow \text{Val}(\Psi_1; \alpha[\sigma]; \Gamma[\sigma])$$

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2. Forcing

$$\text{force} : \text{Val}(\Psi; \alpha; \Gamma) \rightarrow \text{Val}(\Psi; \alpha; \Gamma)$$

Computes delayed substitutions sufficiently to yield a *head normal* value.
See also: notion of forcing in lazy evaluation.

Stability annotations

Forcing has trivial cost on canonical values, for example:

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Our implementation:

- Neutrals are annotated with *blocking sets* of interval variables.
- Only an approximation of precise predicates!
- We can quickly see if a substitution has no action on a neutral.

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Forcing w.r.t. cofibrations

Forcing doesn't just compute substitutions, but *cofibration weakening* as well.

$$\text{let } x := \text{coe } i \, j \, (k. A) \, y \text{ in} \\ \text{hcom } 0 \, 1 \, [i = j \mapsto x] \, z$$

x is first evaluated under some cofibration α , but then mentioned under $\alpha \wedge i = j$.

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Contrast MLTT NbE: weakening of values has no cost!
(if we use a suitable variable representation in values, e.g. De Bruijn levels)

Closures vs. binders

We can't represent all interval binders with closures!

$$\text{coe } r \, r' (i. A \rightarrow B) f \equiv \lambda x. \text{coe } r \, r' (i. B) (f (\text{coe } r' \, r (i. A) x))$$

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- `coe`, `hcom`: we need to peek under interval binders, so we use *explicit weakenings* as semantic binders.
- Other cases (e.g. dependent paths, path abstractions): we use closures.

Defunctionalization (1)

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Defunctionalization: representing higher-order functions with first-order data and a first-order generic application.

Defunctionalization (2)

Interval substitution has action on closures:

$$(\text{eval}_{\text{cl}}(x, \delta, t))[\sigma] \equiv \text{eval}_{\text{cl}}(x, \delta[\sigma], t)$$

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Fun fact: we have **37** different closures in the implementation. It's a bit tedious!

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Ideally, we'd just write higher-order binders in semantics, and automatically generate for each one:

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Seems like a major challenge. In the long term we'd want some *logical framework* for implementing (C)TT evaluation.

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If we don't coerce along Glue, interval substitution only has linear runtime overhead.

Exploiting CTT canonicity (1)

Back to MLTT for a bit:

- Consider closed evaluation of if — then — else.
- The Bool scrutinee is true or false, so we have to evaluate just one of the branches.
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- This is bad.

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The offending rules are precisely the hcom rules for strict inductive types.

$$\text{hcom } r \, r' [\alpha \mapsto i. \text{suc } t] (\text{suc } b) \equiv \text{suc } (\text{hcom } r \, r' [\alpha \mapsto i. t] b)$$

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If there are no fibrant free variables, if we have:

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Then canonicity implies that $t \equiv \text{suc } t'$ for some t' .

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Then canonicity implies that $t \equiv \text{suc } t'$ for some t' .

So we can use this rule instead²:

$$\text{hcom } r \, r' [\alpha \mapsto i. t] (\text{suc } b) \equiv \text{suc } (\text{hcom } r \, r' [\alpha \mapsto i. \text{pred } t] b)$$

pred is a metatheoretic function which unwraps a suc.

²Used in Simon Huber: *Cubical Interpretations of Type Theory*, sec. 7.2

Exploiting CTT canonicity (3)

The pred rule can be generalized for arbitrary strict inductive types.

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In a purely cubical context (no fibrant variables), no computation rule evaluates all components of a system.

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- ① Normalization-by-evaluation for MLTT
- ② NbE for CTT
- ③ Implementation & benchmarks
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Implementation

- <https://github.com/AndrasKovacs/cctt>
- It's called `cctt` because it's a Cartesian CTT.
- ~5000 lines of Haskell.
- Features: path types, line types, bidirectional type inference, strict inductive types, parameterized HITs.
- Design is a mixture of AFH, ABCFHL and `cubicaltt`.
 - Systems and `ghcom` from AFH.
 - Glue type from ABCFHL.
 - HIT implementation from `cubicaltt`.
- No universe checking (type-in-type), no termination checking.
- At least 100 times faster type checking than Agda.

Transporting along Bool negation

Convert Bool negation to a path, compose it with itself N times, transport true over it. Times in seconds.

N	Agda	cctt	Ratio
100	0.29	0.00041	707
250	0.97	0.00095	1021
500	3.36	0.0019	1768
750	7.07	0.0030	2356
1000	12.57	0.0047	2674
10^6	N/A	5.65	N/A

Computing winding numbers

Take an integer, convert it to a path in base $=_{\mathbb{S}^1}$ base, then convert back.
Times in seconds.

N	Agda	cctt	Ratio
100	0.34	0.0005	680
250	1.89	0.0012	1575
500	5.643	0.0023	2453
750	10.37	0.0043	2411
1000	18.52	0.0059	3138
10^6	N/A	7.98	N/A

Brunerie and the issue with hcom-s (1)

We tried the new Brunerie number definition by Ljungström and Mörtberg³.

Problem: we did not have ghcom at that point. We had two extra empty hcom-s for each coercion along univalence.

This caused a mismatch with cubical Agda, the following did not typecheck:

```
brunerie : ℤ :=  
  g10 (g9 (g8 (λ i j. f7 (λ k. η₃ (push (loop1 i) (loop1 j) k)))));
```

³*Formalizing $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ and Computing a Brunerie Number in Cubical Agda*

Brunerie and the issue with hcom-s (2)

Fortunately, I was able to manually insert 18 or 36 Glue types at several places to make it well-typed. One such place:

```
g9' : gbase1'' = gbase1'' → sTrunc  $\mathbb{Z}$  :=  
  λ p.  
    unglue (unglue (unglue (unglue (unglue (unglue (unglue (unglue (unglue (unglue (unglue (  
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- Computes 60 million hcom-s in total.
- Just before the last g10 step, we have the set truncation of -2 wrapped in half million empty hcom-s.

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“Who needs ghcom if we can easily compute a few million empty hcom-s?”

More Brunerie numbers

With the addition of ghcom:

- The Agda-computable Brunerie number definition runs in 0.5 ms, computing a mere 700 hcom-s (~ 1 million times speedup!).

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To do:

- Two more variants from Anders & Axel's paper (β_1 and β_2).
- The infamous older cubicaltt definitions.

Speedup from De Morgan intervals?

Tom Jack has a $\pi_2(\mathbb{S}^3)$ generator definition:

- Computes instantly in `cubicaltt` (De Morgan CTT).
- Computes in 3 minutes in `cctt`, in 96 million `hcom`-s.
(Fun fact: without `ghcom`, it computes in 20 minutes, in **9.5 billion** `hcom`-s.)

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The difference *appears to be* the usage of interval connections.

Could we add some connections to Cartesian CTT?

Or: implement a De Morgan CTT with our basic optimizations.

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How should we associate iterated path composition, e.g. $p \cdot q \cdot r$?

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Can we add this to Agda? Yes. Some things are harder. We'd need a complete rewrite of the Agda Abstract Machine.