INVESTIGATIONS INTO CARTESIAN CUBICAL TYPE THEORY

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Abstract. The goal of this note is to present a type theory for the recent cartesian cubical models. This type theory is inspired by [CCHM18] and can be seen as step towards a "formal" version of the computational higher type theory of [AFH17] and the recent cubical set model of [ABC+17]. The main goal of the note is to present the generalized Kan operations in the cartesian setting and how they are defined for the various type formers. The definitions presented here for univalent universes differ from [AFH17, ABC+17] and is closer to the one in [CCHM18].

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Date: September 10, 2018.

1. Introduction

This note describes a *cartesian* cubical type theory and the key differences compared to the cubical type theory of [CCHM18] (based on De Morgan cubes) are:

(1) Less structure on the interval (no connections or reversals).
(2) Decomposition of Kan composition into coercion and homogeneous composition (similar to [CHM18]).
(3) More general Kan operations (from $r : \mathbb{I}$ to $s : \mathbb{I}$ instead of just 0 to 1).
(4) Diagonal "face-formulas" (i.e., $(i = j)$ in $\mathbb{F}$).

A variation of the type theory presented in this note has been implemented in yacctt.[1] The major differences between the implementation and this note are more or less the same as for cubicaltt and [CCHM18] (e.g. the paper version uses partial elements, etc.).

The type theory in this note is heavily inspired by the semantic presentations of [AFH17] and [ABC+17]. The treatment of the Kan operations for basic type formers (Π, Σ, Path and natural numbers) are the same as in the semantics. This note presents both the V/fcom types of [AFH17] and the Glue-types of [ABC+17] together with (constructive) proofs why they are fibrant (i.e. the definition of hcom/coe for these types). The algorithms for hcom/coe in V/fcom and Glue-types presented here are quite similar (as expected), but they differ from the semantic presentations in various ways. Some parts have been substantially simplified and the algorithms are now closer to the one for Glue-types in [CCHM18], making the connection to the De Morgan cubical set model and type theory clearer.

2. Generalized Kan operations

We assume a discrete countably infinite set of names (written i, j, k) with a total ordering $<$.[2] The interval $\mathbb{I}$ is described by the following grammar:

$$r, s ::= 0 \mid 1 \mid i$$

Given a term u we write $u(r/i)$ for u with r substituted for i.

The face lattice $\mathbb{F}$ is described by:[3]

$$\varphi, \psi ::= 0_{\mathbb{F}} \mid 1_{\mathbb{F}} \mid (i = 0) \mid (i = 1) \mid (i = j) \mid \varphi \vee \psi \mid \varphi \wedge \psi$$

This can be presented as the lattice generated by formal generators $(i = 0)$, $(i = 1)$ and $(i = j)$. For simplicity we assume that $i < j$ whenever we have $(i = j)$. We use the same notations for context restrictions, partial elements, systems and boundary conditions as in [CCHM18]. The exact definition of these are omitted as the main point of this note is to present that Kan operations and these are independent of the various choices that can be made with respect to the treatment of systems

[1] https://github.com/mortberg/yacctt/

[2] The ordering is probably not strictly necessary, but it simplifies the implementation as it allows us to always make a uniform choice of which name to pick when being under a diagonal face formula.

[3] A join semi-lattice might be sufficient if we drop the partial elements.

and related notions (we could for example require fewer equations as in [AFH17] or we could treat them closer to the categorical semantics as in [ABC+17]). We also omit the typing rules for Π, Σ, Path and natural numbers as they are identical to [CCHM18].

Like in [CHM18] the heterogeneous composition operations are decomposed into coercion and homogeneous composition operations. This seems crucial in order to be able to prove that various (parametrized) higher inductive types are fibrant. However, compared to [CHM18], the Kan operations are generalized to allow us to also directly define Kan filling operations (which is necessary as $\mathbb{I}$ does not have connections).

2.1. **Coercion.** Typing rule for coercion:

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \qquad \Gamma \vdash u : A(r/i)}{\Gamma \vdash \mathsf{coe}_i^{r \to s}\, A\, u : A(s/i)[(r = s) \mapsto u]}$$

Note that i is bound in A.

2.2. **Homogeneous composition.** Typing rule for homogeneous composition:

$$\frac{\begin{array}{c}\Gamma \vdash A \qquad \Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \\ \Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash u_0 : A[\varphi \mapsto u(r/i)]\end{array}}{\Gamma \vdash \mathsf{hcom}_i^{r \to s}\, A\, [\varphi \mapsto u]\, u_0 : A[\varphi \mapsto u(s/i), (r = s) \mapsto u_0]}$$

Note that i is only bound in u.

2.3. **Composition.** From this we can derive heterogeneous composition:

$$\frac{\begin{array}{c}\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \\ \Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash u_0 : A(r/i)[\varphi \mapsto u(r/i)]\end{array}}{\begin{array}{c}\Gamma \vdash \mathsf{com}_i^{r \to s}\, A\, [\varphi \mapsto u]\, u_0 := \mathsf{hcom}_i^{r \to s}\, A(s/i)\, [\varphi \mapsto \mathsf{coe}_j^{i \to s}\, A\, u]\, (\mathsf{coe}_i^{r \to s}\, A\, u_0) : \\ A(s/i)[\varphi \mapsto u(s/i), (r = s) \mapsto u_0]\end{array}}$$

where j is a fresh dimension. Note that i is bound in both A and u.

We say that a type is *fibrant* if it can be equipped with a heterogeneous composition operation (it hence suffices to explain how to do coercion and homogeneous composition at each type under consideration). Note that being fibrant in the semantics is a *structure* and not only a property. Furthermore, any fibrant type also has a coercion and homogeneous composition structure.

Remark 1. For efficiency it might be better to define com directly by induction on the type instead of treating it uniformly for all types. Consider for example the case of Path types.

2.4. **Various constructions using the generalized Kan operations.** Coercing and composing to an interval variable lets us define Kan filling and many more things.

Given $j : \mathbb{I}$ we define the "coercion filler" $F := \mathsf{coe}_i^{0 \to j}\, A\, u : A(j/i)$ satisfying $F(0/j) = u$ and $F(1/j) = \mathsf{coe}_i^{0 \to 1}\, A\, u$. Similarly we define a "squeeze" operation as in [CCHM18, CHM18] by $S := \mathsf{coe}_i^{j \to 1}\, A\, u : A(1/i)$ satisfying $S(0/j) = \mathsf{coe}_i^{0 \to 1}\, A\, u$ and $S(1/j) = u$. For simplicity we often use the same name for the fresh dimension and the dimension being bound in the coercion, e.g. $\mathsf{coe}_i^{0 \to i}\, A\, u$. This is justified as we may always rename to a fresh dimension. Given $i : \mathbb{I} \vdash u : A$ we hence get the diagram:

$$\begin{tikzcd}
& u(1/i) \\
u(0/i) \arrow[ur, "u"] \arrow[r, "\mathsf{coe}_i^{0\to i}\,A\,u(0/i)"'] & \mathsf{coe}_i^{0\to 1}\,A\,u(0/i) \arrow[u, "\mathsf{coe}_i^{i\to 1}\,A\,u"'] \\
A(0/i) \arrow[r, "A"'] & A(1/i)
\end{tikzcd}$$

We can also coerce from i to get $F' := \mathsf{coe}_i^{i\to 1}\,A\,u$ such that $F'(0/i) = \mathsf{coe}_i^{0\to 1}\,A\,u$ and $F'(1/i) = u$, etc.

Similarly we define the homogeneous Kan filler $H := \mathsf{hcom}_i^{0\to i}\,A\,[\varphi \mapsto u]\,u_0$ satisfying $H(0/i) = u_0$, $H(1/i) = \mathsf{hcom}_i^{0\to 1}\,A\,[\varphi \mapsto u]\,u_0$ and $H = u$ on the extent φ. We also get the heterogenous Kan filler by composing heterogeneously to a fresh dimension.

Reversals and connections are definable using homogeneous composition:

Lemma 2 (Reversals)**.** *Given* $p : \mathsf{Path}\,A\,a\,b$ *we define* $p^{-1} : \mathsf{Path}\,A\,b\,a$ *as*

$$p^{-1} := \langle i\rangle\,\mathsf{hcom}_j^{0\to 1}\,A\,[(i=0)\mapsto p\,j, (i=1)\mapsto a]\,a$$

Proof. This satisfies $p^{-1}\,0 = p\,1 = b$ and $p^{-1}\,1 = a$, as desired. Pictorially this corresponds to the dashed line:

$$\begin{tikzcd}
b \arrow[r, dashed, "p^{-1}"] & a \\
a \arrow[u, "p"] \arrow[r, "a"'] & a \arrow[u, "a"']
\end{tikzcd}$$

□

Note that we could also take:

$$p^{-1} := \langle i\rangle\,\mathsf{hcom}_j^{1\to 0}\,A\,[(i=0)\mapsto b, (i=1)\mapsto p\,j]\,b$$

which corresponds to the dashed line in:

$$\begin{tikzcd}
b \arrow[r, "b"] & b \\
b \arrow[u, "b"] \arrow[r, dashed, "p^{-1}"'] & a \arrow[u, "p"']
\end{tikzcd}$$

We can also define the connections from [CCHM18] as:

Lemma 3 (Connections)**.** *Given* $p : \mathsf{Path}\,A\,a\,b$ *we can define terms* $\mathsf{connAnd}(p)$ *and* $\mathsf{connOr}(p)$ *with boundaries:*

$$\begin{tikzcd}
a \arrow[r, "p"] & b \\
a \arrow[u, "a"] \arrow[r, "a"'] \arrow[ur, phantom, "\mathsf{connAnd}(p)"] & a \arrow[u, "p"']
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r, "b"] & b \\
a \arrow[u, "p"] \arrow[r, "p"'] \arrow[ur, phantom, "\mathsf{connOr}(p)"] & b \arrow[u, "b"']
\end{tikzcd}$$

Furthermore, we can construct these in such a way that the diagonals are p*.*

Proof (due to Favonia). We define $\mathsf{connAnd}(p)$ as:

$$\begin{aligned}
\langle i\ j\rangle\ \mathsf{hcom}_k^{0\to 1}\,A\,[&(i=0)\mapsto \mathsf{hcom}_l^{1\to 0}\,A\,[(k=0)\mapsto a,(k=1)\mapsto p\,l]\,(p\,k)\\
&,(i=1)\mapsto \mathsf{hcom}_l^{1\to j}\,A\,[(k=0)\mapsto a,(k=1)\mapsto p\,l]\,(p\,k)\\
&,(j=0)\mapsto \mathsf{hcom}_l^{1\to 0}\,A\,[(k=0)\mapsto a,(k=1)\mapsto p\,l]\,(p\,k)\\
&,(j=1)\mapsto \mathsf{hcom}_l^{1\to i}\,A\,[(k=0)\mapsto a,(k=1)\mapsto p\,l]\,(p\,k)\\
&,(i=j)\mapsto \mathsf{hcom}_l^{1\to i}\,A\,[(k=0)\mapsto a,(k=1)\mapsto p\,l]\,(p\,k)]\\
&a
\end{aligned}$$

Similarly we define $\mathsf{connOr}(p)$ as:

$$\begin{aligned}
\langle i\ j\rangle\ \mathsf{hcom}_k^{1\to 0}\,A\,[&(i=0)\mapsto \mathsf{hcom}_l^{0\to j}\,A\,[(k=0)\mapsto p\,l,(k=1)\mapsto b]\,(p\,k)\\
&,(i=1)\mapsto \mathsf{hcom}_l^{0\to 1}\,A\,[(k=0)\mapsto p\,l,(k=1)\mapsto b]\,(p\,k)\\
&,(j=0)\mapsto \mathsf{hcom}_l^{0\to i}\,A\,[(k=0)\mapsto p\,l,(k=1)\mapsto b]\,(p\,k)\\
&,(j=1)\mapsto \mathsf{hcom}_l^{0\to 1}\,A\,[(k=0)\mapsto p\,l,(k=1)\mapsto b]\,(p\,k)\\
&,(i=j)\mapsto \mathsf{hcom}_l^{0\to i}\,A\,[(k=0)\mapsto p\,l,(k=1)\mapsto b]\,(p\,k)]\\
&b
\end{aligned}$$

□

The $\mathsf{connAnd}(p)$ operation corresponds to $\langle i\ j\rangle\,p\,(i\wedge j)$ and $\mathsf{connOr}(p)$ corresponds to $\langle i\ j\rangle\,p\,(i\vee j)$ in [CCHM18]. Furthermore, the diagonal constraints are often not necessary as it very often suffices to just have a square with the appropriate sides.

Note that the operations defined in the above lemmas don't satisfy the judgmental equalities that the corresponding definitions in [CCHM18] do. For example $(p^{-1})^{-1}$ is not judgmentally equal to p. It remains to be seen how this affect practical formalizations, however having the more general Kan operations does simplify some definitions compared to [CCHM18]. For example, given $i : \mathbb{I} \vdash A$ and $a : A(0/i)$ we can form the left line of

$$\begin{tikzcd}
\mathsf{coe}_i^{1\to 0}\,A\,(\mathsf{coe}_i^{0\to 1}\,A\,a) & \\
a \arrow[u] \arrow[r] & \mathsf{coe}_i^{0\to 1}\,A\,a \arrow[ul]
\end{tikzcd}$$

$$A(0/i) \xrightarrow[A]{\qquad\qquad} A(1/i)$$

as $\langle j\rangle\ \mathsf{coe}_i^{j\to 0}\,A\,(\mathsf{coe}_i^{0\to j}\,A\,a)$. The corresponding result in [CCHM18] is quite a bit more involved (it uses 3 heterogeneous compositions)

3. Recursive definition of coercion

We now explain $\mathsf{coe}_i^{r\to s}\,A\,u$ by induction on the type A for natural numbers, Path-types, Σ-types and Π-types. For all of the definitions below we can easily check that they satisfy all of the necessary conditions specified by the typing rules above.

3.1. **Natural Numbers.** For the (strict) natural numbers we can take the identity coercion:

$$\mathsf{coe}_i^{r\to s}\,\mathsf{N}\,u = u$$

3.2. **Dependent Paths.** Let $\Gamma, i : \mathbb{I}, j : \mathbb{I} \vdash A$, $\Gamma, i : \mathbb{I} \vdash v : A(0/j)$ and $\Gamma, i : \mathbb{I} \vdash w : A(1/j)$. We then take:

$$\mathsf{coe}_i^{r\to s}\,(\mathsf{Path}^j\ A\ v\ w)\,u = \langle j\rangle\,\mathsf{com}_i^{r\to s}\,A\,[(j=0)\mapsto v, (j=1)\mapsto w]\,(u\ j)$$

3.3. **Dependent Pairs.** Let $\Gamma, i : \mathbb{I} \vdash A$ and $\Gamma, i : \mathbb{I}, x : A \vdash B$. We then take:

$$\mathsf{coe}_i^{r\to s}\,((x : A) \times B)\,u = (v(s/i), \mathsf{coe}_i^{r\to s}\,B(v/x)\,(u.2))$$

where $v = \mathsf{coe}_i^{r\to i}\,A\,(u.1)$.

3.4. **Dependent Functions.** Let $\Gamma, i : \mathbb{I} \vdash A$ and $\Gamma, i : \mathbb{I}, x : A \vdash B$. Given $u : ((x : A) \to B)(r/i)$ and $v : A(s/i)$ we take:

$$(\mathsf{coe}_i^{r\to s}\,((x : A) \to B)\,u)\ v = \mathsf{coe}_i^{r\to s}\,B(w/x)\,(u\ w(r/i))$$

where $w = \mathsf{coe}_i^{s\to i}\,A\,v$.

4. Recursive definition of homogeneous composition

We now explain $\mathsf{hcom}_i^{r\to s}\,A\,[\varphi \mapsto u]\,u_0$ by induction on the type A for natural numbers, Path-types, Σ-types and Π-types. We omit most of the typing information for the arguments as these can be easily inferred from the typing rules, furthermore the correctness of the below definitions are easily checked and are hence omitted.

4.1. **Natural Numbers.** We define this by recursion on u and u_0:

$$\mathsf{hcom}_i^{r\to s}\,\mathsf{N}\,[\varphi \mapsto 0]\,0 = 0$$
$$\mathsf{hcom}_i^{r\to s}\,\mathsf{N}\,[\varphi \mapsto \mathsf{S}\,u]\,(\mathsf{S}\,u_0) = \mathsf{hcom}_i^{r\to s}\,\mathsf{N}\,[\varphi \mapsto u]\,u_0$$

In a closed context the following would also work:

$$\mathsf{hcom}_i^{r\to s}\,\mathsf{N}\,[\varphi \mapsto u]\,u_0 = u_0$$

4.2. **Dependent Paths.**

$$\begin{aligned}&\mathsf{hcom}_i^{r\to s}\,(\mathsf{Path}^j\,A\,v\,w)\,[\varphi\mapsto u]\,u_0 =\\ &\qquad\qquad\langle j\rangle\,\mathsf{hcom}_i^{r\to s}\,A\,[\varphi\mapsto u\ j,(j=0)\mapsto v,(j=1)\mapsto w]\,(u_0\ j)\end{aligned}$$

4.3. **Dependent Pairs.** Given $\Gamma\vdash A$ and $\Gamma,x:A\vdash B$.

$$\mathsf{hcom}_i^{r\to s}\,((x:A)\times B)\,[\varphi\mapsto u]\,u_0 = (v(s/i),\mathsf{com}_i^{r\to s}\,B(v/x)\,[\varphi\mapsto u.2]\,(u_0.2))$$

where $v=\mathsf{hcom}_i^{r\to i}\,A\,[\varphi\mapsto u.1]\,(u_0.1)$.

4.4. **Dependent Functions.** Given $\Gamma\vdash A$, $\Gamma,x:A\vdash B$ and $\Gamma\vdash v:A$.

$$(\mathsf{hcom}_i^{r\to s}\,((x:A)\to B)\,[\varphi\mapsto u]\,u_0)\ v = \mathsf{hcom}_i^{r\to s}\,B(v/x)\,[\varphi\mapsto u\ v]\,(u_0\ v)$$

Remark 4. Note that there is almost no difference to the definitions in [CCHM18] for any of these types if one spell out the special case of comp that corresponds to coe and hcom from 0 to 1 (and define the filler using a connection instead of coe/hcom to a fresh dimension).

5. Univalent and fibrant universes

To prove univalence we need to be able to turn equivalences into lines. We first only consider a very special case of Glue-types [CCHM18], which we call "V-types" following [AFH17] (Section 5.1). The drawback of not having Glue-types is that we cannot prove the fibrancy of the universe as in [CCHM18] which means that we have to consider fcom-types (Section 5.2). We then also consider adding Glue-types as in [CCHM18] (Section 5.3) which lets us prove both univalence and fibrancy of the universe in one go. The algorithms we give for these types are similar, but slightly more direct than in [AFH17, ABC+17]. The point of including both V/fcom and Glue-types in this note is to be able to compare the algorithms for hcom/coe for the different types and the proofs of univalence and fibrancy of the universe that follow from the two approaches.

We define equivalences as:

$$\begin{aligned}\mathsf{Equiv}\,A\,B &:= (e:A\to B)\times\mathsf{IsEquiv}\,e\\ \mathsf{IsEquiv}\,e &:= (x:B)\to\mathsf{IsContr}\,(\mathsf{Fiber}\,e\,x)\\ \mathsf{IsContr}\,C &:= (x:C)\times((y:C)\to\mathsf{Path}\ C\ y\ x)\\ \mathsf{Fiber}\,e\,x &:= (y:A)\times\mathsf{Path}\ B\ (e\ y)\ x\end{aligned}$$

This differs from [AFH17, CCHM18], but it coincides with both RedPRL and UniMath. The reason for choosing this definition is that it makes it possible to simplify the coercion operation for V-types a lot compared to [AFH17]. For simplicity we just write $e\ u$ for $e.1\ u$.

Remark 5. We will need to compute the homogeneous composition in fiber types, so we expand this here for reference:

$$\begin{aligned}\mathsf{hcom}_i^{r\to s}\,(\mathsf{Fiber}\,e\,x)\,[\varphi\mapsto u]\,u_0 &= \mathsf{hcom}_i^{r\to s}\,((y:A)\times\mathsf{Path}\ B\ (e\ y)\ x)\,[\varphi\mapsto u]\,u_0\\ &= (\mathsf{hcom}_i^{r\to s}\,A\,[\varphi\mapsto u.1]\,u_0.1,\mathsf{com}_i^{r\to s}\,P\,[\varphi\mapsto u.2]\,u_0.2)\end{aligned}$$

where $P=\mathsf{Path}\ B\ (e\ (\mathsf{hcom}_i^{r\to i}\,A\,[\varphi\mapsto u.1]\,u_0.1))\ x$.

The key lemma in the constructions below is:

Lemma 6. *For any $r, s : \mathbb{I}$ and line $i : \mathbb{I} \vdash A$ the map*

$$\mathsf{coe}_i^{r \to s}\, A \, : A(r/i) \to A(s/i)$$

is an equivalence.

We next add a universe U, that is closed under all the type formers (rules omitted), by the rules in Figure 1.

$$\frac{}{\Gamma \vdash \mathsf{U}} \qquad\qquad \frac{\Gamma \vdash A}{\Gamma \vdash A : \mathsf{U}}$$

FIGURE 1. Inference rules for U

Coercion in the universe is the identity function:

$$\mathsf{coe}_i^{r \to s}\, \mathsf{U}\, u = u$$

Homogeneous composition for the universe is slightly more complicated and defined later.

5.1. **V-types.** The rules for V-types are given in Figure 2.

$$\frac{\Gamma \vdash r : \mathbb{I} \qquad \Gamma, (r = 0) \vdash A \qquad \Gamma \vdash B \qquad \Gamma, (r = 0) \vdash e : \mathsf{Equiv}\, A\, B}{\Gamma \vdash \mathsf{V}_r(A, B, e)[(r = 0) \mapsto A, (r = 1) \mapsto B]}$$

$$\frac{\begin{array}{c}\Gamma \vdash r : \mathbb{I} \qquad \Gamma, (r = 0) \vdash u : A \\ \Gamma \vdash v : B[(r = 0) \mapsto e\ u] \qquad \Gamma, (r = 0) \vdash e : \mathsf{Equiv}\, A\, B\end{array}}{\Gamma \vdash \mathsf{Vin}_r\, u\, v : \mathsf{V}_r(A, B, e)[(r = 0) \mapsto u, (r = 1) \mapsto v]}$$

$$\frac{\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash u : \mathsf{V}_r(A, B, e)}{\Gamma \vdash \mathsf{Vproj}_r\, u\, e : B[(r = 0) \mapsto e\ u, (r = 1) \mapsto u]}$$

$$\frac{\Gamma \vdash u : \mathsf{V}_r(A, B, e)}{\Gamma \vdash u = \mathsf{Vin}_r\, u\, (\mathsf{Vproj}_r\, u\, e) : \mathsf{V}_r(A, B, e)}$$

$$\frac{\begin{array}{c}\Gamma \vdash r : \mathbb{I} \qquad \Gamma, (r = 0) \vdash u : A \\ \Gamma \vdash v : B[(r = 0) \mapsto e\ u] \qquad \Gamma, (r = 0) \vdash e : \mathsf{Equiv}\, A\, B\end{array}}{\Gamma \vdash \mathsf{Vproj}_r\, (\mathsf{Vin}_r\, u\, v)\, e = v : B}$$

FIGURE 2. Inference rules for V-types

In the case when r is a dimension variable $i : \mathbb{I}$ the V-type $\mathsf{V}_i(A, B, e)$ can be drawn as the dashed line in:

$$\begin{tikzcd}
A \arrow[d, "e"', "\wr"] \arrow[dr, dashed, "\mathsf{V}_i(A,B,e)"] & \\
B(0/i) \arrow[r, "B"'] & B(1/i)
\end{tikzcd}$$

Semantically V-types correspond to the following special case of Glue-types:

$$\begin{tikzcd}
A \arrow[rr, dashed] \arrow[dr, "e", "\sim"'] \arrow[ddr, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathsf{V}_i(A,B,e) \arrow[dr, dashed, "\mathsf{Vproj}_i"] \arrow[ddr, dashed, two heads] & \\
 & B \arrow[rr] \arrow[d, two heads] \arrow[drr, phantom, "\lrcorner", very near start] & & B \arrow[d, two heads] \\
 & \Gamma, (i = 0) \arrow[rr, tail] & & \Gamma
\end{tikzcd}$$

We now define coercion and homogeneous composition for V-types.

5.1.1. *Homogeneous composition for V-types.* We want to define

$$\Gamma \vdash w := \mathsf{hcom}_j^{r \to s}\, (\mathsf{V}_i(A, B, e))\, [\varphi \mapsto u]\, u_0 : \mathsf{V}_i(A, B, e)[\varphi \mapsto u(s/j), (r = s) \mapsto u_0]$$

Satisfying

$$\begin{aligned}
w(0/i) &= \mathsf{hcom}_j^{r \to s}\, A\, [\varphi \mapsto u]\, u_0 \\
w(1/i) &= \mathsf{hcom}_j^{r \to s}\, B\, [\varphi \mapsto u]\, u_0
\end{aligned}$$

Let

$$\begin{aligned}
\Gamma, (i = 0), j : \mathbb{I} &\vdash h_A := \mathsf{hcom}_j^{r \to j}\, A\, [\varphi \mapsto u]\, u_0 : A \\
\Gamma, (i = 1), j : \mathbb{I} &\vdash h_B := \mathsf{hcom}_j^{r \to j}\, B\, [\varphi \mapsto u]\, u_0 : B
\end{aligned}$$

and

$$\Gamma \vdash v := \mathsf{hcom}_j^{r \to s}\, B\, [\varphi \mapsto \mathsf{Vproj}_i\, u\, e, (i = 0) \mapsto e\ h_A, (i = 1) \mapsto h_B]\, (\mathsf{Vproj}_i\, u_0\, e)$$

We then take

$$w := \mathsf{Vin}_i\, (h_A(s/j))\, v$$

5.1.2. *Coercion for V-types.* Given $\Gamma, (j = 0) \vdash A$, $\Gamma \vdash B$ and $\Gamma, (j = 0) \vdash e : \mathsf{Equiv}\, A\, B$ we write V_j for $\mathsf{V}_j(A, B, e)$. To define coercion in V_j we are given $\Gamma, i : \mathbb{I} \vdash V_j$, $\Gamma \vdash r, s : \mathbb{I}$ and $\Gamma \vdash u : V_j(r/i)$ and the goal is to define

$$\Gamma \vdash w := \mathsf{coe}_i^{r \to s}\, V_j\, u : V_j(s/i)[(r = s) \mapsto u]$$

We do this by cases on whether $i = j$.

Case $i \neq j$: We let

$$\begin{aligned}
\Gamma, (j = 0) &\vdash u_0 := \mathsf{coe}_i^{r \to s}\, A\, u : A(s/i) \\
\Gamma, (j = 0), i : \mathbb{I} &\vdash v_0 := e\ (\mathsf{coe}_i^{r \to i}\, A\, u) : B[(i = r) \mapsto e\ u, (i = s) \mapsto e\ (\mathsf{coe}_i^{r \to s}\, A\, u)] \\
\Gamma, (j = 1), i : \mathbb{I} &\vdash v_1 := \mathsf{coe}_i^{r \to i}\, B\, u : B[(i = r) \mapsto u, (i = s) \mapsto \mathsf{coe}_i^{r \to s}\, B\, u] \\
\Gamma &\vdash v_2 := \mathsf{Vproj}_j\, u\, e(r/i) : B[(j = 0) \mapsto e(r/i)\ u, (j = 1) \mapsto u]
\end{aligned}$$

so that we can take

$$w := \mathsf{Vin}_j\, u_0\, (\mathsf{com}_i^{r \to s}\, B\, [(j = 0) \mapsto v_0, (j = 1) \mapsto v_1]\, v_2)$$

Case $i = j$: [4] We are given $\Gamma \vdash u : V_r$ and want to define:

$$\Gamma \vdash w := \mathsf{coe}_i^{r \to s}\, V_i\, u : V_s[(r = s) \mapsto u]$$

For this let $u' := \mathsf{Vproj}_r\, u\, e(r/i) : B(r/i)[(r = 0) \mapsto e(r/i)\, u, (r = 1) \mapsto u]$. We then coerce this over to $B(s/i)$:

$$\Gamma \vdash\ P := \mathsf{coe}_i^{r \to s}\, B\, u' : B(s/i)$$

[Anders: Maybe we can get uabeta definitionally if we take $P := \mathsf{coe}_i^{r \to k}\, V_i\, u$?]

We write $F := \mathsf{Fiber}\, e(s/i)\, P$ and extract the proof that F is contractible when $s = 0$:

$$\Gamma, (s = 0) \vdash (C_1, C_2) := e(s/i).2\ P : \mathsf{IsContr}\, F$$

so that $C_1 : F$ is the center of contraction and $C_2 : (x : F) \to \mathsf{Path}\ F\ x\ C_1$.

Note that when $s = 0$ and $r = 0$ we have $P = e(s/i)\, u$ so that $(u, \langle \_ \rangle\, P) : F$. We can then extend $C_2\, (u, \langle \_ \rangle\, P)$ to all of r by:

$$\begin{aligned}
\Gamma, (s = 0) \vdash R := \mathsf{hcom}_k^{1 \to 0}\, F\, &[(r = 0) \mapsto C_2\, (u, \langle _ \rangle\, P)\, k]\, C_1 \\
&: F[(r = 0) \mapsto (u, \langle _ \rangle\, P)]
\end{aligned}$$

[Anders: It is not necessary to include (r=s) as when r=s under s=0 then r=0 and the hcom will simplify]

[Anders: If one unfolds this hcom in F then I believe some optimizations can be made]

The first component of this is what we want to output as the first part of a Vin, however the second component is not exactly what we want as it is only defined when $s = 0$, so we need to do a similar extension of this part to all of s.

$$\begin{aligned}
\Gamma \vdash S := \mathsf{hcom}_k^{1 \to 0}\, B(s/i)\, &[(s = 0) \mapsto R.2\, k, (s = 1) \mapsto P \\
&, (r = s) \mapsto \mathsf{Vproj}_s\, u\, e(s/i)]\, P : B(s/i)
\end{aligned}$$

So that we can return:

$$w := \mathsf{Vin}_s\, R.1\, S$$

Note that we are using a diagonal constraint in S to force this coercion to be the identity function when $r = s$. This use of diagonal constraints seem crucial to define coercion for V-types.

[4] This definition differs substantially from what is done in both [AFH17] and RedPRL. In particular we don't case analysis on what r is, but rather define $\mathsf{coe}_i^{r \to s}\, V_i\, u$ directly.

Remark 7. We could avoid the case distinction in coe by using the $\forall i$ operation. This might be desirable as case splitting on names cannot be done in the internal language of the presheaf topos. Combining the two cases using a $\forall i$ might also be desirable as it makes the definition more compact.

5.2. **fcom-types.** For homogeneous composition in the universe we introduce neutral homogeneous compositions of types as constructors in Fig. 3. We then let

$$\frac{\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \qquad \Gamma, \varphi, i : \mathbb{I} \vdash B \qquad \Gamma \vdash A[\varphi \mapsto B(r/i)]}{\Gamma \vdash \mathsf{fcom}_i^{r \to s}\,[\varphi \mapsto B]\,A\,[(r = s) \mapsto A, \varphi \mapsto B(s/i)]}$$

$$\frac{\begin{array}{c}\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \qquad \Gamma, \varphi, i : \mathbb{I} \vdash B \\ \Gamma \vdash A[\varphi \mapsto B(r/i)] \qquad \Gamma, \varphi \vdash u : B(s/i) \qquad \Gamma \vdash u_0 : A[\varphi \mapsto \mathsf{coe}_i^{s \to r}\,B\,u]\end{array}}{\Gamma \vdash \mathsf{box}^{r \to s}\,[\varphi \mapsto u]\,u_0 : \mathsf{fcom}_i^{r \to s}\,[\varphi \mapsto B]\,A[(r = s) \mapsto u_0, \varphi \mapsto u]}$$

$$\frac{\begin{array}{c}\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \\ \Gamma, \varphi, i : \mathbb{I} \vdash B \qquad \Gamma \vdash A[\varphi \mapsto B(r/i)] \qquad \Gamma \vdash u : \mathsf{fcom}_i^{r \to s}\,[\varphi \mapsto B]\,A\end{array}}{\Gamma \vdash \mathsf{cap}_i^{r \leftarrow s}\,[\varphi \mapsto B]\,u : A[(r = s) \mapsto u, \varphi \mapsto \mathsf{coe}_i^{s \to r}\,B\,u]}$$

$$\frac{\begin{array}{c}\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \qquad \Gamma, \varphi, i : \mathbb{I} \vdash B \\ \Gamma \vdash A[\varphi \mapsto B(r/i)] \qquad \Gamma, \varphi \vdash u : B(s/i) \qquad \Gamma \vdash u_0 : A[\varphi \mapsto \mathsf{coe}_i^{s \to r}\,B\,u]\end{array}}{\Gamma \vdash \mathsf{cap}_i^{r \leftarrow s}\,[\varphi \mapsto B]\,(\mathsf{box}^{r \to s}\,[\varphi \mapsto u]\,u_0) = u_0 : A}$$

$$\frac{\begin{array}{c}\Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \\ \Gamma, \varphi, i : \mathbb{I} \vdash B \qquad \Gamma \vdash A[\varphi \mapsto B(r/i)] \qquad \Gamma \vdash u : \mathsf{fcom}_i^{r \to s}\,[\varphi \mapsto B]\,A\end{array}}{\Gamma \vdash \mathsf{box}^{r \to s}\,[\varphi \mapsto u]\,(\mathsf{cap}_i^{r \leftarrow s}\,[\varphi \mapsto B]\,u) = u : \mathsf{fcom}_i^{r \to s}\,[\varphi \mapsto B]\,A}$$

Figure 3. Inference rules for fcom

$$\mathsf{hcom}_i^{r \to s}\,\mathsf{U}\,[\varphi \mapsto B]\,A = \mathsf{fcom}_i^{r \to s}\,[\varphi \mapsto B]\,A$$

As we have added new type formers to the system we now have to explain how to do homogeneous composition and coercion in these.

5.2.1. *Homogeneous composition in fcom.* Homogeneous composition in $\mathsf{fcom}_i^{r \to s}\,[\varphi \mapsto B]\,A$ is quite involved, but does not involve the $\forall i$ operation.

We omit the ambient context Γ and are given:

$$\vdash \varphi : \mathbb{F} \qquad \vdash s, s' : \mathbb{I} \qquad \varphi, j : \mathbb{I} \vdash B \qquad \vdash A[\varphi \mapsto B(s/j)]$$

so that

$$\vdash F := \mathsf{fcom}_j^{s \to s'}\,[\varphi \mapsto B]\,A$$

is a well-formed fcom type such that:

$$\varphi \vdash F = B(s'/j)$$
$$(s = s') \vdash F = A$$

Given

$$\vdash r, r' : \mathbb{I} \qquad \vdash \psi : \mathbb{F} \qquad \psi, i : \mathbb{I} \vdash u : F \qquad \vdash u_0 : F[\psi \mapsto u(r/i)]$$

the goal is to define

$$\vdash w := \mathsf{hcom}_i^{r \to r'}\, F\, [\psi \mapsto u]\, u_0 : F$$

such that

$$\begin{aligned}
\varphi &\vdash w = \mathsf{hcom}_i^{r \to r'}\, B(s'/j)\, [\psi \mapsto u]\, u_0 : B(s'/j) \\
(s = s') &\vdash w = \mathsf{hcom}_i^{r \to r'}\, A\, [\psi \mapsto u]\, u_0 : A \\
\psi &\vdash w = u(r'/i) : F \\
(r = r') &\vdash w = u_0 : F
\end{aligned}$$

Step 1. We first compute the cap of the input:

$$\begin{aligned}
\psi, i : \mathbb{I} &\vdash a := \mathsf{cap}_j^{s \leftarrow s'}\, [\varphi \mapsto B]\, u : A[(s = s') \mapsto u, \varphi \mapsto \mathsf{coe}_j^{s' \to s}\, B\, u] \\
&\vdash a_0 := \mathsf{cap}_j^{s \leftarrow s'}\, [\varphi \mapsto B]\, u_0 : A[(s = s') \mapsto u_0, \varphi \mapsto \mathsf{coe}_j^{s' \to s}\, B\, u_0]
\end{aligned}$$

Note that $\psi \vdash u_0 = u(r/i)$ entails that:

$$\psi \vdash a_0 = \mathsf{cap}_j^{s \leftarrow s'}\, [\varphi \mapsto B]\, u(r/i) = a(r/i)$$

Step 2. We next write:

$$\varphi, i : \mathbb{I} \vdash \tilde{u} := \mathsf{hcom}_i^{r \to i}\, B(s'/j)\, [\psi \mapsto u]\, u_0$$

so that $\tilde{u}(r/i) = u_0$ and $u_1 := \tilde{u}(r'/i) = \mathsf{hcom}_i^{r \to r'}\, B(s'/j)\, [\psi \mapsto u]\, u_0$. This is well-typed as $\varphi \vdash F = B(s'/j)$.

Step 3. We can then compose in A keeping track of the appropriate boundaries:

$$\begin{aligned}
\vdash a_1 := \mathsf{hcom}_i^{r \to r'}\, A\, [\, &\psi \mapsto a \\
&, \varphi \mapsto \mathsf{coe}_j^{s' \to s}\, B\, \tilde{u} \\
&, (s = s') \mapsto \mathsf{hcom}_i^{r \to i}\, A\, [\psi \mapsto u]\, u_0]\, a_0 : A
\end{aligned}$$

so that

$$\begin{aligned}
(r = r') &\vdash a_1 = a_0 \\
\psi &\vdash a_1 = a(r'/i) \\
\varphi &\vdash a_1 = \mathsf{coe}_j^{s' \to s}\, B\, u_1 \\
(s = s') &\vdash a_1 = \mathsf{hcom}_i^{r \to r'}\, A\, [\varphi \mapsto u]\, u_0
\end{aligned}$$

This composition is well-formed as $\psi \vdash a(r/i) = a_0$ by the remark in Step 1, and $\varphi \vdash \mathsf{coe}_j^{s' \to s}\, B\, u_0 = a_0$ by the boundary of a_0. Furthermore, when $(s = s')$ we have $a_0 = u_0$, and $F = A$ so that the final face of the system is well-formed.

Step 4. We now package everything up:

$$\vdash w := \mathsf{box}^{s \to s'} \, [\varphi \mapsto u_1] \, a_1 : F$$

This is well-formed as $\varphi \vdash a_1 = \mathsf{coe}_j^{s' \to s} \, B \, u_1$. Furthermore this satisfies:

$$\varphi \vdash w = u_1 = \mathsf{hcom}_i^{r \to r'} \, B(s'/j) \, [\psi \mapsto u] \, u_0 : B(s'/j)$$

$$(s = s') \vdash w = a_1 = \mathsf{hcom}_i^{r \to r'} \, A \, [\psi \mapsto u] \, u_0 : A$$

$$\begin{aligned}
\psi \vdash w &= \mathsf{box}^{s \to s'} \, [\varphi \mapsto u_1] \, a_1 \\
&= \mathsf{box}^{s \to s'} \, [\varphi \mapsto \tilde{u}(r'/i)] \, a(r'/i) \\
&= \mathsf{box}^{s \to s'} \, [\varphi \mapsto \tilde{u}(r'/i)] \, a(r'/i) \\
&= \mathsf{box}^{s \to s'} \, [\varphi \mapsto \mathsf{hcom}_i^{r \to r'} \, B(s'/j) \, [\psi \mapsto u] \, u_0] \, ((\mathsf{cap}_j^{s \leftarrow s'} \, [\varphi \mapsto B] \, u)(r'/i)) \\
&= \mathsf{box}^{s \to s'} \, [\varphi \mapsto u(r'/i)] \, (\mathsf{cap}_j^{s \leftarrow s'} \, [\varphi \mapsto B] \, u(r'/i)) \\
&= u(r'/i) : F
\end{aligned}$$

$$\begin{aligned}
(r = r') \vdash w &= \mathsf{box}^{s \to s'} \, [\varphi \mapsto u_1] \, a_1 \\
&= \mathsf{box}^{s \to s'} \, [\varphi \mapsto u_0] \, a_0 \\
&= \mathsf{box}^{s \to s'} \, [\varphi \mapsto u_0] \, (\mathsf{cap}_j^{s \leftarrow s'} \, [\varphi \mapsto B] \, u_0) \\
&= u_0 : F
\end{aligned}$$

as desired.

5.2.2. *Coercion in fcom.* The goal of this section is to explain how to compute:

$$\mathsf{coe}_i^{r \to r'} \, (\mathsf{fcom}_j^{s \to s'} \, [\varphi \mapsto B] \, A) \, u_0 : (\mathsf{fcom}_j^{s \to s'} \, [\varphi \mapsto B] \, A)(r'/i)$$

We omit the ambient context Γ and are given:

$$i : \mathbb{I} \vdash \varphi : \mathbb{F} \qquad i : \mathbb{I} \vdash s, s' : \mathbb{I} \qquad i : \mathbb{I}, \varphi, j : \mathbb{I} \vdash B \qquad i : \mathbb{I} \vdash A[\varphi \mapsto B(s/j)]$$

so that

$$i : \mathbb{I} \vdash F := \mathsf{fcom}_j^{s \to s'} \, [\varphi \mapsto B] \, A$$

is a well-formed fcom type such that:

$$i : \mathbb{I}, \varphi \vdash F = B(s'/j)$$
$$i : \mathbb{I}, (s = s') \vdash F = A$$

Given $\vdash r, r' : \mathbb{I}$ and $\vdash u_0 : F(r/i)$ the goal is to define

$$\vdash w := \mathsf{coe}_i^{r \to r'} \, F \, u_0 : F(r'/i)$$

such that

$$\forall i.\varphi \vdash w = \mathsf{coe}_i^{r \to r'} \, B(s'/j) \, u_0 : B(s'/j)(r'/i)$$
$$(r = r') \vdash w = u_0 : F(r/i)$$
$$\forall i.(s = s') \vdash w = \mathsf{coe}_i^{r \to r'} \, A \, u_0 : A(r'/i)$$

Step 1. Compute the cap of u_0:

$$\begin{aligned}\vdash a_0 := \mathsf{cap}_j^{s(r/i)\leftarrow s'(r/i)}\, [\varphi(r/i) \mapsto B(r/i)]\, u_0 \\ : A(r/i)[\,(s(r/i) = s'(r/i)) \mapsto u_0 \\ , \varphi(r/i) \mapsto \mathsf{coe}_j^{s'(r/i)\to s(r/i)}\, B(r/i)\, u_0]\end{aligned}$$

Note that if i does not occur in s and s' then the first restriction reads like $(s = s') \mapsto u_0$.

Step 2. Note that $\varphi \vdash u_0 : B(s'/j)(r/i)$ and we can coerce in $B(s'/j)$ to get a line along i:

$$\forall i.\varphi, i : \mathbb{I} \vdash \tilde{b} := \mathsf{coe}_i^{r\to i}\, B(s'/j)\, u_0 : B(s'/j)$$

so that

$$\begin{aligned}\forall i.\varphi &\vdash \tilde{b}(r/i) = u_0 : B(s'/j)(r/i) \\ \forall i.\varphi &\vdash b_1 := \tilde{b}(r'/i) = \mathsf{coe}_i^{r\to r'}\, B(s'/j)\, u_0 : B(s'/j)(r'/i)\end{aligned}$$

Note that $(r = r') \vdash b_1 = u_0$.

Step 3. We now compute a composition of a_0 with the tubes given by $\tilde{b}$ and a suitable correction:

$$\vdash a_1 := \mathsf{com}_i^{r\to r'}\, A\, [\forall i.\varphi \mapsto \mathsf{coe}_j^{s'\to s}\, B\, \tilde{b}, \forall i.(s = s') \mapsto \mathsf{coe}_i^{r\to i}\, A\, u_0]\, a_0 : A(r'/i)$$

which is well-formed as $\forall i.\varphi \leqslant \varphi(r/i)$ so that

$$\forall i.\varphi \vdash (\mathsf{coe}_j^{s'\to s}\, B\, \tilde{b})(r/i) = \mathsf{coe}_j^{s'(r/i)\to s(r/i)}\, B(r/i)\, u_0 = a_0$$

and

$$\forall i.(s = s') \vdash a_1 = \mathsf{coe}_i^{r\to r}\, A\, u_0 = u_0 = a_0$$

by the remark at the end of Step 1.

Furthermore this satisfies:

$$\begin{aligned}\forall i.\varphi &\vdash a_1 = \mathsf{coe}_j^{s'(r'/i)\to s(r'/i)}\, B(r'/i)\, (\mathsf{coe}_i^{r\to r'}\, B(s'/j)\, u_0) : B(r'/i)(s(r'/i)/j) \\ (r = r') &\vdash a_1 = a_0 : A(r/i) \\ \forall i.(s = s') &\vdash a_1 = \mathsf{coe}_i^{r\to r'}\, A\, u_0 : A(r'/i)\end{aligned}$$

where the first equation makes sense as $\varphi(r'/i) \vdash A(r'/i) = B(s/j)(r'/i) = B(r'/i)(s(r'/i)/j)$.

Step 4. Note that we have an element in the fiber of

$$e := \mathsf{coe}_j^{s'(r'/i)\to s(r'/i)}\, B(r'/i)\ : B(r'/i)(s'(r'/i)/j) \to B(r'/i)(s(r'/i)/j)$$

over a_1 given by:

$$\varphi(r'/i), \forall i.\varphi \vdash (b_1, \langle \_ \rangle\, a_1) : \mathsf{Fiber}\, e\, a_1$$

where

$$\mathsf{Fiber}\, e\, a_1 := (x : B(r'/i)(s'(r'/i)/j)) \times \mathsf{Path}\ B(r'/i)(s(r'/i)/j)\ (e\ x)\ a_1$$

which makes sense by the remark in the end of Step 3 above.

The fact that the pair $(b_1, \langle \_ \rangle\, a_1)$ has type $\mathsf{Fiber}\, e\, a_1$ on this restriction is justified by:

$$\varphi(r'/i), \forall i.\varphi \vdash e\, b_1 = \mathsf{coe}_j^{s'(r'/i)\to s(r'/i)}\, B(r'/i)\, (\mathsf{coe}_i^{r\to r'}\, B(s'/j)\, u_0) = a_1$$

Furthermore, we have:

$$\varphi(r'/i), (r = r') \vdash e\, b_1 = \mathsf{coe}_j^{s'(r/i) \to s(r/i)}\, B(r/i)\, u_0 = a_0 = a_1$$

The second equality holds as $\varphi(r/i) \vdash \mathsf{coe}_j^{s'(r/i) \to s(r/i)}\, B(r/i)\, u_0 = a_0$, and the other equalities hold by the remarks about what happens on $(r = r')$ in Step 2 and 3. We hence get $(b_1, \langle \_ \rangle\, a_1) : \mathsf{Fiber}\, e\, a_1$ on $\varphi(r'/i) \wedge (r = r')$.

Step 5. By Lemma 6 we get $e' : \mathsf{IsEquiv}\, e$ so that we can compute:

$$\varphi(r'/i) \vdash (C_1, C_2) := e'\, a_1$$

Here $C_1 : \mathsf{Fiber}\, e\, a_1$ is the center of contraction and $C_2 : (x : \mathsf{Fiber}\, e\, a_1) \to \mathsf{Path}\ (\mathsf{Fiber}\, e\, a_1)\ x\ C_1$. We use this to extend $(b_1, \langle \_ \rangle\, a_1)$ to all of $\varphi(r'/i)$:

$$\varphi(r'/i) \vdash R := \mathsf{hcom}_k^{1 \to 0}\, (\mathsf{Fiber}\, e\, a_1)\, [\, \forall i.\varphi \vee (r = r') \mapsto C_2\, (b_1, \langle \_ \rangle\, a_1)\, k]\, C_1$$

This is well-formed by the remarks in Step 2 and 3 about $(r = r')$.

Step 6. We can then use the second component of the equivalence to extend this to a total element in $A(r'/i)$:

$$\begin{aligned} \vdash a_1' := \mathsf{hcom}_k^{1 \to 0}\, A(r'/i)\, [\, & \varphi(r'/i) \mapsto R.2\, k \\ , & (r = r') \mapsto a_0 \\ , & \forall i.(s = s') \mapsto a_1]\, a_1 : A(r'/i) \end{aligned}$$

Note that on $\varphi(r'/i)$ we have $R.2\, 1 = a_1$ (as $R : \mathsf{Fiber}\, e\, a_1$) and on $(r = r')$ we have $a_1 = a_0$, making the homogeneous composition well-formed.

Step 7. Finally we combine all the parts we have computed:

$$\vdash w := \mathsf{box}^{s(r'/i) \to s'(r'/i)}\, [\varphi(r'/i) \mapsto R.1]\, a_1' : F(r'/i)$$

To see that w is well-formed note that:

$$\varphi(r'/i) \vdash a_1' = R.2\, 0 = e\, R.1 = \mathsf{coe}_j^{s'(r'/i) \to s(r'/i)}\, B(r'/i)\, R.1$$

where the second equality holds as $R : \mathsf{Fiber}\, e\, a_1$.

To see that w satisfies the three necessary properties note that $\forall i.\varphi \leqslant \varphi(r'/i)$ so that

$$\forall i.\varphi \vdash w = R.1 = (C_2\, (b_1, \langle \_ \rangle\, a_1)\, 0).1 = b_1 = \mathsf{coe}_i^{r \to r'}\, B(s'/j)\, u_0$$

as desired. Furthermore,

$$\begin{aligned} \varphi(r'/i), (r = r') \vdash R.1 &= (C_2\, (b_1, \langle \_ \rangle\, a_1)\, 0).1 = b_1 = u_0 \\ (r = r') \vdash a_1' &= a_0 = \mathsf{cap}_j^{s(r/i) \leftarrow s'(r/i)}\, [\varphi(r/i) \mapsto B(r/i)]\, u_0 \end{aligned}$$

so that

$$\begin{aligned} (r = r') \vdash w = {} & \mathsf{box}^{s(r/i) \to s'(r/i)}\, [\varphi(r/i) \mapsto u_0] \\ & \quad (\mathsf{cap}_j^{s(r/i) \leftarrow s'(r/i)}\, [\varphi(r/i) \mapsto B(r/i)]\, u_0) \\ = {} & u_0 \end{aligned}$$

by the η-rule for box and cap.

Finally, we have:

$$
\begin{aligned}
\forall i.(s = s') \vdash w &= \mathsf{box}^{s(r'/i)\to s'(r'/i)}\,[\varphi(r/i) \mapsto R.1]\,a'_1 \\
&= a'_1 \\
&= a_1 \\
&= \mathsf{coe}_i^{r\to r'}\,A\,u_0
\end{aligned}
$$

5.3. **Glue types.** The typing rules for Glue types are given in Fig. 4. Note that these are similar to $\mathsf{fcom}^{1\mapsto 0}$.

$$\frac{\Gamma \vdash A \qquad \Gamma,\varphi \vdash T \qquad \Gamma,\varphi \vdash e : \mathsf{Equiv}\;\,T\;\,A}{\Gamma \vdash \mathsf{Glue}\,[\varphi \mapsto (T,e)]\,A}$$

$$\frac{\Gamma \vdash b : \mathsf{Glue}\,[\varphi \mapsto (T,e)]\,A}{\Gamma \vdash \mathsf{unglue}\,[\varphi \mapsto e]\,b : A[\varphi \mapsto e\;b]}$$

$$\frac{\Gamma,\varphi \vdash e : \mathsf{Equiv}\,T\,A \qquad \Gamma,\varphi \vdash t : T \qquad \Gamma \vdash a : A[\varphi \mapsto e\;t]}{\Gamma \vdash \mathsf{glue}\,[\varphi \mapsto t]\,a : \mathsf{Glue}\,[\varphi \mapsto (T,e)]\,A}$$

$$\frac{\Gamma \vdash T \qquad \Gamma \vdash e : \mathsf{Equiv}\,T\,A}{\Gamma \vdash \mathsf{Glue}\,[1_{\mathbb{F}} \mapsto (T,e)]\,A = T} \qquad \frac{\Gamma \vdash t : T \qquad \Gamma \vdash e : \mathsf{Equiv}\,T\,A}{\Gamma \vdash \mathsf{glue}\,[1_{\mathbb{F}} \mapsto t]\,(e\;t) = t : T}$$

$$\frac{\Gamma \vdash b : \mathsf{Glue}\,[\varphi \mapsto (T,e)]\,A}{\Gamma \vdash b = \mathsf{glue}\,[\varphi \mapsto b]\,(\mathsf{unglue}\,[\varphi \mapsto e]\,b) : \mathsf{Glue}\,[\varphi \mapsto (T,e)]\,A}$$

$$\frac{\Gamma,\varphi \vdash e : \mathsf{Equiv}\,T\,A \qquad \Gamma,\varphi \vdash t : T \qquad \Gamma \vdash a : A[\varphi \mapsto e\;t]}{\Gamma \vdash \mathsf{unglue}\,[\varphi \mapsto e]\,(\mathsf{glue}\,[\varphi \mapsto t]\,a) = a : A}$$

FIGURE 4. Typing rules for Glue types

5.3.1. *Homogeneous composition for Glue Types.* We omit the ambient context Γ and are given:

$$\vdash A \qquad\qquad \vdash \varphi : \mathbb{F} \qquad\qquad \varphi \vdash T \qquad\qquad \varphi \vdash e : \mathsf{Equiv}\,T\,A$$

so that

$$\vdash G := \mathsf{Glue}\,[\varphi \mapsto (T,e)]\,A$$

is a well-formed Glue type such that:

$$\varphi \vdash G = T$$

Given

$$\vdash r, r' : \mathbb{I} \qquad\qquad \vdash \psi : \mathbb{F} \qquad\qquad \psi, i : \mathbb{I} \vdash u : G \qquad\qquad \vdash u_0 : G[\psi \mapsto u(r/i)]$$

the goal is to define

$$\vdash w := \mathsf{hcom}_i^{r\to r'}\,G\,[\psi \mapsto u]\,u_0 : G$$

such that

$$\begin{aligned}
\varphi &\vdash w = \mathsf{hcom}_i^{r \to r'} T \, [\psi \mapsto u] \, u_0 : T \\
\psi &\vdash w = u(r'/i) : G \\
(r = r') &\vdash w = u_0 : G
\end{aligned}$$

Step 1. We first unglue the input:

$$\begin{aligned}
\psi, i : \mathbb{I} &\vdash a := \mathsf{unglue} \, [\varphi \mapsto e] \, u : A[\varphi \mapsto e \; u] \\
&\vdash a_0 := \mathsf{unglue} \, [\varphi \mapsto e] \, u_0 : A[\varphi \mapsto e \; u_0]
\end{aligned}$$

Note that $\psi \vdash u_0 = u(r/i)$ entails that:

$$\psi \vdash a_0 = \mathsf{unglue} \, [\varphi \mapsto e] \, u(r/i) = a(r/i)$$

Step 2. We then write:

$$\varphi, i : \mathbb{I} \vdash \tilde{u} := \mathsf{hcom}_i^{r \to i} T \, [\psi \mapsto u] \, u_0 : T$$

so that $\tilde{u}(r/i) = u_0$ and $u_1 := \tilde{u}(r'/i) = \mathsf{hcom}_i^{r \to r'} T \, [\psi \mapsto u] \, u_0$. This is well-typed as $\varphi \vdash G = T$.

Step 3. We can then compose homogeneously in A, keeping track of the appropriate boundaries:

$$\vdash a_1 := \mathsf{hcom}_i^{r \to r'} A \, [\psi \mapsto a, \varphi \mapsto e \; \tilde{u}] \, a_0 : A$$

so that

$$\begin{aligned}
(r = r') &\vdash a_1 = a_0 \\
\psi &\vdash a_1 = a(r'/i) \\
\varphi &\vdash a_1 = e \; u_1
\end{aligned}$$

This composition is well-formed as $\psi \vdash a(r/i) = a_0$ by the remark in Step 1 and $\varphi \vdash e \; u_0 = a_0$ by the boundary constraint of a_0.

Step 4. We now package everything up:

$$\vdash w := \mathsf{glue} \, [\varphi \mapsto u_1] \, a_1 : G$$

This is well-formed as $\varphi \vdash a_1 = e\ u_1$. Furthermore this satisfies:

$$\varphi \vdash w = u_1 = \mathsf{hcom}_i^{r\to r'}\, T\, [\psi \mapsto u]\, u_0 : T$$

$$\begin{aligned}
\psi \vdash w &= \mathsf{glue}\, [\varphi \mapsto u_1]\, a_1 \\
&= \mathsf{glue}\, [\varphi \mapsto \tilde{u}(r'/i)]\, a(r'/i) \\
&= \mathsf{glue}\, [\varphi \mapsto \mathsf{hcom}_i^{r\to r'}\, T\, [\psi \mapsto u]\, u_0]\, ((\mathsf{unglue}\, [\varphi \mapsto e]\, u)(r'/i)) \\
&= \mathsf{glue}\, [\varphi \mapsto u(r'/i)]\, (\mathsf{unglue}\, [\varphi \mapsto e]\, u(r'/i)) \\
&= u(r'/i) : G
\end{aligned}$$

$$\begin{aligned}
(r = r') \vdash w &= \mathsf{glue}\, [\varphi \mapsto u_1]\, a_1 \\
&= \mathsf{glue}\, [\varphi \mapsto u_0]\, a_0 \\
&= \mathsf{glue}\, [\varphi \mapsto u_0]\, (\mathsf{unglue}\, [\varphi \mapsto e]\, u_0) \\
&= u_0 : G
\end{aligned}$$

as desired.

5.3.2. *Coercion for Glue types.* We omit the ambient context Γ and are given:

$$i : \mathbb{I} \vdash A \qquad i : \mathbb{I} \vdash \varphi : \mathbb{F} \qquad i : \mathbb{I}, \varphi \vdash T \qquad i : \mathbb{I}, \varphi \vdash e : \mathsf{Equiv}\, T\, A$$

so that

$$i : \mathbb{I} \vdash G := \mathsf{Glue}\, [\varphi \mapsto (T, e)]\, A$$

is a well-formed Glue type such that:

$$i : \mathbb{I}, \varphi \vdash G = T$$

Given $\vdash r, r' : \mathbb{I}$ and $\vdash u_0 : G(r/i)$ the goal is to define

$$\vdash w := \mathsf{coe}_i^{r\to r'}\, G\, u_0 : G(r'/i)$$

such that

$$\begin{aligned}
\forall i.\varphi \vdash w &= \mathsf{coe}_i^{r\to r'}\, T\, u_0 : T(r'/i) \\
(r = r') \vdash w &= u_0 : G(r/i)
\end{aligned}$$

Step 1. First unglue u_0:

$$\vdash a_0 := \mathsf{unglue}\, [\varphi(r/i) \mapsto e(r/i)]\, u_0 : A(r/i)[\varphi(r/i) \mapsto e(r/i)\ u_0]$$

Step 2. Note that $\varphi \vdash u_0 : T(r/i)$ and we can coerce in T to get a line along i:

$$\forall i.\varphi, i : \mathbb{I} \vdash \tilde{b} := \mathsf{coe}_i^{r\to i}\, T\, u_0 : T$$

so that

$$\begin{aligned}
\forall i.\varphi &\vdash \tilde{b}(r/i) = u_0 : T(r/i) \\
\forall i.\varphi &\vdash b_1 := \tilde{b}(r'/i) = \mathsf{coe}_i^{r\to r'}\, T\, u_0 : T(r'/i)
\end{aligned}$$

Note that $(r = r') \vdash b_1 = u_0$.

Step 3. We now compute a composition of a_0 with the tubes given by $e\ \tilde{b}$:

$$\vdash a_1 := \mathsf{com}_i^{r \to r'}\, A\, [\forall i.\varphi \mapsto e\ \tilde{b}]\, a_0 : A(r'/i)$$

which is well-formed as $\forall i.\varphi \leqslant \varphi(r/i)$ so that

$$\forall i.\varphi \vdash (e\ \tilde{b})(r/i) = e(r/i)\ u_0 = a_0$$

Furthermore this satisfies:

$$\forall i.\varphi \vdash a_1 = e(r'/i)\ (\mathsf{coe}_i^{r \to r'}\, T\, u_0) : A(r'/i)$$
$$(r = r') \vdash a_1 = a_0 : A(r/i)$$

Step 4. Note that we have an element in the fiber of $e(r'/i) : T(r'/i) \to A(r'/i)$ over a_1 given by:

$$\varphi(r'/i), \forall i.\varphi \vdash (b_1, \langle \_ \rangle\, a_1) : \mathsf{Fiber}\, e(r'/i)\, a_1$$

where

$$\mathsf{Fiber}\, e(r'/i)\, a_1 := (x : T(r'/i)) \times \mathsf{Path}\ A(r'/i)\ (e(r'/i)\ x)\ a_1$$

The fact that the pair $(b_1, \langle \_ \rangle\, a_1)$ has type $\mathsf{Fiber}\, e(r'/i)\, a_1$ on this restriction is justified by:

$$\varphi(r'/i), \forall i.\varphi \vdash e(r'/i)\, b_1 = e(r'/i)(\mathsf{coe}_i^{r \to r'}\, T\, u_0) = a_1$$

Furthermore, we have:

$$\varphi(r'/i), (r = r') \vdash e(r'/i)\, b_1 = e(r'/i)\ u_0 = a_0 = a_1$$

We hence also get $\varphi(r'/i), (r = r') \vdash (b_1, \langle \_ \rangle\, a_1) : \mathsf{Fiber}\, e(r'/i)\, a_1$.

Step 5. We can then use that e is an equivalence to compute:

$$\varphi(r'/i) \vdash (C_1, C_2) := e(r'/i).2\, a_1 : \mathsf{IsContr}\, (\mathsf{Fiber}\, e(r'/i)\, a_1)$$

Here $C_1 : \mathsf{Fiber}\, e(r'/i)\, a_1$ is the center of contraction and $C_2 : (x : \mathsf{Fiber}\, e(r'/i)\, a_1) \to \mathsf{Path}\ (\mathsf{Fiber}\, e(r'/i)\, a_1)\ x\ C_1$. We use this to extend $(b_1, \langle \_ \rangle\, a_1)$ to all of $\varphi(r'/i)$:

$$\varphi(r'/i) \vdash R := \mathsf{hcom}_k^{1 \to 0}\, (\mathsf{Fiber}\, e(r'/i)\, a_1)\, [\forall i.\varphi \vee (r = r') \mapsto C_2\, (b_1, \langle \_ \rangle\, a_1)\, k]\, C_1 : \mathsf{Fiber}\, e(r'/i)\, a_1$$

Step 6. We can then use the second component to extend this to a total element in $A(r'/i)$:

$$\vdash a_1' := \mathsf{hcom}_k^{1 \to 0}\, A(r'/i)\, [\varphi(r'/i) \mapsto R.2\, k, (r = r') \mapsto a_0]\, a_1 : A(r'/i)$$

Note that on $\varphi(r'/i)$ we have $R.2\, 1 = a_1$ (as $R : \mathsf{Fiber}\, e(r'/i)\, a_1$) and on $(r = r')$ we have $a_1 = a_0$, making the homogeneous composition well-formed.

Step 7. Finally we combine all the parts we have computed:

$$\vdash w := \mathsf{glue}\,[\varphi(r'/i) \mapsto R.1]\,a_1' : G(r'/i)$$

To see that w is well-formed note that:

$$\varphi(r'/i) \vdash a_1' = R.2\,0 = e(r'/i)\,R.1$$

where the second equality holds as $R : \mathsf{Fiber}\,e(r'/i)\,a_1$.

To see that w satisfies the two necessary properties note that $\forall i.\varphi \leqslant \varphi(r'/i)$ so that

$$\forall i.\varphi \vdash w = R.1 = (C_2\,(b_1, \langle \_ \rangle\,a_1)\,0).1 = b_1 = \mathsf{coe}_i^{r \to r'}\,T\,u_0 : T(r'/i)$$

as desired. Furthermore,

$$\begin{aligned} \varphi(r'/i), (r = r') &\vdash R.1 = (C_2\,(b_1, \langle \_ \rangle\,a_1)\,0).1 = b_1 = u_0 \\ (r = r') &\vdash a_1' = a_0 = \mathsf{unglue}\,[\varphi(r/i) \mapsto e(r/i)]\,u_0 \end{aligned}$$

so that

$$(r = r') \vdash w = \mathsf{glue}\,[\varphi(r/i) \mapsto u_0]\,(\mathsf{unglue}\,[\varphi(r/i) \mapsto e(r/i)]\,u_0) = u_0$$

by the η-rule for glue and unglue.

5.4. Fibrancy of the universes using Glue-types.

5.5. **Univalence.** In this section we present two proofs of the univalence axiom. The first one relies on a by now well-known observation that univalence can be derived from the two constants ua and ua_β. The second proof is similar to the proof in [CCHM18] which is based on the observation that unglue is an equivalence.

We will prove the following formulation of the univalence axiom:

Theorem 8 (Univalence)**.** *For any type $A : \mathsf{U}$ the type $(X : U) \times \mathsf{Equiv}\,X\,A$ is an equivalence.*

[Anders: this would look nicer if we swapped X and A, but that seems inconvenient in the second proof with the current formulation of contractibility.]

Proof 1. Given $A, B : \mathsf{U}$ we can define a constant

$$\mathsf{ua} : \mathsf{Equiv}\,A\,B \to \mathsf{Path}\,\mathsf{U}\,A\,B$$

as $\mathsf{ua} := \lambda(e : \mathsf{Equiv}\,A\,B).\langle i \rangle\,\mathsf{V}_i(A, B, e)$. Given $e : \mathsf{Equiv}\,A\,B$ and $a : A$ we can unfold to algorithm for coercion in V-types to see that

$$\begin{aligned} \mathsf{coe}_i^{0 \to 1}\,(\mathsf{ua}\,e\,i)\,a &= \mathsf{coe}_i^{0 \to 1}\,(\mathsf{V}_i(A, B, e))\,a \\ &= \mathsf{coe}_i^{0 \to 1}\,B\,(e\,a) \end{aligned}$$

We can hence define

$$\mathsf{ua}_\beta : \mathsf{Path}\,B\,(\mathsf{coe}_i^{0 \to 1}\,(\mathsf{ua}\,e\,i)\,a)\,(e\,a)$$

as $\mathsf{ua}_\beta := \langle i \rangle\,\mathsf{coe}_j^{i \to 1}\,B\,(e\,a)$. Where j is a fresh dimension. This hence means that the type $\mathsf{Equiv}\,A\,B$ is a retract of $\mathsf{Path}\,\mathsf{U}\,A\,B$ and the proof of the Theorem then follows the standard argument.[5] □

[5] See e.g. https://groups.google.com/forum/#!topic/homotopytypetheory/j2KBIvDw53s

Remark. Note that the proof of ua_β is slightly more direct here than in cubicaltt. Indeed, the algorithm for composition in Glue-types in cubicaltt gives two trivial coercions (i.e. compositions with empty systems) as opposed to only one with the algorithm for V-types. In [BCH17] the authors introduce another type called "G-types" which allow them to get ua_β without any trivial coercions/compositions, however those types do not seem to be definable in a system with diagonals.

Proof 2. □

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Appendix A. Appendix: old version of $\mathsf{coe}_i^{r\to s}\,V_i\,u$

This section contains a variation of coe for V that is closer to what is in [AFH17]. This definition is by cases on r and hence a lot longer than the one above.

We are using the RedPRL definition of equivalences:

$$\begin{aligned}
\mathsf{Equiv}\,A\,B &:= (e : A \to B) \times \mathsf{IsEquiv}\,e \\
\mathsf{IsEquiv}\,e &:= (x : B) \to \mathsf{IsContr}\,(\mathsf{Fiber}\,e\,x) \\
\mathsf{IsContr}\,C &:= (x : C) \times ((y : C) \to \mathsf{Path}\ C\ y\ x) \\
\mathsf{Fiber}\,e\,x &:= (y : A) \times \mathsf{Path}\ B\ (e\ y)\ x
\end{aligned}$$

If $i = j$, we have $\Gamma, i = 0 \vdash A$, $\Gamma \vdash B$ and $\Gamma, i = 0 \vdash e : \mathsf{Equiv}\,A\,B$ and we write V_i for $\mathsf{V}_i(A, B, e)$. To define coercion in V_i we are given $\Gamma, i : \mathbb{I} \vdash V_i$, $\Gamma \vdash r, s : \mathbb{I}$ and $\Gamma \vdash u : V_r$, and the goal is to define

$$\Gamma \vdash w := \mathsf{coe}_i^{r\to s}\,V_i\,u : V_s[r = s \mapsto u]$$

We do this by cases on r:

Case $r = 0$: We are given $\Gamma \vdash u : A(0/i)$ and we can simply take:

$$w := \mathsf{Vin}_s\,u\,(\mathsf{coe}_i^{0\to s}\,B\,(e(0/i)\ u))$$

Case $r = 1$: We are given $\Gamma \vdash u : B(1/i)$ and begin by coercing this to $B(s/i)$:

$$\Gamma \vdash v := \mathsf{coe}_i^{1\to s}\,B\,u : B(s/i)$$

We can then use the equivalence to compute the center of contraction of the fibers of $e(0/i)$ over the $s = 0$ face of v:

$$\Gamma, s = 0 \vdash (O_1, O_2) := (e(s/i).2\ v).1 : \mathsf{Fiber}\,e(s/i)\,v$$

where $O_1 : A(s/i)$ and $O_2 : \mathsf{Path}\ B(s/i)\,(e(s/i)\,O_1)\,v$. We would now like to return $\mathsf{Vin}_s\,O_1\,O_2$, but O_2 is only defined when $s = 0$ so we must first extend it to all of s:

$$\begin{aligned}
\Gamma \vdash P := \mathsf{hcom}_i^{1\to 0}\,B(s/i)\,[s = 0 \mapsto O_2\ i, s = 1 \mapsto \langle_\rangle\,u]\,v :& \\
B(s/i)[s = 0 \mapsto e(s/i)\ O_1, s = 1 \mapsto u]&
\end{aligned}$$

We can now take the following:

$$w := \mathsf{Vin}_s\,O_1\,P$$

Case $r = j$:[6] We have $\Gamma \vdash j : \mathbb{I}$ and given $\Gamma \vdash u : V_j$ we want to define:

$$\Gamma \vdash w := \mathsf{coe}_i^{j\to s}\,V_i\,u : V_s[j = s \mapsto u]$$

For this let $u' := \mathsf{Vproj}_j\,u\,e(j/i) : B(j/i)[j = 0 \mapsto e(j/i)\,u, j = 1 \mapsto u]$. We then compose this over to $B(s/i)$:

$$\Gamma \vdash\ P := \mathsf{com}_i^{j\to s}\,B\,[j = 0 \mapsto \mathsf{coe}_i^{0\to i}\,B\,(e(0/i)\ u), j = 1 \mapsto \mathsf{coe}_i^{1\to i}\,B\,u]\,u'$$

Note that when $s = 0$ we have that P connects $e(0/i)\ u$ to $\mathsf{coe}_i^{1\to 0}\,B\,u$ along j.

We then extract the proof that the fibers of $e(0/i)$ over P are contractible on the $s = 0$ face:

$$\Gamma, s = 0 \vdash (C_1, C_2) := e(0/i).2\ P : \mathsf{IsContr}\,(\mathsf{Fiber}\,e(0/i)\,P)$$

[6] As $i = j$ we forget about the old j and write j for r.

so that $C_1 : \mathsf{Fiber}\, e(0/i)\, P$ and

$$C_2 : (x : \mathsf{Fiber}\, e(0/i)\, P) \to \mathsf{Path}\ (\mathsf{Fiber}\, e(0/i)\, P)\ C_1\ x$$

Note that when $s = 0$ we have $P(0/j) = e(0/i)\, u(0/j)$ so that we easily can define another point in the fiber of $e(0/i)$ over P which is defined on all of $\Gamma, s = 0$:

$$\Gamma, s = 0 \vdash Q := \mathsf{coe}_j^{0\to j}\ (\mathsf{Fiber}\, e(0/i)\, P)\ (u(0/j), \langle \_ \rangle\, P(0/j)) : \mathsf{Fiber}\, e(0/i)\, P$$

such that $Q = (u, \langle \_ \rangle\, e(0/i)\ u)$ when $j = 0$. Using C_2 we define:

$$\Gamma, s = 0 \vdash R := C_2\ Q\ j : \mathsf{Fiber}\, e(0/i)\, P[j = 0 \mapsto Q, j = 1 \mapsto C_1]$$

The first component of this is what we want to output as the first part of a Vin, however the second component is not exactly what we want as it is only defined when $s = 0$ so we need to do a similar extension of this part as in the $r = 1$ case above. For this we first need to do the same kind of extension of $C_1.2$ as when $r = 1$, but this time using a filler:

$$\Gamma, j = 1, k : \mathbb{I} \vdash O := \mathsf{hcom}_k^{1\to k}\, B(s/i)\, [s = 0 \mapsto C_1.2\ k, s = 1 \mapsto \langle \_ \rangle\, u]\, (\mathsf{coe}_i^{1\to s}\, B\, u)$$

Finally we can extend $R.2$ to all of s by:

$$\begin{aligned} \Gamma \vdash S := \mathsf{hcom}_k^{1\to 0}\, B(s/i)\, [&j = 0 \mapsto \langle \_ \rangle\, \mathsf{coe}_i^{0\to s}\, B\, (e(0/i)\ u) \\ &, j = 1 \mapsto O \\ &, j = s \mapsto \langle \_ \rangle\, \mathsf{Vproj}_s\, u\, e(s/i) \\ &, s = 0 \mapsto R.2(k/j)]\, P : B(s/i) \end{aligned}$$

So that we can return:

$$w := \mathsf{Vin}_s\, R.1\, S$$

It is now straightforward to check that w agrees with the above cases when $j = 0$ and $j = 1$. Note that we are using a diagonal constraint in S to force this coercion to be the identity function when $j = s$.

Appendix B. Coercion in fcom: special cases

B.1. **Coercion in fcom: special case with $r = s = 0$ and $r' = s' = 1$.** As I'm only considering the $r = s = 0$ and $r' = s' = 1$ case no diagonal constraints are needed. The goal of this section is hence to explain how to compute:

$$\mathsf{coe}_i^{0\to 1}\, (\mathsf{fcom}_j^{0\to 1}\, [\varphi \mapsto B]\, A)\, u_0 : (\mathsf{fcom}_j^{0\to 1}\, [\varphi \mapsto B]\, A)(1/i)$$

We omit the ambient context Γ and are given:

$$i : \mathbb{I} \vdash \varphi : \mathbb{F} \qquad i : \mathbb{I}, \varphi, j : \mathbb{I} \vdash B \qquad i : \mathbb{I} \vdash A[\varphi \mapsto B(0/j)]$$

so that

$$i : \mathbb{I} \vdash F := \mathsf{fcom}_j^{0\to 1}\, [\varphi \mapsto B]\, A$$

is a well-formed fcom type such that:

$$i : \mathbb{I}, \varphi \vdash F = B(1/j)$$

Given $\vdash u_0 : F(0/i)$ the goal is to define

$$\vdash w := \mathsf{coe}_i^{0\to 1}\, F\, u_0 : F(1/i)$$

such that

$$\forall i.\varphi \vdash w = \mathsf{coe}_i^{0\to 1}\, B(1/j)\, u_0 : B(1/j)(1/i)$$

Step 1. Compute the cap of u_0:

$$\vdash a_0 := \mathsf{cap}_j^{0\leftarrow 1}\,[\varphi(0/i) \mapsto B(0/i)]\,u_0 : A(0/i)[\varphi(0/i) \mapsto \mathsf{coe}_j^{1\to 0}\,B(0/i)\,u_0]$$

Step 2. Note that $\varphi \vdash u_0 : B(1/j)(0/i)$ and we can coerce in $B(1/j)$ to get a line along i:

$$\forall i.\varphi, i : \mathbb{I} \vdash \tilde{b} := \mathsf{coe}_i^{0\to i}\,B(1/j)\,u_0 : B(1/j)$$

so that

$$\forall i.\varphi \vdash \tilde{b}(0/i) = u_0 : B(1/j)(0/i)$$
$$\forall i.\varphi \vdash b_1 := \tilde{b}(1/i) = \mathsf{coe}_i^{0\to 1}\,B(1/j)\,u_0 : B(1/j)(1/i)$$

Step 3. We now compute a composition of a_0 with the tubes given by $\tilde{b}$:

$$\vdash a_1 := \mathsf{com}_i^{0\to 1}\,A\,[\forall i.\varphi \mapsto \mathsf{coe}_j^{1\to 0}\,B\,\tilde{b}]\,a_0 : A(1/i)$$

which is well-formed as $\forall i.\varphi \leqslant \varphi(0/i)$ so that

$$\forall i.\varphi \vdash (\mathsf{coe}_j^{1\to 0}\,B\,\tilde{b})(0/i) = \mathsf{coe}_j^{1\to 0}\,B(0/i)\,u_0 = a_0$$

Furthermore this satisfies:

$$\forall i.\varphi \vdash a_1 = \mathsf{coe}_j^{1\to 0}\,B(1/i)\,(\mathsf{coe}_i^{0\to 1}\,B(1/j)\,u_0) : B(1/i)(0/j)$$

which makes sense as $\varphi(1/i) \vdash A(1/i) = B(1/i)(0/j)$.

Step 4. Note that we have an element in the fiber of

$$e := \mathsf{coe}_j^{1\to 0}\,B(1/i)\ : B(1/i)(1/j) \to B(1/i)(0/j)$$

over a_1 given by:

$$\varphi(1/i), \forall i.\varphi \vdash (b_1, \langle\_\rangle\,a_1) : \mathsf{Fiber}\,e\,a_1$$

where

$$\mathsf{Fiber}\,e\,a_1 := (x : B(1/i)(1/j)) \times \mathsf{Path}\ B(1/i)(0/j)\ (e\ x)\ a_1$$

which makes sense by the remark in the end of Step 3 above.

The fact that the pair $(b_1, \langle\_\rangle\,a_1)$ has type $\mathsf{Fiber}\,e\,a_1$ on this restriction is justified by:

$$\varphi(1/i), \forall i.\varphi \vdash e\,b_1 = \mathsf{coe}_j^{1\to 0}\,B(1/i)\,(\mathsf{coe}_i^{0\to 1}\,B(1/j)\,u_0) = a_1$$

Step 5. By Lemma 6 we get $e' : \mathsf{IsEquiv}\,e$ so that we can compute:

$$\varphi(1/i) \vdash (C_1, C_2) := e'\,a_1$$

Here $C_1 : \mathsf{Fiber}\,e\,a_1$ is the center of contraction and $C_2 : (x : \mathsf{Fiber}\,e\,a_1) \to \mathsf{Path}\ (\mathsf{Fiber}\,e\,a_1)\ x\ C_1$. We use this to extend $(b_1, \langle\_\rangle\,a_1)$ to all of $\varphi(1/i)$:

$$\varphi(1/i) \vdash R := \mathsf{hcom}_k^{1\to 0}\,(\mathsf{Fiber}\,e\,a_1)\,[\forall i.\varphi \mapsto C_2\,(b_1, \langle\_\rangle\,a_1)\,k]\,C_1$$

Step 6. We can then use the second component of the equivalence to extend this to a total element in $A(1/i)$:

$$\vdash a_1' := \mathsf{hcom}_k^{1\to 0}\,A(1/i)\,[\varphi(1/i) \mapsto R.2\,k]\,a_1 : A(1/i)$$

Note that on $\varphi(1/i)$ we have $R.2\,1 = a_1$ (as $R : \mathsf{Fiber}\,e\,a_1$) making the homogeneous composition well-formed.

Step 7. Finally we combine all the parts we have computed:

$$\vdash w := \mathsf{box}^{0\to 1}\,[\varphi(1/i) \mapsto R.1]\,a_1' : F(1/i)$$

To see that w is well-formed note that:

$$\varphi(1/i) \vdash a_1' = R.2\,0 = e\,R.1 = \mathsf{coe}_j^{1\to 0}\,B(1/i)\,R.1$$

where the second equality holds as $R : \mathsf{Fiber}\,e\,a_1$.

Finally, note that $\forall i.\varphi \leqslant \varphi(1/i)$, so that

$$\forall i.\varphi \vdash w = R.1 = (C_2\,(b_1, \langle \_ \rangle\,a_1)\,0).1 = b_1 = \mathsf{coe}_i^{0\to 1}\,B(1/j)\,u_0$$

as desired.

B.2. **Coercion in fcom: special case with $s = 0$ and $s' = 1$.** The of this section is to explain how to compute:

$$\mathsf{coe}_i^{r\to r'}\,(\mathsf{fcom}_j^{0\to 1}\,[\varphi \mapsto B]\,A)\,u_0 : (\mathsf{fcom}_j^{0\to 1}\,[\varphi \mapsto B]\,A)(r'/i)$$

We omit the ambient context Γ and are given:

$$i : \mathbb{I} \vdash \varphi : \mathbb{F} \qquad i : \mathbb{I}, \varphi, j : \mathbb{I} \vdash B \qquad i : \mathbb{I} \vdash A[\varphi \mapsto B(0/j)]$$

so that

$$i : \mathbb{I} \vdash F := \mathsf{fcom}_j^{0\to 1}\,[\varphi \mapsto B]\,A$$

is a well-formed fcom type such that:

$$i : \mathbb{I}, \varphi \vdash F = B(1/j)$$

Given $\vdash r, r' : \mathbb{I}$ and $\vdash u_0 : F(r/i)$ the goal is to define

$$\vdash w := \mathsf{coe}_i^{r\to r'}\,F\,u_0 : F(r'/i)$$

such that

$$\forall i.\varphi \vdash w = \mathsf{coe}_i^{r\to r'}\,B(1/j)\,u_0 : B(1/j)(r'/i)$$
$$(r = r') \vdash w = u_0 : F(r/i)$$

Step 1. Compute the cap of u_0:

$$\vdash a_0 := \mathsf{cap}_j^{0\leftarrow 1}\,[\varphi(r/i) \mapsto B(r/i)]\,u_0 : A(r/i)[\varphi(r/i) \mapsto \mathsf{coe}_j^{1\to 0}\,B(r/i)\,u_0]$$

Step 2. Note that $\varphi \vdash u_0 : B(1/j)(r/i)$ and we can coerce in $B(1/j)$ to get a line along i:

$$\forall i.\varphi, i : \mathbb{I} \vdash \tilde{b} := \mathsf{coe}_i^{r\to i}\,B(1/j)\,u_0 : B(1/j)$$

so that

$$\forall i.\varphi \vdash \tilde{b}(r/i) = u_0 : B(1/j)(r/i)$$
$$\forall i.\varphi \vdash b_1 := \tilde{b}(r'/i) = \mathsf{coe}_i^{r\to r'}\,B(1/j)\,u_0 : B(1/j)(r'/i)$$

Note that $(r = r') \vdash b_1 = u_0$.

Step 3. We now compute a composition of a_0 with the tubes given by $\tilde{b}$:

$$\vdash a_1 := \mathsf{com}_i^{r \to r'}\, A\, [\forall i.\varphi \mapsto \mathsf{coe}_j^{1 \to 0}\, B\, \tilde{b}]\, a_0 : A(r'/i)$$

which is well-formed as $\forall i.\varphi \leqslant \varphi(r/i)$ so that

$$\forall i.\varphi \vdash (\mathsf{coe}_j^{1 \to 0}\, B\, \tilde{b})(r/i) = \mathsf{coe}_j^{1 \to 0}\, B(r/i)\, u_0 = a_0$$

Furthermore this satisfies:

$$\begin{aligned} \forall i.\varphi &\vdash a_1 = \mathsf{coe}_j^{1 \to 0}\, B(r'/i)\, (\mathsf{coe}_i^{r \to r'}\, B(1/j)\, u_0) : B(r'/i)(0/j) \\ (r = r') &\vdash a_1 = a_0 : A(r/i) \end{aligned}$$

where the first equation makes sense as $\varphi(r'/i) \vdash A(r'/i) = B(r'/i)(0/j)$.

Step 4. Note that we have an element in the fiber of

$$e := \mathsf{coe}_j^{1 \to 0}\, B(r'/i)\ : B(r'/i)(1/j) \to B(r'/i)(0/j)$$

over a_1 given by:

$$\varphi(r'/i), \forall i.\varphi \vdash (b_1, \langle \_ \rangle\, a_1) : \mathsf{Fiber}\, e\, a_1$$

where

$$\mathsf{Fiber}\, e\, a_1 := (x : B(r'/i)(1/j)) \times \mathsf{Path}\ B(r'/i)(0/j)\ (e\ x)\ a_1$$

which makes sense by the remark in the end of Step 3 above.

The fact that the pair $(b_1, \langle \_ \rangle\, a_1)$ has type $\mathsf{Fiber}\, e\, a_1$ on this restriction is justified by:

$$\varphi(r'/i), \forall i.\varphi \vdash e\, b_1 = \mathsf{coe}_j^{1 \to 0}\, B(r'/i)\, (\mathsf{coe}_i^{r \to r'}\, B(1/j)\, u_0) = a_1$$

Furthermore, we have:

$$\varphi(r'/i), (r = r') \vdash e\, b_1 = \mathsf{coe}_j^{1 \to 0}\, B(r/i)\, u_0 = a_0 = a_1$$

The second equality holds as $\varphi(r/i) \vdash \mathsf{coe}_j^{1 \to 0}\, B(r/i)\, u_0 = a_0$, and the other equalities hold by the remarks about what happens on $(r = r')$ in Step 2 and 3. We hence get $(b_1, \langle \_ \rangle\, a_1) : \mathsf{Fiber}\, e\, a_1$ on $\varphi(r'/i) \wedge (r = r')$.

Step 5. By Lemma 6 we get $e' : \mathsf{IsEquiv}\, e$ so that we can compute:

$$\varphi(r'/i) \vdash (C_1, C_2) := e'\, a_1$$

Here $C_1 : \mathsf{Fiber}\, e\, a_1$ is the center of contraction and $C_2 : (x : \mathsf{Fiber}\, e\, a_1) \to \mathsf{Path}\ (\mathsf{Fiber}\, e\, a_1)\ x\ C_1$. We use this to extend $(b_1, \langle \_ \rangle\, a_1)$ to all of $\varphi(r'/i)$:

$$\varphi(r'/i) \vdash R := \mathsf{hcom}_k^{1 \to 0}\, (\mathsf{Fiber}\, e\, a_1)\, [\forall i.\varphi \vee (r = r') \mapsto C_2\, (b_1, \langle \_ \rangle\, a_1)\, k]\, C_1$$

This is well-formed by the remarks in Step 2 and 3 about $(r = r')$.

Step 6. We can then use the second component of the equivalence to extend this to a total element in $A(r'/i)$:

$$\vdash a_1' := \mathsf{hcom}_k^{1 \to 0}\, A(r'/i)\, [\, \varphi(r'/i) \mapsto R.2\, k,\ (r = r') \mapsto a_0]\, a_1 : A(r'/i)$$

Note that on $\varphi(r'/i)$ we have $R.2\, 1 = a_1$ (as $R : \mathsf{Fiber}\, e\, a_1$) and on $(r = r')$ we have $a_1 = a_0$, making the homogeneous composition well-formed.

Step 7. Finally we combine all the parts we have computed:

$$\vdash w := \mathsf{box}^{0\to 1}\,[\varphi(r'/i) \mapsto R.1]\,a'_1 : F(r'/i)$$

To see that w is well-formed note that:

$$\varphi(r'/i) \vdash a'_1 = R.2\,0 = e\,R.1 = \mathsf{coe}_j^{1\to 0}\,B(r'/i)\,R.1$$

where the second equality holds as $R : \mathsf{Fiber}\,e\,a_1$.

To see that w satisfies the two necessary properties note that $\forall i.\varphi \leqslant \varphi(r'/i)$ so that

$$\forall i.\varphi \vdash w = R.1 = (C_2\,(b_1, \langle\_\rangle\,a_1)\,0).1 = b_1 = \mathsf{coe}_i^{r\to r'}\,B(1/j)\,u_0$$

as desired. Furthermore,

$$\varphi(r'/i), (r = r') \vdash R.1 = (C_2\,(b_1, \langle\_\rangle\,a_1)\,0).1 = b_1 = u_0$$
$$(r = r') \vdash a'_1 = a_0 = \mathsf{cap}_j^{0\leftarrow 1}\,[\varphi(r/i) \mapsto B(r/i)]\,u_0$$

so that

$$(r = r') \vdash w = \mathsf{box}^{0\to 1}\,[\varphi(r/i) \mapsto u_0]\,(\mathsf{cap}_j^{0\leftarrow 1}\,[\varphi(r/i) \mapsto B(r/i)]\,u_0) = u_0$$

by the η-rule for box and cap.

Appendix C. Summary of algorithms for Glue types

C.1. **hcom in Glue.** The algorithm for computing $w := \mathsf{hcom}_i^{r\to r'}\,(\mathsf{Glue}\,[\varphi \mapsto (T, e)]\,A)\,[\psi \mapsto u]\,u_0$ is:

$$\begin{aligned}
a &:= \mathsf{unglue}\,[\varphi \mapsto e]\,u \\
a_0 &:= \mathsf{unglue}\,[\varphi \mapsto e]\,u_0 \\
\tilde{u} &:= \mathsf{hcom}_i^{r\to i}\,T\,[\psi \mapsto u]\,u_0 \\
a_1 &:= \mathsf{hcom}_i^{r\to r'}\,A\,[\psi \mapsto a, \varphi \mapsto e\ \tilde{u}]\,a_0 \\
w &:= \mathsf{glue}\,[\varphi \mapsto \tilde{u}(r'/i)]\,a_1
\end{aligned}$$

With context and typing information:

$$\begin{aligned}
\psi, i : \mathbb{I} &\vdash a := \mathsf{unglue}\,[\varphi \mapsto e]\,u : A[\varphi \mapsto e\ u] \\
&\vdash a_0 := \mathsf{unglue}\,[\varphi \mapsto e]\,u_0 : A[\varphi \mapsto e\ u_0] \\
\varphi, i : \mathbb{I} &\vdash \tilde{u} := \mathsf{hcom}_i^{r\to i}\,T\,[\psi \mapsto u]\,u_0 : T \\
&\vdash a_1 := \mathsf{hcom}_i^{r\to r'}\,A\,[\psi \mapsto a, \varphi \mapsto e\ \tilde{u}]\,a_0 : A \\
&\vdash w := \mathsf{glue}\,[\varphi \mapsto u_1]\,a_1 : \mathsf{Glue}\,[\varphi \mapsto (T, e)]\,A
\end{aligned}$$

C.2. **coe in Glue.** The algorithm for computing $w := \mathsf{coe}_i^{r \to r'}\,(\mathsf{Glue}\,[\varphi \mapsto (T, e)]\,A)\,u_0$ is:

$$
\begin{aligned}
a_0 &:= \mathsf{unglue}\,[\varphi(r/i) \mapsto e(r/i)]\,u_0 \\
\tilde{b} &:= \mathsf{coe}_i^{r \to i}\,T\,u_0 \\
a_1 &:= \mathsf{com}_i^{r \to r'}\,A\,[\forall i.\varphi \mapsto e\ \tilde{b}]\,a_0 \\
(C_1, C_2) &:= e(r'/i).2\,a_1 \\
R &:= \mathsf{hcom}_k^{1 \to 0}\,(\mathsf{Fiber}\,e(r'/i)\,a_1)\,[\forall i.\varphi \vee (r = r') \mapsto C_2\,(\tilde{b}(r'/i), \langle \_ \rangle\,a_1)\,k]\,C_1 \\
a_1' &:= \mathsf{hcom}_k^{1 \to 0}\,A(r'/i)\,[\varphi(r'/i) \mapsto R.2\,k, (r = r') \mapsto a_0]\,a_1 \\
w &:= \mathsf{glue}\,[\varphi(r'/i) \mapsto R.1]\,a_1'
\end{aligned}
$$

With context and typing information:

$$
\begin{aligned}
&\vdash a_0 := \mathsf{unglue}\,[\varphi(r/i) \mapsto e(r/i)]\,u_0 : A(r/i)[\varphi(r/i) \mapsto e(r/i)\ u_0] \\
\forall i.\varphi, i : \mathbb{I} &\vdash \tilde{b} := \mathsf{coe}_i^{r \to i}\,T\,u_0 : T \\
&\vdash a_1 := \mathsf{com}_i^{r \to r'}\,A\,[\forall i.\varphi \mapsto e\ \tilde{b}]\,a_0 : A(r'/i) \\
\varphi(r'/i) &\vdash (C_1, C_2) := e(r'/i).2\,a_1 : \mathsf{IsContr}\,(\mathsf{Fiber}\,e(r'/i)\,a_1) \\
\varphi(r'/i) &\vdash R := \mathsf{hcom}_k^{1 \to 0}\,(\mathsf{Fiber}\,e(r'/i)\,a_1)\,[\forall i.\varphi \vee (r = r') \mapsto C_2\,(b_1, \langle \_ \rangle\,a_1)\,k]\,C_1 : \mathsf{Fiber}\,e(r'/i)\,a_1 \\
&\vdash a_1' := \mathsf{hcom}_k^{1 \to 0}\,A(r'/i)\,[\varphi(r'/i) \mapsto R.2\,k, (r = r') \mapsto a_0]\,a_1 : A(r'/i) \\
&\vdash w := \mathsf{glue}\,[\varphi(r'/i) \mapsto R.1]\,a_1' : (\mathsf{Glue}\,[\varphi \mapsto (T, e)]\,A)(r'/i)
\end{aligned}
$$