

Canonicity for Indexed Inductive-Recursive Types

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Inductive-Recursive Types

Mutual definition of an inductive type and a function acting on it.

mutual

data Code : Set₀ **where**

Nat' : Code

Π' : (A : Code) → (El A → Code) → Code

El : Code → Set₀

El Nat' = Nat

El (Π' A B) = (a : El A) → El (B a)

Early and informal use by Per Martin-Löf [ML75, ML84].

Formal syntax & semantics developed by Peter and Anton [Dyb00, DS99, DS03, DS06].

Use-cases:

- Metatheory for TTs with various universe hierarchies:
 - Normalization for TTs with countable universes [ML75, AÖV18, PT23, ADE23].
 - Consistency [Kov22] and canonicity [CW25] for notions of first-class universe levels.
- Others: partial functions [BC01], generic programming [BDJ03, Die17], large countable ordinals [Kov23, Kyu25].

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Canonicity: can we evaluate every closed term in $\text{MLTT} + \text{IR}$ to a value?

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In this talk & upcoming paper: formal proof of canonicity.

How to specify an IR type?

The type of **IR signatures** is an inductive type.

$$\begin{aligned} &\mathbf{data} \text{ Sig } i \{j\} (O : \text{Set}_j) : \text{Set}_{(i+1 \sqcup j)} \mathbf{where} \\ &\quad \iota : O \rightarrow \text{Sig } i \ O \\ &\quad \sigma : (A : \text{Set}_i) \rightarrow (A \rightarrow \text{Sig } i \ O) \rightarrow \text{Sig } i \ O \\ &\quad \delta : (A : \text{Set}_i) \rightarrow ((A \rightarrow O) \rightarrow \text{Sig } i \ O) \rightarrow \text{Sig } i \ O \end{aligned}$$

The signature of the previous Code example:

$$\begin{aligned} &\text{SigCode} : \text{Sig } 0 \ \text{Set}_0 \\ &\text{SigCode} :\equiv \sigma \ \text{Bool} \ \lambda t. \mathbf{case} \ t \ \mathbf{of} \\ &\quad \text{true} \rightarrow \iota \ \text{Nat} \\ &\quad \text{false} \rightarrow \delta \ \top \ \lambda EIA. \delta \ (EIA \ \text{tt}) \ \lambda EIB. \iota \ ((x : EIA \ \text{tt}) \rightarrow EIB \ x) \end{aligned}$$

For each signature, we postulate type formation, term formation, elimination and computation rules, for the described IR type.

How to prove canonicity?

Every closed term with IR type should be convertible to an IR constructor. E.g. a closed term $t : \text{Code}$ should be either Nat' or Π' .

Type-theoretic Artin gluing [Coq19, KHS19]:

- A closed type is interpreted as a proof-relevant predicate over its closed terms. The predicate should imply canonicity.
- A closed term is interpreted as a predicate witness.
- Open types & terms are generalized over closed substitutions.

How to prove canonicity?

Notation: we write \mathbf{Ty} for the set of closed types, $\mathbf{Tm}\ A$ for sets of closed terms, and $\mathbf{U} : \mathbf{Ty}$ for object-theoretic universes (omitting levels).

Example: canonicity predicate for natural numbers.

```
data Nat° : Tm Nat → Set where  
  zero° : Nat° zero  
  suc° : (n : Tm Nat) → Nat° n → Nat° (suc n)
```

Generally: the canonicity predicate for an inductive type is a **metatheoretic indexed inductive type**.

How to prove canonicity?

Example: the canonicity predicate for Code is the following **metatheoretic indexed IR type**:

data $\text{Code}^\circ : \text{Tm Code} \rightarrow \text{Set}_0$

$\text{Nat}'^\circ : \text{Code}^\circ \text{Nat}'$

$\Pi'^\circ : \{A : \text{Tm Code}\} (A^\circ : \text{Code}^\circ A)$

$\{B : \text{Tm} (\text{El } A \rightarrow \text{Code})\} (B^\circ : \{a : \text{Tm} (\text{El } a)\} \rightarrow \text{El}^\circ A^\circ a \rightarrow \text{Code}^\circ (B a))$
 $\rightarrow \text{Code}^\circ (\Pi' A B)$

$\text{El}^\circ : \{t : \text{Tm Code}\} \rightarrow \text{Code}^\circ t \rightarrow (\text{Tm} (\text{El } t) \rightarrow \text{Set}_0)$

$\text{El}^\circ \text{Nat}'^\circ = \text{Nat}^\circ$

$\text{El}^\circ (\Pi'^\circ A^\circ B^\circ) = \lambda f. \{a : \text{Tm} (\text{El } A)\} (a^\circ : \text{El}^\circ A^\circ a) \rightarrow \text{El}^\circ (B^\circ a^\circ) (f a)$

If we have $t : \text{Tm Code}$, the gluing interpretation hands us an element of $\text{Code}^\circ t$.

How to prove canonicity?

In the object theory, let's assume general IR types specified by an internal Sig type.

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Interpreting IR types in the glued model:

- We write $\text{IR } S : \text{Tm } U$ for the IR *type formation* rule, for $S : \text{Tm } (\text{Sig } O)$.

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- To interpret $\text{IR } S$:
 - By induction hypothesis, we get $S^\circ : \text{Sig}^\circ S$ witnessing the canonicity of the signature S itself.
 - We compute a metatheoretic indexed IR signature by induction on S° . This yields the canonicity predicate for $\text{IR } S$.

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 - We compute a metatheoretic indexed IR signature by induction on S° . This yields the canonicity predicate for $\text{IR } S$.
- We also need to interpret *term formation*, *elimination* and *computation* rules.
- For these, we have to show that universal properties of IR types are preserved through the indexed IR encoding.

How to prove canonicity?

The construction is a moderately technical “generic programming” exercise, where we have to do some tricky induction over signatures.

We use metatheoretic IR to show canonicity of object-theoretic IR. The metatheory has to have more universes; in our case the metatheory has extra levels ω and $\omega + 1$ ¹.

The canonicity interpretation of IR is formalized in Agda using a shallow embedding of Ty and Tm.

¹But $\omega + 1$ is only used for convenience and could be omitted.

Indexed Induction-Recursion

What about canonicity for **indexed** IR types?

We only show canonicity for plain IR types, but also show that indexed IR types are constructible in $\text{MLTT} + \text{IR}$.

The construction supports strict computation rules for all IIR types that are definable in Agda. For *neutral* signatures we only get propositional computation.

We use the well-known construction that converts indices to parameters and propositional identities.²

Here, since we work in plain MLTT without function extensionality and UIP, we have to do a modest amount of HoTT reasoning.

²Popularized as “fording” by Conor McBride.

- Future work:
 - Normalization for IR types. For this, we first have to show that presheaf models have IR types (which is already quite technical!).
 - Generalize the canonicity proof to a model construction (i.e. extend general type-theoretic gluing with IR types). This seems to be incompatible with first-class signatures.
- Paper: under review. I'll publish a preprint in ~ 2 weeks.

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Thank you!

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