

# Canonicity for Indexed Inductive-Recursive Types

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We prove canonicity for a Martin-Löf type theory that supports a countable universe hierarchy where each universe supports indexed inductive-recursive (IIR) types. We proceed in two steps. First, we construct IIR types from inductive-recursive (IR) types and intensional identity types, in order to simplify the subsequent canonicity proof. The constructed IIR types support the same definitional computation rules that are available in Agda's native IIR implementation. Second, we give a canonicity proof for IR types, building on the well-known method of Artin gluing. The main idea is to encode the canonicity predicate for each IR type using a metatheoretic IIR type.

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## 1 Introduction

Induction-recursion (IR) was first used by Martin-Löf in an informal way [?], then made formal by Dybjer and Setzer [?], who also developed set-theoretic and categorical semantics [?]. A common application of IR is to define custom universe hierarchies inside a type theory. In the proof assistant Agda, we can use IR to define a universe that is closed under our choice of type formers:

```
mutual
  data Code : Set0 where
    Nat' : U
    Π'   : (A : Code) → (El A → Code) → Code

  El : Code → Set0
  El Nat'   = Nat
  El (Π' A B) = (a : El A) → El (B a)
```

Here, `Code` is a type of codes of types which behaves as a custom Tarski-style universe. This universe, unlike the ambient  $\text{Set}_0$  universe, supports an induction principle and can be used to define type-generic functions. *Indexed induction-recursion* (IIR) additionally allows indexing `Code` over some type, which lets us define inductive-recursive predicates [?].

One application of IR has been to develop semantics for object theories that support universe hierarchies. IR has been used in normalization proofs [?], in modeling first-class universe levels [?] and proving canonicity for them [?], and in characterizing domains of partial functions [?]. Another application is to do generic programming over universes of type descriptions [?] or data layout descriptions [?].

IIR has been supported in Agda 2 since the early days of the system [?], and it is also available in Idris 1 and Idris 2 [?]. In these systems, IR has been implemented in the “obvious” way, supporting

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closed program execution in compiler backends and normalization during type checking, but without any formal justification.

Our **main contribution** is to **show canonicity** for a Martin-Löf type theory that supports a countable universe hierarchy, where each universe supports indexed inductive-recursive types. Canonicity means that every closed term is definitionally equal to a canonical term. Canonical terms are built only from constructors; for instance, a canonical natural number term is a numeral. Hence, canonicity justifies evaluation for closed terms. The outline of our development is as follows.

- (1) In Section ?? we specify what it means to support IR and IIR, using Dybjer and Setzer's rules with minor modifications [?]. We use first-class signatures, meaning that descriptions of (I)IR types are given as ordinary inductive types internally.
- (2) In Section ?? we construct IIR types from IR types and other basic type formers. This allows us to only consider IR types in the subsequent canonicity proof, which is a significant simplification. In the construction of IIR types, we lose some definitional equalities when IIR signatures are neutral, but we still get strict computation for canonical signatures. This matches the computational behavior of Agda and Idris, where IIR signatures are second-class and necessarily canonical. We formalize the construction in Agda.
- (3) In Section ??, we give a proof-relevant logical predicate interpretation of the type theory, from which canonicity follows. We follow the well-known method of Artin gluing [?]. The main challenge here is to give a logical predicate interpretation of IR types. We do this by using IIR in the metatheory: from each object-theoretic signature we compute a metatheoretic IIR signature which encodes the canonicity predicate for the corresponding IR type. We formalize the predicate interpretation of IR types in Agda, using a shallow embedding of the syntax of the object theory. Hence, there is a gap between the Agda version and the fully formal construction, but we argue that it is a modest gap.

## 2 Specification for (I)IR types

In this section we describe the object type theory, focusing on the specification of IR and IIR types. We do not yet go into the formal details of structure of the object type theory; instead, we mostly give internal definitions in an Agda-like syntax. We give an overview of the object theory.

We have a countable hierarchy of Russell-style universes, written as  $U_i$ , where  $i$  is an external natural number. We have  $U_i : U_{i+1}$ . We take maxima of universe levels for sizes of type formers. For example, the  $\Pi$ -type  $(x : A) \rightarrow B x$  is in  $U_{\max(i, j)}$  when  $A : U_i$  and  $B : A \rightarrow U_j$ . We also have a lifting operation  $\text{Lift } i j : U_i \rightarrow U_j$  together with  $\uparrow : A \rightarrow \text{Lift } i j A$  and  $\downarrow : \text{Lift } i j A \rightarrow A$  such that  $\uparrow$  and  $\downarrow$  are definitional inverses. We have intensional identity types, as  $t = u : U_i$  for  $t : A : U_i$ . We have  $\top : U_0$  for the unit type with the unique inhabitant  $\text{tt}$ .

### 2.1 IR types

IR signatures are parameterized by the following data:

- The level  $i$  is the size of the IR type that is being specified.
- The level  $j$  is the size of the recursive output type.
- $O : U_j$  is the output type.

IR signatures are specified by the following inductive type. We only mark  $i$  and  $O$  as parameters to  $\text{Sig}$ , since  $j$  is inferable from  $O$ .

```

data Sig $i$   $O : \mathcal{U}_{\max(i+1, o)}$ 
   $\iota : O \rightarrow \text{Sig}$ 
   $\sigma : (A : \mathcal{U}_i) \rightarrow (A \rightarrow \text{Sig}) \rightarrow \text{Sig}$ 
   $\delta : (A : \mathcal{U}_i) \rightarrow ((A \rightarrow O) \rightarrow \text{Sig}) \rightarrow \text{Sig}$ 

```

*Example 2.1.* We can reproduce the Agda example from Figure [?]. First, we need an enumeration type to represent the constructor labels of Code. We assume this as  $\text{Tag} : \mathcal{U}_0$  with constructors  $\text{Nat}'$  and  $\Pi'$ , and we use an informal case splitting operation for it. We also assume  $\text{Nat} : \mathcal{U}_0$  for natural numbers and a right-associative  $\text{\$}$  operator for function application.

```

 $S : \text{Sig}_0 \mathcal{U}_0$ 
 $S := \sigma \text{Tag} \$ \lambda t. \text{case } t \text{ of}$ 
   $\text{Nat}' \rightarrow \iota \text{Nat}$ 
   $\Pi' \rightarrow \delta \top \$ \lambda \text{ELA}. \delta (\text{ELA } \text{tt}) \$ \lambda \text{ELB}. \iota ((x : \text{ELA } \text{tt}) \rightarrow \text{ELB } x)$ 

```

First, we introduce a choice between two constructors by  $\sigma \text{Tag}$ . In the  $\text{Nat}'$  branch, we specify that the recursive function maps the constructor to  $\text{Nat}$ . In the  $\Pi'$  branch, we first introduce a single inductive constructor field by  $\delta \top$ , where  $\top$  represents the number of introduced fields. The naming of the freshly bound  $\text{ELA}$  variable is meant to suggest that it represent the recursive function's output for the inductive field. It has type  $\top \rightarrow \mathcal{U}_0$ . Next, we introduce  $(\text{ELA } \text{tt})$ -many inductive fields, and bind  $\text{ELB} : \text{ELA } \text{tt} \rightarrow \mathcal{U}_0$  to represent the corresponding recursive output. Finally,  $\iota ((x : \text{ELA } \text{tt}) \rightarrow \text{ELB } x)$  specifies the output of the recursive function for a  $\Pi'$  constructor.

This notion of signature is close to Dybjer and Setzer's [?]. There are two main differences. First, Dybjer and Setzer factor out the  $O$  field from the  $\iota$  constructor into a separate annotation on signatures, called "Arg", but this is clearly equivalent to ours. Second, we consider a countable universe hierarchy while Dybjer and Setzer only consider a single universe.

### 3 TODO

- Small IIR paper: reduction of small IIR to IR + identity, similar to mine
- Bove-Capretta: IIR is used to model domains of partial functions.
- Canonicity metatheory: Loic, Anton say it looks OK

a

### References