

# Canonicity for Indexed Inductive-Recursive Types

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We prove canonicity for a Martin-Löf type theory that supports a countable universe hierarchy where each universe supports indexed inductive-recursive (IIR) types. We proceed in two steps. First, we construct IIR types from inductive-recursive (IR) types and intensional identity types, in order to simplify the subsequent canonicity proof. The constructed IIR types support the same definitional computation rules that are available in Agda's native IIR implementation. Second, we give a canonicity proof for IR types, building on the well-known method of Artin gluing. The main idea is to encode the canonicity predicate for each IR type using a metatheoretic IIR type.

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## 1 Introduction

Induction-recursion (IR) was first used by Martin-Löf in an informal way [?], then made formal by Dybjer and Setzer [?], who also developed set-theoretic and categorical semantics [?]. A common application of IR is to define custom universe hierarchies inside a type theory. In the proof assistant Agda, we can use IR to define a universe that is closed under our choice of type formers:

```
mutual
  data Code : Set0 where
    Nat' : U
    Π'   : (A : Code) → (El A → Code) → Code

  El : Code → Set0
  El Nat'   = Nat
  El (Π' A B) = (a : El A) → El (B a)
```

Here, `Code` is a type of codes of types which behaves as a custom Tarski-style universe. This universe, unlike the ambient  $\text{Set}_0$  universe, supports an induction principle and can be used to define type-generic functions. *Indexed induction-recursion* (IIR) additionally allows indexing `Code` over some type, which lets us define inductive-recursive predicates [?].

One application of IR has been to develop semantics for object theories that support universe hierarchies. IR has been used in normalization proofs [?], in modeling first-class universe levels [?] and proving canonicity for them [?], and in characterizing domains of partial functions [?]. Another application is to do generic programming over universes of type descriptions [?] or data layout descriptions [?].

IIR has been supported in Agda 2 since the early days of the system [?], and it is also available in Idris 1 and Idris 2 [?]. In these systems, IR has been implemented in the “obvious” way, supporting

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closed program execution in compiler backends and normalization during type checking, but without any formal justification.

Our **main contribution** is to **show canonicity** for a Martin-Löf type theory that supports a countable universe hierarchy, where each universe supports indexed inductive-recursive types. Canonicity means that every closed term is definitionally equal to a canonical term. Canonical terms are built only from constructors; for instance, a canonical natural number term is a numeral. Hence, canonicity justifies evaluation for closed terms. The outline of our development is as follows.

- (1) In Section ?? we specify what it means to support IR and IIR, using Dybjer and Setzer's rules with minor modifications [?]. We use first-class signatures, meaning that descriptions of (I)IR types are given as ordinary inductive types internally.
- (2) In Section ?? we construct IIR types from IR types and other basic type formers. This allows us to only consider IR types in the subsequent canonicity proof, which is a significant simplification. In the construction of IIR types, we lose some definitional equalities when IIR signatures are neutral, but we still get strict computation for canonical signatures. This matches the computational behavior of Agda and Idris, where IIR signatures are second-class and necessarily canonical. We formalize the construction in Agda.
- (3) In Section ??, we give a proof-relevant logical predicate interpretation of the type theory, from which canonicity follows. We build on the well-known method of Artin gluing [?]. The main challenge here is to give a logical predicate interpretation of IR types. We do this by using IIR in the metatheory: from each object-theoretic signature we compute a metatheoretic IIR signature which encodes the canonicity predicate for the corresponding IR type. We formalize the predicate interpretation of IR types in Agda, using a shallow embedding of the syntax of the object theory. Hence, there is a gap between the Agda version and the fully formal construction, but we argue that it is a modest gap.

## 2 Specification for (I)IR types

In this section we describe the object type theory, focusing on the specification of IR and IIR types. We do not yet go into the formal details; instead, we shall mostly work with internal definitions in an Agda-like syntax. In Section ?? we will give a rigorous specification that is based on categories-with-families.

*Basic type formers.* We have a countable hierarchy of Russell-style universes, written as  $U_i$ , where  $i$  is an external natural number. We have  $U_i : U_{i+1}$ . We have  $\Pi$ -types as  $(x : A) \rightarrow Bx$ , which has type  $U_{\max(i, j)}$  when  $A : U_i$  and  $B : A \rightarrow U_j$ . We use Agda-style implicit function types for convenience, as  $\{x : A\} \rightarrow Bx$ , to mark that a function argument should be inferred from context. We also have a lifting operation  $\text{Lift } i j : U_i \rightarrow U_j$  together with  $\uparrow : A \rightarrow \text{Lift } i j A$  and  $\downarrow : \text{Lift } i j A \rightarrow A$  such that  $\uparrow$  and  $\downarrow$  are definitional inverses. We have intensional identity types, as  $t = u : U_i$  for  $t : A : U_i$ , with a transport operation  $\text{tr} : \{A : U_i\}(P : A \rightarrow U_j)\{x y : A\} \rightarrow x = y \rightarrow Px \rightarrow Py$ . We have  $\top : U_0$  for the unit type with the unique inhabitant  $\text{tt}$ . We have  $\Sigma$ -types, written as  $(x : A) \times Bx$ , where the type also lands in  $U_{\max(i, j)}$ . Finally, we have binary sum types, written as  $A + B$  with constructors  $\text{inj}_1$  and  $\text{inj}_2$ .

### Implicit argument quantification

#### 2.1 IR types

The object theory additionally supports inductive-recursive types. On a high level, the specification consists of the following.

- (1) A type of signatures. Each signature describes an IR type. Also, we internally define some functions on signatures which are required in the specification of other rules.
- (2) Rules for type formation, term formation and the recursive function, with a computation rule for the recursive function.
- (3) The induction principle with a  $\beta$  rule.

2.1.1 *IR signatures.* Signatures are parameterized by the following data:

- The level  $i$  is the size of the IR type that is being specified.
- The level  $j$  is the size of the recursive output type.
- $O : U_j$  is the output type.

IR signatures are specified by the following inductive type. We only mark  $i$  and  $O$  as parameters to  $\text{Sig}$ , since  $j$  is inferable from  $O$ .

```

data  $\text{Sig}_i O : U_{\max(i+1, j)}$  where
   $\iota : O \rightarrow \text{Sig}_i O$ 
   $\sigma : (A : U_i) \rightarrow (A \rightarrow \text{Sig}_i O) \rightarrow \text{Sig}_i O$ 
   $\delta : (A : U_i) \rightarrow ((A \rightarrow O) \rightarrow \text{Sig}_i O) \rightarrow \text{Sig}_i O$ 

```

Formally, we can view  $\text{Sig}$  in two ways. We can either view it as just a particular family of W-types, or as an inductive type that is primitively part of the object theory. The choice is not important, since inductive families are constructible from W-types [Hugunin 2020].

*Example 2.1.* We can reproduce the Agda example from Figure [?]. First, we need an enumeration type to represent the constructor labels of Code. We assume this as  $\text{Tag} : U_0$  with constructors  $\text{Nat}'$  and  $\Pi'$ , and we use an informal case splitting operation for it. We also assume  $\text{Nat} : U_0$  for natural numbers and a right-associative  $\$$ -operator for function application.

```

 $S : \text{Sig}_0 U_0$ 
 $S \equiv \sigma \text{Tag} \$ \lambda t. \text{case } t \text{ of}$ 
   $\text{Nat}' \rightarrow \iota \text{Nat}$ 
   $\Pi' \rightarrow \delta \top \$ \lambda ELA. \delta (ELA \text{tt}) \$ \lambda ELB. \iota ((x : ELA \text{tt}) \rightarrow ELB x)$ 

```

First, we introduce a choice between two constructors by  $\sigma \text{Tag}$ . In the  $\text{Nat}'$  branch, we specify that the recursive function maps the constructor to  $\text{Nat}$ . In the  $\Pi'$  branch, we first introduce a single inductive constructor field by  $\delta \top$ , where  $\top$  represents the number of introduced fields. The naming of the freshly bound variable  $ELA$  is meant to suggest that it represent the recursive function's output for the inductive field. It has type  $\top \rightarrow U_0$ . Next, we introduce  $(ELA \text{tt})$ -many inductive fields, and bind  $ELB : ELA \text{tt} \rightarrow U_0$  to represent the corresponding recursive output. Finally,  $\iota ((x : ELA \text{tt}) \rightarrow ELB x)$  specifies the output of the recursive function for a  $\Pi'$  constructor.

Our signatures are identical to Dybjer and Setzer's [?], except for one difference. We have countable universe levels, while Dybjer and Setzer use a logical framework presentation with only three universes, *set*, *stype* and *type*, where *set* contains the inductively specified type, *stype* contains the non-inductive constructor arguments and *type* contains the recursive output type and the type of signatures.

2.1.2 *Type and term formation.* In this section we also follow Dybjer and Setzer [?], with minor differences of notation, and also accounting for the refinement of universe levels.

First, assuming  $i$  and  $O : U_j$ , a signature  $S : \text{Sig}_i O$  can be interpreted as a function from  $(A : U_i) \times (A \rightarrow O)$  to  $(A : U_i) \times (A \rightarrow O)$ . This can be extended to an endofunctor on the slice

category  $U_i/O$ , but in the following we only need the action on objects. We split this action to two functions, to aid readability:

$$\begin{aligned}
 -_0 &: \text{Sig}_i O \rightarrow (ir : U_i) \rightarrow (ir \rightarrow O) \rightarrow U_i \\
 S_0 (\iota o) \quad ir \, el &\equiv \text{Lift } \top \\
 S_0 (\sigma AS) \, ir \, el &\equiv (a : A) \times (S a)_0 \, ir \, el \\
 S_0 (\delta AS) \, ir \, el &\equiv (f : A \rightarrow ir) \times (S (el \circ f))_0 \, ir \, el \\
 \\ 
 -_1 &: (S : \text{Sig}_i O) \rightarrow S_0 \, ir \, el \rightarrow O \\
 S_1 (\iota o) \quad x &\equiv o \\
 S_1 (\sigma AS) (a, x) &\equiv (S i)_1 x \\
 S_1 (\delta AS) (f, x) &\equiv (S (el \circ f))_1 x
 \end{aligned}$$

Although we use Agda-like pattern matching notation above, these functions are formally defined by the elimination principle of  $\text{Sig}$ . Also note the quantification of the  $i$  and  $j$  universe levels. The object theory does not support universe polymorphism, so this quantification is understood to happen in the metatheory. The introduction rules are the following.

$$\begin{aligned}
 \text{IR} &: (S : \text{Sig}_i O) \rightarrow U_i \\
 \text{El} &: \text{IR } S \rightarrow O \\
 \text{intro} &: S_0 (\text{IR } S) \text{El} \rightarrow \text{IR } S \\
 \text{El-intro} &: \text{El} (\text{intro } x) \equiv S_1 x
 \end{aligned}$$

Above, we leave some rule arguments implicit, like  $S$  in  $\text{El}$ ,  $\text{intro}$  and  $\text{El-intro}$ . The rule  $\text{El-intro}$  specifies a definitional equality. Note that these rules are not internal definitions but part of the specification of the object theory. Hence, they are also assumed to be stable under object-theoretic substitution, formally speaking. On a high level, the introduction rules express the existence of an  $S$ -algebra where we view  $S$  as an endofunctor on  $U_i/O$ .

**2.1.3 Elimination.** Here we follow the specification in [?]. We assume another universe level  $k$  that specifies the size of the type into which we eliminate. We define two additional functions on signatures:

$$\begin{aligned}
 -_{\text{IH}} &: (S : \text{Sig}_i O)(P : ir \rightarrow U_k) \rightarrow S_0 \, ir \, el \rightarrow U_{\max(i, k)} \\
 (\iota o)_{\text{IH}} \quad P x &\equiv \text{Lift } \top \\
 (\sigma AS)_{\text{IH}} P (a, x) &\equiv (S a)_{\text{IH}} P x \\
 (\delta AS)_{\text{IH}} P (f, x) &\equiv ((a : A) \rightarrow P (f a)) \times (S (el \circ f))_{\text{IH}} P x
 \end{aligned}$$

$$\begin{aligned}
 -_{\text{map}} &: (S : \text{Sig}_i O)(P : ir \rightarrow U_k) \rightarrow ((x : ir) \rightarrow P x) \rightarrow (x : S_0 \, ir \, el) \rightarrow S_{\text{IH}} P x \\
 (\iota o)_{\text{map}} \quad P h x &\equiv \uparrow \text{tt} \\
 (\sigma AS)_{\text{map}} P h (a, x) &\equiv (S a)_{\text{map}} P h x \\
 (\delta AS)_{\text{map}} P h (f, x) &\equiv (h \circ f, (S (el \circ f))_{\text{map}} P h x)
 \end{aligned}$$

$-_{\text{IH}}$  stands for “induction hypothesis”: it specifies having a witness of a predicate  $P$  for each inductive field in a value of  $S_0 \, ir \, el$ .  $S_{\text{map}}$  maps over  $S_0 \, ir \, el$ , applying the section  $h : (x : ir) \rightarrow P x$

to each inductive field. Elimination is specified as follows.

$$\begin{aligned} \text{elim} & : (P : \text{IR } S \rightarrow \mathcal{U}_k) \rightarrow ((x : S_0 (\text{IR } S) \text{ El}) \rightarrow S_{\text{IH}} P x \rightarrow P (\text{intro } x)) \rightarrow (x : \text{IR } S) \rightarrow P x \\ \text{elim} - \beta & : \text{elim } P f (\text{intro } x) \equiv f x (S_{\text{map}} P (\text{elim } P f) x) \end{aligned}$$

If we have function extensionality, this specification of elimination can be shown to be equivalent to the initiality of  $(\text{IR } S, \text{El})$  as an  $S$ -algebra [?].

## 2.2 IIR types

We extend Dybjer and Setzer's general IIR signatures [?] with refined universe levels. Since IIR is quite similar to IR, we give a more terse presentation of the rules and definitions in the following.

**2.2.1 Signatures.** We assume levels  $i, j, k$ , an indexing type  $I : \mathcal{U}_k$  and a type family for the recursive output as  $O : I \rightarrow \mathcal{U}_j$ . Signatures are as follows.

$$\begin{aligned} \text{data } \text{Sig}_i I O & : \mathcal{U}_{\max(i+1, j, k)} \text{ where} \\ \iota & : (i : I) \rightarrow O i \rightarrow \text{Sig}_i I O \\ \sigma & : (A : \mathcal{U}_i) \rightarrow (A \rightarrow \text{Sig}_i I O) \rightarrow \text{Sig}_i I O \\ \delta & : (A : \mathcal{U}_i)(ix : A \rightarrow I) \rightarrow (((a : A) \rightarrow O (ix a)) \rightarrow \text{Sig}_i I O) \rightarrow \text{Sig}_i I O \end{aligned}$$

*Example 2.2.* We reproduce length-indexed vectors as an IIR type. We assume  $A : \mathcal{U}_0$  for a type of elements in the vector, and a type  $\text{Tag} : \mathcal{U}_0$  with inhabitants  $\text{Nil}'$  and  $\text{Cons}'$ .

$$\begin{aligned} S & : \text{Sig}_0 \text{Nat } (\lambda \_ . \top) \\ S & \equiv \sigma \text{Tag } \$ \lambda t. \text{case } t \text{ of} \\ \text{Nil}' & \rightarrow \iota \text{zero } \text{tt} \\ \text{Cons}' & \rightarrow \sigma \text{Nat } \$ \lambda n. \sigma A \$ \lambda \_ . \delta \top (\lambda \_ . n) \$ \lambda \_ . \iota (\text{suc } n) \text{tt} \end{aligned}$$

We set  $O$  to be constant  $\top$  because vectors do not have an associated recursive function. In the  $\text{Nil}'$  case, we simply set the constructor index to zero. In the  $\text{Cons}'$  case, we introduce a non-inductive field, binding  $n$  for the length of the tail of the vector. Then, when we introduce the inductive field using  $\delta$ , we use  $(\lambda \_ . n)$  to specify that the length of the (single) inductive field is indeed  $n$ . Finally, the length of the  $\text{Cons}'$  constructor is  $\text{suc } n$ .

**2.2.2 Type and term formation.** The signature actions  $-_0$  and  $-_1$  are similar to before:

$$\begin{aligned} -_0 & : \text{Sig}_i I O \rightarrow (ir : I \rightarrow \mathcal{U}_{\max(i, k)}) \rightarrow (\{i : I\} \rightarrow ir i \rightarrow O i) \rightarrow I \rightarrow \mathcal{U}_{\max(i, k)} \\ S_0 (\iota i' o) \quad ir \text{el } i & \equiv \text{Lift } (i' = i) \\ S_0 (\sigma A S) \quad ir \text{el } i & \equiv (a : A) \times (S a)_0 ir \text{el } i \\ S_0 (\delta A ix S) ir \text{el } i & \equiv (f : (a : A) \rightarrow ir (ix a)) \times (S (el \circ f))_0 ir \text{el } i \\ -_1 & : (S : \text{Sig}_i I O) \rightarrow S_0 ir \text{el } i \rightarrow O i \\ S_1 (\iota i' o) \quad (\uparrow x) & \equiv \text{tr } O x o \\ S_1 (\sigma A S) \quad (a, x) & \equiv (S i)_1 x \\ S_1 (\delta A S) \quad (f, x) & \equiv (S (el \circ f))_1 x \end{aligned}$$

Note the transport in  $\text{tr } O x o$ : this is necessary, since  $o$  has type  $O i'$  while the required type is  $O i$ . The type and term formation rules are the following.

$$\begin{aligned} \text{IR} & : (S : \text{Sig}_i IO) \rightarrow I \rightarrow U_{\max(i, k)} \\ \text{El} & : \text{IR } S i \rightarrow O i \\ \text{intro} & : S_0 (\text{IR } S) \text{El } i \rightarrow \text{IR } S i \\ \text{El-intro} & : \text{El} (\text{intro } x) \equiv S_1 x \end{aligned}$$

**2.2.3 Elimination.**  $-_{\text{IH}}$ ,  $-_{\text{map}}$  and elimination are as follows. We assume a level  $l$  for the target type of elimination.

$$\begin{aligned} -_{\text{IH}} & : (S : \text{Sig}_i IO)(P : \{i : I\} \rightarrow ir i \rightarrow U_l) \rightarrow S_0 ir el i \rightarrow U_{\max(i, l)} \\ (\iota i o)_{\text{IH}} \quad P x & \quad \equiv \text{Lift } \top \\ (\sigma AS)_{\text{IH}} \quad P(a, x) & \quad \equiv (S a)_{\text{IH}} P x \\ (\delta A ix S)_{\text{IH}} P(f, x) & \quad \equiv ((a : A) \rightarrow P(f a)) \times (S(el \circ f))_{\text{IH}} P x \\ -_{\text{map}} & : (S : \text{Sig}_i IO)(P : \{i : I\} \rightarrow ir i \rightarrow U_l) \\ & \rightarrow (\{i : I\}(x : ir i) \rightarrow P x) \rightarrow (x : S_0 ir el i) \rightarrow S_{\text{IH}} P x \\ (\iota o)_{\text{map}} \quad P h x & \quad \equiv \uparrow \text{tt} \\ (\sigma AS)_{\text{map}} P h(a, x) & \quad \equiv (S a)_{\text{map}} P h x \\ (\delta AS)_{\text{map}} P h(f, x) & \quad \equiv (h \circ f, (S(el \circ f))_{\text{map}} P h x) \end{aligned}$$

$$\text{elim} : (P : \text{IR } S \rightarrow U_k) \rightarrow ((x : S_0 (\text{IR } S) \text{El}) \rightarrow S_{\text{IH}} P x \rightarrow P (\text{intro } x)) \rightarrow (x : \text{IR } S) \rightarrow P x$$

$$\text{elim-}\beta : \text{elim } P f (\text{intro } x) \equiv f x (S_{\text{map}} P (\text{elim } P f) x)$$

### 3 TODO

- Small IIR paper: reduction of small IIR to IR + identity, similar to mine
- Bove-Capretta: IIR is used to model domains of partial functions.
- Canonicity metatheory: Loic, Anton says it looks OK
- Rename wrap to intro in Agda

### References

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