Generalizations of Quotient Inductive-Inductive Types *

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Quotient inductive-inductive types (QIITs) are the most general class of inductive types studied thus far in a set-truncated setting, i.e. in the presence of uniqueness of identity proofs (UIP). In the current work, we develop QIITs further, focusing on applications in practical metatheory of type theories. We extend previous work on QIITs [5] with the following:

1. **Large constructors, large elimination** and algebras at different universe levels. This fills in an important formal gap; large models are routinely used in the metatheory of type theories, but they have not been presented explicitly in previous QIIT literature.

2. **Infinitary constructors**. This covers real, surreal [6] and ordinal numbers. Additionally, the theory which describes QII signatures is itself a large and infinitary QIIT, which allows the theory of signatures to describe its own signature (modulo universe levels), and provide its own model theory. This was not possible previously in [5], where only finitary QIITs were described.

3. **Recursive equations**, i.e. equations appearing as assumptions of constructors. These have occurred previously in syntaxes of cubical type theories, as boundary conditions [4, 1, 2].

4. **Sort equations**. Sort equations were included in Cartmell's generalized algebraic theories (GATs) [3], which overlap significantly with finitary QIITs. Sort equations appear to be useful for algebraic presentations of Russell-style and cumulative universes [7].

Self-describing signatures

In the current work, we would also like to streamline and make more rigorous the specification of signatures. Previous descriptions of GATs [3, 7] used raw syntax with well-formedness relations to describe signatures, which is rather unwieldy to formally handle. Also, the precursor of the current work [5] used an ad-hoc QIIT to describe signatures, which did not have a model theory worked out, and its existence was simply assumed.

In contrast, equipped with large elimination and self-description, we are able to specify signatures and develop a model theory for signatures, without ever using raw syntax or assuming the existence of a particular QIIT. We do the following in order.

1. We specify a notion of model for the theory of signatures (ToS); this is a category with family (CwF) extended with several type formers, allowing to represent a signature as a typing context, with types specifying various constructors.

2. We say that a signature is a context in an *arbitrary* model, i.e. a function with type $(M : \mathsf{ToS}) \to \mathsf{Con}_M$. This can be viewed as a fragment of a Church-encoding; here we

*Both authors author were supported by the European Union, co-financed by the European Social Fund (EFOP-3.6.3-VEKOP-16-2017-00002). The second author was supported by by the National Research, Development and Innovation Fund of Hungary, financed under the Thematic Excellence Programme funding scheme, Project no. ED_18-1-2019-0030 (Application-specific highly reliable IT solutions).

do not care about encoding the whole syntax of ToS, nor the initiality of the syntax, we only need a representation of signatures and the ability to interpret a signature in a ToS-model. For example, the signature for natural number algebras is a function

$$\lambda(M : \mathsf{ToS}).\,(\bullet_M \triangleright_M (N : \mathsf{U}_M) \triangleright_M (\mathit{zero} : \mathsf{El}_M\, N) \triangleright_M (\mathit{suc} : N \Rightarrow_M \mathsf{El}_M\, N))$$

which maps every model M to a typing context in M, consisting of the declaration of a sort and two constructors.

3. We give a semantics for signatures, as a particular $M : \mathsf{ToS}$ model which interprets each CwF context Γ as a structured category of Γ-algebras. E.g. for the signature of natural numbers, we get a structured category of $\mathbb{N}$-algebras, with $\mathbb{N}$-homomorphisms as morphisms.

4. We give a (large, infinitary) signature for ToS itself, such that interpreting the signature in the semantic model yields a structured category of ToS-algebras. From this, we acquire notions of *recursion* and *induction*, hence we gain the ability to define further constructions by induction on an assumed initial model of the theory of signatures.

In the above construction, everything is appropriately indexed with universe levels (we omit the details), and there is a "bump" of levels at every instance of self-description.

Extending semantics to infinitary constructors and sort equations.

Previously in [5], contexts in ToS were interpreted as CwFs of algebras with extra structure, substitutions as strict morphisms of such CwFs, and types as displayed CwFs. Infinitary constructors force a major change: substitutions must be interpreted as weak CwF morphisms, and types as CwF isofibrations, which are displayed CwFs with an additional lifting structure for isomorphisms. In short, this means that the semantics of infinitary constructors can be only given mutually with a form of invariance under algebra isomorphisms. Recursive equations similarly require this kind of semantics.

However, strict sort equations are not invariant under isomorphisms. For example, if we have an isomorphism in $\mathbf{Set} \times \mathbf{Set}$ between (A, B) and (A', B'), and we also know that $A = B$ strictly, then it is not necessarily the case that $A' = B'$. This means that a strict semantics for sort equations is incompatible with the isofibration semantics for infinitary constructors. Our current solution is to simply keep the troublesome features apart. Hence we have

1. A theory of signatures supporting recursive equations and infinitary constructors, but no sort equations. This ToS can describe itself, and by a term model construction we can reduce all described QIITs to an assumed syntax of the same ToS. This term model construction is also weakened (i.e. it is up to algebra isomorphisms), hence it is significantly more complicated than in [5].

2. A theory of signatures supporting sort equations, but no recursive equations and infinitary constructors. This ToS is infinitary and has no sort equations, so we can give it a model theory as an infinitary QIIT. This ToS supports a stricter semantics which is not invariant under isomorphisms, and we also have a term model construction. Here, the semantics and the term models are straightforward extensions of [5].

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