Generalizations of Quotient Inductive-Inductive Types*

Ambrus Kaposi and András Kovács

Eötvös Loránd University, Budapest, Hungary
{akaposi|kovacsandras}@inf.elte.hu

Abstract. Quotient inductive-inductive types (QIITs) are generalized inductive types which allow sorts to be indexed over previously declared sorts, and allow usage of equality constructors. QIITs are especially useful for algebraic descriptions of type theories and constructive definitions of real, ordinal and surreal numbers. We develop new metatheory for large QIITs, large elimination, recursive equations, infinitary constructors and equations between sorts. As in prior work, we describe QIITs using a type theory where each context represents a QIIT signature. However, in our case the theory of signatures can also describe its own signature. We use self-description to bootstrap a model theory for the theory of signatures without using preterms or assuming a pre-existing internal syntax for a type theory. We give initial algebra semantics for described QIITs, and we show the equivalence of initiality and induction. We present two extensions of a previous term model construction. The first one constructs all large infinitary QIITs without sort equations from the QIIT of the theory of signatures. The second one constructs all large finitary QIITs with sort equations, from the same syntax. This separation is required because handling infinitary constructors in the term model requires showing a strong form of invariance under algebra isomorphisms, which is violated by sort equations.

1 Introduction

Citations

The aim of this work is to provide theoretical underpinning to a general notion of inductive types, called quotient inductive-inductive types (QIITs). QIITs are of interest because there are many commonly used mathematical structures, which can be conveniently described as QIITs in type theory, but cannot be defined as less general inductive types, or doing so incurs large encoding overhead.

Categories are a good example. Signatures for QIITs allow having multiple sorts, with later ones indexed over previous ones, and equations as well. We need

* This work was supported by EFOP-3.6.3-VEKOP-16-2017-00002 grant and COST Action EUTypes CA15123.

both features in order to write down the signature of categories:

$$\begin{aligned}
&Obj &&: \mathsf{Set} \\
&Mor &&: Obj \to Obj \to \mathsf{Set} \\
&id &&: Mor\,i\,j \\
&- \circ - &&: Mor\,j\,k \to Mor\,i\,j \to Mor\,i\,k \\
&idl &&: id \circ f = f \\
&idr &&: f \circ id = f \\
&ass &&: f \circ (g \circ h) = (f \circ g) \circ h
\end{aligned}$$

The benefit of having a QIIT signature is getting a model theory "for free", from the metatheory of QIITs. This model theory includes a category of algebras which has an initial object and also some additional structure. For the signature of categories, we get the empty category as the initial object, but it is common to consider categories with more structure, which have more interesting initial models.

Algebraic notions of models of type theories are examples for this. Here, initial models represent syntax, and initiality corresponds to induction on syntax. A number of different notions have been used, from contextual categories and comprehension categories to categories with families, but all of these are categories with extra structure.

The main motivation of the current paper is to extend QIITs so that it accommodates all algebraic notions of type theories which have been used in previous works. As a beneficial side effect of fulfilling this goal, infinitary QIITs such as real numbers will be covered as well.

We generalize previous notions of QIITs in the following ways:

1. **Large constructors, large elimination** and models at different universe levels. This feature is routinely used in the metatheory of type theory, but it has not been presented explicitly in previous works about QIITs. In order to handle universe levels gracefully, we develop a set of tools, working inside an extensional type theory with *algebraic cumulativity.*
2. **Infinitary constructors**. This enables defining Cauchy real numbers and surreal numbers as QIITs. Also, the theory of QIIT signatures is itself large and infinitary, thus it can "eat itself", i.e. include its own signature and provide its own metatheory. We use this self-representation to bootstrap the model theory of signatures, without having to assume any pre-existing internal syntax.
3. **Recursive equations**, i.e. equations appearing as assumptions of constructors. These have occurred previously in syntaxes of cubical type theories, as boundary conditions.
4. **Sort equations**, or equations between type constructors. They have been used recently to conveniently represent type-theoretic universes, for example Russell-style universes or cumulative hierarchies.

We also develop semantics. We show that for each signature, there is a cwf (category with families) of algebras, extended with Σ-types, an extensional identity type and constant families. This additional structure corresponds to a type-theoretic flavor of finite limits, and it was shown in [**?**] that the category of such cwfs is biequivalent to the category of finitely complete categories.

As to the existence of initial algebras, we present two different term model constructions, yielding the following statements:

1. All large infinitary QIITs with recursive equations, but *without sort equations*, are reducible to the theory of signatures, i.e. their initial algebras are constructible from the initial algebra for the theory of signatures.
2. All large finitary QIITs with sort equations, but no recursive equations, are reducible to the theory of signatures.

The reason for the two separate constructions is that sort equations are not as well-behaved as the other extensions: they are modelled as strict equalities of sets in algebras, and so they do not respect isomorphism of sets. We make essential use of invariance under set isomorphism in the first term model construction, so we cannot throw sort equations into that mix.

In Section 2 we set up cumulative universes which we use in later sections. In Section 3 we introduce the theory of QIIT signatures. In Section 4 we develop initial algebra semantics, also covering the theory of signatures itself. In Sections 5-6 we present the two term model constructions. We discuss related work and conclusions in Sections 7-8.

2 Cumulative Extensional Type Theory

In the following sections, we will consider QIIT algebras at arbitrary finite levels, along with large eliminations, where the initial algebra can be at a different level than the target algebra. For a simple example, consider natural number algebras at a given level i, given as the Σ-type $NatAlg_i :\equiv (Nat : \mathsf{Set}_i) \times Nat \times (Nat \to Nat)$. The initial such algebra is the set of natural numbers, which is at level 0, but in type theory we often want to eliminate into larger Set_i, for example when computing a Nat-indexed family of types.

Universe levels are usually viewed as a bureaucratic and tedious part of type theory, and are often omitted or mentioned only in passing. Here we intend to model universe levels in a precise way, because large eliminations are an essential part of the practical usage of inductive types. Hence, we need a way to conveniently reason about levels without overmuch administrative burden. Unfortunately, simple universe setups are insufficient for this purpose. We need two features: cumulativity and a way to quantify over finite levels.

2.1 Cumulativity

Cumulativity makes handling of algebras and constructions at different levels much easier. For example, given an algebra $\gamma : NatAlg_i$, cumulativity allows us to also have $\gamma : NatAlg_{i+j}$. In a non-cumulative setting such as Agda, we may only have for each type $A : \mathsf{Set}_i$ a type $\mathsf{Lift}_{i\,j}\,A : \mathsf{Set}_{i+j}$ which is isomorphic to A up to definitional equality.

In the most basic form, cumulativity requires that whenever $A : \mathsf{Set}_i$, then also $A : \mathsf{Set}_{i+j+1}$. However, the $NatAlg$ example above also required a form of cumulativity for Σ and Π types, and we need this in general for QIIT algebras.

We use Sterling's cumulative algebraic type theory [**?**] as the general setting for the rest of the paper. The reason for this is twofold. First, it supports the right kind of cumulativity for our purposes. Second, it is itself a finitary QIIT with sort equations, hence its model theory is covered in this paper[1].

We extend the base theory with an extensional identity type and Σ-types. In [**?**] a proof of canonicity is provided for the base theory, which also includes a standard set-theoretic model. It is straightforward to extend the canonicity proof to cover our additional type formers.

From now on, we refer to this theory as cETT (cumulative extensional type theory). When working in cETT, we use conventional type-theoretic notation. We have Russell-style universes Set_i indexed by natural numbers, dependent functions as $(x : A) \to B$, and dependent pairs as $(x : A) \times B$ with projections $\mathsf{proj}1$ and $\mathsf{proj}2$. We also have a *lifting* operation on types, which introduces cumulativity. The most important rules for lifting are the following:

Universe formation

$$\frac{}{\Gamma \vdash \mathsf{Set}_i : \mathsf{Set}_{i+j+1}}$$

Type lifting

$$\frac{\Gamma \vdash A : \mathsf{Set}_i}{\Gamma \vdash \Uparrow_j A : \mathsf{Set}_{i+j+1}}$$

Universe lifting

$$\Uparrow_j Set_i = Set_{i+j}$$

Term lifting

$$\{t \,|\, \Gamma \vdash t : A\} = \{t \,|\, \Gamma \vdash t : \Uparrow_j A\}$$

Context lifting

$$(\Gamma,\, x : A) = (\Gamma,\, x : \Uparrow_j A)$$

Function lifting

$$\Uparrow_j ((x : A) \to B) = (x : \Uparrow_j A) \to \Uparrow_j B$$

Pair lifting

$$\Uparrow_j ((x : A) \times B) = (x : \Uparrow_j A) \times \Uparrow_j B$$

Fig. 1. Some of the rules for lifting

In Figure 1, we include an excerpt of Sterling's rules, with a slight change of presentation. We use $i + j + 1$ levels instead of assuming $i < j$ and lifting

[1] Which is, of course, somewhat circular, but not more circular than e.g. the study of models of ZFC in ZFC.

to Set_j. We also omit rules for naturality and substitution of lifting. In general, we can expect that lifting never impedes constructions, because it appropriately computes out of the way.

Of special note is the *term lifting rule*. It is a sort equation, an equation between sets of terms, expressing that lifted types have exactly the same terms as unlifted ones. This allows us to have $t : \Uparrow_j A$ whenever $t : A$. Together with the universe lifting rule, this implies $A : Set_{i+j+1}$ whenever $A : Set_i$. Similarly, the lifting rules for functions and pairs give us cumulativity for $NatAlg$.

3 Theory of Signatures

3.1 Bootstrapping

4 Categories with Families of Algebras

5 Infinitary Term Models

5.1 Preservation of Isomorphisms

6 Term Models for Sort Equations

7 Related Work

8 Conclusion