Staged Compilation with Two-Level Type Theory

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18 January 2022, TKP Workshop

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- (Typed) Template Haskell.
- C++ templates.
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- C++ templates.
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Motivations:

- Low-cost abstraction.
- DSLs.
- Inlining & fusion with strong guarantees.

Two-Level Type Theory (2LTT)

Comes from **homotopy type theory**:

- *Voevodsky: A simple type system with two identity types*.
- *Annekov, Capriotti, Kraus, Sattler: Two-Level Type Theory and Applications*.
- Motivation: meta-programming and modular axioms for HoTT.

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   - Including dependent types.
4. Supports efficient *staging-by-evaluation*.

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For **formal details**, see the paper.

Rules of 2LTT

1. Two universes U_0, U_1, closed under arbitrary type formers.
   - U_0 is the universe of runtime (object-level) types.
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Staging runs all metaprograms in splices and inserts their result in the code output.

Inlined definitions

Staging input:

$$\mathsf{two} : {\Uparrow}\mathsf{Nat}_0$$
$$\mathsf{two} = \mathsf{<} \mathsf{suc}_0\,(\mathsf{suc}_0\,\mathsf{zero}_0)\mathsf{>}$$

$$\mathsf{f} : \mathsf{Nat}_0 \to \mathsf{Nat}_0$$
$$\mathsf{f} = \lambda\,\mathsf{x}.\,\mathsf{x} + {\sim}\mathsf{two}$$

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Output:

$$\mathsf{f} : \mathsf{Nat}_0 \to \mathsf{Nat}_0$$
$$\mathsf{f} = \lambda\,\mathsf{x}.\,\mathsf{x} + \mathsf{suc}_0\,(\mathsf{suc}_0\,\mathsf{zero}_0)$$

Compile-time identity function

Input:

$$\mathsf{id} : (\mathsf{A} : \mathsf{U}_1) \to \mathsf{A} \to \mathsf{A}$$
$$\mathsf{id} = \lambda\, \mathsf{A}\, \mathsf{x}.\, \mathsf{x}$$

$$\mathsf{idBool}_0 : \mathsf{Bool}_0 \to \mathsf{Bool}_0$$
$$\mathsf{idBool}_0 = \lambda\, \mathsf{x}.\, {\sim}(\mathsf{id}\, ({\Uparrow}\mathsf{Bool}_0)\, {<}\mathsf{x}{>})$$

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Output:

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An alternative identity function

Input:

$$\mathsf{id}_{\Uparrow} : (A : \Uparrow U_0) \to \Uparrow{\sim}A \to \Uparrow{\sim}A$$
$$\mathsf{id}_{\Uparrow} = \lambda\, A\, x.\, x$$

$$\mathsf{idBool}_0 : \mathsf{Bool}_0 \to \mathsf{Bool}_0$$
$$\mathsf{idBool}_0 = \lambda\, x.\, {\sim}(\mathsf{id}_{\Uparrow} <\mathsf{Bool}_0> <x>)$$

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$$\mathsf{idBool}_0 : \mathsf{Bool}_0 \to \mathsf{Bool}_0$$
$$\mathsf{idBool}_0 = \lambda\,\mathsf{x}.\,{\sim}(\mathsf{id}_{\Uparrow}\,{<}\mathsf{Bool}_0{>}\,{<}\mathsf{x}{>})$$

Note that

$$\begin{aligned} &\mathsf{A} &&: {\Uparrow}\mathsf{U}_0 \\ &{\sim}\mathsf{A} &&: \mathsf{U}_0 \\ &{\Uparrow}{\sim}\mathsf{A} &&: \mathsf{U}_1 \\ &{<}\mathsf{x}{>} &&: {\Uparrow}\mathsf{Bool}_0 \\ &{<}\mathsf{x}{>} &&: {\Uparrow}{\sim}{<}\mathsf{Bool}_0{>} \end{aligned}$$

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Note that

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Output:

$$\mathsf{idBool}_0 : \mathsf{Bool}_0 \to \mathsf{Bool}_0$$
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map with inlining

Input:

$$\mathsf{inlMap} : \{A\,B : \Uparrow U_0\} \to (\Uparrow {\sim}A \to \Uparrow {\sim}B) \to \Uparrow(\mathsf{List}_0\,{\sim}A) \to \Uparrow(\mathsf{List}_0\,{\sim}B)$$
$$\mathsf{inlMap} = \lambda\,f\,as.\;<\mathsf{foldr}_0\,(\lambda\,a\,bs.\;\mathsf{cons}_0\,{\sim}(f\,<a>)\,bs)\;\mathsf{nil}_0\,{\sim}as>$$

$$f\;:\;\mathsf{List}_0\,\mathsf{Nat}_0 \to \mathsf{List}_0\,\mathsf{Nat}_0$$
$$f\,=\,\lambda\,xs.\;{\sim}(\mathsf{inlMap}\,(\lambda\,n.\,<{\sim}n + 2>)\,<xs>)$$

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Inference for staging operations

Lifting preserves negative types up to definitional isomorphism:

$$\Uparrow \top_0 \simeq \top_1$$
$$\Uparrow ((a : A) \to B\,a) \simeq ((a : \Uparrow A) \to \Uparrow(B \sim a))$$
$$\Uparrow ((a : A) \times B\,a) \simeq ((a : \Uparrow A) \times \Uparrow(B \sim a))$$

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We can use **bidirectional elaboration** & **coercive subtyping along isos** to infer most quotes and splices.

$$\mathsf{inlMap} : \{A\,B : \Uparrow U_0\} \to (\Uparrow A \to \Uparrow B) \to \Uparrow (\mathsf{List}_0\,A) \to \Uparrow (\mathsf{List}_0\,B)$$
$$\mathsf{inlMap} = \lambda\, f.\,\mathsf{foldr}_0\,(\lambda\, a\, bs.\,\mathsf{cons}_0\,(f\,a)\,bs)\,\mathsf{nil}_0$$

$$f\ :\ \mathsf{List}_0\,\mathsf{Nat}_0 \to \mathsf{List}_0\,\mathsf{Nat}_0$$
$$f\ =\ \mathsf{inlMap}\,(\lambda\, n.\, n + 2)$$

Staging types

Input:

$$\begin{aligned}
&\mathsf{Vec} : \mathsf{Nat}_1 \to {\Uparrow}\mathsf{U}_0 \to {\Uparrow}\mathsf{U}_0\\
&\mathsf{Vec}\,\mathsf{zero}_1 \quad\;\; \mathsf{A} = <\top_0>\\
&\mathsf{Vec}\,(\mathsf{suc}_1\,\mathsf{n})\,\mathsf{A} = <{\sim}\mathsf{A} \times {\sim}(\mathsf{Vec}\,\mathsf{n}\,\mathsf{A})>
\end{aligned}$$

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&\mathsf{Tuple3} : \mathsf{U}_0 \to \mathsf{U}_0\\
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&\mathsf{Tuple3}\,\mathsf{A} = \mathsf{A} \times (\mathsf{A} \times (\mathsf{A} \times \top_0))
\end{aligned}$$

map for Vec

Input:

$$
\begin{aligned}
&\mathsf{map} : \{A\,B : {\Uparrow}U_0\} \to (n : \mathsf{Nat}_1) \to ({\Uparrow}{\sim}A \to {\Uparrow}{\sim}B) \\
&\qquad\qquad\qquad\quad \to {\Uparrow}(\mathsf{Vec}\,n\,A) \to {\Uparrow}(\mathsf{Vec}\,n\,B) \\
&\mathsf{map}\,\mathsf{zero}_1 \quad\; f\,as = <\mathsf{tt}_0> \\
&\mathsf{map}\,(\mathsf{suc}_1\,n)\,f\,as = <({\sim}(f\,<\mathsf{fst}_0\,{\sim}as>),\ {\sim}(\mathsf{map}\,n\,f\,<\mathsf{snd}_0\,{\sim}as>))>
\end{aligned}
$$

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&f : {\sim}(\mathsf{Vec}\,2\,<\mathsf{Nat}_0>) \to {\sim}(\mathsf{Vec}\,2\,<\mathsf{Nat}_0>) \\
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&f : \sim(\mathsf{Vec}\,2\,<\mathsf{Nat}_0>) \to \sim(\mathsf{Vec}\,2\,<\mathsf{Nat}_0>) \\
&f\,xs = \sim(\mathsf{map}\,2\,(\lambda\,x.\,<\sim x + 2>)\,<xs>)
\end{aligned}
$$

Output:

$$
\begin{aligned}
&f : \mathsf{Nat}_0 \times (\mathsf{Nat}_0 \times \top_0) \to \mathsf{Nat}_0 \times (\mathsf{Nat}_0 \times \top_0) \\
&f\,xs = (\mathsf{fst}_0\,xs + 2,\ (\mathsf{fst}_0\,(\mathsf{snd}_0\,xs) + 2,\ \mathsf{tt}_0))
\end{aligned}
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- Staged foldr/build fusion.
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Possible future research:

- Staging to low-level (e.g. first-order) languages.
- Staged fusion.
- Partially static data types.

Thank you!