A Generalized Logical Framework

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Overview

1. Two-level type theories (2LTT):
   - metaprogramming over a **single model** of a **single type theory**.
   - the chosen model is defined **outside the system**.
   - **only a second-order (“internal”)** view on the model.

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In this talk:

1. A syntax of GLF + examples + increasing amount of syntactic sugar.
2. A short overview of semantics.

GLF basic universes & type formers

$\mathbf{U}$	$: \mathbf{U}$	A universe of that supports ETT.
$\mathbf{Base}$	$: \mathbf{U}$	Type of "base categories".
$\mathbf{1}$	$: \mathbf{Base}$	The terminal category as a base category.
$\mathbf{PSh}$	$: \mathbf{Base} \to \mathbf{U}$	Universes of presheaves. Cumulativity: $\mathsf{PSh}_i \subseteq \mathsf{U}$. Supports ETT.
		We can only eliminate from PSh_i to PSh_i.
$\mathbf{Cat}_i$	$: \mathbf{PSh}_i$	$:=$ *type of categories in* PSh_i
$\mathbf{In}$	$: \mathbf{Cat}_i \to \mathbf{U}$	"Permission token" for working in presheaves over some $\mathbb{C} : \mathsf{Cat}_i$.
$\mathbf{base}$	$: \mathbf{In}\,\mathbb{C} \to \mathbf{Base}$	"Using the permission".

We use type-in-type everywhere for simplicity, i.e. $\mathsf{U} : \mathsf{U}$ and $\mathsf{PSh}_i : \mathsf{PSh}_i$.

Basic things we can do

$$\mathsf{U} : \mathsf{U} \qquad \mathsf{Base} : \mathsf{U} \qquad \mathbf{1} : \mathsf{Base} \qquad \mathsf{PSh} : \mathsf{Base} \to \mathsf{U}$$

$$\mathsf{Cat}_i : \mathsf{PSh}_i := \textit{type of cats in } \mathsf{PSh}_i \qquad \mathsf{In} : \mathsf{Cat}_i \to \mathsf{U} \qquad \mathsf{base} : \mathsf{In}\,\mathbb{C} \to \mathsf{Base}$$

PSh_1 is a universe supporting ETT (semantically: universe of sets).

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We can define some $\mathbb{C} : \mathsf{Cat}_1$, where $\mathsf{Obj}(\mathbb{C}) : \mathsf{PSh}_1$.

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At this point, we have no interesting interaction between PSh_1 and PSh_i.

Syntactic sugar: we'll omit "base" in the following.

Example: embedding pure lambda calculus

A **second-order model of pure LC** in PSh_i consists of:

$$
\begin{aligned}
\mathsf{Tm} &: \mathsf{PSh}_i \\
\mathsf{lam} &: (\mathsf{Tm} \to \mathsf{Tm}) \to \mathsf{Tm} \\
\_\$\_ &: \mathsf{Tm} \to \mathsf{Tm} \to \mathsf{Tm} \\
\beta &: \mathsf{lam}\, f \,\$\, t = f\ t \\
\eta &: \mathsf{lam}\,(\lambda x.\, t \,\$\, x) = t
\end{aligned}
$$

We define $\mathsf{SMod}_i : \mathsf{PSh}_i$ as the above Σ-type.

Example: embedding pure lambda calculus

A **first-order model of pure LC** consists of:

- A category of contexts and substitutions with $\mathsf{Con} : \mathsf{PSh}_i$, $\mathsf{Sub} : \mathsf{Con} \to \mathsf{Con} \to \mathsf{PSh}_i$ and terminal object $\bullet$.
- $\mathsf{Tm} : \mathsf{Con} \to \mathsf{PSh}_i$, plus a term substitution operation.
- A context extension operation $-\triangleright : \mathsf{Con} \to \mathsf{Con}$ such that $\mathsf{Sub}\,\Gamma\,(\Delta\triangleright) \simeq \mathsf{Sub}\,\Gamma\,\Delta \times \mathsf{Tm}\,\Gamma$.
- A natural isomorphism $\mathsf{Tm}\,(\Gamma\triangleright) \simeq \mathsf{Tm}\,\Gamma$ whose components are λ and application.

We define $\mathsf{FMod}_i : \mathsf{PSh}_i$ as the above Σ-type.

FMod is mechanically derivable from SMod.[1]

[1] Ambrus Kaposi & Szumi Xie: *Second-Order Generalised Algebraic Theories*.

Example: embedding pure lambda calculus

GLF Axiom 1

Assuming $M : \mathsf{FMod}_i$ and $j : \mathsf{In}\, M$, we have $\mathsf{S}_j : \mathsf{SMod}_j$.

(In "In M" we implicitly convert M to its underlying category.)

Now we have a 2LTT inside PSh_j:

- ETT type formers in PSh_j comprise the outer level.
- S_j comprises the inner level.

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Y-combinator as example:

$$\mathsf{YC} : \mathsf{Tm}_{\mathsf{S}_j}$$
$$\mathsf{YC} := \mathsf{lam}_{\mathsf{S}_j}(\lambda\, f.\, (\mathsf{lam}_{\mathsf{S}_j}(\lambda x.\, x \,\$_{\mathsf{S}_j}\, x))\, \$_{\mathsf{S}_j}\, (\mathsf{lam}_{\mathsf{S}_j}(\lambda f.\, \mathsf{lam}_{\mathsf{S}_j}(\lambda x.\, f \,\$_{\mathsf{S}_j}\, (x \,\$_{\mathsf{S}_j}\, x)))))$$

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With a reasonable amount of sugar:

$$\mathsf{YC} : \mathsf{Tm}_{\mathsf{S}_j}$$
$$\mathsf{YC} := \mathsf{lam}\, f.\, (\mathsf{lam}\, x.\, x\, x)\, (\mathsf{lam}\, f.\, \mathsf{lam}\, x.\, f\, (x\, x))$$

- More generally, we have the previous axiom for every second-order generalized algebraic theory.
- Hence: all 2LTTs are syntactic fragments of GLF.
- (For each 2LTT, the semantics of GLF restricts to the standard presheaf semantics of the 2LTT.)

Moving between internal & external views

GLF Axiom: Yoneda embedding for pure LC

Assuming $M : \mathsf{FMod}_i$, we have

$$\begin{aligned}
\mathsf{Y} : \mathsf{Con}_M &\to ((j : \mathsf{In}_M) \to \mathsf{PSh}_j) \\
\mathsf{Y} : \mathsf{Sub}_M\,\Gamma\,\Delta &\simeq ((j : \mathsf{In}_M) \to \mathsf{Y}\,\Gamma\,j \to \mathsf{Y}\,\Delta\,j) \\
\mathsf{Y} : \mathsf{Tm}_M\,\Gamma &\simeq ((j : \mathsf{In}_M) \to \mathsf{Y}\,\Gamma\,j \to \mathsf{Tm}_{\mathsf{S}_j})
\end{aligned}$$

such that Y preserves empty context and context extension up to iso:

$$\begin{aligned}
\mathsf{Y} \bullet j &\simeq \top \\
\mathsf{Y}\,(\Gamma \triangleright)\,j &\simeq \mathsf{Y}\,\Gamma\,j \times \mathsf{Tm}_{\mathsf{S}_j}
\end{aligned}$$

and Y preserves all other structure strictly.

Notation: we write Λ for inverse Y.

LC examples, sugar

Y and Λ allow ad-hoc switching between first-order and second-order notation. Let's redefine some operations using second-order notation:

$$\begin{aligned} &\mathsf{id} : \mathsf{Sub}_M\,\Gamma\,\Gamma && \mathsf{comp} : \mathsf{Sub}_M\,\Delta\,\Theta \to \mathsf{Sub}_M\,\Gamma\,\Delta \to \mathsf{Sub}_M\,\Gamma\,\Theta \\ &\mathsf{id} := \Lambda\,(\lambda\,j\,\gamma.\,\gamma) && \mathsf{comp}\,\sigma\,\delta := \Lambda\,(\lambda\,j\,\gamma.\,\mathsf{Y}\,\sigma\,(\mathsf{Y}\,\delta\,\gamma\,j)\,j \end{aligned}$$

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Or even:

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Example for "pattern matching" notation (Y preserves extended contexts):

$$\begin{array}{l} \mathsf{p} : \mathsf{Sub}_M\,(\Gamma \triangleright)\,\Gamma \\ \mathsf{p} := \Lambda\,(\gamma,\,\alpha).\,\gamma \end{array}$$

Second-order named notation

- When working with CwF-s, De Bruijn indices and substitutions can be hard to read.
- Handwaved “named” binders in CwFs have been used a couple of times.
- GLF provides a rigorous implementation of such notation.

Embedding type theories

In a first order model, we have:

$$\begin{aligned}
\mathsf{Con} &: \mathsf{PSh}_i \\
\mathsf{Sub} &: \mathsf{Con} \to \mathsf{Con} \to \mathsf{PSh}_i \\
\mathsf{Ty} &: \mathsf{Con} \to \mathsf{PSh}_i \\
\mathsf{Tm} &: (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{PSh}_i \\
&\dots
\end{aligned}$$

In a second order model, we have

$$\begin{aligned}
\mathsf{Ty} &: \mathsf{PSh}_i \\
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\end{aligned}$$

Sugar for contexts & sorts.

$$(\Gamma \triangleright A \triangleright B) : \mathsf{Con}_M \quad \text{is equal to} \quad \Gamma \triangleright (\Lambda\,\gamma.\mathsf{Y} A\,\gamma) \triangleright (\Lambda\,(\gamma,\,\alpha).\mathsf{Y} B\,(\gamma,\,\alpha))$$

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$$(\Gamma \rhd A \rhd B) : \mathsf{Con}_M \quad \text{is equal to} \quad \Gamma \rhd (\Lambda\,\gamma.\mathsf{Y} A\,\gamma) \rhd (\Lambda\,(\gamma, \alpha).\mathsf{Y} B\,(\gamma, \alpha))$$

This suggests the notation:

$$(\gamma : \Gamma,\, \alpha : \mathsf{Y} A\,\gamma,\, \beta : \mathsf{Y} B\,(\gamma, \alpha)) : \mathsf{Con}_M$$

With implicit Y:

$$(\gamma : \Gamma,\, \alpha : A\,\gamma,\, \beta : B\,(\gamma, \alpha)) : \mathsf{Con}_M$$

Embedding type theories

More "contextual" Sugar for Tm_M. We have

$$\mathsf{Tm}_M\,(\Gamma \triangleright A \triangleright B)\,C \;=\; \mathsf{Tm}_M\,(\Gamma \triangleright A \triangleright B)\,(\Lambda\,(\gamma,\,\alpha,\,\beta).\,B\,(\gamma,\,\alpha,\,\beta))$$

which suggests the notation

$$\mathsf{Tm}_M\,(\gamma : \Gamma,\,\alpha : A\,\gamma,\,\beta : B\,(\gamma,\,\alpha))\,(B\,(\gamma,\,\alpha,\,\beta))$$

Embedding type theories

Example: a construction which looks awful in explicit CwF notation[2]

$$
\begin{aligned}
\mathsf{Con}^\circ\,\Gamma &:= \mathsf{Ty}\,(F\,\Gamma)\\
\mathsf{Ty}^\circ\,\Gamma^\circ\,A &:= \mathsf{Ty}\,(F\,\Gamma \triangleright \Gamma^\circ \triangleright F\,A[\mathsf{p}])\\
\mathsf{Tm}^\circ\,\Gamma^\circ\,A^\circ\,t &:= \mathsf{Tm}\,(F\,\Gamma \triangleright \Gamma^\circ)\,(A^\circ[\mathsf{id},\, F\,t[\mathsf{p}]])\\
\Gamma^\circ \triangleright^\circ A^\circ &:= \Sigma(\Gamma^\circ[\mathsf{p}\circ F_{\triangleright.1}])(A^\circ[\mathsf{p}\circ F_{\triangleright.1}\circ \mathsf{p},\, \mathsf{q},\, \mathsf{q}[F_{\triangleright.1}\circ \mathsf{p}]])\\
&\ldots
\end{aligned}
$$

but is reasonable in sugary GLF notation:

$$
\begin{aligned}
\mathsf{Con}^\circ\,\Gamma &:= \mathsf{Ty}\,(\gamma : F\,\Gamma)\\
\mathsf{Ty}^\circ\,\Gamma^\circ\,A &:= \mathsf{Ty}\,(\gamma : F\,\Gamma,\, \gamma^\circ : \Gamma^\circ\,\gamma,\, \alpha : F\,A\,\gamma)\\
\mathsf{Tm}^\circ\,\Gamma^\circ\,A^\circ\,t &:= \mathsf{Tm}\,(\gamma : F\,\Gamma,\, \gamma^\circ : \Gamma^\circ\,\gamma)\,(A^\circ\,(\gamma,\, \gamma^\circ,\, F\,t\,\gamma))\\
\Gamma^\circ \triangleright^\circ A^\circ &:= \Lambda\,(F_{\triangleright.1}(\gamma,\, \alpha)).\, \Sigma(\gamma^\circ : \Gamma^\circ\,\gamma) \times A^\circ\,(\gamma,\, \gamma^\circ,\, \alpha)\\
&\ldots
\end{aligned}
$$

[2] Kaposi, Huber, Sattler: *Gluing for Type Theory*, Section 5

Semantics

Each PSh_i should be an universe of internal presheaves over an internal category.

We should work with **Cat** somehow, but there are issues with that:

- There's no general Π.
- Π-types of presheaves and universes of presheaves are not stable under reindexing by arbitrary functors.

In GLF, the categorical part (Base, In) is purely for bookkeeping, we can't do synthetic category theory. We can only do interesting things with presheaves.

We use *trees of categories* where tree morphisms only have interesting action on "discrete" parts of the tree.

Semantics

Notation:

- For category C and split fibration A over it, we write $C \triangleright A$ for the total category.
- For presheaf A, we write $\mathrm{Disc}\, A$ for the discrete fibration.

Definition. A *category telescope* is either the terminal category, or (inductively) of the form $C \triangleright \mathrm{Disc}\, A \triangleright B$ where C is a category telescope. We write $C : \mathrm{CatTel}$ for a category telescope.

Definition. A tree of categories is inductively defined as:

$$
\begin{aligned}
&\textbf{data}\ \mathrm{Tree}\,(B : \mathrm{CatTel}) : \mathrm{Set}\ \textbf{where} \\
&\quad \mathrm{node} : (\Gamma : \mathrm{PSh}\, B) \\
&\qquad\qquad (n : \mathbb{N}) \\
&\qquad\qquad (C : \mathrm{Fin}\, n \to \mathrm{Fib}\,(B \triangleright \mathrm{Disc}\, \Gamma)) \\
&\qquad\quad \to ((i : \mathrm{Fin}\, n) \to \mathrm{Tree}\,(B \triangleright \mathrm{Disc}\, \Gamma \triangleright C\, i)) \\
&\qquad\quad \to \mathrm{Tree}\, B
\end{aligned}
$$

Semantics

$$\mathsf{node} : (\Gamma : \mathsf{PSh}\, B)(n : \mathbb{N})(C : \mathsf{Fin}\, n \to \mathsf{Fib}\,(B \triangleright \mathsf{Disc}\,\Gamma)) \to ((i : \mathsf{Fin}\, n) \to \mathsf{Tree}\,(B \triangleright \mathsf{Disc}\,\Gamma \triangleright C\, i)) \to \mathsf{Tree}\, B$$

A GLF context is an element of Tree 1. Some examples for semantic contexts. We have $\mathbb{N}_i : \mathsf{PSh}_i$. We use $- \triangleright -$ for "context extension" in presheaves as well.

$$\begin{aligned}
\bullet \quad & := \mathsf{node}\, 1\, 0\, []\, [] \\
(\bullet \triangleright \mathbb{N}_1) \quad & := \mathsf{node}\,(1 \triangleright \mathbb{N})\, 0\, []\, [] \\
(\bullet \triangleright \mathbb{N}_1 \triangleright \mathsf{In}\, C) \quad & := \mathsf{node}\,(1 \triangleright \mathbb{N})\, 1\, [C]\, [\mathsf{node}\, 1\, 0\, []\, []] \\
(\bullet \triangleright \mathbb{N}_1 \triangleright i : \mathsf{In}\, C \triangleright \mathbb{N}_{(\mathsf{base}\, i)}) \quad & := \mathsf{node}\,(1 \triangleright \mathbb{N})\, 1\, [C]\, [\mathsf{node}\,(1 \triangleright \mathbb{N})\, 0\, []\, []]
\end{aligned}$$

Semantics

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\end{array}$$

- A Base points to a node of the tree.
- An In points to a subtree of a node.
- Extending a context with $A : \mathsf{PSh}_i$ extends the presheaf in node i.
- Extending a context with $j : \mathsf{In}\, C$ for $C : \mathsf{Cat}_j$ adds a new subtree at node j.