Type-Theoretic Signatures for Algebraic Theories and Inductive Types

Theses of the Ph.D. Dissertation

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1 Introduction

The main goal of the thesis is to develop certain type theories as specification languages for algebraic theories and inductive types. In each type theory of signatures presented in the thesis, typing contexts specify algebraic theories by listing sorts, operations and equations.

The use of dependent type theories as specification languages confers significant expressiveness and allows us to develop their metatheory using standard methods from the broader metatheory of type theories.

We present three theories of signatures, in order of increasing expressiveness. In all three cases, there are further possible variations and design choices.

The current results extend and generalize prior work on signatures for inductive types, in the context of type theory. A primary motivation for the current results was to develop more expressive inductive types for future proof assistants. Thus, our syntaxes and semantics of signatures are close to what would be required in practical implementation. However, our results can be also viewed in the broader mathematical context of the study of algebraic theories.

2 Contributions

The main contributions are summarized in the following four theses.

Thesis 1

In Chapter 3 we describe a way to use two-level type theory [ACKS19] as a metalanguage for developing semantics of algebraic signatures. This makes it possible to work in a concise internal notation of a type theory, and at the same time build semantics internally to arbitrary structured categories. For example, the signature for natural number objects can be interpreted in any category with finite products.

Thesis 2

We present syntax and semantics for finitary quotient inductive-inductive (FQII) signatures in Chapter 4 of the thesis. These are close in expressive power to Cartmell's generalized algebraic theories [Car86], but differ in formalization and what kind of semantics results and constructions are built around them.

- FQII signatures can describe most type theories in the wild, thus providing a model theory for them through the semantics of signatures.
- The theory of FQII signatures is specified compactly as a type theory, and it is itself amenable to algebraic specification.
- For each signature a finitely complete category of algebras is given. This category is presented as a cwf (category with families, see [CCD19]) with certain type formers, which makes it possible to exactly compute notions of induction. We show that induction is equivalent to initiality in each category of algebras.
- We show that initial algebras can be constructed from the syntax of FQII signatures, by a term algebra construction. In turn, we show that certain fragments of the syntax of FQII signatures can be reduced to basic type formers, thereby reducing some of the initial algebras to basic type formers.
- We show that substitutions of signatures can be viewed as model constructions, being functors between categories of algebras in the semantics. Additionally, under the assumption that initial FQII-algebras exist, every such functor has a left adjoint.

Thesis 3

In Chapter 5, we modify FQII signatures to obtain infinitary quotient inductive-inductive signatures. This allows us to describe infinitely branching trees as initial algebras.

- Real numbers, surreal numbers, ordinals and the cumulative hierarchy of sets [Uni13] can be now specified using signatures.

- Additionally, theories of FQII and infinitary QII signatures can be themselves described with infinitary QII signatures. This self-description can be utilized to bootstrap the metatheory of theories of signatures, starting from minimal assumptions.

- The semantics of signatures is extended to include *iso-fibrancy* of signature types; this means that every construction in the theory of signatures respects isomorphisms of described algebras.

- We adapt constructions of term algebras and left adjoint functors to the current setting.

- We show that signatures have semantic interpretation internally to the theory of signatures itself. This implies, in particular, that for each signature, the notion of algebra morphisms can be still specified with a signature.

Thesis 4

In Chapter 6, we describe higher inductive-inductive signatures. These differ from the previous signatures mostly in their intended semantics, whose context is now homotopy type theory [Uni13], and which allows specified equalities to be proof-relevant. The higher-dimensional generalization of types and equalities makes semantics more complicated, so we only present enough semantics to specify notions of initiality and induction for each signature. Additionally, we consider two different notions of algebra morphisms: one preserves structure strictly (up to definitional equality), while the other preserves structure up to paths.

3 Publications

The above contributions build on and extend the following previous publications, all coauthored by the thesis' author.

1. *A Syntax for Higher Inductive-Inductive Types* [KK18].

2. *Signatures and Induction Principles for Higher Inductive-Inductive Types* [KK20a].

3. *Constructing Quotient Inductive-Inductive Types* [KKA19].

4. *Large and Infinitary Quotient Inductive-Inductive Types* [KK20b].

5. *For Finitary Induction-Induction, Induction is Enough* [KKL19].

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