Review of "Type-Theoretic Signatures for Algebraic Theories and Inductive Types" by András Kovács

Dr Fredrik Nordvall Forsberg

December 1, 2021

Topic evaluation and summary This is a dissertation in the area of the theory of data types, universal algebra and more generally algebraic specification. This is a foundational topic, underlying almost all uses of type theory and its implementation in proof assistants to formalise mathematics. It is also very timely, because advances in homotopy type theory and cubical type theories in recent years has introduced a new concept of data types in the form of *(quotient) inductive-inductive types* (QIITs) and *higher inductive-inductive types* (HIITs), thus requiring new research into both their syntax and semantics, which is exactly what this dissertation contributes. In simpler logical systems and programming languages, data types are often straightforward to describe, but not so in dependent type theory, where mutual dependency and proof-relevant equations introduces new challenges. This is thus a very interesting topic in type theory.

After a short introduction, the dissertation starts with a pedagogical chapter introducing the concepts and techniques of the dissertation in the setting of simply typed signatures. This is a great help for getting the reader ready for the more complicated constructions in later chapters, as the basic idea stays the same: algebraic structures are specified by typing contexts of models of a theory of signatures, and their semantics is given by interpreting them in certain models of the theory of signatures. This gives a notion of algebra, algebra morphism, induction, etc. It is then shown that an initial term algebra can be constructed, which supports induction.

In the next chapter, the heavily used tool of (a variation of) Two-Level Type Theory (2LTT) is introduced. It is shown how the presheaf semantics of 2LTT can be used to interpret signatures in arbitrary models of type theory, whilst still working in the internal language of the presheaf category, which reduces boilerplate and simplifies reasoning.

In Chapter 4, 2LTT is used to describe a theory of signatures for finitary QIITs. This is a natural and elegant extension of simply typed signatures, which is enough to describe many naturally occurring data types. Most of the chapter is dedicated to constructing a model of type theory consisting of models of type theory itself, which is then used, together with the 2LTT presheaf technique, to interpret finitary QII signatures in arbitrary models. A term model construction is again given, assuming an extensional type theory as a metatheory.

Chapter 5 drops the finitariness assumptions and describes also infinitary QIITs. Again this is a very natural extension at the level of signatures, but requires a careful adaption of the semantics to cope with additional type formers appearing in signatures: these are not preserved on the nose, but only up to isomorphism, and so types need to be interpreted not just as mere families, but as isofibrations, i.e., as families respecting isomorphisms. This again supports a term model construction in extensional type theory. It is shown that the theory of signatures is expressive enough to be self-describing.

In the final chapter, another assumption is dropped, namely that of uniqueness of identity proofs. Again, on the signature side, not much happens, but on the semantic side, the more general setting introduces several complications when dealing with higher coherences in the models. Accordingly, results here are more limited compared to previous chapters: no *category* of algebras is constructed, only a type, and the equivalence of induction and initiality is not proven, nor the construction of a term model. However this is still an impressive finale of an impressive dissertation.

Overall, the dissertation has a logical structure, with complexity building up gradually chapter by chapter. The only thing missing in this regard is a short final concluding chapter which summarises the dissertation and discusses future directions.

Research methods and new scientific discoveries The research methods of using type theories to specify signatures, and working within 2LTT to reduce bureaucracy are modern, novel and very innovative. There are appropriate discussion of and references to related work throughout the dissertation. The following new scientific discoveries are contributed:

- That the notion of algebra homomorphisms are given by logical relations interpretations of signatures (Section 2.2.2), displayed algebras by a logical predicate interpretation (Section 2.2.3), and sections and induction by a "dependent" logical relations interpretation (Section 2.2.4);
- That initial algebras can be constructed using the formal terms of the type theory of signatures (Sections 2.3, 4.4, 5.6);
- That 2LTT can be used to reason about strict (and weak) computation rules when formalising algebras and algebra morphisms, while still working internally in a type theory (Section 3.5);
- That there is an easy to work with axiomatisation of finititary QIITs (Section 4.1), and that it can be given a semantics using a finite limit category with families (flcwf) of flcwfs, and that induction is equivalent to initiality in all such models (Section 4.2);
- Similarly that there is an easy to work with axiomatisation of infinititary QIITs (Section 5.1), and that it can be given a semantics using flcwfs where types are isofibrations (Section 5.2);
- That there also is a syntax for HIITs using a theory of signatures, using logical relations interpretations for their strict or weak semantics (Sections 6.1 and 6.2);
- That if the theory of signatures is closed under unit and dependent pair types, and each signature has an initial algebra, then there is a syntactic presentation of a left adjoint for each syntactically presented functor between algebras (Theorem 4 in Section 4.2 (for finitary QIITs), and Theorem 9 in Section 5.3 (for infinitary QIITs));
- That the theory of infinitary QIIT signatures is self-describing, and if the signature describing the theory of signatures has an initial algebra, then all signatures have an initial algebra (Theorem 6 in Section 4.6); and
- Furthermore that the logical relations interpretations that underlie many of the model constructions in the dissertation can themselves be described syntactically internally to the theory of signatures (Section 5.4).

This is an impressive list of results. However many of them could be better signposted in the text, for example by explicitly stating a theorem at the end of a section.

The attached short theses accurately and adequately describe the above results. They have also been reported in five high-quality publications that have appeared at highly prestigious venues. It is thus clear that the dissertation makes an important contribution to science.

Questions and comments Overall, the dissertation is of a very high standard, but I have some questions and comments that I hope could improve it even further.

1. In Section 2.2.1, you use a "functional" representation of finite products, but in Notation 2 (and Definition 38), you make use of a more first-order presentation. You could prove this equivalence between functional and first-order products explicitly.

2. In Section 3.1, you remark that specifications are more compact if everything is natural by construction, and suggest HOAS as a way to make this idea precise. Could one also imagine a less radical solution making use of a "universe of natural constructions", closed under sensible operations, which could slot in more directly into the 2LTT approach that you follow, rather than adopting a the different framework of HOAS?

3. On page 33, you could note that a CwF supporting constant families implies that it also supports unit types (especially since you implicitly use this later).

4. On page 53, you mention that the recursion principle you get is not the same as the "usual recursion principle", because of the extra dependency on the predecessor natural number. This is in fact the difference between the terminology of primitive recursion and primitive iteration in recursion theory — see for example Kleene, *Recursive function theory* 1981 (DOI: 10.1109/MAHC.1981.10004) for an exposition.

5. On page 54, you remark that inner induction principles are not needed to construct term algebras, but do you need the inner induction principles to actually show that the term algebra is initial?

6. Definition 43 when read "internally" says that the context Γ is inconsistent: every type is inhabitated. Can you comment? There is an echo of this in the proof of Theorem 1 when proving uniqueness: assuming that Γ is inductive, we can make any equation we want true, by using the identity types of the interpretation of Γ. Could it be made clearer that this is the same as the more elementary proof, where we show uniqueness by reducing to the case where the input argument is a constructor, so that the functions agree since they both are algebra morphisms, and then by the induction hypothesis? (I think that follows from the definition of the identity type, see for example on page 82, but it is unclear to me what it means for "false" equations — is the principle then vacuously true, because the equation is false already given algebra as an algebra?)

7. In Section 4.4.1, could you use $\leq$ to simplify type constructors to be "levelwise" only, rather than returning in the lub of levels?

8. In Definition 55, is there any advantages and trade-offs in the bootstrap signature being polymorphic in the level, rather than fixing a level beforehand?

9. At the end of Section 4.5 (also see Section 5.7), you say that we are able to derive all required concepts just from the notion of ToS model; can you clarify if this also includes for example constructing the term algebra, or if this requires additional assumptions such as an induction principle?

10. In Section 5.2, you drop sort equations, because they are not stable under isomorphisms. However, could they instead be interpreted not as identities, but as isomorphisms?

Quality of writing The dissertation is written in a clear way, with good grammar and spelling (a very small list of typos can be found at the end of the review). The subject matter is necessarily technical, but the author has done a very good job of highlighting the important aspects and overall making the dissertation readable. The same applies to the English summary of the dissertation.

Recommendation In my opinion, the dissertation can be accepted already in its current form. However I hope that the candidate would be willing to address (at least some of) the questions and comments above, and to consider writing a short concluding chapter, including a discussion of future work.

List of minor corrections

The following list of corrections should be easy to fix.

Pages 4, 171 Missing reference **[?]** to Agda formalisation.

Page 9 Typo "constructior".

Page 29, Def. 12 Dybjer's terminology is "Category with famil**ies**", not "... with famil**y**" — there are both Ty and Tm families, after all.

Page 46 In the definition of the exponential, there are some shifts in the variable names used; better would be $\mathsf{Sub}(\Xi \otimes \Gamma, \Delta) \cong \mathsf{Sub}(\Xi, \Delta^{\Gamma})$.

Page 66 Typo: "over a inductive" should be "over inductive".

Page 67 Typo: $\mathsf{List}\,\mathsf{Tree}$ should be List.

Page 83 Typo: "On Figure 4.1" should be "In Figure 4.1".

Page 89 $\mathsf{Fin} :\equiv \mathsf{Nat}$ should be $\mathsf{Fin} :\equiv \lambda^{ext}\_.\, \mathsf{Nat}$.

Page 128 In the derivation of the Σ context isomorphism, a first intermediate step like in the K one would help the reader.

References Some references refer to preprint versions rather than published versions (e.g. [AAC+20], [Kov21]); many uppercase letters have been BibTex rendered lowercase (e.g. [AHW16], [AKS15], [AMM19], [CD14], [JVWW06], [Kis14], [PV07], [SAG20], [TS18]); some entries have spurious URLs (e.g. [Awo10], [Hed98], [KK18], [ML98], [Scch17]); entry [CS] is missing a year, and [Hof95] is a PhD thesis. There is an extra "In" in [HS96].