Review of "Type-Theoretic Signatures for Algebraic Theories and Inductive Types" by András Kovács

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Topic evaluation and summary This is a dissertation in the area of the theory of data types, universal algebra and more generally algebraic specification. This is a foundational topic, underlying almost all uses of type theory and its implementation in proof assistants to formalise mathematics. It is also very timely, because advances in homotopy type theory and cubical type theories in recent years has introduced a new concept of data types in the form of *(quotient) inductive-inductive types* (QIITs) and *higher inductive-inductive types* (HIITs), thus requiring new research into both their syntax and semantics, which is exactly what this dissertation contributes. In simpler logical systems and programming languages, data types are often straightforward to describe, but not so in dependent type theory, where mutual dependency and proof-relevant equations introduces new challenges. This is thus a very interesting topic in type theory.

After a short introduction, the dissertation starts with a pedagogical chapter introducing the concepts and techniques of the dissertation in the setting of simply typed signatures. This is a great help for getting the reader ready for the more complicated constructions in later chapters, as the basic idea stays the same: algebraic structures are specified by typing contexts of models of a theory of signatures, and their semantics is given by interpreting them in certain models of the theory of signatures. This gives a notion of algebra, algebra morphism, induction, etc. It is then shown that an initial term algebra can be constructed, which supports induction.

In the next chapter, the heavily used tool of (a variation of) Two-Level Type Theory (2LTT) is introduced. It is shown how the presheaf semantics of 2LTT can be used to interpret signatures in arbitrary models of type theory, whilst still working in the internal language of the presheaf category, which reduces boilerplate and simplifies reasoning.

In Chapter 4, 2LTT is used to describe a theory of signatures for finitary QIITs. This is a natural and elegant extension of simply typed signatures, which is enough to describe many naturally occurring data types. Most of the chapter is dedicated to constructing a model of type theory consisting of models of type theory itself, which is then used, together with the 2LTT presheaf technique, to interpret finitary QII signatures in arbitrary models. A term model construction is again given, assuming an extensional type theory as a metatheory.

Chapter 5 drops the finitariness assumptions and describes also infinitary QIITs. Again this is a very natural extension at the level of signatures, but requires a careful adaption of the semantics to cope with additional type formers appearing in signatures: these are not preserved on the nose, but only up to isomorphism, and so types need to be interpreted not just as mere families, but as isofibrations, i.e., as families respecting isomorphisms. This again supports a term model construction in extensional type theory. It is shown that the theory of signatures is expressive enough to be self-describing.

In the final chapter, another assumption is dropped, namely that of uniqueness of identity proofs. Again, on the signature side, not much happens, but on the semantic side, the more general setting introduces several complications when dealing with higher coherences in the models. Accordingly, results here are more limited compared to previous chapters: no *category* of algebras is constructed, only a type, and the equivalence of induction and initiality is not proven, nor the construction of a term model. However this is still an impressive finale of an impressive dissertation.

Overall, the dissertation is well organised, with clear and helpful sections and subsections, and

complexity building up gradually chapter by chapter. Two appendices summarise the technical definitions underlying the main interpretations of the thesis.

Research methods and new scientific discoveries The research methods of using type theories to specify signatures, and working within 2LTT to reduce bureaucracy are modern, novel and very innovative. There is appropriate discussion of, and references to, related work throughout the dissertation. The following new scientific discoveries are contributed:

- that the notion of algebra homomorphisms are given by logical relations interpretations of signatures (Section 2.2.2), displayed algebras by a logical predicate interpretation (Section 2.2.3), and sections and induction by a "dependent" logical relations interpretation (Section 2.2.4);
- that initial algebras can be constructed using the formal terms of the type theory of signatures (Sections 2.3, 4.4, 5.6);
- that 2LTT can be used to reason about strict (and weak) computation rules when formalising algebras and algebra morphisms, while still working internally in a type theory (Section 3.5);
- that there is an easy to work with axiomatisation of finititary QIITs (Section 4.1), and that it can be given a semantics using a finite limit category with families (flcwf) of flcwfs, and that induction is equivalent to initiality in all such models (Section 4.2);
- similarly that there is an easy to work with axiomatisation of infinititary QIITs (Section 5.1), and that it can be given a semantics using flcwfs where types are isofibrations (Section 5.2);
- that there also is a syntax for HIITs using a theory of signatures, using logical relations interpretations for their strict or weak semantics (Sections 6.1 and 6.2);
- that if the theory of signatures is closed under unit and dependent pair types, and each signature has an initial algebra, then there is a syntactic presentation of a left adjoint for each syntactically presented functor between algebras (Theorem 4 in Section 4.2 (for finitary QIITs), and Theorem 9 in Section 5.3 (for infinitary QIITs));
- that the theory of infinitary QIIT signatures is self-describing, and if the signature describing the theory of signatures has an initial algebra, then all signatures have an initial algebra (Theorem 6 in Section 4.6); and
- furthermore that the logical relations interpretations that underlie many of the model constructions in the dissertation can themselves be described syntactically internally to the theory of signatures (Section 5.4).

This is an impressive list of results. However many of them could be better signposted in the text, for example by explicitly stating a theorem at the end of a section.

The attached short theses accurately and adequately describe the above results. They have also been published in five high-quality articles that have appeared at highly prestigious venues. It is thus clear that the dissertation makes an important contribution to science.

Questions for the disputation

1. The idea of bootstrap signatures is an elegant way to avoid assuming that there exists an initial model of the theory of signatures. However to construct term algebras, you need to assume the additional principle of elimination (Definition 57); how hard would it be to construct a concrete model of ToS with elimination, in a concrete model of the ambient type theory?

2. The Böhm-Berarducci encoding of data types in System F in general only gives weakly initial algebras, but it can be shown that these are actually initial in parametric models (see e.g. Ghani, N.-F. and Simpson, *Comprehensive Parametric Polymorphism*, 2016, 10.1007/978-3-662-49630-5_1). Do you think a similar approach could work for bootstrap signatures?

3. For future work, can you see any limitations to the technique of the theory of signatures? Could you for example imagine treating inductive-recursive definitions in a similar way?

Quality of writing The dissertation is written in a clear way, with good grammar and spelling — I could not find any typos in the latest version of the thesis. The subject matter is necessarily technical, but the author has done a very good job of highlighting the important aspects and overall making the dissertation readable. The same applies to the English summary of the dissertation.

Recommendation I recommend the thesis for public defence. My recommendation would be to award the qualification *summa cum laude*.

Signed:

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Glasgow, 5 August 2022