Some Finite State Aspects of Legged Locomotion*

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ABSTRACT

Animal locomotion systems making use of legs as the basic component for support and propulsion can be studied from the point of view of finite state machine theory by regarding each leg as an elementary two-state sequential machine. The two states are simply the state of being in contact with the supporting surface and the state of being raised above it. This idealization permits the construction of a general theory of locomotion equally applicable to animals and legged locomotion machines. Such a theory can be made sufficiently complete to permit the synthesis of finite control algorithms capable of coordinating limb movements in either animals or machines. The validity of the finite state approach has been established by the construction and testing of an artificial quadruped based entirely upon finite state principles.

INTRODUCTION

Natural and artificial systems for land locomotion have developed along quite different lines. Whereas mechanical systems are based primarily upon the wheel, terrestrial animals generally make use of either legs or segmented bodies to achieve forward motion. Whatever the source of motive power, the forward progress of such animals typically results from a periodic alternation of forward and backward movement of certain parts of the animal relative to the central part of its body. In the case of legged vertebrates and insects, the organs responsible for locomotion have evolved into distinct segmented legs that both support the body of the animal and propel it forward.

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The possession of legs provides an animal with the possibility of using any of a number of distinct modes of locomotion depending on the nature of the terrain being traversed and the speed desired. These modes, ordinarily called *gaits*, are distinguished one from the other by the sequence in which the feet of the animal are placed upon the supporting surface and lifted from it. Despite the very early recognition and naming of such common gaits as the walk, trot, gallop, and so on, a careful scientific investigation of animal locomotion became possible only after the invention of the motion picture camera. The first person to undertake such a study was Eadweard Muybridge who, in 1872, began a research program that resulted eventually in two books that are still regarded as basic authorities on the movements and gaits natural to most animals [1, 2]. Much more recently, Hildebrand [3, 4] has carried out an exhaustive study of quadruped locomotion with modern photographic equipment and obtained results that substantially extend the work of Muybridge [1, 2].

The purpose of this paper is to imbed the results of Muybridge [1, 2], Hildebrand [3, 4], and others in a more general mathematical theory of locomotion. Following the suggestions of Tomovic and Karplus [5, 6], a finite state point of view will be adopted. In particular, the notion of a *binary output state* for each leg will be introduced by recognizing that at any given instant a particular leg is in just one of two conditions: on the ground pushing backward or in the air moving forward. It will be seen that this idealization of the actual complex physical state of a leg permits meaningful questions regarding locomotion to be posed and answered in purely mathematical terms without regard to the characteristics of a particular animal or species. Moreover, it will also be shown that a finite state theory can be made sufficiently complete to permit the analysis, design, and construction of artificial legged locomotion systems [7]. In this paper, no distinction will be made between natural and artificial locomotion machines. The results obtained are intended to apply equally well to either case.

LOCOMOTION AUTOMATA

The construction of a finite state theory of locomotion begins with the definition of a number of concepts. Insofar as possible, the language used in these definitions conforms to the terminology of both physiology and anatomy on the one hand and of the mathematical theory of sequential machines [8] on the other hand.

Definition 1. A *leg* is a sequential machine m with just two *output states*, 1 and 0. By convention, the state $m = 0$ will be taken to represent the state of being in contact with a supporting surface while the state $m = 1$ will correspond to a leg that is raised above this surface [5]. These two states will also be referred to as the *support phase* and the *transfer phase* of a leg cycle [9].

Definition 2. A locomotion automaton M is an indexed set of legs $M = \{m_1, m_2, \ldots, m_k\}$.

Definition 3. A *gait* for a locomotion automaton composed of k legs is a periodic sequence of binary k-tuples representing the successive states of the legs of the automaton. Within any complete cycle of a gait, the state of every leg must change *exactly once* from 0 to 1 and from 1 to 0.

Definition 4. A *gait matrix* G is a k-column matrix whose successive rows are binary k-tuples corresponding to the successive states of a particular gait of a k-legged locomotion automaton and whose total number of rows is equal to the length of one cycle of the gait sequence. Such a matrix will be said to *represent* the associated gait. A gait matrix is in *canonical form* when it is written so that the first row begins with a 0 and the last row begins with a 1.

Definition 5. The *duration vector* $\mathbf{t}$ for an n-row gait matrix G is an $n \times 1$ array whose entries specify the time duration of the corresponding rows of G. The *cycle time* τ for a given duration vector $\mathbf{t}$ is defined as

$$\tau = \sum_{i=1}^{n} t_i, \qquad t_i > 0. \tag{1}$$

Definition 6. A *gait realization* is a specific locomotion matrix G together with a particular duration vector $\mathbf{t}$. Such a pair, denoted $\{G, \mathbf{t}\}$, will be said to *realize* the gait *represented* by G.

Definition 7. The 0-*duration* τ_0 for a leg m_i is the total amount of time that m_i is in a 0 state during one cycle of a gait described by a specific gait realization $\{G, \mathbf{t}\}$. The *duty factor* β is the relative amount of time that a leg spends in the zero state; that is,

$$\beta = \frac{\tau_0}{\tau}. \tag{2}$$

Definition 8. The *transition delay time* d_i for a leg m_i is the time interval between the occurence of the 1-to-0 transition of leg m_1 and the following 1-to-0 transition of m_i within any cycle of a gait specified by a given gait realization. The *relative phase* of leg i is defined by

$$\phi_i = \frac{d_i}{\tau}. \tag{3}$$

Definition 9. A *gait formula* $\mathbf{g}$ for a particular mode of locomotion of a k-legged automaton is a point in a unit $(2k - 1)$-cube defined by

$$\mathbf{g} = (\beta_1, \beta_2, \ldots, \beta_k, \phi_2, \phi_3, \ldots, \phi_k)'. \tag{4}$$

A gait formula will be said to *implement* a particular gait.

Definition 1 was first suggested by Tomovic [10]. Definitions 2 through 8 were proposed by Meisel and McGhee [11]. Definition 9 is a generalization of Hildebrand's notion of a gait formula [3, 4]. It should be pointed out that although Definition 3 seems to be satisfied by all natural locomotion systems, there is no inherent reason why it could not be violated. That is, it should be possible for some legs of a locomotion system to be cycled at a higher rate than others. Since there is no evident reason why this should be done, however, only locomotion modes conforming to Definition 3 will be considered further here.

The foregoing definitions permit the following elementary theorems to be stated [11].

THEOREM 1. *For any specified gait matrix G and duration vector* $\mathbf{t}$ *the period of the cyclic alternation of output states is the same for all legs and is equal to τ, the cycle time of the gait.*

THEOREM 2. *The maximum number of rows in the locomotion matrix for a k-legged automaton is $2k$.*

Theorem 1 follows immediately from the fact that the sum in (1) is independent of the order of summation. Theorem 2 is proved by noting that the definition of a gait requires that each leg change from 0 to 1 and 1 to 0 exactly once in each locomotion cycle.

The following basic theorem permits a gait realization to be described in either of two equivalent ways and establishes a connection between k-column gait matrices and points in a unit $(2k - 1)$ cube.

THEOREM 3. *Let $U = \{G, \mathbf{t}\}$ be the set of all possible k-column canonical gait matrices G and associated duration vectors* $\mathbf{t}$, *and let $V = \{\mathbf{g}, \tau\}$ be the set of all $(2k - 1)$-element gait formulas* $\mathbf{g}$ *together with an arbitrary positive cycle time* τ. *Then the elements of U correspond one-to-one to the elements of V.*

Proof. Given $u \in U$, since the number of rows in G is bounded by $2k$, it is possible to compute both τ and τ_0 for each leg m_i by summing over the appropriate components of $\mathbf{t}$. Consequently, all k of the duty factor variables β_i can be evaluated. In a similar way, the delay time d_i can be obtained for each leg, so that the phase variables ϕ_i are also determined. Obviously, these calculations associate a unique $v \in V$ with the given $u \in U$.

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{t} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}. \qquad \text{(a)}$$

$$\mathbf{g} = \begin{pmatrix} 11/12 \\ 11/12 \\ 11/12 \\ 11/12 \\ 1/2 \\ 3/4 \\ 1/4 \end{pmatrix}, \qquad \tau = 12. \qquad \text{(b)}$$

FIG. 1. Alternate descriptions of a quadruped crawl. (a) Gait matrix and duration vector; (b) gait formula and cycle time.

If $v \in V$ is specified, then if the time origin $t = 0$ is chosen to correspond to the 1-to-0 transition of leg m_1, it is possible to construct $m_i(t)$ for all i and all $t \geqslant 0$. Whenever a change from 0 to 1 or 1 to 0 occurs in any of the k leg states, a new entry in the gait matrix is generated along with the duration of the previous entry. Since $m(t + \tau) = m(t)$, the construction of G and $\mathbf{t}$ can be completed by an examination of leg states only over the interval $0 \leqslant t \leqslant \tau$.

Figure 1 presents both the $\{G, \mathbf{t}\}$ and $\{\mathbf{g}, \tau\}$ descriptions of a particular realization of a slow quadruped walk, sometimes called a *crawl* or *creep* [1, 5]. Figure 2 shows the corresponding state functions $m_i(t)$. This graphical type of representation for a gait realization was introduced by Hildebrand, who called such a figure a "gait diagram" [4].

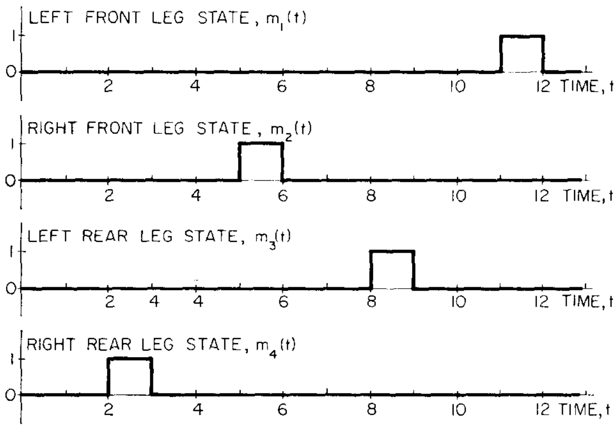

FIG. 2. Leg state functions for one cycle of a quadruped crawl.

From the proof of Theorem 3, the following corollary is immediately apparent.

COROLLARY 1. *A gait realization* $\{G, \mathbf{t}\} = \{\mathbf{g}, \tau\}$ *implies a gait formula* $\mathbf{g}$, *which in turn implies a gait matrix* G.

The converse of this corollary is not generally true; some gaits can be implemented by more than one gait formula [4] and any gait can be realized by some gait formula with an arbitrary positive cycle time τ.

Because of the generality of the definition of locomotion automata, it is difficult to obtain additional results without the introduction of further concepts. In the following discussion, several import types of gaits are defined and the consequences of these definitions are explored.

FURTHER PROPERTIES OF GAITS

The following definitions are relevant to the further consideration of locomotion automata.

Definition 10. A gait formula $\mathbf{g}$ is *regular* if the duty factor of every leg is the same as that of every other leg. A gait represented by a gait matrix G is *regularly realizable* if there exists a $\mathbf{t}$ such that $\{G, \mathbf{t}\}$ implies a regular $\mathbf{g}$.

Definition 11. For a given gait formula $\mathbf{g}$ any two legs of a locomotion automaton constitute a *symmetric pair* if the duty factor of either leg is the same as that of the other leg and the phase shift of one leg relative to the other is exactly equal to one half.

Definition 12. A gait formula $\mathbf{g}$ for a $2K$-legged automaton is *symmetric* if the legs of the automaton can be partitioned into K symmetric pairs. A gait represented by a gait matrix G is *symmetrically realizable* if there exists a $\mathbf{t}$ such that $\{G, \mathbf{t}\}$ implies a symmetric $\mathbf{g}$.

Definition 13. A gait formula $\mathbf{g}$ is *singular* if there exists at least one component of $\mathbf{g}$ such that for an arbitrarily small change in the value of this component, the gait matrix G implied by $\mathbf{g}$ is changed. A *singular gait* is a gait with a gait matrix G such that there exists no duration vector $\mathbf{t}$ with the property that $\{G, \mathbf{t}\}$ implies a nonsingular $\mathbf{g}$.

Definition 14. A gait matrix is *connected* if every row differs in exactly one column from the row just above it and the row just below it. The first and last row must also differ in just one column.

Definition 15. A gait matrix is *Markov* if, when its rows are ordered from top to bottom, every binary k-tuple appearing as a row has a unique predecessor. The first row must be considered to follow the last row in applying this test.

Regularly realizable gaits occur commonly in nature [3, 4]. Such gaits would also seem to be well suited for artificial legged locomotion systems since regular gait formulas imply identical behavior for each leg. Symmetrically realizable gaits are important for similar reasons. Among quadruped gaits, those that are realizable by regular symmetric gait formulas are the ones that have been studied most extensively. This is mainly because such gait formulas involve only two independent variables: the duty factor common to all legs, and the relative phase shift between the front and rear pair of legs. Hildebrand has made use of this simplification to show that there exist exactly 44 quadruped gait matrices that are implied by regular symmetric gait formulas. Each such matrix corresponds to a set of points in the unit square [3, 4].

The properties defined by Definitions 13, 14, and 15 will be seen to be significant with respect to the problem of designing automatic controllers for the coordination of limb movements [7, 12]. The crawl defined by Figs. 1 and 2 is regular, symmetric, and connected. It is neither singular nor Markov.

Because of the sensitivity of singular gaits to phase and duty factor changes, it is clear that they must represent an idealization attainable neither by artificial nor natural legged locomotion systems. The following two theorems are therefore significant in situations where it is important that a single specified gait be maintained during the forward motion of an animal or machine.

THEOREM 4. *The set of all nonsingular gaits is equivalent to the set of all connected gaits.*

Proof. Suppose that a gait is connected. Then for any leg m_i there must exist just one row of G, γ_s, such that $m_i = 0$ in γ_s and $m_i = 1$ in γ_{s-1}. Likewise there must be just one row γ_r such that $m_i = 1$ in γ_r and $m_i = 0$ in γ_{r-1}. Now suppose β_i, the duty factor for m_i, is reduced by an amount ε where $0 \leqslant \varepsilon < t_{r-1}/\tau$. The effect of this change will be to replace t_{r-1} by $t_{r-1} - \tau\varepsilon$. But since $\tau\varepsilon < t_{r-1}$, t_{r-1} remains positive and row $r - 1$ remains in G. A similar argument shows that if $0 \leqslant \varepsilon < t_r/\tau$, then β_i can be increased by ε without affecting G. Similarly, if ϕ_i is replaced by $\phi_i + \varepsilon$, then t_s is changed to $t_s - \varepsilon\tau$ and t_{s-1} becomes $t_{s-1} + \varepsilon\tau$. Since t_s and t_{s-1} are strictly positive, sufficiently small values of ε will leave G unaltered. Since m_i is an arbitrary leg, the same arguments apply to any leg, so every connected gait is nonsingular.

Now suppose that a gait is not connected. Then there must exist at least one row of G, γ_q, in which at least two legs, m_i and m_j, are in a different state than in γ_{q-1}. Suppose that $m_i = 0$ in γ_q. Then from the preceding argument, it is clear that an arbitrarily small change in ϕ_i will cause m_i to change state either before or after m_j changes state, thereby introducing a new state in G. If $m_i = 1$ in γ_q, then an arbitrarily small change in β_i will likewise alter G. Consequently, every gait that is not connected is singular.

THEOREM 5. *For a k-legged locomotion automaton, all gait matrices of maximum length are connected matrices with 2k rows. Moreover, no connected gait matrix exists with fewer than 2k rows. There are $(2k-1)!$ distinct canonical connected gait matrices.*

Proof. If a gait matrix is connected, any given row differs from the preceding row in only one column. Since each column must change from 0 to 1 and 1 to 0 exactly once, there are just two rows associated with each column in which the preceding entry in the selected column differs from the entry in the given row. Since no row may differ from the preceding row in more than one column, there must be exactly $2k$ rows in a k-column connected gait matrix.

If a gait matrix possesses $2k$ rows, then from the proof of Theorem 4 it must be connected, since otherwise additional rows could be created by small changes in phase or duty factor variables and this would result in a matrix of more than the maximum number of rows allowed by Theorem 1.

Since a connected gait matrix is $2k \times k$, there exist $2k!$ ways of choosing 1-to-0 and 0-to-1 transition locations within the columns of the matrix. However, since there exist $2k$ simple cyclic row permutations of a $2k$-row matrix, the number of canonical connected gait matrices is

$$N = \frac{(2k)!}{2k} = (2k-1)! \tag{5}$$

Theorems 4 and 5 show that the properties of maximum length, nonsingularity, and connectedness are all equivalent. Moreover, it is easy to determine by inspection whether or not a given gait possesses these properties. From the definition of Markov gaits, it is also possible to determine immediately if a given matrix G is Markov. The following theorem establishes the lack of a connection between the Markov and connectedness properties.

THEOREM 6. *For an arbitrary gait matrix G the Markov and connectedness properties are independent.*

Proof. Figure 1 shows a gait that is connected but not Markov while the *walk* described by Muybridge [1] is a Markov connected gait. The *two-state quadruped trot* [4] defined by

$$G = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

is Markov, but not connected. Finally, the *four-state slow trot* [4] defined by

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

is neither Markov nor connected.

It is interesting to note that all of the seven basic quadruped gaits described by Muybridge [1] are connected. Of these seven, just four possess regular symmetric realizations. Using more modern equipment and methods, Hildebrand has determined that exactly sixteen regular symmetric connected quadruped gaits are theoretically possible. His records show that at least eleven of these are used by some animal [3, 4].

TESTS FOR REGULAR AND SYMMETRIC REALIZABILITY

While the Markov and connectedness properties apply directly to gait matrices, the properties of symmetry and regularity depend both upon the gait matrix G and duration vector $\mathbf{t}$. It is therefore appropriate to seek conditions on G under which a $\mathbf{t}$ exists such that $\{G, \mathbf{t}\}$ implies a regular or symmetric gait formula. The following theorem [11] shows that regular realizability can be determined by examination of the solution of a quadratic programming problem [13].

THEOREM 7. *If T denotes the half-open unit n cube defined by $0 < t_i \leqslant 1$, $i = 1, \ldots, n$, then an n-row, k-column gait matrix G possesses a regular realization $\{G, \mathbf{t}\} = \{\mathbf{g}, \tau\}$ if and only if there exists a point $\mathbf{t} \in T$ such that*

$$\min_{\mathbf{t} \in T} \sum_{j=1}^{k} \left[\left(\sum_{i=1}^{n} t_i g_{ij}\right) - 1\right]^2 = 0 \tag{8}$$

where t_i are the components of $\mathbf{t}$ and g_{ij} are the elements of G.

Proof. The inner sum in (8) is equal to τ_1, the duration of the one state in a particular realization of a gait. Since a gait formula $\mathbf{g}$ is independent of τ and $\mathbf{g}$ implies G, if any $\mathbf{t}$ exists such that $\{G, \mathbf{t}\} \equiv \{\mathbf{g}, \tau\}$ is regular, then if α is any positive real constant, $\{G, \alpha\mathbf{t}\} \equiv \{\mathbf{g}, \alpha\tau\}$ must also be regular. Consequently, if any regular realization of G exists, there must be a $\mathbf{t}$ such that $\tau_1 = 1$ for every column of G. But that is just the condition imposed by (8). In addition, since $\mathbf{t}$ is constrained to the half-open unit n cube, every component of $\mathbf{t}$ must be strictly positive as required by the definition of the duration vector. Consequently, G possesses a regular realization if and only if (8) is satisfied.

It is easy to show that both regularly realizable and nonregularly realizable gaits exist. Figures 1 and 2 exhibit a regular gait. On the other hand, it is a simple matter to verify that

$$G = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{9}$$

possesses no regular realization.

The symmetric realizability of a gait can also be resolved to a quadratic programming problem. An additional definition is first required, however.

Definition 16. A $2K$-column gait matrix G is *symmetrically partitioned* if every pair of columns m_{2J-1}, m_{2J} ($J = 1, 2, \ldots, K$) is such that the sequence of changes in m_{2J-1}, m_{2J} is of one of three types:

$$00, 01, 00, 10; \tag{10}$$

$$01, 10; \tag{11}$$

$$01, 11, 10, 11; \tag{12}$$

or is a simple cyclic permutation of one of these types.

THEOREM 8. *In order for a gait to be symmetrically realizable, it is necessary that it be possible to put the* $(n \times 2K)$*-gait matrix* G *into symmetrically partitioned form by permutation of its columns. Once this has been done,* G *possesses a symmetric realization if and only if for at least one such partition there exists a* $\mathbf{t}$ *in the half-open unit* K *cube* R *such that*

$$\min_{\mathbf{t} \in R} \sum_{j=1}^{K} \left[\frac{1}{2} - \sum_{i=r_j}^{s_j} t_i \right]^2 = 0 \tag{13}$$

subject to the constraint

$$\sum_{i=1}^{n} t_i = 1 \tag{14}$$

where the integers r_j *and* s_j *refer respectively to the indices of the first and second rows of* G *in which either of* m_{2J} *or* m_{2J-1} *first becomes equal to zero. The first row of* G *must be considered to follow its last row in determining* r_j.

Proof. The relative motion of a symmetric pair of legs is governed by just one variable, the duty factor β, for either leg in the pair. Construction of gait diagrams for such a pair of legs shows immediately that for $0 < \beta < 0.5$, the sequence of changes for a symmetric pair is given by (10). For $\beta = 0.5$, the sequence described by (11) results and for $0.5 < \beta < 1$ (12) is produced. Since β must lie in the open interval (0, 1), only the three patterns of Definition 16 can appear in a symmetric pair.

Equations (13) and (14) amount merely to a restatement of the definition of a symmetric gait formula, so G is symmetrically realizable if and only if these conditions are satisfied.

ALGORITHMIC CONTROL OF CONNECTED GAITS

The treatment of locomotion machines at the level of abstract locomotion automata ignores the problem of limb coordination, since only the output states of legs are considered. However, attainment of a deeper understanding of locomotion processes requires a recognition of the fact that legs also possess *internal states* that must be properly controlled in order to produce the desired output sequence characteristic of a particular gait. In keeping with the aims of this paper, only the finite state aspects of this control problem will be considered. In addition, the discussion will further be restricted to connected gaits, since these are the only ones that are physically attainable. With these restrictions, control laws take the form of *finite algorithms* governing the transitions between discrete internal leg states in such a way as to produce the intended gait.

Shik and Orlovskii [9] have stated two plausible hypotheses for accomplishment of the limb coordination control function in animals. The first is that there could exist a single center of control that operates by transmitting commands to the muscles controlling each joint in the legs of an animal in a predetermined periodic sequence appropriate to the desired mode of locomotion. The second is that it is possible that the legs are controlled by the interaction of autonomous control networks associated with each leg without the existence of a central source of commands. In the terminology of control theory, the first of these two alternatives amounts to *open loop* whereas the second corresponds to *closed loop* control [14]. In sequential machine theory, these two modes of operation are called *synchronous* and *asynchronous*, respectively [8]. Although the evidence in [9] seems to strongly favor an asynchronous closed loop mechanization of the control function in living quadrupeds, no such assumption will be made here. Rather, it will simply be shown by means of examples that both synchronous and asynchronous controllers are capable of producing connected gaits in either machines or animals.

Considering first the case of synchronous control, in order to obtain simple algorithms it is necessary to introduce certain additional concepts and restrictions. First of all, in addressing the control problem, it must be recognized that controlling the sequence of leg internal states results in forward motion of an animal or machine only if dynamic stability is maintained. For the purposes of this paper, it will be assumed that such stability is inherent in the gaits to be considered and active stability control is not required. Obviously, this is a poor assumption for bipeds. For quadrupeds, however, experiments with walking machines have established the existence of such gaits [7]. In particular, a locomotion machine utilizing the quadruped crawl shown in Figs. 1 and 2 can be made inherently stable because it is possible to choose the dimensions and angular stroke of the legs so that the center of gravity of the machine always lies within the triangle formed by the three supporting legs whenever one of the legs is being transferred to its forward position.

Since dynamic stability will be assumed to be attained passively as a result of the relative position of the legs and body of a locomotion machine, it is necessary that some notions of spatial and time synchronization be introduced.

Definition 17. The legs of a locomotion machine are *support phase synchronized* if the relative distance between the feet of every pair of legs m_i and m_j is constant so long as both legs are in the 0 state.

Definition 18. The legs of a locomotion machine are *transfer phase synchronized* if there exists a bound $\tau_r(\theta)$ such that any leg of the machine can rotate through an angle θ measured from the end of the support phase in a time less than or equal to $\tau_r(\theta)$.

Support phase synchronization amounts essentially to a requirement that no foot should skid on the supporting surface. This requirement is almost always satisfied by animals and may result in some gaits simply from limitation of the torque applied at each hip. Transfer phase synchronization can always be obtained by application of a sufficiently large torque to the hip of a leg in the transfer phase. Neither of these types of synchronization requires knowledge of the continuous state of the legs of an animal or machine.

The derivation of explicit coordination control algorithms requires that the characteristics of the legs to be coordinated be defined in further detail. In particular, something must be said about the number of joints and the nature of the muscles or actuators. In the remainder of this paper, legs with just two joints, one at the hip and one at the knee, each powered by a three-state actuator, will be assumed as in [7]. The three states will be taken to be: locked, forward rotation, and rearward rotation, designated by joint states 0, 1, and 2, respectively. The leg angle θ in Definition 18 is therefore the angle formed by the upper portion of such a leg with respect to the body of the machine. The algorithm for knee control will also be assumed to be the same as in [7]. The only problem to be solved with respect to synchronous control is then the derivation of an algorithm for hip actuator coordination. But this is also solved in [7] where it is shown that a synchronous shift register implementation is possible. Figure 3 illustrates such an implementation for the crawl of Figs. 1 and 2.

The success of an open loop or synchronous control algorithm depends on both support phase and transfer phase synchronization. In addition, the rate of rotation of each leg during its support phase must be adjusted so that the amount of rearward rotation that occurs before the initiation of the transfer phase is compatible with the requirement for dynamic stability. Finally, the rate of rotation during the transfer phase must be sufficient to make $\tau_r(0)$ short enough to allow every hip actuator to enter a locked state before initiation of the support phase of the associated leg. Clearly, these conditions are sufficient to insure the proper movement of the legs to achieve the desired gait.

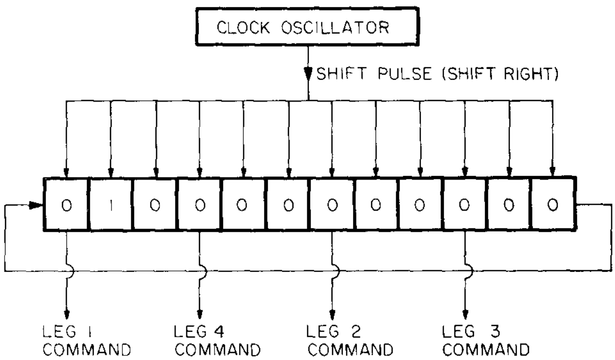

FIG. 3. A shift register synchronous controller for a quadruped crawl.

It should be noted that in the event that a gait is Markov, an alternate realization of synchronous control is possible. Since every state has a unique predecessor in such a gait, the coordination control computer need contain no memory elements; it can be a *combinational* rather than a *sequential* network [8] with the transition from one output state to the next occurring at times determined by a separate timing signal generator. A control scheme of this type was proposed for "creeping" locomotion by Tomovic [5]. Since the quadruped crawl is not a Markov gait, no combinational control algorithm exists for it and this simplest type of controller cannot be used.

An asynchronous control algorithm is more complicated to obtain than a synchronous one since the control function is distributed rather than centralized. Shik and Orlovskii [9] propose a control based upon continuous comparison of actual and desired relative leg positions. While such control can undoubtedly be realized, one of the purposes of this paper is to show that asynchronous control can also be realized by finite algorithms. The notation and terminology to be used for this purpose are taken from Tomovic and McGhee [12]. Once again, the crawl will be used to illustrate the derivation of a control algorithm.

In keeping with [12], the output of each state of the control computer to be designed will be designated by the 4-tuple $H_1H_2H_3H_4$ contained within parentheses in the circle representing each state in the computer state diagram. Each element of this 4-tuple is a ternary variable representing the state of the associated hip actuator. Also as in [12], binary feedback signals from each hip will be assumed. Three such signals will be used: h_1, h_2, h_3. All of these variables are equal to 0 when the hip actuator is rotated to its forward mechanical limit. As the hip of any leg is rotated toward the rear, first h_1, then h_2, and finally h_3 changes from 0 to 1. The 0-to-1 transition of h_1 occurs at the normally fully extended leg position, the 0-to-1 transition of h_2 occurs at the angle at which the leg following the selected leg should enter its support phase, and the 0-to-1 transition of h_3 occurs at the normal rearward limit of the selected leg.

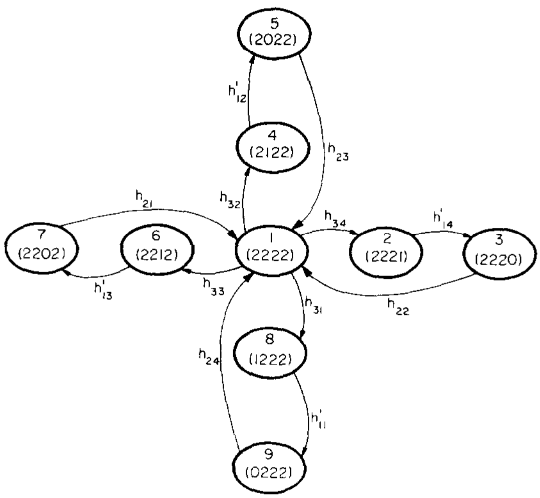

FIG. 4. State diagram for an asynchronous hip control algorithm for a quadruped crawl.

With the preceding definitions, and assuming both support phase and transfer phase leg synchronization, the asynchronous control algorithm illustrated by the state diagram [8, 12] shown in Fig. 4 is capable of achieving leg coordination for the quadruped crawl gait. The second subscript on the feedback signals appearing on this diagram refers to the leg indices defined in Fig. 2. Primes appearing on variables indicate electronic inversion; that is,

$$x' = 1 - x. \tag{15}$$

It should be noted that, unlike a synchronous algorithm, the asynchronous algorithm produces a correct sequence of leg output states regardless of the rate of leg rotation providing only that the initiation of a support phase of a leg is separated from the initiation of the transfer phase by a time exceeding the transfer return time τ_r. Moreover, even this requirement can be eliminated by a slightly more complicated control algorithm.

SUMMARY AND CONCLUSIONS

By adopting a finite state mathematical model for the functioning of legged locomotion systems, the properties of such systems can be studied at a level of abstraction that avoids detailed consideration of dynamical behavior. At the same time, a finite state theory can be made sufficiently complete to allow the synthesis of finite algorithms for controlling the limbs of artificial legged locomotion machines. Further study of such algorithms may shed additional light on the control principles employed by animals to achieve limb coordination in steady gaits.

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