Derived Improved MSE

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$$L(\bar{x}, \vec{x}) = \sum_{i=1}^{n} \frac{(\bar{x} - x_i)^2}{n}$$

Where $\vec{x}$ is a vector of size n

$\bar{x}$ is the mean of $\vec{x}$

$L(\bar{x}, \vec{x})$ is the mean squared error of $\vec{x}$

This function can be rewritten as

$$\Rightarrow L(\bar{x}, \vec{x}) = \frac{1}{n} \sum_{i=1}^{n} (\bar{x} - x_i)^2$$

$$\Rightarrow L(\bar{x}, \vec{x}) = \frac{1}{n} \sum_{i=1}^{n} \bar{x}^2 - 2\bar{x} x_i + x_i^2$$

$$\Rightarrow L(\bar{x}, \vec{x}) = \frac{1}{n} \left[\sum_{i=1}^{n} \bar{x}^2 - \sum_{i=1}^{n} 2\bar{x} x_i + \sum_{i=1}^{n} x_i^2 \right]$$

$$\Rightarrow L(\bar{x}, \vec{x}) = \frac{1}{n} \left[\bar{x}^2 \cdot \sum_{i=1}^{n} 1 - 2\bar{x} \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i^2 \right]$$

$$\Rightarrow L(\bar{x}, \vec{x}) = \frac{1}{n} \left[n \cdot \bar{x}^2 - 2\bar{x} \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i^2 \right]$$

So $L(\bar{x}, \vec{x}) = \frac{1}{n} \left[n \cdot \bar{x}^2 - 2\bar{x} \cdot \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i^2 \right]$

but since $\bar{x}$ can be represented as $\left[\sum_{i=1}^{n} \frac{x_i}{n} \right]$ $L(\bar{x}, \vec{x})$ can be rewritten to $L(\vec{x})$

$$\Rightarrow L(\vec{x}) = \frac{1}{n} \left[n \cdot \left[\sum_{i=1}^{n} \frac{x_i}{n} \right]^2 - 2 \left[\sum_{i=1}^{n} \frac{x_i}{n} \right] \cdot \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i^2 \right]$$

Side note: $\left[\sum_{i=1}^{n} \frac{x_i}{n} \right]^2 = \left[\frac{x_1 + x_2 \ldots + x_n}{n} \right]^2 = \frac{(x_1 + x_2 \ldots + x_n)^2}{n^2} = \frac{1}{n^2} \left[\sum_{i=1}^{n} x_i \right]^2$

$$\Rightarrow L(\vec{x}) = \frac{1}{n} \cdot \left[n \cdot \frac{1}{n^2} \left[\sum_{i=1}^{n} x_i \right]^2 - \frac{2}{n} \left[\sum_{i=1}^{n} x_i \right] \cdot \left[\sum_{i=1}^{n} x_i \right] + \sum_{i=1}^{n} x_i^2 \right]$$

$$\Rightarrow L(\vec{x}) = \frac{1}{n} \cdot \left[\frac{1}{n} \cdot \left[\sum_{i=1}^{n} x_i \right]^2 - \frac{2}{n} \left[\sum_{i=1}^{n} x_i \right]^2 + \sum_{i=1}^{n} x_i^2 \right]$$

$$\Rightarrow L(\vec{x}) = \frac{1}{n} \cdot \left[\frac{-1}{n} \cdot \left[\sum_{i=1}^{n} x_i \right]^2 + \sum_{i=1}^{n} x_i^2 \right]$$

$$\Rightarrow L(\vec{x}) = \frac{-1}{n^2} \cdot \left[\sum_{i=1}^{n} x_i \right]^2 + \frac{1}{n} \cdot \left[\sum_{i=1}^{n} x_i^2 \right]$$

$$\therefore L(\bar{x}, \vec{x}) = \sum_{i=1}^{n} \frac{(\bar{x} - x_i)}{n} = \frac{-1}{n^2} \cdot \left[\sum_{i=1}^{n} x_i \right]^2 + \frac{1}{n} \cdot \left[\sum_{i=1}^{n} x_i^2 \right]$$