$\mathbb{R}^n = X \oplus Y$

$\forall v \in V \ , \ \exists! \ x \in X \ , \ y \in Y \ : \ v = x + y$

$proj_X v = x \ ; \ proj_Y v = y$

projector $\quad \varphi_{X,Y} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ t. linear

$v \longmapsto proj_X v$ ao longo de Y

$\underbrace{P_{n\times n}}_{\text{projector}}$ matriz que representa φ

Pv é a proj. de v em X ao longo de Y

$B_X = \{x_1, \dots, x_r\}$ base de X $\qquad B_Y = \{y_1, \dots, y_{n-r}\}$ base Y

$B_X \cup B_Y$ é base de $\mathbb{R}^n$ $\qquad$ Por hip. $\mathbb{R}^n = X \oplus Y$

$$B_{n\times n} = \left[\begin{array}{cccc|ccc} x_1 & x_2 & \dots & x_r & y_1 & \dots & y_{n-r} \\ | & | & & | & | & & | \end{array}\right]_{n\times n}$$

$car(B) = n$ $\quad$ p/ as n colunas de B são l.i.

Como B é $n \times n$, e $car(B) = n$ então B é invertível

Recordar: $\mathbb{R}^n = X \oplus Y$, $\forall v, \exists! \ x \in X, \ y \in Y : \ v = x + y$

$x \in X$
$y \in Y$

$$x = \underset{X}{x} + \underset{Y}{0} \qquad Px = x$$

$$y = \underset{X}{0} + \underset{Y}{y} \qquad Py = 0$$

$$PB = P\left[\begin{array}{ccc|ccc} x_1 & \cdots & x_r & y_1 & \cdots & y_{n-r} \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} Px_1 & Px_2 & \cdots & Px_r & Py_1 & Py_2 & \cdots & Py_{n-r} \end{array}\right], \quad x_i \in X, \; y_i \in Y$$

$$= \left[\begin{array}{ccc|ccc} x_1 & \cdots & x_r & 0 & \cdots & 0 \end{array}\right] = \left[\begin{array}{c|c} X_{n\times r} & O_{n\times(n-r)} \end{array}\right]$$

i.e., $PB = [X | 0]$

$$\underset{B \text{ invert.}}{\Longrightarrow} \quad P = [X | 0] B^{-1}$$

$$B = [X | Y] \qquad B\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] = [X | Y]\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] = [X | 0]$$

$$PB = [X | 0] = B\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \underset{\text{RHS } B^{-1}}{\Longrightarrow} P = B\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] B^{-1}$$

Pretendemos mostrar que P é o projector de v em X ao longo de Y.

$v = x + y$

$Pv = x$.

Sejam $x = Pv$; $y = (I - P)v$

$$\mathbb{R}^n = \cancel{x} \oplus Y$$
$$v = x + y$$

$$x + y = Pv + (I-P)v = Pv + v - Pv = v$$

$$x \overset{?}{\in} X$$

$$x = Pv = [X \mid 0]B^{-1}v \in \underbrace{CS(X)} = \mathcal{L}(x_1, \dots, x_r) = X$$

espaço das colunas de $X = [x_1 \cdots x_r]$

$$y = (I-P)v = (\underset{= BB^{-1}}{I} - B\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} B^{-1})v$$

$$= \left(BB^{-1} - B\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} B^{-1}\right)v$$

$$= B\left(\begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} - \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\right)B^{-1}v$$

$$= B\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} B^{-1}v = [x_1 \cdots x_r \mid y_1 \cdots y_{n-r}]\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} B^{-1}v$$

$$= [0 \mid y_1 \cdots y_{n-r}]B^{-1}v = [0 \mid Y]B^{-1}v \in CS(Y) = \mathcal{L}(y_1, \dots, y_{n-r}) = Y$$

$$v = \underbrace{Pv}_{\in X} + \underbrace{(I-P)v}_{\in Y}$$

O Projector em X ao longo de Y é único!

Unicidade dos projectores:

P_1, P_2 projectores ; $B = [x_1 \cdots x_r \mid y_1 \cdots y_{n-r}]$

sobre X ao longo de Y $= [X \mid Y]$

$$P_1 B = P_1 [X \mid Y] = [P_1 X \mid P_1 Y] = [X \mid 0]$$

$$P_2 B = P_2 [X \mid Y] = [P_2 X \mid P_2 Y] = [X \mid 0]$$

i.e. $P_1 B = P_2 B \xRightarrow[\text{RHS } B^{-1}]{} P_1 = P_2$

$Q = I - P$ é o projector complementar

(i.e. proj. em Y ao longo de X)

$\rightarrow$ $P^2 = P$

$\rightarrow$ $I - P$ proj. em Y ao longo de X

$\rightarrow$ $CS(P) := \{Pv : v \in \mathbb{R}^n\} = \{x : Px = x\}$

i.e. $CS(P)$ é o subesp. de $\mathbb{R}^n$ dos elementos fixados por P

$\rightarrow$ $CS(P) = \ker(I-P) = X$

$CS(I-P) = CS(Q) = \ker(P) = Y$

$$P = [X \mid 0] \underbrace{[X \mid Y]}_{=B}^{-1} = B \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] [X \mid Y]^{-1}$$

onde as cols. de X são base de X

" " Y " " " Y

$\mathbb{R}^3$ $\qquad X = \mathcal{L}(x_1, x_2)$ $\qquad x_1 = (1, 1, 0)$

$\qquad\qquad\qquad\qquad\qquad\qquad x_2 = (1, 1, 1)$

$Y = \mathcal{L}(y_1)$ $\qquad y_1 = (0, 1, 0)$

$\mathbb{R}^3 = X \oplus Y$ $\qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = B$ $\qquad car(B) = 3$

$\Rightarrow B$ é inv.

P proj. sobre X ao longo de Y

$Q = I - P$ " " Y " " " X

$$[B\,|\,I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\ell_2 - \ell_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\ell_2 \leftrightarrow \ell_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\ell_1 \leftarrow \ell_1 - \ell_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{matrix}}_{= B^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$P = [x\,|\,0]\, B^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\qquad P_v = P \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ é a proj de v em X ao longo de Y

Decomposição ortogonal

M subesp. de $\mathbb{R}^n$

$$M^\perp = \{ v \in \mathbb{R}^n : v \perp x , \forall x \in M \}$$

complemento ortogonal de M

$$\mathbb{R}^n = M \oplus M^\perp$$

$\rightarrow \dim M^\perp = n - \dim M$

$\rightarrow (M^\perp)^\perp = M$

Teorema da decomposição ortogonal

$A_{m \times n}$

$$CS(A)^\perp = \mathrm{Ker}(A^T)$$
$$\mathrm{Ker}(A)^\perp = CS(A^T)$$

$$\mathbb{R}^m = CS(A) \oplus^\perp CS(A)^\perp = CS(A) \oplus^\perp \mathrm{Ker}(A^T)$$

dim

$$\underbrace{\mathbb{R}^n}_{n} = \mathrm{Ker}(A) \oplus^\perp (\mathrm{Ker}(A))^\perp = \underbrace{\mathrm{Ker}(A)}_{\dim \mathrm{Ker}(A)} \oplus^\perp \underbrace{CS(A^T)}_{car(A)}$$

$B_{CS(A)}$ base do espaço das colunas de A

$$B_{CS(A)} = \{ u_1, \dots, u_r \} \qquad r = car(A)$$

$$\mathbb{R}^m = CS(A) \oplus^\perp \mathrm{Ker}(A^T)$$

$$B_{\ker(A^T)} = \{u_{r+1}, \ldots, u_m\}$$

SPG, $B_{CS(A)}$, $B_{\ker(A^T)}$ são bases ortonormadas

$$B_{CS(A^T)}, \quad B_{\ker(A)} \quad ; \quad \mathbb{R}^n = CS(A^T) \oplus^{\perp} \ker(A)$$

SPG, assumem-se ortonormadas

obtemos assim

$B_{CS(A^T)} \cup B_{\ker(A)}$ base de $\mathbb{R}^n$

$B_{CS(A)} \cup B_{\ker(A^T)}$ base de $\mathbb{R}^m$

$$U_{m\times m} = \Big[\underbrace{u_1 \cdots u_r}_{\substack{\text{base de}\\ CS(A)}} \ \underbrace{u_{r+1} \cdots u_m}_{\substack{\text{base de}\\ \ker(A^T)}}\Big]_{m\times m} \quad \text{é invertível}$$

$$V_{n\times n} = \Big[\underbrace{v_1 \cdots}_{\substack{\text{base de}\\ CS(A^T)}} \ \underbrace{\cdots v_n}_{\substack{\text{base de}\\ \ker(A)}}\Big]_{n\times n} \quad \text{é invertível}$$

$U^TU = I \Rightarrow U$ é ortogonal

$(\ldots)$ V é ortogonal

$$R = U^TAV = [r_{ij}]$$

$$R = U^T A V = \begin{bmatrix} — u_1^T — \\ — u_2^T — \\ \vdots \\ — u_m^T — \end{bmatrix} A \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

$$r_{ij} = u_i^T A v_j$$

$A^T u_i = 0$ se $i \geq r+1$

Logo $r_{ij} = 0$ se $i \geq r+1$

$A v_j = 0$ se $j \geq r+1$

Logo $r_{ij} = 0$ se $j \geq r+1$

Portanto

$$R = U^T A V = \left[\begin{array}{cccc|c} u_1^T A v_1 & u_1^T A v_2 & \cdots & u_1^T A v_r & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ u_r^T A v_1 & u_r^T A v_2 & \cdots & u_r^T A v_r & 0 \\ \hline 0 & \cdots & \cdots & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right] \qquad \begin{array}{l} c_{ij} = u_i^T A v_j \\ C \text{ é } r \times r \end{array}$$

Como $car(A) = car(R) = car(C) = r$ então

A é invertível.

$$R = U^T A V \iff A = U R V^T$$

Recíprocamente, uma factorização $A = URV^T$ c/ U, V ortogonais, $R = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right]$, C invertível

induz uma base ortonormada p/ cada subespaço fundamental (i.e. $CS(A)$, $CS(A^T)$, $Ker(A)$, $Ker(A^T)$)

<u>Obs.</u> $A = URV^T$ $\quad R = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right]$

U, V ortog.

$\exists P$ ortogonal t.q. $PA = \begin{bmatrix} B \\ 0 \end{bmatrix}$ $\quad B_{r \times n}$, $car(B) = r$

$\exists Q$ ortogonal t.q. $QB^T = \begin{bmatrix} T \\ 0 \end{bmatrix}$ $\quad T$ triang. super.

(usar QR) $\quad T_{r \times r}$

$$QB^T = \begin{bmatrix} T \\ 0 \end{bmatrix} \Rightarrow B = \left[T^T \,|\, 0 \right] Q$$

$$\Rightarrow \begin{bmatrix} B \\ \hline 0 \end{bmatrix} = \left[\begin{array}{c|c} T^T & 0 \\ \hline 0 & 0 \end{array}\right] Q$$

$$PA = \begin{bmatrix} B \\ 0 \end{bmatrix} = \left[\begin{array}{c|c} T^T & 0 \\ \hline 0 & 0 \end{array}\right] Q \underset{\substack{LHS \\ P^T}}{\Longrightarrow} A = P^T \left[\begin{array}{c|c} T^T & 0 \\ \hline 0 & 0 \end{array}\right] Q$$

T^T triang. inf. invert.

$A_{n \times n}$; $car(A) = r$

São equivalentes:

1) $CS(A) \perp Ker(A)$

2) $CS(A) = CS(A^T)$

3) $Ker(A) = Ker(A^T)$

4) $A = U \left[\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right] U^T$

$C_{r \times r}$

U ortogonal ; C invertível

Neste caso, a matriz A diz-se matriz EP