$v_1 = (1, -1, 0) \; ; \; v_2 = (1, 1, 1)$ $\qquad b = (1, -5, -2)$

$$V = \langle v_1, v_2 \rangle \subseteq \mathbb{R}^3$$

$$\| $$

$$\{ \alpha_1 v_1 + \alpha_2 v_2 : \alpha_1, \alpha_2 \in \mathbb{R} \}$$

$$A = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \qquad R(A) = CS(A) = \{\alpha_1 v_1 + \alpha_2 v_2\}_{\alpha_i \in \mathbb{R}}$$

$$\dim R(A) = \operatorname{car}(A)$$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow[l_2 \leftarrow l_2 + l_1]{l_2 + l_1} \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow[l_3 \leftarrow l_3 - \frac{1}{2} l_2]{l_3 - \frac{l_2}{2}} \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} = U$$

$$\underbrace{\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}}_{E_{32}(-1/2)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}(1)}}_{K} A = U \Rightarrow A = LU$$

com $L = K^{-1}$

$$\boxed{PA = LU}$$

⚠ $R(A) \neq R(U)$ ⚠

Sup. $A = BV$ c/ V invertível

$\dim R(A) = \text{car}(A) = \text{car}(B) = \dim R(B)$

Resta mostrar que $R(A) \subseteq R(B)$

Seja $b \in R(A) \Rightarrow b = \alpha_1 v_1 + \alpha_2 v_2 = \underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}}_{=x}$

$\Rightarrow \exists x : Ax = b$

$\underset{A=BV}{\Rightarrow} \exists x : BVx = b$

$\Rightarrow \underset{y=Vx}{\exists y} : BVV^{-1}y = b$

$\Rightarrow \exists y : By = b \Rightarrow b \in R(B)$

$\therefore R(A) = R(B)$

Sup.

$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$

$b = \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}$ $\quad c = \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}$

$b \overset{?}{\in} R(A)$ $\quad c \overset{?}{\in} R(B)$

$b \in R(A) \Leftrightarrow \exists x: Ax = b$ $\quad c \in R(B) \Leftrightarrow \exists x: Bx = c$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & -5 \\ 0 & 1 & -2 \end{array}\right] \xrightarrow{l_2 \leftarrow l_2 + l_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{array}\right] \xrightarrow[l_3 - \frac{1}{2} l_2]{} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

$\alpha_2 = -2$

$\alpha_1 = 3$

$b = 3v_1 - 2v_2$

$\{v_1, v_2\}$ l.i. logo são base $R(A)$

$\Downarrow$

pq. car$[v_1 \; v_2] = 2$

$\mathcal{B}_{R(A)} = (v_1, v_2)$ base ordenada

$b = 3v_1 + (-2)\, v_2$ $\qquad [b]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$Bx = C \quad (\dots) \qquad [C]_{\mathcal{B}_{R(B)}} = \,?$

Seja V e.v., $\mathcal{B}$ base ordenada de V $= (v_1, \dots, v_n)$

$x \in V$. As coordenadas de x na base $\mathcal{B}$ é $(x)_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$ se

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

Como calcular α_i's ?

(Podemos usar AEG)

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$\quad v_1 \quad v_2$$

$$v_1 \perp v_2$$

$$v_1 \cdot v_2 = v_1^T v_2$$
$$= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= 0$$

$$\mathcal{B}_{R(A)} = \left(\frac{v_1}{\|v_1\|} ; \frac{v_2}{\|v_2\|} \right) = \left(\left(\tfrac{\sqrt{2}}{2} ; -\tfrac{\sqrt{2}}{2} ; 0 \right), \left(\tfrac{\sqrt{3}}{3}, \tfrac{\sqrt{3}}{3}, \tfrac{\sqrt{3}}{3} \right) \right)$$

$$(b)_{\mathcal{B}} = ?$$

No caso geral, $\mathcal{B} = (u_1, u_2, \dots, u_n)$

base (ordenada) ortonormada (i.e., $u_i \perp u_j$
$\|u_i\| = 1$)

$$(x)_{\mathcal{B}} = ? \qquad c/ \quad x = \sum_{i=1}^{n} \alpha_i u_i$$

$$x \cdot u_j = \left(\sum_{i=1}^{n} \alpha_i u_i\right) \cdot u_j = \sum_{i=1}^{n} \alpha_i \, u_i \cdot u_j$$

$$= \alpha_j \underbrace{u_j \cdot u_j}_{= \|u_j\|^2 = 1} = \alpha_j$$

$$\alpha_1 = x \cdot u_1 \; ; \; \alpha_2 = x \cdot u_2 \, , \; \dots \, , \; \alpha_n = x \cdot u_n$$

$$(x)_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} x \cdot u_1 \\ x \cdot u_2 \\ \vdots \\ x \cdot u_n \end{bmatrix}$$

$$x = \sum_{i=1}^{n} \alpha_i u_i = \underbrace{\sum_{i=1}^{n} (x \cdot u_i) \, u_i}_{\text{expansão de Fourier de } x}$$

Coeficientes de Fourier

Gram - Schmidt

Sup. que temos $B = (v_1, v_2, \ldots, v_n)$ base (ordenada) de V, $\dim V = n$

$$u_1 := \frac{v_1}{\|v_1\|}$$

$$u_{k+1} := \frac{v_{k+1} - \sum_{i=1}^{k} (u_i \cdot v_{k+1})\, u_i}{\vartheta_{k+1}}, \quad k \geq 1$$

Onde $\vartheta_{k+1} = \left\| v_{k+1} - \sum_{i=1}^{k} (u_i \cdot v_{k+1})\, u_i \right\|$

$$u_2 = \frac{v_2 - (u_1 \cdot v_2)\, u_1}{\| \quad \|}$$

$$u_2 \cdot u_1 = \frac{v_2 - (u_1 \cdot v_2) \cdot u_1}{\| \quad \|} \cdot \frac{v_1}{\|v_1\|}$$

$$\left(v_2 - (u_1 \cdot v_2)\, u_1\right) \cdot v_1 = v_2 \cdot v_1 - \left(\underbrace{(u_1 \cdot v_2)}_{\in \mathbb{R}} u_1\right) \cdot v_1$$

$$= v_2^T v_1 - \left(u_1 \overbrace{(u_1 \cdot v_2)}^{u_1^T v_2}\right) \cdot v_1$$

$$x \cdot y = x^T y$$

$$= v_2^T v_1 - \left(u_1\, u_1^T v_2\right) \cdot v_1$$

$$(AB)^T = B^T A^T$$

$$= v_2^T v_1 - v_2^T u_1 u_1^T v_1$$

$$u_1^T u_1 = u_1 \cdot u_1 = \|u_1\|^2 = 1$$

$$= v_2^T v_1 - v_2^T u_1 \, u_1^T u_1 \cdot \|v_1\| \qquad u_1^T u_1 = 1$$

pq. $v_1 = u_1 \|v_1\|$

$$= v_2^T v_1 - v_2^T \underbrace{u_1 \|v_1\|}_{= v_1} = v_2^T v_1 - v_2^T v_1 = 0$$

Alternativamente,

$$u_k = \frac{\left(I - U_k U_k^*\right) v_k}{\|\left(I - U_k U_k^*\right) v_k\|}, \quad k \geq 1$$

onde $U_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$, $U_k = \left[u_1 \mid u_2 \mid \cdots \mid u_{k-1} \right]_{n \times (k-1)}$, $k > 1$

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \qquad \text{car}(A) = n$$

Aplicar G-S a $a_1, a_2, \ldots, a_n$ (que são l.i.)

$$q_1 = \frac{a_1}{\underbrace{\|a_1\|}_{r_1}} \quad ; \quad q_k = \frac{a_k - \sum_{i=1}^{k-1} (q_i \cdot a_k)\, q_i}{r_k}$$

$k = 2, 3, \ldots, n$

$$r_k = \left\| a_k - \sum_{i=1}^{k-1} (q_i \cdot a_k)\, q_i \right\|$$

$$a_1 = r_1 q_1$$

$$a_2 = (q_1 \cdot a_2)\, q_1 + r_2 q_2 = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} \begin{bmatrix} q_1 \cdot a_2 \\ r_2 \end{bmatrix}$$

$$a_3 = (q_1 \cdot a_3)\, q_1 + (q_2 \cdot a_3)\, q_2 + r_3 q_3 = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} q_1 \cdot a_3 \\ q_2 \cdot a_3 \\ r_3 \end{bmatrix}$$

$\vdots$

$$a_k = (q_1 \cdot a_k)\, q_1 + (q_2 \cdot a_k)\, q_2 + \cdots + (q_{k-1} \cdot a_k)\, q_{k-1} + r_k q_k$$

$$\underbrace{\begin{bmatrix} | & | & | & & | \\ a_1 & a_2 & a_3 & \cdots & a_n \\ | & | & | & & | \end{bmatrix}}_{=:A} = \underbrace{\begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} r_1 & q_1 \cdot a_2 & q_1 \cdot a_3 & \\ 0 & r_2 & q_2 \cdot a_3 & \\ \vdots & 0 & r_3 & \\ & \vdots & & \ddots \\ 0 & 0 & \cdots & 0 & r_n \end{bmatrix}}_{R}$$

$$A = QR \qquad \text{decomposição } QR \text{ de } A$$

As colunas de Q são ortonormadas 2 a 2

R é uma matriz triang. superior, $n \times n$, c/ elementos diagonais positivos, logo invertível, $R^{-1} = \begin{bmatrix} \diagdown & \\ 0 & \end{bmatrix}$

$$\mathcal{R}(A) = \mathcal{R}(Q)$$

A decomposição QR de $A_{m \times n}$ é única se $car(A) = n$

(I) Sup. $A_{n\times n}$ $\quad car(A) = n$

$$A = QR$$

$$Ax = b \Rightarrow QRx = b \Rightarrow Q^{-1}QRx = Q^{-1}b \quad (1)$$

$$Q^TQ = \begin{bmatrix} - q_1^T - \\ - q_2^T - \\ \vdots \\ - q_n^T - \end{bmatrix} \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}$$

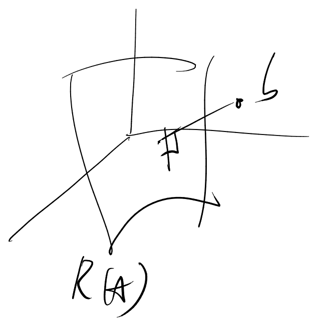
$$\Rightarrow Q^{-1} = Q^T$$

$$(1) \Rightarrow Rx = Q^Tb$$

$$\Rightarrow \begin{bmatrix} \diagdown & \\ 0 & \diagdown \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Q^Tb$$

resolvo por substituição inversa

(II) $A_{m\times n}$, $car(A) = n$



neste caso, a solução no sentido dos mínimos quadrados é única

A solução de $Ax = b$ no sentido dos mínimos quadrados é a (única) solução de

$$(A^T A)x = A^T b \quad ;$$

$A = QR$

$$A^T A = (QR)^T QR = R^T Q^T Q R$$

$$= R^T \begin{bmatrix} — q_1^T — \\ — q_2^T — \\ \vdots \\ — q_n^T — \end{bmatrix}_{n \times m} \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}_{m \times n} R$$

$$= R^T I_n R = R^T R$$

$$(A^T A)x = A^T b \iff \cancel{R^T} R x = \cancel{R^T} Q^T b$$

$$\iff Rx = Q^T b$$

usar substituição inversa

Defn. $U_{n\times n}$ é unitária se as colunas de U formarem uma base ortonormada de $\mathbb{C}^n$

$P_{n\times n}$ é ortogonal se as colunas de P formarem uma base ortonormada de $\mathbb{R}^n$

U é unitária sse $UU^* = I_n$
sse $U^*U = I_n$

P é ortogonal sse $PP^T = I_n$
sse $P^TP = I_n$

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$$

U é isometria: $\|Ux\| = \|x\|$

U é unitária sse $\|Ux\| = \|x\|$ $\forall x \in \mathbb{C}^n$

Projecção ortogonal elementar

$u \in \mathbb{C}^n$; $\|u\| = 1$

$Q = I_n - uu^*$ é proj. ortog. elementar

$u^\perp = \{ v \in \mathbb{C}^n : u \perp v \}$ é subesp. vect. de $\mathbb{C}^n$

denominado complemento ortogonal de u.

$Q = I - uu^*$ é o projector ortogonal ao longo de $u^\perp$

$x = (I - Q)x + Qx$

$((I-Q)x) \perp (Qx)$

$Q^2 = Q$

decomposição de Pierce

$$Q^2 = (I - uu^*)(I - uu^*) = I - uu^* - uu^* + \underbrace{u\underbrace{u^* u}_{=\|u\|^2 = 1} u^*}_{uu^*} = Q$$

$$\big((I - Q)x\big) \cdot (Qx)$$

$$= \big((I-Q)x\big)^* Qx = x^* \underbrace{(I-Q)Q}_{Q-Q} x = 0$$