$p \in \mathcal{V}^{CP}$, $\psi \in \mathcal{F}^{CP}$

$$[\psi/p] : \mathcal{F}^{CP} \longrightarrow \mathcal{F}^{CP}$$
$$\varphi \longmapsto \varphi[\psi/p]$$

resulta de φ por substituição das ocorrências de p por ψ

$[\psi/p]$: definida por recursão estrutural por:

a) $\bot[\psi/p] = \bot$;

b) $p_i[\psi/p] = \begin{cases} \psi & \text{se } p_i = p \\ p_i & \text{se } p_i \neq p \end{cases}$, para todo $i \in \mathbb{N}_0$;

c) $(\neg\varphi_1)[\psi/p] = \neg\varphi_1[\psi/p]$, para todo $\varphi_1 \in \mathcal{F}^{CP}$;

d) $(\varphi_1 \square \varphi_2)[\psi/p] = \varphi_1[\psi/p] \square \varphi_2[\psi/p]$, para todo $\square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, $\varphi_1, \varphi_2 \in \mathcal{F}^{CP}$.

1.2. φ: i) p_{2020} ii) $\neg\bot \vee \bot$ iii) $p_0 \rightarrow (\neg p_0 \rightarrow \neg p_1)$

a) Calcule $\varphi[p_2/p_0]$, $\varphi[p_0 \wedge p_1/p_1]$, $\varphi[p_{2021}/p_{2020}]$

i) $\varphi[p_2/p_0] = p_{2020}$

$\varphi[p_0 \wedge p_1/p_1] = p_{2020}$

$\varphi[p_{2021}/p_{2020}] = p_{2021}$

ii) $\varphi[p_2/p_0] =$

$(\neg\bot)[p_2/p_0] \vee \bot[p_2/p_0]$

$= \neg\bot[p_2/p_0] \vee \bot$

$= \neg\bot \vee \bot$

$\varphi[p_0 \wedge p_1/p_1] = \neg\bot \vee \bot$

$\varphi[p_{2021}/p_{2020}] = \neg\bot \vee \bot$

iii)

$\varphi[p_2/p_0] = p_0[p_2/p_0] \rightarrow (\neg p_0 \rightarrow \neg p_1)[p_2/p_0]$

$= p_2 \rightarrow ((\neg p_0)[p_2/p_0] \rightarrow (\neg p_1)[p_2/p_0])$

$= p_2 \rightarrow (\neg p_0[p_2/p_0] \rightarrow \neg p_1[p_2/p_0])$

$= p_2 \rightarrow (\neg p_2 \rightarrow \neg p_1)$

$\varphi[p_2 \wedge p_1/p_1] = p_0 \rightarrow (\neg p_0 \rightarrow \neg(p_2 \wedge p_1))$

$\varphi[p_{2021}/p_{2020}] = \varphi$

subf : $\mathcal{F}^{CP} \longrightarrow \mathcal{P}(\mathcal{F}^{CP})$

$\varphi \longmapsto subf(\varphi)$ conjunto das subfórmulas de φ

subf é definida, por recursão estrutural, por:

a) $subf(\varphi) = \{\varphi\}$, para todo $\varphi \in \mathcal{V}^{CP} \cup \{\bot\}$;

b) $subf(\neg\varphi) = \{\neg\varphi\} \cup subf(\varphi)$, para todo $\varphi \in \mathcal{F}^{CP}$;

c) $subf(\varphi \square \psi) = \{\varphi \square \psi\} \cup subf(\varphi) \cup subf(\psi)$,

para todo $\square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ e para todo $\varphi, \psi \in \mathcal{F}^{CP}$.

1.2. φ: i) p_{2020} ii) $\neg\bot \vee \bot$ iii) $p_0 \rightarrow (\neg p_0 \rightarrow \neg p_1)$

b) Indique o conjunto das subfórmulas de φ.

$subf(p_{2020}) = \{p_{2020}\}$

$subf(\neg\bot \vee \bot) = \{\neg\bot \vee \bot\} \cup subf(\neg\bot) \cup subf(\bot)$

$= \{\neg\bot \vee \bot\} \cup \{\neg\bot\} \cup subf(\bot) \cup \{\bot\}$

$= \{\neg\bot \vee \bot, \neg\bot, \bot\}$

$subf(p_0 \rightarrow (\neg p_0 \rightarrow \neg p_1)) = \{p_0 \rightarrow (\neg p_0 \rightarrow \neg p_1)\} \cup$

$\cup\, subf(p_0) \cup subf(\neg p_0 \rightarrow \neg p_1) = \{p_0 \rightarrow (\neg p_0 \rightarrow \neg p_1)\} \cup \{p_0\} \cup$

$\cup \{\neg p_0 \rightarrow \neg p_1\} \cup subf(\neg p_0) \cup subf(\neg p_1) = \{p_0 \rightarrow (\neg p_0 \rightarrow \neg p_1),$

$p_0, \neg p_0 \rightarrow \neg p_1\} \cup \{\neg p_0\} \cup subf(p_0) \cup \{\neg p_1\} \cup subf(p_1) =$

$= \{p_0 \rightarrow (\neg p_0 \rightarrow \neg p_1), p_0, \neg p_0 \rightarrow \neg p_1, \neg p_0, \neg p_1, p_0, p_1\}$

1.3.

a) $p : \mathcal{F}^{CP} \longrightarrow \mathbb{N}_0$

$\varphi \longmapsto p(\varphi) =$ nº de ocorrências de parêntesis em φ

$p : \mathcal{F}^{CP} \to \mathbb{N}_0$ é definida, por recursão estrutural, do seguinte modo:

(a) $p(\bot) = 0$;

(b) $p(p_i) = 0$, para todo $i \in \mathbb{N}_0$;

(c) $p((\neg\varphi)) = 2 + p(\varphi), \quad \forall \varphi \in \mathcal{F}^{CP}$

(d) $p((\varphi \square \psi)) = 2 + p(\varphi) + p(\psi),$

$\forall \square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

$\forall \varphi, \psi \in \mathcal{F}^{CP}$.

1.3.

b) $v : \mathcal{F}^{CP} \longrightarrow \mathbb{N}_0$

$\varphi \longmapsto v(\varphi) =$ nº de ocorrências de variáveis proposicionais em φ

$v : \mathcal{F}^{CP} \longrightarrow \mathbb{N}_0$ é definida, por recursão estrutural em $\mathcal{F}^{CP}$, do seguinte modo:

(a) $v(\perp) = 0$

(b) $v(p_i) = 1, \quad \forall i \in \mathbb{N}_0$

(c) $v((\neg\varphi)) = v(\varphi), \quad \forall \varphi \in \mathcal{F}^{CP}$

(d) $v\big((\varphi \square \psi)\big) = v(\varphi) + v(\psi), \quad \forall \square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}, \ \forall \varphi, \psi \in \mathcal{F}^{CP}$

1.3.

$BIN = \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

c) $b : \mathcal{F}^{CP} \rightarrow \mathcal{P}(BIN)$

$$\varphi \longmapsto b(\varphi) = \{\square \in BIN : \square \text{ ocorre em } \varphi\}$$

$b(\varphi)$ = conj. dos conetivos binário que ocorrem em φ

$b : \mathcal{F}^{CP} \rightarrow \mathcal{P}(BIN)$ é definida, por recursão estrutural em fórmulas do CP, do seguinte modo:

(a) $b(\perp) = \emptyset$

(b) $b(p_i) = \emptyset$, $\forall i \in \mathbb{N}_0$

(c) $b((\neg\varphi)) = b(\varphi)$, $\forall \varphi \in \mathcal{F}^{CP}$

(d) $b((\varphi \square \psi)) = \{\square\} \cup b(\varphi) \cup b(\psi)$,

$\forall \square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

$\forall \varphi, \psi \in \mathcal{F}^{CP}$

1.3.

d) $\_[\perp/p_7] : \mathcal{F}^{CP} \to \mathcal{F}^{CP}$

$$\varphi \longmapsto \varphi[\perp/p_7]$$

obtida de φ substituindo todas as ocorrências de p_7 por $\perp$

$\_[\perp/p_7] : \mathcal{F}^{CP} \to \mathcal{F}^{CP}$ é definida, por recursão estrutural em fórmulas do CP, do seguinte modo:

(a) $\perp[\perp/p_7] = \perp$

(b) $p_i[\perp/p_7] = \begin{cases} \perp & \text{se } i = 7 \\ p_i & \text{se } i \in \mathbb{N}_0 \setminus \{7\} \end{cases}$

(c) $(\neg\varphi)[\perp/p_7] = \neg\varphi[\perp/p_7]$, $\forall \varphi \in \mathbb{N}_0$

(d) $(\varphi \square \psi)[\perp/p_7] = \varphi[\perp/p_7] \square \psi[\perp/p_7]$,

$\forall \square \in \{\land, \lor, \to, \leftrightarrow\}$
$\forall \varphi, \psi \in \mathcal{F}^{CP}$

1.4.

a) $\mathcal{P}(\varphi)$: $\boxed{v(\varphi) \geq \# \, var(\varphi)}$

Se

(1) $P(\bot)$;

(2) $P(p)$, para todo $p \in \mathcal{V}^{CP}$;

(3) $P(\psi) \Rightarrow P(\neg\psi)$, para todo $\psi \in \mathcal{F}^{CP}$;

(4) $P(\psi_1)$ e $P(\psi_2) \Rightarrow P(\psi_1 \square \psi_2)$, para todo $\square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ e para todo $\psi_1, \psi_2 \in \mathcal{F}^{CP}$;

então $P(\varphi)$, para todo $\varphi \in \mathcal{F}^{CP}$

v

$v(\bot) = 0$

$v(p_i) = 1$, $\forall i \in \mathbb{N}_0$

$v(\neg\varphi) = v(\varphi)$, $\forall \varphi \in \mathcal{F}^{CP}$

$v(\varphi \square \psi) = v(\varphi) + v(\psi)$, $\forall \square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, $\forall \varphi, \psi \in \mathcal{F}^{CP}$

var

$var(\bot) = \emptyset$

$var(p_i) = \{p_i\}$, $\forall i \in \mathbb{N}_0$

$var(\neg\varphi) = var(\varphi)$, $\forall \varphi \in \mathcal{F}^{CP}$

$var(\varphi \square \psi) = var(\varphi) \cup var(\psi)$, $\forall \square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, $\forall \varphi, \psi \in \mathcal{F}^{CP}$

v	var
$v(\perp) = 0$ $v(p_i) = 1, \forall i \in \mathbb{N}_0$ $v(\neg\varphi) = v(\varphi), \forall \varphi \in \mathcal{F}^{CP}$ $v(\varphi \square \psi) = v(\varphi) + v(\psi)$ $\forall \square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ $\forall \varphi, \psi \in \mathcal{F}^{CP}$	$var(\perp) = \emptyset$ $var(p_i) = \{p_i\}, \forall i \in \mathbb{N}_0$ $var(\neg\varphi) = var(\varphi), \forall \varphi \in \mathcal{F}^{CP}$ $var(\varphi \square \psi) = var(\varphi) \cup var(\psi)$, $\forall \square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ $\forall \varphi, \psi \in \mathcal{F}^{CP}$

(1) $v(\perp) = 0$
$\# var(\perp) = \# \emptyset = 0$. Logo, $\mathcal{P}(\perp)$

(2) $v(p_i) = 1$
$\# var(p_i) = \# \{p_i\} = 1$. Portanto, $\mathcal{P}(p_i)$ (para todo $i \in \mathbb{N}_0$).

(3) Seja $\varphi \in \mathcal{F}^{CP}$ tal que $\mathcal{P}(\varphi)$, ou seja,

$$v(\varphi) \geq \# var(\varphi). \quad (HI)$$

Temos que

$$v(\neg\varphi) = v(\varphi) \underset{HI}{\geq} \# var(\varphi) = \# var(\neg\varphi)$$

Logo, $\mathcal{P}(\neg\varphi)$.

(4) Sejam $\varphi, \psi \in \mathcal{F}^{CP}$ tais que $\mathcal{P}(\varphi)$ e $\mathcal{P}(\psi)$, ou seja, tais que

$$v(\varphi) \geq \# var(\varphi)$$

e

$$v(\psi) \geq \# var(\psi)$$

Temos que

$$\begin{aligned} v(\varphi \square \psi) &= v(\varphi) + v(\psi) \\ &\geq \# var(\varphi) + \# var(\psi) \\ &\geq \# \left(var(\varphi) \cup var(\psi) \right) \\ &= \# var(\varphi \square \psi). \end{aligned}$$

Logo, $P(\varphi \square \psi)$.

Por (1)–(4), pelo Princípio de Indução Estrutural para fórmulas do CP, podemos concluir $P(\varphi)$, para todo $\varphi \in \mathcal{F}^{CP}$.

1.4.

c) $\mathcal{P}(\varphi): \quad \nu(\varphi) \geq \nu(\varphi[\bot/p_7])$

$\nu(\bot) = 0$ $\nu(p_i) = 1, \forall i \in \mathbb{N}_0$ $\nu(\neg\varphi) = \nu(\varphi), \forall \varphi \in \mathcal{F}^{CP}$ $\nu(\varphi \square \psi) = \nu(\varphi) + \nu(\psi),$ $\forall \square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ $\forall \varphi, \psi \in \mathcal{F}^{CP}$ ν	$\bot[\bot/p_7] = \bot$ $p_i[\bot/p_7] = \begin{cases} \bot & \text{se } i = 7 \\ p_i & \text{se } i \neq 7 \end{cases} \quad (i \in \mathbb{N}_0)$ $(\neg\varphi)[\bot/p_7] = \neg\varphi[\bot/p_7], \forall \varphi \in \mathcal{F}^{CP}$ $(\varphi \square \psi)[\bot/p_7] = \varphi[\bot/p_7] \square \psi[\bot/p_7],$ $\forall \square \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ $\forall \varphi, \psi \in \mathcal{F}^{CP}$ $-[\bot/p_7]$

(1) $\nu(\bot) = 0 = \nu(\bot[\bot/p_7])$ Logo, $\mathcal{P}(\bot)$

(2) $\underline{i \in \mathbb{N}_0}$

$\underline{i = 7}$: $\nu(p_i) = 1, \quad \nu(p_i[\bot/p_7]) = \nu(\bot) = 0$
Portanto, $\mathcal{P}(p_7)$

$\underline{i \neq 7}$: $\nu(p_i) = 1, \quad \nu(p_i[\bot/p_7]) = \nu(p_i) = 1$
Logo, $\mathcal{P}(p_i)$

(3) Seja $\varphi \in \mathcal{F}^{CP}$ tal que $\mathcal{P}(\varphi)$, ou seja,

$$\nu(\varphi) \geq \nu(\varphi[\bot/p_7]). \quad (HI)$$

Pretendemos mostrar $\mathcal{P}(\neg\varphi)$, isto é,

$$\nu(\neg\varphi) \geq \nu((\neg\varphi)[\bot/p_7])$$

Temos que

$$\nu(\neg\varphi) = \nu(\varphi) \underset{HI}{\geq} \nu(\varphi[\bot/p_7]) = \nu(\neg\varphi[\bot/p_7])$$
$$= \nu((\neg\varphi)[\bot/p_7])$$

(4) Sejam $\varphi, \psi \in \mathcal{F}^{CP}$ tais que $\mathcal{P}(\varphi)$ e $\mathcal{P}(\psi)$.
Então,

$$v(\varphi) \geq v(\varphi[\bot/p_7])$$

e

$$v(\psi) \geq v(\psi[\bot/p_7]) . \quad \text{(HI)}.$$

Pretendemos mostrar $\mathcal{P}(\varphi \square \psi)$, ou seja, que

$$v(\varphi \square \psi) \geq v((\varphi \square \psi)[\bot/p_7])$$

Temos que

$$v(\varphi \square \psi) = v(\varphi) + v(\psi) \geq v(\varphi[\bot/p_7]) + v(\psi[\bot/p_7]) = v(\varphi[\bot/p_7] \square \psi[\bot/p_7]) = v((\varphi \square \psi)[\bot/p_7]).$$

Portanto, $\mathcal{P}(\varphi \square \psi)$.

Por (1)-(4), pelo Princípio de Indução Estrutural para fórmulas do CP, $\mathcal{P}(\varphi)$, para todo $\varphi \in \mathcal{F}^{CP}$.