Fixamos o espaço vectorial $\mathbb{R}^n$ ou $\mathbb{C}^n$

Definimos o produto interno usual

$x, y \in \mathbb{R}^n$: $\quad x \cdot y = x^T y \qquad \cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$x, y \in \mathbb{C}^n$: $\quad x \cdot y = x^* y \qquad \cdot : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$

$v \in \mathbb{C}^3 \qquad v = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix} \qquad v^T = \begin{bmatrix} 1 & 1+i & i \end{bmatrix}$

$\begin{bmatrix} 1 \\ i \end{bmatrix} \in \mathbb{C}^2 \qquad\qquad v^* = \begin{bmatrix} 1 & 1-i & -i \end{bmatrix}$

$$\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 - \underbrace{i^2}_{=-1} = 1+1 = 2$$

<u>Propriedade</u> 1) $v \cdot v \geq 0$

e $v \cdot v = 0 \iff v = 0$

2) $(v+u) \cdot w = v \cdot w + u \cdot w$

3) $(\alpha v) \cdot w = \alpha (v \cdot w)$, $\alpha \in \mathbb{R}, \mathbb{C}$.

<u>Def</u>. norma associada ao p.i.

$$\|v\| := \sqrt{v \cdot v} \in \mathbb{R}$$

$$\theta = \measuredangle(u,v) \quad : \quad \cos\theta = \frac{u\cdot v}{\|u\|\,\|v\|}$$

$\| \ \|$ norma induzida pelo p.i.

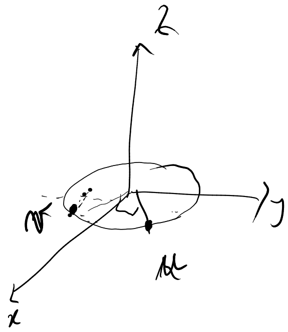
Seja V subesp. vectorial de $\mathbb{R}^n$, $\mathbb{C}^n$

(i.e., $V \neq \emptyset$, $V+V \subseteq V$, $\mathbb{R}V \subseteq V$ (ou $\mathbb{C}V \subseteq V$))

$$B = \{u_1, u_2, \dots, u_k\} \subseteq V$$

$\mathbb{R}^3$; V subesp. de $\mathbb{R}^3$

$$B = \left\{ \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right) \right\}$$



$$\underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}}_{u} \underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}}_{v} = 1$$

$$u \cdot v = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = 0 \qquad u \perp v$$

$B = \{u_1, u_2, \dots, u_k\} \subseteq V$

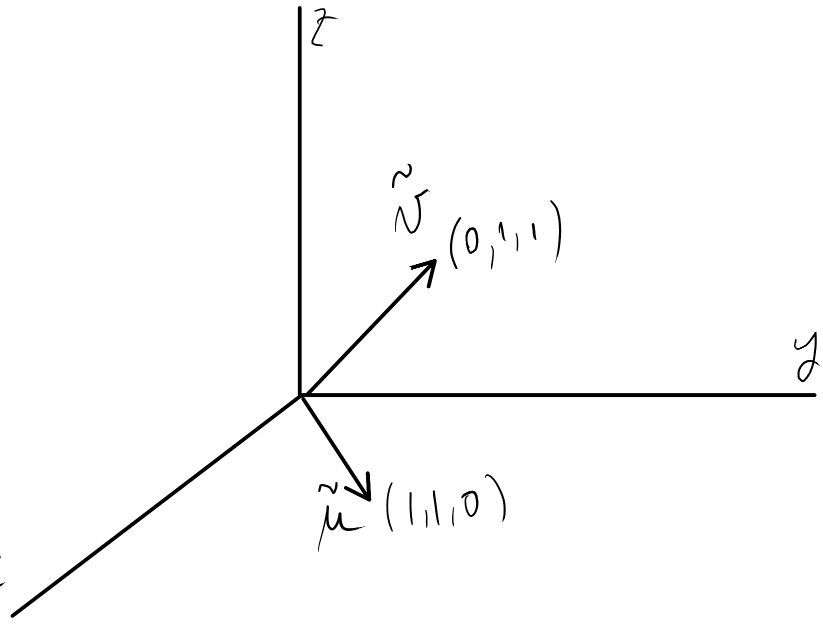
u_i são ortogonais $i \neq j$, $u_i \perp u_j$ $\quad i,j = 1, \dots, k$

então B é conjunto ortogonal.

Se adicionalmente $\|u_i\| = 1$ então

B é conjunto ortonormado

Exemplo.



$$\tilde{u} \cdot \tilde{v} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 1$$

$$B = \{(1,1,0), (0,1,1)\}$$

$$\{\alpha(1,1,0) + \beta(0,1,1)\}_{\alpha,\beta \in \mathbb{R}} =: \langle B \rangle$$

subespaço gerado por B

⚠ B não é ortogonal

Recordar defn. de base

$\{u_1, \dots, u_k\} = B \subseteq V$ é base de V se

B é linearmente independente

i.e., $\sum_{1}^{k} \alpha_i u_i = 0 \Rightarrow \alpha_i = 0$, $i = 1, \dots, k$.

e $\forall v \in V$, $\exists! \, \alpha_1, \dots, \alpha_k \in \mathbb{R}$:

$$v = \sum \alpha_i u_i$$

$k = \dim V$ $\qquad$ k é a dimensão de V

Ex $(1,1,0)$ e $(0,1,1)$ são l.i.

$$x(1,1,0) + y(0,1,1) = (0,0,0)$$

$$\Rightarrow (x, x, 0) + (0, y, y) = (0,0,0)$$

$$\Rightarrow (x, x+y, y) = (0,0,0) \Rightarrow \begin{cases} x = 0 \\ x+y = 0 \\ y = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$B \subseteq V$ é base ortonormada de V se B é base de V e é conjunto ortonormado

__Proposição.__ Se $v_1, \dots, v_k \in \mathbb{R}^n$ são ortonormados então são linearmente independentes.

dem.

Sup. $\|v_i\| = 1$ e $v_i \perp v_j$ $i \neq j$

Pretendemos mostrar que $\sum_{1}^{k} \alpha_i v_i = 0 \Rightarrow \alpha_i = 0$, $i = 1, \dots, k$

$$0 = \left(\sum_{i=1}^{k} \alpha_i v_i \right) \cdot v_1 = \sum_{i=1}^{k} \alpha_i (v_i \cdot v_1)$$

$$= \alpha_1 (v_1 \cdot v_1) = \alpha_1$$

$$0 = \left(\sum \quad \right) \cdot v_2 = \alpha_2 \qquad (\dots)$$

$\square$

$B = \{(1,1,0), (1,-1,1)\}$ conj.to ortogonal

$\tilde{B} = \left\{ \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right), \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right) \right\}$ ortonormado

Ex. $\{(1,-1,0),(1,1,1),(-1,-1,2)\} \subseteq \mathbb{R}^3$ cnj^to ortogonal

$\underbrace{(1,-1,0)}_{u_1}, \underbrace{(1,1,1)}_{u_2}, \underbrace{(-1,-1,2)}_{u_3}$

$u_1 \perp u_2 \qquad u_1 \perp u_3 \qquad u_2 \perp u_3$

$$\left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|} \right\}$$

$$\left(\frac{u_1}{\|u_1\|}\right) \cdot \left(\frac{u_2}{\|u_2\|}\right) = \frac{1}{\|u_1\|\|u_2\|} u_1 \cdot u_2 = 0$$

$$\left\| \frac{u_1}{\|u_1\|} \right\| = \frac{1}{\|u_1\|} \|u_1\| = 1$$

$v = (x_1, x_2, \dots, x_n) \qquad \alpha v = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$

$\|v\|^2 = \sum x_i^2$

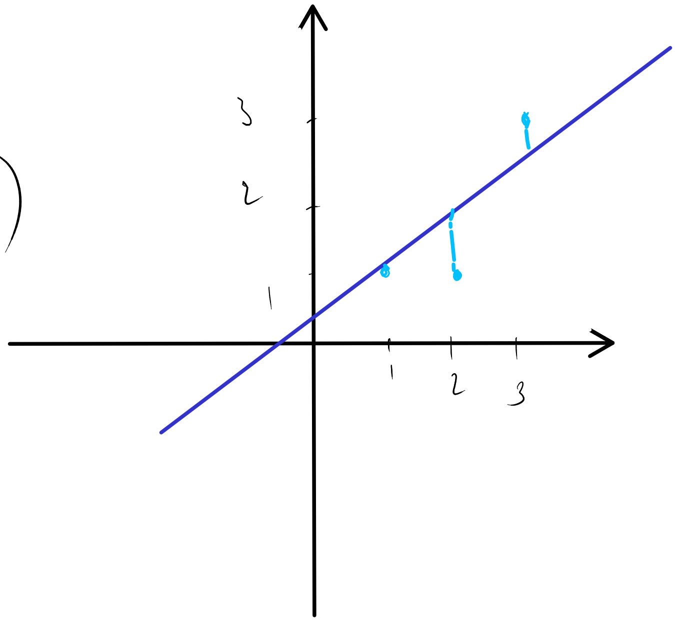
$\|\alpha v\|^2 = \sum (\alpha x_i)^2 = \sum \alpha^2 x_i^2 = \alpha^2 \sum x_i^2$

$= \alpha^2 \|v\|^2$

$\|\alpha v\| = |\alpha| \, \|v\|$

$y = mx + b$

$(1,1), (2,1), (3,3)$



$$\begin{cases} m + b = 1 \\ 2m + b = 1 \\ 3m + b = 3 \end{cases}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}}_{b}$$

$$[A \mid b] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 3 \end{array}\right] \xrightarrow[l_3 - 3l_1]{l_2 - 2l_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[l_3 - 2l_2]{} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{array}\right]$$

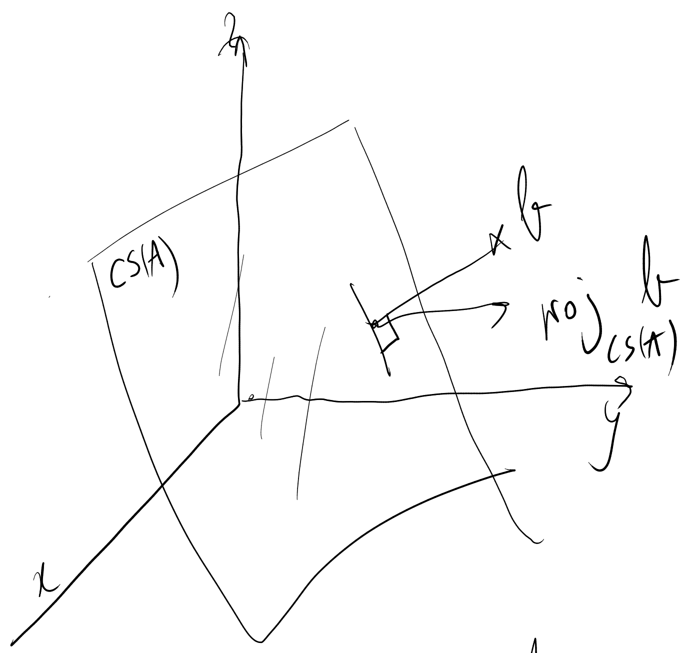
$car(A) = 2 < 3 = car([A \mid b])$

(o sist. eq. lineares é impossível)

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$CS(A) = \{ Av : v \in \mathbb{R}^2 \}$$

$$A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$



$Ax = \text{proj}_{CS(A)}\, b$ é possível

¿Como calcular $\text{proj}_{CS(A)}\, b$?

$\text{proj}_{CS(A)}\, b$ é o único elem^to^ de $CS(A)$ que minimiza $\|b - v\|$

$v \in CS(A)$

$$\langle \underbrace{(1,-1,0)}_{u}, \underbrace{(1,1,1)}_{v} \rangle$$

$u \perp v$ $\qquad A = \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix}$

$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = b \in CS(A)$? $\qquad$ i.e., $Ax = b$ é possível?

$$[A|b] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow[l_2 + l_1]{} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[l_3 - \frac{1}{2} l_2]{} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad \text{car}(A) = 2 < 3 = \text{car}([A|b])$$

<u>NÃO EXISTEM</u> $\alpha, \beta \in \mathbb{R}$: $b = \alpha \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Mas existe x t.q. $Ax = \text{proj}_{CS(A)} b$

Se $x = (x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ então

$$\text{proj}\, b = Ax = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_2$$

$$(\text{proj}\, b) \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_2 \right) \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= x_1 \begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix} + x_2 \underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}_{0}$$

$$= 2x_1$$

$$(\text{proj } b)\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} = 3x_2$$