Dado $x \in \mathbb{R}^n$, $\|x\| = 1$, $x_1 \neq 0$

Objectivo: Construir Q ortogonal t.q. $[Q]_{:,1} = x$

i.e. $Q = \begin{bmatrix} | & | & | & | \\ x & & & \\ | & | & | & | \end{bmatrix}$ usando rotações de Givens

$$\underbrace{P_{1n} \cdots P_{13} P_{12}}_{P} x = \begin{bmatrix} \|x\| \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e_1$$

$\longrightarrow$ 1ª coluna de I_n

$P = \prod_{k=n}^{2} P_{1k}$ ortogonal, i.e., $P^{-1} = P^T$

$$Px = e_1 \Rightarrow x = P^T e_1 = P^T \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= [P^T]_{:,1}$$

Basta tomar $Q = P^T$

REDUÇÃO ORTOGONAL

$A_{m\times n}$ AEG

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ -1 & 0 & -2 \end{bmatrix} \xrightarrow[\ell_3 \leftarrow \ell_3 + \ell_1]{\ell_2 \leftarrow \ell_2 - 2\ell_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -4 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{\ell_2 \leftrightarrow \ell_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{Perm_{2,3}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{E_{31}(1)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}(-2)} A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -4 \end{bmatrix} = U$$

$$\underbrace{\left(\prod E_i\right)}_{K} A = U \Rightarrow A = L\,U$$

$$L = K^{-1} =$$

$$Perm_{2,3}\, E_{31}(1) = Perm_{2,3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Perm_{2,3} = E_{21}(1)\, Perm_{2,3}$$

$$Perm_{2,3}\, E_{31}(1)\, E_{21}(-2) A = E_{21}(1)\, Perm_{2,3}\, E_{21}(-2) A = \underbrace{E_{21}(1)\, E_{31}(-2)}_{K} \underbrace{Perm_{2,3}}_{P} A = U$$

$$KPA = U \Rightarrow PA = K^{-1}U$$

$$\Rightarrow PA = LU$$

$L := K^{-1}$

$U \rightarrow$ forma canónica de linhas

U triang. inf.

Vamos usar transformadas de Householder para "eliminar" as entradas debaixo do pivot.

Seja $A_{m \times n}$

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \quad x := a_1 \quad ; \quad x_1 \neq 0$$

$$R_1 = I - 2\frac{uu^*}{u^*u} \qquad u = a_1 \pm \mu \|a_1\| e_1 = x \pm \mu \|x\| e_1$$

$$\mu = \begin{cases} 1 & \text{se } x_1 \in \mathbb{R} \\ \frac{x_1}{|x_1|} & \text{senão} \end{cases}$$

$$R_1 x = R_1 a_1 = \mp \mu \|x\| e_1 = \begin{bmatrix} t_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$R_1 A = R_1 \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} R_1 a_1 & R_1 a_2 & \cdots & R_1 a_n \end{bmatrix}$$

$$= \left[\begin{array}{c|ccc} t_{11} & t_{12} & \cdots & t_{1n} \\ \hline 0 & & & \\ \vdots & & ? & \\ 0 & & & \end{array}\right] = \left[\begin{array}{c|c} t_{11} & t_1^T \\ \hline 0 & A_2 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ -1 & 0 & -2 \end{bmatrix} \qquad a_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$u = a_1 + \mu \|a_1\| e_1 \quad \left(e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \ \mu = 1$$

$$= \begin{bmatrix} 1+\sqrt{6} \\ 2 \\ -1 \end{bmatrix} \qquad \|a_1\| = \sqrt{6}$$

$$R_1 = I - 2\frac{uu^T}{\|u\|^2}$$

$$\|u\|^2 = 4 + 1 + (1+\sqrt{6})^2 = 12 + 2\sqrt{6}$$

$$\frac{1}{\|u\|^2} = \frac{1}{6+\sqrt{6}}$$

$(\ldots)$

Repete-se o processo relativamente a A_2

$$\hat{R}_2 = \ldots \qquad \hat{R}_2 A_2 = \left[\begin{array}{c|c} t_{22} & t_2^T \\ \hline 0 & A_3 \end{array}\right]$$

Ao fim de k passos

$$R_{k-1} \cdots \underbrace{\left[\begin{array}{c|c} I_2 & \\ \hline & \hat{R}_3 \end{array}\right]}_{R_3} \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \hat{R}_2 \end{array}\right]}_{R_2} R_1 A = \left[\begin{array}{cc|ccc} t_{11} & ? & ? & \cdots \\ \hline 0 & t_{22} & & \\ \hline \vdots & 0 & & \\ \vdots & \vdots & & \hat{A}_k \\ 0 & 0 & & \end{array}\right]$$

Se A tiver característica máxima,

$$R_? \cdots R_2 R_1 A = \left[\begin{array}{cccc} * & & & ? \\ 0 & * & & \\ \vdots & 0 & * & \\ & & & * \\ \hline & & 0 & \end{array}\right]$$

invert.

$$\left[\begin{array}{cccc|c} * & & ? & & \\ & * & & & ? \\ & & * & & \\ 0 & & & * & \end{array}\right]$$

inv.

R_i são ortogonais/unitários

$$\Rightarrow \underbrace{\prod R_i}_{=:P} \text{ é ortogonal/unitária}$$

$$PA = T \Rightarrow A = P^* T$$

Usando rotações de Givens

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ -1 & 0 & -2 \end{bmatrix} \qquad a_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = x$$

$$P_{12} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad c = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} = \frac{1}{\sqrt{5}} \qquad s = \frac{x_2}{\sqrt{x_1^2 + x_2^2}} = \frac{2}{\sqrt{5}}$$

Factorização QR

$A_{n \times n}$ real não-singular, i.e., $car(A) = n$,
i.e., A é invertível

$\exists! Q$ ortogonal
R t. sup. $R_{ii} > 0$

$$A = QR$$

Unicidade: $A = Q_1 R_1 = Q_2 R_2$

$$\Rightarrow Q_2^T Q_1 R_1 = R_2 \longrightarrow Q_2^T Q_1 = R_2 R_1^{-1}$$

LHS Q_2^{-1} RHS R_1^{-1}

$$\underbrace{Q_2^T Q_1}_{\text{ortogon.}} = \underbrace{R_2 R_1^{-1}}_{\text{t. sup. c/ elem}^{\text{tos}}\text{ diag} > 0}$$

$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$ ortogonal que é t. sup. c/ diagon. > 0

$$= \begin{bmatrix} u_{11} & ? & ? & ? \\ 0 & u_{22} & & \\ \vdots & 0 & u_{33} & \\ 0 & \vdots & 0 & u_{nn} \end{bmatrix}$$

$u_i \perp u_j$, $i \neq j$
$\|u_i\| = 1$

$$\|u_1\| = 1 \Rightarrow u_{11} = 1$$

$$U = \begin{bmatrix} 1 & u_{12} & ? & ? & ? \\ 0 & u_{22} & & & \\ \vdots & 0 & & & \\ & \vdots & & & \\ 0 & 0 & & & \end{bmatrix}$$

$$\begin{array}{cc} u_1 & u_2 \end{array}$$

$$u_1 \perp u_2 \implies u_{12} = 0$$

$$U = \begin{bmatrix} 1 & 0 & \\ 0 & u_{22} & \\ 0 & 0 & ? \\ \vdots & \vdots & \end{bmatrix} \qquad \|u_2\| = 1 \implies u_{22} = 1$$

$$\begin{array}{ccc} u_1 & u_2 & \cdots \end{array}$$

$$U = \begin{bmatrix} 1 & 0 & u_{13} & \\ 0 & 1 & u_{23} & ? \\ \vdots & 0 & u_{33} & \\ & \vdots & 0 & \\ & & \vdots & \end{bmatrix} \quad (\cdots)$$

$$U = I$$

$$U = Q_2^T Q_1 = R_2 R_1^{-1} = I$$

$$\implies Q_1 = Q_2 \quad \& \quad R_1 = R_2$$

Subespaços Complementares

V esp. vect. $\qquad X, Y \leq V$

subesp. vect. de V

$V = X + Y$

$X \cap Y = \{0\}$. Neste caso, X e Y dizem-se subesp. complementares

$V = X \oplus Y$

B_X base de X , B_Y base de Y

Então $B_X \cup B_Y$ é base de V

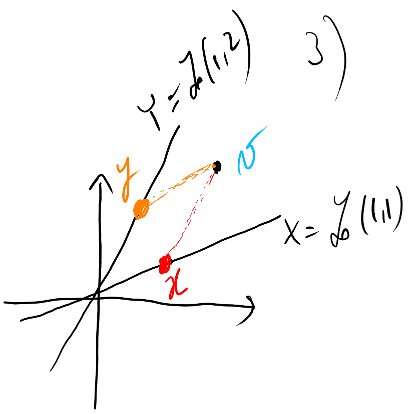
São equivalentes

1) $V = X \oplus Y$

2) $\forall v \in V$, $\exists!\, x \in X$: $v = x + y$
$\qquad y \in Y$

3) $B_X \cup B_Y$ é base de V

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$



$v = x + y$

x é a proj de v em X ao longo de Y

y é a projecção de v em Y ao longo de X

$X \perp Y$ se $\forall x \in X, y \in Y$, $x \perp y$

Se $V = X + Y$ e $X \perp Y$ então a soma é directa, e denota-se por

$$V = X \oplus^{\perp} Y$$

Se $V = X \oplus^{\perp} Y$, $v = x + y$, $x \in X$, $y \in Y$

x é a proj ort. de v em X

y " " " " " v em Y

$\mathbb{R}^n = X \oplus Y$; $v \in \mathbb{R}^n$

$\exists! \; x \in X, y \in Y : x + y = v$

Vamos construir um projector $P_{n \times n}$

t.q. Pv é a projecção de v em X ao longo de Y

$B_X = \{x_1, \ldots, x_r\}$ base de X

$B_Y = \{y_1, \ldots, y_{n-r}\}$ base de Y

Então $B_X \cup B_Y$ é base de $\mathbb{R}^n$

$$B_{n\times n} = \left[x_1 \; x_2 \cdots x_k \,\middle|\, y_1 \; y_2 \cdots y_{n-k} \right]_{n\times n} \quad \text{é invertível}$$

$\mathbb{R}^3$; $X = \mathcal{L}(1,1,1)$; $Y = \mathcal{L}(\{(1,1,0),(1,-1,1)\})$

$$\text{car}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\right) = 3 \Rightarrow \mathbb{R}^3 = X \oplus Y$$

$v = x + y$, $x \in X$, $y \in Y$

proj de v em X ao longo de Y

proj de v em Y ao longo de X

fixando X, Y , $x = \text{proj}_X v$

$y = \text{proj}_Y v$

$$X \ni x = \underset{X}{\underset{\Uparrow}{x}} + \underset{Y}{\underset{\Uparrow}{0}}$$

$\text{proj}_X x = x$

$\text{proj}_Y x = 0$

$\text{proj}_X x_i = x_i$; $\text{proj}_Y x_i = 0$

$\text{proj}_X y_i = 0$; $\text{proj}_Y y_i = y_i$

$$PB = P\left[\begin{array}{cccc|ccc} | & | & & | & | & & | \\ x_1 & x_2 & \cdots & x_r & y_1 & \cdots & y_{n-r} \\ | & | & & | & | & & | \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} Px_1 & Px_2 & \cdots & Px_r & Py_1 & Py_2 & \cdots & Py_{n-r} \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} x_1 & x_2 & \cdots & x_r & 0 & 0 & \cdots & 0 \end{array}\right] = \left[\begin{array}{c|c} X & 0 \end{array}\right]$$