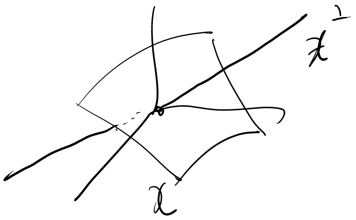
A matriz $m \times n$ $\qquad T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\qquad\qquad\qquad\qquad\qquad v \longmapsto Av$

$CS(A) \leq \mathbb{R}^m$

$Ker(A) = \{ v : Av = 0 \} \leq \mathbb{R}^n$

$\mathcal{X} \leq \mathbb{R}^n \quad , \quad \mathcal{X}^{\perp} = \{ y \in \mathbb{R}^n : x \perp y, \ \forall x \in \mathcal{X} \}$



$$\mathbb{R}^n = \mathcal{X} \oplus^{\perp} \mathcal{X}^{\perp}$$

$$\mathbb{R}^m = CS(A) \oplus^{\perp} Ker(A^T)$$

$$\mathbb{R}^n = Ker(A) \oplus^{\perp} CS(A^T)$$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\qquad car(A) = 2 \Rightarrow \dim CS(A) = 2$

$$B_{CS(A)} = \{ (1,0,0), (1,1,0) \}$$

$$\mathbb{R}^3 = CS(A) \oplus^{\perp} Ker(A^T)$$

$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow[l_2 \leftarrow l_2 - l_1]{AEG} \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ x_1 \ \ x_2 \ \ x_3 \end{array}$

$A^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$

x_3 ff

$$Ker(A^T) = \left\{ \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \forall \in \mathbb{R} \right\}$$

$$= \langle (0,0,1) \rangle$$

$$B_{Ker(A^T)} = \{ (0,0,1) \}$$

$B_{\mathbb{R}^3} = B_{CS(A)} \cup B_{Ker(A^T)}$ $\quad$ pq a soma é directa!

$= \{ (1,0,0), (1,1,0), (0,0,1) \}$

$u_1 = (1,0,0)$

$$u_2 = \frac{(1,1,0) - ((1,1,0)\cdot(1,0,0))\,(1,0,0)}{\|\quad - \quad\|} = (0,1,0)$$

$$U = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad V =$$

em relação a $CS(A^T)$
e $Ker(A)$

$$A = URV^T \quad , U, V \text{ são ortogonais}$$

$$R = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right] \quad C \text{ invert.}$$

A $m \times n$; obtemos então uma factorização

URV ; $A = URV^T$, U, V ortogonais

$$R = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array}\right]$$

$car(A) = r$

As 1.ªs r colunas de U são base o.n. de $CS(A)$
últimas $m-r$ " " " " " " " $Ker(A^T)$
1.ªs r " " V " " " " $CS(A^T)$
últimas $n-r$ " " " " " " " $Ker(A)$

Projecções ortogonais

$M \leq \mathbb{R}^n$, $M^\perp = \{v \in \mathbb{R}^n : v \perp m, \forall m \in M\}$

$\mathbb{R}^n = M \oplus^\perp M^\perp$

$\forall v \in \mathbb{R}^n$, $\exists! \; \begin{matrix} m \in M \\ n \in M^\perp \end{matrix} : v = m + n$

proj de v em M ao longo de $M^\perp$

proj de v em $M^\perp$ ao longo de M

$P = P_M$ projector sobre M (ao longo de $M^\perp$)

as colunas de M base de M, as colunas de N base ortonormada de $M^\perp$

$$P = [M|N] \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right] [M|N]^{-1} = [M|0][M|N]^{-1}$$

M e $M^\perp$ são complementos ortogonais

logo $N^T M = 0 \Rightarrow M^T N = 0$

Sup. $\dim M = r$ então

$\operatorname{car} M = r = \operatorname{car}(M^T M)$, M $n \times r$, $M^T M$ é $r \times r$

Logo $M^T M$ é invertível.

$$\left[\begin{array}{c}(M^TM)^{-1}M^T \\ \hline N^T\end{array}\right][M|N] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I\end{array}\right] \Rightarrow [M|N]^{-1} = \left[\begin{array}{c}(M^TM)^{-1}M^T \\ \hline N^T\end{array}\right]$$

$$P_{\mathcal{M}} = [M|0][M|N]^{-1} = [M|0]\left[\begin{array}{c}(M^TM)^{-1}M^T \\ \hline N^T\end{array}\right] \Longrightarrow$$

$$\Longrightarrow \boxed{P_{\mathcal{M}} = M(M^TM)^{-1}M^T}$$

Em particular, se as colunas de M formam base ortonormada de $\mathcal{M}$, então $M^TM = I$ e

$$P_{\mathcal{M}} = MM^T$$

Mais, se $M = [\underbrace{|\,|\,|}_{\text{base ortonormada de } \mathcal{M}}]$, $N = [\underbrace{|\,|\,|}_{\text{base orton. de } \mathcal{M}^\perp}]$

então $U = [M|N]$ é ortogonal e

$$P_M = U\left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & 0\end{array}\right]U^T \quad \text{onde } n = \dim \mathcal{M}$$

Caso geral. $\quad P_{\mathcal{M}} = M(M^TM)^{-1}M^T$

$$P_{\mathcal{N}^\perp} = N(N^TN)^{-1}N^T$$

$$P_{\mathcal{N}^\perp} = I - P_{\mathcal{N}}$$

$$\mathbb{R}^n = \mathcal{N} \overset{\perp}{\oplus} \mathcal{N}^\perp$$

$b \in \mathbb{R}^n$. Então $\displaystyle\min_{m \in \mathcal{N}} \| b - m \| = \| b - P_{\mathcal{N}} b \| =: \mathrm{dist}(b, \mathcal{N})$

dem. Seja $p = P_{\mathcal{N}} b$. Então $p - m \in \mathcal{N}$, $\forall m \in \mathcal{N}$

$\Rightarrow b - p = b - P_{\mathcal{N}} b = (I - P_{\mathcal{N}}) b \in \mathcal{N}^\perp$

$p - m \in \mathcal{N}$ e $(I - P_{\mathcal{N}}) b \in \mathcal{N}^\perp$

$$\Rightarrow \quad (p-m) \perp \big((I - P_{\mathcal{N}}) b \big)$$

i.e. $(p - m) \perp (b - p)$

$$\| b - m \|^2 = \| b - p + p - m \|^2 = \| b - p \|^2 + \| p - m \|^2$$

Logo, $\displaystyle\min_{m \in \mathcal{N}} \| b - m \| = \| b - p \|$ e é atingido c/ $m = p = P_{\mathcal{N}} b$

O arg min é único!

Sup. $\exists\, \tilde{m} \in \mathcal{N}$ t.q. $\| b - \tilde{m} \|^2 = \| b - p \|^2$

$$\|b-\tilde{m}\|^2 = \|b-p+p-\tilde{m}\|^2 = \|b-p\|^2 + \|p-\tilde{m}\|^2$$

$$\underset{\text{minimização}}{\Longrightarrow} \quad \|p-\tilde{m}\|^2 = 0 \quad \Rightarrow \quad p = \tilde{m}$$

Sup. $Ax = b$

O problema dos mínimos quadrados: encontrar x t.q. $\|b - Ax\|$ é mínima.

$$(Ax-b)^T(Ax-b) = \|Ax-b\|^2 \text{ mínimo.}$$

$$Ax \in CS(A) \quad ; \quad \underset{m \in CS(A)}{\operatorname{argmin}} \|b-m\| = P_{CS(A)}\, b$$

$$Ax = P_{CS(A)}\, b$$

$$\Leftrightarrow P_{CS(A)} Ax = P_{CS(A)}\, b$$

$$\Leftrightarrow P_{CS(A)}(Ax-b) = 0$$

$$\Leftrightarrow Ax-b \in \operatorname{Ker}(P_{CS(A)}) = CS(A)^\perp = \operatorname{Ker}(A^T)$$

$$\Leftrightarrow A^T(Ax-b) = 0 \quad \Leftrightarrow \quad A^TAx = A^Tb$$

$A_{m \times n}$, $car(A) = n$ então $car(A^TA) = n$

A^TA é $n \times n$

i.e., $(A^TA)^{-1}$ existe.

$$A^TAx = A^Tb \Rightarrow x = (A^TA)^{-1}A^Tb$$

SVD

$X_{n \times m}$ $\qquad car(X) = r = car(XX^T) = car(X^TX)$

X^TX é SDP

S simétrica é SDP se $\forall v,\ v \cdot (Sv) \geq 0$

S simétrica

$\Rightarrow \sigma(S) = \{\lambda : |\lambda I - S| = 0\}$

$\subseteq \mathbb{R}$

A é SDP $\Leftrightarrow A^T = A$

e $\sigma(A) \subseteq \mathbb{R}_0^+$

X^TX é $m \times m$ $\qquad \exists\, V$ ortogonal $(V^{-1} = V^T)$

D diagonal t.q.

$$X^TX = VDV^{-1}$$

λ_i valores próprios de X^TX

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_r > 0 = \cdots = 0$$

$$\begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ v_1 & v_2 & v_3 & & v_r \end{array}$$

$\lambda_i \neq \lambda_j \Rightarrow v_i \perp v_j$ $\qquad v_i \in \ker(\lambda_i I - X^TX)$

SPG. podemos assumir que → aplicar G-S, Householder, Givens

$$v_i \perp v_j \quad , \quad \|v_i\| = 1$$

onde $\quad v_i \in \mathrm{Ker}(\lambda_i I - X^T X)$

$$X^T X v_i = \lambda_i v_i$$

(por definição de valor/vector próprios)

$$X^T X v_i = \lambda_i v_i \quad , \quad \lambda_i > 0$$

$i = 1, \dots, r$

$v_i \neq 0$ é vector próp. assoc. ao valor próprio λ_i

se $\quad A v_i = \lambda_i v_i$

i.e., $\quad (\lambda_i I - A) v_i = 0 \quad , v_i \neq 0$

i.e., $\quad v_i \in \mathrm{Ker}(\lambda_i I - A) \setminus \{0\}$

$$X^T X v_i = 0 \, ,$$

$$\Rightarrow v_i \in \mathrm{Ker}(X^T X)$$

$$= \mathrm{Ker}(X)$$

$$\sigma_i = \sqrt{\lambda_i} \quad , \quad i = 1, \dots, r$$

$$u_i = \frac{1}{\sigma_i} X v_i$$

$$\|u_i\| = 1 \qquad (\dots)$$

$$u_i \perp u_j \quad , \quad i \neq j \qquad (\dots)$$

$$u_i = \frac{1}{\sigma_i} X v_i \Rightarrow X v_i = \sigma_i u_i$$

$$u_i \in \ker(X^T) \qquad u_i \perp u_j \;, i \neq j \;, \|u_i\| = 1$$

$$v_i \perp v_j \;, i \neq j \;, \|v_i\| = 1$$

$$\begin{cases} X v_1 = \sigma_1 u_1 \\ X v_2 = \sigma_2 u_2 \\ \vdots \\ X v_r = \sigma_r u_r \end{cases}$$

$$X \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_r \\ | & | & & | \end{bmatrix} = \begin{bmatrix} X v_1 & X v_2 & \cdots & X v_r \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 & \cdots & \sigma_r u_r \end{bmatrix}$$

$$= \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}$$

$$X v_{r+1} = X v_{r+2} = X v_m = 0$$

$$X \underbrace{\begin{bmatrix} v_1 & \cdots & v_r & v_{r+1} & \cdots & v_m \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} u_1 & \cdots & u_r & u_{r+1} & \cdots & u_n \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}}_{\Sigma}$$

$$V^T = V^{-1} \qquad U^T = U^{-1}$$

$$XV = U\Sigma \Rightarrow X = U\Sigma V^{-1}$$

SVD ; decomposició en valores singulares

$$X = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \dots + \sigma_r u_r v_r^T$$

$$= \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

$\tilde{r} \ll r$

$$X \approx \sum_{1}^{\tilde{r}} \sigma_i u_i v_i^T$$