$$x_i = \begin{bmatrix} x_1^1 \\ x_1^2 \\ \vdots \\ x_i^m \end{bmatrix} \qquad x_1, \dots, x_N$$

$$\mu_x = \frac{1}{N} \sum_{k=1}^{N} x_k$$

$$y_i := x_i - \mu_x \qquad B = \begin{bmatrix} | & | & & | \\ y_1 & y_2 & \cdots & y_N \\ | & | & & | \end{bmatrix}_{m \times N}$$

$$\underset{m \times m}{S} = BB^T \quad \text{simétrica} \ \& \ \text{SDP}$$

$\lambda_1, \dots, \lambda_m$ valores próprios, i.e. raízes

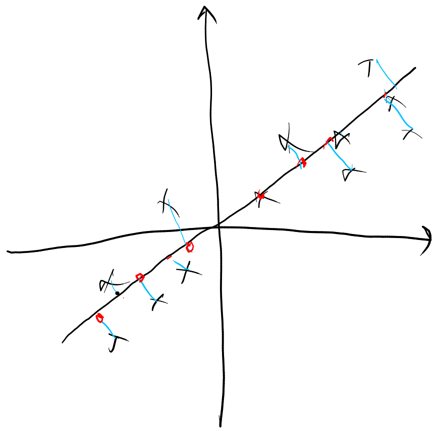
$$\Delta_S(\lambda) = \det(\lambda I - S) = |\lambda I - S|$$

$$\deg \Delta_S = m$$

S simétrica $\rightarrow$ $\lambda_i \in \mathbb{R}$

S simétrica $\lambda_i \in \mathbb{R}_0^+ \iff S$ é SDP

S simétrica $\rightarrow$ vect. prop. assoc. $\lambda_i \neq$'s são $\perp$



Objectivo:

encontrar w c/ $\|w\| = 1$

$$\mathcal{L}_0(w) = \{\alpha w : \alpha \in \mathbb{R}\}$$

$$\text{proj}_w x_i = (w \cdot x_i)\, w$$

Média das projecções: $\frac{1}{N} \sum_{i=1}^{N} \text{proj}_w x_i = \frac{1}{N} \sum_{i=1}^{N} (w \cdot x_i)\, w =$

$w \cdot (u + v)$
$= w \cdot u + w \cdot v$

$$= \underbrace{\Big(w \cdot \underbrace{\Big(\frac{1}{N} \sum_{i=1}^{N} x_i\Big)}_{= 0 \in \mathbb{R}^m}\Big)}_{0 \in \mathbb{R}} w = 0 \in \mathbb{R}^m \quad , m = 2$$

Ou seja, a média das projecções é nula.

Qual w por forma a minimizar a soma dos quadrados das distâncias.

$$\| x_i - \text{proj}_w x_i \|^2 = \| \underbrace{x_i}_{u} - \underbrace{(w \cdot x_i)}_{\alpha} \underbrace{w}_{v} \|^2 =$$

$$\| u - \alpha v \|^2 = (u - \alpha v) \cdot (u - \alpha v) = u \cdot (u - \alpha v) - \alpha (v \cdot (u - \alpha v))$$

$$= u \cdot u - \alpha\, u \cdot v - \alpha\, u \cdot v + \alpha^2\, v \cdot v$$

$$= \|u\|^2 - 2 \alpha\, u \cdot v + \alpha^2 \|v\|^2$$

$$= \|x_i\|^2 - 2\underbrace{(w\cdot x_i)\underbrace{(x_i\cdot w)}_{=w\cdot x_i}}_{=(w\cdot x_i)^2} + (w\cdot x_i)^2$$

$$= \|x_i\|^2 - (w\cdot x_i)^2$$

<u>Obj.</u>: minimizar $\displaystyle Res(w) = \sum_{i=1}^{N} \|x_i - proj_w x_i\|^2$

$$= \sum_{i=1}^{N}\left(\|x_i\|^2 - (w\cdot x_i)^2\right)$$

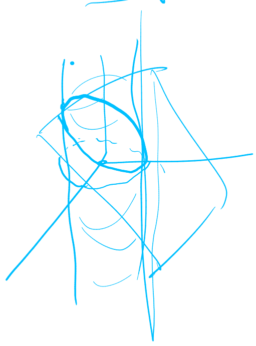
é o mesmo que maximizar $\displaystyle\sum_{i=1}^{N}(w\cdot x_i)^2$

sujeito à condição $\|w\|=1$

<u>Multiplicadores de Lagrange</u>

max $f(x,y) = x+y$

suj. $x^2+y^2=1$



$$u(x,y,\lambda) := f(x,y) - \lambda(x^2+y^2-1)$$

$\nabla u = 0$

$$\frac{\partial u}{\partial x} = 0 \Leftrightarrow x = \frac{1}{2\lambda}$$

$$\frac{\partial u}{\partial y} = \Leftrightarrow y = \frac{1}{2\lambda}$$

$$\frac{\partial u}{\partial \lambda} = 0 \Leftrightarrow x^2+y^2=1 \Leftrightarrow \lambda = \pm\frac{\sqrt{2}}{2}$$

$$\mathcal{L}(w,\lambda) = f(w) - \lambda(g(w) - c)$$

$$f(w) = w^T V w$$

$$g(w) = \|w\|^2 = w \cdot w$$

$$c = 1$$

O que é V?

<u>Recordar</u>: queremos maximizar $\sum_{i=1}^{N} (w \cdot x_i)^2$

é o mesmo que maximizar $\frac{1}{N} \sum (w \cdot x_i)^2$

$$X = \begin{bmatrix} — & x_1^T & — \\ — & x_2^T & — \\ & \vdots & \\ — & x_N^T & — \end{bmatrix}$$

$$Xw = \begin{bmatrix} — & x_1^T & — \\ — & x_2^T & — \\ & \vdots & \\ — & x_N^T & — \end{bmatrix} w = \begin{bmatrix} x_1^T w \\ x_2^T w \\ \vdots \\ x_N^T w \end{bmatrix} = \begin{bmatrix} x_1 \cdot w \\ x_2 \cdot w \\ \vdots \\ x_N \cdot w \end{bmatrix}$$

$$(Xw)^T = [w^T x_1 \quad w^T x_2 \quad \dots \quad w^T x_N] = [w \cdot x_1 \quad w \cdot x_2 \quad \cdots \quad w \cdot x_N]$$

$$\frac{1}{N} \sum (w \cdot x_i)^2 = \frac{1}{N} [w \cdot x_1 \quad w \cdot x_2 \cdots w \cdot x_N] \begin{bmatrix} w \cdot x_1 \\ w \cdot x_2 \\ \vdots \\ w \cdot x_N \end{bmatrix}$$

$$= \frac{1}{N}(Xw)^T (Xw) = \frac{1}{N} w^T \underbrace{(X^T X)} w$$

$$= w^T V w \quad c/ \; V = \frac{1}{N} X^T X$$

No objectivo: maximizar $w^T V w$

suj. $\|w\|^2 = 1$

$$\mathcal{L}(w, \lambda) = f(w) - \lambda(g(w) - c)$$

$f(w) = w^T V w$

$g(w) = \|w\|^2 = w \cdot w$

$c = 1$

$w = (w_1, \dots, w_m) \qquad V = [v_{ij}]_{m \times m} = V^T$

$$\boxed{\frac{\partial (w^T V w)}{\partial w} = 2 w^T V}$$

$y = \psi(x) \qquad x = (x_1, \dots, x_n)$

$y = (y_1, \dots, y_m)$

$$\frac{\partial y}{\partial x} = J(\psi(x)) =$$

$$= \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

$$= \left[\frac{\partial y_i}{\partial x_j}\right]$$

$$Vw = [v_{ij}] \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{m} v_{1k} w_k \\ \vdots \\ \sum_{k=1}^{m} v_{mk} w_k \end{bmatrix}$$

$$w^T V w = [w_1 \cdots w_m] \begin{bmatrix} \Sigma \\ \vdots \\ \Sigma \end{bmatrix}$$

$$= w_1 \sum_{k=1}^{m} v_{1k} w_k + w_2 \sum_{k=1}^{m} v_{2k} w_k + \dots + w_m \sum_{k=1}^{m} v_{mk} w_k$$

$$= \sum_{i=1}^{m} w_i \sum_{k=1}^{m} v_{ik} w_k = \sum_{i=1}^{m} \sum_{k=1}^{m} v_{ik} w_k w_i$$

$$\frac{\partial\left(\qquad\right)}{\partial w_j} = \frac{\partial\left(\sum_{i} \sum_{k} v_{ik} w_i w_k\right)}{\partial w_j}$$

$$= \frac{\partial\left(v_{jj} w_j^2\right)}{\partial w_j} + \overset{i=j}{\frac{\partial\left(\sum_{k,\, k \neq j} v_{jk} w_j w_k\right)}{\partial w_j}} + \overset{k=j}{\frac{\partial\left(\sum_{i,\, i \neq j} v_{ji} w_j w_i\right)}{\partial w_j}}$$

$$= 2 v_{jj} w_j + \sum_{k,\, k \neq j} v_{jk} w_k + \sum_{k,\, k \neq j} v_{jk} w_k$$

$$= 2\left(v_{jj} w_j + \sum_{k,\, k \neq j} v_{jk} w_k\right) = 2 \sum_{k=1}^{m} v_{jk} w_k$$

$$= 2 \begin{bmatrix} w_1 & \cdots & w_m \end{bmatrix} \begin{bmatrix} v_{j1} \\ v_{j2} \\ \vdots \\ v_{jm} \end{bmatrix}$$

$$\frac{\partial \varphi(w)}{\partial w} = \begin{bmatrix} \frac{\partial\left(\quad\right)}{\partial w_1} & \cdots & \frac{\partial\left(\quad\right)}{\partial w_m} \end{bmatrix} = 2 w^T V$$

Logo $\quad \dfrac{\partial (w^T V w)}{\partial w} = 2 w^T V$

$$\frac{\partial (w^T w)}{\partial w} = 2 w^T$$

$$\frac{\partial u}{\partial w} = 2 w^T V - \lambda 2 w^T = 0$$

$$\Leftrightarrow w^T V = \lambda w^T$$

$$\Leftrightarrow \boxed{V w = \lambda w}$$

λ é valor próprio assoc. V, que queremos max.

w é um vector. próp. λ c/ $\|w\| = 1$

$$\underbrace{w^T V w}_{\max} = w^T(\lambda w) = \lambda w^T w = \lambda \|w\|^2 = \underbrace{\lambda}_{\max}$$

$V_{m \times m}$ SDP, $\sigma(V) \subseteq \mathbb{R}_0^+$

vect. próp. assoc. val. próp. $\perp$ 2 a 2

Componentes principais

$V_{m\times m}$ $\quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_m \geq 0$

$k < m \quad \dfrac{\lambda_1+\lambda_2+\dots+\lambda_k}{\lambda_1+\lambda_2+\dots+\lambda_m} \leq 1$

Recordar

1) MM^T é SDP i.e., simétrica e $\sigma(MM^T) \subseteq \mathbb{R}_0^+$

2) $tr(A) = \sum\limits_{\lambda_i \in \sigma(A)} \lambda_i \to$ possivelmente repetidos

soma dos elementos diagon.

3) $M_{n\times n}$ val. prop $\lambda_1, \dots, \lambda_n$ então

$\alpha M_{n\times n}$ val prop $\alpha\lambda_1, \dots, \alpha\lambda_n$

$\alpha \neq 0$ $\quad v_1, \dots, v_n$ vect. prop. de M assoc.

val. prop. $\lambda_1, \lambda_2, \dots, \lambda_n$ então, em relação αM,

$v_1, v_2, \dots, v_n$ são vect. pr. assoc.

val. prop. $\alpha\lambda_1, \alpha\lambda_2, \dots, \alpha\lambda_n$ (resp.)

4) $\sigma(MM^T)\setminus\{0\} = \sigma(M^TM)\setminus\{0\}$

Se v é vect. próp. de MM^T assoc. a $\lambda \neq 0$

então Mv é " " " M^TM " " $\lambda \neq 0$

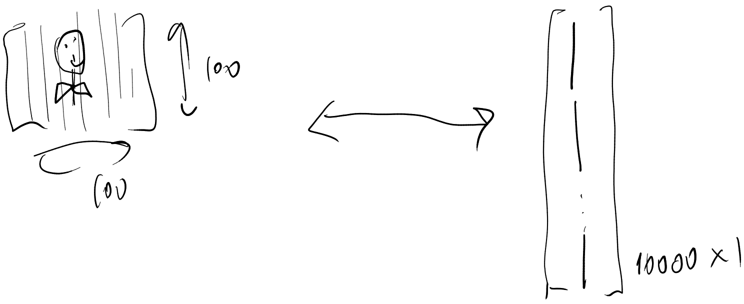
Se w é vect. próp. de M^TM assoc. $\lambda \neq 0$

então M^Tw é " " " MM^T " $\lambda \neq 0$

---

400 fotografias, cada foto 100×100

$n = 40$



$B = \begin{bmatrix} | & | & | & | & \cdots & | \end{bmatrix}$ 10000×400