Projecção ortogonal elementar

$u \in \mathbb{C}^n \quad , \quad \|u\| = 1$

$Q = I - uu^*$ proj. ortogonal elementar

$v^\perp = \{ w \in \mathbb{C}^n : w \perp v \}$ Complemento ortogonal de v

$v = (1,2,3) \in \mathbb{R}^3 \qquad v^\perp = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2, x_3) \cdot (v) = 0 \}$
$= \ker(M)$

$0 = (x_1, x_2, x_3)^T (1,2,3) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x_1 + 2x_2 + 3x_3$

$\underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{M} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad A_{m \times n}$

$n = \text{car } A + \dim \ker(A)$

$\dim \ker(M) = \dim v^\perp = 2$

$\mathbb{R}^n : \quad Q = I - uu^T \quad , \quad \|u\| = 1$

$x = (I - Q)x + Qx \qquad ((I-Q)x) \perp (Qx) \; , \; Q^2 = Q$

$$(I-Q)x = (I-(I-uu^T))x = uu^Tx = u\underbrace{(u^Tx)}_{\in\mathbb{R}} \in \mathcal{L}(u)$$

$$Qx \overset{?}{\in} u^\perp$$

$$(Qx)\cdot u = \left((I-uu^T)x\right)\cdot u$$

$$= \left((I-uu^T)x\right)^T u$$

$$= x^T(I-uu^T)u$$

$$= x^Tu - x^Tu\underbrace{\cancel{(u^Tu)}}_{=\|u\|^2=1}$$

$$= 0$$

$$(I-Q)^2 = (I-Q)(I-Q)$$

$$= I - Q - Q + Q^2 = I - Q$$

$$Q^2 = (I-uu^T)^2 = (I-uu^T)(I-uu^T)$$

$$= I - uu^T - uu^T + u\underbrace{(u^Tu)}_{=\|u\|^2=1}u^T$$

$$= I - uu^T = Q$$

$$\mathbb{R}^n = \mathcal{L}(u) \oplus^\perp u^\perp$$

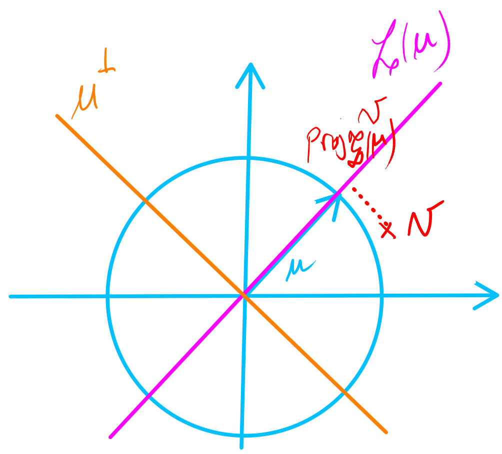
$$I - Q = uu^T \quad ; \quad (I-Q)x = \underbrace{uu^Tx}_{\in\mathbb{R}} = (u^Tx)u \in \mathcal{L}(u)$$

$$\Downarrow$$

$$= I - uu^T$$

$$\mathbb{R}^2 \quad ; \quad u = \left(\frac{\sqrt{2}}{2} ; \frac{\sqrt{2}}{2}\right)$$

$$v = \left(1, \frac{1}{2}\right)$$



$$\text{proj}_{\mathcal{L}(u)} v = \text{proj}_u v = (I-Q)v = uu^Tv =$$

$$= \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} = (u^T v)\, u$$

$$= \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = \underbrace{\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4}\right)}_{\text{Comprimento da proj. ortogonal em } \mathcal{L}(u)} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$u \in \mathbb{C}^n \setminus \{0\}$ $\qquad Q = I - \dfrac{u}{\|u\|}\dfrac{u^*}{\|u\|} = I - \dfrac{uu^*}{\|u\|^2} = I - \dfrac{uu^*}{u^*u}$

$$\text{proj}_{\mathcal{L}(u)} v = \text{proj}_u v = \frac{uu^*}{u^*u}\, v$$

Refletores elementares; Transformações de Householder

$u \neq 0$ $\qquad R = I - 2\dfrac{uu^*}{u^*u}$

$\|u\| = 1$ $\qquad R = I - 2uu^*$

$\mathbb{R}^n$

Sup. $\|u\| = 1$ $\qquad Q = I - uu^T$ proj. sobre $u^\perp$

$Qx = \text{proj}_{u^\perp} x$. $\qquad$ Onde está $Rx = (I - 2uu^T)\, x$?

$$Q(Rx) = (I - uu^T)(I - 2uu^T)x = \underbrace{(I - uu^T)}_{=Q} x = Qx$$

As proj ort. de x e de Rx sobre $u^\perp$ são iguais.

$$\|x - Qx\| = \|(I-Q)x\| = \| u \underbrace{u^T x}_{\in \mathbb{R}} \| \underset{\|u\|=1}{=} |u^T x|$$

$$\|Qx - Rx\| = \|(Q-R)x\| = \|((I - uu^T) - (I - 2uu^T))x\|$$
$$= \|u \underbrace{u^T x}_{\in \mathbb{R}}\| \underset{\|u\|=1}{=} |u^T x|$$

Propriedades

1) R é unitária, hermítica e involutória

i.e. $R = R^* = R^{-1}$; $R^2 = I$

2) $x \in \mathbb{C}^n$; $x = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \end{bmatrix}$, $x_1 \neq 0$; $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}$

$u = x \pm \mu \|x\| e_1$ $\qquad \mu = \begin{cases} 1 & \text{se } x_1 \in \mathbb{R} \\ \dfrac{x_1}{|x_1|} & \text{se } x_1 \notin \mathbb{R} \end{cases}$

$$R = I - 2\frac{uu^*}{u^*u}$$

Então $Rx = \mp \mu \|x\| e_1$

i.e., R "reflecte" na 1ª coordenada

$$Rx = \left(I - 2\frac{uu^*}{u^*u}\right)x = x - \frac{2u\overbrace{u^*x}^{\in \mathbb{C}}}{u^*u} = x - \left(\frac{2u^*x}{u^*u}\right)u$$

$$2u^*x = u^*u$$

$$2u^*x = 2(x^* \pm \bar{\mu}\|x\|e_1^*)x = 2x^*x \pm 2\bar{\mu}\|x\|\underbrace{e_1^*x}_{=x_1}$$

$$= 2x^*x \pm 2\frac{\bar{x}_1}{|x_1|}x_1\|x\|$$

$$u^*u = (x \pm \mu\|x\|e_1)^*(x \pm \mu\|x\|e_1) =$$

$$= \underbrace{x^*x}_{=\|x\|^2} \pm x^*\mu\|x\|e_1 \pm \bar{\mu}\|x\|e_1^*x + \underbrace{\bar{\mu}\mu}_{=1}\|x\|^2\underbrace{e_1^*e_1}_{=1}$$

$$= 2x^*x \pm 2\frac{\bar{x}_1}{|x_1|}x_1\|x\|$$

$$Rx = \left(I - 2\frac{uu^*}{u^*u}\right)x = x - \frac{2u\overbrace{u^*x}^{\in \mathbb{C}}}{u^*u} = x - \frac{2u^*x}{u^*u}u \quad = u^*u$$

$$= x - \cancel{\frac{u^*u}{u^*u}}^{1}u = x - \underset{= x \pm \mu\|x\|e_1}{u} = \mp \mu\|x\|e_1$$

Exemplo: Seja $\|x\|=1$, $x \in \mathbb{C}^n$, $x_1 \neq 0$

Vamos construir uma base ortonormada de $\mathbb{C}^n$ que contenha x.

Seja $u = x \pm \mu \|x\| e_1 = x \pm \mu e_1$, $\mu = \begin{cases} 1 & \text{se } x_1 \in \mathbb{R} \\ \frac{x_1}{|x_1|} & \text{senão} \end{cases}$

$$R = I - 2\frac{uu^*}{u^*u} \quad ; \quad Rx = \mp \mu e_1$$

$$Rx = \mp \mu e_1 \Rightarrow \underbrace{R^2}_{=I} x = \mp \mu R e_1 \Rightarrow x = \mp \mu R e_1$$

$$\Rightarrow x = \left[\mp \mu R\right]_{:,1}$$

$$U = \mp \mu R \quad ; \text{ tais que } x = [U]_{:,1}$$

$$U^*U = (\mp \mu R)^*(\mp \mu R) = \mp(\mp R^* \underbrace{\bar{\mu}\mu}_{=1} R) = R^*R = I$$

Logo U é unitária

$$U = \begin{bmatrix} | & | & | & & | \\ x & u_2 & u_3 & \cdots & u_n \\ | & | & | & & | \end{bmatrix} \qquad I = U^*U = \begin{bmatrix} - & x^* & - \\ - & u_2^* & - \\ & \vdots & \\ - & u_n^* & - \end{bmatrix} \begin{bmatrix} | & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} x^*x & x^*u_2 & \cdots & x^*u_n \\ u_2^*u_1 & u_2^*u_2 & \cdots & u_2^*u_n \\ \vdots & & & \end{bmatrix}$$

$$u_1 := x \qquad u_i^* u_j = 0 \ , \ i \neq j \ ,$$

$$\|u_i\| = 1$$

Portanto, as colunas de U formam uma base ortonormada de $\mathbb{C}^n$

Exemplo: $x = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \end{bmatrix}$; $\mu = 1$

$$u = x - e_1 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -4 \\ 2 \\ 0 \\ -2 \end{bmatrix}$$

$$R = I - \frac{uu^T}{u^T u} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 0 & -2 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ -2 & 1 & 0 & 2 \end{bmatrix}$$

$$U = +R = \begin{bmatrix} -\frac{1}{3} & | & | & | \\ 2/3 & & & \\ 0 & & & \\ -2/3 & | & | & | \end{bmatrix}$$

As colunas de U formam uma base ortonormada de $\mathbb{R}^4$, e x é um dos elementos da base

Rotações

Rotação em $\mathbb{R}^3$

OX $\qquad P_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$

OY $\qquad P_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

OZ $\qquad P_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

No caso geral

$$P_{ij} = \begin{array}{c} \left[\begin{array}{ccccc} 1 & & & & \\ & c & \cdots & s & \\ & \vdots & 1 & \vdots & \\ & -s & \cdots & c & \\ & & & & 1 \end{array}\right] \begin{array}{l} \\ \to i \\ \\ \to j \\ \\ \end{array} \\ \begin{array}{ccc} \downarrow & \quad & \downarrow \\ i & & j \end{array} \end{array} \qquad \text{t.q. } c^2 + s^2 = 1$$

$P_x = P_{23}$; $P_y = P_{13}$; $P_z = P_{12}$

São as matrizes de rotação no plano, ou

$\boxed{\text{ROTAÇÕES DE GIVENS}}$

Para $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \neq 0$, $x \in \mathbb{R}^n$

$$P_{ij}\,x = \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & c & & s \\ & & & 1 & \\ & & -s & & c \\ & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ c x_i + s x_j \\ x_{i+1} \\ \vdots \\ -s x_i + c x_j \\ \vdots \\ x_n \end{bmatrix} \begin{array}{c} \\ \\ \to i \\ \\ \\ \to j \\ \\ \\ \end{array}$$

$$\begin{array}{c} \quad\quad\quad\quad\quad \downarrow \quad\quad \downarrow \\ \quad\quad\quad\quad\quad i \quad\quad j \end{array}$$

x_i, x_j n.s.n. (não simultaneamente nulos)

$$c = \frac{x_i}{\sqrt{x_i^2 + x_j^2}} \quad ; \quad s = \frac{x_j}{\sqrt{x_i^2 + x_j^2}} \quad ; \quad c^2 + s^2 = 1$$

$$P_{ij}\,x = P_{ij} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ \sqrt{x_i^2 + x_j^2} \\ \vdots \\ 0 \\ \vdots \\ x_n \end{bmatrix} \begin{array}{c} \\ \\ \to i \\ \\ \to j \\ \\ \\ \end{array}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad x_1 \neq 0 \; ; \qquad P_{12}\,x = \begin{bmatrix} \sqrt{x_1^2 + x_2^2} \\ 0 \\ * \\ * \\ \vdots \\ * \end{bmatrix} , \qquad P_{13}(P_{12}\,x) = \begin{bmatrix} \sqrt{x_1^2 + x_2^2 + x_3^2} \\ 0 \\ 0 \\ * \\ * \\ \vdots \\ * \end{bmatrix}$$

$$P_{1n} \cdots P_{14} P_{13} P_{12} x = \begin{bmatrix} \|x\| \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$P = \prod_{k=n}^{2} P_{1,k}$$

e' ortogonal

$$P_{ij} P_{ij}^T = I \Rightarrow P_{ij}^{-1} = P^T$$

$\Rightarrow P_{ij}$ ortogonai