Lema 2.7. *Sejam c e n_0 constantes positivas, e $f(n)$ uma função tal que $f(2n) \leq c.f(n), \forall n \geq n_0$. Então $f(2^k n) \leq c^k.f(n), \forall n \geq n_0$ e $k \geq 1$.*

Teorema 2.8. *Seja $f(n)$ uma função suave. Então para qualquer $b \geq 2$ fixado,*

$$f(b.n) = \Theta(f(n))$$

O teorema a seguir é conhecido como *regra da suavização*

Teorema 2.9. *Seja $T(n)$ uma função eventualmente não-decrescente, e $f(n)$ uma função suave. Se $T(n) = \Theta(f(n))$ para valores de n que são potências de b $(b \geq 2)$, então*

$$T(n) = \Theta(f(n)), \forall n.$$

$b^k < n \leq b^{k+1}$

Prova do lema 2.7: Indução em K.

$(K=1)$: Trivial

$(K>1)$: $f(2^k.n) = f(2.2^{k-1}.n) \overset{h.p.}{\leq} c.f(2^{k-1}.n) \overset{h.i.}{\leq} c.c^{k-1}.f(n) = c^k.f(n), \forall n \geq n_0$ $\boxtimes$

Prova do Teorema 2.8: Mostraremos $f(b.n) = O(f(n))$, onde $f(n)$ é suave. Queremos encontrar constantes positivas c e n_0 tais que $f(b.n) \leq c.f(n), \forall n \geq n_0$. Como $f(n)$ é suave, temos $f(2.n) = \Theta(f(n))$. Em particular, $f(2.n) = O(f(n))$, i.e. existem constantes positivas c_1 e n_1 tais que $f(2.n) \leq c_1.f(n), \forall n \geq n_1$. Sabemos que $2^k \leq \underbrace{b < 2^{k+1}}$ $(k \geq 1)$

$\Downarrow$

$\underbrace{b.n < 2^{k+1}.n}$ $(n>0)$

$\Downarrow$ f é eventualmente não decrescente: $\exists n_0' > 0$

$f(b.n) \leq f(2^{k+1}.n)$, $\forall n \geq n_0'$

Logo, $f(b.n) \leq f(2^{k+1}.n) \overset{(2.7)}{\leq} c_1^{k+1}.f(n)$, $\forall n \geq \max(n_0', n_1)$.

Para $c = c_1^{k+1}$ e $n_0 = \max(n_0', n_1)$, temos a desigualdade desejada, i.e. $f(b.n) = O(f(n))$.

Exercício: Mostrar que $f(b.n) = \Omega(f(n))$.

Prova do Teorema 2.9 (Regra da suavização)

$T(n)$ é eventualmente não-decrescente

$f(n)$ é suave.

$T(b^k) = \Theta(f(b^k)) \overset{(k>0)}{\Longrightarrow} \exists c_1 > 0$ e $n_1 > 0$: $T(b^k) \leq c_1.f(b^k), \forall n \geq n_1$ (*)

Queremos mostrar que $T(n) = \Theta(f(n))$. Mostraremos que $T(n) = O(f(n))$, i.e. encontraremos constantes positivas c e n_0 tais que $T(n) \leq c \cdot f(n), \forall n \geq n_0$.

Sabemos que $\exists r > 0 : \underbrace{b^r \leq n < b^{r+1}}$.

$\Downarrow$ T é event. $\bar{n}$-dec. $\exists n_2 > 0$:

$$T(n) \leq T(b^{r+1}), \forall n \geq n_2.$$

$$\overset{(*)}{\leq} c_1 \cdot f(b^{r+1}), \forall n \geq \max(n_1, n_2)$$

$$= c_1 \cdot f(b \cdot b^r), \forall n \geq \max(n_1, n_2)$$

$\overset{2.8}{\equiv}$ $\exists n_3 > 0$:

$$\leq c_1 \cdot c_b \cdot f(b^r), \forall n \geq n_0$$

onde $n_0 = \max(n_3, \max(n_1, n_2))$

$$\leq c_1 . c_b \cdot f(n), \forall n \geq n_0 \ldots \boxed{\text{QED}}$$

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Exercício: $T(n) = \Omega(f(n)), \forall n$

THEOREM 3 Let $f(n)$ be a smooth function as just defined. Then, for any fixed integer $b \geq 2$,

$$f(bn) \in \Theta(f(n)),$$

i.e., there exist positive constants c_b and d_b and a nonnegative integer n_0 such that

$$d_b f(n) \leq f(bn) \leq c_b f(n) \quad \text{for } n \geq n_0.$$

(The same assertion, with obvious changes, holds for the O and Ω notations.)

PROOF We will prove the theorem for the O notation only; the proof of the Ω part is the same. First, it is easy to check by induction that if $f(2n) \leq c_2 f(n)$ for $n \geq n_0$, then

$$f(2^k n) \leq c_2^k f(n) \quad \text{for } k = 1, 2, \ldots \text{ and } n \geq n_0.$$

The induction basis for $k = 1$ checks out trivially. For the general case, assuming that $f(2^{k-1}n) \leq c_2^{k-1} f(n)$ for $n \geq n_0$, we obtain

$$f(2^k n) = f(2 \cdot 2^{k-1}n) \leq c_2 f(2^{k-1}n) \leq c_2 c_2^{k-1} f(n) = c_2^k f(n).$$

Consider now an arbitrary integer $b \geq 2$. Let k be a positive integer such that $2^{k-1} \leq b < 2^k$. We can estimate $f(bn)$ above by assuming without loss of generality that $f(n)$ is nondecreasing for $n \geq n_0$:

$$f(bn) \leq f(2^k n) \leq c_2^k f(n).$$

Hence, we can use c_2^k as a required constant for this value of b to complete the proof. ■

The importance of the notions introduced above stems from the following theorem.

THEOREM 4 ***(Smoothness Rule)*** Let $T(n)$ be an eventually nondecreasing function and $f(n)$ be a smooth function. If

$$T(n) \in \Theta(f(n)) \quad \text{for values of } n \text{ that are powers of } b,$$

where $b \geq 2$, then

$$T(n) \in \Theta(f(n)).$$

(The analogous results hold for the cases of O and Ω as well.)

PROOF We will prove just the O part; the Ω part can be proved by the analogous argument. By the theorem's assumption, there exist a positive constant c and a positive integer $n_0 = b^{k_0}$ such that

$$T(b^k) \leq cf(b^k) \quad \text{for } b^k \geq n_0,$$

$T(n)$ is nondecreasing for $n \geq n_0$, and $f(bn) \leq c_b f(n)$ for $n \geq n_0$ by Theorem 3. Consider an arbitrary value of n, $n \geq n_0$. It is bracketed by two consecutive powers of b: $n_0 \leq b^k \leq n < b^{k+1}$. Therefore,

$$T(n) \leq T(b^{k+1}) \leq cf(b^{k+1}) = cf(bb^k) \leq cc_b f(b^k) \leq cc_b f(n).$$

Hence, we can use the product cc_b as a constant required by the $O(f(n))$ definition to complete the O part of the theorem's proof. ■