UNKNOWN

THE FRAMEWORK: DIFFERENTIAL PROBLEMS

- CONSIDER AN EQUATION (PDE/ODE), DEPENDING ON SOME PARAMETERS $\underline{\theta} \in \Theta$, DEFINED ON A DOMAIN D.
- LET:

$$u: \Theta \times D \longrightarrow \mathbb{R}^n$$
$$(\underline{\theta}, \underline{x}) \longmapsto u(\underline{\theta}, \underline{x})$$

BE THE SOLUTION OF SUCH EQUATION.

- SUPPOSE WE OBSERVE $\{(\underline{x}_i, y_i)\}_{i=1}^{N}$ WITH
    - $\underline{x}_i \in D$
    - $y_i = u(\underline{\theta}, \underline{x}_i) + \varepsilon_i, \quad \varepsilon_i \overset{iid}{\sim} N(0, \tau^2)$

$$\Longrightarrow y_i | \underline{\theta} \overset{ind}{\sim} N(u(\underline{\theta}, \underline{x}_i), \tau^2) \quad (\tau^2 > 0)$$

- MODEL A PRIORI KNOWLEDGE ABOUT $\underline{\theta}$ THROUGH A PRIOR DISTRIBUTION $\pi(\underline{\theta})$, AND ESTIMATE $\pi(\underline{\theta}, \underline{y})$

SLIDE 01

THE METHOD: MULTILEVEL MCMC

- COMPUTING LIKELIHOOD $L(\underline{\theta}^*, \underline{x})$ REQUIRES SOLUTION $u(\underline{\theta}^*, \underline{x})$: COMPUTATIONALLY INTENSIVE
- ML MCMC SOLVES THE EQUATION AT DIFFERENT LEVELS OF ACCURACY.

$$\boxed{\text{COARSE LEVEL}} \xrightarrow{\text{PROPOSES SAMPLES TO}} \boxed{\text{FINE LEVEL}}$$

COARSE LEVEL:
- FAST
- LESS ACCURATE

FINE LEVEL:
- SLOW
- MORE ACCURATE

- ACHIEVE MORE EFFICIENT SAMPLINGS
    - HIGHER ESS (AND ESS/sec)
    - LESS CORRELATED SAMPLES

SLIDE 02

CASE STUDIES (OUTLINE)

A. PDE: COMET EQUATION

	coarse	fine
A1	FEM (LOOSE GRID)	FEM (DENSE GRID)
A2	SURROGATE (NN)	FEM (DENSE GRID)

B. ODE: SEIR EPIDEMIOLOGICAL MODELS

	coarse	fine
B1	scipy (LARGE TIME STEP)	scipy (SMALL TIME STEP)
	MERGE E, I (SIR) + scipy	scipy (SEIR)

SLIDE 03 | a

CASE STUDIES (OUTLINE)

A. PDE: COMET EQUATION

A1: COARSE AND FINE LEVEL DIFFER IN THE GRID REFINEMENT

A2: COARSE LEVEL FEATURES A SURROGATE MODEL (A TRAINED NN)

B. ODE: SEIR EPIDEMIOLOGICAL MODEL

B1: COARSE AND FINE DIFFER IN THE SOLVER TIME-STEP

B2: COARSE LEVEL MERGES 2 COMPARTMENTS (E, I) $\Longrightarrow$ SIR MODEL

SLIDE 03 | b

A. COMET EQUATION

IT'S A LINEAR ADVECTION - DIFFUSION PDE, FEATURING 2 PARAMETER WITH A CLEAR PHYSICAL INTERPRETATIONS

COMET EQUATION

$$-\mu \Delta u + 10(\cos(\theta), \sin(\theta)) \cdot \nabla u = 10 e^{-50\|\underline{x} - \underline{x}_0\|}$$

- $\mu \in$
- $\theta \in (0, 2\pi)$
- $x_0 \in (1/2, 1/2)$

DATA IS PRODUCED THROUGH SIMULATION WITH FIXED VALUES FOR THE PARAMETERS, AND FOR THE ERROR SCALE AND KNOWN

- $\mu^* = 2$
- $\theta^* = 2\pi$
- $\Sigma^* = \ldots$

SLIDE 04

B. SEIR EPIDEMIOLOGICAL MODEL

SEIR MODEL

$$\dot{S} = -\beta S I$$
$$\dot{E} = \beta S I - \sigma E$$
$$\dot{I} = \sigma E - \gamma I$$
$$\dot{R} = \gamma I$$
$$S + E + I + R = 1$$
$$(S, E, I, R)(0) = (S_0, E_0, I_0, R_0)$$

- $\beta \in (0, +\infty)$ INFECTION RATE
- $\sigma \in (0, +\infty)$ INCUBATION RATE
- $\gamma \in (0, +\infty)$ RECOVERY RATE
- $\Sigma \in (0, +\infty)$ NOISE SCALE

DATA IS SIMULATED WITH FIXED AND KNOWN VALUES FOR THE PARAMETERS

SLIDE 05

B1: SEIR: DIFFERENT TIME-STEPS

- LET $\underline{\theta} = (\beta, \sigma, \gamma)$ MODEL PARAMETERS
  - $X_s^{(\ell)}(\underline{\theta}, t)$ THE STATE X_s OF THE SCIPY SOLUTION FOR THE ODE AT LEVEL ℓ AT TIME t
  - X_s AVAILABLE DATA ABOUT STATE X_s AT TIME t_i
- WE HAVE: $X_{i,s} \mid \underline{\theta}, \Sigma \sim N(X_s^{(\ell)}(\underline{\theta}, t_i), \Sigma)$ $\quad s = 2,3,4 \quad i = 1, \ldots n$
- COARSE AND FINE MODELS DIFFER IN THE TIME STEP FOR THE SOLUTION: $\Delta t_0 \gg \Delta t_1$

SLIDE 06

UNDER REASONABLE CONDITIONS ON THE INITIAL STATES

B2. SEIR: MERGING 2 COMPARTMENTS

LET $\underline{X}(\underline{\theta},t) = (E(\underline{\theta},t), I(\underline{\theta},t), R(\underline{\theta},t))$ BE SOLUTION OF SEIR MODEL FOR $\underline{\theta}$. WE CAN APPROXIMATE $(E+I, R)(t)$ THROUGH A SIR MODEL.

SIR MODEL

$\simeq \quad \simeq$

ENFORCING:

- SAME ASYMPTOTIC VALUE OF $R(t)$
- SAME PEAK TIME FOR $E(t)/I(t)$

$\Longrightarrow \tilde{\underline{\theta}}(\underline{\theta}) = (\tilde{\beta}(\underline{\theta}), \tilde{\gamma}(\underline{\theta})) \rightarrow \tilde{\beta}(\underline{\theta}) = \dots$

$\tilde{\gamma}(\underline{\theta}) = \dots$

$\Longrightarrow E(t)+I(t) \simeq \tilde{I}(t), \ \tilde{R}(t) \simeq R(t)$

SLIDE 07

◆ AT COARSE LEVEL:

$E_i + I_i \mid \underline{\theta} \overset{ind}{\sim} N(\tilde{I}(\tilde{\underline{\theta}}(\underline{\theta}), t_i), 2\Sigma_0)$ $\quad i=1,\dots n$

$R_i \mid \underline{\theta} \overset{ind}{\sim} N(\tilde{R}(\tilde{\underline{\theta}}(\underline{\theta}), t_i), \Sigma_0)$

($\forall i \ E_i+R_i \perp\!\!\!\perp R_i$, CONDITIONALLY ON $\underline{\theta}$)

◆ AT FINE LEVEL:

$X_{i,s} \mid \underline{\theta} \sim N(X_s^{(0)}(\underline{\theta}, t_i), \Sigma_1)$ $\quad s=2,3,4$ $\quad i=1,\dots n$

$(\Sigma^* < \Sigma_1 < \Sigma_0)$

SLIDE 08