

THE FRAMEWORK: DIFFERENTIAL PROBLEMS

- CONSIDER AN EQUATION (PDE/ODE), DEPENDING ON SOME PARAMETERS $\theta \in \Theta$, DEFINED ON A DOMAIN D . LET

LET: $u: \Theta \times D \rightarrow \mathbb{R}^n$
 $(\theta, x) \mapsto u(\theta, x)$

BE THE SOLUTION OF SUCH EQUATION.

- WE CAN ALSO MODEL THE EQUATION AS A STOCHASTIC PROCESS

- SUPPOSE WE OBSERVE $\{(x_i, y_i)\}_{i=1}^N$ WITH

- $x_i \in D$
- $y_i = u(\theta, x_i) + \epsilon_i, \epsilon_i \stackrel{\text{iid}}{\sim} N(0, \Sigma)$
- $\Rightarrow y_i | \theta \stackrel{\text{ind}}{\sim} N(u(\theta, x_i), \Sigma) \quad (\Sigma > 0)$

- MODEL A PRIORI KNOWLEDGE ABOUT θ THROUGH A PRIOR DISTRIBUTION $\pi(\theta)$, AND ESTIMATE $\pi(\theta, y)$

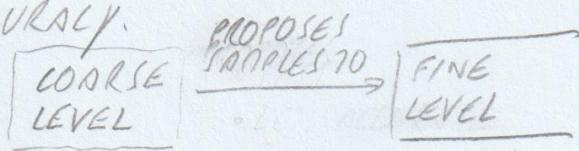
UNKNOWN

SLIDE 01

THE METHOD: MULTILEVEL MCMC

- COMPUTING LIKELIHOOD $L(\theta^*, x)$ REQUIRES SOLUTION $u(\theta^*, x)$: COMPUTATIONALLY INTENSIVE

- MCMC SOLVES THE EQUATION AT DIFFERENT LEVELS OF ACCURACY.



SLIDE 02

- ACHIEVE MORE EFFICIENT SAMPLINGS

- HIGHER ESS (AND ESS/SEC)
- LESS CORRELATED SAMPLES

CASE STUDIES (OUTLINE)

A. PDE: ~~SEIR~~ COMET EQUATION

A1

| | COARSE | FINE |
|----|------------------|------------------|
| A1 | FEN (LOOSE GRID) | FEN (DENSE GRID) |
| A2 | SURROGATE (NN) | FEN (DENSE GRID) |

SLIDE 03

2

B. ODE: ~~SEIR~~ EPIDEMIOLOGICAL MODELS

| | COARSE | FINE |
|----|---|---|
| B1 | SCIPY (LARGE TIME STEP) MERGE E, I (SIR) + SCIPY | SCIPY (SMALL TIME STEP) SCIPY (SEIR) |

CASE STUDIES (OUTLINE)

A. PDE: COMET EQUATION

A1: COARSE AND FINE LEVEL DIFFER IN THE GRID REFINEMENT

A2: COARSE LEVEL FEATURES A SURROGATE MODEL (A TRAINED NN)

B. ODE: SEIR EPIDEMIOLOGICAL MODEL

B1: COARSE AND FINE DIFFER IN THE SOLVER TIME-STEP ~~SCIPY~~

B2: COARSE LEVEL MERGES 2 COMPARTMENTS (E, I) \Rightarrow SIR MODEL

SLIDE 03

6

A. COMET EQUATION

IT'S A LINEAR ADVECTION-DIFFUSION PDE,
FEATURING 2 PARAMETERS WITH A CLEAR
PHYSICAL INTERPRETATIONS

COMET EQUATION

$$-\mu \partial u + 10(\cos(\theta), \sin(\theta)) \cdot \nabla u = 10e^{-50\|x-x_0\|}$$

SLIDE 04

- $\mu \in$
- $\theta \in (0, 2\pi)$
- $x_0 \in (\frac{1}{2}, \frac{1}{2})$

DATA IS PRODUCED THROUGH SIMULATION WITH ~~A~~
FIXED VALUES FOR THE PARAMETERS, AND FOR THE ERROR RATE

AND KNOWN $\mu^* = 2$ $\theta^* = 2\pi$ $x_0^* = \dots$

B. SEIR EPIDEMIOLOGICAL MODEL

SEIR MODEL

$$\dot{S} = -\beta SI$$

$$\dot{E} = \beta SI - \sigma E$$

$$\dot{I} = \sigma E - \gamma I$$

$$\dot{R} = \gamma I \quad \langle(S, E, I, R)(0) = (S_0, E_0, I_0, R_0) \rangle \\ S+E+I+R = 1$$

SLIDE 05

- $\beta \in (0, \infty)$ INFECTION RATE
- $\sigma \in (0, \infty)$ INCUBATION RATE
- $\gamma \in (0, \infty)$ RECOVERY RATE
- $\epsilon \in (0, \infty)$ NOISE SCALE

DATA IS SIMULATED WITH FIXED AND KNOWN
VALUES FOR THE PARAMETERS

B1: SEIR: DIFFERENT TIME-STEP

LET $\underline{\theta} = (\theta, \sigma, \gamma)$ MODEL PARAMETERS

- $X_j^{(0)}(\underline{\theta}, t)$ THE STATE X_j OF THE SCIPY SOLUTION
FOR THE ODE AT LEVEL l AT TIME t
- X_j AVAILABLE DATA ABOUT STATE X_j AT TIME t_i

WE HAVE: $X_{i,j}(\underline{\theta}, \Sigma) \sim N(X_j^{(0)}(\underline{\theta}, t_i), \Sigma) \quad j=2, 3, 4 \\ i=1, \dots, n$

SLIDE 06

COARSE AND FINE MODELS DIFFER ONE IN THE TIME
STEP ~~FOR~~ FOR THE SOLUTION: $\Delta t_0 \gg \Delta t_1$

UNDER REASONABLE CONDITIONS
ON THE INITIAL STATES

B2). SEIR: MERGING 2 COMPARTMENTS

LET $\underline{X}(\underline{\theta}, t) = (E(\underline{\theta}, t), I(\underline{\theta}, t), R(\underline{\theta}, t))$ BE SOLUTION OF
SEIR MODEL FOR $\underline{\theta}$. WE CAN APPROXIMATE $(E+I, R)(t)$
THROUGH A SIR MODEL.

SIR MODEL

$$\approx \approx$$

(SLIDE 07)

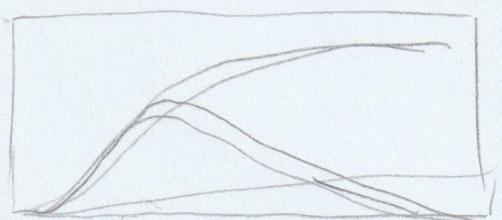
ENFORCING:

- SAME ASYMPTOTIC VALUE OR $R(t)$

- SAME PEAK TIME FOR $E(t)/I(t)$

$$\Rightarrow \tilde{\underline{\theta}}(\underline{\theta}) = (\tilde{I}(\underline{\theta}), \tilde{S}(\underline{\theta})) \Rightarrow \tilde{I}(\underline{\theta}) = \dots$$

$$\Rightarrow E(t) + I(t) \approx \tilde{I}(t), \tilde{R}(t) \approx R(t) \quad \tilde{S}(t) = \dots$$



(SLIDE 08)

AT COARSE LEVEL:

$$E_i + I_i | \underline{\theta} \stackrel{\text{ind}}{\sim} N(\tilde{I}(\tilde{\underline{\theta}}(\underline{\theta}), t_i), 2\sigma_i^2) \quad i=1, \dots, n$$

$$R_i | \underline{\theta} \stackrel{\text{ind}}{\sim} N(\tilde{R}(\tilde{\underline{\theta}}(\underline{\theta}), t_i), \sigma_i^2)$$

($\forall i$) $E_i + R_i \perp\!\!\!\perp R_i$, CONDITIONALLY ON $\underline{\theta}$)

AT FINE LEVEL:

$$X_{i,j} | \underline{\theta} \sim N(X_j^{(i)}(\underline{\theta}, t_i), \tau_i) \quad j=2, 3, 4 \quad i=1, \dots, n$$

$$(T^* < T \leq T_0)$$