

CS 281: Advanced Machine Learning

taught by Sasha Rush

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Lecture 1: Discrete Models

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- Discrete models take values from a countable set, e.g. $\{0,1\}$, $\{\text{cold, flu, asthma}\}$ and are simpler than continuous models.
- We will use simple discrete models to develop our tactics such as marginalization and conditioning.
- Today, we will focus coins as a real-world example.

1.1 Bernoulli model

The likelihood is of the form $p(\text{heads}) = \theta$.

Easy Prior

Assume we know the coin came from one of 3 unknown manufacturers (later, we'll have mixture model estimation, but for now assume these probabilities come from an oracle).

1. $\theta = 0.4$ with probability .1
2. $\theta = 0.5$ with probability .8
3. $\theta = 0.6$ with probability .1

$$p(\theta) = 0.1 \cdot \delta(\theta = 0.4) + 0.8 \cdot \delta(\theta = 0.5) + 0.1 \cdot \delta(\theta = 0.6)$$

Likelihood

Likelihood = $p(\text{data}|\text{parameters})$. For the coin example,

$$p(\text{coin flips}|\theta) = \text{Bin}(N_1|N, \theta) = \binom{N}{N_1} \theta^{N_1} (1 - \theta)^{N - N_1} \quad \text{where } N = N_0 + N_1 = \text{number of flips}$$

Note that the last two terms, the "score", is our focus since they are the only terms that depend on θ . The first term normalizes the distribution.

1.2 Inference

Inference 1: $p(\theta|x)$ ($x \in N_0, N_1$). How can we estimate θ ?

Maximum Likelihood Estimation (MLE)

$$\theta_{MLE} = \text{argmax}_{\theta} p(N_0, N_1|\theta) = \text{argmax}_{\theta} \log [p(N_0, N_1|\theta)]$$

$$\theta_{MLE} = \text{argmax}_{\theta} \log \binom{N}{N_1} + N_1 \log \theta + N_0 \log (1 - \theta) \quad \text{Because the first term is not a function of } \theta, \text{ we can ignore it.}$$

$$\frac{d}{d\theta} = \frac{N_1}{\theta} + \frac{N_0}{1 - \theta} \cdot (-1) \rightarrow \theta_{MLE} = \frac{N_1}{N_0 + N_1}$$

Note that Inference \neq Decision Making. If we asked you to make a bet on the coin, based on this you could either

1. Always take heads if $\theta > .5$. In this case, $p(\text{win}) = \theta$
2. Take heads with probability $= \theta$. In this case, $p(\text{win}) = \theta^2 + (1 - \theta)^2 [p(\text{is heads}) * p(\text{choose heads}) + \dots]$

If $\theta = 0.6$, for option 1, $p(\text{win}) = \theta = 0.6$. For option 2, $p(\text{win}) = \theta^2 + (1 - \theta)^2 = 0.52$. In this case, the additional information used in the calculation of option 2 does not result in a better decision.

Maximizing the Posterior (MAP)

Bayes Rule : $p(\theta|\text{data}) \propto p(\text{data}|\theta)p(\theta)$

- Posterior: $p(\theta|x)$
- Likelihood: $p(x|\theta)$
- Prior: $p(\theta)$

$$\theta_{MAP} = \text{argmax}_{\theta} p(\theta|x) = \text{argmax}_{\theta} \log [p(x|\theta)p(\theta)] \quad \text{from Bayes' Rule: } p(\theta|x) \propto p(x|\theta)p(\theta)$$

Consider an example:

$$p(\theta = 0.4|N_0, N_1) \propto \binom{N}{N_1} (.4)^{N_1} (1 - .4)^{N_0} (0.1)$$

$$p(\theta = 0.45|N_0, N_1) = 0 \quad \text{Due to the sparsity of the prior - similar result for } \theta = 0.5 \text{ and } 0.6$$

$\theta_{MAP} = \theta_{MLE}$ when we have a uniform prior since the MLE calculation does not explicitly factor a prior into its calculation.

Full Posterior

Partition or Marginal Likelihood: $p(N_0, N_1) = \int_{\theta} p(N_0, N_1, \theta)$.

$$p(\theta|N_0, N_1) = \frac{p(x|\theta)p(\theta)}{p(N_0, N_1)} \quad \text{Note that } p(N_0, N_1) \text{ is a very difficult term to compute.}$$

Beta Prior

$$p(\theta|\alpha_0, \alpha_1) = \frac{\Gamma(\alpha_0 + \alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)} \theta^{\alpha_1-1} (1 - \theta)^{\alpha_0-1} \quad \text{support} \in [0, 1]$$

From the image of the beta function for different parameters, we can see that it can either be balanced, skewed to one side, or tend toward infinity on one side.

With a beta prior:

$$p(\theta|N_0, N_1) = \frac{\Gamma(\alpha_0 + \alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)} \theta^{\alpha_1-1} (1 - \theta)^{\alpha_0-1} \theta^{N_1} (1 - \theta)^{N_0} \cdot (\text{constant normalization term w.r.t } \theta)$$

The key insight is that we get additive terms in the exponent and the resulting distribution looks like another beta. The prior "counts" (pseudocounts) from the hyperparameters can be interpreted as counts we have beforehand.

$$p(\theta|N_0, N_1) = \frac{\Gamma(\alpha_0 + \alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)} \theta^{N_1 + \alpha_1 - 1} (1 - \theta)^{N_0 + \alpha_0 - 1} \cdot (\text{constant normalization term w.r.t } \theta)$$

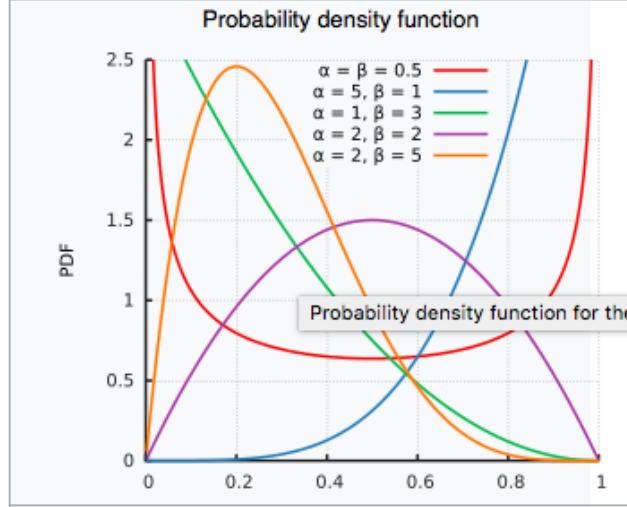


Figure 1.1: Beta Params

To make this distribution sum to 1, use the known beta normalizer

$$p(\theta|N_0, N_1) = \frac{\Gamma(\alpha_0 + \alpha_1 + N_0 + N_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)\Gamma(N_0)\Gamma(N_1)} \theta^{N_1+\alpha_1-1} (1-\theta)^{N_0+\alpha_0-1} \sim \text{Beta}(\theta|N_0 + \alpha_0, N_1 + \alpha_1) \quad (\text{posterior})$$

The mode of the Beta gives us back θ_{MAP} , but with additional information about the shape of the distribution. What does the prior that tends to infinity at 1 imply? That in the absence of other information, the coin is definitely heads.

Predictive Distribution

$$\begin{aligned} p(\hat{x}|N_0, N_1) &= \int_{\theta} p(x|\theta, N_0, N_1) p(\theta|N_0, N_1) d\theta \\ &= \int_{\theta} \theta p(\theta|N_0, N_1) d\theta \\ &= \mathbb{E}_{\theta \sim p(\theta|N_0, N_1)} \theta \end{aligned}$$

This is the expectation under the posterior of θ which is the mean of the Beta distribution. Feel free to prove this as an exercise.

Marginal Likelihood

$$\begin{aligned} p(N_0, N_1) &= \int_{\theta} p(x_1, \dots, x_n|\theta) p(\theta) d\theta \\ &= \int_{\theta} \frac{\Gamma(\alpha_1 + \alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)} \theta^{\alpha_1+N_1-1} (1-\theta)^{\alpha_0+N_0-1} \end{aligned}$$

The first term can be moved outside, as it does not depend on θ . After introducing our normalization term and making the distribution inside sum to 1,

$$p(N_0, N_1) = \frac{\Gamma(\alpha_1 + \alpha_1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)} \frac{\Gamma(N_0 + \alpha_0)\Gamma(N_1 + \alpha_1)}{\Gamma(N_0 + N_1 + \alpha_0 + \alpha_1)}$$

1.3 Extensions on the Coin Flip Model: Super Coins

- Many correlated coins: models of binary data, important for discrete graphical models
- Many-sided coins aka dice: models of categorical data, generalization of Bernoulli

1.4 Other Distributions

$$\text{Bernoulli}(x|\theta) = \theta^x(1 - \theta)^{1-x}$$

$$\text{Categorical}(x|\theta) = \prod_k \theta_k^{x_k} \quad \text{generalization of Bernoulli}$$

$$\text{Multinomial}(x|\theta) = \frac{(\sum_k x_k)!}{\prod_k x_k!} \prod_k \theta_k^{x_k} \quad \text{generalization of Binomial}$$

$$\text{Dirichlet}(x|\alpha) = \frac{\Gamma(\sum_k \alpha_k)}{\prod_k \Gamma(\alpha_k)} \prod_k \theta_k^{\alpha_k - 1} \quad \text{generalization of Beta, often used as a prior}$$

Note that the Dirichlet distribution is the conjugate prior of the Categorical and Multinomial distributions.

1.5 Example notebook

See [Beta.ipynb](#)

Lecture 3: Multivariate Normal Distributions

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3.6 Examples

Multivariate gaussians are used for modeling in various applications, where knowing mean and variance is useful:

- radar: mean and variance of approaching objects (like invading aliens)
- weather forecasting: predicting the position of a hurricane, where the uncertainty in the storm's position increases for timepoints farther away
- tracking the likely outcome of a sports game: last year's superbowl is an example of a failure of modeling with multivariate gaussians as the Patriots still won after a large Falcons' lead

3.7 Review: Eigendecomposition

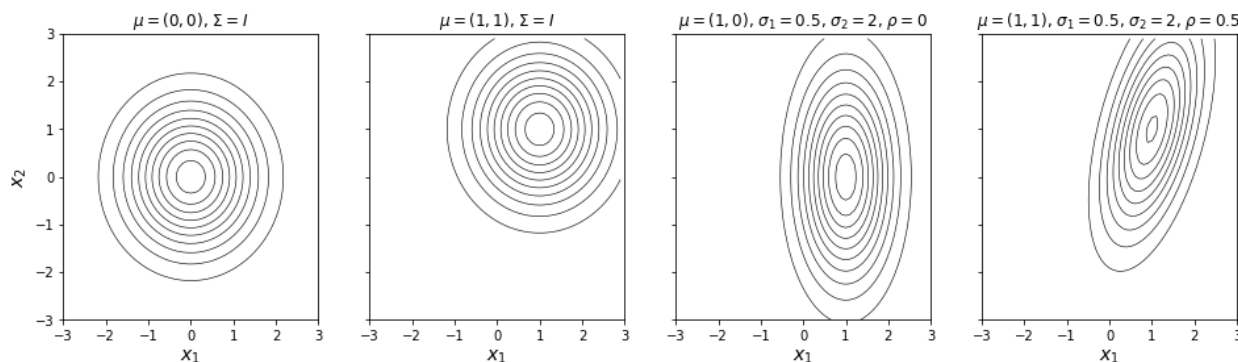
Let Σ be a square, symmetric matrix. Then its eigendecomposition is given by $\Sigma = \mathbf{U}^T \Lambda \mathbf{U}$, where \mathbf{U} is an orthogonal matrix and Λ is a diagonal matrix. In the special case that Σ is positive semidefinite (as is the case for covariance matrices), denoted $\Sigma \succeq 0$, all its eigenvalues are nonnegative, $\Lambda_{ii} \geq 0$, and we can decompose its inverse as $\Sigma^{-1} = \mathbf{U}^T \Lambda^{-1} \mathbf{U}$, where $\Lambda_{ii}^{-1} = 1/\Lambda_{ii}$.

3.8 Multivariate Normal Distributions (MVNs)

Let X be a D -dimensional MVN random vector with mean μ and covariance matrix Σ , denoted $X \sim \mathcal{N}(\mu, \Sigma)$. Then the pdf of X is

$$p(x) = (2\pi)^{-D/2} |\Sigma|^{-1/2} \exp\left[-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right],$$

where for many problems we focus on the quadratic form $(x - \mu)^T \Sigma^{-1}(x - \mu)$ (which geometrically can be thought of as distance) and ignore the normalization factor $(2\pi)^{-D/2} |\Sigma|^{-1/2}$. The figure below plots the contours of a bivariate Normal for various μ and Σ (in the figure, ρ denotes the off-diagonal elements of Σ , given by the covariance of x_1 and x_2).



Note that we can decompose Σ as

$$\begin{aligned}
\Sigma &= (x - \mu)^T \Sigma^{-1} (x - \mu) \\
&= (x - \mu)^T \left(\mathbf{U}^T \Lambda^{-1} \mathbf{U} \right) (x - \mu) \\
&= (x - \mu)^T \left(\sum_d \frac{1}{\lambda_d} U_d U_d^T \right) (x - \mu) \\
&= \sum_d \frac{1}{\lambda_d} (x - \mu)^T U_d U_d^T (x - \mu),
\end{aligned}$$

where $(x - \mu)^T U_d$ can be interpreted as the projection of $(x - \mu)$ onto U_d (which can each be thought of as univariate gaussians), the eigenvector corresponding to the eigenvalue λ_d . Since Σ is the weighted sum of the dot product of such projections (with weights being given by $1/\lambda_d$, which can be thought of as the scale $1/\sigma^2$), we can describe the MVN as tiling of univariates.

3.8.1 Manipulating MVNs: Stretches, Rotations, and Shifts

Let $x \sim \mathcal{N}(0, I)$ and $y = Ax + b$. We want to consider two ways of obtaining the complete distribution of y .

- 'Overkill': We can perform a change of variables¹. Here, we have $x = A^{-1}y$ and $|dx/dy| = |A^{-1}|$, leading to

$$\begin{aligned}
p(y) &= \mathcal{N}(A^{-1}(y - b) | 0, I) |A^{-1}| \\
&= \frac{1}{z} \exp[(A^{-1}(y - b))^T (A^{-1}(y - b))] \\
&= \frac{1}{z} \exp[(y - b)^T (A^{-1})^T (A^{-1})(y - b)] \\
&= \mathcal{N}(y | b, AA^T),
\end{aligned}$$

where z is the normalizing constant.

- Using the properties of MVN, we know that y is also MVN, so is completely specified by its mean and covariance matrix which can easily be derived,

$$\mathbb{E}(y) = \mathbb{E}(Ax + b) = A\mathbb{E}(x) + b \quad \text{cov}(y) = AA^T.$$

Thus, we can generate MVN from $\mathcal{N}(0, I)$ via the transformation $y = Ax + b$, where we set $A = \mathbf{U}\Lambda^{1/2}$, leading to $\Sigma_Y = \mathbf{U}^T \Lambda \mathbf{U}$. Then shifts are represented by b , stretches by Λ , and rotations by \mathbf{U} .

3.8.2 Detour: MVN in High-Dimensions ($D \gg 0$)

Let x be a D -dimensional random vector, distributed as $\mathcal{N}(0, I/D)$, where I is the identity. The expected length of x is given by

$$\mathbb{E}(\|x\|^2) = \mathbb{E}\left(\sum_d x_d^2\right) = D\sigma_d^2 = 1,$$

which means that x is expected to be on the boundary of a unit sphere centered at the origin. Moreover, the variance of the length is

$$\text{var}(\|x\|^2) = D \cdot \left(\mathbb{E}(x^4) - \mathbb{E}(x^2)^2 \right) = D \cdot (3\sigma^4 - \sigma^4) = 2D/D^2 = 2/D$$

¹A change of variables can be done in the following way: Let $y = f(x)$ and assume f is invertible so that $x = f^{-1}(y)$. Then $p(y) = p(x)|dx/dy|$. This is a technique which will be used often in this course

Thus, it is not only expected that x lies on the boundary but as D increases most of its realizations will in fact fall on the boundary².

3.8.3 Key Formulas for MVN: Marginalization and Conditioning

Let $X \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

Note that $\boldsymbol{\Sigma}$ is written in block matrix form, rather than scalar entries. It turns out that the marginals, X_1 and X_2 , are also MVN, and their mean and covariance matrices are given by $\boldsymbol{\mu}_1$ and $\boldsymbol{\Sigma}_{11}$ and $\boldsymbol{\mu}_2$ and $\boldsymbol{\Sigma}_{22}$ respectively. A sketch of the proof is provided below.

$$p(x_1) = \int_{x_2} N(x|\boldsymbol{\mu}, \boldsymbol{\Sigma}) dx_2,$$

which can be written as

$$0.5 \int_{x_2} \exp \left[(x_1 - \boldsymbol{\mu}_1)^T \boldsymbol{\Sigma}_{11}^{-1} (x_1 - \boldsymbol{\mu}_1) + 2(x_1 - \boldsymbol{\mu}_1)^T \boldsymbol{\Sigma}_{12}^{-1} (x_2 - \boldsymbol{\mu}_2) + (x_2 - \boldsymbol{\mu}_2)^T \boldsymbol{\Sigma}_{22}^{-1} (x_2 - \boldsymbol{\mu}_2) \right] dx_2.$$

Note that this equals

$$p(x_1) \int_{x_2} p(x_2|x_1) dx_2,$$

implying that $X_1 \sim \mathcal{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$.

While the marginals have a simple form, the conditionals are more complicated. (For a complete derivation, which requires matrix inversion lemmas, refer to Murphy.) It can be shown that $X_1|X_2 \sim \boldsymbol{\mu}_{\infty|\in}, \boldsymbol{\Sigma}_{\infty|\in}$ with

$$\boldsymbol{\mu}_{1|2} = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} (x_2 - \boldsymbol{\mu}_2), \quad \boldsymbol{\Sigma}_{1|2} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{21}.$$

3.8.4 Information Form

An alternative formulation, called information form, uses the precision matrix (inverse variance) $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}^{-1}$. Partitioning $\boldsymbol{\Lambda}$ as

$$\boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{\Lambda}_{11} & \boldsymbol{\Lambda}_{12} \\ \boldsymbol{\Lambda}_{21} & \boldsymbol{\Lambda}_{22} \end{pmatrix},$$

the covariance matrices of the conditional distributions have a simple form. For example, the covariance matrix of X_1 given X_2 is $\boldsymbol{\Lambda}_{1|2} = \boldsymbol{\Lambda}_{11}$. However, the simplicity of the conditional precision comes at the cost of marginalization (which was simple when using $\boldsymbol{\Sigma}$) becoming a more complicated expression (see Murphy subsection 4.3 for more details).

²It is left as an exercise to show that this formula holds. Hint: Use the fact that we assumed no covariance.