TPC#1
Data de entrega: 21 de Março de 2022
Autores: André Couto#1 (100%); Diogo Reis#2 (100%); Diogo Santiago#3 (100%)

1. **a)**

A_k uma matriz real, de ordem n, invertível logo $A_k{}^{-1}A_k = I = A_kA_k{}^{-1}$
$\sigma = 1 + v^TA_k{}^{-1}u \neq 0$
$[A_k + uv^T]$ é invertível se $det[A_k + uv^T] \neq 0$.
Ora $det[A_k + uv^T] = det[A_k + A_k(A_k^{-1}uv^T)] = det[A_k(1 + A_k^{-1}uv^T)]$
$= detA_kdet[1 + A_k^{-1}uv^T] = detA_k\sigma \neq 0$ pois A_k invertível e $\sigma \neq 0$ logo é invertível.

Agora verifique-se que:

$$(A_k{}^{-1} - \frac{1}{\sigma}A_k{}^{-1}uv^TA_k{}^{-1})(A_k + uv^T) = (A_k + uv^T)(A_k{}^{-1} - \frac{1}{\sigma}A_k{}^{-1}uv^TA_k{}^{-1}) = I$$

$$(A_k{}^{-1} - \frac{1}{\sigma}A_k{}^{-1}uv^TA_k{}^{-1})(A_k + uv^T) =$$

$$= A_k{}^{-1}A_k + A_k{}^{-1}uv^T - \frac{A_k{}^{-1}uv^TA_k{}^{-1}A_k}{1 + v^TA_k{}^{-1}u} - \frac{A_k{}^{-1}uv^TA_k{}^{-1}uv^T}{1 + v^TA_k{}^{-1}u}$$

$$= I + A_k{}^{-1}uv^T - \frac{A_k{}^{-1}uv^T + A_k{}^{-1}uv^TA_k{}^{-1}uv^T}{1 + v^TA_k{}^{-1}u}$$

$$= I + A_k{}^{-1}uv^T - \frac{A_k{}^{-1}u(v^T + v^TA_k{}^{-1}uv^T)}{1 + v^TA_k{}^{-1}u}$$

$$= I + A_k{}^{-1}uv^T - \frac{A_k{}^{-1}u(1 + v^TA_k{}^{-1}u)v^T}{1 + v^TA_k{}^{-1}u}$$

$$= I + A_k{}^{-1}uv^T - A_k{}^{-1}uv^T = I$$

$$(A_k + uv^T)(A_k{}^{-1} - \frac{1}{\sigma}A_k{}^{-1}uv^TA_k{}^{-1}) =$$

$$= A_kA_k{}^{-1} - \frac{A_kA_k{}^{-1}uv^TA_k{}^{-1}}{1 + v^TA_k{}^{-1}u} + uv^TA_k{}^{-1} - \frac{uv^TA_k{}^{-1}uv^TA_k{}^{-1}}{1 + v^TA_k{}^{-1}u}$$

$$= I - \frac{uv^TA_k{}^{-1} + uv^TA_k{}^{-1}uv^TA_k{}^{-1}}{1 + v^TA_k{}^{-1}u} + uv^TA_k{}^{-1}$$

$$= I - \frac{u(v^TA_k{}^{-1} + v^TA_k{}^{-1}uv^TA_k{}^{-1})}{1 + v^TA_k{}^{-1}u} + uv^TA_k{}^{-1}$$

$$= I - \frac{u(1 + v^TA_k{}^{-1}u)v^TA_k{}^{-1}}{1 + v^TA_k{}^{-1}u} + uv^TA_k{}^{-1}$$

$$= I - uv^TA_k{}^{-1} + uv^TA_k{}^{-1)} = I$$

Assim, prova-se o pretendido.

b)

Sabendo que

$$(A_k + uv^T)^{-1} = A_k{}^{-1} - \frac{1}{\sigma} A_k{}^{-1} u v^T A_k{}^{-1}$$

$$A_{k+1} = A_k + \frac{1}{s_k^T s_k}(y_k - A_k s_k) s_k^T$$

$$s_k = x_{k+1} - x_k$$

$$y_k = F(x_{k+1}) - F(x_k)$$

$$u = \frac{1}{s_k^T s_k}(y_k - A_k s_k)$$

$$B_k = A_k^{-1}$$

Queremos provar que

$$B_{k+1} = B_k + \frac{1}{s_k^T B_k y_k}(s_k - B_k y_k) s_k^T B_k,$$

ou seja, que

$$A_{k+1}^{-1} = A_k^{-1} + \frac{1}{s_k^T A_k^{-1} y_k}(s_k - A_k^{-1} y_k) s_k^T A_k^{-1}$$

Ora,

$$A_{k+1}^{-1} = (A_k + uv^T)^{-1} = A_k^{-1} - \frac{1}{\sigma} A_k{}^{-1} u v^T A_k{}^{-1}$$

$$= A_k^{-1} - \frac{1}{1 + s_k^T A_k^{-1} \frac{1}{s_k^T s_k}(y_k - A_k s_k)} A_k^{-1} \frac{1}{s_k^T s_k}(y_k - A_k s_k) s_k^T A_k^{-1} = por[1]$$

$$= A_k^{-1} - \frac{s_k^T s_k}{s_k^T A_k^{-1} y_k} A_k^{-1} \frac{1}{s_k^T s_k}(y_k - A_k s_k) s_k^T A_k^{-1}$$

$$= A_k^{-1} - \frac{1}{s_k^T A_k^{-1} y_k}[A_k^{-1} y_k s_k^T A_k^{-1} - s_k s_k^T A_k^{-1}]$$

$$= A_k^{-1} - \frac{1}{s_k^T A_k^{-1} y_k}(A_k^{-1} y_k - s_k) s_k^T A_k^{-1}$$

$$= A_k^{-1} + \frac{1}{s_k^T A_k^{-1} y_k}(s_k - A_k^{-1} y_k) s_k^T A_k^{-1}$$

$$= B_k + \frac{1}{s_k^T B_k y_k}(s_k - B_k y_k) s_k^T B_k$$

$$[1] : s_k^T A_k^{-1} \frac{1}{s_k^T s_k}(y_k - A_k s_k) = s_k^T A_k^{-1} \frac{1}{s_k^T s_k} y_k - \frac{1}{s_k^T s_k} s_k^T A_k^{-1} A_k s_k = s_k^T A_k^{-1} \frac{1}{s_k^T s_k} y_k - 1$$

.

c)

$F(x) = \begin{bmatrix} x_1^2 + x_2^2 - 1 \\ 5x_1^2 -_2^2 -2 \end{bmatrix}$

$x_0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$A_0 = I_2$

$B_k = A_k^{-1}$

$s_k = -B_k F(x_k)$ logo $s_0 = -B_0 F(x_0)$

$B_0 = A_0^{-1} = I_2$

$F(x_0) = \begin{bmatrix} 2^2 + 2^2 - 1 \\ 5 * 2^2 - 2^2 - 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$ logo $s_0 = - \begin{bmatrix} 7 \\ 14 \end{bmatrix}$

$x_{k+1} = x_k + s_k$ logo $x_1 = x_0 + s_0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ 14 \end{bmatrix} = \begin{bmatrix} 161 \\ -35 \end{bmatrix}$

$$B_{k+1} = B_k + \frac{1}{s_k^T B_k y_k}(s_k - B_k y_k) s_k^T B_k \quad logo$$

$$B_1 = B_0 + \frac{1}{s_0^T B_0 y_0}(s_0 - B_0 y_0) s_0^T B_0 = I + \frac{(\begin{bmatrix} -7 \\ -14 \end{bmatrix} - I \begin{bmatrix} 161 \\ -35 \end{bmatrix})[-7 - 14] I}{[-7 - 14] I \begin{bmatrix} 161 \\ -35 \end{bmatrix}}$$

$$= I + \frac{\begin{bmatrix} -168 \\ 21 \end{bmatrix}[-7 - 14]}{-637} = I - \frac{1}{637}\begin{bmatrix} 1176 & 2352 \\ -147 & -294 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -\frac{24}{13} & -\frac{48}{13} \\ \frac{3}{13} & \frac{6}{13} \end{bmatrix} = \begin{bmatrix} -\frac{11}{13} & -\frac{48}{13} \\ \frac{3}{13} & \frac{19}{13} \end{bmatrix}$$

$$s_1 = -B_1 F(x_1) = \begin{bmatrix} \frac{11}{13} & \frac{48}{13} \\ -\frac{3}{13} & -\frac{19}{13} \end{bmatrix} \begin{bmatrix} 168 \\ -21 \end{bmatrix} = \begin{bmatrix} \frac{840}{13} \\ -\frac{105}{13} \end{bmatrix}$$

$$x_2 = x_1 + s_1 = - \begin{bmatrix} 5 \\ 12 \end{bmatrix} + \begin{bmatrix} \frac{840}{13} \\ -\frac{105}{13} \end{bmatrix} = \begin{bmatrix} \frac{775}{13} \\ -\frac{261}{13} \end{bmatrix}$$

$$y_1 = F(x_2) - F(x_1) = \begin{bmatrix} (\frac{775}{13})^2 + (-\frac{261}{13})^2 - 1 \\ 5(\frac{775}{13})^2 - (-\frac{261}{13})^2 - 2 \end{bmatrix} - \begin{bmatrix} 168 \\ -21 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{51428}{13} \\ 17364,88757 \end{bmatrix} - \begin{bmatrix} 168 \\ -21 \end{bmatrix} = \begin{bmatrix} \frac{49245}{13} \\ 17385,88757 \end{bmatrix}$$

$$B_2 = B_1 + \frac{1}{s_1^T B_1 y_1}(s_1 - B_1 y_1) s_1^T B_1$$

$$= \begin{bmatrix} -\frac{11}{13} & -\frac{48}{13} \\ \frac{3}{13} & \frac{19}{13} \end{bmatrix} + \frac{(\begin{bmatrix} \frac{840}{13} \\ -\frac{105}{13} \end{bmatrix} - \begin{bmatrix} -\frac{11}{13} & -\frac{48}{13} \\ \frac{3}{13} & \frac{19}{13} \end{bmatrix} \begin{bmatrix} \frac{49245}{13} \\ 17385,88757 \end{bmatrix})[\frac{840}{13} - \frac{105}{13}] \begin{bmatrix} -\frac{11}{13} & -\frac{48}{13} \\ \frac{3}{13} & \frac{19}{13} \end{bmatrix}}{[\frac{840}{13} - \frac{105}{13}] \begin{bmatrix} -\frac{11}{13} & -\frac{48}{13} \\ \frac{3}{13} & \frac{19}{13} \end{bmatrix} \begin{bmatrix} \frac{49245}{13} \\ 17385,88757 \end{bmatrix}}$$

$$= \begin{bmatrix} -\frac{11}{13} & -\frac{48}{13} \\ \frac{3}{13} & \frac{19}{13} \end{bmatrix} + \frac{\begin{bmatrix} 67463,95765 \\ -26292,39184 \end{bmatrix} [-\frac{735}{13} \quad -\frac{3255}{13}]}{-4567330,814}$$

$$= \begin{bmatrix} -\frac{11}{13} & -\frac{48}{13} \\ \frac{3}{13} & \frac{19}{13} \end{bmatrix} + \frac{\begin{bmatrix} -38143308,375 & -16891957,09 \\ 1486531,388 & 6583210,481 \end{bmatrix}}{-456730,814}$$

$$= \begin{bmatrix} -0,011025 & 0,000612 \\ -0.094701 & 0,020169 \end{bmatrix}$$

.

2. **a)** $X_{k+1} = X_k + X_k(I - AX_k)$

$R_k = I - AX_k$

$E_k = A^{-1} - X_k$

Queremos mostrar que $R_{k+1} = R_k^2$ e $E_{k+1} = E_k A_k E_k$

Ora

$$R_k^2 = (I - AX_k)^2 = I^2 - 2IAX_k + A^2X_k^2 = I - 2AX_k + A^2X_k^2$$

e

$$R_{k+1} = I - AX_{k+1} = I - A(X_k + X_k(I - AX_k)) = I - A(X_k + X_k - X_kAX_k)$$
$$= I - AX_k - AX_k + AX_kAX_k = I - 2AX_k + A^2X_k^2 = R_k^2$$

Ora

$$E_{k+1} = A^-1 - X_{k+1} = A^{-1} - X_k - X_k(I - AX_k) = A - X_k - X_k + X_kAX_k = A^{-1} - 2X_k + X_kAX_k$$

E

$$E_K AE_k = (A^{-1} - X_K)A(A^{-1} - X_k) = (A^{-1}A - X_kA)(A^{-1} - X_k) = (I - X_kA)(A^{-1} - X_k)$$
$$= A^{-1} - X_k - X_kAA^{-1} + X_kAX_k = A^{-1} - 2X_k + X_kAX_k = E_{k+1}$$

Uma sucessão $\{X_n\}$ converge quadraticamente para X_* se existe $M > 0$ tal que

$$\|X_{k+1} - X_*\| \leq M \|X_k - X_*\|, \forall k > k_0$$

Ora $X_* = A^{-1}$ logo

$$\left\|X_{k+1} - A^{-1}\right\| = \left\|A^{-1} - X_{k+1}\right\| = \|E_{k+1}\| = \|E_kAE_k\|$$
$$\leq \|E_k\| \|A\| \|E_k\| = \|A\| \|E_k\|^2 = \|A\| \left\|X_k - A^{-1}\right\|$$

Prova-se então a convergência para $M = \|A\|$.

b) Com o objetivo de calcular a inversa de uma matriz A usando o método da alínea anterior construímos o seguinte programa:

```
clc
clear
close all;
A=[9 3 18; 4 4 4 ;2 6 2]; % Definir uma matriz
X=A'/(norm(A,1)*norm(A,inf)) %Definir o X0 como pretendido

[m , n]= size(A); % Definir o tamanho da matriz para poder usar na matriz identidade

while (A*X ~= eye(n))

    X=X+X*(eye(n)-A*X) % Método iterativo pretendido
end
```

Assim para a matriz que lá aparece encontramos

```
X =

   -0.1111    0.7083   -0.4167
   -0.0000   -0.1250    0.2500
    0.1111   -0.3333    0.1667
```

Para a matriz $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ encontramos:

```
X =

   -2.0000    1.0000
    1.5000   -0.5000
```

E por fim para a matriz $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ encontramos

```
X =

    1.0000   -0.5000   -0.5000    0.0000
   -0.0000    0.5000   -0.5000   -0.0000
    0.0000   -0.0000    1.0000   -1.0000
   -0.0000    0.0000    0.0000    1.0000
```

A utilização deste programa é simples e eficaz o que o torna mais vantajoso em comparação com outros métodos.