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André Abreu Couto

A Decision Support Tool for Multi-Criteria Analysis

Dissertation in the context of the Master in Data Science and Engineering , advised by Professor Samuel Moniz and Professor Luís Paquete presented to the Department of Informatics Engineering of the Faculty of Sciences and Technology of the University of Coimbra.

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Uma Ferramenta de Apoio à Decisão para Análise Multicritério

Dissertação no âmbito do Mestrado em Engenharia e Ciência de Dados, , orientada pelo Professor Samuel Moniz e pelo Professor Luís Paquete apresentada ao Departamento de Engenharia Informática da Faculdade de Ciências e Tecnologia da Universidade de Coimbra.

Julho 2024

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Abstract

Decision making is a process that involves evaluating multiple alternatives across various criteria. To facilitate this process, Multi-Criteria Decision Making techniques serve as powerful tools for decision-makers. However, these techniques can raise some doubt among users unfamiliar with them, and existing software applications for their implementation may be overly technical and overwhelming. Additionally, decision-makers must assign preferences among criteria, and these preferences need to be consistent to ensure that the output of the techniques robustly reflects their decisions. Existing methods for automatically adjusting preferences often result in preferences that deviate significantly from what the user initially envisioned.

In this thesis, we propose an approach that addresses both challenges. First, we create a visualization tool that employs storytelling, allowing users to navigate the decision-making process. This tool integrates *Analytic Hierarchy Process* for determining criterion weights and *Technique for Order Preference by Similarity to Ideal Solution* for ranking alternatives. Second, we introduce a novel automatic adjustment approach that involves solving a minimum feedback arc set problem to handle intransitivities in preferences. Subsequently, using a perturbation map, we make small adjustments until consistent preferences are achieved. Our approach maintains preferences closer to the user's initial intentions than existing methods found in the literature, highlighting the potential of this methodology.

Keywords

Multi-criteria · Consistency · Preferences · Visualization · Decision-making · Storytelling

Resumo

A tomada de decisão é um processo que envolve a avaliação de várias alternativas com base em vários critérios. Para facilitar este processo, as técnicas de Tomada de Decisão Multicritério servem como ferramentas poderosas para os decisores. No entanto, estas técnicas podem levantar algumas dúvidas entre os utilizadores que não estão familiarizados com elas, e as aplicações de software existentes para a sua implementação podem ser demasiado técnicas e confusas. Além disso, os decisores devem atribuir preferências entre os critérios, e essas preferências precisam de ser consistentes para garantir que os resultados reflitam de forma robusta as suas decisões. Os métodos existentes para ajustar automaticamente as preferências muitas vezes resultam em preferências que se desviam significativamente do que o utilizador inicialmente idealizou.

Nesta tese, propomos uma abordagem que aborda ambos os desafios. Primeiro, criamos uma ferramenta de visualização que utiliza storytelling, permitindo aos utilizadores navegar pelo processo de tomada de decisão. Esta ferramenta integra o *Analytic Hierarchy Process* para determinar os pesos dos critérios e a técnica *Technique for Order Preference by Similarity to Ideal Solution* para classificar as alternativas. Em segundo lugar, introduzimos uma nova abordagem de ajuste automático de preferências que envolve a resolução do problema do Minimum Feedback Arc Set para lidar com as intransitividades. Seguidamente, utilizando um mapa de perturbações, fazemos pequenos ajustes até que sejam alcançadas preferências consistentes. A nossa abordagem mantém as preferências mais próximas das intenções iniciais do utilizador do que os métodos existentes na literatura, destacando o potencial desta metodologia.

Palavras-Chave

Multicritério · Consistência · Preferências · Visualização · Tomada de decisão · Storytelling

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Chapter 1

Introduction

1.1 Motivation

Multi-criteria decision problems involve selecting the best alternative based on a set of criteria. These criteria are parameters or measures used to evaluate the alternatives, determined by the decision-maker and dependent on the specific context of the decision problem. The alternatives are the various options available for consideration, each examined in relation to every criterion. The decision-maker must often define preferences for each pair of criteria, specifying which is more important and by how much. These preferences can be organized into a preference matrix. The alternative that best meets the criteria, according to his preferences, is considered the optimal solution. These problems are structured to allow a detailed and systematic comparison of the available options, aiding decision-makers in various fields, from business to public policy. In real life, multi-criteria decision problems are crucial as they enable complex decisions to be made in a rational and well-founded manner. In situations where multiple factors need to be considered, such as choosing a location for a new factory or selecting an investment portfolio, multi-criteria techniques provide a clear framework for evaluating and comparing alternatives, taking into account all relevant variables.

With the exponential increase in the volume of data being created, captured, and replicated worldwide (see Figure 1.1), decision-makers are often faced with the challenge of information overload. This overload occurs when the volume of input data exceeds the processing capacity of the decision-makers, leading to a decline in the quality of decisions [Schulman and Gross, 1966]. Hal Varian, Chief Economist of Google, highlighted this issue by stating, "The ability to take data - to be able to understand it, to process it, to extract value from it, to visualize it, to communicate it - that's going to be a hugely important skill in the next decades." Indeed, data visualization has emerged as a necessity for processing data and making more informed decisions. Consequently, the ability to process, extract value from, visualize, and communicate data plays a crucial role in today's world.

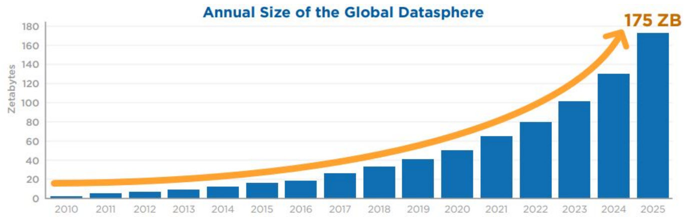


Figure 1.1: Annual Size of the Global Datasphere [Reinsel et al., 2018].

This is particularly relevant to the problem we are addressing, as the decision-making process requires handling large amounts of data. This necessitates the use of visualizations to represent complex information, providing decision-makers with an easy and quick way to understand the results of the models. Computer-based visualization systems offer visual representations of datasets, helping people perform tasks more effectively [Munzner, 2014]. Thus, visualization can mitigate limitations, whether they are human (cognition and memory) or computational (processing time and system memory). Indeed, according to [Card et al., 1999], mentally multiplying two numbers is five times slower than using a paper and pen, where the numbers can be visually represented. In this context, multi-criteria techniques and the software that implements them become essential tools for transforming vast amounts of data into actionable insights. These software tools, while powerful, are often overly technical and lack storytelling elements, making it difficult for users without prior knowledge to understand and effectively use them [Cinelli et al., 2020; Greene et al., 2011]. This is the first problem we address in our research.

Another significant issue in multi-criteria decision-making is the consistency of preferences. When using techniques such as the *Analytic Hierarchy Process* (AHP), the decision-maker must assign preferences to each pair of criteria. These preferences are organized into a preference matrix and reflected in the calculation of the techniques, but they can sometimes be inconsistent. Inconsistencies occur when the comparisons made by the user are not logically coherent, leading to cycles. For example, if the user prefers A over B, B over C, but also C over A, this results in an inconsistent preference matrix. AHP addresses this issue by calculating a consistency ratio (CR) from the preference matrix's eigenvalue. The consistency ratio is a measure of how consistent the judgments are. A CR less than or equal to 0.10 is considered acceptable; if the CR exceeds this threshold, it indicates significant inconsistencies in the preferences, suggesting the need for revision. There are existing methods that attempt to automatically adjust preferences to improve consistency, but these methods can diverge significantly from the user's original preferences, compromising the integrity of the decisions. These methods often adjust the preferences in ways that depart from the user's initial intentions, potentially leading to less satisfactory results. This is the second problem we address in our research.

In conclusion, our research tackles two major challenges in multi-criteria decision-making: the lack of storytelling and effective visualization on decision support systems and addressing inconsistencies in preference matrices. By developing tools and methods that make these processes more intuitive and reliable, we aim to enhance the overall quality and efficacy of decision-making in complex scenarios.

1.2 Highlights

Considering the current state-of-the-art, we developed a visualization tool incorporating storytelling elements to enhance the decision-making process, making it more engaging and comprehensible for users. This tool aims to simplify interactions for decision-makers by providing intuitive guidance and recommending adjustments to their initially supplied preferences.

To address preference inconsistencies, we developed a method that maintains preferences as close as possible to those initially defined by the user. To achieve this, we divide the problem into two phases: addressing ordinal inconsistencies first, followed by cardinal inconsistencies. Ordinal inconsistencies occur when the relative ranking of preferences is not logically coherent and creates a cycle. To handle ordinal inconsistencies, we formulate our problem as a feedback arc set problem, a well-known problem in graph theory. This approach involves identifying and breaking the minimum number of arcs in a directed graph to eliminate cycles. A cardinal inconsistency exists if, for some criteria i, j, and k, the preference value assigned directly between i and k does not match the product of the preference values assigned between i and j, and j and k. For addressing cardinal inconsistencies, we suggest an improvement over the method presented in [Saaty, 2003], employing a map of small perturbations.

By integrating a visualization tool with storytelling and a refined mathematical approach for adjusting preferences, we offer a user-friendly solution that helps decision-makers make more informed and consistent choices. This tool was tested and presented at IO2024 - XXIII Congresso da Associação Portuguesa de Investigação Operacional. It was selected for EstudIO, a section highlighting the top 10 works by bachelor and master's students in operational research, where it received positive feedback for its utility and clarity from both specialists and non-specialists. Additionally, we compared our method of adjusting preferences with existing ones in the literature and found that our method produced better results in most cases.

1.3 Organization

The rest of the thesis is organized as follows. Chapter 2 presents a literature review on multi-criteria decision-making, exploring various techniques employed in the field. It also discusses the rationale behind choosing AHP and TOPSIS, along with their mathematical formulations. Chapter 3 reviews existing software tools that implement multi-criteria techniques, identifies their main disadvantages, and describes how our visualization tool was developed to address these shortcomings. Chapter 4 details the methods used for adjusting preferences, introduces our proposed method, and compares the results obtained from our approach with those reported in the existing literature using distance metrics. Finally, Chapter 5 identifies the research objectives, highlights the results achieved, and outlines potential directions for future research.

Chapter 2

Multi-Criteria Decision Making

This chapter is structured as follows: Section 2.1 presents the most commonly used multi-criteria techniques in the literature, with particular emphasis on AHP and TOPSIS. Section 2.2 provides a brief framework on how to select appropriate multi-criteria techniques. Finally Section 2.3 concludes with the mathematical formulation of AHP and TOPSIS.

2.1 The different multi-criteria techniques

In multi-criteria problems, the alternatives and their performances according to multiple evaluation criteria are explicitly known. In these context, it is very common to use the process of Multi-Criteria Decision Analysis (MCDA), or Multi-Criteria Decision Making (MCDM), specially when the decision making process involves multiple perspectives, constraints, variables, and stakeholders [Cinelli et al., 2020]. To ensure that decision-makers make the most informed decisions, various techniques have emerged. These techniques combine information from various criteria and alternatives and return to the user three different types of output: Choice, if the goal is to select the *best* alternative or a reduced set of *best* alternatives; Sorting, if the goal is to assign the alternatives to predefined ordered classes (categories) of merit; and finally, Ranking, when the goal is the same as this master's dissertation, which is to rank the alternatives from the *best* one to the *worst* one.

The use of these techniques has been growing significantly over the years due to their ability to address complex problems that involve considering various viewpoints and trade-offs [Cinelli et al., 2020]. Furthermore, the areas they cover are vast, going from environment [Dias et al., 2018; Volkart et al., 2016], technology [Barker and Zabinsky, 2011; Bertola et al., 2019], supply chain management [Govindan et al., 2017; Jasiński et al., 2018], to politics [Reale et al., 2017] demonstrating that these tools are capable of leading the way to the best decision.

The choice of the best technique to use varies from problem to problem, with each one having its advantages and disadvantages. Several studies have been developed to compare the more than 50 different existing multi-criteria methods

[Cinelli et al., 2020; Wątróbski et al., 2019], providing the best practices to use when choosing the method, always considering the type of problem, preferences, uncertainty, and the desired outcome. However, there are techniques that are already well consolidated in the literature and are commonly used for their simplicity, capacity, and acceptance by the research community, such as AHP, TOPSIS, MAUT, PROMETHEE, and ELECTRE family. Table 2.1 shows some of the key articles that have employed the mentioned techniques across various fields.

Table 2.1: Multi-criteria decision analysis.

Paper	**AHP**	**TOPSIS**	**MAUT**	**ELECTRE**	**PROMETHEE**
[Saaty, 2004]	•				
[Aras et al., 2004]	•				
[Norese, 2006]				•	
[Yong, 2006]		•			
[Canbolat et al., 2007]			•		
[Ertuğrul and Karakaşoğlu, 2008]	•	•			
[Barker and Zabinsky, 2011]	•				
[Espitia-Escuer et al., 2015]				•	
[Govindan et al., 2017]					•
[Dias et al., 2018]				•	
[Zaharah Allah Bukhsh, 2019]			•		
[Bertola et al., 2019]					•

The *Analytic Hierarchy Process* (AHP), developed by T. L. Saaty between 1971 and 1975, is today one of the most widely used techniques in multi-criteria analysis [Zyoud and Fuchs-Hanusch, 2017]. This technique was developed with the aim of combining both physical and psychological events [Saaty, 1987]. In other words, it combines tangible information, which is some form of objective reality outside the individual conducting the measurement, and intangible information, exploring subjective ideas and beliefs that the user wants to use in the decision-making process [Saaty, 1987]. To combine this information, the AHP structures the problem hierarchically. The top level of the hierarchy is the goal of the decision, followed by levels of criteria, sub-criteria, and finally, the list of alternatives at the bottom level [Saaty, 1987]. This process will be analyzed later in Section 2.3. One of the main advantages of AHP over other decision-making tools, such as *Multi-Attribute Utility Theory* (MAUT), is its ease of use. Additionally, AHP does not assume away intransitivities or inconsistencies in judgments but rather deals directly with these inconsistencies and even measures the level of inconsistency in one's judgment [Harker, 1987]. Just like all techniques, AHP has been the subject of criticism over several years, involving both accusers [Bana e Costa and Vansnick, 2008] and defenders [Bozóki et al., 2015; Saaty and Hu, 1998]. For the purpose of this project, entering the ongoing debate is not necessary. This is especially true considering that the choice of technique for this thesis is primarily influenced by its potential for visual tracking and its range of use. Another factor influencing this choice is the simplicity of understanding the technique. An integrated approach combining tracking, storytelling, and visualizations is essential for effective decision-making, addressing a gap in the works developed in [Aras et al., 2004; Barker and Zabinsky, 2011; Ertuğrul and Karakaşoğlu, 2008; Saaty, 2004].

The *Technique for Order of Preference by Similarity to Ideal Solution* (TOPSIS), developed by Hwang and Yoon in 1981, is a method used to evaluate, assess, and rank alternatives across a variety of industries [Ertuğrul and Karakaşoğlu, 2008; Yong, 2006]. The standard TOPSIS method requires defining criteria as either benefit or cost. Benefit criteria are those where higher values are preferred, while costl criteria are those where lower values are preferred. Thus, the objective is to identify alternatives that are closest to the positive ideal solution and furthest from the negative ideal solution. The positive ideal solution seeks to maximize the benefit criteria and minimize the cost criteria, while the negative ideal solution does the opposite [Behzadian et al., 2012]. TOPSIS has been applied in numerous fields, including Supply Chain Management [Yong, 2006], Product Design [Lin et al., 2008], and Business [Peng et al., 2011], among others. The main advantage of this technique lies in its simplicity in concept and application. It effectively utilizes attribute information and provides a cardinal ranking of alternatives without the need for independent attribute preferences [Behzadian et al., 2012]. However, it is important to note that TOPSIS requires numeric attribute values with commensurable units, which may not always be available or applicable.

MAUT is another technique used to evaluate and choose among various alternatives. It considers utility functions to represent the decision-maker's preferences. This process involves identifying relevant criteria and assigning a utility function to each criterion. These utilities are then aggregated to obtain an overall score for each alternative [Figueira et al., 2005]. MAUT has the advantage of providing a robust structure for making complex decisions [Canbolat et al., 2007; Zaharah Allah Bukhsh, 2019]. However, the lack of visualizations to accompany the process remains a persistent issue. Furthermore, it requires that each of the utility functions accurately reflects the decision-maker's preferences, which can be hard to accomplish.

The Elimination and Choice Expressing Reality (ELECTRE) methods, first proposed in [Roy, 1968], are a family of outranking methods that have continuously evolved over time. These methods, which include ELECTRE I, II, III, IV, and TRI, each have operational differences and are suited for different types of decision problems such as choice, ranking, or sorting [Figueira et al., 2005; Govindan and Jepsen, 2016]. The primary objective of these methods is to use the concept of outranking to compare alternatives. This is particularly useful in situations where there is uncertainty or a lack of data [Figueira et al., 2005]. The numerous variations of these methods allow for significant flexibility, enabling adaptation to a wide range of problems [Espitia-Escuer et al., 2015; Norese, 2006]. Each

ELECTRE method has its own set of advantages and disadvantages. These often relate to their ability to handle complex decision-making scenarios, the ease of understanding for decision-makers, and the specific requirements of different problems. For example, ELECTRE IV has had few applications overall. It has less flexibility than the other methods, but it does not require the definition of criteria weights, which could make it more suitable than other methods in some cases [Govindan and Jepsen, 2016]. ELECTRE TRI has been successfully applied to a number of risk-related problems in financial management and land use management, indicating that it may be particularly well suited for kind of problems [Govindan and Jepsen, 2016]. ELECTRE III has various applications in the areas of energy management and natural resources and environmental management, but it may be overlooked in some areas. However, as previously mentioned, their complexity can introduce some degree of uncertainty into the decision-making process for a decision-maker who is not an expert in multi-criteria analysis [Govindan and Jepsen, 2016].

The *Preference Ranking Organization Method for Enrichment Evaluations* (PROMETHEE) is also a family of outranking methods for MCDA, first developed by J.P. Brans (PROMETHEE I and PROMETHEE II) and introduced in 1982. Over the years, various variations have emerged, but the primary objective remains to rank alternatives based on multiple conflicting criteria [Figueira et al., 2005]. For instance, PROMETHEE II starts with a phase of pairwise comparisons, followed by the use of relevant preference functions for each criterion. It then calculates the global preference index, culminating in the calculation of positive and negative outranking flows for each alternative and partial ranking [Behzadian et al., 2010]. One of PROMETHEE's main strengths lies in its ability to handle both qualitative and quantitative data. Existing software packages, such as PROMCALC and Decision Lab, also facilitate the process. However, the method can lead to potential bias due to subjective weight assignments, difficulty in handling a large number of criteria or alternatives, and the need for decision-makers to define preference functions and thresholds [Behzadian et al., 2010].

To summarize, Table 2.2 compares the described multi-criteria techniques, focusing on how preferences are provided, their advantages, and their disadvantages. AHP employs pairwise comparisons, making it easy to understand and apply. Its significant advantage lies in providing a consistency ratio, which ensures the logical coherence of the decision-maker's preferences. However, it can be time-consuming and may result in inconsistent matrices if not carefully managed (a problem that we are going to solve in Section 4.2). TOPSIS, on the other hand, requires weights for each criterion for every alternative. Its primary strengths are its computational simplicity and intuitive appeal, making it accessible and easy to use. Nonetheless, TOPSIS is sensitive to the relative scaling of criteria, which can affect the accuracy of the results, but can be solved using the weights provided by AHP.

Compared to other techniques like MAUT, which provides a robust decision framework but struggles with the accurate definition of utility functions, AHP and TOPSIS offer more straightforward and practical approaches. ELECTRE handles uncertainty well but is complex and requires an understanding of outranking, whereas PROMETHEE, despite its flexibility and ability to handle various data types, can introduce bias through subjective weight assignments. In conclusion, AHP and TOPSIS provide accessible, reliable, and practical methods for multi-criteria decision-making, making them favorable choices over more complex or subjective techniques.

Table 2.2: Comparison of Multi-Criteria Techniques

Technique	**How Preferences are Provided**	**Advantages**	**Disadvantages**
AHP	Pairwise comparisons	Easy to understand and apply; provides a consistency ratio	Can be time-consuming; may lead to inconsistent matrices
TOPSIS	Scores for each criterion for each alternative	Simple to compute; intuitive appeal	Sensitive to the relative scaling of criteria
MAUT	Utility functions for each criterion	Provides robust decision framework	Difficult to accurately define utility functions
ELECTRE	Outranking relations	Handles uncertainty well; flexible	Complex; requires understanding of outranking
PROMETHEE	Pairwise comparisons	Flexible; handles a wide range of data types	Subjective weight assignments can introduce bias

The combination of AHP and TOPSIS

Different techniques have different capabilities and can serve different purposes, often leading to the need to combine two or more techniques. Among the previously discussed techniques, two stand out as particularly suitable for combined use: AHP and TOPSIS. Observing Figure 2.1, we can see the growth associated with the number of published articles using both AHP and TOPSIS. In a study developed in [Zyoud and Fuchs-Hanusch, 2017], the clear emergence of these two methods is notable, both in the variety of areas where they can operate and in their combined capacity to solve complex problems.

Indeed, according to [Behzadian et al., 2012], of the articles published up until 2012 that mentioned TOPSIS, about 22% used it in combination with AHP. In the field of Supply Chain Management and Logistics, [Büyüközkan et al., 2008] uses the combination of AHP and TOPSIS to select a suitable partner for a strategic alliance in a logistic value chain. [Ertuğrul and Karakaşoğlu, 2008] uses it to select the facility location of a textile company. Additionally, [Kandakoglu et al., 2009] ranks the shipping registry alternatives in the maritime transportation industry based on a SWOT analysis. In the area of Design, Engineering, and Manufacturing Systems, [Chang, 2012] uses this approach to select an optimal wire saw in photovoltaic wafer manufacturing. Also, [Lin et al., 2008] performs competitive benchmarking to identify the most competitive design alternative for further detailed design. Finally, in the area of Business and Environment Management, [Işıklar and Büyüközkan, 2007] and [Soltanmohammadi et al., 2010] emerge as interesting works in their respective areas. The first one aims to evaluate mobile phone options according to users' preference orders, and the last one to determine a preference order of post-mining land uses. A variety of other areas can be found in [Behzadian et al., 2012], showing that the AHP - TOPSIS combination has significant potential, using AHP to determine the weights of the criteria and TOPSIS to determine the ranking of the alternatives.

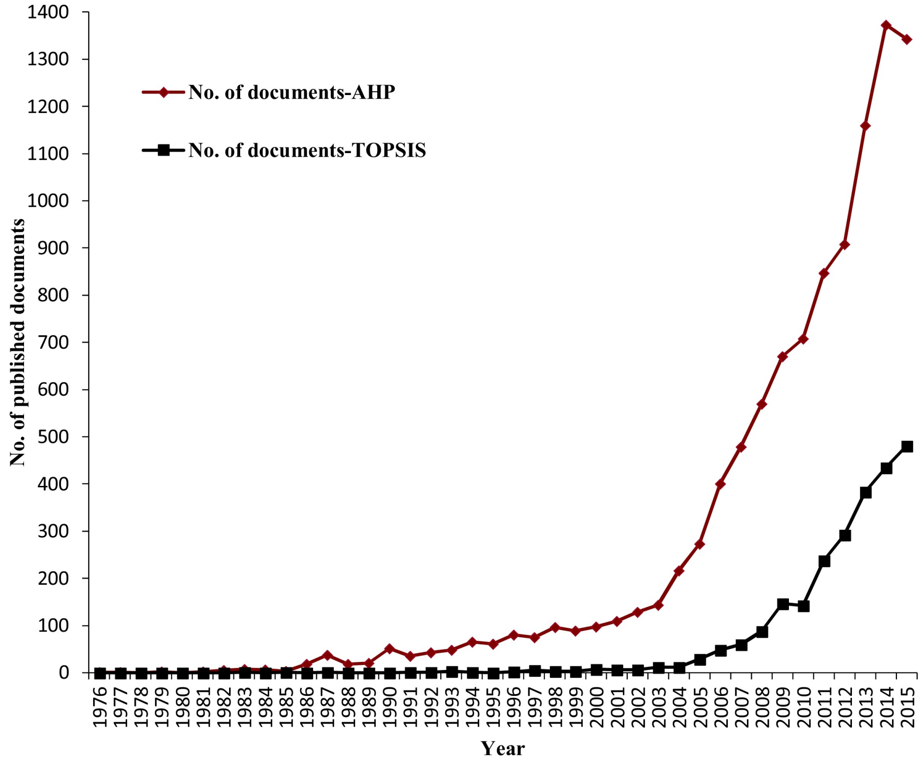


Figure 2.1: Number of published articles and review articles from the global in AHP & TOPSIS methods [Zyoud and Fuchs-Hanusch, 2017].

2.2 Choice of multi-criteria techniques

The initial step towards making an informed decision involves structuring the problem in the most effective manner. Figure 2.2 illustrates the potential scenarios that a multi-criteria problem can take. The choice of technique should be based on the nature of the decision and the criteria used to evaluate alternatives. This process involves two main considerations: first, distinguishing the type of recommendation the model seeks (e.g., choice, ranking, sorting) and whether the set of alternatives is stable or dynamic; second, assessing the structure of criteria, which can be either flat or hierarchical, and their evaluation based on ordinal, interval, or ratio scales. Additionally, performance evaluation can further be categorized as deterministic, involving precise values, or non-deterministic, involving uncertain values [Cinelli et al., 2020].

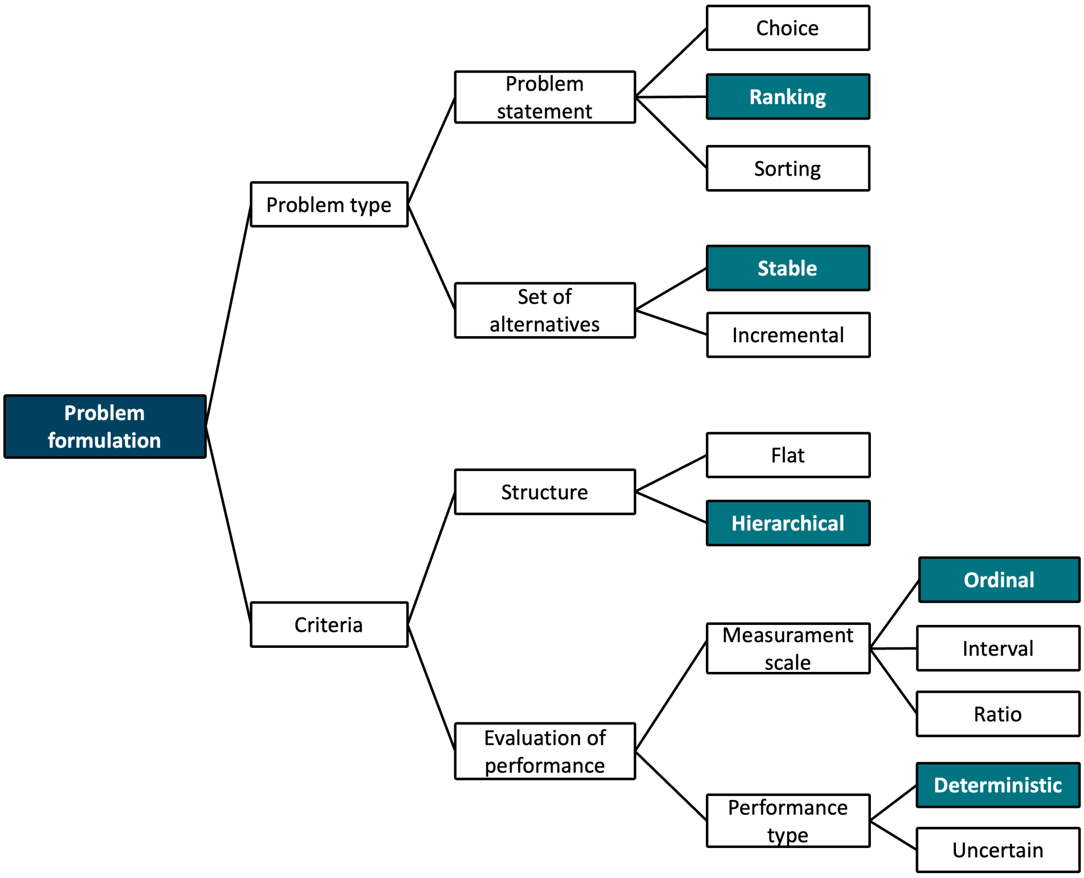


Figure 2.2: Problem formulation. Adapted from [Cinelli et al., 2020]

The type of recommendation desired by the user, or the problem statement, can typically be categorized as choice, ranking, or sorting, as previously discussed. According to the review in [Cinelli et al., 2020], 75% of studies focus on ranking, which involves ordering alternatives from best to worst and highlighting significant differences among them, a task well-suited for TOPSIS. Regarding the stability of the set of alternatives, 9% of the reviewed articles maintain a stable number of alternatives, while another 9% incrementally increase the set. However, when the set of alternatives is evolving, relative comparisons among them become less consistent. This suggests a preference towards sorting techniques, which is not the aim of this project. Therefore, we have chosen to work with a stable set of alternatives.

The distinguishing characteristics of criteria can be understood through their structure, measurement scale, and type of performance used as input data. In many multi-criteria projects, structuring criteria hierarchically is essential, a choice made by 32% of studies, where AHP is particularly effective. Regarding measurement scales, studies are evenly divided among ordinal, interval, and ratio scales. Given the unsupervised nature of our tool, we find the ordinal scale most appropriate, aligning well with TOPSIS methodology. Lastly, in terms of the type of performance for alternatives, two categories can be distinguished: deterministic and uncertain. Uncertain performance is prevalent due to challenges in obtaining precise data and the desire to assess results stability amid data variability. Although Fuzzy and probabilistic approaches are common for handling uncertainty, they are not the focus of this study. Consequently, since the performance values for each alternative and criterion are known, AHP and TOPSIS are identified as the most suitable techniques for our purposes.

When selecting a MCDM method, it is essential to consider the characteristics outlined in Figure 2.3. These factors primarily include the mathematical literacy of decision-makers/stakeholders, the availability of time and resources, the technical complexity and familiarity with the chosen MCDM method, and the software's capability for implementation and visualizing results.

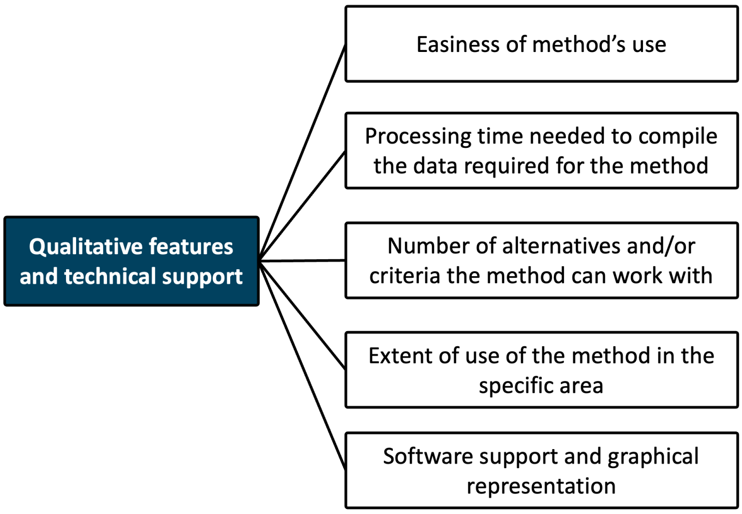


Figure 2.3: Evaluation of MCDM techniques. Adapted from [Cinelli et al., 2020].

The review in [Cinelli et al., 2020] reveals that up to 36% of the included studies consider characteristics that are not specific to the method, but rather subjective. These characteristics often relate to the diverse backgrounds and mathematical competencies of the stakeholders involved in the decision-making process. The primary qualitative consideration that several authors mention when selecting a MCDM method is the understandability or ease of use of the method. This refers to the time required for interaction with the DM to explain the method and obtain preference information, as well as the level of input required from the stakeholders. This is one of the main reasons for the creation of our storytelling tool, which engages the user in the decision-making process without being overly technical or tiring.

The processing time and effort needed to compile the data required for the method are other important characteristics. For instance, the TOPSIS method requires initial weights that can be determined by the AHP method, instead of being determined by the user, which can be a difficult decision. The effort here lies in assigning preferences and dealing with inconsistencies, something that can be done interactively and simply, as we will see in Section 4.2. The complexity also varies according to the number and type of criteria the method can handle. One way to deal with this complexity is to ensure that the user only uses necessary information, which can be done through storytelling regardless of the technique used, as we will see later. Another subjective consideration added to this set is the extent of use of the method in the specific context/area. The assumption is that the more a method is used in a certain area, the more it is seen as the correct method. However, as we saw in the literature review section, both AHP and TOPSIS have the ability to adapt to any resolution area, so this is not a limitation of this project.

The processing time and effort required to compile necessary data are significant considerations for selecting a method. For example, the TOPSIS method necessitates initial weights that can be derived from the AHP method instead of being determined by the user, which can be a challenging decision. Effort is also involved in assigning preferences and addressing inconsistencies, tasks that can be simplified through interactive methods as discussed in Section 4.2. Method complexity also varies depending on the number and type of criteria it handles. Managing this complexity involves ensuring users only engage with essential information, a process facilitated through storytelling regardless of the technique employed, as detailed later. Another subjective consideration is the method's extent of use within a specific context or area. The assumption is that methods frequently used in a particular domain are perceived as more appropriate. However, both AHP and TOPSIS are versatile and applicable across various domains, as noted in Section 2.1. Therefore, this project is not constrained by limitations related to specific application areas.

The final characteristic pertains to the availability of software to implement the method(s) and its capacity for graphical representation and exploration of results. These features continue to drive the recommendation of MCDM methods. This observation is linked to the fact that software tools implementing MCDM methods, and presenting results with a range of customization options, can assist analysts in computing results and presenting them to DMs/stakeholders in a structured, compelling, and traceable manner [Cinelli et al., 2020]. More specifically, these tools can enhance the understandability of the decision recommendation through advanced exploration of results and graphical visualization capabilities. These features include explanations of which criteria most influence the final results, and which portions of certain performance and/or preferences lead to a certain ranking, classification, or choice of alternatives. In Section 3, we have already analyzed the main software that handles this characteristic, and all of them lacked storytelling support. We have observed that to our knowledge, there is no software currently available that integrates both TOPSIS and AHP techniques. This suggests a promising area for exploration, complementing the considerations we have discussed regarding methodological characteristics and the specific problem formulation we aim to address. Even in Section 2.1, we realized that the emergence of these two techniques is notable, and the need to address them together arises when dealing with complex problems. Thus, the combination of AHP and TOPSIS seems to be the most suitable approach for this project.

2.3 Mathematical formulation of AHP and TOPSIS

As previously mentioned, the goal of a decision-making problem can be to assign a ranking to a set of alternatives. In our case, AHP will be used to determine the weights associated with each criterion. These weights will then be used in TOPSIS to assign the final ranking. The mathematical formulation of these techniques is as follows:

2.3.1 AHP

AHP can be described by the following steps.

1. **Define the preference matrix:** For each pair of criteria (i, j), $i, j \in \{1, \ldots, n\}$ and $i \neq j$, where n is the number of criteria, define which one is considered more important under that criterion and how much more (using the fundamental scale values from Table 2.3). This gives a_{ij} if criterion i is more important than criterion j, or a_{ji} otherwise. The reciprocal value is then automatically entered for the transpose.

   With these preferences, define the preference matrix $A = [a_{ij}]$, $i, j \in \{1, \ldots, n\}$ and $i \neq j$, based on the pairwise comparisons between criteria, and define $a_{ij} = 1$ for $i = j$.

Table 2.3: The fundamental AHP scale. This table is adapted from [Saaty, 1987]

Intensity of importance on an absolute scale	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective
3	Moderate importance of one over another	Experience and judgment moderately favor one activity over another
5	Essential or strong importance	Experience and judgment strongly favor one activity over another
7	Very strong importance	An activity is strongly favored and its dominance demonstrated in practice
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation
2,4,6,8	Intermediate values between the two adjacent judgments	When compromise is needed
Reciprocals	If activity i as one of the above numbers assigned to it when compared with activity j, then j has the reciprocal value when compared with i	

2. **Determine the consistency ratio:** Once the matrix A is defined, calculate

the principal eigenvalue λ_{max} (the highest real eigenvalue - see Definition A.0.2). Then, determine the consistency ratio CR of the defined preferences:

$$CR = \frac{CI}{RI},$$

where CI is a consistency index given by

$$CI = \frac{\lambda_{max} - n}{n - 1},$$

and RI is a Random Index that varies according to the number of criteria (see Table 2.4).

Table 2.4: Random Index values

n	1	2	3	4	5	6	7	8	9	10
RI	0.00	0.00	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49

3. **Obtain the associated weights:** Obtain the weights for preferences by normalizing the principal eigenvector $\vec{w} = (w_1, \ldots, w_n)$ associated with λ_{max}.

When $CR < 0.1$, we can proceed to use TOPSIS. Otherwise, we encounter the inconsistency problem discussed in Section 4.1. We will address this using the minimum feedback arc set problem along with a perturbation map. The method is discussed in detail in Section 4.2.

2.3.2 TOPSIS

After obtaining the final weights with AHP, TOPSIS is used to rank the alternatives. In this context, the process begins with the definition of an initial decision matrix, which is then normalized. Subsequently, a weighted normalized decision matrix is constructed using the weights derived from AHP. This step is followed by the determination of the positive and negative ideal solutions. After this, separation measures for each alternative are calculated. The final step is the computation of the relative closeness coefficient. The alternatives, or candidates, can then be ranked in descending order based on this coefficient.

TOPSIS can then be described by the following steps.

1. **Definition of the decision matrix:** For each pair of alternative-criterion (i, j), for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, where m is the number of alternatives, define the x_{ij} value of the i-th alternative on the j-th criterion. Then define the decision matrix $B = [x_{ij}]$.

2. **Normalization:** Determine the normalized decision matrix $B_1 = [r_{ij}]$, where r_{ij} is calculated as follows:

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}. \tag{2.1}$$

3. **Weighted normalization:** Determine the weighted normalized decision matrix $B_2 = [v_{ij}]$, where v_{ij} is calculated as follows:

$$v_{ij} = w_j \cdot r_{ij}, \tag{2.2}$$

   where w_j is the AHP weight of the j-th criterion.

4. **Determination of the ideal and negative-ideal solutions:** The ideal solution $A^* = (v_1^*, v_2^*, ..., v_n^*)$ and negative-ideal solution $A^- = (v_1^-, v_2^-, ..., v_n^-)$ are defined as follows:

$$v_j^* = \begin{cases} \max v_{ij}, & \text{if } j \text{ is a benefit criterion} \\ \min v_{ij}, & \text{if } j \text{ is a cost criterion} \end{cases} \tag{2.3}$$

$$v_j^- = \begin{cases} \min v_{ij}, & \text{if } j \text{ is a benefit criterion} \\ \max v_{ij}, & \text{if } j \text{ is a cost criterion} \end{cases} \tag{2.4}$$

5. **Calculation of separation measures:** Determine the separation of each alternative from the ideal solution S_i^* and the negative-ideal solution S_i^- as follows:

$$S_i^* = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^*)^2} \tag{2.5}$$

$$S_i^- = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^-)^2} \tag{2.6}$$

6. **Calculation of relative closeness to the ideal solution:** Determine the relative closeness of the alternative a_i with respect to A^* as follows:

$$C_i = \frac{S_i^-}{S_i^* + S_i^-}. \tag{2.7}$$

The ranking of alternatives is obtained by ordering the C_i in descending order, meaning the alternative with the highest C_i is the best one.

2.4 Discussion

In selecting a technique, we considered factors such as mathematical literacy of stakeholders, time and resource availability, technical complexity, and software capabilities for implementation and visualization. Our discussion highlighted the strengths and weaknesses of established methods like AHP, TOPSIS, MAUT, PROMETHEE, and ELECTRE, acknowledging their capacity to address complex problems across diverse fields. We made a conscious decision to focus on AHP and TOPSIS due to their widespread acceptance and the potential for visual tracking. AHP's hierarchical structuring of decision problems and its capacity to handle both tangible and intangible information make it a powerful tool for combining various criteria. Its principal advantage lies in its ability to manage inconsistencies in judgment, providing a measure of consistency through its consistency ratio. On the other hand, TOPSIS offers a straightforward concept that effectively utilizes criteria information to rank alternatives. It distinguishes itself with its simplicity in application, requiring only the definition of criteria as either benefit or cost and then identifying alternatives closest to the positive ideal solution.

The combination of AHP and TOPSIS was identified as a powerful approach, leveraging AHP to determine criteria weights and TOPSIS to rank alternatives. This integrated method has shown significant potential in various studies, proving effective in strategic alliance selection and facility location, among others.

Chapter 3

A new decision support system

This chapter presents the first contribution of our thesis: a visualization tool accompanied by storytelling elements, created with the aim of mitigating the lack of a tool that guides the decision-maker to the best decision. However, to achieve this, it was first necessary to conduct a brief review of the software currently used to apply multi-criteria techniques. Thus, the chapter is structured as follows: Section 3.1 presents the most commonly used software applications recognized by the MCDM community. Section 3.2 then introduces the tool developed in this thesis incorporating storytelling into the AHP and TOPSIS methodologies.

3.1 Existing software related to MCDM

As previously noted, the need to develop software to assist in the decision-making process became evident, and several tools have been proposed. These software solutions are accompanied by visualizations that enable users to describe their problems, solve them, and interpret the results (see Table 3.1).

Table 3.1: Available software for MCDM.

Paper	**Software**	**Techniques used**
[Mustajoki and Hämäläinen, 2000]	Web-HIPRE	AHP, SMART, SWING, SMARTER and MAVT
[Dias and Mousseau, 2003]	IRIS	ELECTRE TRI
[Mustajoki and Hämäläinen, 2007]	Smart-Swaps	Even swaps
[Siraj et al., 2015]	PriEsT	AHP
[de Almeida et al., 2016]	FITradeoff	FITradeoff
[Nitzsch et al., 2020]	Entscheidunfsnavi	MAUT
[Bohanec, 2022]	DEX	DEXi

However, the complexity of these techniques and the level of user engagement in their implementation can lead to uncertainties or inefficiencies in their utilization [Cinelli et al., 2020; Li et al., 2020].

Web-HIPRE

One of the most powerful software tools in this field is Web-HIPRE, developed as described in [Mustajoki and Hämäläinen, 2000]. This software utilizes techniques like AHP, SMART, SWING, SMARTER, and MAVT to aid in problem structuring, prioritization, and result analysis. Web-HIPRE facilitates the visual structuring of decision problems through value trees or hierarchies, employing various weighting and rating methods. It offers visual aids such as bar graphs and sensitivity analysis windows to present composite priorities and assess the impact of criteria weights or alternative ratings (see Figure 3.1).

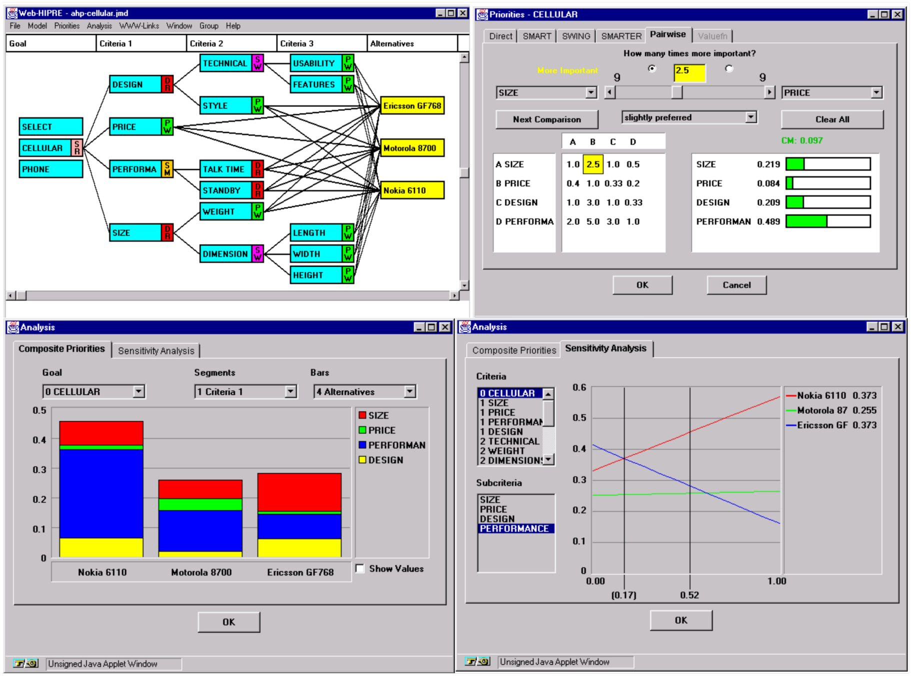


Figure 3.1: Web-HIPRE software overview [Mustajoki and Hämäläinen, 2000].

However, it is crucial to recognize that using this software requires users to possess a solid understanding of decision analysis to apply these methods correctly. It also necessitates careful consideration by the user to ensure accurate interpretation of the results.

IRIS

IRIS (Interactive Robustness analysis and parameters' Inference for multi-criteria Sorting problems) is another Decision Support System (DSS) specifically tailored for sorting problems, utilizing the ELECTRE TRI method [Dias and Mousseau, 2003]. This system integrates parameter inference with robustness analysis, facilitating the interactive construction of a sorting model. IRIS employs linear programming and Monte-Carlo simulation to handle imprecise information and propose parameter values that align with the decision-maker's constraints. Its user interface adopts a "notebook with multiple tabs" metaphor, dividing the screen into input and output areas. It visualizes sorting ranges, inferred sorting, and parameter values, highlighting inconsistencies and robust conclusions (see Figure 3.2). However, it is important to note that the system may lack a storytelling aspect, potentially making it overly technical for users unfamiliar with the method.

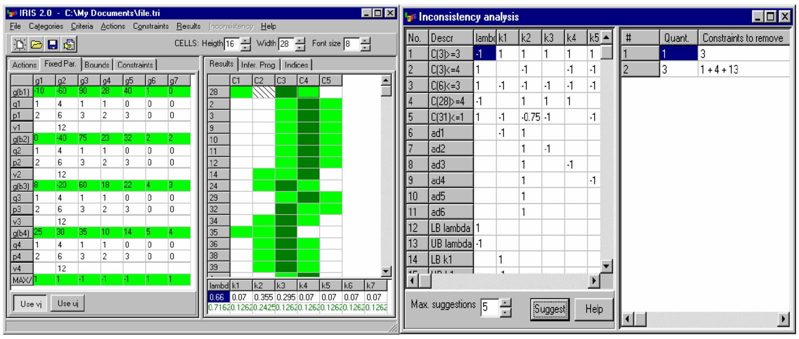


Figure 3.2: IRIS software overview [Dias and Mousseau, 2003].

Smart-Swaps

Smart-Swaps is a web-based decision support tool that employs the even swaps method [Mustajoki and Hämäläinen, 2007]. This method facilitates DMs in making consistent decisions by utilizing an elimination process based on value trade-offs. The user interface of Smart-Swaps is organized into tab panels that correspond to the PrOACT (Problem, Objectives, Alternatives, Consequences, and Trade-offs) phases. It visually presents the performance of each alternative across attributes and uses a white-yellow color scale to indicate rankings in the consequences table (see Figure 3.3).

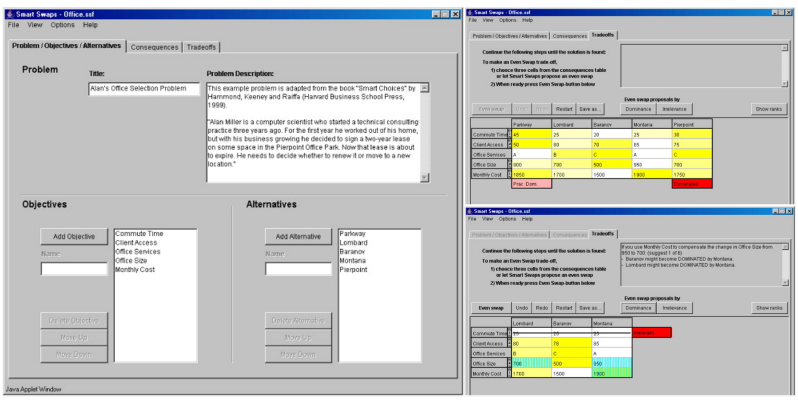


Figure 3.3: Smart-Swaps software overview [Mustajoki and Hämäläinen, 2007].

However, similar to the previous two software tools, the process can be complex, especially for users unfamiliar with decision analysis techniques. This under-scores the importance of having a clearer and more intuitive interface that guides DMs through the decision-making process in a straightforward manner.

PriEsT

When it comes to employing the AHP technique, PriEsT software stands out as particularly effective in identifying both ordinal and cardinal inconsistencies (see Figure 3.4). PriEsT utilizes mathematical formulas and algorithms to analyze pairwise comparison data. It incorporates measures of consistency, congruence, and dissonance to evaluate the coherence of the provided judgments [Siraj et al., 2015]. Despite its robust analytical capabilities, PriEsT lacks in the storytelling aspect. Users may need additional explanations to fully grasp the analytical processes and results. Furthermore, although it correctly identifies inconsistencies, it does not offer suggestions for new preferences. This limitation highlights a significant issue with AHP, where users must repeatedly revise their preferences until consistency is achieved (something that we will solve on Chapter 4.

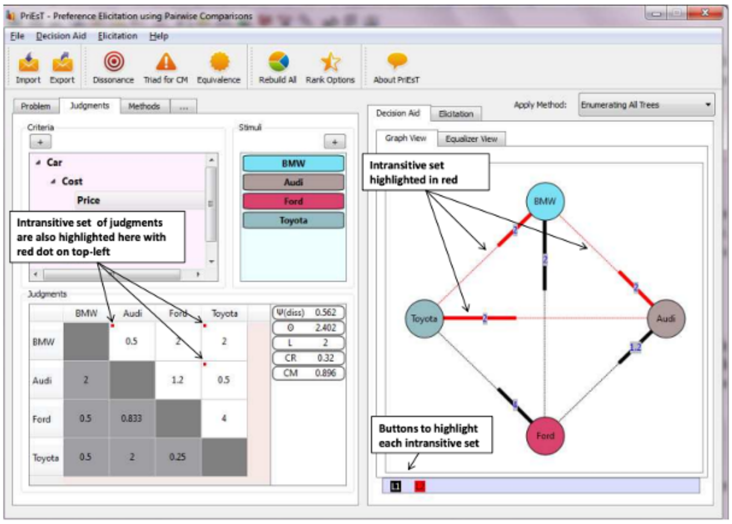


Figure 3.4: PriEsT software overview [Siraj et al., 2015].

FITradeoff

FITradeoff is a software that provides extensive visualizations to enhance the decision-making process, making it highly reliable and user-friendly (see Figure 3.5). According to [de Almeida et al., 2016], it employs a flexible elicitation process that does not require complete information upfront. The software aims to gather comprehensive information through a tradeoff elicitation procedure but can also function with incomplete information if a unique solution is identified or if the decision maker cannot provide additional information. The DSS interacts with the decision maker to systematically evaluate the feasibility of finding a solution during the elicitation process. Overall, this software is well-executed, and the methodology implemented is well-explained, minimizing the need for intervention in this project.

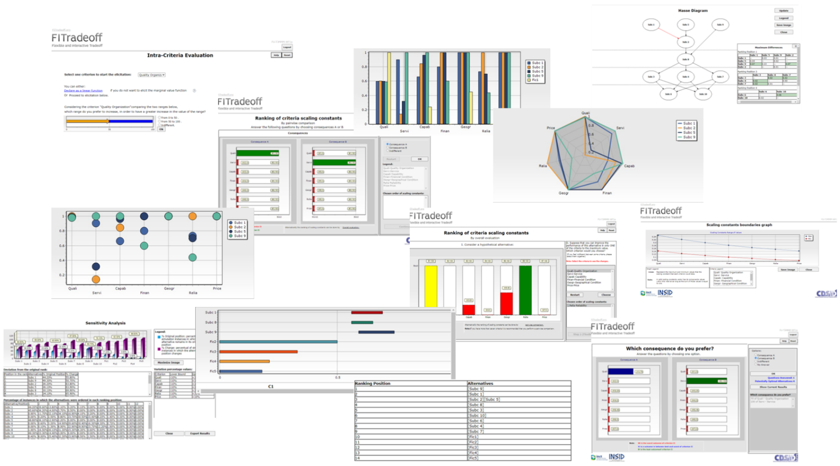


Figure 3.5: FITradeoff software overview [de Almeida et al., 2016].

Entscheidungsnavi

The Entscheidungsnavi software is a DSS that integrates VFT and MAUT [Nitzsch et al., 2020]. VFT aids in structuring the decision problem, while MAUT provides an analytical evaluation of alternatives. The software offers various visualizations, including a graphical interface to visualize objectives and graphical representations to analyze preferences with utility functions (refer to Figure 3.6). Despite incorporating a Monte Carlo simulator to assess decision robustness, the software faces inherent challenges associated with this technique. These challenges include defining utility functions and determining attribute weights. Therefore, its optimal use typically requires guidance from a decision analysis expert, particularly in professional applications. Nonetheless, Entscheidungsnavi has been successfully applied in diverse educational and professional contexts to enhance decision-making processes.

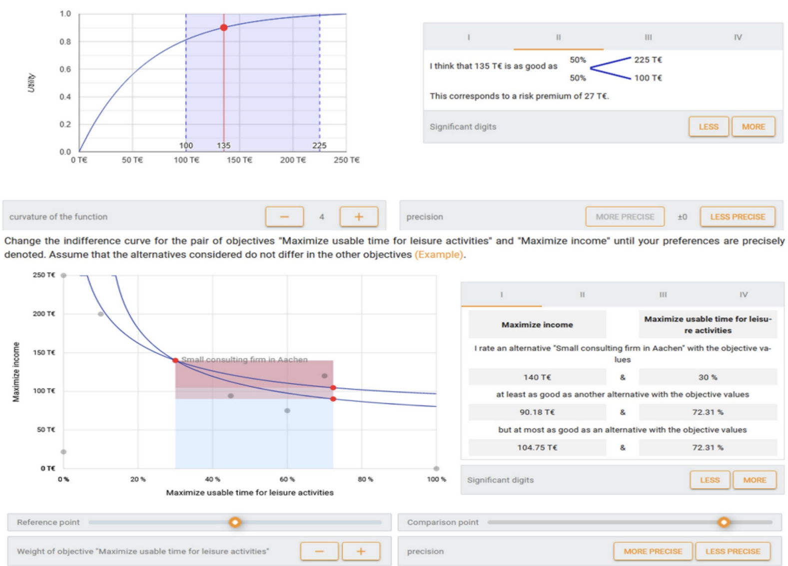


Figure 3.6: Entscheidungsnavi software overview [Nitzsch et al., 2020].

DEX

Finally, DEX is a software that handles hierarchical models, similar to AHP, but also incorporates decision rules represented in decision tables [Bohanec, 2022]. However, the software's visualizations are primarily confined to tables and a 3D graph for illustrating complex rules (see Figure 3.7). This limitation leads to a deficiency in storytelling and may fail to inspire confidence in users due to its restricted visual presentation. Therefore, while DEX is a functional tool, it could greatly benefit from improved visualization techniques to enhance user comprehension and trust.

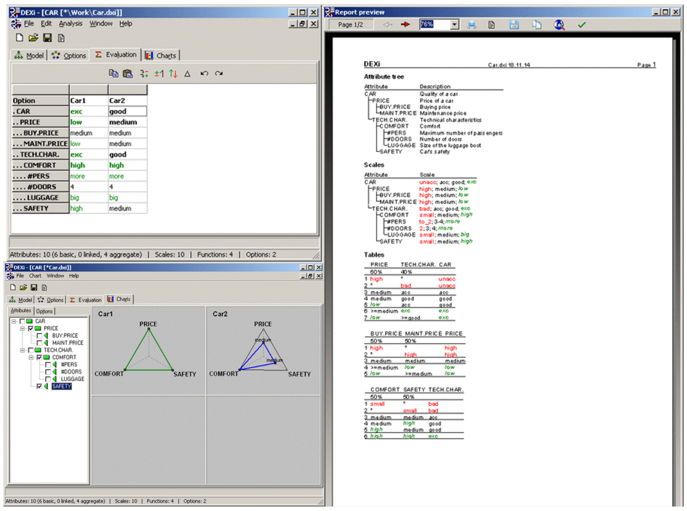


Figure 3.7: DEX software overview [Bohanec, 2022].

Summary

As we can see, several software tools exist to implement multi-criteria decision analysis. While technically powerful, they do not fulfill the purpose of this project, which is to visually deconstruct a multi-criteria technique, functioning as a storytelling. Storytelling, in this context, refers to the ability to present the decision-making process in a narrative form that is easy to understand and follow. It involves breaking down complex processes into simpler parts and presenting them in a logical manner. Storytelling can help decision-makers better understand the process, comprehend how various factors influence the decision, and grasp the rationale behind the final decision. Therefore, there is a pressing need for tools that can simplify the implementation of MCDM techniques and enhance the decision-maker's understanding in the results. Such tools should focus on clarity, simplicity, and the ability to visually communicate complex decision processes effectively. Among the software reviewed, TOPSIS has not been widely implemented in existing tools. Additionally, suggesting new preferences in AHP is not a feature that, to our knowledge, has been incorporated into any current software.

3.2 Our Storytelling Dashboard

To adress the limitiations described above the primary objective of this visualization tool is to offer a robust framework that guides users through a structured decision-making process, effectively combining the strengths of both AHP and TOPSIS. To achieve this, the dashboard allows users to input relevant data and criteria, adjust their preferences until a consistent matrix is obtained, calculate the weights of these criteria using the AHP method, and then apply these weights in the TOPSIS method to rank the given alternatives. Furthermore, the storytelling component of the dashboard is designed to guide users through each step of the decision-making process. The dashboard presents the results through intuitive visualizations and a narrative format, helping users to interpret the weights of the criteria and the ranking of alternatives effectively. The dashboard was implemented using Dash, a web-based application framework that facilitates the creation of analytical web applications, and Plotly, a graphing library integrated with Dash, which is used to create dynamic and responsive visualizations.

To better understand the dashboard and its organization, we formulated an hypothetical example problem: IKEA needs to choose a new store and has three possible alternatives, Store A, B and C. These alternatives are evaluated based on the following criteria: Foot Traffic (pp per day), Rental Cost ($ per month), Proximity (km) and Competition (number of stores). Figure 3.8 gives an overview of the dashboard for this example.

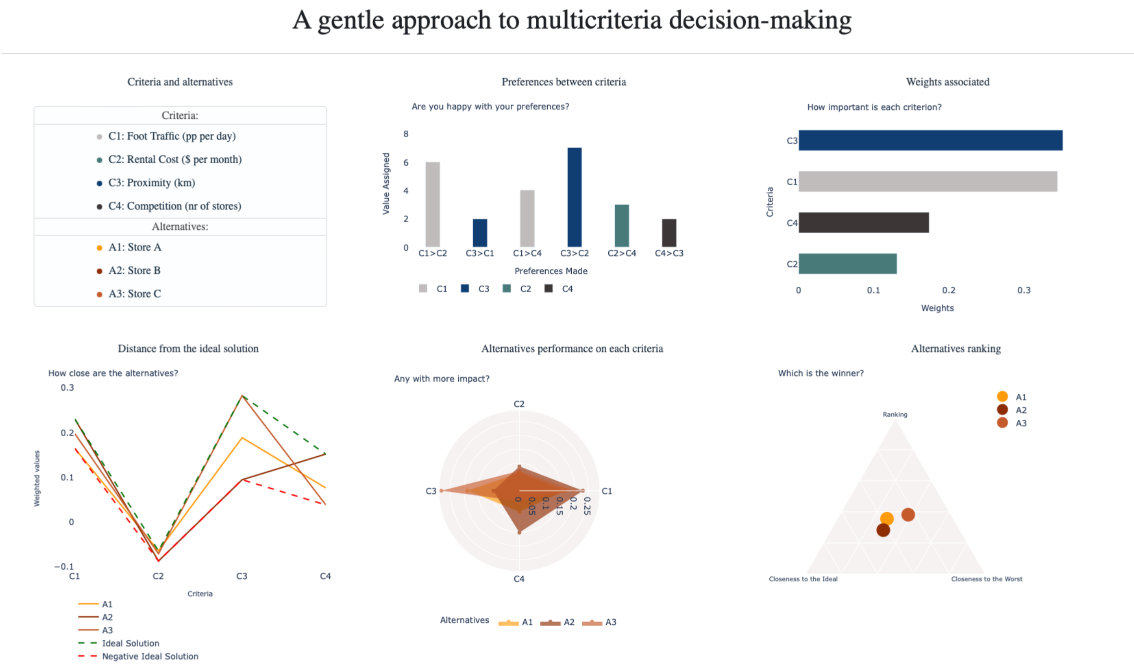


Figure 3.8: Overall view of the dashboard.

Figure 3.8 reflects the final product of the tool. However, it is necessary to embark on a lengthy journey to make the most accurate decision. Metaphorically, the user must climb a mountain to make a decision. At each step of the mountain, the tool explains and shows the user their current stage in the journey, illustrating how it can facilitate their path to the summit. Initially, the tool presents itself as if it were telling a story. The purpose of the tool is explained on the first page (see Figure 3.9). On the second page (see Figure 3.10), the user can describe their problem, creating a sense of closeness and engagement.

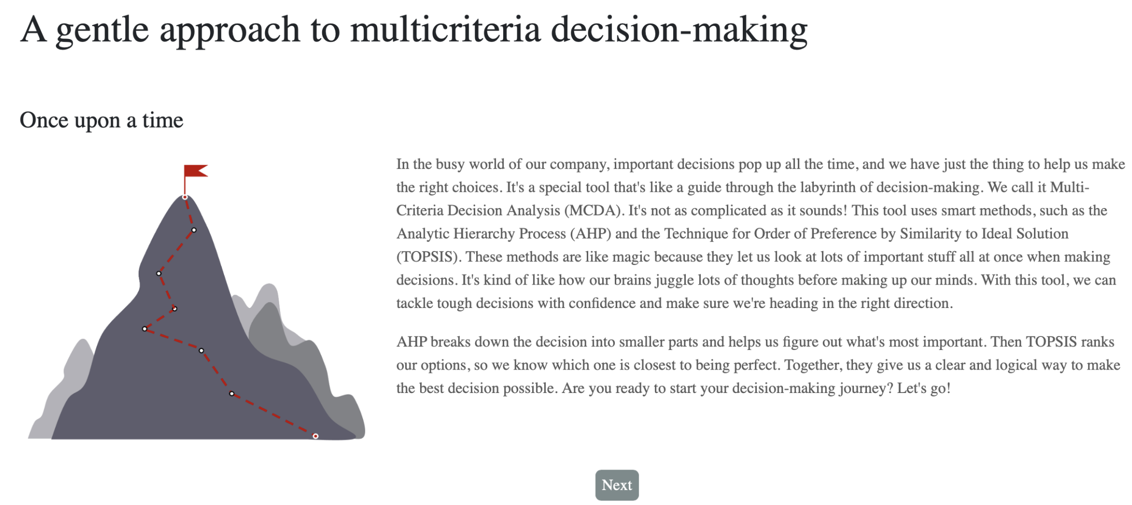


Figure 3.9: Introduction page.

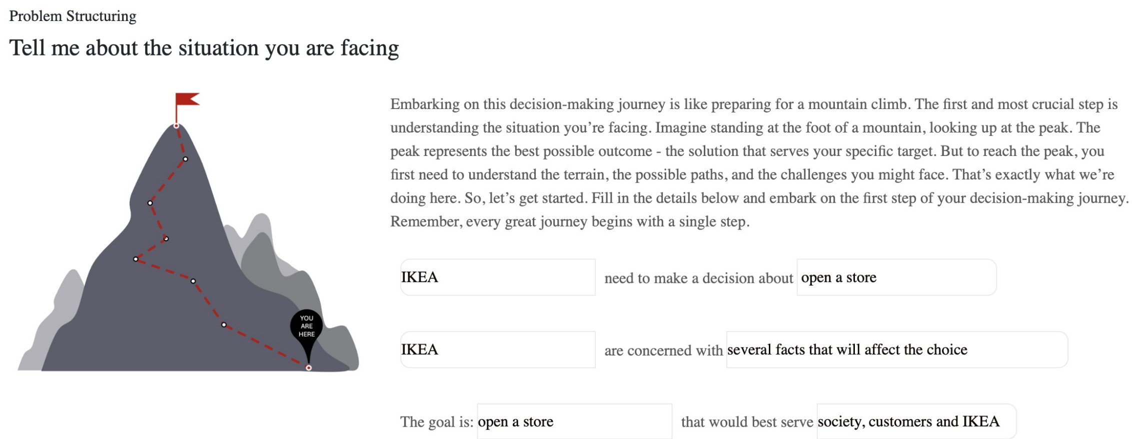


Figure 3.10: Problem structuring.

Subsequently, the user begins to input the necessary information to apply multi-criteria techniques, specifically the required criteria (see Figure 3.11). The user enters the number of criteria, the name of each one, and the type of criterion—whether it is a benefit or cost. If it is a benefit, the goal is to maximize the criterion, and if it is a cost, the goal is to minimize it. Following this, it is necessary to input the preferences between the criteria (see Figure 3.12). The user is asked to input the preferences by evaluating each pair of criteria on the Saaty scale (see Table 2.3). When the user presses the validate button, the tool will search for cycles. If it finds any, it will solve the minimum feedback arc set problem and present suggestions to eliminate intransitivities in the user's preferences (see Figure 3.13). This allows the user to modify their preferences or proceed as they wish. It is recommended that the preference matrix be consistent so that the weights associated with each criterion accurately reflect the decision-maker's intentions. Accepting these suggestions would help achieve that goal. A complete description of the method to suggest new preferences is presented on (4.2).

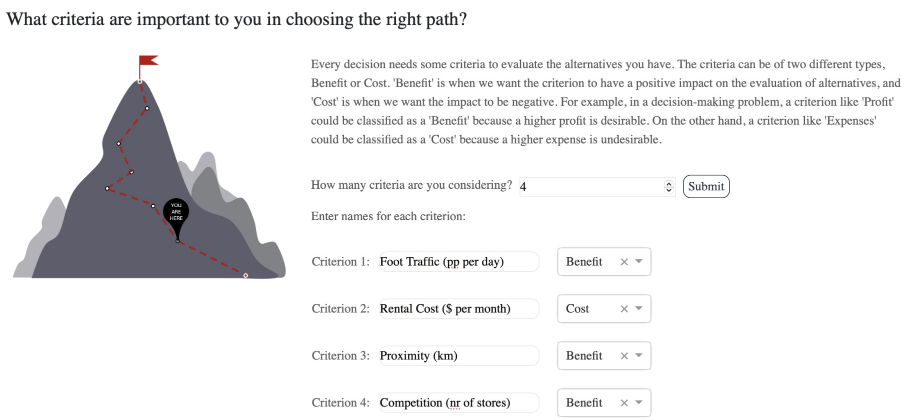


Figure 3.11: Insert criteria.

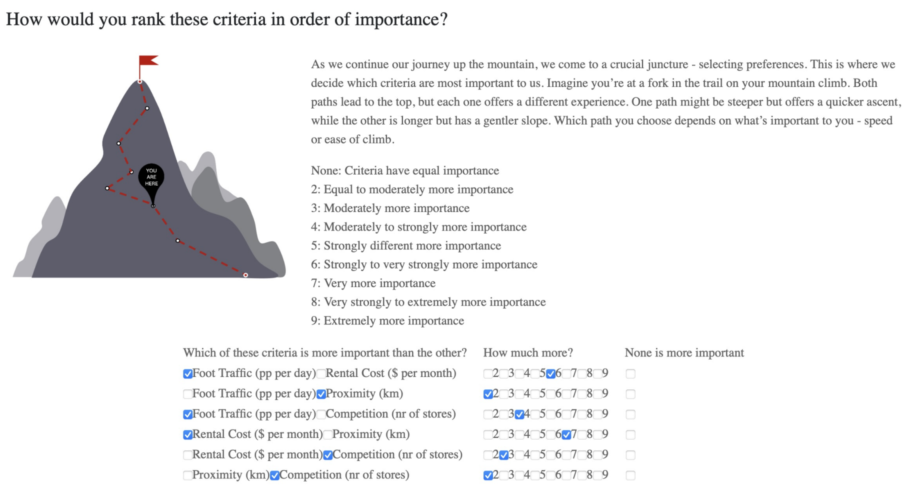


Figure 3.12: Insert criteria preferences.

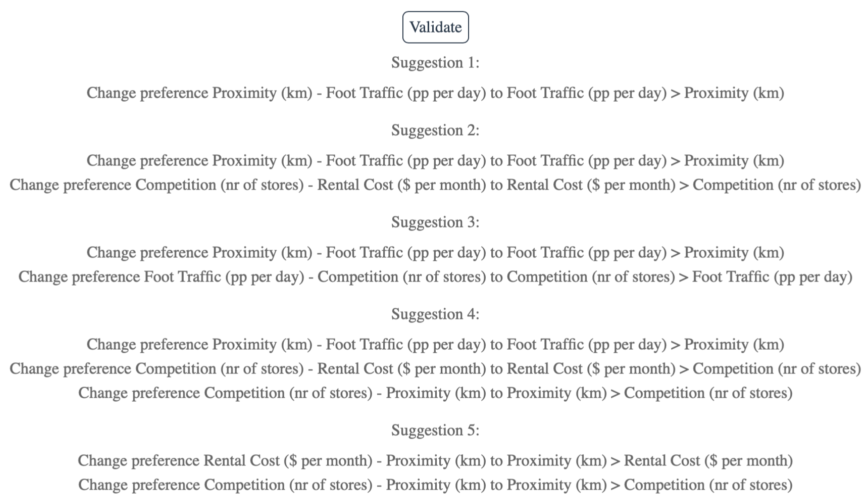


Figure 3.13: Suggestions for new preferences, if necessary.

On the next page (Figure 3.14), the user can understand how the problem is structured. The idea is that the area of each circle represents the sum of the preferences made on that criterion, enabling the user to identify and potentially remove criteria that do not hold any degree of importance. In fact, in the example figure, the "Proximity" criterion carries such little weight that it could be removed without significantly impacting the final decision.

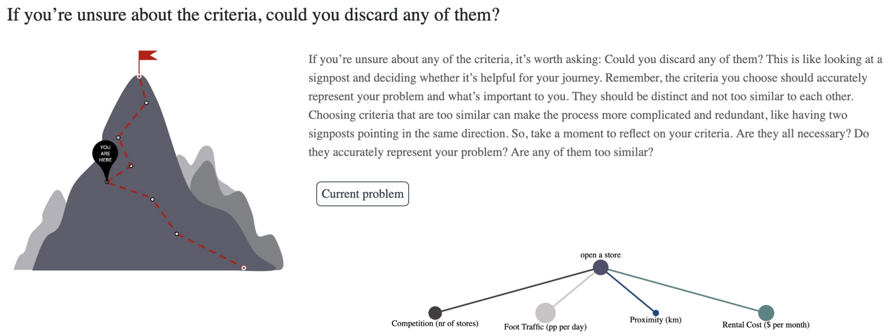


Figure 3.14: Criteria structure.

Another crucial piece of information that needs to be entered is the list of alternatives. In this context, alternatives represent the various paths the user can take to reach the top of the mountain. Therefore, the user needs to input the names of these alternatives (see Figure 3.15) and also their performance on each criterion (see Figure 3.16). Throughout this initial data entry phase, there is accompanying narrative that emphasizes the importance of providing consistent, realistic, and numerical information.

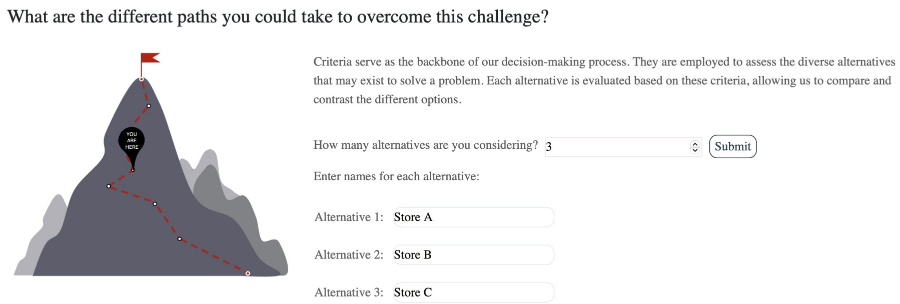


Figure 3.15: Insert alternatives.

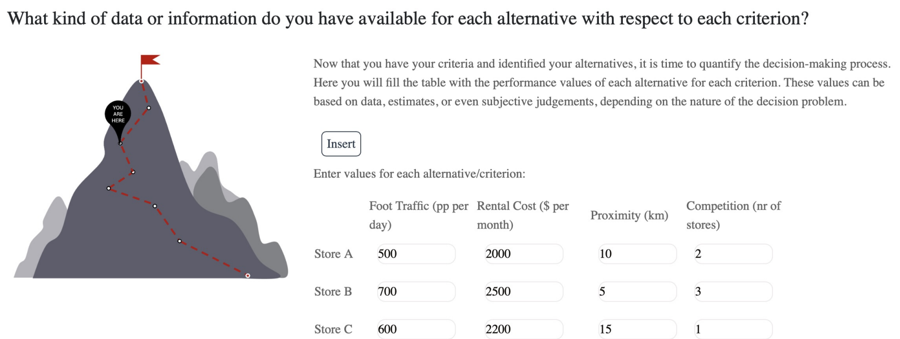


Figure 3.16: Insert alternatives performances.

Once this information is entered, the user can view how their problem is organized on the following page (see Figure 3.17). Here, the general map of their problem is presented with the problem statement at the top, followed by criteria and the performance of each alternative in each criterion. The size of each element reflects its importance or presence. This concise layout allows the user to see their entire problem described comprehensively, enabling them to proceed to the application of techniques.

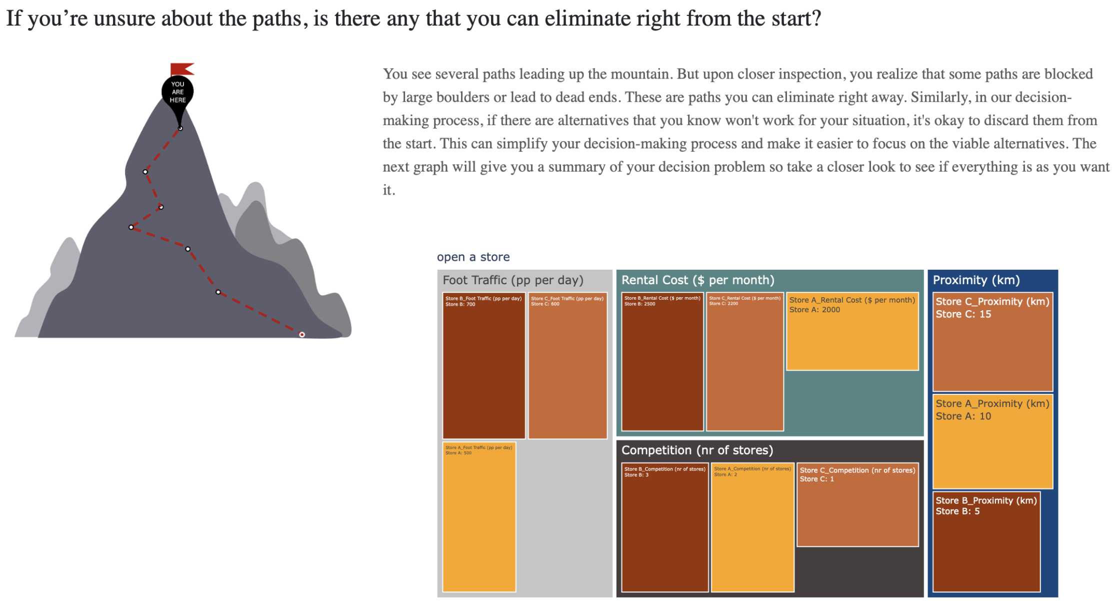


Figure 3.17: Summary of the problem.

In terms of using the AHP, the user can observe significant moments in the process. Specifically, they can see how their preferences are reflected in the matrix, as well as the associated Principal Eigenvalue and Eigenvectors (see Figure 3.18). These pieces of information are displayed because they are crucial steps in the AHP method, enabling the understanding of the associated consistency ratio. This ratio is further explained on the following page, where it is also detailed how it is obtained (see Figure 3.19).

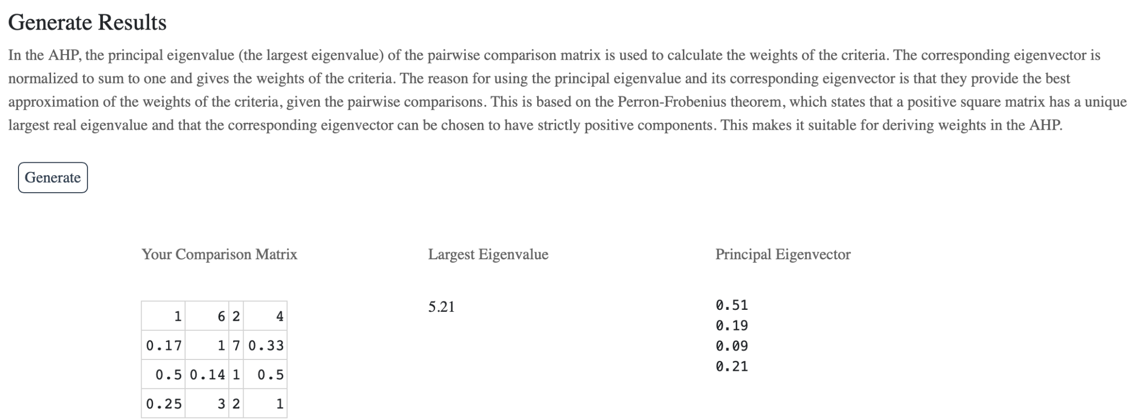


Figure 3.18: Initial phase of AHP performance.

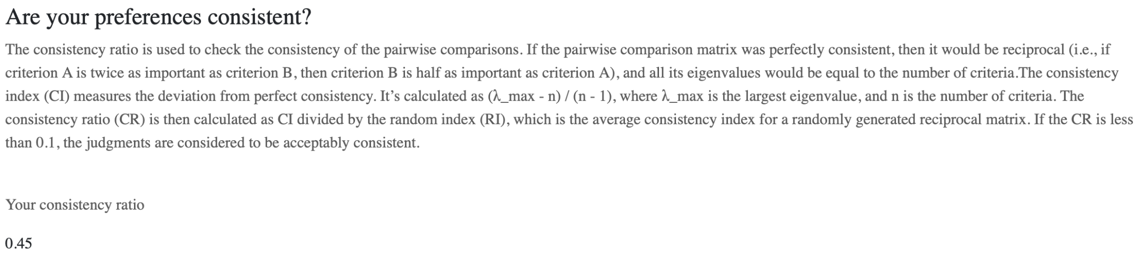


Figure 3.19: Consistency Ratio view.

Since the consistency ratio is not as desired (below 0.1), it becomes necessary to adjust the preferences until a consistent matrix is achieved. The user can track the modifications made to the matrix across various AHP iterations needed to achieve consistency through a bar chart (refer to Figure 3.20). By selecting two bars, the user can compare different iterations, understand the changes in the matrices, and comprehend their implications on the weights associated with the criteria (refer to Figure 3.21). Also, through a radar chart, it is possible to see if the associated weights have changed significantly after the AHP method has been applied. It is important to note that most of the time, the radar charts show very little variation from the first to the last iteration. This suggests that despite making adjustments in preferences and achieving consistency, the weights associated with the criteria do not change significantly from the user's initial intentions. Nonetheless, the user must select the weights associated with an iteration to proceed with the application of the TOPSIS method. It is advisable to choose the most consistent iteration so that the weights accurately reflect the user's preferences.

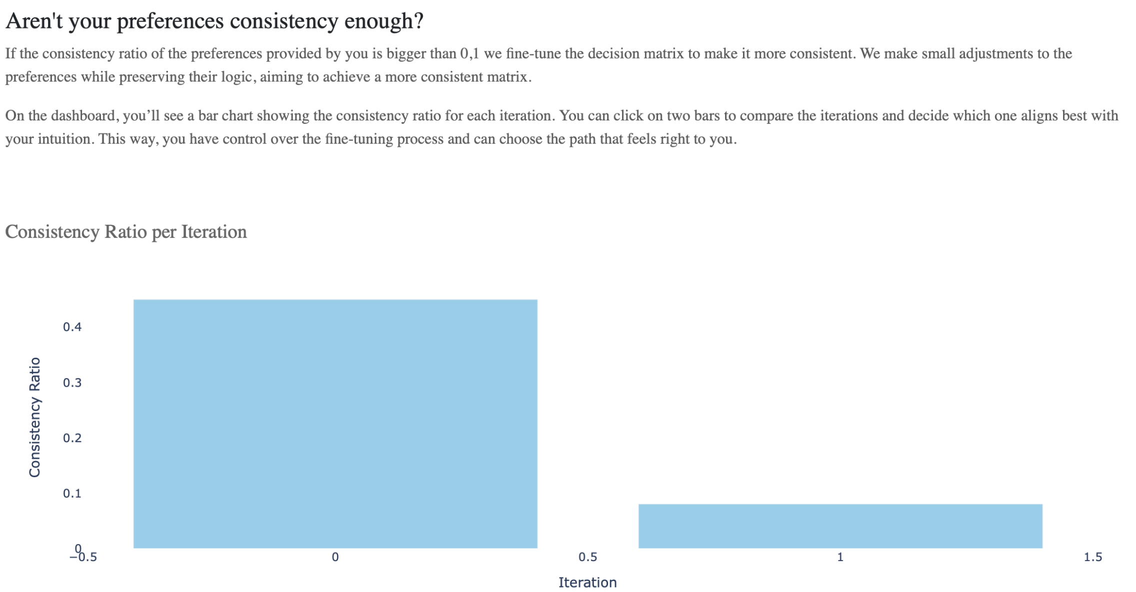


Figure 3.20: Consistency Ratio per iteration.

Figure 3.21: Comparison of iterations.

Upon completion of the AHP process, the user can finally view the final dashboard being constructed (refer to Figure 3.22). The dashboard allows the user to identify their criteria and alternatives, each represented with its respective color and abbreviated for easier identification in subsequent visualizations. A vertical bar chart displays the final preferences, where each bar's color represents the dominant criterion, and its size corresponds to the assigned value. Meanwhile, a horizontal bar chart depicts the final weights of each criterion, which will be used to rank the alternatives.

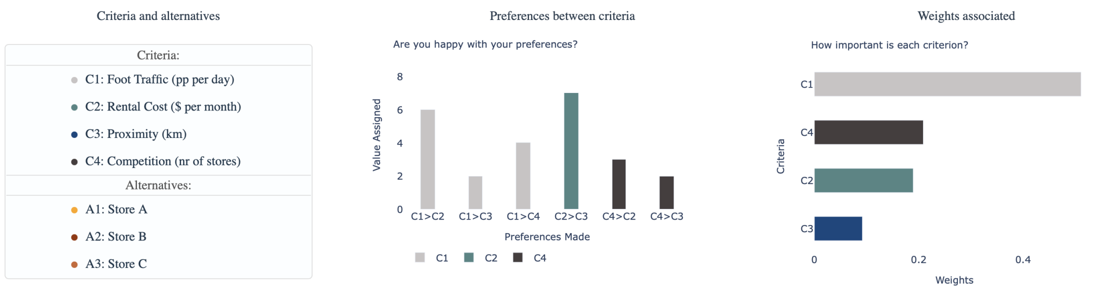
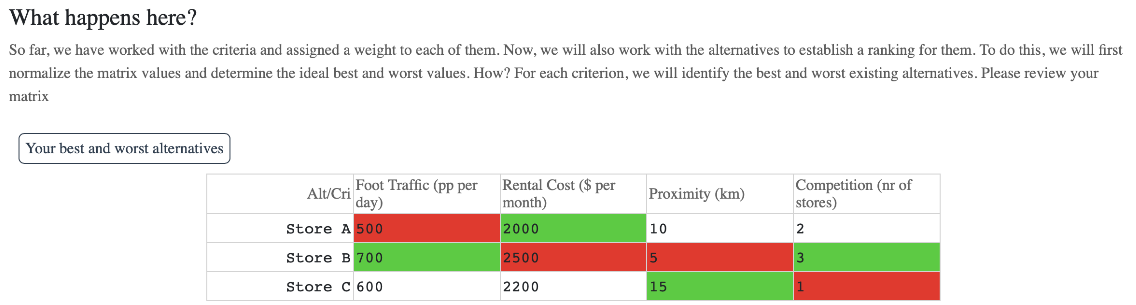


Figure 3.22: First half of the final dashboard.

Finally, with the obtained weights, the user can understand how their alternatives are evaluated. The initial visualization allows the user to comprehend how the TOPSIS method will rank their alternatives based on their distance from the ideal solution and the worst solution. Thus, the table in Figure 3.23 identifies the best and worst alternatives for each criterion, providing the user with crucial insights into the strengths and weaknesses of each alternative.



What happens here?

So far, we have worked with the criteria and assigned a weight to each of them. Now, we will also work with the alternatives to establish a ranking for them. To do this, we will first normalize the matrix values and determine the ideal best and worst values. How? For each criterion, we will identify the best and worst existing alternatives. Please review your matrix

Your best and worst alternatives

Alt/Cri	Foot Traffic (pp per day)	Rental Cost ($ per month)	Proximity (km)	Competition (nr of stores)
Store A	500	2000	10	2
Store B	700	2500	5	3
Store C	600	2200	15	1

Figure 3.23: Best and worst values identification.

With this setup complete, the user can fully construct the dashboard and interpret the three visualizations built with the steps of the TOPSIS method (see Figure 3.24). The first visualization displays the already weighted normalized values to show how each alternative deviates from the ideal solution and the worst solution. This visualization also helps the user understand if the alternatives are closely ranked, as illustrated in the example figure.

The second visualization is a radar chart that facilitates the analysis of each alternative's performance across criteria. Simultaneously, the horizontal bar chart displaying the weights (as described earlier) aids the user in understanding the most critical criteria. This dual approach allows for effective evaluation of each alternative within the radar chart.

The final visualization used in the dashboard is the ranking of alternatives, which guides the user in making a decision. Alternatives are presented in ascending order of their assigned ranking, with the highest-ranked alternative positioned at the top, indicating it as the recommended choice. Additionally, alternatives are shifted to the right if they are closer to the ideal solution and to the left if they are closer to the worst solution. This visual arrangement helps users distinguish the relative performance of each alternative. In case of a tie in the ranking, the user can then choose the alternative they prefer, supported by additional information provided by their positions relative to the ideal and worst solutions.

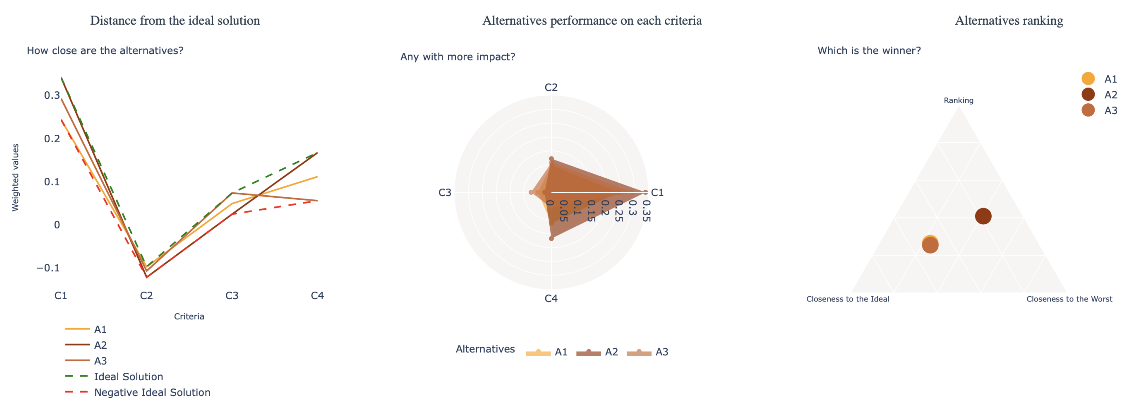


Figure 3.24: Second half of the final dashboard.

We believe that the initial storytelling for inputting the necessary information, combined with the tool's graphical and interactive features, is essential. The layout allows users to revisit important parts of the process easily. The visualizations used throughout the AHP and TOPSIS stages, along with the automatic adjustment of preferences—where users can accept or reject suggestions—are all critical components. Together, these elements provide a simple and engaging user experience that enables informed decision-making. Complex problems are explained clearly and simply, helping any user make a well-informed decision.

3.3 Discussion

In developing our visualization tool, we made a comprehensive exploration of existing software applications within the MCDM community. Our investigation revealed a variety of tools, each with unique visualizations and techniques. We particularly appreciated the visual support provided by FITradeoff and Entscheidungsnavi, as they offered compelling visualizations to accompany their techniques. These tools demonstrated the power of visual aids in enhancing user understanding and engagement, despite not utilizing AHP or TOPSIS methods. The remaining tools not fully addressed the need for a user-friendly decision support system that incorporates storytelling elements.

Our contribution is a visualization tool that integrates storytelling with AHP and TOPSIS methodologies, aiming to mitigate the challenges associated with existing software. We developed a dashboard that not only facilitates the input of relevant data and criteria but also guides users through a structured decision-making process. The tool's interactive nature allows users to adjust their preferences until a consistent matrix is obtained, ensuring that the weights of criteria accurately reflect their intentions.

The storytelling aspect of our dashboard is crucial, as it narrates the decision-making journey, making complex processes more accessible. We believe that this narrative approach, combined with the tool's graphical and interactive features, is essential for a simple and engaging user experience that enables informed decision-making. By breaking down complex problems and presenting them clearly, we help users make well-informed decisions, regardless of their familiarity with MCDM techniques. Our tool stands out by suggesting new preferences in AHP, a feature not commonly found in current software, and by implementing TOPSIS in a way that enhances user comprehension and trust in the results.

Chapter 4

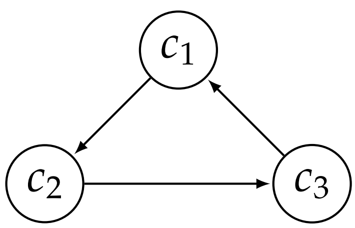
A new method for adjusting preferences

In this chapter, we present the second contribution of our thesis: a new method for adjusting user preferences when they are inconsistent. To achieve this, we first reflect on the problem of inconsistencies and discuss the most commonly used methods to adjust user preferences, as outlined in Section 4.1. Next, Section 4.2 introduces the method we developed to address this issue. Finally, Section 4.3 presents the results and compares them with the methods discussed in the literature.

4.1 The existing methods to adjust preferences

Since human judgments and preferences are often inaccurate and subsequently inconsistent, decision makers have to handle inconsistencies in the evaluation procedure [Magnot et al., 2023].

AHP uses user preferences on pairs of criteria to construct a pairwise comparison matrix. It then calculates the principal eigenvector and determines the consistency ratio. If the consistency ratio exceeds 0.1, the matrix is considered inconsistent, and the DM is advised to reconsider their preferences. There are several reasons for this to happen, one of which is intransitivities in the preferences, known as ordinal inconsistencies. This could happen if for example the DM judged criterion c_1 to be more preferable than criterion c_2 and criterion c_2 to be more preferable than criterion c_3, but then judged criterion c_3 to be more preferable than criterion c_1 (see the next graph).



On the other hand, the value assigned to a preference significantly influences the calculation of the consistency ratio. This can lead to inefficiencies as decision-makers may need to continually adjust their preferences to achieve a consistent matrix. Furthermore, the process of adjusting preferences can be time-consuming, especially when dealing with a large number of criteria, and considering that the number of necessary preferences is given by $\binom{n}{2}$. Given that achieving consistency is crucial for the weights returned by the AHP to accurately reflect the user's preferences, there is a clear need for an automated method to adjust preferences. This method should aim to achieve a consistent matrix while maintaining the minimum possible distance from the initial matrix provided by the user.

Fortunately, there have been numerous attempts to adjust preferences using various methods (see Table 4.1).

Table 4.1: Methods to adjust DM's preferences.

Paper	**Method**
[Saaty, 2003]	Perron Eigenvalue
[Li and Ma, 2007]	Gower plot
[Cao et al., 2008]	Heuristic
[Benitez et al., 2010]	Linearization
[Benítez et al., 2011]	Linearization
[Ergu et al., 2011]	Inconsistent elements identification
[Bozóki et al., 2015]	Nonlinear mixed-integer optimization
[Magnot et al., 2023]	Gradient

In [Saaty, 2003], it is proven that calculating the $n(n-1)/2$ values $v_i w_j - a_{ji}^2 v_j w_i$, where $v = [v_i]$ is the unique positive eigenvector of the normalized positive matrix A^T and $w = (w_1, ..., w_n)$ are the weights from the eigenvector associated with λ_{max}, identifies the entry of A whose adjustment maximizes the rate of change of λ_{max}, thus highlighting the element causing inconsistency (see Appendix A for details). Following this, it is suggested to the user to change the preference of the cell whose calculated value is greater in absolute value. However, this suggested change can be too abrupt. For instance, we might go from a preference where the user initially decided that criterion 1 was extremely more important than criterion 2, to criterion 2 being strongly more important than criterion 1. This could be highly undesirable for a decision-maker.

Another approach to improve the consistency of the pairwise comparison matrix is developed in [Li and Ma, 2007]. This method uses a Gower plot to visually identify both ordinal and cardinal inconsistencies in a preference matrix. Ordinal inconsistencies occur when the preference matrix lacks transitivity, while cardinal inconsistencies occur when for some i, j, $k \in \{1, 2, \ldots, n\}$ there exists

$a_{ik} \neq a_{ij} \times a_{jk}$, with n is the number of criteria and $i \neq j \neq k$. In the Gower plot, if the points representing alternatives do not form a semi-circle or have unequal angles between them, it indicates the presence of ordinal inconsistencies. For instance, in Figure 4.1, we observe ordinal inconsistency (a) as A_4 falls outside the semi-circle, and cardinal inconsistencies (b) since the points are not perfectly aligned with the line. On the other hand, in Figure 4.2, there are no ordinal inconsistencies in (a), but cardinal inconsistencies (b) are recorded. The method then applies optimization models to suggest adjustments. The decision maker interacts with the process, making iterative revisions based on these suggestions until a consistent ranking of alternatives is achieved. In (c), the points are perfectly aligned with the line.

However, the decision maker must actively participate in the iterative revision process. This can be time consuming and may lead to subjective adjustments that deviate from the original judgments. Therefore, there is a need for improvement of a method that automatically suggests the nearest consistent matrix to the one given by the user.

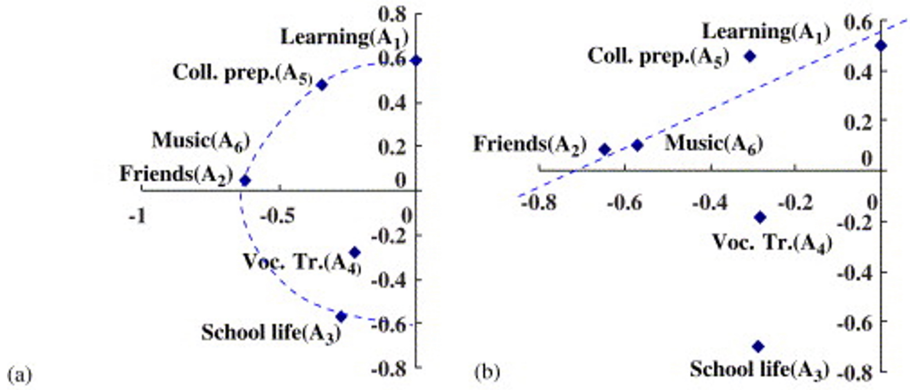


Figure 4.1: Ordinal and cardinal inconsistencies found [Li and Ma, 2007].

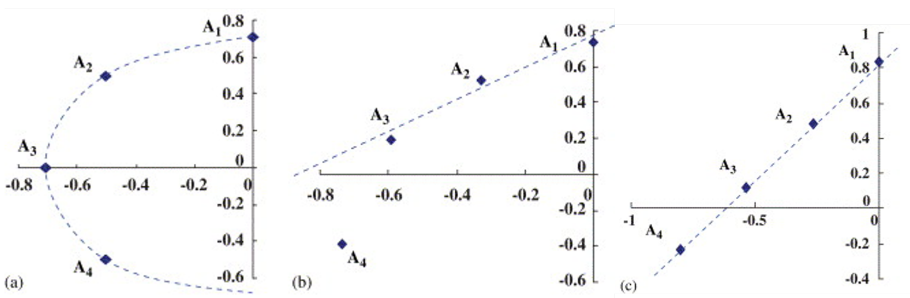


Figure 4.2: Cardinal inconsistencies found in (a) and (b), and solved in (c) [Li and Ma, 2007].

In the study by [Cao et al., 2008], a method is proposed that automatically generates a consistent matrix. This is achieved by representing the inconsistent matrix as a deviation matrix and iteratively adjusting it to enhance the consistency ratio. The degree of preservation is evaluated using two distinct metrics. The first, denoted as maximum difference (4.1) quantifies the maximum disparity between the entries of the original and modified matrix preferences. The second metric, referred to as the sum of squared differences (4.2), computes the root mean square of the differences between the entries of the original and modified matrices, thereby providing a measure of overall variation. Note that $a_{ij}^{(m)}$ represents the element a_{ij} of the modified matrix m. While the results are interesting when compared with other existing methods in the literature, it is important to note that the effectiveness of this method can significantly depend on the parameter value selected for the modification process.

$$\delta = \max_{i,j} |a_{ij}^{(m)} - a_{ij}| \tag{4.1}$$

$$\sigma = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij}^{(m)} - a_{ij}\right)^2 / n} \tag{4.2}$$

Benitez on [Benitez et al., 2010] and [Benítez et al., 2011] introduces a linearization process. This process begins with a logarithmic transformation of the matrix. Following this transformation, the matrix is projected onto a linear space using the Frobenius norm. The goal of this projection is to identify the matrix that is most consistent with a given non-consistent matrix. The authors argue that this approach discard the need for iterative techniques, which they deem to be inefficient. However, it is important to note that the resulting consistent matrix must be validated through sensitivity analyses and feedback. This implies that the method, while promising, may not always yield an optimal result.

Another approach is developed in [Ergu et al., 2011]. This method begins by identifying the location of the inconsistent element with the largest absolute value in the induced pairwise comparison matrix. Following this, potential inconsistent elements are identified using the bias identifying vector. Only then are the inconsistent elements identified through various methods. Despite being a direct application method that is mathematically well-supported and effectively reveals inconsistent elements, it encounters problems when many criteria are inserted. It requires the decision-maker to adjust their preferences according to the identified inconsistent elements. This approach, while robust, necessitates careful consideration and adjustment based on the identified inconsistencies.

Approaching this issue from an optimization perspective, [Bozóki et al., 2015] aimed to either identify the minimal number of matrix elements that need alteration to achieve consistency, or to attain the lowest level of inconsistency by modifying a limited number of elements. In both scenarios, the solution is derived from nonlinear mixed-integer optimization methods. These methods initially transform the problems into convex optimization problems in the logarithmic space, allowing them to be solved using available software and methods. However, the use of this method can lead to multiple consistent matrices. This can complicate the decision-making process regarding which elements to modify, as the final decision on revisions relies on the evaluator.

Finally, the most recent strategy developed in [Magnot et al., 2023] applies a mathematically robust minimization method to reduce inconsistency in pairwise comparison matrices. This method involves adjusting the matrix elements in the direction of the gradient of an inconsistency indicator. This direction points towards the most efficient path to achieve a more consistent matrix. However, this method can be mathematically complex and may require an advanced understanding of calculus and optimization. This complexity can pose challenges for a decision-maker who is not familiar with these concepts. Additionally, the outcome is sensitive to the choice of inconsistency indicator, which can lead to different results.

4.2 A new approach for adjusting preferences

The AHP-TOPSIS combination works exceptionally well when the preference matrix provided by the user is consistent. However, as we have seen, human imprecision often leads to inconsistencies in preferences, complicating the method's procedure. Although there are already several techniques for adjusting preferences, the final results often deviate significantly from the user's decisions or are overly complex, making them difficult for the user to follow. Therefore, there is a need for an automatic adjustment method that maintains the integrity of the user's original preferences to the greatest extent possible.

In that context, we propose a method for adjusting user preferences when they are inconsistent, until a consistency ratio less than 0.1 is achieved. The method is succinctly represented on Figure 4.3. The user defines their preferences for each pair of criteria. The method checks if there are any ordinal inconsistencies in the preferences. If ordinal inconsistencies are found, it solves the Minimum Feedback Arc Set problem to remove the inconsistencies. Then it checks for cardinal inconsistencies, so it calculates the consistency ratio, and if it is higher than 0.1, the method uses a perturbation map to adjust the preference values. Once all inconsistencies are resolved, the method determines the weights for each criterion. These weights can then be used in TOPSIS to rank the alternatives.

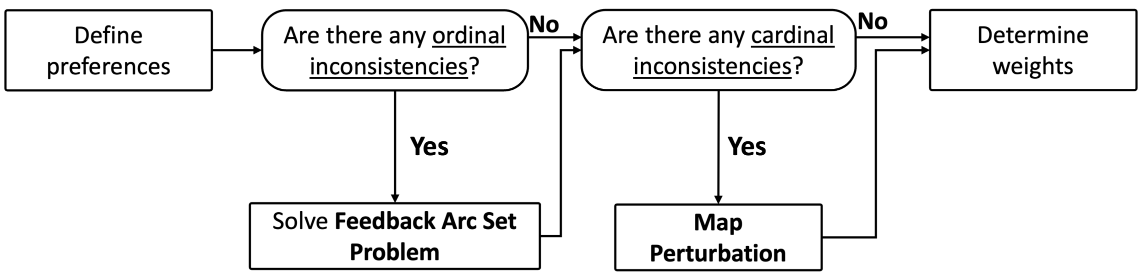


Figure 4.3: Framework of our proposed method.

Ordinal Inconsistencies

To formalize our method, let us denote the set of n criteria as $C = \{c_1, c_2, \ldots, c_n\}$. The first step is to define the preference matrix A as described for AHP. Then, we construct a weighted directed graph $G(V, E)$, where each vertex in V corresponds to a criterion in C, and each edge in E corresponds to the preferences a_{ij} where $i \neq j$ and where $a_{ij} \geq 1$. Therefore, the weight of each edge (i, j) is equal to a_{ij}, and the direction of the edge represents the preference direction, that is, if criterion i is more important than criterion j, the edge is directed from i to j. It is important to note that if the user decides that two criteria have the same importance, the associated weight is 1, and the edge is bidirectional.

Once the preferences are represented in the graph, we look for cycles. If a cycle is detected, it indicates the presence of intransitivities (ordinal inconsistencies) that need to be resolved.

Resolving the intransitivities is equivalent to solving the Minimum Feedback Arc Set (MFAS) problem, which aims to find the smallest set of edges with the least total weight that, when reversed, makes the graph acyclic. To the best of our knowledge, solving the MFAS has never been used to eliminate inconsistencies in preference matrices, such as in the AHP. This approach is innovative, likely because the MFAS is an NP-hard problem, thus presenting a limitation. However, as we are dealing with a relatively small number of criteria, solving it turns out to be a good idea.

There are numerous methods in the literature to solve the MFAS problem. However, since we are dealing with a small number of criteria, we can solve it using enumeration or an integer programming model.

For enumeration, the idea is to linearize the graph by considering all possible combinations of criteria. For each combination, we draw the edges forward or backward based on their direction. We also recognize that we do not need to perform $n!$ iterations to calculate all possible permutations. For each permutation, the inverse permutation has the "backward" edges reversed, and the total weight is the total weight of the graph minus the total feedback weight of the permutation of which this is the inverse. To better understand the enumeration process, we provide an example later in the text. After enumerating all possible permutations, we decided to select the top five to suggest to the user. These suggestions involve reversing the direction of the backward edges.

However, a better way to look at this problem is to consider the integer programming formulation with triangle inequalities provided in [Baharev et al., 2021].

The goal is to find the minimum cost ordering p^* of the nodes in $G = (V, E)$. Let c_{ij} represent the cost associated with the directed edge $(i, j) \in E$, and $c_{ij} = 0$ if $(i, j) \notin E$. To minimize the cardinality of the feedback edge set, we set c_{ij} equal to the weight of the edge (i, j). Additionally, let the binary variables y_{ij} associated with a given ordering p be defined as follows: $y_{ij} = 0$ if node i precedes node j in p, and $y_{ij} = 1$ otherwise. Each ordering p uniquely determines a corresponding y. This leads to the following integer programming formulation:

$$
\begin{aligned}
\min_y \sum_{j=1}^{n} & \left(\sum_{k=1}^{j-1} c_{kj} y_{kj} + \sum_{\ell=j+1}^{n} c_{\ell j}(1 - y_{j\ell}) \right) \\
\text{subject to} \quad & y_{ij} + y_{jk} - y_{ik} \leq 1, \quad 1 \leq i < j < k \leq n \\
& - y_{ij} - y_{jk} + y_{ik} \leq 0, \quad 1 \leq i < j < k \leq n \\
& y_{ij} \in \{0, 1\}, \quad 1 \leq i < j \leq n.
\end{aligned}
$$

The idea behind this formulation is to halve the number of variables, given that the objective function of the model is

$$\min_y \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} y_{ij} \text{ for } i \neq j.$$

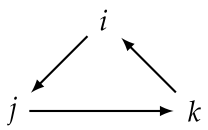
However, in this case, i can be divided into $k + l$, which allows us to obtain

$$\min_y \sum_{j=1}^{n} \left(\sum_{k=1}^{j-1} c_{kj} y_{kj} + \sum_{\ell=j+1}^{n} c_{\ell j} y_{\ell j} \right).$$

Thus, as $y_{ij} = 1 - y_{ji}$, we can obtain the final formulation:

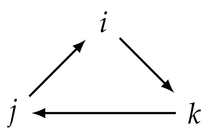
$$\min_y \sum_{j=1}^{n} \left(\sum_{k=1}^{j-1} c_{kj} y_{kj} + \sum_{\ell=j+1}^{n} c_{\ell j}(1 - y_{j\ell}) \right).$$

Regarding the constraints, the idea of using triangular inequalities is to prevent cycles of the form:



Thus: $y_{ij} + y_{jk} - y_{ik} \leq 1$

And



Thus: $-y_{ij} - y_{jk} + y_{ik} \leq 0$

Any y that satisfies the triangle inequalities must correspond to an ordering [Baharev et al., 2021; Harker, 1987].

To solve this problem we use Gurobi, which is an optimization solver used to find optimal solutions for various mathematical problems, including linear programming and mixed-integer programming. As we aim to generate five suggestions, it is necessary to utilize Gurobi's parameters, namely *PoolSolutions* and *PoolSearchMode*, to search for multiple solutions. With this approach, the first solution found is likely to be the optimal solution or very close to it. The subsequent solutions may not be optimal, but they represent the best alternatives that Gurobi identified following the discovery of the optimal solution.

Directly, the output of the solution does not allow us to make suggestions as it only returns the indication of the position of any node i in relation to node j. Therefore, to make a suggestion, we need to know which edges have j before i ($y_{ij} = 1$), and whether this edge is in the preferences made by the user. If it is, it indicates that the edge in this permutation is behind and is suggested to be changed.

Any suggestion that the decision-maker accepts will break the intransitivity in their preferences, thus allowing them to proceed to the analysis of cardinal inconsistencies.

Cardinal Inconsistencies

After resolving the intransitivities, we analyze the consistency of the preferences. If the preferences are still inconsistent, it indicates the presence of cardinal inconsistencies. To handle this, we will initially use the method described in Appendix A to detect the cell that may be causing the most inconsistency in the matrix. Thus, we consider w the vector of the weights, and v the unique positive eigenvector of the positive matrix A^T that is normalized so that $v^T w = 1$, and we use the $\frac{n(n-1)}{2}$ values $\left\{ v_i w_j - \frac{v_j w_i}{a_{ji}^2} \right\}$ and select any one with the largest absolute value. We then adjust the preferences in that cell using the map (Table 4.2). The concept behind this map is that the preferences, when altered, should not deviate significantly. In other words, for each element, the maximum change was chosen to be 2. The algorithm replace the preference value with all corresponding values, calculate the CR, and choose the one that reduces the CR the most. Next, we move to the cell with the second largest absolute value and repeat the calculation and selection process. This process is repeated until we obtain a matrix with a CR less than 0.1.

Table 4.2: Map values for perturbations

Values	1	2	3	4	5	6	7	8	9
Changes	1	3, 4	2, 4, 5	2, 3, 5, 6	3, 4, 6, 7	4, 5, 7, 8	5, 6, 8, 9	6, 7, 9	7, 8

Once the algorithm finds a matrix with a consistency ratio less than 0.1, the user can finally proceed with the multi-criteria analysis process using TOPSIS. Our method of adjusting preferences is thus completed.

Practical Example

To illustrate the application of this method, we will walk through a practical example. Let us consider a decision problem with three criteria c_1, c_2 and c_3. The decision-maker expresses the following preferences based in Table 2.3:

1. c_1 is preferred over c_2 with value assigned equal to 4

2. c_3 is preferred over c_1 with value assigned equal to 6

3. c_2 is preferred over c_3 with value assigned equal to 2

When we incorporate these preferences into the comparison matrix, it appears as follows:

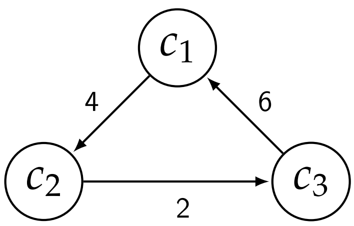
	c_1	c_2	c_3
c_1	1	4	1/6
c_2	1/4	1	2
c_3	6	1/2	1

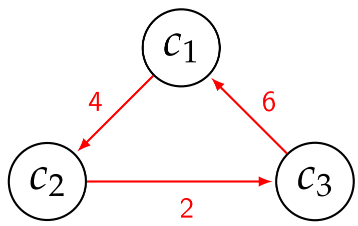
Upon calculating the principal eigenvalue, we obtain a value of 4.91. This leads us to the calculation of the consistency ratio. Given $RI = 0.58$ for $n = 3$ (see Table 2.4) we have:

$$CI = \frac{\lambda_{max} - n}{n - 1} = \frac{4.91 - 3}{3 - 1} = 0.955,$$

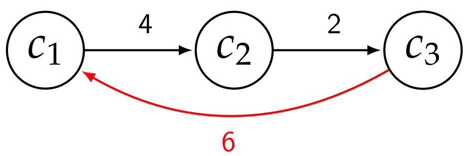
$$CR = \frac{CI}{RI} = \frac{0.955}{0.58} = 1.65.$$

Given that the consistency ratio is greater than 0.1 $(CR = 1.65)$, it indicates that the matrix is inconsistent. The first thing we will do is to understand if the user's preferences do not have any type of intransitivity, that is, ordinal inconsistencies. To achieve this, we will transform the user's preferences into a complete graph and search for cycles. The presence of cycles indicates intransitivities in the preferences.

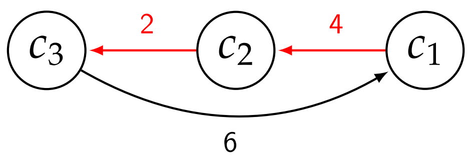




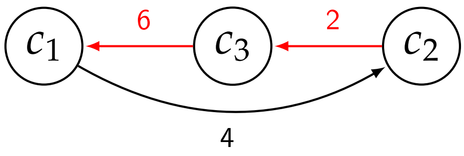
As we can see from the graph above, the preferences form a cycle, indicating intransitivity and explaining the high consistency ratio. In order to decide which preferences should be altered, we can solve the MFAS problem exactly, through enumeration.



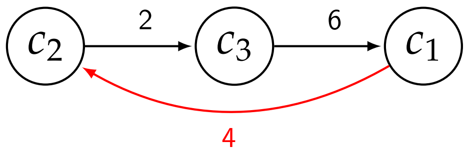
Total feedback weight = 6



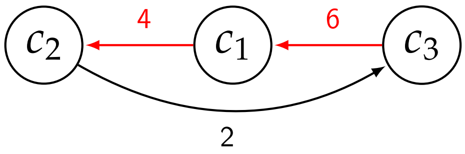
Total feedback weight = 6



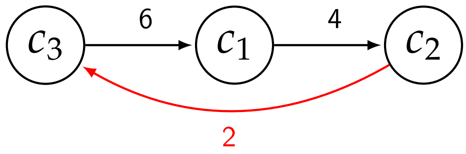
Total feedback weight = 8



Total feedback weight = 4



Total feedback weight = 10



Total feedback weight = 2

Therefore, in this case, the suggestions to the user are as follows:

1. Suggestion 1: c_3 should be preferred over c_2 with value assigned equal to 2
2. Suggestion 2: c_2 should be preferred over c_1 with value assigned equal to 2
3. Suggestion 3: c_1 should be preferred over c_3 with value assigned equal to 2
4. Suggestion 4: c_3 should be preferred over c_2 with value assigned equal to 2 and c_2 should be preferred over c_1 with value assigned equal to 2
5. Suggestion 5: c_1 should be preferred over c_3 with value assigned equal to 2 and c_3 should be preferred over c_2 with value assigned equal to 2

However, as mentioned, a more robust and widely accepted approach in the multi-criteria community is to use the mathematical formulation described above. After defining the model, Gurobi's optimization algorithm find the optimal solution, where suggestions returned by the model for the example we are working on were exactly the same, as expected.

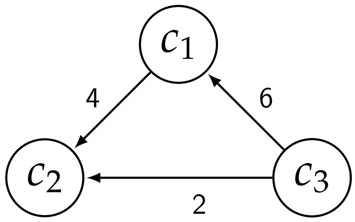
Let us assume then that the decision-maker accepts the first suggestion, their preferences would then become:

1. c_1 is preferred over c_2 with value assigned equal to 4
2. c_3 is preferred over c_1 with value assigned equal to 6
3. c_3 is preferred over c_2 with value assigned equal to 2

Now the comparison matrix appears as follows:

	c_1	c_2	c_3
c_1	1	4	1/6
c_2	1/4	1	1/2
c_3	6	2	1

The consistency ratio associated is 0.63, indicating that there are still cardinal inconsistencies, as there are no ordinal inconsistencies:



Thus, for the matrix

	c_1	c_2	c_3
c_1	1	4	1/6
c_2	1/4	1	1/2
c_3	6	2	1

we have $w = \begin{bmatrix} 0.24 \\ 0.14 \\ 0.62 \end{bmatrix}$ and $v = \begin{bmatrix} 0.31 \\ 0.55 \\ 0.12 \end{bmatrix}$.

Now we are in conditions to use the $\frac{n(n-1)}{2}$ values $\left\{ v_i w_j - \frac{v_j w_i}{a_{ji}^2} \right\}$ and select any one with the largest absolute value:

$$\begin{bmatrix} 1 & 0.035 & \mathbf{-0.844} \\ & 1 & 0.273 \\ & & 1 \end{bmatrix}$$

Then, we go to that cell and we use the map to adjust preferences on that cell (Table 4.2).

So, given that the value is 6, we will examine all the elements of the map corresponding to 6 to determine if the consistency ratio is already acceptable. If not, we will select the one that has reduced the consistency ratio the most and move on to the cell with the second highest absolute value. We will continue this process until we find a matrix with a consistency ratio less than 0.1. Thus, we obtain:

$$\text{Initial matrix: } \begin{bmatrix} 1 & 4 & 1/6 \\ 1/4 & 1 & 1/2 \\ 6 & 2 & 1 \end{bmatrix} \quad \text{Consistency Ratio: } 0.63$$

$$\text{First change: } \begin{bmatrix} 1 & 4 & \mathbf{1/4} \\ 1/4 & 1 & 1/2 \\ \mathbf{4} & 2 & 1 \end{bmatrix} \quad \text{Consistency Ratio: } 0.43$$

$$\text{Second change: } \begin{bmatrix} 1 & 4 & 1/4 \\ 1/4 & 1 & \mathbf{1/4} \\ 4 & \mathbf{4} & 1 \end{bmatrix} \quad \text{Consistency Ratio: } 0.19$$

$$\text{Third change: } \begin{bmatrix} 1 & \mathbf{2} & 1/4 \\ \mathbf{1/2} & 1 & 1/4 \\ 4 & 4 & 1 \end{bmatrix} \quad \text{Consistency Ratio: } 0.05$$

With the consistent matrix, the user can confidently proceed to TOPSIS, assured by the robustness of the method.

4.3 Comparison with the literature

Regarding the method presented for adjusting user preferences, it is crucial to note that users are not obligated to accept our suggestions; they retain the freedom to decline, as is typical in preference adjustment methodologies. The goal is to assess the effectiveness of this methodology in aligning with the user's originally decided preferences. We will employ the metrics (4.1) and (4.2) introduced in [Cao et al., 2008] to compare the results obtained in previous studies with those achieved using our approach on the same problem. This comparison will provide a clear understanding of the effectiveness of our method.

Benitez et al. [2010]

Let us consider that, after evaluation and following Saaty's scale, we obtain matrix A as presented in Table 4.3. This matrix reflects the opinions of a panel of experts regarding the relative importance among seven criteria, as discussed in [Benitez et al., 2010].

	C1	C2	C3	C4	C5	C6	C7
C1	1	1/3	1/5	1	1/4	2	3
C2	3	1	1/2	2	1/3	3	3
C3	5	2	1	4	5	6	5
C4	1	1/2	1/4	1	1/4	1	2
C5	4	3	1/5	4	1	3	1
C6	1/2	1/3	1/6	1	1/3	1	1/3
C7	1/3	1/3	1/5	1/2	1	3	1

Table 4.3: Matrix of preferences [Benitez et al., 2010]

This matrix is inconsistent, as indicated by the consistency ratio of 0.11. Therefore, it is necessary to apply a method to make it consistent.

The solution obtained in [Benitez et al., 2010] is given in Table 4.4.

For this consistent matrix, the closeness metrics results are:

$$\delta = \max_{i,j} |a_{ij}^{(m)} - a_{ij}| = 5.28$$

$$\sigma = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij}^{(m)} - a_{ij} \right)^2 / n} = 1.19$$

With our approach, we obtain the matrix given in Table 4.5.

	C1	C2	C3	C4	C5	C6	C7
C1	1	0.526	0.154	0.794	0.471	1.738	1.17
C2	1.902	1	0.293	1.51	0.896	3.306	2.225
C3	6.487	3.411	1	5.149	3.055	11.28	7.59
C4	1.26	0.662	0.194	1	0.593	2.19	1.474
C5	2.123	1.116	0.327	1.685	1	3.691	2.484
C6	0.575	0.302	0.089	0.457	0.271	1	0.673
C7	0.855	0.449	0.132	0.678	0.403	1.486	1

Table 4.4: Consistent matrix closest to the preferences obtained in [Benitez et al., 2010].

	C1	C2	C3	C4	C5	C6	C7
C1	1	1/3	1/5	1	1/4	2	3
C2	3	1	1/2	2	**1/2**	3	3
C3	5	2	1	4	5	6	5
C4	1	1/2	1/4	1	1/4	1	2
C5	4	**2**	1/5	4	1	3	1
C6	1/2	1/3	1/6	1	1/3	1	1/3
C7	1/3	1/3	1/5	1/2	1	3	1

Table 4.5: Consistent matrix closest to [Benitez et al., 2010] obtained by our approach

For our consistent matrix, the closeness metrics results are:

$$\delta = \max_{i,j} |a_{ij}^{(m)} - a_{ij}| = 1$$

$$\sigma = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \left(a_{ij}^{(m)} - a_{ij}\right)^2 / n} = 0.14,$$

which are significantly better than those reported in the work by [Benitez et al., 2010]. This improvement can be attributed to the near-consistency of the matrix, where adjusting a single preference was sufficient to eliminate the inconsistency.

[Benítez et al., 2011]

The authors of the previous example improved their method using an iterative feedback process in [Benítez et al., 2011], applying their methodology to address a significant challenge for water supply managers: minimizing water loss. The criteria used to decide on the alternatives are multiple, but decision-makers consider the following:

- C1: planning development cost and its implementation;
- C2: damage to property and other service networks;
- C3: effects (cost or compensations) of supply disruptions;

- C4: inconveniences caused by closed or restricted streets;
- C5: water extractions (benefits for aquifers, wetlands, or rivers);
- C6: construction of tanks and reservoirs (environmental and recreational impacts);
- C7: CO2 emissions (related to energy used in pumping stations).

Upon evaluation using the Saaty's scale, the matrix in Table 4.6 was produced.

	C1	C2	C3	C4	C5	C6	C7
C1	1	7	9	5	7	5	3
C2	1/7	1	5	9	5	7	5
C3	1/9	1/5	1	7	3	7	3
C4	1/5	1/9	1/7	1	7	5	5
C5	1/7	1/5	1/3	1/7	1	9	7
C6	1/5	1/7	1/7	1/5	1/9	1	5
C7	1/3	1/5	1/3	1/5	1/7	1/5	1

Table 4.6: Matrix of preferences [Benítez et al., 2011].

This matrix is inconsistent since the consistency ratio is 0.59. The solution obtained in [Benítez et al., 2011] is given in Table 4.7

	C1	C2	C3	C4	C5	C6	C7
C1	1	1.28	1.17	1.77	2.12	4.49	9.89
C2	0.78	1	0.92	1.39	1.66	3.52	7.74
C3	0.85	1.09	1	1.51	1.81	3.83	8.42
C4	0.57	0.72	0.66	1	1.20	2.54	5.59
C5	0.47	0.60	0.55	0.83	1	2.16	4.66
C6	0.22	0.28	0.26	0.39	0.47	1	2.20
C7	0.10	0.13	0.12	0.18	0.22	0.45	1

Table 4.7: Consistent matrix closest to the preferences obtained in [Benítez et al., 2011].

For this consistent matrix, the closeness metrics results are:

$$\delta = \max_{i,j} |a_{ij}^{(m)} - a_{ij}| = 7.83$$

$$\sigma = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij}^{(m)} - a_{ij}\right)^2 / n} = 3.05$$

Using our approach we obtain the matrix given in Table 4.8.

For our consistent matrix, the closeness metrics results are:

$$\delta = \max_{i,j} |a_{ij}^{(m)} - a_{ij}| = 5$$

	C1	C2	C3	C4	C5	C6	C7
C1	1	4	4	5	7	8	7
C2	1/4	1	2	4	5	7	5
C3	1/4	1/2	1	4	3	7	3
C4	1/5	1/4	1/4	1	4	5	2
C5	1/7	1/5	1/3	1/4	1	4	4
C6	1/8	1/7	1/7	1/5	1/4	1	1/2
C7	1/7	1/5	1/3	1/2	1/4	2	1

Table 4.8: Consistent matrix closest to [Benítez et al., 2011] obtained by our approach.

$$\sigma = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij}^{(m)} - a_{ij}\right)^2 / n} = 1.77,$$

which are better than those reported in [Benítez et al., 2011]. Furthermore, it is noted that we have maintained the preferences as in Table 2.3, whereas matrices obtained in both [Benitez et al., 2010] and [Benítez et al., 2011] present irrational values that may pose greater difficulty for the decision-maker to comprehend. Therefore, our approach ensures that preferences are preserved in a more intuitive and user-friendly format.

[Cao et al., 2008]

In [Cao et al., 2008], it is applied a heuristic designed to find a consistent matrix using a parameter γ. The closer γ is to 1, the more original information is retained. The authors applied this methodology to a situation where a company is selecting a trucking company to ship its goods. The selection of a trucking company is based on the performance of the following eight criteria: punctuality, delivery time, temperature control, track and trace, error rate, service reputation, damage or loss, and GPS features. The preference matrix obtained by the decision-makers is presented in Table 4.9.

	C1	C2	C3	C4	C5	C6	C7	C8
C1	1	5	3	7	6	1/3	1/4	1/5
C2	1/5	1	1/3	5	3	3	1/5	1/7
C3	1/3	3	1	6	3	4	1/5	1/6
C4	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8
C5	1/6	1/3	1/3	3	1	1/2	1/5	1/6
C6	1/6	1/3	1/4	4	2	1	1/5	1/6
C7	3	5	5/6	7	5	5	1	1/2
C8	4	7	5	8	6	6	2	1

Table 4.9: Matrix of preferences [Cao et al., 2008].

This matrix has a consistency ratio of 0.17. The authors examine solutions for two cases, one for $\gamma = 0.5$ and another for $\gamma = 0.98$. For $\gamma = 0.5$, the matrix is presented in Table 4.10.

	C1	C2	C3	C4	C5	C6	C7	C8
C1	1.0000	4.1027	1.9599	8.1951	5.7867	5.3817	0.5045	0.3375
C2	0.2437	1.0000	0.3101	4.0415	2.3693	2.2430	0.2472	0.1518
C3	0.5102	3.2244	1.0000	7.7000	4.0131	4.5138	3.5638	0.2954
C4	0.1220	0.2474	0.1299	1.0000	0.4190	0.3292	0.1239	0.0888
C5	0.1728	0.4221	0.2492	2.3866	1.0000	0.6309	0.1930	0.1299
C6	0.1858	0.4458	0.2215	3.0373	1.5850	1.0000	0.2085	0.1378
C7	1.9821	4.0455	0.2806	8.0710	5.1803	4.7966	1.0000	0.5004
C8	2.9629	6.5865	3.3856	11.2647	7.6971	7.2548	1.9986	1.0000

Table 4.10: Consistent matrix closest to the preferences obtained in [Cao et al., 2008] for $\gamma = 0.5$.

For $\gamma = 0.98$, the resulting matrix is presented in Table 4.11.

	C1	C2	C3	C4	C5	C6	C7	C8
C1	1.0000	4.4412	2.3682	7.6743	5.8559	5.6079	0.4201	0.2968
C2	0.2252	1.0000	0.3210	4.4224	2.6175	2.5392	0.2268	0.1486
C3	0.4223	3.1151	1.0000	6.9149	3.5351	4.2774	4.5105	0.2487
C4	0.1303	0.2261	0.1446	1.0000	0.3805	0.2927	0.1313	0.1030
C5	0.1708	0.3820	0.2829	2.6278	1.0000	0.5722	0.1960	0.1444
C6	0.1783	0.3938	0.2338	3.4166	1.7478	1.0000	0.2055	0.1494
C7	2.3804	4.4099	0.2217	7.6136	5.1012	4.8661	1.0000	0.5004
C8	3.3689	6.7293	4.0215	9.7130	6.9235	6.6949	1.9984	1.0000

Table 4.11: Consistent matrix closest to the preferences obtained in [Cao et al., 2008] for $\gamma = 0.98$

With our approach we obtain the matrix given in Table 4.12.

	C1	C2	C3	C4	C5	C6	C7	C8
C1	1	5	2	7	6	6	2	1/3
C2	0.2	1	1/3	5	3	3	1/3	1/7
C3	1/2	3	1	6	3	4	6	1/3
C4	1/7	1/5	1/6	1	1/3	1/4	1/7	1/9
C5	1/6	1/3	1/3	3	1	1/2	1/5	1/6
C6	1/6	1/3	1/4	4	2	1	1/5	1/6
C7	1/2	3	1/6	7	5	5	1	1/2
C8	3	7	3	9	6	6	2	1

Table 4.12: Consistent matrix closest to [Cao et al., 2008] obtained by our approach.

Regarding the results obtained, for $\gamma = 0.5$ the values for δ and σ were 3.265 and 0.7863, respectively. This is expected as γ was far from 1. For $\gamma = 0.98$, the values were 1.713 and 0.448 for δ and σ , respectively. However, as can be seen from the table, the resulting matrix has non-rational values, similar to the results seen in previous works. As for our results, the value for δ was 2.5 and for σ was 0.56, revealing better results than those obtained with $\gamma = 0.5$ and slightly worse than for $\gamma = 0.98$. However, for problems where preferences exhibit numerous cases of ordinal inconsistencies, the approach developed in [Cao et al., 2008] cannot effectively address these inconsistencies.

[Ergu et al., 2011]

In [Ergu et al., 2011], the authors identify and swap the most inconsistent elements of the matrix through a bias vector, applying their methodology to two examples. The first example pertains to the matrix in Table 4.13, which represents the preferences of a decision-maker regarding four criteria. This matrix is inconsistent, as the consistency ratio is 0.173

	C1	C2	C3	C4
C1	1	1/9	3	1/5
C2	9	1	5	2
C3	1/3	1/5	1	1/2
C4	5	1/2	2	1

Table 4.13: Matrix of preferences [Ergu et al., 2011].

Through their methodology, the element identified for modification corresponded to the element a_{13}, which was changed from 3 to 1/2, thus reversing the user's preference (see Table 4.14).

	C1	C2	C3	C4
C1	1	1/9	1/2	1/5
C2	9	1	5	2
C3	2	1/5	1	1/2
C4	5	1/2	2	1

Table 4.14: Consistent matrix closest to the preferences obtained in [Ergu et al., 2011].

For this consistent matrix, the closeness metrics results are: $\delta = 2.5$ and $\sigma = 0.75$.

If we use our approach we obtain the matrix given in Table 4.15.

	C1	C2	C3	C4
C1	1	1/7	3	1/3
C2	7	1	5	2
C3	1/3	1/5	1	1/3
C4	3	1/2	3	1

Table 4.15: Consistent matrix closest to [Ergu et al., 2011] obtained by our approach.

For our consistent matrix, the closeness metrics results are: $\delta = 2$, which is slightly better than those reported in the article, and $\sigma = 0,75$ which is the same of their work. However, since there are no ordinal inconsistencies in those preferences, we did not invert any preferences; we only made small perturbations.

Regarding the second example (see Table 4.16), it records a consistency ratio of 1.02, indicating a high degree of inconsistency.

	C1	C2	C3	C4
C1	1	2	4	1/8
C2	1/2	1	2	4
C3	1/4	1/2	1	2
C4	8	1/4	1/2	1

Table 4.16: Matrix of preferences, [Ergu et al., 2011], second example.

To solve this, the authors identified the most inconsistent elements again. In this case, there were two elements, so it was necessary to apply the algorithm twice and manually select the best solution, rather than relying on automatic selection. The element to be inverted was a_{14}, which changed from 1/8 to 8. Theoretically, what happened was that criterion 1 (C1) ceased to be 8 times more important than criterion 4 (C4), which was the preference chosen by the user. At the end of the algorithm, the suggestion was that C4 is 8 times more important than C1. Hence, the value of δ was 7.88 and the value of σ was 2.78.

With our approach, we first solved the ordinal inconsistencies by reversing the preferences (C1,C2) and (C3,C4), which together weigh less than (C1,C4), thus breaking the cycles. This allowed us to make small perturbations until we found the matrix presented in Table 4.17.

	C1	C2	C3	C4
C1	1	1/3	2	1/6
C2	3	1	4	2
C3	1/2	1/4	1	1/3
C4	6	1/2	3	1

Table 4.17: Consistent matrix closest to the second example of [Ergu et al., 2011] obtained by our approach.

For our consistent matrix, the closeness metrics results are: $\delta = 2.5$ and $\sigma = 1.46$, which are both better than those reported in the article, demonstrating the potential of this approach.

[Saaty, 2003]

The last work that requires comparison was developed by [Saaty, 2003]. This is the work that we have improved upon. In [Saaty, 2003], the authors only identify the cell that is causing the most inconsistency with the v elements, similar to our approach. However, we first deal with ordinal inconsistencies and then adjust the preferences automatically, starting with the cell that has the highest absolute value of v. This leads to better results than those obtained by [Saaty, 2003], as we will see next.

To illustrate the method, the authors consider the matrix in Table 4.18, which pertains to the judgments of individuals looking to purchase a house. The considered criteria are Size, Transport, Neighborhood, Age, Yard, Modernity, Condition, and Finance.

	C1	C2	C3	C4	C5	C6	C7	C8
C1	1	5	3	7	6	6	1/3	1/4
C2	1/5	1	1/3	5	3	3	1/5	1/7
C3	1/3	3	1	6	3	4	6	1/5
C4	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8
C5	1/6	1/3	1/3	3	1	1/2	1/5	1/6
C6	1/6	1/3	1/4	4	2	1	1/5	1/6
C7	3	5	1/6	7	5	5	1	1/2
C8	4	7	5	8	6	6	2	1

Table 4.18: Matrix of preferences [Saaty, 2003].

As the matrix was not consistent (CR=0.17), the authors applied their methodology and identified the element a_{37} as the cause of the inconsistency. Then, they applied modifications until consistency was achieved. This resulted in the preference (C3, C7) changing from 6 to 1/2, thus completely reversing the user's preference (see Table 4.19).

	C1	C2	C3	C4	C5	C6	C7	C8
C1	1	5	3	7	6	6	1/3	1/4
C2	1/5	1	1/3	5	3	3	1/5	1/7
C3	1/3	3	1	6	3	4	1/2	1/5
C4	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8
C5	1/6	1/3	1/3	3	1	1/2	1/5	1/6
C6	1/6	1/3	1/4	4	2	1	1/5	1/6
C7	3	5	2	7	5	5	1	1/2
C8	4	7	5	8	6	6	2	1

Table 4.19: Consistent matrix closest to the preferences obtained in [Saaty, 2003].

Thus, the recorded values are 5.5 for δ and 0.72 for σ , as the only preference changed was one with a high value.

With our approach, we obtained the matrix presented in Table 4.20.

	C1	C2	C3	C4	C5	C6	C7	C8
C1	1	5	3	7	6	6	2	1/2
C2	1/5	1	1/3	5	3	3	1/3	1/7
C3	1/3	3	1	6	3	4	6	1/3
C4	1/7	0.2	1/6	1	1/3	1/4	1/7	1/9
C5	1/6	1/3	1/3	3	1	1/2	1/5	1/6
C6	1/6	1/3	1/4	4	2	1	1/5	1/6
C7	1/2	3	1/6	7	5	5	1	1/2
C8	2	7	3	9	6	6	2	1

Table 4.20: Consistent matrix closest to [Saaty, 2003] obtained by our approach.

For our consistent matrix, the closeness metrics results are: $\delta = 2.5$ and $\sigma = 0,58$ which are both better than those reported , demonstrating once again the potential of this approach.

4.4 Discussion

We can see from Table 4.21 that, overall, the results obtained with the observed metrics are better than those in some articles that attempted to adjust users' inconsistent preferences. It is important to note that, besides yielding positive results, our method also maintains the mathematical structure of the returned values, making it easier for users to understand the changes. The fact that the method is integrated into a visualization tool is an added advantage compared to other methods. Additionally, users can choose iterations of the method, allowing them to work with matrices that are not entirely consistent. This flexibility lets users prioritize their specific intentions, even if it slightly compromises the robustness of the method.

Table 4.21: Review of the results.

Papers	Others approach		Our approach	
	δ	σ	δ	σ
[Benitez et al., 2010]	5.28	1.19	**1**	**0.14**
[Benítez et al., 2011]	7.93	3.05	**5**	**1.77**
[Cao et al., 2008], $\gamma = 0.5$	3.265	0.783	**2.5**	**0.56**
[Cao et al., 2008], $\gamma = 0.98$	1.713	0.448	2.5	0.56
[Ergu et al., 2011], First	2.5	0.75	**2**	0.75
[Ergu et al., 2011], Second	7.88	2.78	**2.5**	**1.46**
[Saaty, 2003]	5.5	0.72	**2.5**	**0.58**

Chapter 5

Conclusion

The primary objective of this thesis was to develop a gentle approach to multi-criteria decision making, focusing on aligning preference consistency with visual support. Through the integration of AHP and TOPSIS, and the introduction of a novel automatic adjustment approach using the minimum feedback arc set problem, the thesis aimed to simplify and improve the decision-making process. We have explored the complexities involved in MCDM, particularly the challenges faced by decision-makers in maintaining consistent preferences. Existing methods often result in preferences that deviate significantly from what users initially envisioned. Our proposed methodology addresses this issue with a novel adjustment approach integrated into a visualization tool. This tool employs storytelling, allowing users to navigate the decision-making process intuitively while ensuring that preferences remain close to the user's original input.

The main points of the proposal are centered on enhancing user experience and decision accuracy. The visualization tool simplifies the decision-making process by presenting it in a more understandable and engaging format. The automatic adjustment approach ensures that preferences are consistently aligned with the user's original intentions, thereby improving the robustness of the decision-making outcomes.

Key results and insights from this research include the successful implementation of the visualization tool and the novel adjustment method. The tool was tested at IO2024 - XXIII Congresso da Associação Portuguesa de Investigação Operacional and was selected for presentation at EstudIO, a section of the congress highlighting the top 10 works developed by bachelor and master's students in operational research. This provided an opportunity to showcase the application to both specialists and non-specialists, receiving positive feedback regarding its utility and clarity. Additionally, the methodology for adjusting preferences showed promising results in maintaining preference consistency while simplifying the MCDM process. The approach demonstrated its potential in various scenarios, proving to be more effective than existing methods in the literature.

The relevance of these contributions lies in their potential to bridge the gap between complex decision-making methodologies and user-friendly applications. By aligning preference consistency with visual support, the proposed approach makes MCDM accessible to a broader audience, including those unfamiliar with technical decision-making techniques. The novel adjustment approach ensures that user preferences are consistent and closer to the ones provided initially, thus providing more reliable decision-making outcomes.

For future research, plausible guidelines include expanding the visualization tool to incorporate other MCDM techniques and improving its adaptability to different decision-making contexts. Furthermore, as part of our future work, we are currently developing a scientific paper on this topic, aiming to explore and share the method. The idea is also to apply the tool in a case study to evaluate how satisfied decision-makers are with the final decision.

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Appendices

Appendix A

Some definitions are necessary to make the required adjustments to transform an inconsistent matrix into one with a consistency ratio less than 0.1 (near consistent). This is based on the work developed on [Harker, 1987; Saaty, 2003]:

Definition A.0.1 (Eigenvalue [Santana and Queiró, 2010])**.** *Let A be a real or complex $n \times n$ matrix. A number λ is said to be an eigenvalue of A if there exists a non-zero vector v satisfying $Av = \lambda v$. Under these conditions, v is said to be an eigenvector of A associated with λ.*

Definition A.0.2 (Perron root or Principal eigenvalue)**.** *The Perron root of a matrix A, denoted on this thesis as λ_{max}, is the largest eigenvalue in magnitude of A, and it is given by:*

$$\lambda_{max} = \max\{Re(\lambda) \mid f_A(\lambda) = 0\}.$$

Here, $f_A(\lambda)$ represents the characteristic polynomial of matrix A.

Definition A.0.3 (Right Perron Vector)**.** *The right Perron vector, denoted by $x(A)$, is defined as the eigenvector corresponding to the Perron root of a matrix A, satisfying:*

$$Ax(A) = \lambda_{max} x(A).$$

Definition A.0.4 (Left Perron Vector)**.** *The left Perron vector, denoted by $y(A)$, is defined as the eigenvector corresponding to the Perron root of the transpose of matrix A, satisfying:*

$$y(A)^T A = \lambda_{max} y(A)^T.$$

Definition A.0.5 (Positive Reciprocal Matrix)**.** *A positive reciprocal matrix A is a square matrix where each element $a_{ij} > 0$ and $a_{ij} = \frac{1}{a_{ji}}$ for all $i \neq j$:*

$$A = (a_{ij}) \in \mathbb{R}^{n \times n} \mid a_{ij} > 0, \quad a_{ij} = \frac{1}{a_{ji}}, \quad \forall i \neq j$$

Note that for $i = j$, $a_{ij} = 1$

To perturb a positive reciprocal matrix, we only need to consider the $\frac{n(n-1)}{2}$ elements of the upper triangular part of the matrix, as there is a clear inverse relationship with the remaining elements.

These perturbations should be made taking into account the direction of the partial derivatives, as the largest absolute value reflects the greatest change in the value of λ_{max} , which can identify the value that is causing the greatest inconsistency. Thus, the partial derivatives of the Perron root with respect to the matrix elements a_{ij} and a_{kl} need to be defined.

Definition A.0.6 (First Partial Derivative)**.** *The first partial derivative of the Perron root with respect to the matrix element a_{ij} is given by:*

$$\tilde{D_1^A} = \frac{\partial \lambda_{max}}{\partial a_{ij}}$$

To proceed with his calculation it is necessary to take some lemmas and theorems first.

Lemma A.0.1. *Let $A \in \mathcal{A}_{n \times n}$. Then for $i > j$, $\frac{\partial A}{\partial a_{ij}}$ is an $n \times n$ upper triangular matrix of the form*

$$\left(\frac{\partial A}{\partial a_{ij}}\right)_{kl} = \begin{cases} 1 & \textit{if } k = i, l = j, \\ -\frac{1}{(a_{ij})^2} & \textit{if } k = j, l = i, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. For $k = i, l = j$:

$$\frac{\partial A}{\partial a_{ij}} = \frac{\partial}{\partial a_{ij}}(a_{ij}) = 1$$

For $k = j, l = i$:

$$a_{kl} = a_{ji} = \frac{1}{a_{ij}}$$

$$\frac{\partial a_{kl}}{\partial a_{ij}} = -\frac{1}{(a_{ij})^2}.$$

For $(i, j) \neq (k, l)$:

$$\frac{\partial A}{\partial a_{ij}} = 0$$

since it is a constant. □

Lemma A.0.2. *Let $A \in \mathcal{A}_{n \times n}$. Then for $i > j$ and $k > l$, $\frac{\partial^2 A}{\partial a_{ij} \partial a_{kl}}$ is an $n \times n \times n$ matrix of the form*

$$\left(\frac{\partial^2 A}{\partial a_{ij} \partial a_{kl}}\right)_{pq} = \begin{cases} \frac{2}{(a_{ij})^3} & \textit{if } p = j = l \textit{ and } q = i = k, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The only nonconstant value in $\frac{\partial A}{\partial a_{ij}}$ is the $(j, i)^{th}$ element, $-\frac{1}{(a_{ij})^2}$. On differentiating this matrix, the result is obtained. □

Theorem A.0.1. *Let $A \in \mathcal{A}_{n\times n}$.Then $\tilde{D_1^A}$ is an $n \times n$ upper triangular matrix of the form*

$$\tilde{D_1^A} = \frac{\partial \lambda_{max}}{\partial a_{ij}}_{i>j} = \left([y(A)_i x(A)_j] - \frac{[y(A)_j x(A)_i]}{(a_{ij})^2} \right)$$

where x(A) and y(A) are, respectively, the right and left Perron vectors of A.

Proof. (Summary) Since λ_{max} is a simple eigenvalue of A, there exists a neighborhood N_A of A such that each matrix $B \in N_A$ has a simple eigenvalue $\lambda(B)$ and such that $\lambda(B) = \lambda_{max_B}$ when $B \in N_A \cap \mathcal{A}_{n\times n}$. For each $B \in N_A$, let $x(B)$ be the right eigenvector of B corresponding to $\lambda(B)$. The proof involves differentiating the eigenvalue equation and using the properties of the right and left Perron vectors. Please check [Harker, 1987] for more details. □

Now we are in conditions to state that given a positive reciprocal matrix $A = [a_{ij}]$ and a distinct pair of indices $k > l$, with $A(t) = [a_{ij}(t)]$ and $A(0) = A$. Let $\lambda_{\max}(t)$ denote the Perron eigenvalue of $A(t)$ for all t in a neighborhood of $t = 0$ that is sufficiently small to ensure all entries of the reciprocal matrix $A(t)$ are positive. Finally, let $v = [v_i]$ be the unique positive eigenvector of the positive matrix A^T normalized such that $v^T w = 1$. Then using Theorem A.0.1

$$\left.\frac{d\lambda_{\max}(t)}{dt}\right|_{t=0} = \frac{v^T A'(0) w}{v^T w} = v^T A'(0) w = v_k w_l \left(-\frac{1}{a_{kl}^2} \right) v_l w_k.$$

We conclude that

$$\frac{\partial \lambda_{\max}}{\partial a_{ij}} = v_i w_j - \frac{v_j w_i}{a_{ji}^2} \quad \text{for all } i, j = \{1, \ldots, n\}.$$

And,

$$\frac{\partial \lambda_{\max}}{\partial a_{ji}} = -\frac{\partial \lambda_{\max}}{\partial a_{ij}} \quad \text{for all } i \text{ and } j.$$

Thus, to identify the entry of A whose adjustment within the class of reciprocal matrices would result in the largest rate of change in $\lambda_{\max}$, we should examine the $\frac{n(n-1)}{2}$ values $\left\{ v_i w_j - \frac{v_j w_i}{a_{ji}^2} \right\}$, $i > j$, and select any one with the largest absolute value.