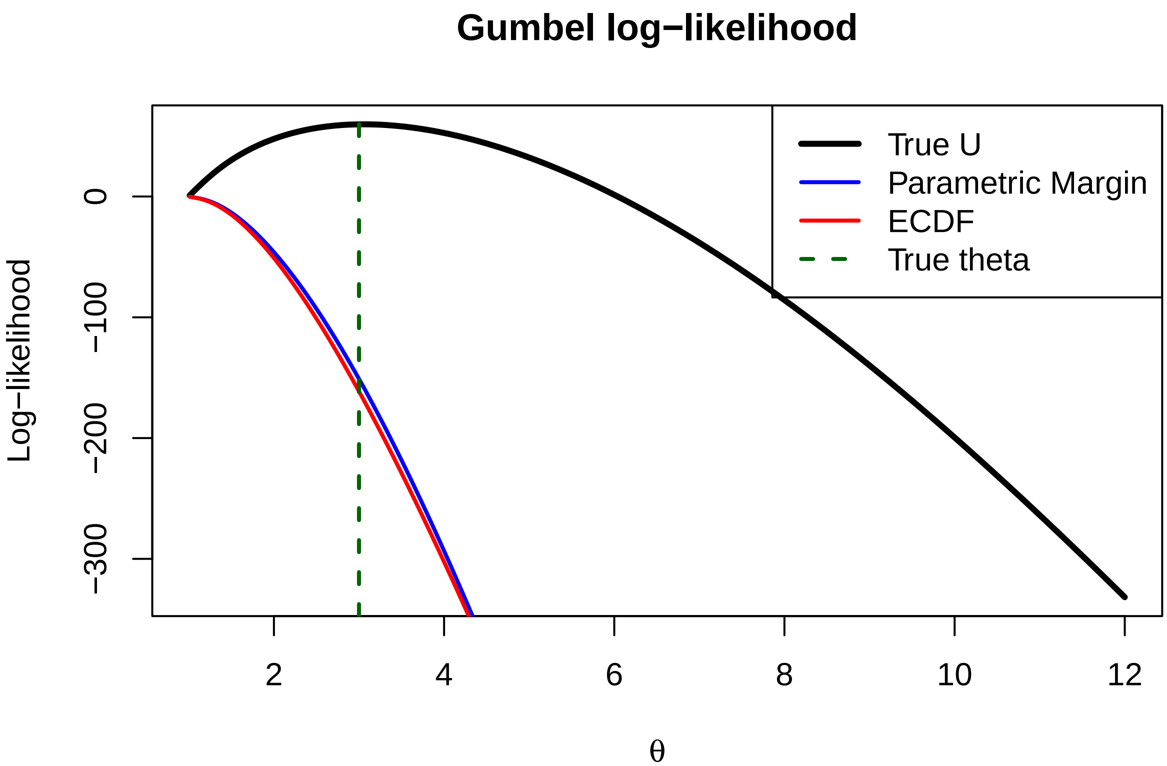
Joint Copula Bayesian Modelling

1. Motivation

We are assuming the following joint distribution for a sequence of random variables $X_1, \ldots, X_n \sim C_\theta(F(x_1), \ldots, F(x_n))$, with C a parametric copula (potentially archimedean) and F the marginal of each X_i.

We know from simulation results that the likelihood peaks at the correct copula generating parameter if we take the realisations $U_1, \ldots, U_n \sim C_\theta$. However, estimating the margin F from a realisation of $X_1, \ldots, X_n$, to obtain the pseudo observations $\hat{U}_i = \hat{F}(X_i)$, leads to a likelihood which peaks at $\theta = 1$, the independence case in the Gumbel copula. This could be understood by realising that we are using estimators for the margin which would work under independence, but which could fail under the dependent setup at hand.



An idea is to avoid the 2 steps procedure by performing estimation on the margin and the copula parameter simultaneously. A fully bayesian model would inherently consider both objects as targets, and hopefully it might be able to consider the dependence structure while estimating the margin.

2. Bayesian Model with Parametric Margin

The simplest approach consists in assuming a parametric form for the margin, that is to assume $X_i \sim F_\varphi$, with φ the parameters of the distribution. Hierarchically, this yields the following structure:

$$\varphi \sim \pi_\varphi\,, \quad \theta \sim \pi_\theta$$
$$X_1, \dots, X_n |\, \theta, \varphi \sim C_\theta(F_\varphi(x_1), \dots, F_\varphi(x_n))$$

2.1 Adaptive Block Metropolis–Hastings Algorithm

We aim to sample from the joint posterior

$$\pi(\theta, \varphi \mid x_{1:n}) \;\propto\; \pi_\theta(\theta)\, \pi_\varphi(\varphi)\, c_\theta(F_\varphi(x_1), \dots, F_\varphi(x_n)) \prod_{i=1}^{n} f_\varphi(x_i),$$

where the full parameter vector is

$$\boldsymbol{\phi} = (\theta, \varphi) \in \mathbb{R}^d.$$

Algorithm 1: Block Adaptive Metropolis–Hastings

1. **Initialisation:**
   - Set the initial parameter vector $\boldsymbol{\phi}^{(0)}$.
   - Compute the initial log-posterior $\ell_{\text{curr}} = \ell(\boldsymbol{\phi}^{(0)})$.
   - Set the proposal covariance $C_0 = \varepsilon_0 I_d$.
   - Let $C \leftarrow C_0$.
   - Set adptation time t_{adapt} and frequency Δ_{adapt}
2. **For** $t = 1, \dots, N$:
   (a) **Proposal:**
   Draw
   $$\boldsymbol{\phi}^\star \sim \mathcal{N}\Big(\boldsymbol{\phi}^{(t-1)},\, C\Big)\,.$$
   (b) **Evaluate proposed log-posterior:**
   Compute $\ell^\star = \ell(\boldsymbol{\phi}^\star)$.
   (c) **Acceptance probability:**
   $$\alpha = \min\{1,\, \exp(\ell^\star - \ell_{\text{curr}})\}.$$
   (d) **Accept/reject:**
   Draw $u \sim \text{Unif}(0, 1)$.
   - If $u < \alpha$: set $\boldsymbol{\phi}^{(t)} = \boldsymbol{\phi}^\star$ and $\ell_{\text{curr}} = \ell^\star$.
   - Else: set $\boldsymbol{\phi}^{(t)} = \boldsymbol{\phi}^{(t-1)}$.

   (e) **Adaptive covariance update:**
   If $t > t_{\text{adapt}}$ and $t \equiv 0 \pmod{\Delta_{\text{adapt}}}$, then:
   $$\widehat{\Sigma}_{1:t} = \text{Cov}\Big(\boldsymbol{\phi}^{(1)}, \dots, \boldsymbol{\phi}^{(t)}\Big)\,,$$
   $$C \leftarrow \frac{2.38^2}{d}\, \widehat{\Sigma}_{1:t} + \varepsilon I_d.$$