

# Ordinary Differential Equations and Dynamical Systems

## Part I

## Modeling

### Fundamentals

Formally, an ordinary differential equation is an equation, in which a function and its derivatives and the independent variable appear.

An (implicit) *ordinary differential equation* of order  $n$  is an equation of the form

$$F(x, y, y', y'', \dots, y^{(n)}) = 0. \quad (1)$$

An example of an explicit ODE of order  $n$  is of the form

$$y^n = G(x, y, y', y'', \dots, y^{n-1}). \quad (2)$$

Remark: The solution of an ODE is not a number, but rather a function, or more precisely: a set of functions.

The set of all solutions of an ordinary differential equation is the *general solution* of an ODE. The solution of an initial value problem is a *special* or *particular solution* of the ODE.

### Classification

Differential equations can be classified according to various criteria. Besides the order of an ODE we are also interested in whether an ODE is linear, homogeneous, has constant coefficient, is separable or autonomous.

#### Linearity

An  $n$ -th order ODE is *linear*, if it is of the form

$$a_n(x) \cdot y^{(n)} + \dots + a_1(x) \cdot y' + a_0(x) \cdot y = g(x) \quad (3)$$

where  $a_n(x), \dots, a_1(x), a_0(x)$  are  $g(x)$  fixed functions. Or in other words: A differential equation is linear if the dependant variable and all of its derivatives appear in a linear fashion (i.e., they are not multiplied together or squared for example or they are not part of transcendental functions such as sins, cosines, exponentials, etc.)

#### Homogeneity

A linear ODE is *homogeneous*, if  $g(x) = 0$  for all  $x$ ; otherwise the ODE is *inhomogeneous*, and  $g(x)$  is the *inhomogeneity* or *source* term.

The distinction between *homogeneous* and *inhomogeneous* ODEs can also be understood physically: Homogeneous ODEs describe systems which act without external input, whereas systems which are driven by an external force are generally described by inhomogeneous ODEs.

#### Exactness

A first-order ODE of the form

$$M(x, y) + N(x, y) \cdot y'(x) = 0 \quad (4)$$

is called *exact*, if the partial derivatives of  $M(x, y)$  and  $N(x, y)$  satisfy the condition

$$M_y(x, y) = N_x(x, y). \quad (5)$$

#### Constant coefficient

A linear ODE has *constant coefficients*, if it is of the form

$$a_n \cdot y^{(n)} + \dots + a_1 \cdot y' + a_0 \cdot y = g(x), \quad (6)$$

with  $a_n \neq 0$  (the source term  $g(x)$  does not have to be constant).

#### Separability

The ODE is *separable*, if  $F(x, y)$  can be written as a product of a  $x$ - and  $y$ -dependent term, i.e. if the ODE is of the form

$$y' = g(x) \cdot h(y) \quad (7)$$

for some functions  $g(x)$  and  $h(y)$ .

#### Autonomy

An ODE is *autonomous*, if  $F(x, y)$  only depends on  $y$ , i.e. if the ODE is of the form

$$y' = h(y) \quad (8)$$

for some function  $h(y)$ .

Every autonomous ODE is separable with  $g(x) = 1$ .

#### Examples

$$y' = f(x)$$

Inhomogeneous linear ODE for  $y(x)$  with source term  $f(x)$

$$m \cdot \dot{v} = m \cdot g - k \cdot v^2$$

Nonlinear ODE for  $v(t)$

$$l \cdot \ddot{\Phi} + g \cdot \sin(\Phi) = 0$$

Nonlinear ODE for  $\Phi(t)$

$$l \cdot \ddot{\Phi} + g \cdot \phi$$

Homogeneous linear ODE for  $\Phi(t)$

$$l \cdot \ddot{\Phi} + g \cdot \phi = \sin(\omega t)$$

Inhomogeneous linear ODE for  $\Phi(t)$  with source term  $\sin(\omega t)$

$$i'' + \frac{R}{L}i' + \frac{1}{LC}i = 0$$

Homogeneous linear ODE for  $i(t)$

### Systems of differential equations

If several systems are coupled with each other and mutually influence each other, one often obtains a system of ODE's.

A *system of differential equations* of first order is a system

$$\begin{aligned} y_1' &= f_1(x, y_1, \dots, y_n) \\ &\vdots \\ y_n' &= f_n(x, y_1, \dots, y_n) \end{aligned} \quad (9)$$

of ODE's for unknown functions  $y_1(x), \dots, y_n(x)$ .

Using the vectorial notation

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}) \quad (10)$$

An ODE of  $n$ -th order is equivalent to a system of first-order ODE's.

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_n' &= f(x, y_1, \dots, y_n) \end{aligned} \quad (11)$$

#### Example: Higher-order ODE to system of first-order ODE's

We want to rewrite the ODE

$$y^4 + \sin(y'') \cdot e^{(x^2+y)} = 0 \quad (12)$$

as a first order system. The solution is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= -\sin(y'') \cdot e^{(x^2+y)} = -\sin(y_3) \cdot e^{(x^2+y_1)} \end{aligned} \quad (13)$$

Where  $y_1 = y$ ,  $y_2 = y'$ ,  $y_3 = y''$ ,  $y_4 = y'''$ .

To make this system autonomous we introduce  $x = y_0$ , plug it into the last equation and add  $y_0' = 1$  to the system above.

#### Example: 2nd-order ODE to system of first-order ODE's

We want to rewrite the following 2nd order ODE into a system of first-order ODE's.

$$\ddot{x}(t) + 2\delta\dot{x}(t) + \omega_0^2 x(t) = f(t) \quad (14)$$

If we introduce the vector-valued function

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \Rightarrow \dot{\mathbf{y}} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} \quad (15)$$

rewriting:

$$\begin{aligned} \dot{\mathbf{y}} &= \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -2\delta\dot{x}(t) - \omega_0^2 x(t) + f(t) \end{pmatrix} \\ \dot{\mathbf{y}} &= \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\delta \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \\ \dot{\mathbf{y}} &= \mathbf{A}\mathbf{y} + \mathbf{b} \end{aligned} \quad (16)$$

#### Example: Coupled second order system to first-order ODE's

Rewrite the second order system (coupled pendulums)

$$\begin{aligned} \ddot{\varphi}_1(t) &= -\beta\varphi_1 + \alpha \cdot (\varphi_2 - \varphi_1) \\ \ddot{\varphi}_2(t) &= -\beta\varphi_2 + \alpha \cdot (\varphi_1 - \varphi_2) \end{aligned} \quad (17)$$

as a system of first order ODE.

With the new variables  $\varphi_3 = \dot{\varphi}_1$ ,  $\varphi_4 = \dot{\varphi}_2$  the system becomes

$$\begin{aligned} \dot{\varphi}_1(t) &= \varphi_3 \\ \dot{\varphi}_2(t) &= \varphi_4 \\ \dot{\varphi}_3(t) &= -\beta\varphi_1 + \alpha \cdot (\varphi_2 - \varphi_1) \\ \dot{\varphi}_4(t) &= -\beta\varphi_2 + \alpha \cdot (\varphi_1 - \varphi_2) \end{aligned} \quad (18)$$

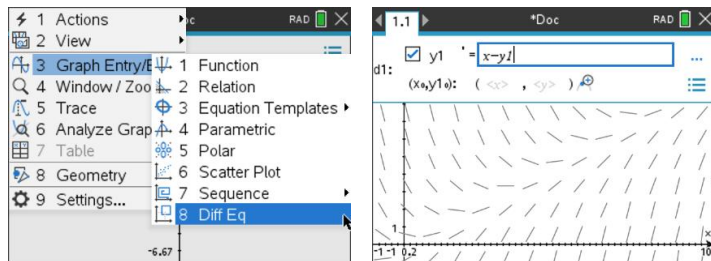
## Slope field

Slope fields are often lead to a good qualitative understanding of the situation described by the ODE under consideration. Slope field can be understood in the following way: To each point  $(x, y)$  in the region  $B$  under consideration,  $F(x, y)$  is a value which describes the slope of the solution curve passing through the point  $(x, y)$ .

### Example with calculator

We want to plot the slope field of the following ODE

$$y' = x - y \quad (19)$$



1. Select: Menu, 3: Graph Entry/Edit, 8: Diff Eq.
2. Write down the ODE

## Part II

# Ordinary Differential Equations

## Analytical methods for first-order ODE's

### Overview

For some types of explicit first-order ODE's there exist analytical solution methods:

#### Separable ODE's

We discuss this type of ODE's in detail. The main reason why separable ODE's are interesting is the fact that the special cases "Indefinite integral" and "Autonomous ODE" are of this type, and many important examples from applications are autonomous ODE's.

#### Linear ODE's

We discuss linear ODE's in the following section in the context of the discussion of linear ODE's of arbitrary order. For linear ODE's the distinction between homogeneous and inhomogeneous equations is crucial. A homogeneous first-order ODE is separable and can therefore be solved by the standard procedure for separable ODE's. For inhomogeneous ODE's, one often arrives at a solution by assuming an ansatz of the form of the inhomogeneity.

### Separable ODE's

How to find the general solution of a separable ODE is best explained with an example:

We compute the general solution of the ODE

$$y' = -\frac{x}{y} \quad (20)$$

We write the equation as

$$\frac{dy}{dx} = -\frac{x}{y} \quad (21)$$

We bring all  $x$ -dependent terms to the right hand side and all  $y$ -dependent terms on the left hand side:

$$y \, dy = -x \, dx \quad (22)$$

We integrate on both sides and get

$$\int y \, dy = -\int x \, dx \Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C, \quad C \in \mathbb{R} \quad (23)$$

We solve for  $y$  and get

$$y = \pm \sqrt{K - x^2}, \quad K \in \mathbb{R} \quad (\text{where } K = 2C) \quad (24)$$

## Analytical methods for linear ODE's

### Overview

We differentiate between first-order linear ODE's and higher-order ODE's as well as between homogeneous and inhomogeneous ODE's.

The general solution of the inhomogeneous ODE is the sum

$$y = y_h + y_s, \quad (25)$$

where  $y_h$  is the general solution of the homogeneous ODE and  $y_s$  any special solution of the inhomogeneous ODE.

### First-order linear ODE's

To solve a first-order linear ODE we thus have to find  $y_h$  and  $y_s$

**$y_h$ :** a homogeneous first-order ODE is separable and can therefore be solved by the standard procedure for separable ODE's described above.

**$y_s$ :** To find a special solution of an ODE, there are several possibilities. In the case of an ODE with constant coefficients, it usually suffices to choose for  $y_s$  an *ansatz of the form of the source term*  $g(x)$ . In the case of non-constant coefficients, the method *variation of constants* usually works better.

#### Example using the ansatz

We solve the ODE

$$y' + ay = b \quad (26)$$

To determine  $y_h$ , we integrate the homogeneous ODE

$$y' + ay = 0 \quad (27)$$

by separation of variables we obtain the solution

$$y_h = C \cdot e^{-ax}, \quad C \in \mathbb{R}. \quad (28)$$

For finding  $y_s$  we choose the ansatz in the form of the source term. In this case the source term is constant,  $g(x) = b$ . Therefore we assume that the special solution  $y_s$  is constant as well, i.e. we make an ansatz  $y_s = c$  (and thus  $y'_s = 0$ ). We plug this ansatz and its derivative into the inhomogeneous ODE and obtain the special solution  $y_s$ .

$$y'_s + ay_s = b \Rightarrow y_s = \frac{a}{b} \quad (29)$$

The general solution therefore is

$$y = C \cdot e^{-ax} + \frac{a}{b}, \quad C \in \mathbb{R} \quad (30)$$

#### Ansatz functions for the solution of the inhomogeneous first-order ODE

Source term  $g(x)$

$$g(x) = b_0$$

$$g(x) = b_1x + b_0$$

$$g(x) = b_2x^2 + b_1x + b_0$$

$$g(x) = \sum_{i=0}^n b_i x^i$$

$$g(x) = A \sin(\omega x) + B \cos(\omega x)$$

$$g(x) = Ae^{bx}$$

Ansatz  $y_s$

$$y_s = c_0$$

$$y_s = c_1x + c_0$$

$$y_s = c_2x^2 + c_1x + c_0$$

$$y_s = \sum_{i=0}^n c_i x^i$$

$$y_s = C_1 \sin(\omega x) + C_2 \cos(\omega x)$$

$$y_s = C \sin(\omega x + \varphi)$$

$$y_s = \begin{cases} \frac{A}{b+a} e^{bx} & \text{for } b \neq -a \\ A x e^{-ax} & \text{for } b = -a \end{cases}$$

In addition, the following rules must be followed:

**Linearity** If  $g(x)$  is a linear combination of several source terms, one has to assume as ansatz for  $y_s(x)$  a corresponding linear combination of several ansatz terms.

**Resonance** If the source term  $g(x)$  is itself already a solution of the homogeneous ODE, the corresponding ansatz for  $y_s$  has to be multiplied with  $x$ . So if for example  $y_h = Ce^x$ , and  $g(x) = e^x$ , the ansatz  $y_s = x \cdot e^x$  is chosen.

#### Example using Variation of constants

The idea behind the variation of constants is to start from the solution  $y = K \cdot e^{-F(x)}$  of the homogeneous ODE and plug the ansatz

$$y = K(x) \cdot e^{-F(x)} \quad (31)$$

into the inhomogeneous ODE.

As an example we want to solve the inhomogeneous linear ODE

$$3y' + 5y = 7e^{\frac{1}{3}x}. \quad (32)$$

We first find the homogeneous solution  $y_h$  by separation of variables.

$$\frac{1}{y} dy = -\frac{5}{3} dx \Rightarrow y_h = e^{\frac{5}{3}x} \cdot K \quad (33)$$

Next we calculate the ansatz and its derivative

$$\begin{aligned} y_s &= e^{-\frac{5}{3}x} \cdot K(x) \\ y'_s &= K(x)' e^{-\frac{5}{3}x} + K(x) \cdot -\frac{5}{3} e^{-\frac{5}{3}x} \end{aligned} \quad (34)$$

The ansatz is then plugged into the inhomogeneous ODE

$$3 \left( K(x)' e^{-\frac{5}{3}x} + K(x) \cdot -\frac{5}{3} e^{-\frac{5}{3}x} \right) + 5 \left( e^{-\frac{5}{3}x} \cdot K(x) \right) = 7 e^{\frac{1}{3}x} \quad (35)$$

We solve for  $K(x)'$  ( $K(x)$  usually disappears)

$$3 \cdot K(x)' e^{-\frac{5}{3}x} = 7 e^{\frac{1}{3}x} \Rightarrow K(x)' = \frac{7}{3} e^{\frac{1}{3}x + \frac{5}{3}x} = \frac{7}{3} e^{2x} \quad (36)$$

We integrate and find  $K(x)$

$$K(x) = \frac{7}{6} e^{2x} \quad (37)$$

Plugin  $K(x)$  into the ansatz we get the special solution  $y_s$

$$y_s = e^{-\frac{5}{3}x} \cdot \frac{7}{6} e^{2x} = \frac{7}{6} e^{\frac{1}{3}x} \quad (38)$$

The solution  $y_h + y_s$  is then

$$y = K \cdot e^{-\frac{5}{3}x} + \frac{7}{6} e^{\frac{1}{3}x} \quad (39)$$

If the problem is an initial value problem; plug in values for  $x$  and  $y$  and solve for  $K$ .

## Higher-order linear ODE's

The fact that the solution of an inhomogeneous ODE is the sum of the solution of the homogeneous ODE and the special solution (equation 25) still holds true for higher-order linear ODE's. For solving such higher-order linear ODE's we distinguish between the homogeneous and the inhomogeneous case

### The homogeneous case

The function  $y = e^{\lambda x}$  is a solution of the homogeneous ODE if and only if  $\lambda$  is a root of the characteristic polynomial, i.e. if

$$P(\lambda) = 0 \quad (40)$$

We now have to distinguish between real and complex and between single and multiple roots of the characteristic polynomial.

**Real Roots:** If  $\lambda$  is a real root of  $P(\lambda)$  of multiplicity  $m$ , then the functions

$$y_1 = e^{\lambda x}, y_2 = x e^{\lambda x}, \dots, y_m = x^{m-1} e^{\lambda x} \quad (41)$$

are distinct linearly independent solutions. i.e. if  $m = 1$  and  $\lambda$  is a simple root then  $y = C_1 e^{\lambda x}$  is a solution.

Example: The homogeneous ODE

$$y^{(3)} - 3y'' - 4y' = 0 \quad (42)$$

has the characteristic polynomial

$$P(\lambda) = \lambda^3 - 3\lambda^2 - 4\lambda. \quad (43)$$

The roots of this polynomial are  $\lambda_1 = 0, \lambda_2 = 4, \lambda_3 = -1$ , hence the general solution of the ODE is

$$y = C_1 + C_2 e^{4x} + C_3 e^{-x} \quad (44)$$

**Complex Roots:** If  $\lambda = \alpha \pm j\beta$  are two complex roots of  $P(\lambda)$  of multiplicity  $m$  then

$$\begin{aligned} y_1 &= e^{\alpha x} \cos(\beta x) & y_2 &= e^{\alpha x} \sin(\beta x) \\ y_3 &= x e^{\alpha x} \cos(\beta x) & y_4 &= x e^{\alpha x} \sin(\beta x) \\ \vdots & & \vdots & \\ y_{2m-a} &= x^{m-1} e^{\alpha x} \cos(\beta x) & y_{2m} &= x^{m-1} e^{\alpha x} \sin(\beta x) \end{aligned}$$

are  $m$  linearly independent solutions.

If  $m = 1$  and  $\lambda_{1,2} = \alpha \pm j\beta$  then  $y_1 = C_1 e^{\alpha x} \cos(\beta x)$  and  $y_2 = C_2 e^{\alpha x} \sin(\beta x)$  are two linearly independent solutions.

Example: The homogeneous ODE

$$y'' + y = 0 \quad (45)$$

has the characteristic polynomial

$$P(\lambda) = \lambda^2 + 1. \quad (46)$$

The roots of this polynomial are  $\lambda_{1,2} = \pm j$  ( $\alpha = 0, \beta = 1$ ), hence the general solution of the ODE is

$$y = C_1 \cos(x) + C_2 \sin(x) \quad (47)$$

### The inhomogeneous case

We now determine special solutions of the inhomogeneous ODE's. The best method for solving such ODE's is as above using an ansatz of the form of the source term, with undetermined coefficients. These coefficients are then determined by plugging the ansatz into the equation.

Example: Consider the inhomogeneous ODE

$$y^{(3)} + y'' + y' + y = 2x + 5 \quad (48)$$

The general solution of the homogeneous ODE is (The complex roots are  $-1, -j, j$ )

$$y_h = C_1 e^{-x} + C_2 \cos(x) + C_3 \sin(x) \quad (49)$$

To find a special solution of the inhomogeneous ODE, we choose the ansatz

$$y_s = b_1 x + b_0 \quad (50)$$

Plugin this into the initial ODE leads to

$$b_1 + (b_1 x + b_0) = 2x + 5 \quad (51)$$

which leads to the following linear system of equations for  $a$  and  $b$

$$\begin{vmatrix} 1 & b_1 \\ b_1 & b_0 \end{vmatrix} = \begin{vmatrix} 2 \\ 5 \end{vmatrix} \Rightarrow b_1 = 2, \quad b_0 = 3 \quad (52)$$

The desired special solution hence is

$$y_s = 2x + 3 \quad (53)$$

and the general solution therefore is

$$y = C_1 e^{-x} + C_2 \cos(x) + C_3 \sin(x) + 2x + 3 \quad (54)$$

### Ansatz functions for the solution of the inhomogeneous higher order ODE

Source term $g(x)$	Ansatz $y_p$
$g(x) = \sum_{i=0}^2 a_i x^i$	$\begin{cases} P_n(x) & \text{if } b \neq 0 \\ x P_n(x) & \text{if } a \neq 0, b = 0 \\ x^2 P_n(x) & a = b = 0 \end{cases}$
$g(x) = B e^{cx}$	$\begin{aligned} &c \text{ is not a solution of } \lambda^2 + a\lambda + b = 0 \\ &y_p = A e^{cx} \\ &c \text{ is a simple solution of } \lambda^2 + a\lambda + b = 0 \\ &y_p = A x e^{cx} \\ &c \text{ is a double solution of } \lambda^2 + a\lambda + b = 0 \\ &y_p = A x^2 e^{cx} \end{aligned}$
$g(x) = C_1 \sin(\beta x) + C_2 \cos(\beta x)$	$\begin{aligned} &i\beta \text{ is not a solution of } \lambda^2 + a\lambda + b = 0 \\ &y_p = A \sin(\beta x) + B \cos(\beta x) \text{ or} \\ &y_p = C x \sin(\beta x + \varphi) \\ &i\beta \text{ is a solution of } \lambda^2 + a\lambda + b = 0 \\ &y_p = x (A \sin(\beta x) + B \cos(\beta x)) \text{ or} \\ &y_p = C x \sin(\beta x + \varphi) \end{aligned}$

### Calculator-Example: Finding roots of Polynomial

The roots of a characteristic polynomial can be found with the calculator command `cPolyRoots()`.

Example: We want to find the roots of the following ODE:

$$y'' + 2y' + 2y = 0 \quad (55)$$

The characteristic polynomial is

$$P(\lambda) = \lambda^2 + 2\lambda + 2 \quad (56)$$

Using the command `cPolyRoots(y''+2y'+2, y)` returns a list of the roots of the characteristic polynomial  $\{-1-i, -1+i\}$ .

$$\begin{aligned} h &= t_{k+1} - t_k \\ t_k &= t_0 + k \cdot h, \quad k \in \mathbb{N} \end{aligned} \quad (57)$$

## Explicit Eulers Method

$$x_{k+1} = x_k + h \cdot f(t_{k+1}, x_{k+1}) \quad (58)$$

## Implizit Eulers Method

$$\begin{aligned} \dot{x} &= f(t, x) \\ x_{k+1} &= x_k + h \cdot f(t_k, x_k) \end{aligned} \quad (59)$$

## Midpoint Method

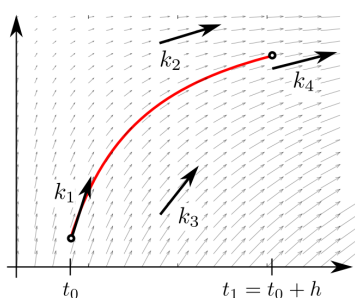
$$\begin{aligned} x_{k+\frac{1}{2}} &= x_k + \frac{h}{2} \cdot f(t_k, x_k) \\ x_{k+1} &= x_k + h \cdot f(t_k + \frac{h}{2}, x_{k+\frac{1}{2}}) \end{aligned} \quad (60)$$

## Heun Method

$$\begin{aligned} k_1 &= f(t_k, x_k) \\ k_2 &= x_k + h \cdot f(t_k, x_k) \\ x_{k+1} &= x_k + \frac{h}{2} (k_1 + k_2) \end{aligned} \quad (61)$$

## Runge-Kutta Method of Fourth Order

$$\begin{aligned} k_1 &= f(t_k, x_k) \\ k_2 &= f(t_k + \frac{h}{2}, x_k + \frac{h}{2} k_1) \\ k_3 &= f(t_k + \frac{h}{2}, x_k + \frac{h}{2} k_2) \\ k_4 &= f(t_k + h, x_k + h k_3) \\ x_{k+1} &= x_k + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4) \end{aligned} \quad (62)$$



## Error Order

The (global) error orders of the various methods discussed before are:

- Explicit Euler method:  $p = 1$
- Midpoint rule:  $p = 2$
- Heun method:  $p = 2$
- Classical Runge-Kutta method:  $p = 4$

The higher the error order of a method, the smaller thus the error is which one has to take into account by using the method. On the other hand, the computational cost for the preciser methods is often higher as well.

# Part III

# System of Differential Equations

## Analytical methods for linear systems

By the definition given above, by a system of ODE's we mean the following system of explicit firstorder ODE's:

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, \dots, x_n) \end{aligned} \quad (63)$$

## Overview

A system of linear first-order ODE's has the form

$$\begin{aligned} \dot{x}_1 &= a_{11}(t)x_1 + \dots + a_{1n}(t)x_n + b_1(t) \\ &\vdots \\ \dot{x}_n &= a_{n1}(t)x_1 + \dots + a_{nn}(t)x_n + b_n(t) \end{aligned} \quad (64)$$

or in matrix-vector notation

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{b}t \quad (65)$$

The general solution of the inhomogeneous system is the sum

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_s \quad (66)$$

where  $x_h$  is the general solution of the homogeneous system and  $x_s$  any special solution of the inhomogeneous system.

The set of solutions of the homogeneous system is an  $n$ -dimensional vector space. This is equivalent to the following two statements:

- Any linear combination of solutions is again a solution, i.e. if  $x_1$  and  $x_2$  are solutions, then  $C_1x_1 + C_2x_2$  is also a solution for any  $C_1, C_2 \in \mathbb{R}$
- There exist precisely  $n$  linearly independent solution  $x_1, \dots, x_n$ . Such a set  $\{x_1, \dots, x_n\}$  of linearly independent solutions is also called a *fundamental system of solutions*. Algebraically, a fundamental system thus is a basis of the vector space of solutions.

## Homogeneous linear systems

Since in the scalar case the solution of a homogeneous linear ODE is given by  $x = e^{at}$ , we try in the vectorial case an ansatz of the form

$$\mathbf{x} = e^{\lambda t} \cdot \mathbf{c} = \begin{pmatrix} e^{\lambda t} c_1 \\ \vdots \\ e^{\lambda t} c_n \end{pmatrix} \quad (67)$$

Plugging the ansatz into the ODE leads to  $e^{\lambda t} \lambda \mathbf{c} = e^{\lambda t} A \mathbf{c}$  and we thus get for  $\lambda$  and  $\mathbf{c}$  the equation

$$A \mathbf{c} = \lambda \mathbf{c} \quad (68)$$

This precisely means that  $\mathbf{c}$  is an *eigenvector* of  $A$  to the *eigenvalue*  $\lambda$ . We distinguish between the following cases for  $\lambda$

### Real eigenvalues

If all eigenvalues of  $A$  are real and  $A$  has  $n$  linear independent eigenvectors then the general solution is a linear combination of

$$\mathbf{x}(t) = e^{\lambda t} \cdot \mathbf{c} \quad (69)$$

Example: We compute the general solution of the homogeneous system

$$\begin{aligned} \dot{x}_1 &= 2x_1 - x_2 \\ \dot{x}_2 &= -x_1 + 2x_2 \end{aligned} \quad (70)$$

or in the form  $\dot{\mathbf{x}} = A\mathbf{x}$  with

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad (71)$$

The matrix  $A$  has the eigenvalues  $\lambda_1 = 1, \lambda_2 = 3$  with corresponding eigenvectors  $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ . Hence the general solution  $\mathbf{x}(t)$  of the homogeneous system is

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (72)$$

### Complex eigenvalues

If there is at least one pair of complex conjugate eigenvalues and  $A$  has  $n$  linear independent eigenvectors then the general (complex) solution is a linear combination of  $\mathbf{x} = e^{\lambda t} \cdot \mathbf{c}$ . Real and complex parts of complex solutions are real solutions

If  $\lambda = \mu + j\nu$  is a simple complex eigenvalue of  $A$  and  $\mathbf{c} = \mathbf{a} + j\mathbf{b}$  a corresponding complex eigenvector of  $A$ , then from the complex solution

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{c} = e^{(\mu+j\nu)t} (\mathbf{a} + j\mathbf{b}) \quad (73)$$

we get two linearly independent real solutions by separating the complex solution into real and imaginary parts:

$$\begin{aligned} \mathbf{z}_1(t) &= \text{Re}(e^{\lambda t} \mathbf{c}) = e^{\mu t} (\mathbf{a} \cos(\nu t) - \mathbf{b} \sin(\nu t)) \\ \mathbf{z}_2(t) &= \text{Im}(e^{\lambda t} \mathbf{c}) = e^{\mu t} (\mathbf{a} \sin(\nu t) + \mathbf{b} \cos(\nu t)) \end{aligned} \quad (74)$$

The complex conjugate of  $\lambda$ , namely  $c = a - jb$  does not have to be considered because it would again lead to the same solutions.

The solution of the linear system is then  $x = z_1 + z_2$

Example: We compute the general solution of a homogeneous system of ODE's where the matrix  $A$  is

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \quad (75)$$

The eigenvalues of  $A$  are  $\lambda_{1,2} = 1 \pm 2j$ , with eigenvectors  $\nu_{1,2} = \begin{pmatrix} 1 \pm j \\ 2 \pm 0 \end{pmatrix}$ .

$$\mathbf{x} = C_1 e^{1t} \begin{pmatrix} 1 \cos(2t) - 1 \sin(2t) \\ 2 \cos(2t) - 0 \sin(2t) \end{pmatrix} + C_2 e^{1t} \begin{pmatrix} 1 \sin(2t) + 1 \cos(2t) \\ 2 \sin(2t) + 0 \cos(2t) \end{pmatrix} \quad (76)$$

### Polynomial expressions

If  $A$  has multiple eigenvalues and less than  $n$  linear independent eigenvectors (More eigenvalues than eigenvectors), then not all solutions are of the form  $\mathbf{x}(t) = e^{\lambda t} \mathbf{c}$ . Additional solutions involve polynomial expressions. These polynomials are of the form

$$\begin{aligned} p_0(t) &= e^{\lambda t} \mathbf{v}_1 \\ p_1(t) &= e^{\lambda t} (t \mathbf{v}_1 + \mathbf{v}_2) \\ p_2(t) &= e^{\lambda t} (t^2 \mathbf{v}_1 + 2t \mathbf{v}_2 + 2 \mathbf{v}_3) \end{aligned} \quad (77)$$

Example: The only eigenvalue of the matrix  $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$  is  $\lambda = 2$ , with the single (linearly independent) eigenvector  $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . Let  $\mathbf{v}_1 = \mathbf{v}$ . To find a generalized eigenvector  $\mathbf{v}_2$ , we solve the equation

$$(A - \lambda E_n) \mathbf{v}_2 = \mathbf{v}_1 \quad (78)$$

$$\begin{aligned} \left( \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \mathbf{v}_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{v}_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned} \quad (79)$$

And therefore

$$\mathbf{p}_1(t) = t \mathbf{v}_1 + \mathbf{v}_2 = t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix} \quad (80)$$

The general solution therefore is

$$\begin{aligned} \mathbf{x}(t) &= C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} t \\ 1 \end{pmatrix} \\ \mathbf{x}(t) &= \begin{pmatrix} C_1 e^{2t} + C_2 t e^{2t} \\ C_2 e^{2t} \end{pmatrix} \end{aligned} \quad (81)$$

### Inhomogeneous linear systems

We now discuss the inhomogeneous system

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t) \quad (82)$$

with a constant coefficient matrix  $A$  and a nonconstant source term  $\mathbf{b}(t)$ . We will look at the following two methods for obtaining the special solution of the homogeneous system.

- Elimination of variables
- Using the decomposition  $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_s$  and choosing an ansatz for  $\mathbf{x}_s$  of a similar type as the source term.

### Elimination of variables

This method can be understood as the inversion of what we've done before. We transform a system of first-order ODE's into a single ODE of higher order. The method is not recommended for ODE systems of dimension 3 or higher. This method is best explained with an example.

We solve the IVP

$$\begin{aligned} \dot{x}_1 &= 2x_1 - x_2 + 1, & x_1(0) &= 0 \\ \dot{x}_2 &= -x_1 + 2x_2 - 1, & x_2(0) &= 1 \end{aligned} \quad (83)$$

To eliminate  $x_2(t)$ , we write the first equation of as

$$x_2 = -\dot{x}_1 + 2x_1 + 1 \quad (84)$$

If this term and its derivative are plugged into the second equation, we obtain for  $x_1$  the second order linear ODE

$$\ddot{x}_1 - 4\dot{x}_1 + 3x_1 + 1 = 0 \quad (85)$$

This ODE can be solved with the method described before, and one gets the general solution

$$x_1 = C_1 e^t + C_2 e^{3t} - \frac{1}{3} \quad (86)$$

If this solution and its derivative are plugged into the second initial equation, one gets for  $x_2$  the general solution

$$x_2 = C_1 e^t - C_2 e^{3t} + \frac{1}{3} \quad (87)$$

Observe that the constants  $C_1$  and  $C_2$  have to be the same in both equations. Now the initial conditions can be plugged in, and we obtain the system of equations

$$\begin{cases} C_1 e^t + C_2 - \frac{1}{3} = 0 \\ C_1 - C_2 + \frac{1}{3} = 1 \end{cases} \quad (88)$$

for  $C_1$  and  $C_2$ . The solution of this system is

$$C_1 = \frac{1}{2}, \quad C_2 = -\frac{1}{6} \quad (89)$$

The IVP thus has the unique solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} e^t - \frac{1}{6} e^{3t} - \frac{1}{3} \\ \frac{1}{2} e^t + \frac{1}{6} e^{3t} + \frac{1}{3} \end{pmatrix} \quad (90)$$

### Choice of an ansatz

As in the scalar case, we choose as ansatz for a special solution  $\mathbf{x}_s$  an ansatz of the same type as the source term  $\mathbf{g}(t)$  with coefficients which are yet undetermined. These coefficients are then determined by plugging the ansatz into the inhomogeneous system. The choice of a suitable ansatz can be made according to the Table above, if one reads the entries of the table vectorially.

Observe that the choice of an ansatz always has to be made globally, and not componentwise: If there are different source terms in the various ODE's of the system, then these component source terms all have to be taken into account in the choice of the ansatz for  $\mathbf{x}_s$ .

**Example** (Continuation). Consider the IVP

$$\begin{aligned} \dot{x}_1 &= 2x_1 - x_2 + 1, & x_1(0) &= 0 \\ \dot{x}_2 &= -x_1 + 2x_2 - 1, & x_2(0) &= 1 \end{aligned} \quad (91)$$

The general solution of the homogeneous system is given by

$$\mathbf{x}_h = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} \quad (92)$$

To find a special solution, we select an ansatz of the type of the source term  $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ , i.e. the constant ansatz

$$\mathbf{x}_p = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad (93)$$

By plugging this ansatz into the initial IVP, we obtain the system of equations

$$\begin{cases} 0 &= 2A_1 - A_2 + 1 \\ 0 &= -A_1 + 2A_2 - 1 \end{cases} \quad (94)$$

In this example the left sides of the equations are zero because the derivative of the ansatz is zero! With the system of equations above it follows that

$$A_1 = -\frac{1}{3}, \quad A_2 = \frac{1}{3} \quad (95)$$

and thus the special solution

$$\mathbf{x}_s(t) = \frac{1}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (96)$$

Combining  $\mathbf{x}_h$  and  $\mathbf{x}_s$ , this leads to the general solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + \frac{1}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (97)$$

To solve the IVP we plug in the initial conditions into the general solution and get the same solution as above.

## Part IV

# Dynamical Systems

## Overview

A *dynamical system* is a time-dependent process which is described by a mathematical model and whose temporal evolution is completely determined by its initial state.

The set of all possible states of a dynamical system is its *phase space*, and the temporal evolution in phase space is the *flow* of the dynamical system.

- A *continuous* dynamical system is a system of differential equations of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R} \quad (98)$$

Time in such systems is in principle measured continuously.

The solutions of these systems are differentiable functions. Solutions are also called *trajectories* or *orbits* of the system.

- A *discrete* dynamical system is a system of difference equations of the form

$$x_{n+1} = f(x_n) \quad (99)$$

Time in physical systems sometimes also has to be measured in discrete steps.

The solutions of this equation are sequences  $(x_n)$ .

## One-Dimensional Systems

### Fixed Points in Continuous Dynamical Systems

A *fixed point* of the a continuous dynamical system is an  $x^*$  with the property

$$\dot{x} = f(x^*) = 0 \quad (100)$$

If  $x^*$  is a fixed point, then  $x(t) = x^*$  is a constant solution of the system.

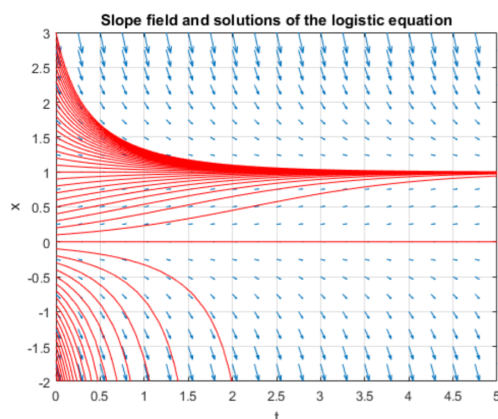
For the solution  $x(t)$  of a one-dimensional system there are the following possibilities:

- The solution is a fixed point
- The solution grows monotonically, i.e.  $\dot{x}(t) > 0$
- The solution decays monotonically, i.e.  $\dot{x}(t) < 0$

Example: We consider the ODE of limited growth

$$\dot{N} = rN \left(1 - \frac{N}{K}\right) \quad (101)$$

If  $K = 1$ ,  $\dot{N}$  is 0 if  $N = 0$  and  $N = 1$  i.e. the fixed points for this equation are  $x_1 = 0$  and  $x_2 = 1$  (or  $x_2 = K$ ). These fixed points are also visible in the following figure.



### Continuous Linear stability analysis

Let  $x^*$  be a fixed point of the continuous dynamical system  $\dot{x} = f(x)$ . The fixed point  $x^*$  is

- stable if  $f'(x^*) < 0$ ,
- unstable, if  $f'(x^*) > 0$ ,
- stable or unstable, if  $f'(x^*) = 0$

Example (continuation): the derivative of the logistic equation above ( $f(x) = rx(1 - \frac{x}{K})$ ) is

$$f'(x) = r \left(1 - \frac{2x}{K}\right) \quad (102)$$

Hence

$$\begin{aligned} f'(x_1) &= f'(0) = r > 0 \\ f'(x_2) &= f'(K) = -r < 0 \end{aligned} \quad (103)$$

Hence  $x_1 = 0$  is an unstable fixed point, and  $x_2 = K$  is a stable fixed point, which is also visible in the figure above.

### Fixed points in Discrete Dynamical Systems

A *fixed point* of the discrete dynamical system  $x_{n+1} = f(x_n)$  is a  $x^*$  with the property

$$x^* = f(x^*) \quad (104)$$

If  $x^*$  is a fixed point of  $x_{n+1} = f(x_n)$ , then  $x_n = x^*$  is a constant solution of that system.

Example: We want to find the fixpoint of the discrete dynamical system

$$x_{n+1} = \cos(x_n) \quad (105)$$

Using the formula above:

$$x^* = \cos(x^*) \Rightarrow x^* \approx 0.739 \quad (106)$$

### Discrete Linear stability analysis

Let  $x^*$  be a fixed point of the discrete dynamical system  $x_{n+1} = f(x_n)$ . The fixed point  $x^*$  is

- stable, if  $|f'(x^*)| < 1$ ,
- unstable, if  $|f'(x^*)| > 1$ ,
- stable or unstable, if  $|f'(x^*)| = 1$

$$\eta_{n+1} = \eta_n \cdot f'(x^*) \Rightarrow \eta_n = \eta_0 \cdot (f'(x^*))^n \quad (107)$$

Example: (continuation): We want to analyse the fixpoint  $x^* \approx 0.739$  of the example above.

$$|f'(x^*)| = |\sin(x^*)| \approx |-0.674| < 1 \quad (108)$$

Hence  $x^*$  is a stable fixed point of the system.

### Bifurcations in Continous Systems

#### Fundamentals

A qualitative change of the dynamics of a dynamical system

$$\dot{x} = f(x, r) \quad \text{or} \quad x_{n+1} = f(x_n, r) \quad (109)$$

depending on a parameter  $r$  is a bifurcation of the system

#### Standard Bifurcations in discrete systems

##### Saddle-node Bifurcation

The saddle-node bifurcation is a bifurcation in which fixed points are created or destroyed by the variation of a system parameter.

Example: Dynamical system, depending on the parameter  $r$

$$\dot{x} = r + x^2 \quad (110)$$

Fixed points:

$$\begin{aligned} r < 0 : x_{1,2}^* &= \pm\sqrt{-r} \\ r = 0 : x^* &= 0 \\ r > 0 : &\text{none} \end{aligned} \quad (111)$$

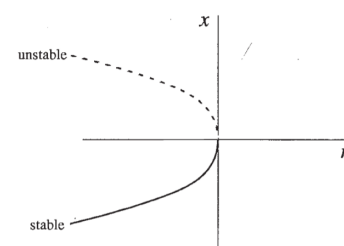
Stability of the fixed points for  $r < 0$ :

$$f(x) = r + x^2, \quad f'(x) = 2x \quad (112)$$

hence

$$\begin{aligned} x_1^* &= \sqrt{-r} > 0 : f'(x_1^*) > 0, \text{unstable} \\ x_2^* &= -\sqrt{-r} < 0 : f'(x_2^*) < 0, \text{stable} \end{aligned} \quad (113)$$

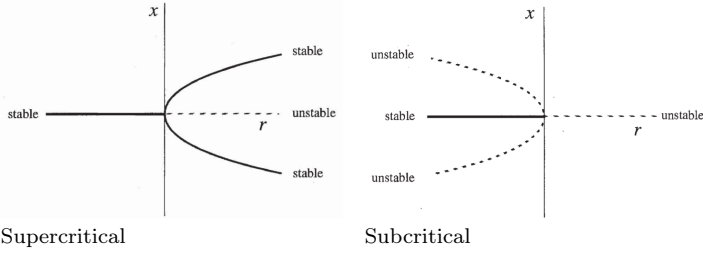
Fixed points plotted in dependence of  $r$ :





### Pitchfork Bifurcation

The pitchfork bifurcation is a bifurcation in which a fixed point changes its stability and two new fixed points are created.

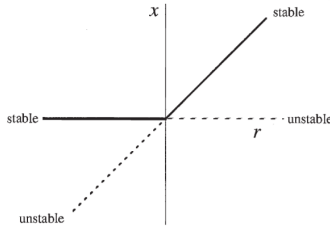


Supercritical: Additional fixed points only for parameter values greater (“super”) than the bifurcation point.

Subcritical: Additional fixed points only for parameter values smaller (“sub”) than the bifurcation point.

### Transcritical Bifurcation

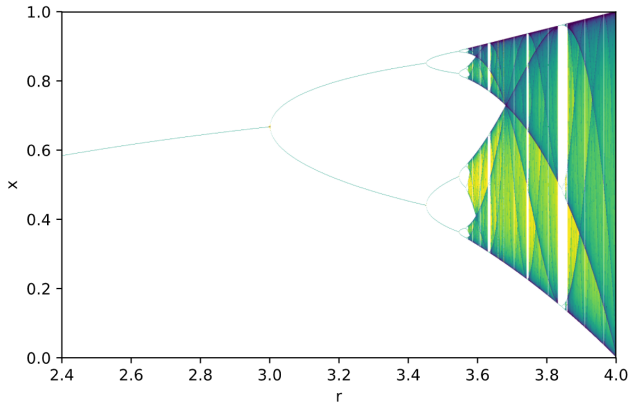
A transcritical bifurcation is a bifurcation in which the stability of two fixed points is interchanged.



## Bifurcations in Discrete Systems

The bifurcations which we introduced in continuous systems can also occur in discrete systems.

Discrete systems can (or do allways?) show a chaotic behaviour in the parameter space:



Example: Consider the discrete dynamical system depending on the parameters  $\mu \geq 0, \varepsilon \geq 0$  (Adler system)

$$x_{n+1} = x_n + \mu + \varepsilon \sin(x_n) \quad (\mu \geq 0, \varepsilon \geq 0) \quad (114)$$

Fixed points  $x^*$  are solutions of the equation  $\mu + \varepsilon \sin(x) = 0$ , hence

$$x_{1,k}^* = -\arcsin\left(\frac{\mu}{\varepsilon}\right) + k \cdot 2\pi, \quad x_{2,k}^* = (2k+1)\pi + \arcsin\left(\frac{\mu}{\varepsilon}\right) \quad (115)$$

The fixed points only exist in the case  $\mu \leq \varepsilon$

For  $\mu = 1$ , the fixed points only exist in the case  $\varepsilon \geq 1$ . We have  $f(x) = x + \mu + \varepsilon \sin(x)$ , hence  $f'(x) = 1 + \varepsilon \cos(x)$ . It follows that for  $\mu = 1$

$$\begin{aligned} f'(x_{1,k}^*) &= 1 + \varepsilon \cos\left(-\arcsin\left(\frac{1}{\varepsilon}\right)\right) \\ &= 1 + \varepsilon \sqrt{1 - \frac{1}{\varepsilon^2}} \\ &= 1 + \sqrt{\varepsilon^2 - 1} \geq 1 \end{aligned} \quad (116)$$

Hence the fixed points  $x_{1,k}^*$  are unstable.

$$\begin{aligned} f'(x_{2,k}^*) &= 1 + \varepsilon \cos\left(\pi + \arcsin\left(\frac{1}{\varepsilon}\right)\right) \\ &= 1 - \varepsilon \cos\left(\arcsin\left(\frac{1}{\varepsilon}\right)\right) \\ &= 1 - \varepsilon \sqrt{1 - \frac{1}{\varepsilon^2}} \\ &= 1 + \sqrt{\varepsilon^2 - 1} \end{aligned} \quad (117)$$

For the stability of  $x_{2,k}^*$  it is necessary that  $|1 - \sqrt{\varepsilon^2 - 1}| < 1$  holds. This is true if and only if  $0 < \sqrt{\varepsilon^2 - 1} < 2$ , hence in the case  $1 < \varepsilon^2 < 5$  resp.  $1 < \varepsilon < \sqrt{5}$ . The fixed points  $x_{2,k}^*$  are thus stable in the case  $1 < \varepsilon < \sqrt{5}$ .

## Two-Dimensional Systems

We now consider the systmes for arbitrary  $n \in (N)$ , i.e.

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \dots, x_n) \\ \dot{x}_2 &= f_2(x_1, x_2, \dots, x_n) \\ &\dots \\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n) \end{aligned} \quad (118)$$

resp. in vector notation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad (\mathbf{x} \in (R)^n) \quad (119)$$

### Definitions

A point  $x^* \in \mathbb{R}^n$  is a *fixed point* of the dynamical system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ , if

$$\mathbf{f}(\mathbf{x}^*) = \mathbf{0} \quad (120)$$

If  $\mathbf{x}^*$  is a *fixed point* of the dynamical system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ , then

$$\mathbf{x}(t) = \mathbf{x}^* \quad (121)$$

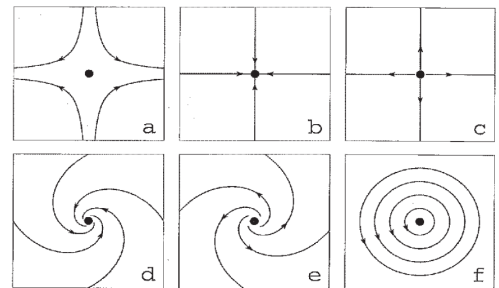
is a constant solution.

A fixed point  $x^* \in \mathbb{R}^n$  of the dynamical system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ , is

- *stable*, if the solution starting close to  $x^*$  stays close to  $x^*$  for all  $t \geq 0$ .
- *asymptotically stable*, if the solution starting close to  $x^*$  converge to  $x^*$  for  $t \rightarrow \infty$ .
- *unstable*, if the solution starts close to  $x^*$  and diverging from  $x^*$  for  $t \rightarrow \infty$ .

## Linear Systems

Overview over the various types of fixed points and the corresponding geometry of the system near the fixed point:



- |                  |                                |
|------------------|--------------------------------|
| a) saddle point  | e) unstable spiral             |
| b) stable node   | f) center/elliptic fixed point |
| c) unstable node | g) degenerate cases            |
| d) stable spiral |                                |

A two-dimensional system can be characterized with the eigenvalues like so:

- $\lambda_1 < 0, \lambda_2 < 0$ : stable node
- $\lambda_1 > 0, \lambda_2 < 0$ : saddle point
- $\lambda_1 > 0, \lambda_2 > 0$ : unstable point
- $\lambda_1 = 0, \lambda_2 = 0$ : non-isolated fixed point
- $\lambda_{1,2} \neq \mathbb{R}, \text{Re}(\lambda_{1,2}) = 0$ : center

- $\lambda_{1,2} \notin \mathbb{R}, \operatorname{Re}(\lambda_{1,2}) > 0$ : unstable spiral
- $\lambda_{1,2} \notin \mathbb{R}, \operatorname{Re}(\lambda_{1,2}) < 0$ : stable spiral

Alternativ characterization instead of with the eigenvalues  $\lambda_1, \lambda_2$  with trace and determinant of A:

$$\begin{aligned}\tau &= \operatorname{tr}(A) = a_{11} + a_{22} \\ \Delta &= \det(A) = a_{11}a_{22} - a_{12}a_{21}\end{aligned}\quad (122)$$

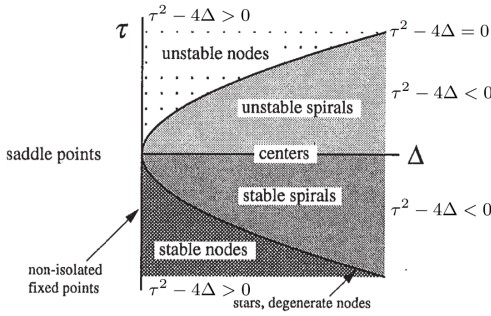
Connection with the eigenvalues:

$$\lambda_{1,2} = \frac{1}{2} \left( \tau \pm \sqrt{\tau^2 - 4\Delta} \right) \quad (123)$$

and

$$\Delta = \lambda_1 \lambda_2, \quad \tau = \lambda_1 + \lambda_2 \quad (124)$$

In the following figure the classification of fixed points based on the properties of  $\tau$  and  $\Delta$  is shown graphically.



If one is only interested in the stability of  $\mathbf{x}^*$  (and not in a further classification into nodes, spirals, etc.), the fixed point  $\mathbf{x}^* = 0$  of a dynamical system is

- asymptotically stable, if

$$\operatorname{Re}(\lambda_i) < 0 \quad (125)$$

holds for all eigenvalues  $\lambda_i$  of A

- stable, if

$$\operatorname{Re}(\lambda_i) \leq 0 \quad (126)$$

holds for all eigenvalues  $\lambda_i$  of A

- unstable, if

$$\operatorname{Re}(\lambda_i) > 0 \quad (127)$$

holds for at least one eigenvalue  $\lambda_i$  of A

## Nonlinear Systems

In order to use the analysis of linear systems, we linearize the nonlinear system near the fixed point  $x^*$ . To this end, we consider small perturbations  $u = x_1 - x_1^*, v = x_2 - x_2^*$  and deduce equations for  $u$  and  $v$  by expanding  $f_1$  and  $f_2$  into Taylor series. The temporal evolution of  $(u, v)$  is thus given by

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}_{(x_1^*, x_2^*)} \cdot \begin{pmatrix} u \\ v \end{pmatrix} + \text{higher-order terms} \quad (128)$$

The matrix

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}_{(x_1^*, x_2^*)} \quad (129)$$

is the Jacobi matrix of the system near the fixed point  $\mathbf{x}^* = (x_1^*, x_2^*)$ .

Example: Consider the system

$$\begin{aligned}\dot{x} &= -x + x^3 \\ \dot{y} &= -2y\end{aligned}\quad (130)$$

fixed points are:

$$P_1 = (0, 0), \quad P_2 = (1, 0), \quad P_3 = (-1, 0). \quad (131)$$

The Jacobi matrix of the system is

$$A = \begin{pmatrix} -1 + 3x^2 & 0 \\ 0 & -2 \end{pmatrix} \quad (132)$$

The eigenvalues of A are:

$$P_1 : \lambda_1 = -1, \lambda_2 = -2; \quad P_2, P_3 : \lambda_{1,2} = \pm 2 \quad (133)$$

Therefore  $P_1$  is a stable node, and  $P_2$  and  $P_3$  are saddle points of the linearized system.

Intuitively: The linearization leads to the correct classification of the fixed points of nonlinear systems, if the fixed point type is described by an inequality and not an equality.

### Hartman-Grobman theorem

If both eigenvalues  $\lambda_1, \lambda_2$  of the corresponding system matrix A fulfill the condition  $\operatorname{Re}(\lambda_i) \neq 0$ , then the behaviour of the nonlinear system is for sufficiently small perturbations topologically equivalent to the behaviour of the linearized system.

### Classification of fixed points into robust and marginal cases

*Robust*: Nodes, spirals, saddle points

*Marginal*: centers/elliptic fixed points, degenerate cases

### Conserved Quantity

Let  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$  be an autonomous dynamical system.

A *conserved quantity* (or integral) of the system is a continuous nonconstant function  $E$ , which is constant on solutions of the system above, i.e. if  $t \mapsto x(t)$  is a solution of the system, then we have

$$\frac{d}{dt} E(\mathbf{x}(t)) = 0 \quad (134)$$

The system above is *conservative*, if there is a conserved quantity in the system.

It is in general difficult to decide, whether there is a conserved quantity in a given system. However, if there is a fixed point in the system which is of another type than saddle point or center, the system cannot be conservative.

### Example: Checking if a quantity is conserved

We consider the dynamical system

$$\begin{aligned}\dot{x} &= \sin(y) \\ \dot{y} &= \sin(x)\end{aligned}\quad (135)$$

We want to check that  $E(x, y) = \cos(x) - \cos(y)$  is a conserved quantity of the system.

$$\begin{aligned}\frac{dE}{dt} &= \frac{\partial E}{\partial x} \cdot \dot{x} + \frac{\partial E}{\partial y} \cdot \dot{y} \\ &= -\sin(x) \cdot \sin(y) + \sin(y) \cdot \sin(x) \\ &= 0\end{aligned}\quad (136)$$

We conclude that the quantity is conserved.

### Example: Find conserved quantity

We consider the dynamical system  $\ddot{x} = x^3 - x$

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x^3 - x\end{aligned}\quad (137)$$

In order to find a conserved quantity, we write the system in the form  $\ddot{x} + V'(x) = 0$ . This is true for  $V(x) = -\frac{1}{4}x^4 + \frac{1}{2}x^2$  ( $V(x)$  is the integral of the negative right side of the initial equation), and a conserved quantity is the total energy

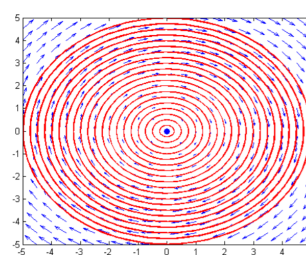
$$E(x, y) = \frac{1}{2}y^2 + V(x) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4 \quad (138)$$

(Control:  $\frac{d}{dt} E(x, y) = \frac{\partial E}{\partial x} \dot{x} + \frac{\partial E}{\partial y} \dot{y} = \dots = 0$ )

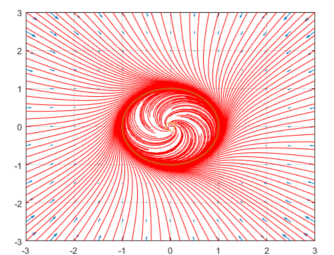
### Limit cycles

If a fixed point is of the type “center”, it is surrounded by nothing but closed trajectories. If there is a single isolated trajectory in the system, one speaks of a limit cycle.

A limit cycle of a dynamical system is an isolated closed trajectory of the system.



Non-isolated closed orbits



Isolated closed orbits, limit cycle



Limit cycles are important structuring elements of the phase space of a 2D dynamical system, in addition to fixed points

#### Methods for detection (Poincaré-Bendixson theorem)

Let  $\dot{\mathbf{x}} = f(\mathbf{x})$  be a dynamical system in a bounded and closed subset  $G$  of  $\mathbb{R}^2$ . If the system does not have a fixed point in  $G$  and if there exists a trajectory  $C$ , which stays in  $G$  for all  $t \geq t_0$ , then  $C$  is either a closed trajectory, or it converges to a closed trajectory. If a trajectory stay confined to a bounded closed region without fixed point, it has to converge to a closed trajectory, if it is not already a closed trajectory itself. In particular, a chaotic behaviour is impossible! Note that the result holds only for  $n = 2$ .

Example: Consider the dynamical system: (image on the top right)

$$\begin{aligned}\dot{x} &= x + y - x(x^2 + y^2) \\ \dot{y} &= -x + y - y(x^2 + y^2)\end{aligned}\quad (139)$$

In order to apply the *Poincaré-Bendixson* theorem we choose

$$G = \{(x, y), \in \mathbb{R}^2 | 0.5 \leq \sqrt{x^2 + y^2} \leq 2\}.$$
 (140)

i.e. a torus with inner radius of 0.5 and an outer radius of 2. The system has no fixed point in  $G$  since the fixed point lays at the origin, which is not contained in  $G$ .

If the equations are formulated in polar coordinates one gets

$$\begin{aligned}\dot{r} &= r(1 - r^2) \\ \dot{\theta} &= -1\end{aligned}\quad (141)$$

For  $r = 0.5$  we have  $\dot{r}(t) > 0$ , whereas for  $r = 2$  have the reverse estimate  $\dot{r}(t) < 0$ . Therefore  $r(t)$  *increases* for a trajectory of the system at the inner limit of the torus  $G$ , whereas  $r(t)$  *decreases* for a trajectory at the outer limit of the region. This implies that a trajectory starting in  $G$  must stay in  $G$  for all times. The conditions for applying the *Poincaré-Bendixson* theorem are thus fulfilled, and it follows from the theorem that the system has a limit cycle.

#### Bifurcations

A bifurcation occurs in the system  $\dot{x} = f(x, r)$ , if the phase portrait changes its topological structure as the parameter  $r$  is varied.

All bifurcations occurring in 1D systems also occur in 2D systems.

In addition, there exists a new type of bifurcation which occurs only in systems of dimension  $n \leq 2$ : the Hopf bifurcation.

##### Hopf Bifurcation

Creation/deletion of a limit cycle by varying a system parameter. This describes the ways in which oscillations in a system can be turned on or of.

If in a linear stability analysis  $\Delta > 0$  (the fixed point is a center) it is possible for a Hopf bifurcation to occur.

- if  $\tau < 0$ , the fixed point is stable and thus, no limit cycle can occur.
- if  $\tau > 0$ , the fixed point is unstable and thus, limit cycles can occur. This has to be hecked with the *Poincaré-Bendixson theorem*.

### Three-Dimensional Systems and Chaos

Chaos is aperiodic long-term behaviour in a deterministic system with sensitive dependence on the initial conditions.

Dimension  $n \geq 3$ : Trajectories can remain bounded without converging to a fixed point or a closed orbit. Possibly sensitive dependence on the initial conditions, i.e. slightly differing initial conditions lead to a totally different long-term behaviour

#### Rössler System

The following nonlinear system was constructed by Rössler in 1976:

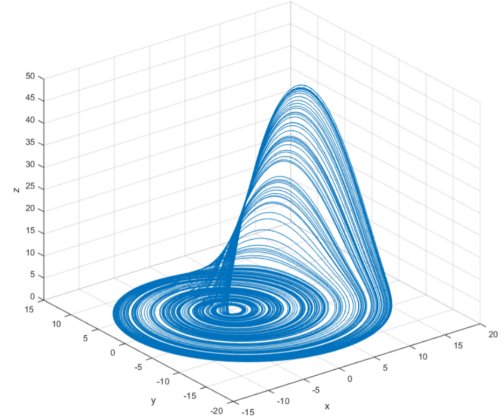
$$\begin{aligned}\dot{x} &= -(x + z) \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned}\quad (142)$$

It depends on the 3 parameters  $a, b, c$ ; we set  $a = 0.2, b = 0.2$  and consider the system for varying values of  $c$ . Depending on the parameter values, the system has at most two fixed points. Depending on the parameter values, the system has at most two fixed points, namely

$$P_{1,2} = \left( \frac{1}{2} \left( c \mp \sqrt{c^2 - 4ab} \right), \frac{1}{2a} \left( -c \pm \sqrt{c^2 - 4ab} \right), \frac{1}{2a} \left( c \mp \sqrt{c^2 - 4ab} \right) \right) \quad (143)$$

The stability of these fixed points can be determined by considering the eigenvalues of the Jacobian of the system. Empirical investigations show that for most parameter values for which the fixed points  $P_{1,2}$  exist, at least one of the eigenvalues of the Jacobian matrix has a positive real part and is therefore unstable.

The behaviour of the Rössler system can vary greatly, depending on the values of the parameters. For  $c = 2.3$  and  $c = 2.9$ , the system converges to a period-one and a periodtwo limit cycle, respectively. For  $c = 8.5$ , we observe a chaotic attractor.



#### Lorenz System

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy\end{aligned}\quad (144)$$

We investigate the Lorenz system for the parameter values  $\sigma = 10, b = \frac{8}{3}$ , and various values of  $r$ , in most cases  $r = 28$ .

We first determine fixed points of the system. To this end, we solve the system of equations and obtain the three fixed points

$$P_1^* = (0, 0, 0), \quad P_{2,3}^* = \left( \pm \sqrt{b(r-1)}, \pm \sqrt{b(r-1)}, r-1 \right) \quad (145)$$

The fixed points  $P_2^*$  and  $P_3^*$  only exist in the case  $r \geq 1$ ; at the parameter value  $r = 1$  there is a bifurcation. The Jacobian of the Lorenz system is

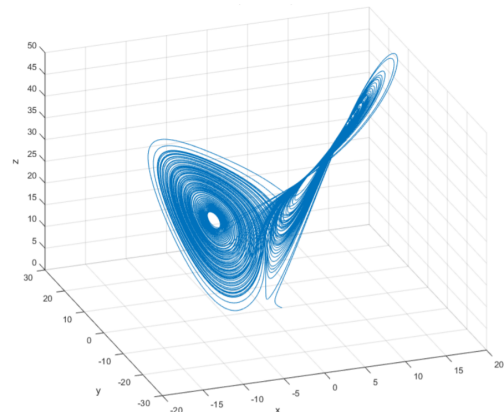
$$J(x, y, z) = \begin{pmatrix} \sigma & -\sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix} \quad (146)$$

At  $P_1^*$  the eigenvalues of  $J$  are, depending on  $r$ , with  $\sigma = 10$  and  $b = \frac{8}{3}$ ,

$$\lambda_1 = -\frac{8}{3}, \quad \lambda_{2,3} = \frac{1}{2} (-11 \pm \sqrt{81 + 40r}) \quad (147)$$

For  $r < 1$  all three eigenvalues are negative and  $P_1^*$  therefore a stable fixed point; for  $r > 1$ , we have  $\lambda_2 > 0$ , and  $P_1^*$  is therefore unstable. Concerning  $P_2^*$  and  $P_3^*$ , from a corresponding analysis of the eigenvalues we see that these points are stable for  $0 < r < r_1 \approx 24.737$  and unstable for  $r > r_1$ .

The greatest degree of instability thus exists in the case  $r > r_1$ , e.g. for the typical value  $r = 28$ .



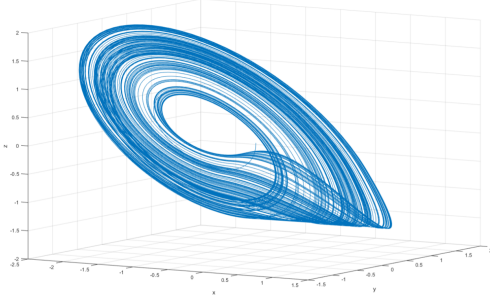
## Linz/Sprott System

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One of the simplest examples of a 3-dimensional continuous system exhibiting chaotic dynamics is the following nonlinear third-order ODE.

$$x^{(3)} + a\ddot{x} + b\dot{x} - |x| + 1 = 0 \quad (148)$$

Note that the absolute value  $|x|$  is the only nonlinearity on the system. A simulation of the system for  $a = 0.6, b = 1$  (and the initial conditions  $x(0) = -0.1, y(0) = z(0) = 0$ ) shows that the system converges to a chaotic attractor which somewhat resembles a Möbius strip:



## Discrete Dynamical Systems

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### Lyapunov exponent

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The Lyapunov exponent  $\lambda$  of the discrete dynamical system  $x_{n+1} =$

$f(x_n)$  for an orbit starting at  $x_0$  is

$$\lambda = \lim_{n \rightarrow \infty} \left( \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \right) \quad (149)$$

Significance of the sign of  $\lambda$ :

- $\lambda > 0$ : The system behaves chaotically
- $\lambda < 0$ : The system does not behave chaotically
- $\lambda = 0$ : The system is at a bifurcation point

### Mandelbrot set

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The Julia set  $J$  of the complex dynamical system  $z_{n+1} = f(z_n)$  is the boundary between the set of starting points with bounded orbits and the set of starting points with unbounded orbits.

### Mandelbrot set

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The Mandelbrot set  $M$  of the complex discrete dynamical system

$$z_{n+1} = z_n^2 + c \quad (150)$$

is the set of parameters  $c \in C$ , for which the set  $(z_n)_n \in N$  of points generated by the dynamics with initial value  $z_0 = 0$  remains bounded:

$$M = \{c \in C | f_c^n(0) \not\rightarrow \infty, \text{ for } n \rightarrow \infty\} \quad (151)$$

Remark. The Mandelbrot set thus is a set of parameters, for which the equation above has a certain properties, namely that the sequence of points starting at the origin and evolving by the dynamics remains bounded.