Ordinary Differential Equations and Dynamical Systems

Part I

Modeling

Fundamentals

Formally, an ordinary differential equation is an equation, in which a function and its derivates and the independent variable appear.

An (implicit) *ordinary differential equation* of order n is an equation of the form

$$F(x, y, y', y'', ..., y^{(n)}) = 0. \tag{1}$$

An example of an explicit ODE of order n is of the form

$$y^n = G(x, y, y', y'', ..., y^{n-1}). \tag{2}$$

Classification

Differential equations can be classified according to various criteria. Besides the order of an ODE we are also interested in whether an ODE is linear, homogeneous, has constant coefficient, is separable or autonomous.

Linearity
An *n-th* order ODE is *linear*, if it is of the form:

$$a_n(x) \cdot y^{(n)} + ... + a_1(x) \cdot y' + a_0(x) \cdot y = g(x) \tag{3}$$

where $a_n(x), ..., a_1(x), a_0(x)$ are $g(x)$ fixed functions. Or in other words: A differential equation is linear if the dependant variable and all of its derivatives appear in a linear fashion (i.e., they are not multiplied together or squared for example or they are not part of transcendental functions such as sins, cosines, exponentials, etc.)

Homogenity
A lienar ODE is *homogeneous*, if $g(x) = 0$ for all x; otherwise the ODE is *inhomogeneous*, and $g(x)$ is the *inhomogeneity* or *source* term.

Constant coefficient
A linear ODE has *constant coefficients*, if it is of the form

$$a_n \cdot y^{(n)} + ... + a_1 \cdot y' + a_0 \cdot y = g(x), \tag{4}$$

with $a_n \neq 0$ (the source term $g(x)$ does not have to be constant).

Separability
The ODE is *separable*, if $F(x, y)$ can be written as a product of a x- and y-dependent term, i.e. if the ODE is of the form

$$y' = g(x) \cdot h(y) \tag{5}$$

Autonomity
The ODE (1.28) is *autonomous*, if $F(x, y)$ only depends on y, i.e. if the ODE is of the form

$$y' = h(y) \tag{6}$$

Every autonomous ODE is separable with $g(x) = 1$.

Examples

$y' = f(x)$	Inhomogeneous linear ODE for $y(x)$ with source term $f(x)$
$m \cdot \dot{v} = m \cdot g - k \cdot v^2$	Nonlinear ODE for $v(t)$
$l \cdot \ddot{\Phi} + g \cdot sin(\Phi) = 0$	Nonlinear ODE for $\Phi(t)$
$l \cdot \ddot{\Phi} + g \cdot \phi$	Homogeneous linear ODE for $\Phi(t)$
$l \cdot \ddot{\Phi} + g \cdot \phi = sin(\omega t)$	Inhomogeneous linear ODE for $\Phi(t)$ with source term $sin(\omega t)$
$i'' + \frac{R}{L} i' + \frac{1}{LC} i = 0$	Homogeneous linear ODE for $i(t)$

Systems of differential equations

If several systems are coupled with each other and mutually influence each other, one often obtains a system of ODE's.
A *system of differential equations* of first oder is a system

$$\begin{aligned} y_1' &= f_1(x, y_1, ..., y_n) \\ &\vdots \\ y_n' &= f_n(x, y_1, ..., y_n) \end{aligned} \tag{7}$$

of ODE's for unknown functions $y_1(x), ..., y_n(x)$.
Using the vectorial notation

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}) \tag{8}$$

An ODE of n-th order is equivalent to a system of first-order ODE's.

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_n' &= f(x, y_1, ..., y_n) \end{aligned} \tag{9}$$

Example: 2nd-order ODE to system of first-order ODE's
We want to rewrite the following 2nd order ODE into a system of first-order ODE's.

$$\ddot{x}(t) + 2\delta\dot{x}(t) + \omega_0^2 x(t) = f(t) \tag{10}$$

If we introduce the vector-valued function

$$\mathrm{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \quad \Rightarrow \quad \dot{\mathrm{y}} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} \tag{11}$$

rewriting:

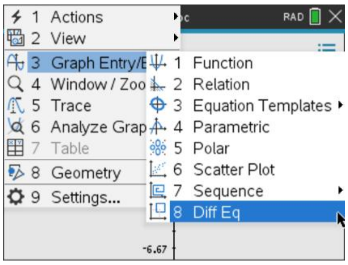
$$\begin{aligned} \dot{\mathrm{y}} &= \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -2\delta\dot{x}(t) - \omega_0^2 x(t) + f(t) \end{pmatrix} \\ \dot{\mathrm{y}} &= \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\delta \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \\ \dot{\mathrm{y}} &= \mathrm{A}\mathbf{y} + \mathbf{b} \end{aligned} \tag{12}$$

Slope field

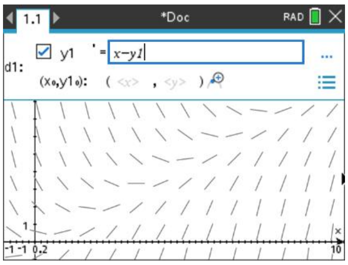
Slope fields are often loead to a good qualitative understanding of the situation described by the ODE under consideration. Slope field can be understood in the following way: To each point (x, y) in the region B under consideration, $F(x, y)$ is a value which discribes the slope of the solution curve passing through the point (x, y).

Example with calculator
We want to plot the slope field of the following ODE

$$y' = x - y \tag{13}$$



Select: Menu, 3: Graph Entry/Edit, 8: Diff Eq.



Write down the ODE

Solvability

Two solution curves of an ODE cannot cross.

Part II

Ordinary Differential Equations

Analytical methods for first-order ODE's

Overview

Separable ODE's
Linear ODE's
Exact ODE's

Separable ODE's

Example
We compute the general solution of the ODE

$$y' = -\frac{x}{y} \tag{14}$$

- We write the eqution as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} \tag{15}$$

-We bring all x-dependent terms to the left hand side and all y-dependent terms on the right hand side:

$$y\,\mathrm{d}y = -x\,\mathrm{d}x \tag{16}$$

-We integrate on both sides and get

$$\int y\,\mathrm{d}y = -\int x\,\mathrm{d}x \Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C, \quad C \in \mathbb{R} \tag{17}$$

- We solve for y and get

$$y = \pm\sqrt{K - x^2}, \quad K \in \mathbb{R} \;\; (\text{where } K = 2C) \tag{18}$$

Example of substitution
We consider the ODE

$$y' = (x+y)^2 \tag{19}$$

Exact ODE's

Analytical methods for linear ODE's

Overview

We differentiate bewteen first-order linear ODE's and higher-oder ODE's as well as between homogeneous and inhomogeneous ODE's.

The general solution of the inhomogeneous ODE is the sum

$$y = y_h + y_s, \tag{20}$$

where y_h is the general solution of the homogeneous ODE and y_s any special solution of the inhomogeneous ODE.

First-order linear ODE's

To solve a first-order linear ODE we thus have to find y_h and y_s
$\mathbf{y_h}$: a homogeneous first-order ODE is separable and can therefore be solved by the standard procedure for separable ODE's described above.
$\mathbf{y_s}$: To find a special solution of and ODE, there are several possibilities. In the case of an ODE with constant coefficients, it usually suffices to choose for y_s an *ansatz of the form of the source term $g(x)$*. In the case of non-constant coefficients, the method *variation of constants* usually works better.

Example using the ansatz
We solve the ODE

$$y' + ay = b \tag{21}$$

- To determine y_h, we integrate the homogeneous ODE

$$y' + ay = 0 \tag{22}$$

by separation of variables we obtain the solution

$$y_h = C \cdot e^{-ax}, \quad C \in \mathbb{R}. \tag{23}$$

- For finding y_s we choose the ansatz in the form of the source term. In this case the source term is constant, $g(x) = b$. Therefore we assume that the special solution y_s is constant as well, i.e. we make an ansatz $y_s = c$. We plug this ansatz into the inhomogeneous ODE and obtain the special solution y_s.

$$y_s' + ay_s = b \quad \Rightarrow \quad y_s = \frac{a}{b} \tag{24}$$

The general solution therefore is

$$y = C \cdot e^{-ax} + \frac{a}{b}, \quad C \in \mathbb{R} \tag{25}$$

Ansatz functions for the solution of the inhomogeneous first-order ODE

Source term $g(x)$	Ansatz y_s
$g(x) = b_0$	$y_s = c_0$
$g(x) = b_1x + b_0$	$y_s = c_1x + c_0$
$g(x) = b_2x^2 + b_1x + b_0$	$y_s = c_2x^2 + c_1x + c_0$
$g(x) = \sum_{i=0}^n b_i x^i$	$y_s = \sum_{i=0}^n c_i x^i$
$g(x) = A\,sin(\omega x) + B\,cos(\omega x)$	$y_s = C_1 sin(\omega x) + C_2 cos(\omega x)$ $y_s = C sin(\omega x + \varphi)$
$g(x) = Ae^{bx}$	$y_s = \begin{cases} \frac{A}{b+a}e^{bx} & \text{for } b \neq -a \\ Axe^{-ax} & \text{for } b = -a \end{cases}$

In addition, the following rules must be followed:
Linearity If $g(x)$ is a linear combination of several source terms, one has to assume as ansatz for $y_s(x)$ a correponding linear combination of several ansatz terms.
Resonance If the source term $g(x)$ is itself already a solution of the homogeneous ODE, the correponding ansatz for y_s has to be multiplied with x. So if for example $y_h = Ce^x$, and $g(x) = e^x$, the ansatz $y_s = \mathbf{x} \cdot e^x$ is choosen.

Example using Variation of constants
The idea behind the variation of constants is to start from the solution $y = K \cdot e^{-F(x)}$ of the homogeneous ODE and plug the ansatz

$$y = K(x) \cdot e^{-F(x)} \tag{26}$$

into the inhomogeneous ODE.
As an example we want to solve the inhomogeneous linear ODE

$$3y' + 5y = 7e^{\frac{1}{3}x}. \tag{27}$$

- We first find the homogeneous solution y_h by separation of variables.

$$\frac{1}{y}\mathrm{d}y = -\frac{5}{3}x\,\mathrm{d}x \quad \Rightarrow \quad y_h = e^{\frac{5}{3}x} \cdot K \tag{28}$$

- Next we calculate the ansatz and its derivative

$$\begin{aligned} y_s &= e^{-\frac{5}{3}x} \cdot K(x) \\ y_s' &= K(x)'e^{-\frac{5}{3}x} + K(x) \cdot -\frac{5}{3}e^{-\frac{5}{3}x} \end{aligned} \tag{29}$$

- The ansatz is then pluged into the inhomogeneous ODE

$$3\left(K(x)'e^{-\frac{5}{3}x} + K(x) \cdot -\frac{5}{3}e^{-\frac{5}{3}x}\right) + 5\left(e^{-\frac{5}{3}x} \cdot K(x)\right) = 7e^{\frac{1}{3}x} \tag{30}$$

- We solve for $K(x)'$ ($K(x)$ usually dissapears)

$$3 \cdot K(x)'e^{-\frac{5}{3}x} = 7e^{\frac{1}{3}x} \quad \Rightarrow \quad K(x)' = \frac{7}{3}e^{\frac{1}{3}x + \frac{5}{3}x} = \frac{7}{3}e^{2x} \tag{31}$$

- We integrate and find $K(x)$

$$K(x) = \frac{7}{6}e^{2x} \tag{32}$$

- Plugin $K(x)$ into the ansatz we get the special solution y_s

$$y_s = e^{-\frac{5}{3}x} \cdot \frac{7}{6}e^{2x} = \frac{7}{6}e^{\frac{1}{3}x} \tag{33}$$

- The solution $y_h + y_s$ is then

$$y = K \cdot e^{-\frac{5}{3}x} + \frac{7}{6}e^{\frac{1}{3}x} \tag{34}$$

If the problem is an initial value problem; plug in values for x and y and solve for K.

Higher-order linear ODE's

The fact that the solution of an inhomogeneous ODE is the sum of the solution of the homogeneous ODE and the special solution (equation 20) still holds true for higher-order linear ODE's
The homogeneous case
The function $y = e^{\lambda x}$ is a solution of the homogeneous ODE if and only if λ is a root of the characteristic polynomial, i.e. if

$$P(\lambda) = 0 \tag{35}$$

We now have to distinguish between real and complex and between single and multiple roots of the characteristic polynomial.
Real Roots If λ is a real root of $P(\lambda)$ of multiplicity m, then the functions

$$y_1 = e^{\lambda x},\ y_2 = xe^{\lambda x}, ..., \ y_m = x^{m-1}e^{\lambda x} \tag{36}$$

are distinct linearly independent solutions. i.e. if $m = 1$ and λ is a simple root then $y = C_1\, e^{\lambda x}$ is the solution.
Example: The homogeneous ODE

$$y^{(3)} - 3y'' - 4y' = 0 \tag{37}$$

has the characteristic polynomial

$$P(\lambda) = \lambda^3 - 3\lambda^2 - 4\lambda. \tag{38}$$

The roots of this polynomial are $\lambda_1 = 0, \lambda_2 = 4, \lambda_3 = -1$, hence the general solution of the ODE is

$$y = C_1 + C_2 e^{4x} + C_3 e^{-x} \tag{39}$$

Complex Roots If $\lambda = \alpha \pm j\beta$ are two complex roots of $P(\lambda)$ of multiplicity m then

$$\begin{array}{ll} y_1 = e^{\alpha x} cos(\beta x) & y_2 = e^{\alpha x} sin(\beta x) \\ y_3 = x e^{\alpha x} cos(\beta x) & y_4 = x e^{\alpha x} sin(\beta x) \\ \vdots & \vdots \\ y_{2m-a} = x^{m-1} e^{\alpha x} cos(\beta x) & y_{2m} = x^{m-1} e^{\alpha x} sin(\beta x) \end{array}$$

are m linearly independent solutions.
If $m = 1$ and $\lambda_{1,2} = \alpha \pm \beta$ then $y_1 = C_1 e^{ax} cos(\beta x)$ and $y_2 = C_2 e^{ax} sin(\beta x)$ are two linearly independent solutions.
Example: The homogeneous ODE

$$y'' + y = 0 \tag{40}$$

has the characteristic polynomial

$$P(\lambda) = \lambda^2 + 1. \tag{41}$$

The roots of this polynomial are $\lambda_1 = j, \lambda_2 = -j$, hence the general solution of the ODE is

$$y = C_1 cos(x) + C_2 sin(x) \tag{42}$$

The inhomogeneous case
We now determine special solutions of the inhomogeneous ODE's. The best method for solving such ODE's is as above using an ansatz of the form of the source term, with undetermined coefficients. These coefficients are then determined by plugging the ansatz into the equation.
Example: Consider the inhomogeneous ODE

$$y^{(3)} + y'' + y' + y = 2x + 5 \tag{43}$$

The general solution of the homogeneous ODE is (The complex roots are $-1, -j, j$)

$$y_h = C_1 e^{-x} + C_2 cos(x) + C_3 sin(x) \tag{44}$$

To find a special solution of the inhomogeneous ODE, we choose the ansatz

$$y_s = b_1 x + b_0 \tag{45}$$

Plugin this into the initial ODE leads to

$$b_1 + (b_1 x + b_0) = 2x + 5 \tag{46}$$

which leas to the following linear system of equations for a and b

$$\left| \begin{array}{ccccc} & & b_1 & = & 2 \\ b_1 & + & b_0 & = & 5 \end{array} \right| \Rightarrow \quad b_1 = 2, \quad b_0 = 3 \tag{47}$$

The desired special solution hence is

$$y_s = 2x + 3 \tag{48}$$

and the general solution therefore is

$$y = C_1 e^{-x} + C_2 cos(x) + C_3 sin(x) + 2x + 3 \tag{49}$$

Numerical Methods

Single-setp methods

Part III
System of Differential Equations

Analytical methods for linear systems

By the definition given above, by a system of ODE's we mean the following system of explicit firstoder ODE's:

$$\begin{array}{rcl} \dot{x}_1 & = & f_1(t, x_1, ..., x_n) \\ \vdots & & \vdots \\ \dot{x}_n & = & f_n(t, x_1, ..., x_n) \end{array} \tag{50}$$

Overview

A system of linear first-order ODE's has the form

$$\begin{array}{rcl} \dot{x}_1 & = & a_{11}(t)x_1 + ... + a_{1n}(t)x_n + b_1(t) \\ \vdots & & \vdots \\ \dot{x}_n & = & a_{n1}(t)x_1 + ... + a_{nn}(t)x_n + b_n(t) \end{array} \tag{51}$$

or in matrix-vector notation

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{b}t \tag{52}$$

The general solution of the inhomogeneous system is the sum

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_s \tag{53}$$

where x_h is the general solution of the homogeneous system and x_s any special solution of the inhomogeneous system.

The set of solutions of the homogeneous system is an n-dimensional vector space. This is equivalent to the following two statements:
- Any linear combination of solutions is again a solution, i.e. if x_1 and x_2 are solutions, then $C_1 x_1 + C_2 x_2$ is also a solution for any $C_1, C_2 \in \mathbb{R}$
- There exist precisely n linearly independent solution $x_1, ..., x_n$. Such a set $\{x_1, ..., x_n\}$ of linearly independent solutions is also called a *fundamental system of solutions*. Algebraically, a fundamental system thus is a basis of the vector space of solutions.

Homogeneous linear systems

Since in the scalar case the solution of a homogeneous linear ODE is given by $x = e^{at}$, we try in the vectorial case an ansatz of the form

$$\mathbf{x} = e^{\lambda t} \cdot \mathbf{c} = \begin{pmatrix} e^{\lambda t} c_1 \\ \vdots \\ e^{\lambda t} c_n \end{pmatrix} \tag{54}$$

Plugging the ansatz into the ODE leads to $e^{\lambda t} \lambda \mathbf{c} = e^{\lambda t} A \mathbf{c}$ and we thus get for λ and $\mathbf{c}$ the equation

$$A\mathbf{c} = \lambda \mathbf{c} \tag{55}$$

This precisely means that $\mathbf{c}$ is an *eigenvector* of A to the *eigenvalue* λ. We destinguish between the following cases for λ

Real eigenvalues
If all eigenvalues of A are real and A has n linear independent eigenvectors then the general solution is a linear combination of

$$\mathbf{x}(t) = e^{\lambda t} \cdot \mathbf{c} \tag{56}$$

Example: We compute the general solution of the homogeneous system

$$\begin{array}{l} \dot{x}_1 = 2x_1 - x_2 \\ \dot{x}_2 = -x_1 + 2x_2 \end{array} \tag{57}$$

or in the form $\dot{\mathbf{x}} = A\mathbf{x}$ with

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{58}$$

The matrix A has the eigenvalues $\lambda_1 = 1, \lambda_2 = 3$ with corresponding eigenvectors $v_1 = \binom{1}{1}, v_2 = \binom{1}{-1}$. Hence the general solution $\mathbf{x}(t)$ of the homogeneous system is

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{59}$$

Complex eigenvalues
If there is at least one pair of complex conjugate eigenvalues and A has n linear independent eigenvectors then the general (complex) solution is a linear combination of $\mathbf{x} = e^{\lambda t} \cdot \mathbf{c}$. Real and complex parts of complex solutions are real solutions

If $\lambda = \mu + j\nu$ is a simple complex eigenvalue of A and $\mathbf{c} = \mathbf{a} + j\mathbf{b}$ a corresponding complex eigenvector of A, then from the complex solution

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{c} = e^{(\mu + j\nu)t}(\mathbf{a} + j\mathbf{b}) \tag{60}$$

we get two linearly independent real solutions by separating the complex solution into real and imaginary parts:

$$\begin{aligned} \mathbf{z}_1(t) &= \mathrm{Re}(e^{\lambda t}\mathbf{c}) = e^{\mu t}(\mathbf{a}\,cos(\nu t) - \mathbf{b}\,sin(\nu t)) \\ \mathbf{z}_2(t) &= \mathrm{Im}(e^{\lambda t}\mathbf{c}) = e^{\mu t}(\mathbf{a}\,sin(\nu t) + \mathbf{b}\,cos(\nu t)) \end{aligned} \tag{61}$$

The complex conjugate of λ, namely $c = a - jb$ does not have to be considered because it would again lead to the same solutions.
x is then z1 + z2 ???

Example: We compute the general solution of a homogeneous system of ODE's where the matrix A is

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \tag{62}$$

The eigenvalues of A are $\lambda_{1,2} = 1 \pm 2j$, with eigenvectors $\lambda_{1,2} = \begin{pmatrix} 1 \pm j \\ 2 \end{pmatrix}$.

$$\mathbf{x} = C_1 e^t \begin{pmatrix} 1\cos(2t) - 1\sin(2t) \\ 2\cos(2t) - 0\sin(2t) \end{pmatrix} + C_2 e^t \begin{pmatrix} 1\sin(2t) + 1\cos(2t) \\ 2\sin(2t) + 0\cos(2t) \end{pmatrix} \tag{63}$$

Polynomial expressions
If A has multiple eigenvalues and less than n linear independent eigenvectors (More eigenvalues than eigenvectors), then not all solutions are of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{c}$. Additional solutions involve polynomial expressions. These polynomials are of the form

$$\begin{aligned} p_0(t) &= e^{\lambda t}\mathbf{v}_1 \\ p_1(t) &= e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2) \\ p_2(t) &= e^{\lambda t}(t^2\mathbf{v}_1 + 2t\mathbf{v}_2 + 2\mathbf{v}_3) \end{aligned} \tag{64}$$

Example: The only eigenvalue of the matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is $\lambda = 2$, with the single (linearly independent) eigenvector $\mathbf{v} = \binom{1}{0}$. Let $\mathbf{v}_1 = \mathbf{v}$
To find a generalized eigenvector $\mathbf{v}_2$, we solve the equation

$$(A - \lambda E_n)\mathbf{v}_2 = \mathbf{v}_1 \tag{65}$$

$$\begin{aligned} \left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right)\mathbf{v}_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{v}_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned} \tag{66}$$

And therefore

$$\mathbf{p}_1(t) = t\mathbf{v}_1 + \mathbf{v}_2 = t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix} \tag{67}$$

The general solution therefore is

$$\begin{aligned} \mathbf{x}(t) &= C_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} t \\ 1 \end{pmatrix} \\ \mathbf{x}(t) &= \begin{pmatrix} C_1 e^{2t} + C_2 e^{2t} \\ C_2 e^{2t} \end{pmatrix} \end{aligned} \tag{68}$$

Inhomogeneous linear systems

We now discuss the inhomogeneous system

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}t \tag{69}$$

with a constant coefficient matrix A and a nonconstant source term $\mathbf{b}(t)$. We will look at the following two methods for obtaining the special solution of the homogeneous system.

- Elimination of variables
- Using the decomposition $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_s$ and choosing an ansatz for $\mathbf{x}_s$ of a similar type as the source term.

Elemination of variables
This method can be understood as the inversion of what we've done before. We transform a system of first-order ODE's into a single ODE of higher order. The method is not recommended for ODE systems of dimension 3 or higher. This method is best explained with an example.

We solve the IVP

$$\begin{aligned} \dot{x}_1 &= 2x_1 - x_2 + 1, \quad x_1(0) = 0 \\ \dot{x}_2 &= -x_1 + 2x_2 - 1, \quad x_2(0) = 1 \end{aligned} \tag{70}$$

To eliminate $x_2(t)$, we write the first equation of as

$$x_2 = -\dot{x}_1 + 2x_1 + 1 \tag{71}$$

If this term and its derivative are plugged into the second equation, we obtain for x_1 the second order linear ODE

$$\ddot{x}_1 - 4\dot{x}_1 + 3x_1 + 1 = 0 \tag{72}$$

This ODE can be solved with the method described before, and one gets the general solution

$$x_1 = C_1 e^t + C_2 e^{3t} - \frac{1}{3} \tag{73}$$

If this solution and its derivative are plugged into the second initial equation, one gets for x_2 the general solution

$$x_2 = C_1 e^t - C_2 e^{3t} + \frac{1}{3} \tag{74}$$

Observe that the constants C_1 adn C_2 have to be the same in both equations. Now the initial conditions can be pluged in, and we obtain the system of equations

$$\begin{vmatrix} C_1 e^t + C_2 - \frac{1}{3} = 0 \\ C_1 - C_2 + \frac{1}{3} = 1 \end{vmatrix} \tag{75}$$

for C_1 and C_2. The solution of this system is

$$C_1 = \frac{1}{2}, \qquad C_2 = -\frac{1}{6} \tag{76}$$

The IVP thus has the unique solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^t - \frac{1}{6}e^{3t} - \frac{1}{3} \\ \frac{1}{2}e^t + \frac{1}{6}e^{3t} + \frac{1}{3} \end{pmatrix} \tag{77}$$

Choice of an ansatz
As in the scalar case, we choose as ansatz for a special solution $\mathbf{x}_s$ an ansatz of the same type as the source term $\mathbf{g}(t)$ with coefficients which are yet undetermined. These coefficients are then determined by plugging the ansatz into the inhomogeneous system. The choice of a suitable ansatz can be made according to the Table above, if one reads the entries of the table vectorially.
Observe that the choice of an ansatz always has to be made globally, and not componentwise: If there are different source terms in the various ODE's of the system, then these component source terms all have to be taken into account in the choice of the ansatz for $\mathbf{x}_s$.

Example (Continuation). Consider the IVP

$$\begin{aligned} \dot{x}_1 &= 2x_1 - x_2 + 1, \quad x_1(0) = 0 \\ \dot{x}_2 &= -x_1 + 2x_2 - 1, \quad x_2(0) = 1 \end{aligned} \tag{78}$$

The general solution of the homogeneous system is given by

$$\mathbf{x}_h = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{3t} \tag{79}$$

To find a special solution, we select an ansatz of the type of the source term $\mathbf{b} = \binom{1}{-1}$, i.e. the constant ansatz

$$\mathbf{x}_p = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{80}$$

By plugging this ansatz into the initial IVP, we obtain the system of equations

$$\begin{vmatrix} 0 & = 2A_1 - A_2 + 1 \\ 0 & = -A_1 + 2A_2 - 1 \end{vmatrix} \tag{81}$$

hence

$$A_1 = -\frac{1}{3}, \qquad A_2 = \frac{1}{3} \tag{82}$$

and thus the special solution

$$\mathbf{x}_s(t) = \frac{1}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{83}$$

Combining $\mathbf{x}_h$ and $\mathbf{x}_s$, this leads to the general solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{3t} + \frac{1}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{84}$$

To solve the IVP we plug in the initial conditions into the general solution and get the same solution as above.

Part IV

Dynamical Systems

Overview

A *dynamical system* is a time-dependent process which is described by a mathematical model and whose temporal evolution is completely determined by its initial state.
The set of all possible states of a dynamical system is its *phase space*, and the temporal evolution in phase space is the *flow* of the dynamical system.

- A *continuous* dynamical system is a system of differential equations of the form
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R} \tag{85}$$
Time in such systems is in principle measured continuously.
The solutions of these systems are differentiable functions. Solutions are also called *trajectories* or *orbits* of the system.
- A *discrete* dynamical system is a system of difference equations of the form
$$x_{n+1} = f(x_n) \tag{86}$$
Time in physical systems sometimes also has to measured in discrete steps.
The solutions of this equation are sequences (x_n).

One-Dimensional Systems

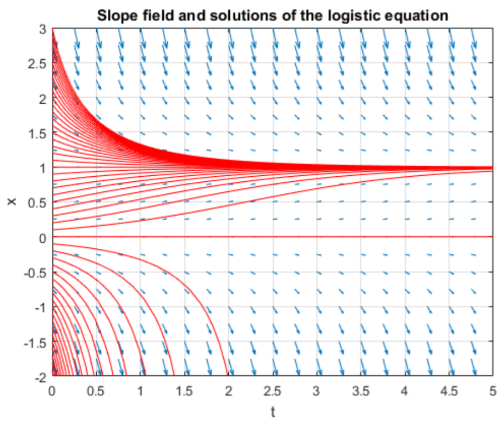
Fixed points in Continuous Dynamical Systems

A *fixed point* of the a continuous dynamical system is an x^* with the property
$$\dot{x} = f(x^*) = 0 \tag{87}$$
If x^* is a fixed point, then $x(t) = x^*$ is a constant solution of the system.

For the solution $x(t)$ of a one-dimensional system there are the following possibilites:

- The solution is a fixed point
- The solution grows monotonically, i.e. $\dot{x}(t) > 0$
- The solution decays monotonically, i.e. $\dot{x}(t) < 0$

Example: We consider the ODE of limited growth
$$\dot{N} = rN\left(1 - \frac{N}{K}\right) \tag{88}$$
If $K = 1$, $\dot{N}$ is 0 if $N = 0$ and $N = 1$ i.e. the fixed points for this equation are $x_1 = 0$ and $x_2 = 1$ (or $x_2 = K$). These fixed points are also visible in the following figure.



Continuous Linear stability analysis

Let x^* be a fixed point of the continuous dynamical system $\dot{x} = f(x)$. The fixed point x^* is

- stable if $f'(x^*) < 0$,
- unstable, if $f'(x^*) > 0$,
- stable or unstable, if $f'(x^*) = 0$

Example (continuation): the derivative of the logistic equation above $\left(f(x) = rx\left(1 - \frac{x}{K}\right)\right)$ is
$$f'(x) = r\left(1 - \frac{2x}{K}\right) \tag{89}$$
Hence
$$\begin{aligned} f'(x_1) &= f'(0) = r > 0 \\ f'(x_2) &= f'(K) = -r < 0 \end{aligned} \tag{90}$$
Hence $x_1 = 0$ is an unstable fixed point, and $x_2 = K$ is a stable fixed point, which is also visible in the figure above.

Fixed points in Discrete Dynamical Systems

A *fixed point* of the discrete dynamical system $x_{n+1} = f(x_n)$ is a x^* with the property
$$x^* = f(x^*) \tag{91}$$
If x^* is a fixed point of $x_{n+1} = f(x_n)$, then $x_n = x^*$ is a constant solution of that system.

Example: We want to find the fixpoint of the discrete dynamical system
$$x_{n+1} = \cos(x_n) \tag{92}$$
Using the formula above:
$$x^* = \cos(x^*) \quad \Rightarrow \quad x^* \approx 0.739 \tag{93}$$

Discrete Linear stability analysis

Let x^* be a fixed point of the discrete dynamical system $x_{n+1} = f(x_n)$. The fixed point x^* is

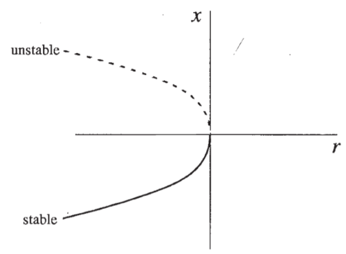
- stable, if $|f'(x^*)| < 1$,
- unstable, if $|f'(x^*)| > 1$,
- stable or unstable, if $|f'(x^*)| = 1$

$$\eta_{n+1} = \eta_n \cdot f'(x^*) \quad \Rightarrow \quad \eta_n = \eta_0 \cdot \left(f'(x^*)\right)^n \tag{94}$$
Example: (continuation): We want to analyse the fixpoint $x^* \approx 0.739$ of the example above.
$$|f'(x^*)| = |\sin(x^*)| \approx |-0.674| < 1 \tag{95}$$
Hence x^* is a stable fixed point of the system.

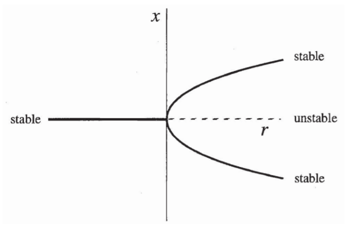
Bifurcations in Continous Systems

Fundamentals
A qualitative change of the dynamics of a dynamical system
$$\dot{x} = f(x, r) \quad \text{or} \quad x_{n+1} = f(x_n, r) \tag{96}$$
depending on a parameter r is a bifurcation of the system

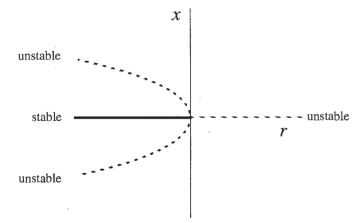
Standard Bifurcations in descrete systems
Saddle-node Bifurcation
The saddle-node bifurcation is a bifurcation in which fixed points are created or destroyed by the variation of a system parameter.
Example: Dynamical system, depending on the parameter r
$$\dot{x} = r + x^2 \tag{97}$$
Fixed points:
$$\begin{aligned} &r < 0 : x^*_{1,2} = \pm\sqrt{-r} \\ &r = 0 : x^* = 0 \\ &r > 0 : \text{none} \end{aligned} \tag{98}$$
Stability of the fixed points for $r < 0$:
$$f(x) = r + x^2, \quad f'(x) = 2x \tag{99}$$
hence
$$\begin{aligned} x^*_1 = \sqrt{-r} > 0 : f'(x^*_1) > 0, \mathit{unstable} \\ x^*_2 = -\sqrt{-r} < 0 : f'(x^*_2) < 0, \mathit{stable} \end{aligned} \tag{100}$$



Pitchfork Bifurcation
The pitchfork bifurcation is a bifurcation in which a fixed point changes its stability and two new fixed points are created.



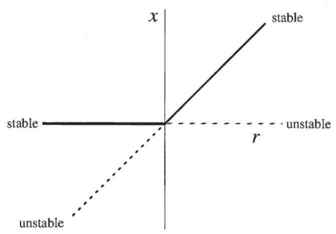
Supercritical



Subcritical

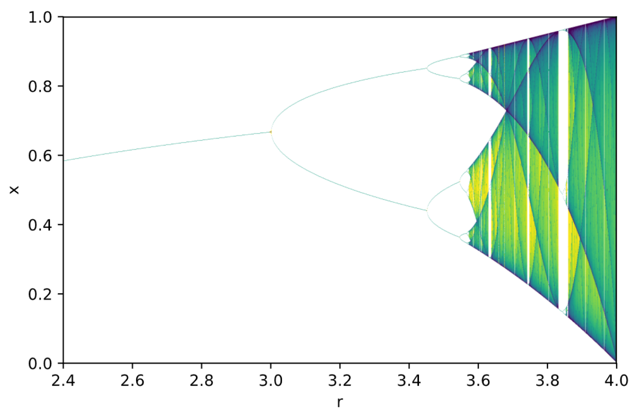
Supercritical: Additional fixed points only for parameter values greater ("super") than the bifurcation point.
Subcritical P-B: Additional fixed points only for parameter values smaller ("sub") than the bifurcation point.

Transcritical Bifurcation
A transcritical bifurcation is a bifurcation in which the stability of two fixed points is interchanged.



Bifurcations in Discrete Systems

The bifurcations which we introduced in continuous systems can also occur in discrete systems.
Discrete systems can (or do allways?) show a chaotic behaviour in the parameter space:



Example: Consider the discrete dynamical system depending on the parameters $\mu \geq 0, \varepsilon \geq 0$ (Adler system)

$$x_{x+1} = x_n + \mu + \varepsilon \sin(x_n) \quad (\mu \geq 0, \varepsilon \leq 0) \tag{101}$$

Fixed points x^* are solutions of the equation $\mu + \varepsilon \sin(x) = 0$, hence

$$x^*_{1,k} = -\arcsin\left(\frac{\mu}{\varepsilon}\right) + k \cdot 2\pi, \quad x^*_{2,k} = (2k+1)\pi + \arcsin\left(\frac{\mu}{\varepsilon}\right) \tag{102}$$

The fixed points only exist in the case $\mu \leq \varepsilon$
For $\mu = 1$, the fixed points only exist in the case $\varepsilon \geq 1$. We have $f(x) = x + \mu + \varepsilon \sin(x)$, hence $f'(x) = 1 + \varepsilon \cos(x)$. It follows that for $\mu = 1$

$$\begin{aligned} f'(x^*_{1,k}) =& 1 + \varepsilon \cos\left(-\arcsin\left(\frac{1}{\varepsilon}\right)\right) \\ =& 1 + \varepsilon\sqrt{1 - \frac{1}{\varepsilon^2}} \\ =& 1 + \sqrt{\varepsilon^2 - 1} \geq 1 \end{aligned} \tag{103}$$

Hence the fixed points $x^*_{1,k}$ are unstable.

$$\begin{aligned} f'(x^*_{2,k}) =& 1 + \varepsilon \cos\left(\pi + \arcsin\left(\frac{1}{\varepsilon}\right)\right) \\ =& 1 - \varepsilon \cos\left(\arcsin\left(\frac{1}{\varepsilon}\right)\right) \\ =& 1 - \varepsilon\sqrt{1 - \frac{1}{\varepsilon^2}} \\ =& 1 + \sqrt{\varepsilon^2 - 1} \end{aligned} \tag{104}$$

For the stability of $x^*_{2,k}$ it is necessary that $|1 - \sqrt{\varepsilon^2 - 1} < 1$ holds. This is true if and only if $0 < \sqrt{\varepsilon^2 - 1} < 2$, hence in the case $1 < \varepsilon^2 < 5$ resp. $1 < \varepsilon < \sqrt{5}$. The fixed points $x^*_{2,k}$ are thus stable in the case $1 < \varepsilon < \sqrt{5}$.

Two-Dimensional Systems

We now consider the systmes for arbitrary $n \in (N)$, i.e.

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, ..., x_n) \\ \dot{x}_2 &= f_2(x_1, x_2, ..., x_n) \\ &... \\ \dot{x}_n &= f_n(x_1, x_2, ..., x_n) \end{aligned} \tag{105}$$

resp. in vector notation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad (\mathbf{x} \in (R)^n) \tag{106}$$

Definitions
A point $x^* \in \mathbb{R}^n$ is a *fixed point* of the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, if

$$\mathbf{f}(\mathbf{x}^*) = \mathbf{0} \tag{107}$$

If $\mathbf{x}^*$ is a *fixed point* of the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then

$$\mathbf{x}(t) = \mathbf{x}^* \tag{108}$$

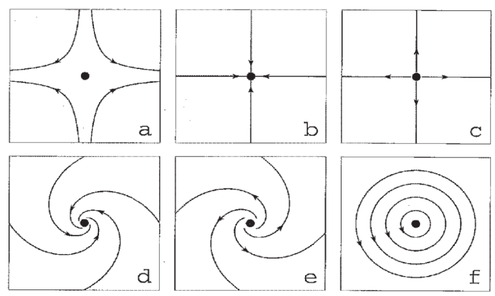
is a constant solution.

A fixed point $x^* \in \mathbb{R}^n$ of the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, is

- *stable*, if the solution starting close to x^* stays close to x^* for all $t \geq 0$.
- *asymptotically stable*, if the solution starting close to x^* converge to x^* for $t \to \infty$.
- *unstable*, if the solution starts close to x^* and diverging from x^* for $t \to \infty$.

Linear Systems

Overview over the various types of fixed points and the corresponding geometry of the system near the fixed point:



a) saddle point
b) stable node
c) unstable node
d) stable spiral
e) unstable spiral
f) center/elliptic fixed point
g) degenerate cases

A two-dimensional system can be characterized with the eigenvalues like so:

- $\lambda_1 < 0, \lambda_2 < 0$: stable node
- $\lambda_1 > 0, \lambda_2 < 0$: saddle point
- $\lambda_1 > 0, \lambda_2 > 0$: usntable point
- $\lambda_1 = 0, \lambda_2 = 0$: non-isolated fixed point
- $\lambda_{1.2} \neq \mathbb{R}, \mathrm{Re}(\lambda_{1,2}) = 0$: center
- $\lambda_{1.2} \neq \mathbb{R}, \mathrm{Re}(\lambda_{1,2}) > 0$: unstable spiral
- $\lambda_{1.2} \neq \mathbb{R}, \mathrm{Re}(\lambda_{1,2}) < 0$: stable spiral

Alternativ characterization instead of with the eigenvalues λ_1, λ_2 with trace and determinant of A:

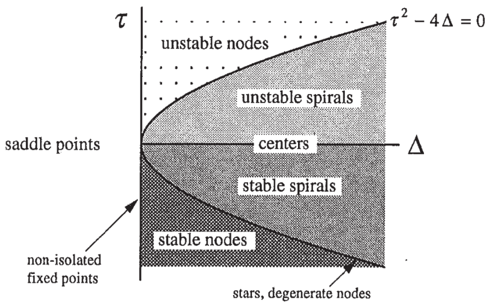
$$\begin{aligned} \tau &= \mathrm{tr} = a_{11} + a_{22} \\ \Delta &= \det(A) = a_{11}a_{22} - a_{12}a_{21} \end{aligned} \tag{109}$$

Connection with the eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right) \tag{110}$$

and

$$\Delta = \lambda_1\lambda_2, \quad \tau = \lambda_1 + \lambda_2 \tag{111}$$



If one is only interested in the stability of $\mathbf{x}^*$ (and not in a further classification into nodes, spirals, etc.), the fixed point $\mathbf{x}^* = 0$ of a dynamical system is

- asymptotically stable, if
$$\mathrm{Re}(\lambda_i) < 0 \tag{112}$$
holds for all eigenvalues λ_i of A
- stable, if
$$\mathrm{Re}(\lambda_i) \leq 0 \tag{113}$$
holds for all eigenvalues λ_i of A
- usntable, if
$$\mathrm{Re}(\lambda_i) > 0 \tag{114}$$
holds for at least one eigenvalue λ_i of A

Nonlinear Systems

find fixpoints calculate jacobi-matrix insert fixpoitns and analyse matrix
Example:

$$\begin{aligned} \dot{x} &= -x + x^3 \\ \dot{y} &= -2y \end{aligned} \tag{115}$$

fixed points are:

$$P_1 = (0,0), \quad P_2 = (1,0), \quad P_3 = (-1,0). \tag{116}$$

The Jacobi matrix of the system is

$$A = \begin{vmatrix} -1 + 3x^2 & 0 \\ 0 & -2 \end{vmatrix} \tag{117}$$

The eigenvalues of A are:

$$P_1 : \lambda_1 = -1, \lambda_2 = -2; \quad P_2, P_3 : \lambda_{1,2} = \pm 2 \tag{118}$$

Therefore P_1 is a stable node, and P_2 and P_3 are saddle points of the linearized system.

Bifurications

Three-Dimensional Systems and Chaos