Ordinary Differential Equations and Dynamical Systems

Part I
Modeling

Fundamentals

Formally, an ordinary differential equation is an equation, in which a function and its derivates and the independent variable appear.

An (implicit) *ordinary differential equation* of order n is an equation of the form

$$F(x, y, y', y'', ..., y^{(n)}) = 0. \tag{1}$$

An example of an explicit ODE of order n is of the form

$$y^n = G(x, y, y', y'', ..., y^{n-1}). \tag{2}$$

Remark: The solution of an ODE is not a number, but rather a function, or more precisely: a set of functions.
The set of all solutions of an ordinary differential equation is the *general solution* of an ODE. The solution of an initial value problem is a *special* or *particular solution* of the ODE.

Classification

Differential equations can be classified according to various criteria. Besides the order of an ODE we are also interested in whether an ODE is linear, homogeneous, has constant coefficient, is separable or autonomous.

Linearity
An n-th order ODE is *linear*, if it is of the form

$$a_n(x) \cdot y^{(n)} + ... + a_1(x) \cdot y' + a_0(x) \cdot y = g(x) \tag{3}$$

where $a_n(x), ..., a_1(x), a_0(x)$ are $g(x)$ fixed functions. Or in other words: A differential equation is linear if the dependant variable and all of its derivatives appear in a linear fashion (i.e., they are not multiplied together or squared for example or they are not part of transcendental functions such as sins, cosines, exponentials, etc.)

Homogenity
A linear ODE is *homogeneous*, if $g(x) = 0$ for all x; otherwise the ODE is *inhomogeneous*, and $g(x)$ is the *inhomogeneity* or *source* term.
The distinction between *homogeneous* and *inhomogeneous* ODEs can also be understood physically: Homogeneous ODEs describe systems which act without external input, whereas systems which are driven by an external force are generally described by inhomogeneous ODEs.

Constant coefficient
A linear ODE has *constant coefficients*, if it is of the form

$$a_n \cdot y^{(n)} + ... + a_1 \cdot y' + a_0 \cdot y = g(x), \tag{4}$$

with $a_n \neq 0$ (the source term $g(x)$ does not have to be constant).

Separability
The ODE is *separable*, if $F(x, y)$ can be written as a product of a x- and y-dependent term, i.e. if the ODE is of the form

$$y' = g(x) \cdot h(y) \tag{5}$$

for some functions $g(x)$ and $h(y)$.

Autonomity
An ODE is *autonomous*, if $F(x, y)$ only depends on y, i.e. if the ODE is of the form

$$y' = h(y) \tag{6}$$

for some function $h(y)$.
Every autonomous ODE is separable with $g(x) = 1$.

Examples

$y' = f(x)$	Inhomogeneous linear ODE for $y(x)$ with source term $f(x)$
$m \cdot \dot{v} = m \cdot g - k \cdot v^2$	Nonlinear ODE for $v(t)$
$l \cdot \ddot{\Phi} + g \cdot sin(\Phi) = 0$	Nonlinear ODE for $\Phi(t)$
$l \cdot \ddot{\Phi} + g \cdot \phi$	Homogeneous linear ODE for $\Phi(t)$
$l \cdot \ddot{\Phi} + g \cdot \phi = sin(\omega t)$	Inhomogeneous linear ODE for $\Phi(t)$ with source term $sin(\omega t)$
$i'' + \frac{R}{L} i' + \frac{1}{LC} i = 0$	Homogeneous linear ODE for $i(t)$

Systems of differential equations

If several systems are coupled with each other and mutually influence each other, one often obtains a system of ODE's.
A *system of differential equations* of first order is a system

$$\begin{aligned} y_1' &= f_1(x, y_1, ..., y_n) \\ &\vdots \\ y_n' &= f_n(x, y_1, ..., y_n) \end{aligned} \tag{7}$$

of ODE's for unknown functions $y_1(x), ..., y_n(x)$.
Using the vectorial notation

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}) \tag{8}$$

An ODE of n-th order is equivalent to a system of first-order ODE's.

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_n' &= f(x, y_1, ..., y_n) \end{aligned} \tag{9}$$

Example: 2nd-order ODE to system of first-order ODE's
We want to rewrite the following $2nd$ order ODE into a system of first-order ODE's.

$$\ddot{x}(t) + 2\delta\dot{x}(t) + \omega_0^2 x(t) = f(t) \tag{10}$$

If we introduce the vector-valued function

$$\mathrm{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \quad \Rightarrow \quad \dot{\mathrm{y}} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} \tag{11}$$

rewriting:

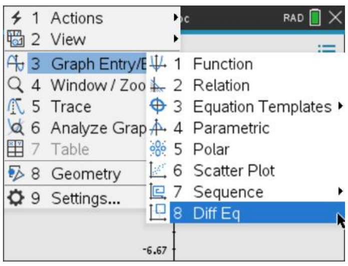
$$\begin{aligned} \dot{\mathrm{y}} &= \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -2\delta\dot{x}(t) - \omega_0^2 x(t) + f(t) \end{pmatrix} \\ \dot{\mathrm{y}} &= \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\delta \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \\ \dot{\mathrm{y}} &= \mathrm{A}\mathbf{y} + \mathbf{b} \end{aligned} \tag{12}$$

Slope field

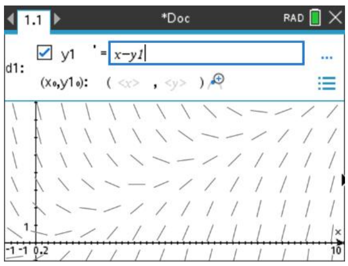
Slope fields are often loead to a good qualitative understanding of the situation described by the ODE under consideration. Slope field can be understood in the following way: To each point (x, y) in the region B under consideration, $F(x, y)$ is a value which discribes the slope of the solution curve passing through the point (x, y).

Example with calculator
We want to plot the slope field of the following ODE

$$y' = x - y \tag{13}$$



1. Select: Menu, 3: Graph Entry/Edit, 8: Diff Eq.



2. Write down the ODE

Part II
Ordinary Differential Equations

Analytical methods for first-order ODE's

Overview

For some types of explicit first-oder ODE's there exist analytical solution methods: **Separable ODE's**
Linear ODE's
Exact ODE's

Separable ODE's

Example
We compute the general solution of the ODE

$$y' = -\frac{x}{y} \tag{14}$$

We write the eqution as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} \tag{15}$$

We bring all x-dependent terms to the left hand side and all y-dependent terms on the right hand side:

$$y\,\mathrm{d}y = -x\,\mathrm{d}x \tag{16}$$

We integrate on both sides and get

$$\int y\,\mathrm{d}y = -\int x\,\mathrm{d}x \Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C, \quad C \in \mathbb{R} \tag{17}$$

We solve for y and get

$$y = \pm\sqrt{K - x^2}, \quad K \in \mathbb{R} \ \ (\text{where } K = 2C) \tag{18}$$

Example of substitution
We consider the ODE

$$y' = (x+y)^2 \tag{19}$$

Exact ODE's

Analytical methods for linear ODE's

Overview

We differentiate bewteen first-order linear ODE's and higher-oder ODE's as well as between homogeneous and inhomogeneous ODE's.

The general solution of the inhomogeneous ODE is the sum

$$y = y_h + y_s, \tag{20}$$

where y_h is the general solution of the homogeneous ODE and y_s any special solution of the inhomogeneous ODE.

First-order linear ODE's

To solve a first-order linear ODE we thus have to find y_h and y_s
$\mathbf{y_h}$: a homogeneous first-order ODE is separable and can therefore be solved by the standard procedure for separable ODE's described above.
$\mathbf{y_s}$: To find a special solution of and ODE, there are several possibilities. In the case of an ODE with constant coefficients, it usually suffices to choose for y_s an *ansatz of the form of the source term* $g(x)$. In the case of non-constant coefficients, the method *variation of constants* usually works better.

Example using the ansatz
We solve the ODE

$$y' + ay = b \tag{21}$$

To determine y_h, we integrate the homogeneous ODE

$$y' + ay = 0 \tag{22}$$

by separation of variables we obtain the solution

$$y_h = C \cdot e^{-ax}, \quad C \in \mathbb{R}. \tag{23}$$

For finding y_s we choose the ansatz in the form of the source term. In this case the source term is constant, $g(x) = b$. Therefore we assume that the special solution y_s is constant as well, i.e. we make an ansatz $y_s = c$. We plug this ansatz into the inhomogeneous ODE and obtain the special solution y_s.

$$y_s' + ay_s = b \quad \Rightarrow \quad y_s = \frac{a}{b} \tag{24}$$

The general solution therefore is

$$y = C \cdot e^{-ax} + \frac{a}{b}, \quad C \in \mathbb{R} \tag{25}$$

Ansatz functions for the solution of the inhomogeneous first-order ODE

Source term $g(x)$	Ansatz y_s
$g(x) = b_0$	$y_s = c_0$
$g(x) = b_1x + b_0$	$y_s = c_1x + c_0$
$g(x) = b_2x^2 + b_1x + b_0$	$y_s = c_2x^2 + c_1x + c_0$
$g(x) = \sum_{i=0}^n b_ix^i$	$y_s = \sum_{i=0}^n c_ix^i$
$g(x) = A\ sin(\omega x) + B\ cos(\omega x)$	$y_s = C_1sin(\omega x) + C_2cos(\omega x)$ $y_s = Csin(\omega x + \varphi)$
$g(x) = Ae^{bx}$	$y_s = \begin{cases} \frac{A}{b+a}e^{bx} & \text{for } b \neq -a \\ Axe^{-ax} & \text{for } b = -a \end{cases}$

In addition, the following rules must be followed:
Linearity If $g(x)$ is a linear combination of several source terms, one has to assume as ansatz for $y_s(x)$ a correponding linear combination of several ansatz terms.
Resonance If the source term $g(x)$ is itself already a solution of the homogeneous ODE, the correponding ansatz for y_s has to be multiplied with x. So if for example $y_h = Ce^x$, and $g(x) = e^x$, the ansatz $y_s = \mathbf{x} \cdot e^x$ is choosen.

Example using Variation of constants
The idea behind the variation of constants is to start from the solution $y = K \cdot e^{-F(x)}$ of the homogeneous ODE and plug the ansatz

$$y = K(x) \cdot e^{-F(x)} \tag{26}$$

into the inhomogeneous ODE.
As an example we want to solve the inhomogeneous linear ODE

$$3y' + 5y = 7e^{\frac{1}{3}x}. \tag{27}$$

We first find the homogeneous solution y_h by separation of variables.

$$\frac{1}{y}\mathrm{d}y = -\frac{5}{3}x\ \mathrm{d}x \quad \Rightarrow \quad y_h = e^{\frac{5}{3}x} \cdot K \tag{28}$$

Next we calculate the ansatz and its derivative

$$\begin{aligned} y_s &= e^{-\frac{5}{3}x} \cdot K(x) \\ y_s' &= K(x)'e^{-\frac{5}{3}x} + K(x) \cdot -\frac{5}{3}e^{-\frac{5}{3}x} \end{aligned} \tag{29}$$

The ansatz is then pluged into the inhomogeneous ODE

$$3\left(K(x)'e^{-\frac{5}{3}x} + K(x) \cdot -\frac{5}{3}e^{-\frac{5}{3}x}\right) + 5\left(e^{-\frac{5}{3}x} \cdot K(x)\right) = 7e^{\frac{1}{3}x} \tag{30}$$

We solve for $K(x)'$ ($K(x)$ usually dissapears)

$$3 \cdot K(x)'e^{-\frac{5}{3}x} = 7e^{\frac{1}{3}x} \quad \Rightarrow \quad K(x)' = \frac{7}{3}e^{\frac{1}{3}x + \frac{5}{3}x} = \frac{7}{3}e^{2x} \tag{31}$$

We integrate and find $K(x)$

$$K(x) = \frac{7}{6}e^{2x} \tag{32}$$

Plugin $K(x)$ into the ansatz we get the special solution y_s

$$y_s = e^{-\frac{5}{3}x} \cdot \frac{7}{6}e^{2x} = \frac{7}{6}e^{\frac{1}{3}x} \tag{33}$$

The solution $y_h + y_s$ is then

$$y = K \cdot e^{-\frac{5}{3}x} + \frac{7}{6}e^{\frac{1}{3}x} \tag{34}$$

If the problem is an initial value problem; plug in values for x and y and solve for K.

The fact that the solution of an inhomogeneous ODE is the sum of the solution of the homogeneous ODE and the special solution (equation 20) still holds true for higher-order linear ODE's

The homogeneous case

The function $y = e^{\lambda x}$ is a solution of the homogeneous ODE if and only if λ is a root of the characteristic polynomial, i.e. if

$$P(\lambda) = 0 \tag{35}$$

We now have to distinguish between real and complex and between single and multiple roots of the characteristic polynomial.

Real Roots If λ is a real root of $P(\lambda)$ of multiplicity m, then the functions

$$y_1 = e^{\lambda x},\ y_2 = xe^{\lambda x}, ...,\ y_m = x^{m-1}e^{\lambda x} \tag{36}$$

are distinct linearly independent solutions. i.e. if $m = 1$ and λ is a simple root then $y = C_1\ e^{\lambda x}$ is the solution.

Example: The homogeneous ODE

$$y^{(3)} - 3y'' - 4y' = 0 \tag{37}$$

has the characteristic polynomial

$$P(\lambda) = \lambda^3 - 3\lambda^2 - 4\lambda. \tag{38}$$

The roots of this polynomial are $\lambda_1 = 0, \lambda_2 = 4, \lambda_3 = -1$, hence the general solution of the ODE is

$$y = C_1 + C_2e^{4x} + C_3e^{-x} \tag{39}$$

Complex Roots If $\lambda = \alpha \pm j\beta$ are two complex roots of $P(\lambda)$ of multiplicity m then

$$\begin{array}{ll} y_1 = e^{\alpha x}cos(\beta x) & y_2 = e^{\alpha x}sin(\beta x) \\ y_3 = xe^{\alpha x}cos(\beta x) & y_4 = xe^{\alpha x}sin(\beta x) \\ \vdots & \vdots \\ y_{2m-a} = x^{m-1}e^{\alpha x}cos(\beta x) & y_{2m} = x^{m-1}e^{\alpha x}sin(\beta x) \end{array}$$

are m linearly independent solutions.

If $m = 1$ and $\lambda_{1,2} = \alpha \pm \beta$ then $y_1 = C_1e^{ax}cos(\beta x)$ and $y_2 = C_2e^{ax}sin(\beta x)$ are two linearly independent solutions.

Example: The homogeneous ODE

$$y'' + y = 0 \tag{40}$$

has the characteristic polynomial

$$P(\lambda) = \lambda^2 + 1. \tag{41}$$

The roots of this polynomial are $\lambda_1 = j, \lambda_2 = -j$, hence the general solution of the ODE is

$$y = C_1cos(x) + C_2sin(x) \tag{42}$$

The inhomogeneous case

We now determine special solutions of the inhomogeneous ODE's. The best method for solving such ODE's is as above using an ansatz of the form of the source term, with undetermined coefficients. These coefficients are then determined by plugging the ansatz into the equation.

Example: Consider the inhomogeneous ODE

$$y^{(3)} + y'' + y' + y = 2x + 5 \tag{43}$$

The general solution of the homogeneous ODE is (The complex roots are $-1, -j, j$)

$$y_h = C_1e^{-x} + C_2cos(x) + C_3sin(x) \tag{44}$$

To find a special solution of the inhomogeneous ODE, we choose the ansatz

$$y_s = b_1x + b_0 \tag{45}$$

Plugin this into the initial ODE leads to

$$b_1 + (b_1x + b_0) = 2x + 5 \tag{46}$$

which leas to the following linear system of equations for a and b

$$\begin{vmatrix} & & b_1 & = & 2 \\ b_1 & + & b_0 & = & 5 \end{vmatrix} \quad \Rightarrow \quad b_1 = 2, \quad b_0 = 3 \tag{47}$$

The desired special solution hence is

$$y_s = 2x + 3 \tag{48}$$

and the general solution therefore is

$$y = C_1e^{-x} + C_2cos(x) + C_3sin(x) + 2x + 3 \tag{49}$$

Numerical Methods

Single-setp methods

Part III

System of Differential Equations

Analytical methods for linear systems

By the definition given above, by a system of ODE's we mean the following system of explicit firstoder ODE's:

$$\begin{array}{rcl} \dot{x}_1 & = & f_1(t, x_1, ..., x_n) \\ \vdots & & \vdots \\ \dot{x}_n & = & f_n(t, x_1, ..., x_n) \end{array} \tag{50}$$

Overview

A system of linear first-order ODE's has the form

$$\begin{array}{rcl} \dot{x}_1 & = & a_{11}(t)x_1 + ... + a_{1n}(t)x_n + b_1(t) \\ \vdots & & \vdots \\ \dot{x}_n & = & a_{n1}(t)x_1 + ... + a_{nn}(t)x_n + b_n(t) \end{array} \tag{51}$$

or in matrix-vector notation

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{b}t \tag{52}$$

The general solution of the inhomogeneous system is the sum

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_s \tag{53}$$

where x_h is the general solution of the homogeneous system and x_s any special solution of the inhomogeneous system.

The set of solutions of the homogeneous system is an n-dimensional vector space. This is equivalent to the following two statements:

- Any linear combination of solutions is again a solution, i.e. if x_1 and x_2 are solutions, then $C_1x_1 + C_2x_2$ is also a solution for any $C_1, C_2 \in \mathbb{R}$
- There exist precisely n linearly independent solution $x_1, ..., x_n$. Such a set $\{x_1, ..., x_n\}$ of linearly independent solutions is also called a *fundamental system of solutions*. Algebraically, a fundamental system thus is a basis of the vector space of solutions.

Homogeneous linear systems

Since in the scalar case the solution of a homogeneous linear ODE is given by $x = e^{at}$, we try in the vectorial case an ansatz of the form

$$\mathbf{x} = e^{\lambda t} \cdot \mathbf{c} = \begin{pmatrix} e^{\lambda t}c_1 \\ \vdots \\ e^{\lambda t}c_n \end{pmatrix} \tag{54}$$

Plugging the ansatz into the ODE leads to $e^{\lambda t}\lambda\mathbf{c} = e^{\lambda t}A\mathbf{c}$ and we thus get for λ and $\mathbf{c}$ the equation

$$A\mathbf{c} = \lambda\mathbf{c} \tag{55}$$

This precisely means that $\mathbf{c}$ is an *eigenvector* of A to the *eigenvalue* λ. We destinguish between the following cases for λ

Real eigenvalues

If all eigenvalues of A are real and A has n linear independent eigenvectors then the general solution is a linear combination of

$$\mathbf{x}(t) = e^{\lambda t} \cdot \mathbf{c} \tag{56}$$

Example: We compute the general solution of the homogeneous system

$$\begin{array}{l} \dot{x}_1 = 2x_1 - x_2 \\ \dot{x}_2 = -x_1 + 2x_2 \end{array} \tag{57}$$

or in the form $\dot{\mathbf{x}} = A\mathbf{x}$ with

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{58}$$

The matrix A has the eigenvalues $\lambda_1 = 1, \lambda_2 = 3$ with corresponding eigenvectors $v_1 = \binom{1}{1}, v_2 = \binom{1}{-1}$. Hence the general solution $\mathbf{x}(t)$ of the homogeneous system is

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{59}$$

Complex eigenvalues
If there is at least one pair of complex conjugate eigenvalues and A has n linear independent eigenvectors then the general (complex) solution is a linear combination of $\mathbf{x} = e^{\lambda t} \cdot \mathbf{c}$. Real and complex parts of complex solutions are real solutions
If $\lambda = \mu + j\nu$ is a simple complex eigenvalue of A and $\mathbf{c} = \mathbf{a} + j\mathbf{b}$ a corresponding complex eigenvector of A, then from the complex solution

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{c} = e^{(\mu + j\nu)t}(\mathbf{a} + j\mathbf{b}) \tag{60}$$

we get two linearly independent real solutions by separating the complex solution into real and imaginary parts:

$$\begin{aligned} \mathbf{z}_1(t) &= \mathrm{Re}(e^{\lambda t}\mathbf{c}) = e^{\mu t}(\mathbf{a}\, cos(\nu t) - \mathbf{b}\, sin(\nu t)) \\ \mathbf{z}_2(t) &= \mathrm{Im}(e^{\lambda t}\mathbf{c}) = e^{\mu t}(\mathbf{a}\, sin(\nu t) + \mathbf{b}\, cos(\nu t)) \end{aligned} \tag{61}$$

The complex conjugate of λ, namely $c = a - jb$ does not have to be considered because it would again lead to the same solutions.
x is then z1 + z2 ???

Example: We compute the general solution of a homogeneous system of ODE's where the matrix A is

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \tag{62}$$

The eigenvalues of A are $\lambda_{1,2} = 1 \pm 2j$, with eigenvectors $\lambda_{1,2} = \begin{pmatrix} 1 \pm j \\ 2 \end{pmatrix}$.

$$\mathbf{x} = C_1 e^t \begin{pmatrix} 1\cos(2t) - 1\sin(2t) \\ 2\cos(2t) - 0\sin(2t) \end{pmatrix} + C_2 e^t \begin{pmatrix} 1\sin(2t) + 1\cos(2t) \\ 2\sin(2t) + 0\cos(2t) \end{pmatrix} \tag{63}$$

Polynomial expressions
If A has multiple eigenvalues and less than n linear independent eigenvectors (More eigenvalues than eigenvectors), then not all solutions are of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{c}$. Additional solutions involve polynomial expressions. These polynomials are of the form

$$\begin{aligned} p_0(t) &= e^{\lambda t}\mathbf{v}_1 \\ p_1(t) &= e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2) \\ p_2(t) &= e^{\lambda t}(t^2\mathbf{v}_1 + 2t\mathbf{v}_2 + 2\mathbf{v}_3) \end{aligned} \tag{64}$$

Example: The only eigenvalue of the matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is $\lambda = 2$, with the single (linearly independent) eigenvector $\mathbf{v} = \binom{1}{0}$. Let $\mathbf{v}_1 = \mathbf{v}$
To find a generalized eigenvector $\mathbf{v}_2$, we solve the equation

$$(A - \lambda E_n)\mathbf{v}_2 = \mathbf{v}_1 \tag{65}$$

$$\begin{aligned} \left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right)\mathbf{v}_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{v}_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned} \tag{66}$$

And therefore

$$\mathbf{p}_1(t) = t\mathbf{v}_1 + \mathbf{v}_2 = t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix} \tag{67}$$

The general solution therefore is

$$\begin{aligned} \mathbf{x}(t) =& C_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} t \\ 1 \end{pmatrix} \\ \mathbf{x}(t) =& \begin{pmatrix} C_1 e^{2t} + C_2 e^{2t} \\ C_2 e^{2t} \end{pmatrix} \end{aligned} \tag{68}$$

Inhomogeneous linear systems

We now discuss the inhomogeneous system

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}t \tag{69}$$

with a constant coefficient matrix A and a nonconstant source term $\mathbf{b}(t)$. We will look at the following two methods for obtaining the special solution of the homogeneous system.

- Elimination of variables
- Using the decomposition $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_s$ and choosing an ansatz for $\mathbf{x}_s$ of a similar type as the source term.

Elemination of variables
This method can be understood as the inversion of what we've done before. We transform a system of first-order ODE's into a single ODE of higher order. The method is not recommended for ODE systems of dimension 3 or higher. This method is best explained with an example.

We solve the IVP

$$\begin{aligned} \dot{x}_1 &= 2x_1 - x_2 + 1, \quad x_1(0) = 0 \\ \dot{x}_2 &= -x_1 + 2x_2 - 1, \quad x_2(0) = 1 \end{aligned} \tag{70}$$

To eliminate $x_2(t)$, we write the first equation of as

$$x_2 = -\dot{x}_1 + 2x_1 + 1 \tag{71}$$

If this term and its derivative are plugged into the second equation, we obtain for x_1 the second order linear ODE

$$\ddot{x}_1 - 4\dot{x}_1 + 3x_1 + 1 = 0 \tag{72}$$

This ODE can be solved with the method described before, and one gets the general solution

$$x_1 = C_1 e^t + C_2 e^{3t} - \frac{1}{3} \tag{73}$$

If this solution and its derivative are plugged into the second initial equation, one gets for x_2 the general solution

$$x_2 = C_1 e^t - C_2 e^{3t} + \frac{1}{3} \tag{74}$$

Observe that the constants C_1 adn C_2 have to be the same in both equations. Now the initial conditions can be pluged in, and we obtain the system of equations

$$\begin{vmatrix} C_1 e^t + C_2 - \frac{1}{3} = 0 \\ C_1 - C_2 + \frac{1}{3} = 1 \end{vmatrix} \tag{75}$$

for C_1 and C_2. The solution of this system is

$$C_1 = \frac{1}{2}, \qquad C_2 = -\frac{1}{6} \tag{76}$$

The IVP thus has the unique solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^t - \frac{1}{6}e^{3t} - \frac{1}{3} \\ \frac{1}{2}e^t + \frac{1}{6}e^{3t} + \frac{1}{3} \end{pmatrix} \tag{77}$$

Choice of an ansatz
As in the scalar case, we choose as ansatz for a special solution $\mathbf{x}_s$ an ansatz of the same type as the source term $\mathbf{g}(t)$ with coefficients which are yet undetermined. These coefficients are then determined by plugging the ansatz into the inhomogeneous system. The choice of a suitable ansatz can be made according to the Table above, if one reads the entries of the table vectorially.
Observe that the choice of an ansatz always has to be made globally, and not componentwise: If there are different source terms in the various ODE's of the system, then these component source terms all have to be taken into account in the choice of the ansatz for $\mathbf{x}_s$.

Example (Continuation). Consider the IVP

$$\begin{aligned} \dot{x}_1 &= 2x_1 - x_2 + 1, \quad x_1(0) = 0 \\ \dot{x}_2 &= -x_1 + 2x_2 - 1, \quad x_2(0) = 1 \end{aligned} \tag{78}$$

The general solution of the homogeneous system is given by

$$\mathbf{x}_h = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} \tag{79}$$

To find a special solution, we select an ansatz of the type of the source term $\mathbf{b} = \binom{1}{-1}$, i.e. the constant ansatz

$$\mathbf{x}_p = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{80}$$

By plugging this ansatz into the initial IVP, we obtain the system of equations

$$\begin{vmatrix} 0 & = 2A_1 - A_2 + 1 \\ 0 & = -A_1 + 2A_2 - 1 \end{vmatrix} \tag{81}$$

hence

$$A_1 = -\frac{1}{3}, \qquad A_2 = \frac{1}{3} \tag{82}$$

and thus the special solution

$$\mathbf{x}_s(t) = \frac{1}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{83}$$

Combining $\mathbf{x}_h$ and $\mathbf{x}_s$, this leads to the general solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + \frac{1}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{84}$$

To solve the IVP we plug in the initial conditions into the general solution and get the same solution as above.

Part IV
Dynamical Systems

Overview

A *dynamical system* is a time-dependent process which is described by a mathematical model and whose temporal evolution is completely determined by its initial state.
The set of all possible states of a dynamical system is its *phase space*, and the temporal evolution in phase space is the *flow* of the dynamical system.

- A *continuous* dynamical system is a system of differential equations of the form
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R} \tag{85}$$
Time in such systems is in principle measured continuously.
The solutions of these systems are differentiable functions. Solutions are also called *trajectories* or *orbits* of the system.
- A *discrete* dynamical system is a system of difference equations of the form
$$x_{n+1} = f(x_n) \tag{86}$$
Time in physical systems sometimes also has to measured in discrete steps.
The solutions of this equation are sequences (x_n).

One-Dimensional Systems

Fixed points in Continuous Dynamical Systems

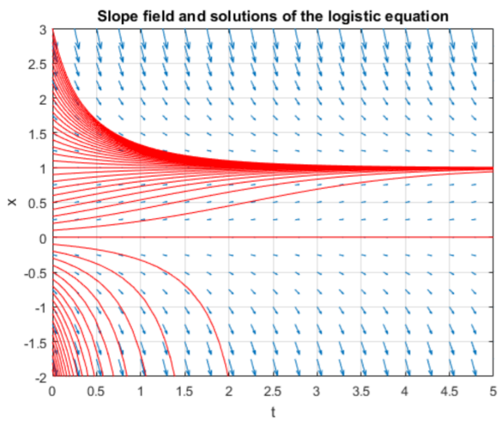
A *fixed point* of the a continuous dynamical system is an x^* with the property
$$\dot{x} = f(x^*) = 0 \tag{87}$$
If x^* is a fixed point, then $x(t) = x^*$ is a constant solution of the system.

For the solution $x(t)$ of a one-dimensional system there are the following possibilites:

- The solution is a fixed point
- The solution grows monotonically, i.e. $\dot{x}(t) > 0$
- The solution decays monotonically, i.e. $\dot{x}(t) < 0$

Example: We consider the ODE of limited growth
$$\dot{N} = rN\left(1 - \frac{N}{K}\right) \tag{88}$$

If $K = 1$, $\dot{N}$ is 0 if $N = 0$ and $N = 1$ i.e. the fixed points for this equation are $x_1 = 0$ and $x_2 = 1$ (or $x_2 = K$). These fixed points are also visible in the following figure.



Continuous Linear stability analysis

Let x^* be a fixed point of the continuous dynamical system $\dot{x} = f(x)$. The fixed point x^* is

- stable if $f'(x^*) < 0$,
- unstable, if $f'(x^*) > 0$,
- stable or unstable, if $f'(x^*) = 0$

Example (continuation): the derivative of the logistic equation above $\left(f(x) = rx\left(1 - \frac{x}{K}\right)\right)$ is
$$f'(x) = r\left(1 - \frac{2x}{K}\right) \tag{89}$$
Hence
$$\begin{aligned} f'(x_1) &= f'(0) = r > 0 \\ f'(x_2) &= f'(K) = -r < 0 \end{aligned} \tag{90}$$
Hence $x_1 = 0$ is an unstable fixed point, and $x_2 = K$ is a stable fixed point, which is also visible in the figure above.

Fixed points in Discrete Dynamical Systems

A *fixed point* of the discrete dynamical system $x_{n+1} = f(x_n)$ is a x^* with the property
$$x^* = f(x^*) \tag{91}$$
If x^* is a fixed point of $x_{n+1} = f(x_n)$, then $x_n = x^*$ is a constant solution of that system.

Example: We want to find the fixpoint of the discrete dynamical system
$$x_{n+1} = \cos(x_n) \tag{92}$$
Using the formula above:
$$x^* = \cos(x^*) \quad \Rightarrow \quad x^* \approx 0.739 \tag{93}$$

Discrete Linear stability analysis

Let x^* be a fixed point of the discrete dynamical system $x_{n+1} = f(x_n)$. The fixed point x^* is

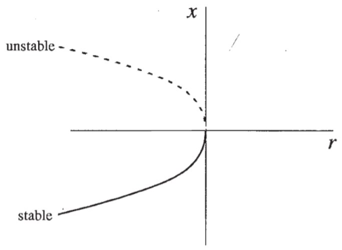
- stable, if $|f'(x^*)| < 1$,
- unstable, if $|f'(x^*)| > 1$,
- stable or unstable, if $|f'(x^*)| = 1$

$$\eta_{n+1} = \eta_n \cdot f'(x^*) \quad \Rightarrow \quad \eta_n = \eta_0 \cdot \left(f'(x^*)\right)^n \tag{94}$$
Example: (continuation): We want to analyse the fixpoint $x^* \approx 0.739$ of the example above.
$$|f'(x^*)| = |\sin(x^*)| \approx |-0.674| < 1 \tag{95}$$
Hence x^* is a stable fixed point of the system.

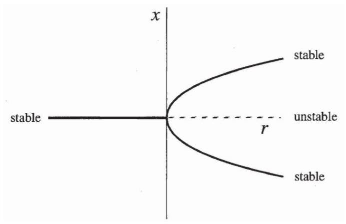
Bifurcations in Continous Systems

Fundamentals
A qualitative change of the dynamics of a dynamical system
$$\dot{x} = f(x, r) \quad \text{or} \quad x_{n+1} = f(x_n, r) \tag{96}$$
depending on a parameter r is a bifurcation of the system

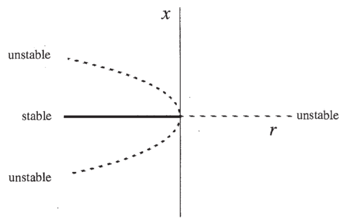
Standard Bifurcations in descrete systems
Saddle-node Bifurcation
The saddle-node bifurcation is a bifurcation in which fixed points are created or destroyed by the variation of a system parameter.
Example: Dynamical system, depending on the parameter r
$$\dot{x} = r + x^2 \tag{97}$$
Fixed points:
$$\begin{aligned} &r < 0 : x^*_{1,2} = \pm\sqrt{-r} \\ &r = 0 : x^* = 0 \\ &r > 0 : \text{none} \end{aligned} \tag{98}$$
Stability of the fixed points for $r < 0$:
$$f(x) = r + x^2, \quad f'(x) = 2x \tag{99}$$
hence
$$\begin{aligned} x^*_1 = \sqrt{-r} > 0 : f'(x^*_1) > 0, \textit{unstable} \\ x^*_2 = -\sqrt{-r} < 0 : f'(x^*_2) < 0, \textit{stable} \end{aligned} \tag{100}$$



Pitchfork Bifurcation
The pitchfork bifurcation is a bifurcation in which a fixed point changes its stability and two new fixed points are created.



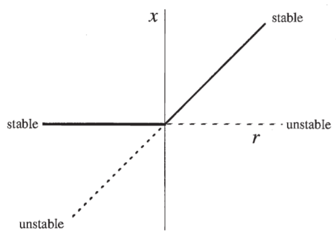
Supercritical



Subcritical

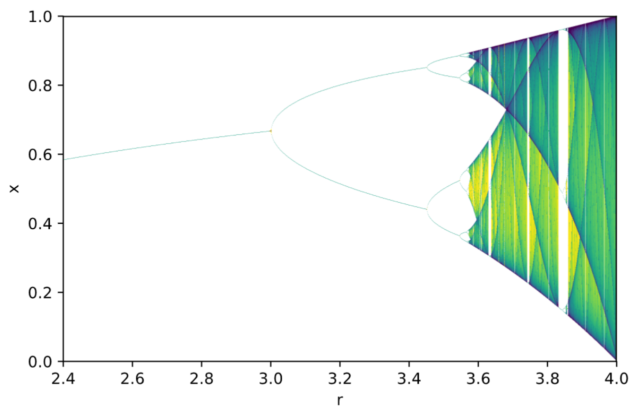
Supercritical: Additional fixed points only for parameter values greater ("super") than the bifurcation point.
Subcritical P-B: Additional fixed points only for parameter values smaller ("sub") than the bifurcation point.

Transcritical Bifurcation
A transcritical bifurcation is a bifurcation in which the stability of two fixed points is interchanged.



Bifurcations in Discrete Systems

The bifurcations which we introduced in continuous systems can also occur in discrete systems.
Discrete systems can (or do allways?) show a chaotic behaviour in the parameter space:



Example: Consider the discrete dynamical system depending on the parameters $\mu \geq 0, \varepsilon \geq 0$ (Adler system)

$$x_{x+1} = x_n + \mu + \varepsilon \sin(x_n) \quad (\mu \geq 0, \varepsilon \leq 0) \tag{101}$$

Fixed points x^* are solutions of the equation $\mu + \varepsilon \sin(x) = 0$, hence

$$x^*_{1,k} = -\arcsin\left(\frac{\mu}{\varepsilon}\right) + k \cdot 2\pi, \quad x^*_{2,k} = (2k+1)\pi + \arcsin\left(\frac{\mu}{\varepsilon}\right) \tag{102}$$

The fixed points only exist in the case $\mu \leq \varepsilon$
For $\mu = 1$, the fixed points only exist in the case $\varepsilon \geq 1$. We have $f(x) = x + \mu + \varepsilon \sin(x)$, hence $f'(x) = 1 + \varepsilon \cos(x)$. It follows that for $\mu = 1$

$$\begin{aligned} f'(x^*_{1,k}) =& 1 + \varepsilon \cos\left(-\arcsin\left(\frac{1}{\varepsilon}\right)\right) \\ =& 1 + \varepsilon\sqrt{1 - \frac{1}{\varepsilon^2}} \\ =& 1 + \sqrt{\varepsilon^2 - 1} \geq 1 \end{aligned} \tag{103}$$

Hence the fixed points $x^*_{1,k}$ are unstable.

$$\begin{aligned} f'(x^*_{2,k}) =& 1 + \varepsilon \cos\left(\pi + \arcsin\left(\frac{1}{\varepsilon}\right)\right) \\ =& 1 - \varepsilon \cos\left(\arcsin\left(\frac{1}{\varepsilon}\right)\right) \\ =& 1 - \varepsilon\sqrt{1 - \frac{1}{\varepsilon^2}} \\ =& 1 + \sqrt{\varepsilon^2 - 1} \end{aligned} \tag{104}$$

For the stability of $x^*_{2,k}$ it is necessary that $|1 - \sqrt{\varepsilon^2 - 1} < 1$ holds. This is true if and only if $0 < \sqrt{\varepsilon^2 - 1} < 2$, hence in the case $1 < \varepsilon^2 < 5$ resp. $1 < \varepsilon < \sqrt{5}$. The fixed points $x^*_{2,k}$ are thus stable in the case $1 < \varepsilon < \sqrt{5}$.

Two-Dimensional Systems

We now consider the systmes for arbitrary $n \in (N)$, i.e.

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, ..., x_n) \\ \dot{x}_2 &= f_2(x_1, x_2, ..., x_n) \\ &... \\ \dot{x}_n &= f_n(x_1, x_2, ..., x_n) \end{aligned} \tag{105}$$

resp. in vector notation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad (\mathbf{x} \in (R)^n) \tag{106}$$

Definitions
A point $x^* \in \mathbb{R}^n$ is a *fixed point* of the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, if

$$\mathbf{f}(\mathbf{x}^*) = \mathbf{0} \tag{107}$$

If $\mathbf{x}^*$ is a *fixed point* of the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then

$$\mathbf{x}(t) = \mathbf{x}^* \tag{108}$$

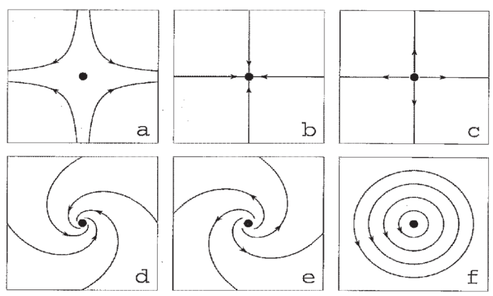
is a constant solution.

A fixed point $x^* \in \mathbb{R}^n$ of the dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, is

- *stable*, if the solution starting close to x^* stays close to x^* for all $t \geq 0$.
- *asymptotically stable*, if the solution starting close to x^* converge to x^* for $t \to \infty$.
- *unstable*, if the solution starts close to x^* and diverging from x^* for $t \to \infty$.

Linear Systems

Overview over the various types of fixed points and the corresponding geometry of the system near the fixed point:



a) saddle point
b) stable node
c) unstable node
d) stable spiral
e) unstable spiral
f) center/elliptic fixed point
g) degenerate cases

A two-dimensional system can be characterized with the eigenvalues like so:

- $\lambda_1 < 0, \lambda_2 < 0$: stable node
- $\lambda_1 > 0, \lambda_2 < 0$: saddle point
- $\lambda_1 > 0, \lambda_2 > 0$: usntable point
- $\lambda_1 = 0, \lambda_2 = 0$: non-isolated fixed point
- $\lambda_{1.2} \neq \mathbb{R}, \mathrm{Re}(\lambda_{1,2}) = 0$: center
- $\lambda_{1.2} \neq \mathbb{R}, \mathrm{Re}(\lambda_{1,2}) > 0$: unstable spiral
- $\lambda_{1.2} \neq \mathbb{R}, \mathrm{Re}(\lambda_{1,2}) < 0$: stable spiral

Alternativ characterization instead of with the eigenvalues λ_1, λ_2 with trace and determinant of A:

$$\begin{aligned} \tau &= \mathrm{tr} = a_{11} + a_{22} \\ \Delta &= \det(A) = a_{11}a_{22} - a_{12}a_{21} \end{aligned} \tag{109}$$

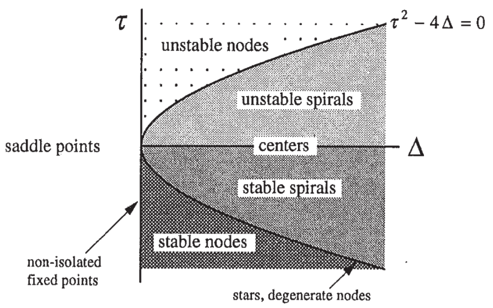
Connection with the eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right) \tag{110}$$

and

$$\Delta = \lambda_1\lambda_2, \quad \tau = \lambda_1 + \lambda_2 \tag{111}$$

In the following figure the classification of fixed points based on the properties of τ and Δ is shown graphically.



If one is only interested in the stability of $\mathbf{x}^*$ (and not in a further classification into nodes, spirals, etc.), the fixed point $\mathbf{x}^* = 0$ of a dynamical system is

- asymptotically stable, if
$$\mathrm{Re}(\lambda_i) < 0 \tag{112}$$
holds for all eigenvalues λ_i of A
- stable, if
$$\mathrm{Re}(\lambda_i) \leq 0 \tag{113}$$
holds for all eigenvalues λ_i of A
- usntable, if
$$\mathrm{Re}(\lambda_i) > 0 \tag{114}$$
holds for at least one eigenvalue λ_i of A

Nonlinear Systems

In order to use the analysis of linear systems, we linearize the nonlinear system near the fixed point x^*. To this end, we consider small perturbations $u = x_1 - x_1^*$, $v = x_2 - x_2^*$ and deduce equations for u and v by expanding f_1 and f_2 into Taylor series. The temporal evolution of (u, v) is thus given by

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \end{pmatrix}_{(x_1^*, x_2^*)} \cdot \begin{pmatrix} u \\ v \end{pmatrix} + \text{higher-order terms} \tag{115}$$

The matrix

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \end{pmatrix}_{(x_1^*, x_2^*)} \tag{116}$$

is the Jacobi matrix of the system near the fixed point $\mathbf{x}^* = (x_1^*, x_2^*)$.

Example: Consider the system

$$\begin{aligned} \dot{x} &= -x + x^3 \\ \dot{y} &= -2y \end{aligned} \tag{117}$$

fixed points are:

$$P_1 = (0,0), \quad P_2 = (1,0), \quad P_3 = (-1,0). \tag{118}$$

The Jacobi matrix of the system is

$$A = \begin{pmatrix} -1 + 3x^2 & 0 \\ 0 & -2 \end{pmatrix} \tag{119}$$

The eigenvalues of A are:

$$P_1 : \lambda_1 = -1, \lambda_2 = -2; \qquad P_2, P_3 : \lambda_{1,2} = \pm 2 \tag{120}$$

Therefore P_1 is a stable node, and P_2 and P_3 are saddle points of the linearized system.

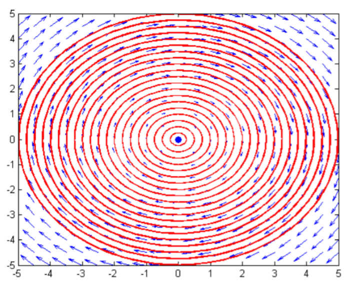
Intuitively: The linearization leads to the correct classification of the fixed points of nonlinear systems, if the fixed point type is described by an inequality and not an equality.

Hartman-Grobman theorem
If both eigenvalues λ_1, λ_2 of the corresponding system matrix A fulfill the condition $\mathrm{Re}(\lambda_i) \neq 0$, then the behaviour of the nonlinear system is for sufficiently small perturbations topologically equivalent to the behaviour of the linearized system.

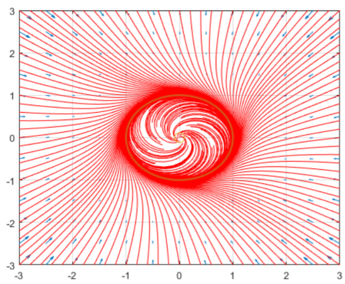
Classification of fixed points into robust and marginal cases
Robust: Nodes, spirals, saddle points
Marginal: centers/elliptic fixed points, degenerate cases

Limit cycles

If a fixed point is of the type "center", it is surrounded by nothing but closed trajectories. If there is a single isolated trajectory in the system, one speaks of a limit cycle.
A limit cycle of a dynamical system is an isolated closed trajectory of the system.



Non-isolated closed orbits



Isolated closed orbits, limit cycle

Limit cycles are important structuring elements of the phase space of a 2D dynamical system, in addition to fixed points

Methods for detection (Poincaré-Bendixson theorem)
Let $\dot{\mathbf{x}} = f(x)$ be a dynamical system in a bounded and closed subset G of $\mathbb{R}^2$. If the system does not have a fixed point in G and if there exists a trajectory C, which stays in G for all $t \geq t_0$, then C is either a closed trajectory, or it converges to a closed trajectory.
If a trajectory stay confined to a bounded closed region without fixed point, it has to converge to a closed trajectory, if it is not already a closed trajectory itself. In particular, a chaotic behaviour is impossible! Note that the result holds only for $n = 2$.

Example: Consider the dynamical system: (image on the top right)

$$\begin{aligned} \dot{x} &= x + y - x(x^2 + y^2) \\ \dot{y} &= -x + y - y(x^2 + y^2) \end{aligned} \tag{121}$$

In order to apply the *Poincaré-Bendixson* theorem we choose

$$G = \left\{(x, y), \in \mathbb{R}^2 | 0.5 \leq \sqrt{x^2 + y^2} \leq 2\right\}. \tag{122}$$

i.e. a torus with inner radius of 0.5 and an outer radius of 2. The system has no fixed point in G since the fixed point lays at the origin, which is not contained in G.
If the equations are formulated in polar coordinates one gets

$$\begin{aligned} \dot{r} &= r(1 - r^2) \\ \dot{\theta} &= -1 \end{aligned} \tag{123}$$

For $r = 0.5$ we have $\dot{r}(t) > 0$, whereas for $r = 2$ have the reverse estimate $\dot{r}(t) < 0$. Therefore $r(t)$ *increases* for a trajectory of the system at the inner limit of the torus G, whereas $r(t)$ *decreases* for a trajectory at the outer limit of the region. This implies that a trajectory starting in G must stay in G for all times. The conditions for applying the *Poincaré-Bendixson* theorem are thus fulfilled, and it follows from the theorem that the system has a limit cycle.

Bifurications

A bifurcation occurs in the system $\dot{x} = f(x, r)$, if the phase portrait changes its topological structure as the parameter r is varied.

All bifurcations occuring in 1D systems also occur in 2D systems.
In addition, there exists a new type of bifurcation which occurs only in systems of dimension $n \leq 2$: the Hopf bifurcation.

Hopf Bifurication

Creation/deletion of a limit cycle by varying a system parameter. This describes the ways in which oscillations in a system can be turned on or of.

If in a linear stability analysis $\Delta > 0$ (the fixed point is a center) it is possible for a Hopf bifurcation to occur.

- if $\tau < 0$, the fixed point is stable and thus, no limit cycle can occur.
- if $\tau > 0$, the fixed point is unstable and thus, limit cycles can occur. This has to be hecked with the *Poincaré-Bendixson theorem.*

Three-Dimensional Systems and Chaos

Chaos is aperiodic long-term behaviour in a deterministic system with sensitive dependence on the initial conditions.

Dimension $n \geq 3$: Trajectories can remain bounded without converging to a fixed point or a closed orbit. Possibly sensitive dependence on the initial conditions, i.e. slightly differing initial conditions lead to a totally different long-term behaviour

Rössler System

The following nonlinear system was constructed by R¨ossler in 1976:

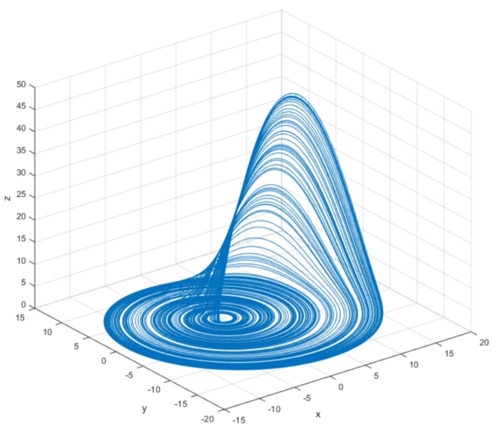
$$\begin{aligned} \dot{x} &= -(x+z) \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x-c) \end{aligned} \tag{124}$$

It depends on the 3 parameters a, b, c; we set $a = 0.2, b = 0.2$ and consider the system for varying values of c. Depending on the parameter values, the system has at most two fixed points. Depending on the parameter values, the system has at most two fixed points, namely

$$P_{1,2} = \left(\frac{1}{2}\left(c \mp \sqrt{c^2 - 4ab}\right), \frac{1}{2a}\left(-c \pm \sqrt{c^2 - 4ab}\right), \frac{1}{2a}\left(c \mp \sqrt{c^2 - 4ab}\right)\right) \tag{125}$$

The stability of these fixed points can be determined by considering the eigenvalues of the Jacobian of the system. Empirical investigations show that for most parameter values for which the fixed points $P_{1,2}$ exist, at least one of the eigenvalues of the Jacobian matrix has a positive real part and is therefore unstable.

The behaviour of the Rössler system can vary greatly, depending on the values of the parameters. For $c = 2.3$ and $c = 2.9$, the system converges to a period-one and a periodtwo limit cycle, respectively. For $c = 8.5$, we observe a chaotic attractor.



Lorenz System

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \end{aligned} \tag{126}$$

We investigate the Lorenz system for the parameter values $\sigma = 10, b = \frac{8}{3}$, and various values of r, in most cases $r = 28$.

We first determine fixed points of the system. To this end, we solve the system of equations and obtain the three fixed points

$$P_1^* = (0,0,0), \quad P_{2,3}^* = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right) \tag{127}$$

The fixed points P_2^* and P_3^* only exist in the case $r \geq 1$; at the parameter value $r = 1$ there is a bifurcation. The Jacobian of the Lorenz system is

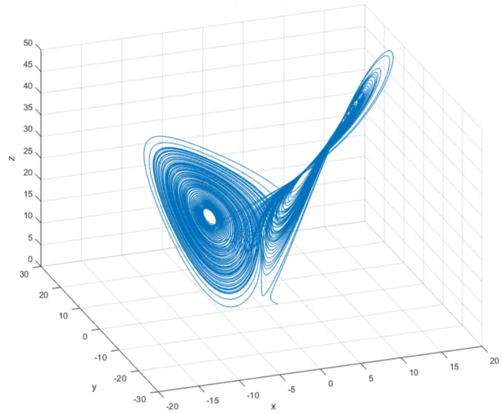
$$J(x,y,z) = \begin{pmatrix} \sigma & -\sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix} \tag{128}$$

At P_1^* the eigenvalues of J are, depending on r, with $\sigma = 10$ and $b = \frac{8}{3}$,

$$\lambda_1 = -\frac{8}{3}, \quad \lambda_{2,3} \frac{1}{2}\left(-11 \pm \sqrt{81 + 40r}\right) \tag{129}$$

For $r < 1$ all three eigenvalues are negative and P_1^* therefore a stable fixed point; for $r > 1$, we have $\lambda_2 > 0$, and P_1^* is therefore unstable. Concerning P_2^* and P_3^* , from a corresponding analysis of the eigenvalues we see that these points are stable for $0 < r < r_1 \approx 24.737$ and unstable for $r > r_1$.

The greatest degree of instability thus exists in the case $r > r_1$, e.g. for the typical value $r = 28$.

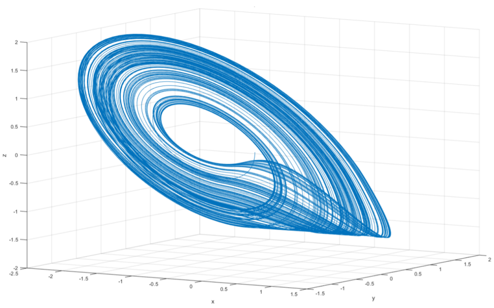


Linz/Sprott System

One of the simplest examples of a 3-dimensional continuous system exhibiting chaotic dynamics is the following nonlinear third-order ODE.

$$x^{(3)} + a\ddot{x} + b\dot{x} - |x| + 1 = 0 \tag{130}$$

Note that the absolute value $|x|$ is the only nonlinearity on the system. A simulation of the system for $a = 0.6, b = 1$ (and the initial conditions $x(0) = -0.1, y(0) = z(0) = 0$) shows that the system converges to a chaotic attrator which somewhat resembles a Möbius strip:



Discrete Dynamical Systems

Lyapunov exponent

The Lyapunov exponent λ of the discrete dynamical system $x_{n+1} = f(x_n)$ for an orbit starting at x_0 is

$$\lambda = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1} \ln|f'(x_i)|\right) \tag{131}$$

Significance of the sign of λ:

- $\lambda > 0$: The system behaves chaotically
- $\lambda < 0$: The system does not behave chaotically
- $\lambda = 0$: The system is at a bifurcation point

Mandelbrot set

The Julia set J of the complex dynamical system $z_{n+1} = f(z_n)$ is the boundary between the set of starting points with bounded orbits and the set of starting points with unbounded orbits.

Mandelbrot set

The Mandelbrot set M of the complex discrete dynamical system

$$z_{n+1} = z_n^2 + c \tag{132}$$

is the set of parameters $c \in C$, for which the set $(z_n)_n \in N$ of points generated by the dynamics with initial value $z_0 = 0$ remains bounded:

$$M = \{c \in C | f_c^n(0) \not\to \infty, \text{for } n \to \infty\} \tag{133}$$

Remark. The Mandelbrot set thus is a set of parameters, for which the equation above has a certain properties, namely that the sequence of points starting at the origin and evolving by the dynamics remains bounded.