Ordinary Differential Equations and Dynamical Systems

Part I

Modeling

Fundamentals

Formally, an ordinary differential equation is an equation, in which a function and its derivates and the independent variable appear.

An (implicit) *ordinary differential equation* of order n is an equation of the form

$$F(x, y, y', y'', ..., y^{(n)}) = 0. \tag{1}$$

An example of an explicit ODE of order n is of the form

$$y^n = G(x, y, y', y'', ..., y^{n-1}). \tag{2}$$

Classification

Differential equations can be classified according to various criteria. Besides the order of an ODE we are also interested in whether an ODE is linear, homogeneous, separable or autonomous.

Linearity
An *n-th* order ODE is *linear*, if it is of the form:

$$a_n(x) \cdot y^{(n)} + ... + a_1(x) \cdot y' + a_0(x) \cdot y = g(x) \tag{3}$$

where $a_n(x), ..., a_1(x), a_0(x)$ are $g(x)$ fixed functions. Or in other words: A differential equation is linear if the dependant variable and all of its derivatives appear in a linear fashion (i.e., they are not multiplied together or squared for example or they are not part of transcendental functions such as sins, cosines, exponentials, etc.)

Homogenity
A lienar ODE is *homogeneous*, if $g(x) = 0$ for all x; otherwise the ODE is *inhomogeneous*, and $g(x)$ is the *inhomogeneity* or *source* term.

Constant coefficient
A linear ODE has *constant coefficients*, if it is of the form

$$a_n \cdot y^{(n)} + ... + a_1 \cdot y' + a_0 \cdot y = g(x), \tag{4}$$

with $a_n \neq 0$ (the source term $g(x)$ does not have to be constant).

Separability
The ODE is *separable*, if $F(x, y)$ can be written as a product of a x- and y-dependent term, i.e. if the ODE is of the form

$$y' = g(x) \cdot h(y) \tag{5}$$

Autonomity
The ODE (1.28) is *autonomous*, if $F(x, y)$ only depends on y, i.e. if the ODE is of the form

$$y' = h(y) \tag{6}$$

Every autonomous ODE is separable with $g(x) = 1$.

Examples

$y' = f(x)$	Inhomogeneous linear ODE for $y(x)$ with source term $f(x)$
$m \cdot \dot{v} = m \cdot g - k \cdot v^2$	Nonlinear ODE for $v(t)$
$l \cdot \ddot{\Phi} + g \cdot sin(\Phi) = 0$	Nonlinear ODE for $\Phi(t)$
$l \cdot \ddot{\Phi} + g \cdot \phi$	Homogeneous linear ODE for $\Phi(t)$
$l \cdot \ddot{\Phi} + g \cdot \phi = sin(\omega t)$	Inhomogeneous linear ODE for $\Phi(t)$ with source term $sin(\omega t)$
$i'' + \frac{R}{L} i' + \frac{1}{LC} i = 0$	Homogeneous linear ODE for $i(t)$

Systems of differential equations

If several systems are coupled with each other and mutually influence each other, one often obtains a system of ODE's.
A *system of differential equations* of first oder is a system

$$\begin{aligned} y_1' &= f_1(x, y_1, ..., y_n) \\ &\vdots \\ y_n' &= f_n(x, y_1, ..., y_n) \end{aligned} \tag{7}$$

of ODE's for unknown functions $y_1(x), ..., y_n(x)$.
Using the vectorial notation

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}) \tag{8}$$

An ODE of n-th order is equivalent to a system of first-order ODE's.

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_n' &= f(x, y_1, ..., y_n) \end{aligned} \tag{9}$$

Example We want to rewrite the following 2nd order ODE into a system of first-order ODE's.

$$\ddot{x}(t) + 2\delta\dot{x}(t) + \omega_0^2 x(t) = f(t) \tag{10}$$

If we introduce the vector-valued function

$$\mathrm{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \quad \Rightarrow \quad \dot{\mathrm{y}} = \begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} \tag{11}$$

rewriting:

$$\begin{aligned} \dot{\mathrm{y}} &= \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -2\delta\dot{x}(t) - \omega_0^2 x(t) + f(t) \end{pmatrix} \\ \dot{\mathrm{y}} &= \begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\delta \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \\ \dot{\mathrm{y}} &= \mathrm{A}\mathbf{y} + \mathbf{b} \end{aligned} \tag{12}$$

Slope field