

# Ordinary Differential Equations and Dynamical Systems

## Part I

## Modeling

### Fundamentals

Formally, an ordinary differential equation is an equation, in which a function and its derivatives and the independent variable appear.

An (implicit) *ordinary differential equation* of order  $n$  is an equation of the form

$$F(x, y, y', y'', \dots, y^{(n)}) = 0. \quad (1)$$

An example of an explicit ODE of order  $n$  is of the form

$$y^n = G(x, y, y', y'', \dots, y^{n-1}). \quad (2)$$

### Classification

Differential equations can be classified according to various criteria. Besides the order of an ODE we are also interested in whether an ODE is linear, homogeneous, has constant coefficient, is separable or autonomous.

#### Linearity

An  $n$ -th order ODE is *linear*, if it is of the form:

$$a_n(x) \cdot y^{(n)} + \dots + a_1(x) \cdot y' + a_0(x) \cdot y = g(x) \quad (3)$$

where  $a_n(x), \dots, a_1(x), a_0(x)$  are  $g(x)$  fixed functions. Or in other words: A differential equation is linear if the dependant variable and all of its derivatives appear in a linear fashion (i.e., they are not multiplied together or squared for example or they are not part of transcendental functions such as sins, cosines, exponentials, etc.)

#### Homogeneity

A linear ODE is *homogeneous*, if  $g(x) = 0$  for all  $x$ ; otherwise the ODE is *inhomogeneous*, and  $g(x)$  is the *inhomogeneity* or *source* term.

#### Constant coefficient

A linear ODE has *constant coefficients*, if it is of the form

$$a_n \cdot y^{(n)} + \dots + a_1 \cdot y' + a_0 \cdot y = g(x), \quad (4)$$

with  $a_n \neq 0$  (the source term  $g(x)$  does not have to be constant).

#### Separability

The ODE is *separable*, if  $F(x, y)$  can be written as a product of a  $x$ - and  $y$ -dependent term, i.e. if the ODE is of the form

$$y' = g(x) \cdot h(y) \quad (5)$$

#### Autonomy

The ODE (1.28) is *autonomous*, if  $F(x, y)$  only depends on  $y$ , i.e. if the ODE is of the form

$$y' = h(y) \quad (6)$$

Every autonomous ODE is separable with  $g(x) = 1$ .

#### Examples

$y' = f(x)$	Inhomogeneous linear ODE for $y(x)$ with source term $f(x)$
$m \cdot \dot{v} = m \cdot g - k \cdot v^2$	Nonlinear ODE for $v(t)$
$l \cdot \ddot{\Phi} + g \cdot \sin(\Phi) = 0$	Nonlinear ODE for $\Phi(t)$
$l \cdot \ddot{\Phi} + g \cdot \phi$	Homogeneous linear ODE for $\Phi(t)$
$l \cdot \ddot{\Phi} + g \cdot \phi = \sin(\omega t)$	Inhomogeneous linear ODE for $\Phi(t)$ with source term $\sin(\omega t)$
$i'' + \frac{R}{L}i' + \frac{1}{LC}i = 0$	Homogeneous linear ODE for $i(t)$

### Systems of differential equations

If several systems are coupled with each other and mutually influence each other, one often obtains a system of ODE's.

A *system of differential equations* of first order is a system

$$\begin{aligned} y_1' &= f_1(x, y_1, \dots, y_n) \\ &\vdots \\ y_n' &= f_n(x, y_1, \dots, y_n) \end{aligned} \quad (7)$$

of ODE's for unknown functions  $y_1(x), \dots, y_n(x)$ .

Using the vectorial notation

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}) \quad (8)$$

An ODE of  $n$ -th order is equivalent to a system of first-order ODE's.

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_n' &= f(x, y_1, \dots, y_n) \end{aligned} \quad (9)$$

**Example: 2nd-order ODE to system of first-order ODE's**

We want to rewrite the following 2nd order ODE into a system of first-order ODE's.

$$\ddot{x}(t) + 2\delta\dot{x}(t) + \omega_0^2 x(t) = f(t) \quad (10)$$

If we introduce the vector-valued function

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \Rightarrow \dot{\mathbf{y}} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} \quad (11)$$

rewriting:

$$\begin{aligned} \dot{\mathbf{y}} &= \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -2\delta\dot{x}(t) - \omega_0^2 x(t) + f(t) \end{pmatrix} \\ \dot{\mathbf{y}} &= \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\delta \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \\ \dot{\mathbf{y}} &= \mathbf{A}\mathbf{y} + \mathbf{b} \end{aligned} \quad (12)$$

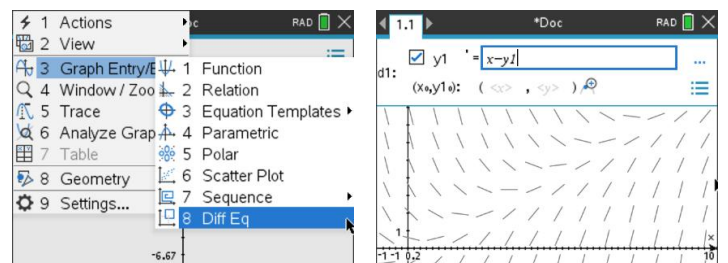
### Slope field

Slope fields are often lead to a good qualitative understanding of the situation described by the ODE under consideration. Slope field can be understood in the following way: To each point  $(x, y)$  in the region  $B$  under consideration,  $F(x, y)$  is a value which describes the slope of the solution curve passing through the point  $(x, y)$ .

#### Example with calculator

We want to plot the slope field of the following ODE

$$y' = x - y \quad (13)$$



Select: Menu, 3: Graph Entry/Edit, 8: Diff Eq.

Write down the ODE

### Solvability

Two solution curves of an ODE cannot cross.

## Part II

## Ordinary Differential Equations

### Analytical methods for first-order ODE's

#### Overview

Separable ODE's  
Linear ODE's  
Exact ODE's

## Separable ODE's

### Example

We compute the general solution of the ODE

$$y' = -\frac{x}{y} \quad (14)$$

- We write the equation as

$$\frac{dy}{dx} = -\frac{x}{y} \quad (15)$$

- We bring all  $x$ -dependent terms to the left hand side and all  $y$ -dependent terms on the right hand side:

$$y \, dy = -x \, dx \quad (16)$$

- We integrate on both sides and get

$$\int y \, dy = - \int x \, dx \Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C, \quad C \in \mathbb{R} \quad (17)$$

- We solve for  $y$  and get

$$y = \pm \sqrt{K - x^2}, \quad K \in \mathbb{R} \quad (\text{where } K = 2C) \quad (18)$$

### Example of substitution

We consider the ODE

$$y' = (x + y)^2 \quad (19)$$

## Exact ODE's

## Analytical methods for linear ODE's

### Overview

We differentiate between first-order linear ODE's and higher-order ODE's as well as between homogeneous and inhomogeneous ODE's.

The general solution of the inhomogeneous ODE is the sum

$$y = y_h + y_s, \quad (20)$$

where  $y_h$  is the general solution of the homogeneous ODE and  $y_s$  any special solution of the inhomogeneous ODE.

### First-order linear ODE's

To solve a first-order linear ODE we thus have to find  $y_h$  and  $y_s$

**$y_h$ :** a homogeneous first-order ODE is separable and can therefore be solved by the standard procedure for separable ODE's described above.

**$y_s$ :** To find a special solution of an ODE, there are several possibilities. In the case of an ODE with constant coefficients, it usually suffices to choose for  $y_s$  an *ansatz of the form of the source term*  $g(x)$ . In the case

of non-constant coefficients, the method *variation of constants* usually works better.

### Example using the ansatz

We solve the ODE

$$y' + ay = b \quad (21)$$

- To determine  $y_h$ , we integrate the homogeneous ODE

$$y' + ay = 0 \quad (22)$$

by separation of variables we obtain the solution

$$y_h = C \cdot e^{-ax}, \quad C \in \mathbb{R}. \quad (23)$$

- For finding  $y_s$  we choose the ansatz in the form of the source term. In this case the source term is constant,  $g(x) = b$ . Therefore we assume that the special solution  $y_s$  is constant as well, i.e. we make an ansatz  $y_s = c$ . We plug this ansatz into the inhomogeneous ODE and obtain the special solution  $y_s$ .

$$y'_s + ay_s = b \Rightarrow y_s = \frac{a}{b} \quad (24)$$

The general solution therefore is

$$y = C \cdot e^{-ax} + \frac{a}{b}, \quad C \in \mathbb{R} \quad (25)$$

### Ansatz functions for the solution of the inhomogeneous first-order ODE

Source term  $g(x)$

$$g(x) = b_0$$

$$g(x) = b_1x + b_0$$

$$g(x) = b_2x^2 + b_1x + b_0$$

$$g(x) = \sum_{i=0}^n b_i x^i$$

$$g(x) = A \sin(\omega x) + B \cos(\omega x)$$

$$g(x) = Ae^{bx}$$

Ansatz  $y_s$

$$y_s = c_0$$

$$y_s = c_1x + c_0$$

$$y_s = c_2x^2 + c_1x + c_0$$

$$y_s = \sum_{i=0}^n c_i x^i$$

$$y_s = C_1 \sin(\omega x) + C_2 \cos(\omega x)$$

$$y_s = C \sin(\omega x + \varphi)$$

$$y_s = \begin{cases} \frac{A}{b+a} e^{bx} & \text{for } b \neq -a \\ A x e^{-ax} & \text{for } b = -a \end{cases}$$

In addition, the following rules must be followed:

**Linearity** If  $g(x)$  is a linear combination of several source terms, one has to assume as ansatz for  $y_s(x)$  a corresponding linear combination of several ansatz terms.

**Resonance** If the source term  $g(x)$  is itself already a solution of the homogeneous ODE, the corresponding ansatz for  $y_s$  has to be multiplied with  $x$ . So if for example  $y_h = Ce^x$ , and  $g(x) = e^x$ , the ansatz  $y_s = x \cdot e^x$  is chosen.

### Example using separation of variables