A posteriori error estimation for multiphysics

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Contents

1. Error estimation

We wish to solve the linear system of equations

$$Au = f \tag{1}$$

and determine a linear functional of solution u, namely the “quantity of interest”

$$Q = (\psi, u). \tag{2}$$

Assume that we have an approximate solution $\hat{u} \approx u$, such that

$$A\hat{u} = \hat{f}. \tag{3}$$

The error in the quantity of interest Q is

$$(\psi, u) - (\psi, \hat{u}) = (\psi, u - \hat{u}) = (\psi, e), \tag{4}$$

where $e = u - \hat{u}$.

In order to construct a *computable* estimate for the error in the QoI, we define and solve the *adjoint* equation

$$A^\top \phi = \psi, \tag{5}$$

for adjoint solution ϕ, and observe that

$$(\psi, e) = (A^\top \phi, e) = (\phi, Ae) = (\phi, A(u - \hat{u})) = (\phi, f - \hat{f}) = (\phi, r), \tag{6}$$

where the *residual* $r = f - \hat{f}$.

2. Iterative solvers

We wish to solve the linear system of equations

$$Au = f.$$

Assuming it is not possible to invert A directly, we seek to compute an approximation $\bar{u}$ to solution u via an iterative procedure. Let $\bar{u}$ be an approximation to u, such that

$$A\bar{u} = \bar{f}. \tag{7}$$

We seek to iteratively improve this approximation. Formally,

$$u - \bar{u} = A^{-1}f - A^{-1}\bar{f} = A^{-1}r$$

where again the *residual* $r = f - \bar{f}$.

Let B be any approximation to A^{-1} and construct the iteration: Given $u^{\{0\}}$, find

$$\begin{aligned} u^{\{k\}} &= u^{\{k-1\}} + Br = u^{\{k-1\}} + B\left(f - Au^{\{k-1\}}\right), \quad k = 1, 2, \ldots, \\ &= (I - BA)u^{\{k-1\}} + Bf, \quad k = 0, 1, \ldots. \end{aligned} \tag{8}$$

Let $u^{\{k\}}$ be the analytic or exact solution to an iterative procedure, and $\hat{u}^{\{k\}}$ be an approximation to $u^{\{k\}}$. We partition the total error into the error arising due to the iterative process, and the error due to the inexact solution of the iterative equations, i.e.,

$$\begin{aligned} \text{Total error, } e^{\{k\}} &= u - \hat{u}^{\{k\}} \\ &= (u - u^{\{k\}}) + (u^{\{k\}} - \hat{u}^{\{k\}}) \\ &= \text{Iteration error} + \text{Discretization error}. \end{aligned} \tag{9}$$

The total error in the QoI can be partitioned similarly, i.e.,

$$\begin{aligned} \text{Total error in QoI, } (e^{\{k\}}, \psi) &= (u - \hat{u}^{\{k\}}, \psi) \\ &= (u - u^{\{k\}}, \psi) + (u^{\{k\}} - \hat{u}^{\{k\}}, \psi) \\ &= \text{Iteration error in QoI} + \text{Discretization error in QoI}. \end{aligned} \tag{10}$$

Remark 1. *We require the spectral radius of $(I - BA)$ to be less than one to ensure convergence. See [1, 2, 3] for details of the convergence of additive and restricted additive Schwarz methods.*

3. Overlapping Schwarz iterative solvers

We wish to solve the linear system of equation

$$Au = f$$

where $A \in \mathbb{R}^{n \times n}$, $u \in \mathbb{R}^n$ and $f \in \mathbb{R}^n$ by an overlapping Schwarz method and estimate the error in a quantity of interest

$$\mathrm{QoI} = (\psi, u)\,,$$

where $\psi \in \mathbb{R}^n$.

Index set $\mathcal{I}$: Let $\mathcal{I}$ be the index set

$$\mathcal{I} = \{1, 2, \cdots, n\}.$$

Overlapping decomposition of $\mathcal{I}$: We construct an overlapping decomposition of the index set $\mathcal{I}$ given by $\mathcal{I}_i, i = 1, 2, \ldots, p$ such that

$$\cup_{i=1}^{p} \mathcal{I}_i = \mathcal{I}$$

where $|\mathcal{I}_i| = m_i$ and $\sum_{i=1}^{p} m_i > n$.

Subdomain restriction operators: We define the subdomain *restriction* operators (matrices) R_i,

$$R_i \in \mathbb{R}^{m_i \times n},\ i = 1, 2, \ldots, p,$$

to be rank m_i matrices with rows $e_j^\top$ for $j \in \mathcal{I}_i$, where $e_j \in \mathbb{R}^n$ is a vector of zeros with a single one in column j.

Partition of $\mathcal{I}$: Construct an non-overlapping partition of the index set $\mathcal{P}$ given by $\mathcal{P}_i, i = 1, 2, \ldots, p$ such that

$$\cup_{i=1}^{p} \mathcal{P}_i = \mathcal{I}$$

and

$$\mathcal{P}_i \cap \mathcal{P}_j = \phi,\ \text{for all } i \neq j,$$

where $|\mathcal{P}_i| = q_i$ and $\sum_{i=1}^{p} q_i = n$.

Partition restriction operators: We define the partition *restriction* operators (matrices) $\widetilde{R}_i$,

$$\widetilde{R}_i \in \mathbb{R}^{m_i \times n},\ i = 1, 2, \ldots, p,$$

to be rank q_i matrices with rows $e_j^\top$ for $j \in \mathcal{I}_i \cap \mathcal{P}_i$. The partition restriction operator $\widetilde{R}_i$ is constructed from the subdomain restriction operator R_i by removing the single one (in column j) in the rows of R_i for which $j \in \mathcal{I}_i$, but $j \notin \mathcal{P}_i$.

4. Additive Schwarz

4.1. Forward problem

We define

$$\begin{aligned} A_i &= R_i A R_i^\top, \in \mathbb{R}^{m_i \times m_i},\ i = 1, 2, \ldots, p, \\ B_i &= R_i^\top A_i^{-1} R_i = R_i^\top \left(R_i A R_i^\top\right)^{-1} R_i \in \mathbb{R}^{n \times n},\ i = 1, 2, \ldots, p, \\ C_i &= B_i A \in \mathbb{R}^{n \times n},\ i = 1, 2, \ldots, p, \\ f_i &= B_i f \in \mathbb{R}^{n},\ i = 1, 2, \ldots, p. \end{aligned} \tag{11}$$

Single iteration: The additive Schwarz algorithm is: Given $u^{\{0\}}$ and $\alpha < 1$, for $k = 1, 2, \ldots$

$$\begin{aligned} u^{\{k\}} &= u^{\{k-1\}} + \alpha \sum_{i=1}^{p} B_i \left(f - A u^{\{k-1\}}\right), \\ &= \left(I - \alpha \sum_{i=1}^{p} B_i A\right) u^{\{k-1\}} + \alpha \sum_{i=1}^{p} B_i f \\ &= \left(I - \alpha \sum_{i=1}^{p} C_i\right) u^{\{k-1\}} + \alpha \sum_{i=1}^{p} f_i \\ &= D u^{\{k-1\}} + g, \end{aligned} \tag{12}$$

where

$$D = I - \alpha \sum_{i=1}^{p} C_i \quad \text{and} \quad g = \alpha \sum_{i=1}^{p} f_i. \tag{13}$$

Note that relaxation is required.

K iterations: Using this notation, $K = 6$ iterations of additive Schwarz can be written as the nK dimensional system of equations

$$U = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ -D & I & 0 & 0 & 0 & 0 \\ 0 & -D & I & 0 & 0 & 0 \\ 0 & 0 & -D & I & 0 & 0 \\ 0 & 0 & 0 & -D & I & 0 \\ 0 & 0 & 0 & 0 & -D & I \end{bmatrix} \begin{pmatrix} u^{\{1\}} \\ u^{\{2\}} \\ u^{\{3\}} \\ u^{\{4\}} \\ u^{\{5\}} \\ u^{\{6\}} \end{pmatrix} = \begin{pmatrix} g \\ g \\ g \\ g \\ g \\ g \end{pmatrix}. \tag{14}$$

This system nK linear equations can be solved (blockwise) by forward substitution. Furthermore, the subdomain components of the n-dimensional systems of equations within each block can be solved in parallel.

4.2. Adjoint equations

The adjoint system of equations of size nK to (14) is

$$U^\top x = \begin{bmatrix} I & -D^\top & 0 & 0 & 0 \\ 0 & I & -D^\top & 0 & 0 \\ 0 & 0 & I & -D^\top & 0 \\ 0 & 0 & 0 & I & -D^\top \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{pmatrix} \phi^{\{1\}} \\ \phi^{\{2\}} \\ \phi^{\{3\}} \\ \phi^{\{4\}} \\ \phi^{\{5\}} \\ \phi^{\{6\}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \psi \end{pmatrix},$$

where $\psi \in \mathbb{R}^n$. This system nK linear equations can be solved (blockwise) by backward substitution. Furthermore, the n-dimensional systems of equations within each block can be solved in parallel. Finally, note that

$$\begin{aligned} \phi^{\{K\}} &= \psi \\ \phi^{\{K-1\}} &= D^\top \phi^{\{K\}} = D^\top \psi \\ &\vdots \\ \phi^{\{1\}} &= \underbrace{D^\top \dots D^\top}_{k \text{ times}} \psi. \end{aligned} \tag{15}$$

4.3. Discretization error representation

The error in the QoI defined by (3) is given by the inner product

$$\text{Discretization error in QoI} = (\psi, u^{\{K\}} - \hat{u}^{\{K\}}) = (R, \Phi), \tag{16}$$

where R is nK-dimensional vector of residuals and

$$\Phi = \begin{pmatrix} \phi^{\{1\}} \\ \phi^{\{2\}} \\ \vdots \\ \phi^{\{K\}} \end{pmatrix}. \tag{17}$$

4.4. Total error representation

Assuming the global adjoint problem (5) can be solved for global adjoint solution ϕ, i.e., solving (5)

$$A^\top \phi = \psi,$$

and defining residual

$$r = f - A\hat{u}^{\{K\}}, \tag{18}$$

then

$$\text{Total error in QoI} = (r, \phi) \tag{19}$$

and

$$\text{Iterative error in QoI} = \text{Total error in QoI} - \text{Discretization error in QoI}. \tag{20}$$

4.5. Results

Input parameters are $K, \alpha, restricted(0/1)$ and $iprint$.

```
>> Additive_Schwarz(2,0.5,0,0)
```

norm(exact - iterative) solutions = 6.937325e-02

Performing C iteration with error	
QoI_discretization_error	8.076575e-03
QoI_discretization_error_estimate	8.076575e-03
QoI_total_error	2.632966e-02
QoI_total_error_estimate	2.632966e-02
QoI_iteration_error	1.825309e-02
Solving U system with error	
QoI_discretization_error_estimate	-5.580385e-04
QoI_discretization_error	-5.580385e-04
QoI_total_error	1.769505e-02
QoI_total_error_estimate	1.769505e-02
QoI_iteration_error	1.825309e-02

Table 1: Additive Schwarz: Two DD iterations

```
>> Additive_Schwarz(8,0.5,0,0)
```

norm(exact - iterative) solutions = 1.991954e-03

Performing C iteration with error	
QoI_discretization_error	3.418924e-02
QoI_discretization_error_estimate	3.418924e-02
QoI_total_error	3.377603e-02
QoI_total_error_estimate	3.377603e-02
QoI_iteration_error	-4.132041e-04
Solving U system with error	
QoI_discretization_error_estimate	-2.564992e-03
QoI_discretization_error	-2.564992e-03
QoI_total_error	-2.978196e-03
QoI_total_error_estimate	-2.978196e-03
QoI_iteration_error	-4.132041e-04

Table 2: Additive Schwarz: Eight DD iterations

5. Restricted Additive Schwarz

5.1. Forward problem

We define

$$\begin{aligned}
A_i &= R_i A R_i^\top, \in \mathbb{R}^{m_i \times m_i},\ i = 1, 2, \dots, p, \\
\widetilde{B}_i &= \widetilde{R}_i^\top A_i^{-1} R_i = \widetilde{R}_i^\top \left(R_i A R_i^\top\right)^{-1} R_i \in \mathbb{R}^{n\times n},\ i = 1, 2, \dots, p, \\
\widetilde{C}_i &= \widetilde{B}_i A \in \mathbb{R}^{n\times n},\ i = 1, 2, \dots, p, \\
\widetilde{f}_i &= \widetilde{B}_i f \in \mathbb{R}^{n},\ i = 1, 2, \dots, p.
\end{aligned} \tag{21}$$

Single iteration: The additive Schwarz algorithm is: For $k = 0, 1, \dots$

$$\begin{aligned}
u^{\{k\}} &= u^{\{k-1\}} + \sum_{i=1}^{p} \widetilde{B}_i \left(f - A u^{\{k-1\}}\right), \\
&= \left(I - \sum_{i=1}^{p} \widetilde{B}_i A\right) u^{\{k-1\}} + \sum_{i=1}^{p} \widetilde{B}_i f \\
&= (I - \widetilde{C}) u^{\{k-1\}} + \sum_{i=1}^{p} \widetilde{f}_i \\
&= \widetilde{D} u^{\{k-1\}} + \widetilde{g}
\end{aligned} \tag{22}$$

where

$$\widetilde{C} = \sum_{i=1}^{p} \widetilde{B}_i A, \quad \widetilde{D} = I - \widetilde{C} \text{ and } \widetilde{g} = \sum_{i=1}^{p} \widetilde{f}. \tag{23}$$

K iterations: $K = 6$ iterations of restricted additive Schwarz can be written as the nK dimensional system of equations

$$U = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ -\widetilde{D} & I & 0 & 0 & 0 & 0 \\ 0 & -\widetilde{D} & I & 0 & 0 & 0 \\ 0 & 0 & -\widetilde{D} & I & 0 & 0 \\ 0 & 0 & 0 & -\widetilde{D} & I & 0 \\ 0 & 0 & 0 & 0 & -\widetilde{D} & I \end{bmatrix} \begin{pmatrix} u^{\{1\}} \\ u^{\{2\}} \\ u^{\{3\}} \\ u^{\{4\}} \\ u^{\{5\}} \\ u^{\{6\}} \end{pmatrix} = \begin{pmatrix} \widetilde{g} \\ \widetilde{g} \\ \widetilde{g} \\ \widetilde{g} \\ \widetilde{g} \\ \widetilde{g} \end{pmatrix}. \tag{24}$$

This system nK linear equations can be solved (blockwise) by forward substitution. Furthermore, the subdomain components of the n-dimensional systems of equations within each block can be solved in parallel.

5.2. Adjoint equations

The adjoint system of equations of size nK to (24) is

$$U^\top x = \begin{bmatrix} I & -\widetilde{D}^\top & 0 & 0 & 0 \\ 0 & I & -\widetilde{D}^\top & 0 & 0 \\ 0 & 0 & I & -\widetilde{D}^\top & 0 \\ 0 & 0 & 0 & I & -\widetilde{D}^\top \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{pmatrix} \phi^{\{1\}} \\ \phi^{\{2\}} \\ \phi^{\{3\}} \\ \phi^{\{4\}} \\ \phi^{\{5\}} \\ \phi^{\{6\}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \psi \end{pmatrix},$$

where $\psi \in \mathbb{R}^n$. This system nK linear equations can be solved (blockwise) by backward substitution. Furthermore, the n-dimensional systems of equations within each block can be solved in parallel. Finally, note that

$$\begin{aligned}
\phi^{\{K\}} &= \psi \\
\phi^{\{K-1\}} &= \widetilde{D}^\top \phi^{\{K\}} = \widetilde{D}^\top \psi \\
&\vdots \\
\phi^{\{1\}} &= \underbrace{\widetilde{D}^\top \dots \widetilde{D}^\top}_{k \text{ times}} \psi.
\end{aligned}$$

5.3. Discretization error representation

The error in the QoI defined by (3) is given by the inner product

$$\text{Discretization error} = (\psi, u^{\{K\}} - \hat{u}^{\{K\}}) = (R, \Phi),$$

where R is nK-dimensional vector of residuals and

$$\Phi = \begin{pmatrix} \phi^{\{1\}} \\ \phi^{\{2\}} \\ \vdots \\ \phi^{\{K\}} \end{pmatrix}.$$

5.4. Total error representation

The total error and iterative error are computed as in §4.4.

5.5. Results

Input parameters are $K, \alpha, restricted(0/1)$ and $iprint$.

```
>> Additive_Schwarz(2,0.5,1,0)
```

norm(exact - iterative) solutions = 2.187820e-02

Performing C iteration with error	
QoI_discretization_error	-1.399007e-02
QoI_discretization_error_estimate	-1.399007e-02
QoI_total_error	-2.064950e-02
QoI_total_error_estimate	-2.064950e-02
QoI_iteration_error	-6.659424e-03
Solving U system with error	
QoI_discretization_error_estimate	-5.580385e-04
QoI_discretization_error	-5.580385e-04
QoI_total_error	-7.217463e-03
QoI_total_error_estimate	-7.217463e-03
QoI_iteration_error	-6.659424e-03

Table 3: Restricted Additive Schwarz: Two DD iterations

```
>> Additive_Schwarz(8,0.5,1,0)
```

norm(exact - iterative) solutions = 4.910251e-06

Performing C iteration with error	
QoI_discretization_error	7.436259e-02
QoI_discretization_error_estimate	7.436259e-02
QoI_total_error	7.435635e-02
QoI_total_error_estimate	7.435635e-02
QoI_iteration_error	-6.247458e-06
Solving U system with error	
QoI_discretization_error_estimate	-2.564992e-03
QoI_discretization_error	-2.564992e-03
QoI_total_error	-2.571240e-03
QoI_total_error_estimate	-2.571240e-03
QoI_iteration_error	-6.247458e-06

Table 4: Restricted Additive Schwarz: Eight DD iterations

6. Multiplicative Schwarz

The notation for multiplicative Schwarz becomes more complicated if every global solution is stored. There are p such global solutions per "sweep" and pK such solutions for K sweeps of multiplicative Schwarz.

$$
\begin{aligned}
&u \in \mathbb{R}^n, v \in \mathbb{R}^{np}, w \in \mathbb{R}^{npK},\\
&f \in \mathbb{R}^n, g \in \mathbb{R}^{np}, h \in \mathbb{R}^{npK},\\
&\phi \in \mathbb{R}^n, \Phi \in \mathbb{R}^{np}, \Theta \in \mathbb{R}^{npK},\\
&\psi \in \mathbb{R}^n, \Psi \in \mathbb{R}^{np}, \Xi \in \mathbb{R}^{npK}.
\end{aligned}
$$

6.1. *Forward problem*

We define

$$
\begin{aligned}
A_i &= R_i A R_i^\top, \in \mathbb{R}^{m_i \times m_i},\ i = 1, 2, \ldots, p,\\
B_i &= R_i^\top A_i^{-1} R_i = R_i^\top \left(R_i A R_i^\top\right)^{-1} R_i \in \mathbb{R}^{n\times n},\ i = 1, 2, \ldots, p,\\
C_i &= B_i A \in \mathbb{R}^{n\times n},\ i = 1, 2, \ldots, p,\\
D_i &= (I - C_i) \in \mathbb{R}^{n\times n},\ i = 1, 2, \ldots, p,\\
f_i &= B_i f \in \mathbb{R}^n,\ i = 1, 2, \ldots, p.
\end{aligned}
\tag{25}
$$

Single iteration: The multiplicative Schwarz algorithm is: Given $u^{\{0\}}$, for $k = 0, 1, \ldots$

$$
\begin{aligned}
u^{\{k+i/p\}} &= u^{\{k+(i-1)/p\}} + B_i\left(f - A u^{\{k+(i-1)/p\}}\right),\ i = 1, 2, \ldots, p,\\
&= (I - B_i A) u^{\{k+(i-1)/p\}} + B_i f\\
&= (I - C_i) u^{\{k+(i-1)/p\}} + B_i f\\
&= D_i u^{\{k+(i-1)/p\}} + f_i
\end{aligned}
\tag{26}
$$

Assume $p = 5$ and let

$$
\begin{aligned}
S &= \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ -D_2 & I & 0 & 0 & 0 \\ 0 & -D_3 & I & 0 & 0 \\ 0 & 0 & -D_4 & I & 0 \\ 0 & 0 & 0 & -D_5 & I \end{bmatrix} \in \mathbb{R}^{np \times np}, \\
T &= \begin{bmatrix} 0 & 0 & 0 & 0 & -D_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{np \times np}, \\
g &= \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} \in \mathbb{R}^{np}, \\
v^{\{k\}} &= \begin{pmatrix} u^{\{(k-1)+1/5\}} \\ u^{\{(k-1)+2/5\}} \\ u^{\{(k-1)+3/5\}} \\ u^{\{(k-1)+4/5\}} \\ u^{\{k\}} \end{pmatrix} \in \mathbb{R}^{np},
\end{aligned} \tag{27}
$$

then, defining

$$
v^{\{0\}} = \begin{pmatrix} 0 \\ \vdots 0 \\ u^{\{0\}} \end{pmatrix},
$$

a single "sweep" of multiplicative Schwarz can be written as

$$
Sv^{\{k\}} + Tv^{\{k-1\}} = g, \quad k = 1, 2, \ldots. \tag{28}
$$

K iterations: $K = 6$ iterations of multiplicative Schwarz can be written as the npK dimensional system of equations

$$
\begin{bmatrix} S & 0 & 0 & 0 & 0 & 0 \\ T & S & 0 & 0 & 0 & 0 \\ 0 & T & S & 0 & 0 & 0 \\ 0 & 0 & T & S & 0 & 0 \\ 0 & 0 & 0 & T & S & 0 \\ 0 & 0 & 0 & 0 & T & S \end{bmatrix} \begin{pmatrix} v^{\{1\}} \\ v^{\{2\}} \\ v^{\{3\}} \\ v^{\{4\}} \\ v^{\{5\}} \\ v^{\{6\}} \end{pmatrix} = \begin{pmatrix} g \\ g \\ g \\ g \\ g \\ g \end{pmatrix}. \tag{29}
$$

This system npK linear equations can be solved (blockwise) by forward substitution. Furthermore, the np-dimensional systems of equations within each block can be solved by forward substitution. Note that the solution of each block changes only values in subdomain i.

Now write (29) as the npK-dimensional system

$$
Uw = h, \tag{30}
$$

where

$$U = \begin{bmatrix} S & 0 & 0 & 0 & 0 & 0 \\ T & S & 0 & 0 & 0 & 0 \\ 0 & T & S & 0 & 0 & 0 \\ 0 & 0 & T & S & 0 & 0 \\ 0 & 0 & 0 & T & S & 0 \\ 0 & 0 & 0 & 0 & T & S \end{bmatrix} \in \mathbb{R}^{npK \times npK},$$

$$w = \begin{pmatrix} v^{\{1\}} \\ v^{\{2\}} \\ v^{\{3\}} \\ v^{\{4\}} \\ v^{\{5\}} \\ v^{\{6\}} \end{pmatrix} \in \mathbb{R}^{npK}, \tag{31}$$

$$h = \begin{pmatrix} g \\ g \\ g \\ g \\ g \\ g \end{pmatrix} \in \mathbb{R}^{npK}.$$

6.2. Adjoint equations

The adjoint system of equations to (29) can be constructed in a straightforward manner by defining

$$S^\top = \begin{bmatrix} I & -D_2^\top & 0 & 0 & 0 \\ 0 & I & -D_3^\top & 0 & 0 \\ 0 & 0 & I & -D_4^\top & 0 \\ 0 & 0 & 0 & I & -D_5^\top \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

$$\text{and} \quad T^\top = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -D_1^\top & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The adjoint system of size npK is

$$\begin{bmatrix} S^\top & T^\top & 0 & 0 & 0 & 0 \\ 0 & S^\top & T^\top & 0 & 0 & 0 \\ 0 & 0 & S^\top & T^\top & 0 & 0 \\ 0 & 0 & 0 & S^\top & T^\top & 0 \\ 0 & 0 & 0 & 0 & S^\top & T^\top \\ 0 & 0 & 0 & 0 & 0 & S^\top \end{bmatrix} \begin{pmatrix} \Phi^{\{1\}} \\ \Phi^{\{2\}} \\ \Phi^{\{3\}} \\ \Phi^{\{4\}} \\ \Phi^{\{5\}} \\ \Phi^{\{6\}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \Psi \end{pmatrix},$$

where $\Phi^{\{k\}} \in \mathbb{R}^{np}, k = 1, 2, \cdots, K$. Here

$$\Phi^{\{k\}} = \begin{pmatrix} \phi^{\{(k-1)+1/5\}} \\ \phi^{\{(k-1)+2/5\}} \\ \phi^{\{(k-1)+3/5\}} \\ \phi^{\{(k-1)+4/5\}} \\ \phi^{\{k\}} \end{pmatrix},$$

where $\phi^{\{(k-1)+i/p\}} \in \mathbb{R}^n$, and

$$\Psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \psi \end{pmatrix},$$

$\Psi \in \mathbb{R}^{np}$, where $\psi \in \mathbb{R}^n$.

This system npK linear equations can be solved (blockwise) by backward substitution. Furthermore, the np-dimensional systems of equations within each block can be solved by backward substitution. This is particularly simple due to the identity matrices on the diagonal of S and therefore of $S^\top$. Solving the adjoint system,

$$\begin{aligned} \phi^{\{K\}} &= \psi \\ \phi^{\{K-1+(p-1)/p\}} &= D_p^\top \psi \\ \phi^{\{K-1+(p-2)/p\}} &= D_{p-1}^\top D_p^\top \psi \\ &\vdots \\ \phi^{\{K-1+1/p\}} &= D_2^\top \cdots D_{p-1}^\top D p^\top \psi = D^\top \psi \\ \phi^{\{K-1\}} &= D_1^\top D_2^\top \cdots D_{p-1}^\top D_p^\top \psi = D_1^\top E^\top \psi \\ &\vdots \\ \phi^{\{1\}} &= E^\top \underbrace{\left(D_1^\top E^\top\right)\left(D_1^\top E^\top\right) \cdots \left(D_1^\top E^\top\right)}_{k-1 \text{ times}} \psi \end{aligned} \tag{32}$$

where we have defined $E^\top = D_2^\top \cdots D_{p-1}^\top D_p^\top$.

We write this compactly as

$$U^\top \Theta = \Xi, \tag{33}$$

where

$$\Theta = \begin{pmatrix} \Phi^{\{1\}} \\ \Phi^{\{2\}} \\ \Phi^{\{3\}} \\ \Phi^{\{4\}} \\ \Phi^{\{5\}} \\ \Phi^{\{6\}} \end{pmatrix} \in \mathbb{R}^{npK},$$

$$\Xi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \Psi \end{pmatrix} \in \mathbb{R}^{npK}. \tag{34}$$

6.3. Discretization error representation

The error in the QoI defined by (3) is given by the inner product

$$\text{Discretization error} = (u^{\{K\}} - \hat{u}^{\{K\}}, \psi) = (R, \Theta),$$

where R is npK-dimensional vector of residuals and

$$\Theta = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_K \end{pmatrix}.$$

6.4. Total error representation

The total error and iterative error are computed as in §4.4.

6.5. Results

Input parameters are K and *iprint*.

```
>> Multiplicative_Schwarz(2,0)
```

norm(exact - iterative) solutions = 3.986929e-03

Performing D iteration with error	
QoI_discretization_error	1.468705e-02
QoI_discretization_error_estimate	1.468705e-02
QoI_total_error	7.373839e-03
QoI_total_error_estimate	7.373839e-03
QoI_iteration_error	-7.313208e-03
Solving U system with error	
QoI_discretization_error	-5.733520e-04
QoI_discretization_error_estimate	-5.733520e-04
QoI_total_error	-7.886560e-03
QoI_total_error_estimate	-7.886560e-03
QoI_iteration_error	-7.313208e-03

Table 5: Multiplicative Schwarz: Two DD iterations

```
>> Multiplicative_Schwarz(8,0)
```

norm(exact - iterative) solutions = 5.517062e-10

Performing D iteration with error	
QoI_discretization_error	4.576417e-02
QoI_discretization_error_estimate	4.576417e-02
QoI_total_error	4.576417e-02
QoI_total_error_estimate	4.576417e-02
QoI_iteration_error	-6.351389e-10
Solving U system with error	
QoI_discretization_error	1.308595e-03
QoI_discretization_error_estimate	1.308595e-03
QoI_total_error	1.308595e-03
QoI_total_error_estimate	1.308595e-03
QoI_iteration_error	-6.351389e-10

Table 6: Multiplicative Schwarz: Eight DD iterations

7. One-way coupled systems solved using DD

NOTE: We will use $n_i,\ i = 1, 2, \ldots$ to denote the sizes of blocks of coupled systems of equations, and $m_i,\ i = 1, 2, \ldots, p$ to denote the sizes of subdomains within blocks.

7.1. The basic problem

We seek to solve the coupled linear system

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{35}$$

where $A_{11} \in \mathbb{R}^{n_1 \times n_1},\ A_{21} \in \mathbb{R}^{n_2 \times n_1},\ A_{22} \in \mathbb{R}^{n_2 \times n_2}$ and estimate the error in the quantity of interest

$$\mathrm{QoI} = (\psi_1, u_1) + (\psi_2, u_2)\,. \tag{36}$$

We solve by forward substitution.

$$\begin{aligned} A_{11}u_1 &= f_1 \\ A_{22}u_2 &= f_2 - A_{21}u_1, \end{aligned} \tag{37}$$

i.e.,

1. Solve $A_{11}u_1 = f_1$,
2. Solve $A_{22}u_2 = f_2 - A_{21}u_1$.

Let $\hat{u}_1$ and $\hat{u}_2$ approximate u_1 and u_2 respectively.

7.2. Adjoint system

The adjoint system is

$$\begin{bmatrix} A_{11}^\top & A_{21}^\top \\ 0 & A_{22}^\top \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

which we solve by backward substitution, i.e.,

1. Solve $A_{22}^\top \phi_2 = \psi_2$,
2. Solve $A_{11}^\top \phi_1 = \psi_1 - A_{21}^\top \phi_2$.

The error in the QoI is given by the inner product

$$(R_1, \phi_1) + (R_2, \phi_2),$$

where

$$R_1 = f_1 - A_{11}\hat{u}_1, \qquad R_2 = f_2 - A_{21}\hat{u}_1 - A_{22}\hat{u}_2$$

7.3. Solving component systems using iterative methods

Let both systems be solved by multiplicative Schwarz.

First system: We solve the first system of equation using K_1 iterations of multiplicative Schwarz with p_1 overlapping subdomains. The first system of equations

$$A_{11}u_1 = f_1,$$

becomes the $n_1p_1K_1$-dimensional system

$$U_{11}w_1 = h_1,$$

where U_{11}, h_1 and w_1 are defined analogously to U, h and w in equation (31) in §6.

Second system: The second equation is

$$A_{22}u_2 = f_2 - A_{21}u_1 = f_2 - A_{21}P_1w_1 = \widetilde{f}_2.$$

Here $P_1 \in \mathbb{R}^{n_1\times n_1p_1K_1}$ is the rank n_1 matrix that *selects the final solution u from the rows from w_1*.

Remark 2. *Depending upon how the solution is stored, construction of P_1 (and later P_2 in two-way coupling) is non-trivial. If the solutions on all p subdomains are stored, components of u are distributed among all p subdomains and must be selected appropriately, taking account of the overlaps and the sequence in which the subdomains are computed. Similarly, when computing the np-vector equivalent to an n-vector of adjoint data, the adjoint "data" must be distributed to the appropriate degrees of freedom which exist in each of the p subdomains. This is accomplished using the transpose $P_i^\top$, , $i = 1, 2$.* `This works, but is it necessary?`

Following the notation in §6, let $i = 1, \dots p_2$, let

$$B_{2,i}\widetilde{f}_2 = B_{2,i}(f_2 - A_{21}P_1w_1) = f_{2,i} - \widetilde{A}_{21,i}w_1$$

where $\widetilde{A}_{21,i} = B_{2,i}A_{21}P_1 \in \mathbb{R}^{n_2\times n_1p_1K_1}$. For $i = 1, \dots, p_2$, we have

$$\begin{aligned}
u_2^{\{k+i/p_2\}} &= u_2^{\{k+(i-1)/p_2\}} + B_{2,i}\left(\widetilde{f}_2 - A_{22}u_2^{\{k+(i-1)/p_2\}}\right) \\
&= (I - B_{2,i}A_{22})\,u_2^{\{k+(i-1)/p_2\}} + f_{2,i} - \widetilde{A}_{21,i}w_1 \\
&= (I - C_{2,i})\,u_2^{\{k+(i-1)/p_2\}} + f_{2,i} - \widetilde{A}_{21,i}w_1 \\
&= D_{2,i}\;u_2^{\{k+(i-1)/p_2\}} + f_{2,i} - \widetilde{A}_{21,i}w_1.
\end{aligned} \tag{38}$$

Each "sweep" of multiplicative Schwarz can be written as

$$S_2v_2^{\{k\}} + T_2v_2^{\{k-1\}} = g_2 - Q_{21}w_1$$

where $S_2, T_2 \in \mathbb{R}^{n_2p_2\times n_2p_2}$ and $g_2, v_2 \in \mathbb{R}^{n_2p_2}$ are defined analogously to S, T, g, v in equation (27) in §6 and

$$Q_{21} = \begin{pmatrix} \widetilde{A}_{21,1} \\ \widetilde{A}_{21,2} \\ \vdots \\ \widetilde{A}_{21,p_2} \end{pmatrix} \tag{39}$$

where $Q_{21} \in \mathbb{R}^{n_2p_2 \times n_1p_1K_1}$.

K_2 "sweeps" of multiplicative Schwarz may be written as

$$U_{22}w_2 = h_2 - U_{21}w_1.$$

where $U_{22} \in \mathbb{R}^{n_2p_2K_2 \times n_2p_2K_2}$ and $h_2 \in \mathbb{R}^{n_2p_2K_2}$ are defined analogously to U, h in §6, $w_i \in \mathbb{R}^{n_ip_iK_i}$, $i = 1, 2$, and

$$U_{21} = \begin{pmatrix} Q_{21} \\ Q_{21} \\ \vdots \\ Q_{21} \end{pmatrix}, \quad \in \mathbb{R}^{n_2p_2K_2 \times n_1p_1K_1}.$$

Forward and adjoint systems: The construction and solution of the adjoint system follows the prescription in §7.2, but with the one-way coupled system

$$\begin{bmatrix} U_{11} & 0 \\ U_{21} & U_{22} \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \tag{40}$$

where matrices

- $U_{11} \in \mathbb{R}^{n_1p_1K_1 \times n_1p_1K_1}$, $U_{22} \in \mathbb{R}^{n_2p_2K_2 \times n_2p_2K_2}$,
- $U_{21} \in \mathbb{R}^{n_2p_2K_2 \times n_1p_1K_1}$
- $h_1 \in \mathbb{R}^{n_1p_1K_1}$, $h_2 \in \mathbb{R}^{n_2p_2K_2}$

and solutions

- $w_1 \in \mathbb{R}^{n_1p_1K_1}$, $w_2 \in \mathbb{R}^{n_2p_2K_2}$.

The adjoint system is

$$\begin{bmatrix} U_{11}^\top & U_{21}^\top \\ 0 & U_{22}^\top \end{bmatrix} \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} = \begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix}, \tag{41}$$

where

$$\Xi_1 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \Psi_1 \end{pmatrix} \in \mathbb{R}^{n_1p_1K_1}, \qquad \Psi_1 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \psi_1 \end{pmatrix} \in \mathbb{R}^{n_1p_1}, \qquad \psi_1 \in \mathbb{R}^{n_1},$$

$$\Theta_1 = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_{K_1} \end{pmatrix} \in \mathbb{R}^{n_1p_1K_1},$$

and

$$\Xi_2 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \Psi_2 \end{pmatrix} \in \mathbb{R}^{n_2 p_2 K_2}, \qquad \Psi_2 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \psi_2 \end{pmatrix} \in \mathbb{R}^{n_2 p_2}, \qquad \psi_2 \in \mathbb{R}^{n_2},$$

$$\Theta_2 = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_{K_2} \end{pmatrix} \in \mathbb{R}^{n_2 p_2 K_2}.$$

7.4. Results

Input parameters are $K1, K2$ and $iprint$.

```
>> one_way_coupling(1,1,0)
```

norm(u-[u1;u2])	3.162493e-02
QoI_discretization_error	-5.879139e-04
QoI_discretization_error_estimate	-5.879139e-04
QoI_total_error	2.430081e-02
QoI_total_error_estimate	2.430081e-02
QoI_iteration_error	2.488872e-02

```
>> one_way_coupling(3,3,0)
```

norm(u-[u1;u2])	1.473453e-04
QoI_discretization_error	2.916209e-04
QoI_discretization_error_estimate	2.916209e-04
QoI_total_error	5.559130e-04
QoI_total_error_estimate	5.559130e-04
QoI_iteration_error	2.642921e-04

```
>> one_way_coupling(6,6,0)
```

norm(u-[u1;u2])	5.824097e-08
QoI_discretization_error	-5.094151e-03
QoI_discretization_error_estimate	-5.094151e-03
QoI_total_error	-5.094092e-03
QoI_total_error_estimate	-5.094092e-03
QoI_iteration_error	5.921197e-08

```
>> one_way_coupling(9,9,0)
```

norm(u-[u1;u2])	2.338602e-11
QoI_discretization_error	4.700422e-03
QoI_discretization_error_estimate	4.700422e-03
QoI_total_error	4.700422e-03
QoI_total_error_estimate	4.700422e-03
QoI_iteration_error	2.758764e-11

```
>> one_way_coupling(12,12,0)
```

norm(u-[u1;u2])	9.840486e-15
QoI_discretization_error	-2.842613e-03
QoI_discretization_error_estimate	-2.842613e-03
QoI_total_error	-2.842613e-03
QoI_total_error_estimate	-2.842613e-03
QoI_iteration_error	1.009609e-14

Remark 3. *Iteration error is smaller for six DD iterations compared with three as expected, but why is the total error larger for a larger number of DD iterations?*

8. Two way coupled algebraic systems

The notation for multiplicative Schwarz becomes more complicated if every global solution is stored. There are p such global solutions per "sweep".

$$
\begin{aligned}
&u \in \mathbb{R}^n, v \in \mathbb{R}^{np}, w \in \mathbb{R}^{npK}, z \in \mathbb{R}^{(n_1p_1K_1+n_2p_2K_2)L}\\
&f \in \mathbb{R}^n, g \in \mathbb{R}^{np}, h \in \mathbb{R}^{npK}, q \in \mathbb{R}^{(n_1p_1K_1+n_2p_2K_2)L}\\
&\phi \in \mathbb{R}^n, \Phi \in \mathbb{R}^{np}, \Theta \in \mathbb{R}^{npK}, \Upsilon \in \mathbb{R}^{(n_1p_1K_1+n_2p_2K_2)L}\\
&\psi \in \mathbb{R}^n, \Psi \in \mathbb{R}^{np}, \Xi \in \mathbb{R}^{npK}, \zeta \in \mathbb{R}^{(n_1p_1K_1+n_2p_2K_2)L}
\end{aligned}
$$

8.1. The basic problem

We seek to solve the 2×2 block linear system

$$
\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{42}
$$

where $A_{11} \in \mathbb{R}^{n_1 \times n_1}$, $A_{12} \in \mathbb{R}^{n_1 \times n_2}$, $A_{21} \in \mathbb{R}^{n_2 \times n_1}$, $A_{22} \in \mathbb{R}^{n_2 \times n_2}$ and estimate the error in the quantity of interest

$$
\mathrm{QoI} = (\psi_1, u_1) + (\psi_2, u_2)\,. \tag{43}
$$

We employ an "outer" Gauss-Seidel iteration with L iterations. Given $u_2^{\{0\}}$, for $l = 1, 2, \ldots, L$ we solve

$$
\left.\begin{aligned}
A_{11}u_1^{\{l\}} &= f_1 - A_{12}u_2^{\{l-1\}}\\
A_{22}u_2^{\{l\}} &= f_2 - A_{21}u_1^{\{l\}}
\end{aligned}\right\}. \tag{44}
$$

Assuming $L = 5$, then the Gauss-Seidel outer iteration can be written as a $(n_1+n_2)L$-dimensional system of equations

$$
M\boldsymbol{y} = \boldsymbol{z}, \tag{45}
$$

where

$$
M = \left[\begin{array}{cc:cc:cc:cc:cc}
A_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
A_{21} & A_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hdashline
0 & A_{12} & A_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & A_{21} & A_{22} & 0 & 0 & 0 & 0 & 0 & 0\\
\hdashline
0 & 0 & 0 & A_{12} & A_{11} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & A_{21} & A_{22} & 0 & 0 & 0 & 0\\
\hdashline
0 & 0 & 0 & 0 & 0 & A_{12} & A_{11} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & A_{21} & A_{22} & 0 & 0\\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{12} & A_{11} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{21} & A_{22}
\end{array}\right],
$$

$$
\boldsymbol{y} = \begin{pmatrix} u_1^{\{1\}} \\ u_2^{\{1\}} \\ \hdashline u_1^{\{2\}} \\ u_2^{\{2\}} \\ \hdashline u_1^{\{3\}} \\ u_2^{\{3\}} \\ \hdashline u_1^{\{4\}} \\ u_2^{\{4\}} \\ \hdashline u_1^{\{5\}} \\ u_2^{\{5\}} \end{pmatrix}, \qquad \text{and} \qquad \boldsymbol{z} = \begin{pmatrix} f_1 \\ f_2 \\ \hdashline f_1 \\ f_2 \\ \hdashline f_1 \\ f_2 \\ \hdashline f_1 \\ f_2 \\ \hdashline f_1 \\ f_2 \end{pmatrix},
$$

and solve by forward substitution.

8.2. The adjoint problem

The adjoint system of equation is

$$
M^\top \boldsymbol{\phi} = \boldsymbol{\psi}, \tag{46}
$$

where

$$
M^\top = \left[\begin{array}{cc:cc:cc:cc:cc}
A_{11}^\top & A_{21}^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_{22}^\top & A_{12}^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline
0 & 0 & A_{11}^\top & A_{21}^\top & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_{22}^\top & A_{12}^\top & 0 & 0 & 0 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0 & A_{11}^\top & A_{21}^\top & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_{22}^\top & A_{12}^\top & 0 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 0 & 0 & 0 & A_{11}^\top & A_{21}^\top & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^\top & A_{12}^\top & 0 \\ \hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{11}^\top & A_{21}^\top \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22}^\top
\end{array}\right],
$$

$$
\boldsymbol{\phi} = \begin{pmatrix} \phi_1^{\{1\}} \\ \phi_2^{\{1\}} \\ \hdashline \phi_1^{\{2\}} \\ \phi_2^{\{2\}} \\ \hdashline \phi_1^{\{3\}} \\ \phi_2^{\{3\}} \\ \hdashline \phi_1^{\{4\}} \\ \phi_2^{\{4\}} \\ \hdashline \phi_1^{\{5\}} \\ \phi_2^{\{5\}} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\psi} = \begin{pmatrix} 0 \\ 0 \\ \hdashline 0 \\ 0 \\ \hdashline 0 \\ 0 \\ \hdashline 0 \\ 0 \\ \hdashline P_1^\top \psi_1 \\ P_2^\top \psi_2 \end{pmatrix},
$$

which we solve using backward substitution.

The error in the QoI is given by the inner product

$$
\sum_{l=1}^{L} \left[\left(R_1^{\{l\}}, \phi_1^{\{l\}} \right) + \left(R_2^{\{l\}}, \phi_2^{\{l\}} \right) \right].
$$

8.3. Results for coupled algebraic systems

One-way coupled

L	verr	Disc Err Est	Eff	Total Err Est	Eff	Iter Err
1	1.00e-03	-1.1990e-03	1.00	-1.1990e-03	1.00	5.8113e-17
1	1.00e-05	-1.1990e-05	1.00	-1.1990e-05	1.00	5.7680e-17

Table 7: One way coupling (algebraic)

Two-way coupled

L	verr	Disc Err Est	Eff	Total Err Est	Eff	Iter Err
1	1.00e-03	-1.1990e-03	1.00	-1.1999e-02	1.00	-1.0800e-02
3	1.00e-03	-6.2295e-04	1.00	-6.8218e-04	1.00	-5.9235e-05
6	1.00e-03	-5.7339e-03	1.00	-5.7340e-03	1.00	-2.7691e-08
9	1.00e-03	-7.2662e-03	1.00	-7.2662e-03	1.00	3.0424e-12
12	1.00e-03	-7.8301e-03	1.00	-7.8301e-03	1.00	-1.1779e-15
1	1.00e-05	-1.1990e-05	1.00	-1.0812e-02	1.00	-1.0800e-02
3	1.00e-05	-6.2295e-06	1.00	-6.5464e-05	1.00	-5.9235e-05
6	1.00e-05	-5.7339e-05	1.00	-5.7367e-05	1.00	-2.7691e-08
9	1.00e-05	-7.2662e-05	1.00	-7.2662e-05	1.00	3.0424e-12
12	1.00e-05	-7.8301e-05	1.00	-7.8301e-05	1.00	-1.1779e-15

Table 8: Two way coupling (algebraic)

8.4. Two-way DD systems

We solve both systems of equations of (44) using multiplicative Schwarz.

First system of equations: Following the description in §7, the first system of equations

$$A_{11}u_1 = f_1 - A_{12}u_2,$$

becomes the $n_1p_1K_1$-dimensional system

$$U_{11}w_1 = h_1 - U_{12}w_2$$

where $u_2 = P_2w_2$. In particular

$$\widetilde{A}_{12,i} = B_iA_{12}P_2 \in \mathbb{R}^{n_1\times n_2p_2K_2},$$
$$Q_{12} = \begin{pmatrix} \widetilde{A}_{12,1} \\ \widetilde{A}_{12,2} \\ \vdots \\ \widetilde{A}_{12,p_1} \end{pmatrix}, \quad \in \mathbb{R}^{n_1p_1\times n_2p_2K_2},$$
$$U_{12} = \begin{pmatrix} Q_{12} \\ Q_{12} \\ \vdots \\ Q_{12} \end{pmatrix}, \quad \mathbb{R}^{n_1p_1K_1\times n_2p_2K_2}.$$

Second system of equations: Following the description in §7, the second system of equations

$$A_{22}u_2 = f_2 - A_{21}u_1,$$

becomes the $n_2p_2K_2$-dimensional system

$$U_{22}w_2 = h_2 - U_{21}w_1,$$

where

In particular

$$\widetilde{A}_{21,i} = B_iA_{21}P_1 \in \mathbb{R}^{n_2\times n_1p_1K_1},$$
$$Q_{21} = \begin{pmatrix} \widetilde{A}_{21,1} \\ \widetilde{A}_{21,2} \\ \vdots \\ \widetilde{A}_{21,p_2} \end{pmatrix}, \quad \in \mathbb{R}^{n_2p_2\times n_1p_1K_1},$$
$$U_{21} = \begin{pmatrix} Q_{21} \\ Q_{21} \\ \vdots \\ Q_{21} \end{pmatrix}, \quad \mathbb{R}^{n_2p_2K_2\times n_1p_1K_1}.$$

Gauss-Seidel iteration: The forward system is

$$V\boldsymbol{z} = \boldsymbol{q}, \tag{47}$$

where

$$V = \left[\begin{array}{cc:cc:cc:cc:cc} U_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ U_{21} & U_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & U_{12} & U_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & U_{21} & U_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & U_{12} & U_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & U_{21} & U_{22} & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & U_{12} & U_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U_{21} & U_{22} & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & U_{12} & U_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U_{21} & U_{22} \end{array}\right],$$

$$\boldsymbol{z} = \left(\begin{array}{c} w_1^{\{1\}} \\ w_2^{\{1\}} \\ \hdashline w_1^{\{2\}} \\ w_2^{\{2\}} \\ \hdashline w_1^{\{3\}} \\ w_2^{\{3\}} \\ \hdashline w_1^{\{4\}} \\ w_2^{\{4\}} \\ \hdashline w_1^{\{5\}} \\ w_2^{\{5\}} \end{array}\right), \qquad \text{and} \qquad \boldsymbol{q} = \left(\begin{array}{c} h_1 \\ h_2 \\ \hdashline h_1 \\ h_2 \\ \hdashline h_1 \\ h_2 \\ \hdashline h_1 \\ h_2 \\ \hdashline h_1 \\ h_2 \end{array}\right),$$

where $V \in \mathbb{R}^{(n_1p_1K_1+n_2p_2K_2)L\times(n_1p_1K_1+n_2p_2K_2)L}$. We solve via forward substitution or given $w_2^{\{0\}}$, for $l = 1, 2, \ldots, L$ solve

$$\left.\begin{aligned} U_{11}w_1^{\{l\}} &= h_1 - U_{12}w_2^{\{l-1\}} \\ U_{22}w_2^{\{l\}} &= h_2 - U_{21}w_1^{\{l\}} \end{aligned}\right\}. \tag{48}$$

<u>The adjoint system:</u> The adjoint system of equations is

$$V^\top \Xi = \zeta, \tag{49}$$

where

$$V^\top = \left[\begin{array}{cc:cc:cc:cc:cc} U_{11}^\top & U_{21}^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & U_{22}^\top & U_{12}^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & U_{11}^\top & U_{21}^\top & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & U_{22}^\top & U_{12}^\top & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & U_{11}^\top & U_{21}^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & U_{22}^\top & U_{12}^\top & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & U_{11}^\top & U_{21}^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & U_{22}^\top & U_{12}^\top & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U_{11}^\top & U_{21}^\top \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U_{22}^\top \end{array}\right],$$

$$
\begin{tikzcd}
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \arrow[r, "GS"] \arrow[d, "DD"] & \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & A_{12} & A_{11} & 0 \\ 0 & 0 & A_{21} & A_{22} \end{bmatrix} \arrow[d, "DD"] \\
U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \arrow[r, "GS"] & \begin{bmatrix} U_{11} & 0 & 0 & 0 \\ U_{21} & U_{22} & 0 & 0 \\ 0 & U_{12} & U_{11} & 0 \\ 0 & 0 & U_{21} & U_{22} \end{bmatrix}
\end{tikzcd}
$$

Figure 1: Commuting diagram

$$
\Upsilon = \begin{pmatrix} \Theta_1^{\{1\}} \\ \Theta_2^{\{1\}} \\ \hdashline \Theta_1^{\{2\}} \\ \Theta_2^{\{2\}} \\ \hdashline \Theta_1^{\{3\}} \\ \Theta_2^{\{3\}} \\ \hdashline \Theta_1^{\{4\}} \\ \Theta_2^{\{4\}} \\ \hdashline \Theta_1^{\{5\}} \\ \Theta_2^{\{5\}} \end{pmatrix} \qquad \text{and} \qquad \zeta = \begin{pmatrix} \Xi_1^{\{1\}} \\ \Xi_2^{\{1\}} \\ \hdashline \Xi_1^{\{2\}} \\ \Xi_2^{\{2\}} \\ \hdashline \Xi_1^{\{3\}} \\ \Xi_2^{\{3\}} \\ \hdashline \Xi_1^{\{4\}} \\ \Xi_2^{\{4\}} \\ \hdashline \Xi_1^{\{5\}} \\ \Xi_2^{\{5\}} \end{pmatrix}.
$$

We solve via back substitution, or given $\phi_1^{\{L+1\}} = 0$, for $l = 0, 1, 2, \ldots, L-1$ solve

$$
\left.\begin{aligned} U_{22}^\top \Theta_2^{\{L-l\}} &= \Xi_2^{\{L-l\}} - U_{12}^\top \Theta_1^{\{L-l+1\}} \\ U_{11}^\top \Theta_1^{\{L-l\}} &= \Xi_1^{\{L-l\}} - U_{21}^\top \Theta_2^{\{L-l\}} \end{aligned} \quad \right\}. \tag{50}
$$

<u>The challenge:</u> The challenge is how to distinguish the iterative errors arising due to the "inner" domain decomposition iterations and the iteration error arising due to the "outer" Gauss-Seidel iteration.

In order to separately identify the two different sources of iterative error we note that

$$
u - U_{MS,GS} = (u - u_{MS}) + (u_{MS} - u_{MS,GS}) + (u_{MS,GS} - U_{MS,GS}), \tag{51}
$$

or equivalently,

$$
u - U_{MS,GS} = (u - u_{GS}) + (u_{GS} - u_{MS,GS}) + (u_{MS,GS} - U_{MS,GS}). \tag{52}
$$

All effectivity ratios when computing the discretization and total errors are 1.00

One-way coupled: solving individual components with DD

Two-way coupled: solving individual components with DD

Here $n_1 = n_2 = 10$, $p_1 = p_2 = 4$, u implies a quantity computed using exact arithmetic and U a quantity computed with random error of magnitude $verr$. The errors are the errors in the quantity of interest given by $\psi = (1, 1, \ldots, 1)^\top$ where $\psi \in \mathbb{R}^{n_1+n_2}$.

K1	K2	L	$u - u_{MS}$	Disc$(u_{MS} - U_{MS,GS})$	Iter$(u_{MS} - U_{MS,GS})$	$u - U_{MS,GS}$
2	2	2	3.408833e-05	-6.924492e-02	6.066375e-03	-6.314445e-02
2	2	6	3.408833e-05	-1.796226e-02	1.511355e-07	-1.792803e-02
6	6	2	-2.144859e-08	-9.797164e-02	6.057039e-03	-9.191462e-02
6	6	6	-2.144859e-08	-2.089675e-04	1.646794e-07	-2.088243e-04

Table 9: Two-way DD: $verr = 0.01$

K1	K2	L	$u - u_{MS}$	Disc$(u_{MS} - U_{MS,GS})$	Iter$(u_{MS} - U_{MS,GS})$	$u - U_{MS,GS}$
2	2	2	3.408833e-05	-6.924492e-01	6.066375e-03	-6.863487e-01
2	2	6	3.408833e-05	-1.796226e-01	1.511355e-07	-1.795884e-01
6	6	2	-2.144859e-08	-9.797164e-01	6.057039e-03	-9.736594e-01
6	6	6	-2.144859e-08	-2.089675e-03	1.646794e-07	-2.089532e-03

Table 10: Two-way DD: $verr = 0.1$

K1	K2	L	$u - u_{MS}$	Disc$(u_{MS} - U_{MS,GS})$	Iter$(u_{MS} - U_{MS,GS})$	$u - U_{MS,GS}$
2	2	2	3.408833e-05	-6.924492e-05	6.066375e-03	6.031219e-03
2	2	6	3.408833e-05	-1.796226e-05	1.511355e-07	1.627720e-05
6	6	2	-2.144859e-08	-9.797164e-05	6.057039e-03	5.959046e-03
6	6	6	-2.144859e-08	-2.089675e-07	1.646794e-07	-6.573669e-08

Table 11: Two-way DD: $verr = 0.0001$

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