The Reducibility of Maps.

By George D. Birkhoff.

Introduction.

It is known that any map on a surface of genus zero, such as the sphere, can be so colored in at most five colors that no two regions with a common boundary line are of the same color; but it is not known whether or not only four colors always suffice.* In the present paper we restrict ourselves to maps on such a surface and to the use of four colors, a, b, c, d.

Substantially all that has been done toward the solution of this four-color problem is contained in the following four reductions:

If more than three boundary lines meet at any vertex of a map, the coloring of the map may be reduced to the coloring of a map of fewer regions.† For, not every pair of regions meeting at such a vertex have a boundary line in common, as is the case when three boundary lines meet at a vertex. Thus we may join two of these regions without a common boundary line, by opening the vertex, so as to form a map in one less region. When this simple map is colored, the original map is obtained, properly colored, by merely restoring the vertex to its original form.

If any region of a map is multiply-connected, the coloring of the map may be reduced to the coloring of maps of fewer regions. For we may color the partial maps which arise when all but one of the parts into which such a multiply-connected region separates the surface are erased, and afterward give the same color to this region in all of the partial maps by a permutation of the colors. By effecting a superposition of the partial maps, we obtain a coloring of the original map in the same colors employed in the partial maps.

If two or three regions of a map form a multiply-connected region, the coloring of the map may be reduced to the coloring of maps of fewer regions.

* A set of references may be found in the *Encyklopädie der mathematischen Wissenschaften*, III A, B 3, pp. 177–178. The question admits of transformation to several remarkable forms; in particular see the recent paper by O. Veblen, *Annals of Mathematics*, Ser. 2, Vol. XIV (1912), pp. 86–94.

† Throughout this article we use the term *boundary line* or *side* to mean one of the continuous lines in common to two distinct regions, and the term *vertex* to mean a point in common to three or more distinct regions. A region may have only a single boundary line or side.

In fact, these regions separate the surface into two or more parts. If we color the partial maps which arise when all but one of these parts are contracted to a point, and by a permutation then make the colors of the two or three regions the same on each of the partial maps, the original map may be colored by a superposition of the partial maps.

If the map contains any 1-, 2-, 3- *or* 4-*sided region, the coloring of the map may be reduced to the coloring of a map of fewer regions.* If a 1-, 2- or 3-sided region be present, we may shrink this region to a point, color the resulting map in one less region, and then introduce this region again in a color different from the colors of the regions to which it is adjacent along a boundary line. If a 4-sided region be present, at least one of the two pairs of regions which abut on opposite sides are distinct and have no boundary line in common. Let two opposite regions of this kind coalesce with the 4-sided region. If the resulting map in two less regions is colored, the original map may be colored by inserting the 4-sided region in a color different from that of the two or three colors of the regions adjacent to it along a boundary line.

By a succession of reductions of the above types it is clear that we are brought either to a map of one region consisting of the entire surface, which can be colored in one color, or to *regular maps* in which (1) all vertices are triple (*i. e.*, belong to three regions), (2) every region is simply-connected, (3) no two or three regions form a multiply-connected region, (4) every region has at least five sides.*

The purpose of the present paper is to show that there exist a number of further reductions which may be effected with the aid of the notion of *chains* due to A. B. Kempe.†

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* In the case where five colors are given, an additional reduction of 5-sided regions is possible. Namely, let coalesce with any 5-sided region two distinct regions having a boundary line in common with this region but themselves without a common boundary line, and color the resulting map which has two less regions. Then introduce the 5-sided region in a color different from the four or fewer colors of the regions which are adjacent to it along a boundary line.

This reduction shows that the coloring in five colors can always be effected since there can not exist a regular map in which there are no 5-sided regions. For, let f_5, f_6, ... denote the number of 5-sided, 6-sided, ... regions respectively in a regular map. By Euler's theorem on polyhedra we have for this map

$$\text{number of regions} + \text{number of vertices} = \text{number of sides} + 2,$$

or

$$f_5 + f_6 + \ldots\ldots + \tfrac{1}{3}\,[5f_5 + 6f_6 + \ldots\ldots] = \tfrac{1}{2}\,(5f_5 + 6f_6 + \ldots\ldots) + 2,$$

whence

$$f_5 = f_7 + 2f_8 + \ldots\ldots + 12,$$

so that there are at least twelve 5-sided regions in any regular map.

† Kempe employed this notion in attempting to prove that every map on the sphere can be colored in only four colors. See AMERICAN JOURNAL OF MATHEMATICS, Vol. II (1879), pp. 193–200. The error was pointed out later by P. J. Heawood, *Quarterly Journal of Mathematics*, Vol. XXIV (1890), pp. 332–338.

§ 1. *The Kempe Chains and Reducibility.*

Suppose that a map M is colored in four colors, a, b, c, d, and select a pair of these as a, b. Consider any region colored in one of this pair of colors together with all of the regions in these two colors, adjacent to it, or connected with it by a set of regions in the two colors. Such a set of regions will be called an *a, b chain*. Obviously, the same a, b chain is defined by any region in the chain.

A fundamental property of the chain is that if the two colors on the regions of a single chain, or of any set of these chains in the same colors, be all interchanged, a new coloring of the map results, since no region in one of these two colors becomes adjacent to another of the same color by such a transposition of colors; the colors on any set of the chains in the complementary pair of colors may be interchanged at the same time.

Every region on the map lies on an a, b chain or c, d chain, an a, c chain or b, d chain, an a, d chain or b, c chain. By successive permutations of the colors on the chains a number of different colorings may be obtained from a single one. It is only when there is but one chain of each description that the new colorings obtained differ from the original one by a mere permutation of the colors throughout the map.

Consider now in a regular map a cyclical arrangement of n $(n>3)$ regions $\mu_1, \mu_2, \ldots, \mu_n$ such that each of these regions has a boundary line in common with the one preceding and following it in cyclical order, but with no other region of the set.

A *ring* R of regions of this kind divides the map into two sets of regions M_1 and M_2 which together with R make up all the regions of the map M.

The partial maps $M_1 + R$ and $M_2 + R$ are bordered by the ring R of n regions. If it is possible to color $M_1 + R$ and $M_2 + R$ so that the arrangement of colors on R is the same in both cases except for a mere permutation of the colors, it is clearly possible to take the coloring on R the same in both the partial maps and by a superposition color the map $M = M_1 + M_2 + R$.

Take now one of the partial maps $M_1 + R$, and consider any pair of its regions on the ring R, either region having one of the colors a, b, say. If these two regions are on the same a, b chain, this fact may be indicated by joining the two regions by an *a, b line*, marked a, b and lying within the chain. Let all such lines be drawn and marked with their pairs of colors, and at the same time let the colors on R be indicated: thus a *scheme* on the ring R is formed.

No two lines marked in complementary pairs of colors will cross since lines in complementary pairs of colors lie in different regions.

If a chain (found in M_1+R) joins a region α of R to a region β of R, and if also a chain in the same colors joins the region β to the region γ of R, then a chain in these colors joins α to γ. Consequently, if in any scheme a line marked in two of the colors joins α to β, and a line marked in the same colors joins β to γ, such a line will also join α to γ.

Two regions α and β of R in one or both of any pair of colors are either joined by a chain in these colors, or else a pair of regions γ and δ of R, such that $\alpha, \gamma, \beta, \delta$ occur in cyclical order on R, are joined by a chain in the complementary pair of colors. For, if the chain in the first pair of colors which contains α does not contain β, this chain abuts on a chain in the complementary pair of colors which joins a region γ to a region δ. Consequently, either a line marked with this pair of colors connects α and β, or a line in the complementary pair of colors connects a pair of regions γ and δ.

These three properties of schemes will serve to define them completely for our purposes.

As we have observed, it is possible to derive from the given coloring of M_1+R a second coloring by transposing the pairs of colors at pleasure on any two complementary sets of chains. Returning now to the corresponding scheme, we see that we effect a transposition of complementary pairs of colors as a, b and c, d on a set of regions of R and that we transpose together the colors of all regions of R connected by a, b and c, d lines respectively. The a, b and c, d lines of the new scheme will be the same as in the old, but the lines of the four other types may be altered.

We are thus led to the following formal principle: *A given scheme on R gives rise to a new scheme in which two complementary sets of lines are unaltered and the corresponding colors are permuted on R in any way so that all those connected by these lines are transposed together.*

Suppose that in M we replace M_2+R by a set of regions $\mathfrak{M}_2$ of not more than a certain number k of regions but with the same n peripheral boundary lines as R; in this manner we form a map $M_1+\mathfrak{M}_2$ which we assume to be colored. For each of the finite number of essentially different choices of $\mathfrak{M}_2$ we get a certain number of choices of colors on the periphery of R, which are possible for M_1, and thence, by permutation, a certain number of sets I of schemes for M_1+R with these same colorings on R.

Likewise, by forming a map $\mathfrak{M}_1$ and then $M_2+\mathfrak{M}_1$ we get a certain number of sets II of schemes for M_2+R.

These two sets are closed under the application of the above formal principle. If it can be shown that for some k any two sets of schemes I and II have at least one coloring in common, we will say that the ring of n regions is *reducible*; and if k is the least integer for which this is true, we will say that $\phi(n) = k - n$ is the *reducing number* of R.

It is clear that if in the map M there exists a reducible ring of n regions having more than $\phi(n)$ regions on both sides of it, then, by coloring the maps $M_1 + \mathfrak{M}_2$ and $M_2 + \mathfrak{M}_1$ of fewer regions than M in all possible ways, one will always be led to two such sets of schemes, at least unless one of these maps can not be colored. By definition, one coloring at least in each of the two sets will be essentially the same. Hence, by a superposition one may color M, provided all of the auxiliary maps employed in this process can be colored. Since these auxiliary maps each contain fewer regions than M, the question of the coloring of M has been reduced to that of coloring maps in fewer regions. This fact justifies the use of the word *reducible* here.

The definition of reducible rings is entirely independent of whether or not a coloring of the map M is possible. We note, however, that if there is a map M which can not be colored but which contains a reducible ring of n regions having more than $\phi(n)$ regions on both sides of it, then one of the auxiliary maps $M_1 + \mathfrak{M}_2$ or $M_2 + \mathfrak{M}_1$ can not be colored; and that the map of the least number of regions which can not be colored, if there is such a map, contains no ring of this kind.

It is proved in the present paper that rings for $n = 4$ or 5 are reducible. However, for $n = 6$ and greater values of n the question is left unsettled.

Suppose now that a ring R of n regions is given, which is such that any possible set of schemes deduced from $M_2 + \mathfrak{M}_1$ as before, where $\mathfrak{M}_1$, however, has fewer regions than $M_1 + R$, always contains at least one coloring for R suitable for $M_1 + R$.

The question of coloring a map M of this kind reduces to that of coloring maps $M_2 + \mathfrak{M}_1$ in fewer regions than M, and in these circumstances the ring R will be termed *reducible with respect to* M_1.

It is clear that in this case the map M can be colored if all the maps $M_2 + \mathfrak{M}_1$ in fewer regions can be colored, and that the map of the least number of regions which can not be colored contains no reducible ring of this type.

§ 2. *Rings of Four Regions.*

We can prove at once that in a regular map *a ring of four regions is reducible and the reducing number is* 0.*

* In this proof and later proofs we use the notation of § 1 in our consideration of rings; in particular, R denotes the ring, M_1 and M_2 the two aggregates of regions on opposite sides of the ring, and M denotes the map $M_1 + M_2 + R$.

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the four regions of the ring taken in cyclical order. Consider the maps $M_1 + \mathfrak{M}_2$ and $M_2 + \mathfrak{M}_1$, where $\mathfrak{M}_2$ and $\mathfrak{M}_1$ are formed from $M_2 + R$ and $M_1 + R$ by shrinking M_2 and M_1, respectively, to a point and joining α_1 and α_3. The possibilities for the colors are essentially either

$$a, b, a, b \quad \text{or} \quad a, b, a, c,$$

for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, respectively. If the colors are the same in both sets, we have a coloring for R the same in both sets of schemes. Otherwise we have

$$a, b, a, b \text{ for } M_1 + R \quad \text{and} \quad a, b, a, c \text{ for } M_2 + R, \text{ say.}$$

Now form a second choice of $\mathfrak{M}_1$ by shrinking M_1 to a point and joining α_2 to α_4, giving

$$a, b, a, b \quad \text{or} \quad a, b, c, b \quad \text{for} \quad M_2 + R.$$

The only case necessary for us to consider is the second of these two since the first is listed for $M_1 + R$. Thus we have the set of colors a, b, a, b in the schemes for $M_1 + R$; and the sets a, b, a, c and a, b, c, b in the schemes for $M_2 + R$.

In the scheme a, b, a, b for $M_1 + R$ either an a, d line connects the regions α_1, α_3, in which case we obtain a, b, a, c by a permutation, or else a b, c line connects the regions colored in b, and we get a, b, d, b by a permutation. In either case we get an arrangement of colors already essentially found in the set of schemes for $M_2 + R$.

Hence, the ring of four regions is reducible. The reducing number is 0 since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ did not contain in any case more than four regions.

§ 3. *Rings of Five Regions.*

Any regular map contains a 5-sided region* and, after the reductions of § 2, contains a ring of five regions about each such region. If rings of five regions were reducible with a reducing number 0, one could therefore conclude that the coloring of every map is reducible to that of maps of fewer regions, and hence that all maps could be colored in four colors. As a matter of fact, however, we can prove that *a ring of five regions is reducible and has a reducing number* 1, *that is* $\phi(5) = 1$.

Let us first prove that $\phi(5)$ does not exceed 1. We need to show that any two possible sets of schemes for $M_1 + R$ and $M_2 + R$ have at least one common coloring for R, if for $\mathfrak{M}_2$ and $\mathfrak{M}_1$ are permitted all maps of less than six regions.

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* See foot-note, p. 116.

We can then take $\mathfrak{M}_1$ and $\mathfrak{M}_2$ to consist of the five regions of R and a single contained region. This leads us to schemes on R which are in at most three colors (those different from the color of the contained region); and, since the colors can not alternate on five regions, there are precisely three colors, at most two of any color; we have then essentially two regions in a, two regions in b and one in c. This gives arrangements c, a, b, a, b for the regions in R taken in cyclical order. We will call the region colored c the *marked region;* it may be any one of the five regions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ of R.

If the marked region is the same in a scheme of each set, we can make the colors on the ring the same for $M_1 + R$ and $M_2 + R$ by a permutation, so that this case is disposed of.

Now in every case either a marked region in set I, of schemes for $M_1 + R$, and set II, of schemes for $M_2 + R$, must fall at adjacent places, or a coloring for R in I and II is the same. For suppose that this is not so. It is then no restriction to assume that α_1 and α_3 are marked regions for I and II, so that we have I$(c\,a\,b\,a\,b)$ [*i. e.*, c, a, b, a, b for $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ in I] and II$(a\,b\,c\,a\,b)$ [*i. e.*, a, b, c, a, b for $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ in II].

In the second of these schemes we may suppose that a b, c line joins the regions α_2, α_3 to the region α_5, for otherwise we would be led to II$(a\,c\,b\,a\,b)$ by a transposition. This is a scheme for II in which the marked region is on α_2 adjacent to α_1, contrary to assumption. Hence a b, c line joins these regions and we may change a on α_4 to d, obtaining II$(a\,b\,c\,d\,b)$.

Now, form a second map $\mathfrak{M}_2$ by shrinking M_2 to a point and joining α_2, α_5, so that $\mathfrak{M}_2$ consists of four regions. We are then led to a scheme I$(*\,a\,*\,*\,a)$ where the $*$-colors are different from a. The last two $*$-colors may be taken to be b and c, and thus the cases arise:

$$\text{I}(b\,a\,b\,c\,a), \quad \text{I}(c\,a\,b\,c\,a), \quad \text{I}(d\,a\,b\,c\,a).$$

The first of these has a marked region α_4 adjacent to α_3, a marked region in II. Hence this case is excluded. The second case is of the same type but with marked region α_3. This is the case disposed of at the outset, namely that of the same marked region in I and II. The last case I$(d\,a\,b\,c\,a)$ must then arise, giving essentially the same coloring as II$(a\,b\,c\,d\,b)$ obtained earlier. Thus a contradiction arises and the statement is proved.

If, however, marked regions in I and in II are adjacent, the two schemes again have a coloring for R in common.

To prove this, let us take these marked elements to be α_1 and α_2 respectively so that we have I$(c\,a\,b\,a\,b)$ and II$(b\,c\,a\,b\,a)$. Consider the regions in the colors

b, c in II$(b\,c\,a\,b\,a)$. If a b, c line does not connect the regions α_1, α_2, and α_4, we obtain II$(c\,b\,a\,b\,a)$ with marked region at α_1, a type already listed in I. In this case the statement is true. When a b, c line does connect these regions we obtain II$(b\,c\,d\,b\,a)$ by changing a on α_3 to d.

Now, form $\mathfrak{M}_2$ by shrinking M_2 to a point and joining α_1 to α_4 in R. This gives I$(b**b*)$, which leads to one of the types of schemes:

$$\mathrm{I}(b\,c\,d\,b\,a),\quad \mathrm{I}(b\,c\,d\,b\,c),\quad \mathrm{I}(b\,c\,d\,b\,d).$$

The first of these has just been listed in II, and the third has a marked region at α_2 so that this type also appears in II. Thus we are led to I$(b\,c\,d\,b\,c)$ with marked region at α_3.

Consequently, if the two sets of schemes have no common coloring for R, we infer that I and II have a marked element at α_1 and α_2 respectively, and thence that I has a marked region at α_3.

By a repetition of this argument, starting with a marked region α_2 of II and α_3 of I, we prove that α_4 is a marked region for II. By another repetition we prove that α_5 is a marked region for I, and by yet another repetition we prove that α_1 is a marked region for II as well as for I.

Hence, in every case there is a coloring common to the two sets of schemes, and this shows that the ring is reducible, with a reducing number which does not exceed 1.

To show that the reducing number $\phi(5)$ is 1, it is required to exhibit two sets of schemes containing essentially different colorings for R, and such that all the permutations of either set of schemes yield another scheme of the same set. These sets of schemes must have the further property that some coloring in each set is consistent with any possible choice of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of less than six regions.

A simple reckoning shows that all the schemes a, b, a, c, d for R in which the regions in b, c and b, d lie on b, c and b, d lines respectively, suffice for one of these sets of schemes; and that all the schemes c, a, b, a, b on R in which the regions in a and b lie on an a, d and b, d line suffice for the other set of schemes. Hence we must have $\phi(5)=1$.

§ 4. *Regular Maps under the Reductions of* § 2 *and* § 3.

It is interesting to inquire as to the kind of map to which we are led by the above reductions. What is the nature of regular maps M containing no rings of four regions, or of five regions except about a single region?

To answer this question we consider an arbitrary k-sided ($k \geqq 5$) region A of the map adjacent to the regions $B_1, B_2, \ldots, B_k$, where these regions are taken in cyclical order, one for each boundary line of A. These regions are all distinct, since otherwise a region B_i and A would together form a multiply-connected region, a case excluded by the definition of a regular map. Likewise, no region B_i has a side in common with any region B_j except the two which precede and follow it in cyclical order, for otherwise B_i, B_j, A together form a multiply-connected region. Consequently the regions $B_1, \ldots, B_k$ form a ring about A.

Consider next the regions $C_1, \ldots, C_l$ which abut on the outer edge of the ring $B_1, \ldots, B_k$. These are taken in cyclical order, one for each boundary line directed outward from the ring $B_1, \ldots, B_k$ and l is at least as great as k since each region B_i has at least five sides. All of the regions $C_1, \ldots, C_l$ are distinct; otherwise we should have a ring of four regions formed by regions C_i, B_j, A, B_k, or a multiply-connected region formed by C_i, B_j, B_k.

If two regions C_i, not adjacent along a boundary line directed outward from a region B, had a side in common, we would then have a ring of five regions of the form C_i, C_j, B_k, A, B_l, or a ring of four regions C_i, C_j, B_k, B_l, or a multiply-connected region C_i, C_j, B_k. The last two cases are excluded by our hypothesis about M. In the first case, such a ring of five regions would contain one region only on one side by our hypothesis, and this region is a region B, of course. The regions C_i, C_j would then be adjacent to each other along a boundary line directed outward from a region B. Thus the first case is excluded as well as the other two.

Therefore we conclude that *in a regular map not subject to the reductions of* § 2, § 3 *the neighborhood of any region of the map is formed by a ring* $B_1, B_2, \ldots, B_k$ ($k \geqq 5$) *enclosing* A *and a second ring* $C_1, C_2, \ldots, C_l$ ($l \geqq k$) *enclosing* $A, B_1, B_2, \ldots, B_k$. Conversely, it is obvious that if a regular map has this property it will contain no rings of four regions, or rings of five regions except about a single region.

A simple example of such a map is the map of twelve 5-sided regions formed by the faces of a dodecahedron.

§ 5. *Rings and the Four-Color Problem.*

After the preceding reductions and the application of them to normalizing the form of the map, it is natural to inquire whether or not rings of more than five regions are reducible and have such reducing numbers as to enable one to extend the preceding conclusions about the structure of a completely reduced map.

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For example, if rings of six regions were reducible with a reducing number 3 we could infer at once that in a map not subject to reduction there was a third set $D_1, \ldots, D_m$ of cyclically ordered regions about $C_1, C_2, \ldots, C_l$, *all of which were distinct.**

It is apparent, however, that the ring of six regions is of a decidedly different character from the ring of five regions. In fact, we may have a regular map, each region of which is surrounded by three successive rings of regions, in which, however, a ring of six regions containing an arbitrary number of 5-sided regions may be found (see fig. 1, p. 125, in which the dotted line indicates the ring). It will be seen later, that if there are more than three such regions, the ring of six regions is reducible. In so far as there is an analogy between the ring of five regions and that of six regions, the number 3 may be expected to play the same rôle for the latter as the reducing number 1 did for the former.

In all the cases which I have considered, the ring R of six regions containing more than three regions in M_1 is reducible with respect to M_1.

It may easily be seen that if a ring R of six regions, or more generally of n regions, contains a sufficiently large number of regions in M_1, it will necessarily be reducible with respect to M_1. In fact, all the possible choices of regions $\mathfrak{M}_1$ to replace $M_1 + R$ (where we do not now limit the number of regions in $\mathfrak{M}_1$) only yield a finite number of different aggregates of colorings for R. Let $t_1, t_2, \ldots, t_k$ be the number of regions in maps furnishing respectively each of the different aggregates of colorings. Clearly if t be the greatest of these numbers, the ring R of n regions will be reducible with respect to M_1 if $M_1 + R$ contains more than t regions.

The importance of reduction by means of rings may be explained as follows: In any map the direct and successful method of coloring is to start with a single region and color the regions adjoining, in succession, by trial. If a ring of k regions, where k is small, occurs somewhere in the map, and one comes to the regions of the ring from one side, it is impossible to determine what coloring to give to the regions of the ring unless one has considered the very complex map that may be on the other side.

On account of the curious results which a careful analysis of the ring of six regions gives, it is not probable that the ring of more than five regions can be eliminated as has been done with the ring of five or fewer regions, and this fact seems to me to make plausible any one of the following three alternatives:

* See § 4.

1. There exist maps which can not be colored in four colors, a leading feature of the simplest one of them perhaps being a ring of six regions with more than three regions on either side. By the method of reduction one will always be led either to a coloring of the given map, or to one or more maps that can not be colored.

2. All maps can be colored in four colors and a set of reducible rings can be found, one of which exists in every map.

3. All maps can be colored in four colors, but only by means of reductions of a more extensive character, applicable to sets of regions bounded by any number of rings.

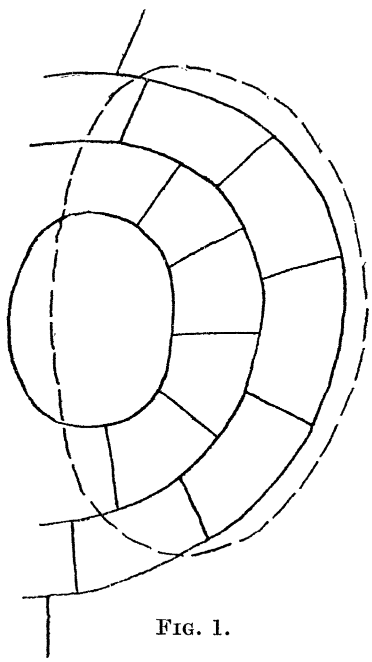

FIG. 1.

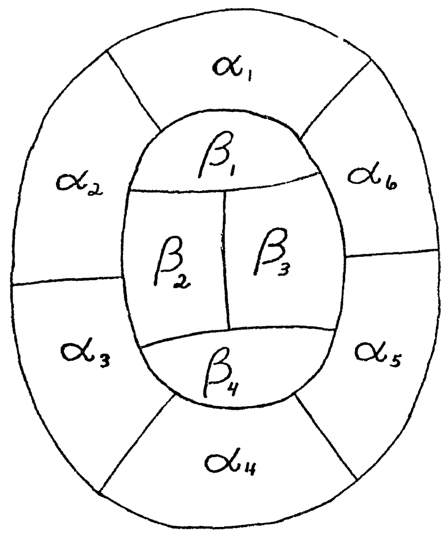

FIG. 2.

§ 6. *Rings of Six Regions.*

I shall not attempt to give an exhaustive analysis of the ring of six regions, such as I have actually carried out, but will confine attention to the more interesting special results that arise.

To begin with, let us show that *a ring R of six regions is reducible with respect to M_1 if M_1 consists of four* 5*-sided regions* (fig. 2).

Let β_1, β_2, β_3, β_4 be the 5-sided regions of M_1, taken to abut on the regions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ of the ring as indicated in the figure.

Our method of proof is to assume R not reducible with respect to M_1, so that by hypothesis one has not necessarily one of the sets of schemes for $M_2 + R$ suitable for coloring R of $M_1 + R$, and to show that a contradiction results.

Let β_1, β_2, β_3, β_4, α_3, α_5 of $M_1 + R$ coalesce to form $\mathfrak{M}_1$. With this choice of $\mathfrak{M}_1$ we obtain a coloring $b, *, a, *, a, *$ of R in the scheme for $M_2 + R$.

The only essentially different cases are then the six following:

$$b, c, a, b, a, c; \quad b, c, a, c, a, c; \quad b, c, a, d, a, c;$$
$$b, c, a, b, a, d; \quad b, c, a, c, a, d; \quad b, c, a, d, a, d.$$

Direct trial shows that it is possible to color $M_1 + R$ starting with any of these colorings for R except b, c, a, c, a, c which must then be listed in the set of schemes for $M_2 + R$.

A c, d line will connect all of the regions colored c in b, c, a, c, a, c, for otherwise we should be led to one of the schemes just excluded by a transposition of the colors c, d. Hence any permutation of the colors a, b is permissible, and one of these gives b, c, b, c, a, c, which is a suitable arrangement of colors on R for $M_1 + R$.

Thus in every case we are led to a scheme for $M_2 + R$ having a coloring on R suitable for $M_1 + R$, so that the initial hypothesis is not correct.

We may generalize the result as follows: *If all the regions M_1 interior to a ring R of six regions are 5-sided, and if there are more than three such regions then R is reducible.* For $M_1 + R$ then contains a configuration of the kind just seen to be reducible. This follows from the fact that the interior arrangement of 5-sided regions is of the simple nature indicated in fig. 1. This suffices to show that R is reducible as stated.

We can show immediately that *a ring R of six regions is reducible with respect to M_1 if M_1 consists of six 5-sided regions about a 6-sided region.* In fact coalesce M_1 and alternate regions of R into a single region, thus forming a possible choice of $\mathfrak{M}_1$ and a scheme $a, *, a, *, a, *$. Taking account of the circular symmetry, we obtain only the cases a, b, a, b, a, b; a, b, a, b, a, c; a, b, a, c, a, d, all of which are suitable colorings for $M_1 + R$.

These are the simplest types of reducible rings of six regions containing more than three regions in M_1.

The first of the above reductions is interesting since it has the consequence that *in a completely reduced map no boundary line can be enclosed by four 5-sided regions.* In fact, four such regions necessarily are surrounded by a ring of six regions in the map if the reductions of § 2, § 3 have been made. If we recall (foot-note, p. 116) that the number of 5-sided regions in a regular map exceeds the number of 7-, 8-, ...-sided regions combined, we see that this reduction applies to a variety of regular maps.

If the ring of six regions is reducible, the reducing number exceeds 3. To prove this statement it is necessary to show that there exist two sets of schemes I and II without a common coloring for R, such that a suitable coloring for every $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of nine or fewer regions exists in both schemes and such that every permutation of either scheme leads to another of the same scheme.

The colorings of R may be taken to be

$$\text{I.}\left\{\begin{array}{lllll} a,b,a,b,a,b; & a,b,a,b,a,c; & a,b,a,c,a,b; & a,c,a,b,a,b; & a,b,c,a,c,b; \\ c,b,a,b,c,a; & c,a,c,b,a,b; & a,b,c,a,d,b; & d,b,a,b,c,a; & c,a,d,b,a,b; \\ b,c,d,a,b,a; & a,b,a,b,c,d; & a,b,c,a,b,d; & a,d,b,a,c,b; & a,b,c,a,b,c. \end{array}\right.$$

$$\text{II.}\left\{\begin{array}{lllll} a,b,a,c,a,d; & b,a,c,a,c,a; & b,a,c,a,b,a; & b,a,b,a,c,a; & b,a,c,a,d,a; \\ a,c,b,a,b,d; & b,d,a,c,b,a; & b,a,b,d,a,c; & c,d,a,b,a,b; & d,a,b,a,b,c; \\ a,b,c,d,a,b; & b,a,b,c,d,a; & a,b,c,d,c,b; & a,c,b,d,c,b. & \end{array}\right.$$

It is not necessary to specify the lines in the schemes but only to verify that for each coloring in I and II and for some arrangement of the lines all the permutations deduced by the formal principle of § 1 lead to colorings in I and II respectively.

We will not verify this property of I and II, or that colorings in I and II exist suitable for all choices of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of not more than nine regions. It may be assumed that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are not subject to the earlier reductions, and this limits the total set of possibilities for $\mathfrak{M}_1$ and $\mathfrak{M}_2$ to 38.

§ 7. *Rings of n Regions.*

It is possible to find reducible rings of n regions. I shall only consider two of the simplest cases.

A ring R of n regions in a map $M_1 + R + M_2$, where M_1 consists of a ring of 5-sided regions about a single region, is reducible with respect to M_1. Let $\alpha_1, \ldots, \alpha_n$ be the n regions of R in cyclical order, and let $\beta_1, \ldots, \beta_n$ be the 5-sided regions of M_1 which abut on (α_1, α_2), (α_2, α_3), $\ldots$, (α_n, α_1) respectively, and let γ be the remaining region of M_1. To form $\mathfrak{M}_1$, let alternate regions $\alpha_1, \alpha_3, \ldots, \alpha_{n-1}$ coalesce with M_1 if n is even, and alternate regions $\alpha_1, \alpha_3, \ldots, \alpha_{n-2}$ with M_1 if n is odd. We thus obtain a coloring in a scheme for $M_2 + R$

$$a, *, a, *, \ldots, a, b, \quad \text{or} \quad a, *, a, \ldots, a, b, c.$$

In either case we can color $M_1 + R$ with such a coloring for R. In the first case if the $*$-colors are all one color b we can color $\beta_1, \ldots, \beta_n$ in c and d

and then γ in a. If this is not so, by taking account of the circular symmetry we may assume that we have

$$a,\ c,\ a,\ *,\ \ldots,\ a,\ b;$$

i. e., we may assume α_2 and α_n are not of the same color. Now color β_1 which abuts on α_1, α_2 in b, β_2 in d, β_3 in a color different from that of α_3, β_2, α_4, and so on, until we reach β_n, which must be colored in a different color than that of α_n, β_{n-1}, α_1 and β_1. But α_n and β_1 are both colored in b, so that this is possible. Having colored $\beta_1, \ldots, \beta_n$ in the three colors different from a, we may color γ in a. In the second case we can employ a similar process as follows. First color β_n in b. If then α_2 is also colored in b, we can color $\beta_{n-1}, \ldots, \beta_1$ and γ as in the first case. Otherwise α_2 is in c or d, and β_1 may be colored in d or c, and β_2 in b. Here again, if α_4 is colored in b, we may color $\beta_{n-1}, \ldots, \beta_3$ as in the first case. Otherwise α_4 is in c or d, and β_3 may be colored in d or c, and β_4 in b. In this way we either color $\beta_1, \ldots, \beta_n$ in b, c, d and γ in a or we come to a coloring

$$c \text{ or } d,\ b,\ c \text{ or } d,\ b,\ \ldots,\ b,\ *,\ *,\ b,$$

where the $*$-colors may be taken to be c and d, and γ may be colored in a.

When combined with the reductions of § 2, § 3 of a regular map, this reduction obviously eliminates any region of n sides surrounded by a ring of 5-sided regions, for this region is surrounded by a second ring. The simplest example not previously covered is the case $n = 7$.

A ring R of $4n$ regions in a map $M_1 + R + M_2$, where M_1 consists of a set of $2n$ 6-sided regions about a single region, is reducible with respect to M_1. Let $\alpha_1, \alpha_2, \ldots, \alpha_{4n}$ be the regions of R, and $\beta_1, \ldots, \beta_{2n}$ the 6-sided regions, so that $\alpha_1, \ldots, \alpha_{4n}$ abut on β_1, (β_1, β_2), β_2, (β_2, β_3), $\ldots$, (β_{2n}, β_1), respectively, and let γ be the inner region. To form $\mathfrak{M}_1$ let $\alpha_2, \alpha_4, \ldots, \alpha_{4n}$ coalesce with M_1 and obtain a coloring $*, a, *, a, *, \ldots, a$, for R. The regions β each touch two regions α_{2i} and one $*$-region. If the $*$-regions are all in one color b, we may color the regions β_i in c and d, alternately, and the n-sided region in a or b. If this is not the case, we may assume that we have c, a, $\ldots$, b, a for $\alpha_1, \alpha_2, \ldots, \alpha_{4n}$ and proceed to color β_1 (in b), $\beta_2, \ldots, \beta_{2n}$, successively, and then γ just as we did in the preceding reduction.

The simplest application is to a 6-sided region surrounded by a ring of six 6-sided regions.