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New Korn Inequalities: application to Buckling of Shells and Fracture of Plates

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by

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Acknowledgements

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Profile Summary

I'm a skilled **Math Ph.D.** candidate with 5 years experience in creating accurate mathematical models to describe and predict deformations in physical materials, capable of adapting quickly to new challenges and working on self-directed projects.

I am committed to expanding my skill set, evidenced by my participation in multiple internships that improved my data analytics skills and blockchain knowledge. My ability to thrive in team settings is demonstrated through active involvement in college and graduate school hockey teams, various leadership roles within clubs, and experience as CFO for a €1,000,000 sports council budget.

I am seeking a [JOB TITLE] position to apply my robust quantitative and computational skills, and to develop data-driven solutions for complex challenges in the [INDUSTRY TYPE] industry.

Data & Analysis Skills

- **Programing Languages:** Proficient **-** Python, Matlab, Typescript**.** Working Knowledge - SQL, C, RStudio, Mathematica
- **Theoretical Foundation:** Machine Learning, Stochastic Calculus, Financial Math, and Statistics.
- **Os and Platforms:** GitHub, AWS LightSail, MongodB, Flask web, Visual Studio Code
- **Language:** Portuguese- Native, English and Spanish- Fluent

Work Experience

Blockchain Researcher Intern

Nethermind, London, England — July 2022-Present

- Meticulously reviewed the entire GitHub repository written in GO of one of the biggest 2nd layer blockchain, analyzed and handpicked data to identify and resolve operational issues with a third-party validator, resulting in a **twofold profit** increase.
- Uncovered a previously unnoticed bug in the protocol consensus mechanism. Through simulations in **Python**, illustrated that exploiting this bug could hike validator profits by over 17x, endangering more than $1 billion locked in the network.
- Presented the discovered vulnerability and proposed a solution to leading validators and high-profile CEOs in the industry.

Research and Development Engineering Intern

Uneven Labs, Santa Barbara, California — June 2022-September 2022

- Hosted an **NFT trading bot** on **AWS Lightsail** to improve market liquidity, leveraging TypeScript for blockchain interaction, **SQL** for handling big datasets, and Flask web app to create an API to interact with the ML model's algorithm.
- Gained expertise in crypto and economic concepts, including **liquidity pools**, **tokenomics**, and **arbitrage** models through independent research and presented findings in company meetings, fostering knowledge sharing and collaboration.
- Identified multiple API bugs, vulnerabilities in mathematical models, and proposed new strategies for providing liquidity to the **NFT marke**t, demonstrating resourcefulness and commitment beyond the scope of the initial role in the **start-up**.

Graduate Researcher

University of California, Santa Barbara — September 2017-**October 2023 (Expected)**

- Developed innovative mathematical models to elucidate and predict deformation changes and fracture in physical materials under different loads, resulting in one published paper, and a second in preparation.
- Presented findings at a conference and multiple seminars, showcasing effective communication and subject matter expertise.
- Completed numerous coding and data science projects utilizing pandas and tidyverse, focusing on data visualization and implementing algorithms such as including logistic regression, **GLM, LDA, k-NN, PCA, Random Forest,** and used s**ymbolic programing** to automate computational part of the research.

Teacher Assistant

University of California, Santa Barbara — September 2017-September 2023

- Review and refine thousands of lines of Python code during 9 quarters of teaching **Numerical Analysis** classes.
- Design and **lead** a custom class for over **90** students across 2 summers, enhancing leadership and organizational skills while communicating complex mathematical concepts through accessible visual, written, and oral formats for a diverse audience.
- Mentored 4 undergraduate students on year-long research projects, focusing on the complexities of **blockchain mining** and **game theory**, helping them with their academic growth.

Education

University of California, Santa Barbara (UCSB), Santa Barbara, CA — Expected September 2023

Ph.D. Mathematics

- Dissertation Topic: **PDE's, Measure Theory and Material Science**

University of Coimbra (UC), Coimbra, Portugal — July 2016

Master's in Science, Mathematics (19/20), Bachelor's in Sience, Mathematics with Minor in Physics (18/20)

- Placed in top 3% of Best Students of University 4 years and was Awarded **"Young Talents in Mathematics"** fellowship
- Dissertation: "Regularity properties for the porous medium equation".

Abstract

New Korn Inequalities: application to Buckling of Shells and Fracture of Plates

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Korn's type inequalities, $\|\nabla \boldsymbol{u} - A\| \leq K_1 \|\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u}\|$, originating from Korn's 1906 contributions, have been instrumental in analyzing boundary value problems and linear elasticity. They are pivotal in establishing the existence of energy minimizers and have been fundamental in both linear and nonlinear shell theories, where the control of the asymptotic of the Korn constant K_1 where shown to also be of extreme importance. However it does not stop here, recently, it has been understood that, in fact, they also play a central role in fundamental classical questions in fracture mechanics, in particular in Griffith's model.

The first part of this thesis is dedicated to the study of the buckling of cylindrical shells under axial compression. While Koiter's theoretical formula predicts a linear relationship between the buckling load $\lambda(h)$ of the shell thickness h ($h > 0$ is a small parameter), experimental data have consistently pointed to $\lambda(h) \sim h^{3/2}$; *i.e.*, the shell buckles at much smaller loads for small thickness. This discrepancy, largely attributed to the shell's sensitivity to imperfections, is rigorously investigated. Our findings assert that the buckling load, when subjected to axial compression, is linked to the curvature of its cross-sectional curve. Specifically, when the cross-section is a convex curve with uniformly positive curvature, then $\lambda(h) \sim h$, and when the cross-section curve has positive curvature except at finitely many points, then $C_1 h^{8/5} \leq \lambda(h) \leq C_2 h^{3/2}$ for h small thickness $h > 0$. This result in particular shows that the load is in fact not sensitive to symmetry breaking in the shell geometry.

The second focus of this work revolves around the introduction of new weighted variants of the Poincaré and Korn inequalities, adapted specifically for plates. These tailored inequalities prove invaluable in scenarios that demand the use of polar coordinates or require a clear demarcation between longitudinal and transverse directions, such as in the study of junctions of massive bodies and thin rods. Furthermore, better control around the solid's boundary paves the way for sophisticated localization techniques and problems with different boundary conditions.

Lastly, in the concluding part, we introduce a groundbreaking Korn inequality for plates for special functions of bounded deformation (SBD). This advancement is essential for analyzing fractures in thin domains and for deriving dimension reduction theories via Griffith's model, similar to traditional approaches in elasticity. The work on SBD functions and fracture mechanics has taken significant strides only in the past decade, particularly with the emergence of Korn-type inequalities for general domains. Drawing from these recent advances, this thesis demonstrates how to manage the Korn constant K_1 as the thickness h of a plate, denoted as $\Omega_h = [0,1]^2 \times [-h,h]$, approaches zero. We prove, the expected result, that the constant asymptotic in SBD is the same as in the Sobolev space, *i. e.*, $K_1 \sim \frac{1}{h}$. This result is a significant step towards the development of a full-fledged dimension reduction theory for fracture mechanics in thin domains.

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2.1 Introduction

Solid bodies, in general, are not absolutely rigid, so when suitable forces are applied both size and shape can change. When the induced changes are considerable, the body might not return to its original shape. However, depending on the material, the body might return to its original shape when the forces are removed. This property is called elasticity. The literature in Elastic Materials is very rich, and there are several introductory books that the reader can explore [**MS1**, **MS2**, **MS3**, **MS4**], with personal preference for the 3 book Series on Mathematical Elasticity from Phillipe G. Ciarlet [**MSP**, **MSP2**, **MSP3**].

Understanding and predicting the deformation of elastic materials when subjected to external/internal forces has long been a fundamental pursuit in science and engineering. Cauchy started this field in 1825, and after that several models have been proposed to describe the behavior of materials, but the most significant ones boiled down to minimization techniques. Any body can possess energy in various way. If it is moving, it has kinetic energy. If it is in a gravitational field, it has gravitational potential energy. If it is stretched, compressed, or deformed in any way it has strain energy. So to predict how a body will deform we need to be able to minimize the total energy of the system.

Modeling the energy of a body is not a trivial task, and it depends on the type of system we are trying to study. Throughout this thesis, we will focus only on elastic bodies, an idealization of real materials, that when subjected to small external force, the body deforms, but once that force is removed, the body regains its original shape. This behavior contrasts with plastic deformation, where the material does not return to its original shape after the force is removed.

Traditional linear models for the elastic energy, have proven successful in describing the behavior of many materials under small deformations. However, these models fail to capture the intricate responses observed in substances that undergo large deformations. To overcome this problem, around 1845, Green introduced "Green elasticity", also known as Hyperelasticity. This model is able to offer a more robust framework by incorporating nonlinear strain-energy functions that better represent the complex behavior of these materials, however, the mathematical analysis of this model is more challenging.

As we will see in the next section, this model still has a lot of limitations and can't model more complex behaviors like fracture of the materials. However, it is a good starting point to understand the minimization techniques used in the field of elasticity. For more on this topic, Lanczos, C. does a great job explaining the connection of Calculus of Variations and Mechanics in his book [**MSVariational**].

Even if the definition of an elastic material is questionable on many grounds, its use has nevertheless led to so many achievements in the analysis of structures, and its mathematical analysis has led to many challenging problems (some of them yet unsolved), therefore it still stands as one of the major achievements of continuum mechanics.

In this chapter, we will introduce the main concepts from material science that will be used in the rest of the Thesis. In section 2.2, we will introduce the concept of hyperelasticity rigorously. After that, in section 2.3 we will introduce Korn inequalities and their non-linear version, geometric rigidity inequality. We will focus on their importance in the field of elasticity, particularly on the development of plate and shell theory. Finally, in section 2.4 we will introduce the concept of fracture mechanics.

2.2 Hyperelastic bodies and Equilibrium equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply-connected, and open set with a Lipschitz boundary, representing an elastic body in its natural state, and $\Omega^{\boldsymbol{y}}$ the deformed state represented by $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$, with $\boldsymbol{x} \in \Omega$. Consider two close points in Ω, $\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{\delta x}$, then applying Taylor expansion to $\boldsymbol{y}(\boldsymbol{x} + \boldsymbol{\delta x})$ we get

$$|\boldsymbol{y}(\boldsymbol{x} + \delta \boldsymbol{x}) - \boldsymbol{y}(\boldsymbol{x})|^2 = \delta \boldsymbol{x}^T \nabla \boldsymbol{y}^T(\boldsymbol{x}) \nabla \boldsymbol{y}(\boldsymbol{x}) \delta \boldsymbol{x} + o(\delta \boldsymbol{x}^2),$$

The symmetric tensor $\boldsymbol{C} := \nabla \boldsymbol{y}^T(\boldsymbol{x}) \nabla \boldsymbol{y}(\boldsymbol{x})$ from the above expression is normally called in elasticity as the Cauchy-Green tensor, and it is very important to measure the strain of a deformed body. To understand why, consider a rigid deformation, *i.e.*, a deformation that does not change the shape of the body, and its strain energy is zero. Then the deformation as the form

$$\boldsymbol{y}^r = \boldsymbol{a} + R\boldsymbol{x}, \; a \in \mathbb{R}^3, \; R \in SO(3),$$

and the Cauchy-Green tensor is given by

$$\boldsymbol{C}^r = R^T R = I,$$

so the first possible strain energy density function we can consider would be the so-called Green-St Venant strain tensor:

$$W(\nabla \boldsymbol{u}) = \frac{1}{2}(C - I) = \frac{1}{2}(\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T \nabla \boldsymbol{u}), \tag{2.2.1}$$

where u is the displacment vector, $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}(\boldsymbol{x}) - \boldsymbol{x}$.

However this model is not general enough, so in hyperelasticity, it's normally considered more general density energy functions. More details for this model can be seen in chapter 4 of [**MSP**] or in Part1 Section 6 of the original Ph.D. Thesis of Robert Kohn [**KohnThesis**]. In general, the energy of a hyperelastic body is given by the following functional:

$$E(\boldsymbol{y}) = \int_{\Omega} W(\nabla \boldsymbol{y}(\boldsymbol{x}))d\boldsymbol{x} - \int_{\partial\Omega} \boldsymbol{y} \cdot \boldsymbol{t}(\boldsymbol{x})dS(\boldsymbol{x}), \tag{2.2.2}$$

where $W(\boldsymbol{F})\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ is the elastic energy density function of the body and $\boldsymbol{t}(\boldsymbol{x})$ are dead traction loads applied to the boundary. Additionally, we will assume that W is of class C^3 in some neighborhood of the identity matrix $\boldsymbol{I}$, and the symbols $W_{\boldsymbol{F}}$ and $W_{\boldsymbol{FF}}$ will denote the gradient and the Hessian of W respectively, i.e.,

$$W_{\boldsymbol{F}}(\boldsymbol{F}) = \left(\frac{\partial W}{\partial f_{ij}}(\boldsymbol{F})\right)_{1\le i,j\le 3} \quad \text{and} \quad W_{\boldsymbol{FF}}(\boldsymbol{F}) = \left(\frac{\partial^2 W}{\partial f_{ij}\partial f_{kl}}(\boldsymbol{F})\right)_{1\le i,j,k,l\le 3},$$

where $\boldsymbol{F} = (f_{ij})_{1\le i,j\le 3.}$ Finally, W will satisfy the following fundamental properties that will be essential for the rest of this Thesis:

(P1) Positivity: $W(\boldsymbol{F}) \ge 0$ for all $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$, and $W(\boldsymbol{I}) = 0$.

(P2) Absence of prestress: $W_{\boldsymbol{F}}(\boldsymbol{I}) = \boldsymbol{0}$.

Note that this condition follows from (P1) and the fact that W is C^3-regular at $\boldsymbol{I}$. However, this condition is traditionally mentioned as it has the mechanical meaning of the absence of prestress.

(P3) Frame indifference: $W(\boldsymbol{RF}) = W(\boldsymbol{F})$ for every $\boldsymbol{R} \in SO(3)$.

This condition is a general axiom in physics that asserts that any "observable quantity" i.e., any quantity with an intrinsic character, such as a mass density, an acceleration vector, etc., must be independent of the particular orthogonal basis in which it is computed. This condition is also known as material symmetry, and it means that the energy density function is invariant under rigid rotations of the body.

(P4) Local stability of the trivial deformation $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{x}$: $(\boldsymbol{L}_0\boldsymbol{\xi}, \boldsymbol{\xi}) \geq 0$ for any $\boldsymbol{\xi} \in \mathbb{R}^{3\times 3}$, where $\boldsymbol{L}_0 = W_{\boldsymbol{FF}}(\boldsymbol{I})$ is the linearly elastic tensor of material properties.

(P5) Non-degeneracy: $(\boldsymbol{L}_0\boldsymbol{\xi}, \boldsymbol{\xi}) = 0$ if and only if $\boldsymbol{\xi}^T = -\boldsymbol{\xi}$.

In addition to the above standard properties (P1)-(P5), we will assume that the material is isotropic,*i.e.*, at a given point, the response of the material is the same in all directions. More formally, the energy density W satisfies the additional property (P6) below.

(P6) Isotropy: $W(\boldsymbol{FR}) = W(\boldsymbol{F})$ for every $\boldsymbol{R} \in SO(3)$.

Especially in Chapter 3, this condition will be essential, since it is known in mechanics that homogeneous deformations are possible for isotropic materials. Additionally, we will also see in Section 3.4.1 that this assumption simplifies the job of proving the existence of a trivial branch.

A very important consequence of frame indifference (P3), is that there exists energy function W can actually be represented as a function that only depends on the Cauchy-Green tensor $\boldsymbol{C}$, *i.e.*, there exist $\tilde{W}$ such that

$$W(\boldsymbol{F}) = \tilde{W}(\boldsymbol{C}).$$

Another advantage of hyperelasticity is that the traditional equilibrium equations can be derived in a very simple way from the energy function. In fact, for some admissible set $\mathcal{A}$ of the deformations $\boldsymbol{y}(\boldsymbol{x})$ (normally a subspace of $W^{1,p}(\Omega)$, $1 < p$, that satisfy some Dirichlet or/and natural Neumann boundary conditions on some complementary portions of $\partial\Omega$ yielded from the loading), the resulting deformation $\bar{\boldsymbol{y}}(\boldsymbol{x})$ must be local minimizer of the energy $E(\boldsymbol{y})$. So the minimizer $\bar{\boldsymbol{y}}$ must satisfy the Euler-Lagrange equations that match exactly the traditional equilibrium equations together with the natural boundary conditions:

$$\nabla \cdot \boldsymbol{P}(\nabla \bar{\boldsymbol{y}}(\boldsymbol{x})) = 0, \qquad \boldsymbol{x} \in \Omega, \tag{2.2.3}$$

$$\boldsymbol{P}(\nabla \bar{\boldsymbol{y}})\boldsymbol{n} = \boldsymbol{t}(\boldsymbol{x}) \qquad \boldsymbol{x} \in \partial\Omega, \tag{2.2.4}$$

where $\boldsymbol{P}(\boldsymbol{F}) = W_{\boldsymbol{F}}(\boldsymbol{F})$ is the Piola-Kirchhoff stress tensor.

2.3 Importance of Korn Inequalities in Elasticity

2.3.1 Origin of Korn inequalities and Uniform Rigidity

Since Korn's original contributions in 1906 [**Korn1**, **Korn2**], Korn's inequality has played a central role in the analysis of boundary value problems and linear elasticity. Later in 1979, Robert Kohn, in his PhD thesis [**KohnThesis**] generalized and popularized this inequality. The non-linear elasticity field was still very recent, no general existence results were known and the existence of local minimizers under very strong conditions was just being published [**john**]. However, for hyperelasticity models to be useful, these important results needed to be proved. Kohn then introduced the first version of Korn inequalities, hoping that would lead to more advances in the field and he was right. Since then Korn inequalities have been useful not only in connection with basic theoretical issues such as existence and uniqueness but also in a variety of applications, for example, fundamental studies on the mathematical foundations of finite elements, stability theory, a priori estimates for solutions in terms of boundary data, fracture of bodies, dimension reduction of thin structures *etc.* Additionally, information on the dimensionless optimal constants appearing in the inequalities, the Korn constants were shown to also be of importance in many applications, and in the last decades, a lot of effort has been put into finding the optimal constants for different geometries and boundary conditions [**bib:Gra.Har.1**, **bib:Harutyunyan.1**, **bib:Gra.Har.4**, **bib:Harutyunyan.2**, **andre**].

For more information on the history of Korn's inequalities, we refer the reader to the survey paper [**bib:Horgan**] and the references therein. In this work, we will focus on Korn's inequalities for slender bodies (section 2.3.2), some weighted generalizations (chapter 4), and their importance in fracture mechanics (section 2.4 and chapter 5).

Korn inequalities can take different forms, however, in this thesis, we will use the following statement:

THEOREM **2.3.1** (First and Second Korn Inequality)**.** *Let $n \geq 3$,$1 < p < \infty$, and let the domain $\Omega \subset \mathbb{R}^n$ be open, bounded, connected, and Lipschitz.There there exists constants $C_1 = C_1(\Omega, p)$ and $C_2 = C_2(\Omega, p)$,such that for any vector field $\boldsymbol{u} \in W^{1,p}(\Omega : \mathbb{R}^n)$ there are $R \in \mathbb{R}^{n\times n}_{skw}$ such that:*

$$\|\nabla \boldsymbol{u} - R\|^p_{L^p(\Omega)} \leq C_1 \|e(\boldsymbol{u})\|^p_{L^p(\Omega)}. \tag{2.3.5}$$

and

$$\|\nabla \boldsymbol{u}\|^p_{L^p(\Omega)} \leq C_2 \left(\|e(\boldsymbol{u})\|^p_{L^p(\Omega)} + \|\boldsymbol{u}\|^p_{L^p(\Omega)}\right), \tag{2.3.6}$$

where $e(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right)$ is the symmetric part of the gradient.

The First and second Korn constants would normally be defined as the smallest constant that satisfies 2.3.5 and 2.3.6 respectively *i.e.*:

$$K_1(\Omega, p) = \inf_{\substack{\boldsymbol{u}\in W^{1,p} \\ R\in\mathbb{R}^{n\times n}_{skw}}} \frac{\|\nabla \boldsymbol{u} - R\|^p_{L^p(\Omega)}}{\|e(\boldsymbol{u})\|^p_{L^p(\Omega)}} \qquad K_2(\Omega, p) = \inf_{\boldsymbol{u}\in W^{1,p}} \frac{\|\nabla \boldsymbol{u}\|^p_{L^p(\Omega)}}{\left(\|e(\boldsymbol{u})\|^p_{L^p(\Omega)} + \|\boldsymbol{u}\|^p_{L^p(\Omega)}\right)}. \tag{2.3.7}$$

and both constants are invariant under rigid motions of the domain Ω. This result shows up often with $R = 0$, however for that, we need to restrain the function $\boldsymbol{u}$ to a smaller set, for example, $W^{1,p}_0(\Omega)$. There are several proofs of these and similar inequalities, some of them can be found in [**KohnThesis**, **KornProof1**, **KornProof2**, **conti0**, **KornProof3**].

When studying boundary-value problems in elasticity and solution of the equilibrium equations, the Korn inequalities are essential to prove the coercivity of the bilinear form, and thus existence of a solution. In fact, to construct the solution from a minimizing sequence one needs to bound $\|\nabla \boldsymbol{u_k}\|_{L^p(\Omega)}$ for a minimizing sequence. However, from the fact that the total potential energy approaches its lower limit, one can immediately only conclude that the expression $\|e(\boldsymbol{u}_k)\|$ remains bounded, which will be enough after proving the desired Korn inequalities.

In view of the fundamental importance of Korn's inequality in linear elasticity, it is not surprising that a suitable nonlinear version would play a central role in models in nonlinear elasticity. In fact, the first nonlinear Korn inequality was proved by Friesecke, James and Muller [**bib:Fri.Jam.Mue.1**] in 2002, and it has been essential to develop the theory of thin structures. This result is commonly known as Geometric Rigidity, and it is stated as follows:

THEOREM **2.3.2.** *Let $n \geq 2$, $1 < p < \infty$, and let the domain $\Omega \subset \mathbb{R}^n$ be open, bounded, connected, and Lipschitz.There exists a constant $C(\Omega)$ with the following property: For each $\boldsymbol{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$, there exists a rotation $S \in SO(n)$ such that:*

$$\|\nabla \boldsymbol{u} - S\|^p_{L^p(\Omega)} \leq C \| \operatorname{dist}(\nabla \boldsymbol{u}, SO(n))\|^p_{L^p(\Omega)}. \tag{2.3.8}$$

In other words, this means that if a gradient is close to the set of all rotations (the deformation is close to a rigid motion), then the deformation is close to a rigid motion.

2.3.2 Korn inequalities for slender bodies

The derivation of plate/shell theories is a problem that has a long history with major contributions from Euler, D. Bernoulli, Cauchy, Kirchhoff, Love, E. and F. Cosserat, Von Karman, and many other recent mathematicians. Most of the theories were based on Γ-convergence for a controlled asymptotic dimension reduction, and then study the stability problem in a low-dimensional setting. However, this type of work couldn't capture all types of energy. In fact, the first rigorous derivation of the Kirchhoff-Love plate theory from three-dimensional nonlinear elasticity was obtained by Friesecke, James and Muller [**bib:Fri.Jam.Mue.1**], using the geometric rigidity result 2.3.8 as a key ingredient. From this paper was also possible to conclude the first asymptotics for the first Korn Constant:

THEOREM **2.3.3.** *Let $\Omega_h = S \times [-h, h]$ be a thin domain, where S is a bounded with a Lipschitz boundary, representing the mid-surface of the shell. Then, for $u \in W^{1,2}$, there* *exists a constant skew-symmetric matrix* A *a constant* $C > 0$ *such that for all* $h \in (0, 1)$

$$\|\nabla \boldsymbol{u} - A\| \leq \frac{C(S)}{h}\|e(\boldsymbol{u})\|. \tag{2.3.9}$$

For general domains, the value of the Korn constant is not very important, however, it is well known in continuum mechanics that the rigidity of a shell $\Omega_h = S \times [-h, h]$ is closely related to the asymptotic of the optimal Korn's constants. The energy of the plate will, of course, depend on h, in fact, we have that $E(\Omega_h) \sim h^\alpha$ where α depends on the time of deformation (stretching corresponds to $\alpha = 1$, buckling to $\alpha = 2$ and bending to $\alpha = 3$ for example). While most of the first dimension reduction theory is based on $\Gamma-$convergence, the techniques used only work for $5/3 \leq \alpha < 2$. However, with the right Korn constants it is possible to extend this theory as we can see in [**bib:Fri.Jam.Mue.1**].

To understand a bit more about the importance of the asymptotics of the Korn constants, consider:

1. A minimizing sequence, $\boldsymbol{u}_h$, for the energy introduced before,
2. $\|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)} \sim E(\boldsymbol{u}_h)^p \sim h^\alpha$ for some $\alpha > 0$,
3. $K_1^p \sim \frac{1}{h^\beta}$ for some $\beta > 0$.

Then, applying a change of coordinates in the thin direction we have that

$$\begin{aligned}
\|\nabla \tilde{\boldsymbol{u}}_h - \tilde{A}_h\|^p_{L^p(\Omega_1)} &= \frac{C}{h}\|\nabla \boldsymbol{u}_h - A_h\|^p_{L^p(\Omega_h)} \\
&\leq \frac{C}{h^{\beta+1}}\|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)} \\
&\leq Ch^{\alpha-\beta-1}.
\end{aligned}$$

So to be able to control the gradient of the deformation in the limit and find a weakly convergent subsequence, we need to have that $\alpha - \beta - 1 \geq 0$. As we will understand in Chapter 3, when studying different type the deformations and consequently different possible α's, we are also changing the set of admissable deformations, so the asymptotic of the Korn constant can also change accordingly, so the study of critical buckling loads is closely related to the asymptotic of the first Korn constant [**bib:Gra.Tru.**, **bib:Gra.Har.2**, **bib:Gra.Har.3**, **bib:Harutyunyan.3**].

Thus finding the optimal constants in Korn's inequalities is a central task in problems concerning thin domains in general. Davit Harutyunyan and Yury Grabovsky, have been making great progress in this area, by proving the optimal constants for different Gaussian curvatures of middle surfaces and different boundary conditions [**bib:Gra.Har.1**, **bib:Gra.Har.4**, **bib:Gra.Har.1**, **bib:Harutyunyan.2**], we can recap most of the important results in the following theorem:

Theorem **2.3.4** (Harutyunyan and Grabovsky, 2014/2017/2018)**.** *Let K_G, κ_θ and κ_z be the Gaussian curvature and the principal curvatures of the mid-surface S. Then, for $u \in W^{1,2}_{=}$, there exists a constant $C > 0$ such that for all $h \in (0, 1)$*

1. *If $k_z = k_\theta = 0$ in an open set*

$$\|\nabla \boldsymbol{u}\|^2 \leq \frac{C}{h^2} \|e(\boldsymbol{u})\|^2.$$

2. *If $K_G > 0$,*

$$\|\nabla \boldsymbol{u}\|^2 \leq \frac{C}{h} \|e(\boldsymbol{u})\|^2.$$

3. *If $K_G < 0$,*

$$\|\nabla \boldsymbol{u}\|^2 \leq \frac{C}{h^{4/3}} \|e(\boldsymbol{u})\|^2.$$

4. *If $k_z = 0$ and $k_\theta > 0$*

$$\|\nabla \boldsymbol{u}\|^2 \leq \frac{C}{h^{3/2}} \|e(\boldsymbol{u})\|^2.$$

Additionally, the author of this thesis in joint work with Harutyunyan also proved in [**andre**], as demonstrated in the next chapter:

Theorem **2.3.5.** *Let K_G, κ_θ and κ_z be the Gaussian curvature and the principal curvatures of the mid-surface S. Then there exists a constant $C > 0$ such that for all $h \in (0,1)$. If $\kappa_z = 0$ and $\kappa_\theta >= 0$, with $\kappa_\theta(z,\theta) = \kappa_\theta(\theta)$, having finitely many zeros, then*

$$C_1 h^{12/7} \leq \inf_{\boldsymbol{\phi} \in V} \frac{\|e(\boldsymbol{\phi})\|^2}{\|\nabla \boldsymbol{\phi}\|^2} \leq C_2 h^{5/3}.$$

2.4 Variational Formulation of Fracture Mechanics

The study of the fracture of materials is a very important topic in material science, and it has been studied for a long time. In fact, the first studies on the fracture of materials date back to the 17th century, when Robert Hooke studied the fracture of glass and metals. However, the first mathematical model for the fracture of materials was proposed by Griffith in 1921 [**FracGrif**], and it was the first theory based on energy considerations for the fracture of brittle materials. Even if the formulation used today is not exactly the same as the one introduced by Griffith, this is still the most used model nowadays. Several books and papers discuss the variational approach to Fracture Mechanics [**FracBook**, **Frac2**, **Frac3**], *etc*, but in this section, we will follow the approach of Francfort and Marigo [**Frac1**].

Extending the approach from elasticity to fracture mechanics is not trivial, and it has two main challenges. First, we need to reconsider the space of admissible deformations, since we need to allow discontinuities, and second, we need to take into account that it takes energy to create and expand the crack. In the spirit of Griffith, if $k(\boldsymbol{x}) > 0$ represents the energy required to create an "infinitesimal" crack at the point $\boldsymbol{x}$, then the energy of a crack Γ is given by

$$E_s(\boldsymbol{u}) = \int_\Gamma k(\boldsymbol{x}) d\mathcal{H}^{n-1}(\boldsymbol{x}),$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure (see [**evansGa**] for more details). Additionally, provided that Γ is a $\mathcal{H}^{n-1}$-rectifiable, this can be generalized a bit more to incorporate the effect of toughness anisotropy and the size of the jump by considering the following generalized energy :

$$E_s(\boldsymbol{u}) = \int_{J_{\boldsymbol{u}}} k(\boldsymbol{x}, \nu(\boldsymbol{x}), [\boldsymbol{u}(\boldsymbol{x})] \cdot \nu(\boldsymbol{x})) d\mathcal{H}^{n-1}(\boldsymbol{x}),$$

where $[\boldsymbol{u}(\boldsymbol{x})]$ denotes the jump in the direction orthogonal to the crack, $\nu(\boldsymbol{x})$, and in this case the crack is represented by the jump set of the deformation $J_{\boldsymbol{u}}$ (more details in the definition of the jump set will be given in section 5.2).

Away from the crack, the energy is simply given by the elastic energy, so the total energy of the system is just the sum of the two energies:

$$E(\boldsymbol{u}) = \int_{\Omega} W(\nabla \boldsymbol{u}(\boldsymbol{x})) d\boldsymbol{x} + \int_{J_{\boldsymbol{u}}} k(\boldsymbol{x}, \nu(\boldsymbol{x}), [\boldsymbol{u}(\boldsymbol{x})] \cdot \nu(\boldsymbol{x})) d\mathcal{H}^{n-1}(\boldsymbol{x}).$$

The first approaches to minimize this energy considered the crack Γ and the deformation $\boldsymbol{u}$ different objects, bringing extra complexity to the problem. However, this can be simplified if we choose the correct space of admissible function. In section 5.2, this exposure will be presented more rigorously, however, the goal here is to provide the intuition on how functions of bounded deformations have desired properties.

They are integrable functions where the symmetric part of the gradient only exists in the sense of distributions. Additionally, it was proven that their jump set is a $\mathcal{H}^{n-1}$-rectifiable set as desirable and its symmetric gradient can be decomposed into an absolutely continuous part and a singular part by Lebesgue decomposition, where the singular part is responsible for the crack. So for a bounded deformation function, we have that:

$$\int_{\Omega} e(\boldsymbol{u}) = \int_{\Omega} e_{ac}(\boldsymbol{u}) d\boldsymbol{x} + \int_{J_{\boldsymbol{u}}} e_s(\boldsymbol{u}) d\mathcal{H}^{n-1}(\boldsymbol{x}),$$

where e_{ac} absolutely continuous part of $e(\boldsymbol{u})$ and e_s is the singular part.

Although his space fits the problem perfectly, their mathematical properties have only been studied very recently [**Ambrosio1997**, **RogerBook**, **RogerPaper**], and to be able to approach this minimization problem with rigor we need several approximations, compactness, and Korn-type results. In fact, in the last decade, several authors have been working on this, and we can now prove the existence of minimizers away from the jump set [**FracExistence**].

Finally, several Korn inequalities have also been proven for general domains [**kornBD1**, **kornBD2**, **kornBD3**], but no work for plates has been done. Sp in the last chapter of this Thesis we will introduce the first Korn inequality for bounded deformations in thin domains, which will be essential to the development of fracture mechanics theory of plates.

Chapter 3

The Problem of Shells Under Axial Compression

3.1 Introduction

Thin-walled shells are, in general, highly efficient structures. To produce reliable designs and avoid unexpected catastrophic failures, one needs to understand buckling. Buckling occurs in a thin structure under loading, when the structure undergoes an overall change in configuration instead of acting in the primary fashion intended by its designers, leading to the failure of the structures.

Physically speaking, buckling in a thin shell occurs, when the shell can absorb a great deal of membrane strain energy without deforming too much but it must deform much more in order to absorb an equivalent amount of bending strain energy. When this stored energy is converted into bending energy, buckling occurs, creating a visible change in the geometry of the shell (typically in the form of several dimples) to accommodate all the energy, e.g., Figure 3.1.

Mathematically speaking, buckling can be interpreted as the instability of the equilibrium state that for a certain load will have two possible trajectories to follow, i.e., when in the stress-strain (or stress-deformation) diagram a bifurcation occurs. This phenomenon is mathematically described as the loss of positivity of the second variation of the total energy of the system.

In engineering, it is essential to have a good estimate of the critical stress that will trigger buckling. In this chapter, we revisit the problem of buckling of cylindrical shells under axial compression. In that problem, one starts applying a homogeneous load of magnitude λ to the top of a cylindrical shell that is resting on a substrate, where λ is increased continuously from zero.

It is observed that at very small load magnitudes λ, the cylindrical shell will undergo a homogeneous deformation with no visible geometric changes. Then at some critical value $\lambda = \lambda(h)$, the shell will buckle, producing a variety of deformation patterns, typically in the form of several (or single) dimples ([**bib:Yoshimura**],[**bib:Bud.Hut.**],[**bib:Bushnell**],[**bib:Lan.Ca**
shown in Figure 3.1. The dimple (dimples) typically appear with a "click" and drop in the load magnitude (which corresponds to the bifurcation point), and disappear when unloading the shell. Some less common buckling patterns, such as the formation of waves in the longitudinal direction or periodic-like wrinkling are also possible, see [**bib:Xu.Mic.**] and Figure 14 of [**bib:Xu.Mic.**] for more details.

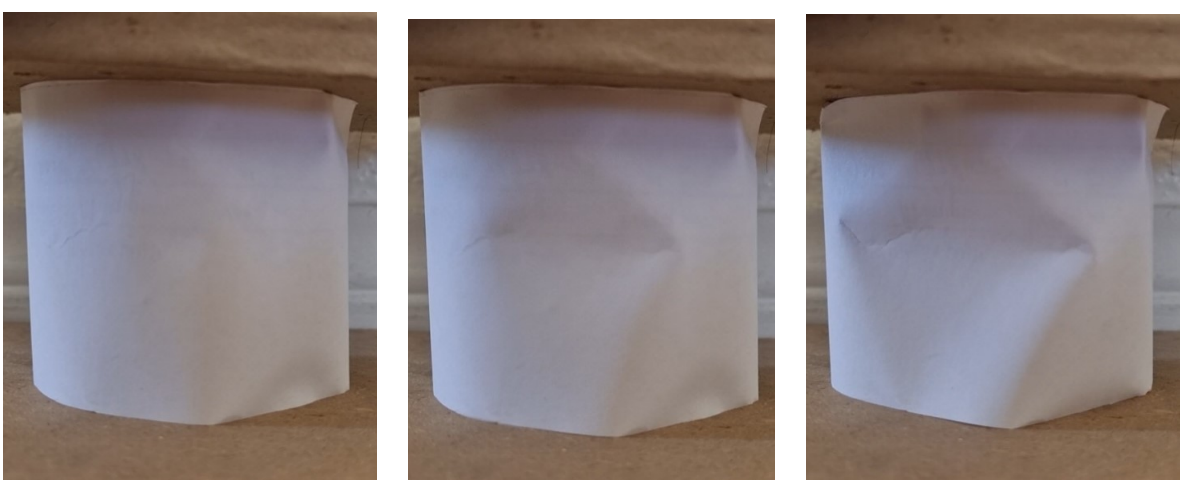


Figure 3.1: Buckled paper cylindrical shells under vertical load, however the second and the third cylinders are apparently deeper into the post-buckling regime.

Despite being a very old problem with a lot of data available in the literature, (studies and experiments have been made since the mid-18th century), it still contains several unsolved puzzles even for the simplest geometry of perfectly circular cylindrical shells. For the case of circular cylindrical shells, it has been observed that the buckling load measured by experiments has a large discrepancy with the theoretical predictions made by the classical asymptotic formula [**bib:Koiter**, **bib:Lorenz**, **bib:Timoshenko**, **bib:Tim.Woi.**]

$$\lambda(h) = \frac{Eh}{\sqrt{3\left(1-\nu^2\right)}}, \tag{3.1.1}$$

which predicts a linear relation between the thickness of the cylinder and the critical buckling load $\lambda(h)$.

Here E and ν are the Young modulus and the Poisson ratio of the material, respectively, and $h = t/R$ is the shell dimensionless thickness, i.e., the ratio of the cylinder wall thickness t to the cylinder radius R. Note that in fact formula (3.1.1) was first derived by Lorenz [**bib:Lorenz**] in 1911 and independently by Timoshenko [**bib:Timoshenko**] in 1914, but sometimes in the literature it also carries the name of Koiter, as Koiter derived the so-called Koiter circle associated to (3.1.1) in his Ph.D. thesis [**bib:Koiter**] in 1945, see also [**bib:Gra.Har.3**] for Koiter's circle.

On the experimental side, plenty of experiments since the 1930s show that the experimental critical stress is actually much lower than the theoretical formula (3.1.1), scaling like $h^{3/2}$ with h, e.g., [**bib:Lan.Cal.Pal.**, **bib:Zhu.Man.Cal.**]. It has been believed in the engineering and applied mathematics communities, that such a paradoxical behavior is in general due to the fact that the buckling load may be highly sensitive to shape or load imperfections ([**bib:Almroth**],[**bib:Tennyson**],[**bib:Wei.Mor.Sei.**],[**bib:Gor.Eva.**],[**bib:Yamaki**],[b

Note that *general geometric symmetry breaking or (even a very small) preexisting deformation have been believed to be important factors in the asymptotics of the critical buckling load.* These questions have been addressed in the works ([**bib:Calladine**],[**bib:Hor.Lor.Pel.**],[in an attempt to resolve the paradox using mainly numerical approach and/or reduced shell theory equations. One possible gap in these approaches may be whether the utilized reduced shell theory equations are indeed applicable to the problem under consideration and capture the sought parameters within the acceptable error. This concern is based in particular on the works ([**bib:Fri.Jam.Mor.Mue.**],[**bib:Hor.Lew.Pak.**],[**bib:Lew.Mor.Pak.**]), that reveal whether a specific reduced shell theory holds in a specific applied load and elastic energy regime. In the meantime the answer to those specific and important questions is unknown. Another weakness was the presence of some heuristic arguments.

On the rigorous side, Grabovsky and Harutyunyan proved in [**bib:Gra.Har.3**] that indeed Koiter's asymptotic formula (3.1.1) must hold in the case of perfect cylindrical shells and perfect axial homogeneous loading. This was achieved by the improvement and application of the "Thin structure buckling theory" by Grabovsky and Truskinovsky [**bib:Gra.Tru.**]. Another crucial component of the analysis in [**bib:Gra.Har.3**] was the derivation of the optimal asymptotic constants (not only the asymptotics but also the leading term in it) in Korn and generalized Korn inequalities for circular cylindrical shells. Then Grabovsky and Harutyunyan went on to prove in [**bib:Gra.Har.2**] that in fact if even a very small twist (in the angular direction) is present in the shell loading, then the asymptotics of the buckling load has to drop to $h^{5/4}$, see Table 3.1 below.

Cross-section (C.-S.) and load	Circular C.-S., vertical load $\boldsymbol{\alpha}(\theta)=(\cos(\theta),\sin(\theta))$, $\boldsymbol{t}=-\lambda\boldsymbol{e}_z$	Circular C.-S., twist in the load $\boldsymbol{\alpha}(\theta)=(\cos(\theta),\sin(\theta))$, $\boldsymbol{t}=-\lambda(\boldsymbol{e}_z+\epsilon\boldsymbol{e}_\theta)$
Buckling load asymptotics	$\lambda(h)=\frac{Eh}{\sqrt{3(1-\nu^2)}}$	$\lambda(h)=Ch^{5/4}$

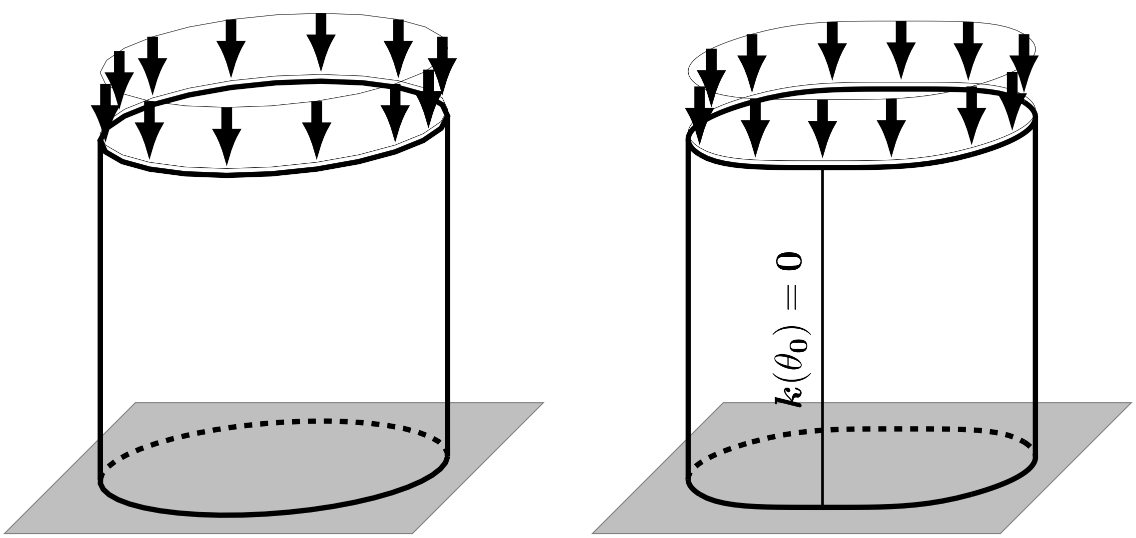
Table 3.1: The dependence of the critical buckling load of circular cylindrical shells on the type of loading. Vertical load versus vertical load with a small twist.

This analysis, to some extent, gave an explanation of the fact of sensitivity of the buckling load to load imperfections. Also, a somewhat less rigorous argument in [**bib:Gra.Har.2**] demonstrated why one should expect the buckling load to drop to $h^{3/2}$ in the presence of some small dimples in the shell.

Our task in this chapter is to analyze the "sensitivity to imperfections" for some other kind of shape imperfections, that are diversions from the perfect cylindrical shell. Namely, we will consider cylindrical shells, generated by cylindrical surfaces, that are non-circular but have convex cross sections: Two main families will be analyzed.

(i) The first family contains cylindrical surfaces with convex cross sections that have uniformly positive curvature (when regarded from the exterior of the curve). An illustrative example of such a curve is given in the left half of Figure 2.

(ii) The second family contains cylindrical surfaces with convex cross sections that have uniformly positive curvature (when regarded from the exterior of the curve) except on finitely many points on the curve, where the curvature vanishes. An illustrative example of such a curve is given in the right half of Figure 2.



(a) Case (i) in Theorem 3.1: uniformly positive positive curvature, $k_\theta > k > 0$ for all θ.

(b) Case (ii) in Theorem 3.1: curvature vanish in 2 isolated points. Around θ_0 the curve behaves like $\boldsymbol{\alpha}(\theta) = (\theta, \theta^4)$.

Figure 3.2: Possible Cross-sections, with positive curvatures

Note that in both cases the cylinder cross-section does not have to have any geometric symmetry property, but rather only has to be convex with some imposed curvature condition. We will prove that in case (i) the critical buckling load $\lambda(h)$ has the asymptotics Ch, and in case (ii) we will prove the bounds $C_1 h^{8/5} \leq \lambda(h) \leq C_2 h^{3/2}$ in the vanishing thickness regime $h \to 0$. The result in part (i) in particular disproves the longstanding belief that geometric symmetry breaking would lead to the drop in the asymptotics of $\lambda(h)$, i.e., there is an $\epsilon > 0$, such that $\lambda(h) \leq Ch^{1+\epsilon}$. Also, the result in part (ii) provides new evidence on how a geometric shape imperfection, which is a diversion from the perfect cylindrical shell may lead to the drop in the asymptotics of $\lambda(h)$ to at least $h^{3/2}$.

As already pointed out, we will be working in the framework of the (improved) "thin structure buckling theory" of Grabovsky and Truskinovsky [**bib:Gra.Tru.**] rigorously derived from three-dimensional nonlinear hyper-elasticity. Some of the main components in the analysis will be asymptotically sharp Korn inequalities for the displacement gradient components, proven for the shells under consideration.

The rest of this chapter will be organized as follows: In Section 3.2 we present a brief introduction to the general theory of slender structure buckling [**bib:Gra.Har.2**, **bib:Gra.Tru.**]. In Section 3.3 we will be formulating the main results of the chapter. In order to apply the theory in Section 3.2, on one hand, one needs to determine (with some amount of proximity) the so-called "trivial branch", and the asymptotic stress tensor, which will be done in Section 3.2.1. Additionally, one needs to prove asymptotically sharp Korn and Korn-type inequalities for the displacement gradient components, which will be formulated and proved in in Sections 3.2.2-3.2.3. Finally, in Section 3.4 we will prove the main results on the buckling load, and we will finish with a particular application for Neo-Hookean materials in Section 3.5.

3.2 Buckling of slender Structures

For the convenience of the reader, this section is devoted to the presentation of the general theory "Buckling of slender structures" by Grabovsky and Truskinovsky [**bib:Gra.Tru.**], that was later extended by Grabovsky and the Harutyunyan in [**bib:Gra.Har.2**]. In this presentation, there will be a small amount of focus on the buckling of cylindrical shells, which is the subject of the chapter. While we will try to keep it as self-contained as possible, we will avoid going into the technical details and proofs referring the reader to the papers [**bib:Gra.Tru.**, **bib:Gra.Har.2**] for details.

3.2.1 Trivial branch

The general theory of buckling of slender structures originated in [**bib:Gra.Tru.**], one considers a sequence of *slender* domains Ω_h parametrized by a small parameter h. The notion of slenderness used in this work will be given a precise definition later in Section 3.2.3. In the case when Ω_h is a shell, the parameter h typically represents the thickness of the shell, or the dimensionless thickness, i.e., the thickness divided by one of the in-plane parameters. In our case, h will represent the thickness of the non-circular cylinders Ω_h under consideration, with constant height L. Consider a loading program

$$\boldsymbol{t}(\boldsymbol{x}, h, \lambda) = \lambda \boldsymbol{t}_h(\boldsymbol{x}) + O(\lambda^2), \qquad \boldsymbol{x} \in \Gamma_h^1, \tag{3.2.1}$$

applied to the part Γ_h^1 of the boundary of Ω_h, where λ is the load magnitude and $\boldsymbol{t}_h$ is the load direction. For the problem of cylindrical shell axial compression, the part Γ_h^1 would be the top and the bottom parts of the boundary only. Assume for some $\lambda_0 > 0$ the loading (3.2.1) results in a family of Lipschitz deformations $\boldsymbol{y}(\boldsymbol{x}; h, \lambda) \in W^{1,\infty}(\Omega_h, \mathbb{R}^3)$ for all $\lambda \in [0, \lambda_0]$, where as mentioned above, the field $\boldsymbol{y}(\boldsymbol{x}; h, \lambda)$ is a stable deformation for given boundary conditions, i.e., it is a weak local minimizer of the elastic energy

$$E_{el.}(\boldsymbol{y}) = \int_\Omega W(\nabla \boldsymbol{y}(\boldsymbol{x}))d\boldsymbol{x}$$

in the admissible set

$$\mathcal{A}_\lambda = \{\boldsymbol{y}(\boldsymbol{x}, h, \lambda) \in W^{1,\infty}(\Omega_h, \mathbb{R}^3) \ : \ \boldsymbol{y}(\boldsymbol{x}, \lambda, h) = \boldsymbol{g}(\boldsymbol{x}, \lambda, h) \text{ on } \ \Gamma_h^1 \text{ in the trace sence}\}, \tag{3.2.2}$$

for all $\lambda \in [0, \lambda_0]$, where the vector field $\boldsymbol{g}(\boldsymbol{x}, \lambda, h) \in W^{1,\infty}(\Omega_h; \mathbb{R}^3)$ represents the boundary conditions on the part Γ_h^1 where the load is applied, and it is typically defined specifically for each problem and results from the loading program (3.2.1). The family of deformations $\boldsymbol{y}(\boldsymbol{x}; h, \lambda)$ is then called a *trivial branch.* Next, one defines the so-called *linearly elastic trivial branch.*

Definition 3.2.1. *A family of stable (within the admissible set $\mathcal{A}_\lambda$) Lipschitz deformations $\boldsymbol{y}(\boldsymbol{x}, h, \lambda) \in W^{1,\infty}(\Omega_h, \mathbb{R}^3)$ is called a* ***linearly elastic trivial branch,*** *if there exist $h_0 > 0$, so that for every $h \in [0, h_0]$ and $\lambda \in [0, \lambda_0]$ one has:*

(i) $\boldsymbol{y}(\boldsymbol{x}; h, 0) = \boldsymbol{x}$.

(ii) There exist a family of Lipschitz functions $\boldsymbol{u}^h(\boldsymbol{x})$, independent of λ, such that

$$\left\|\nabla \boldsymbol{y}(\boldsymbol{x}; h, \lambda) - \boldsymbol{I} - \lambda \nabla \boldsymbol{u}^h(\boldsymbol{x})\right\|_{L^\infty(\Omega_h)} \leq C\lambda^2, \tag{3.2.3}$$

(iii)

$$\left\|\frac{\partial(\nabla \boldsymbol{y})}{\partial \lambda}(\boldsymbol{x}; h, \lambda) - \nabla \boldsymbol{u}^h(\boldsymbol{x})\right\|_{L^\infty(\Omega_h)} \leq C\lambda. \tag{3.2.4}$$

Here the constant $C > 0$ is independent of h and λ.

We remark that neither uniqueness nor general stability of the trivial branch is assumed. It is important to note that, here, the term general stability is stability without the boundary conditions in (3.2.2). It is worth mentioning that as understood in [**bib:Gra.Tru.**], usually the family $\boldsymbol{y}(\boldsymbol{x}, h, \lambda)$ is not stable in general due to the possibility of infinitesimal flips. This is always the case for cylindrical shell compression problems due to possible infinitesimal rotations in the cross-section plane. Additionally the leading order term $\lambda \boldsymbol{u}_h(\boldsymbol{x})$ of the nonlinear trivial branch is nothing else but the linear elastic displacement, that can be calculated solving the equations of linear elasticity $\nabla \cdot \left(\boldsymbol{L}_0 e\left(\boldsymbol{u}^h\right)\right) = \boldsymbol{0}$ (the Lamé system) with the imposed boundary conditions, where

$$e\left(\lambda \boldsymbol{u}^h\right) = \frac{\lambda}{2}(\nabla \boldsymbol{u}^h + \left(\nabla \boldsymbol{u}^h\right)^T)$$

is the linear strain.

3.2.2 The buckling load and buckling modes

Buckling of the thin structure Ω_h occurs when the trivial branch $\boldsymbol{y}(\boldsymbol{x}; h, \lambda)$ becomes unstable for some value $\lambda = \lambda_{crit}(h)$ for the first time. This happens because it becomes energetically more favorable to bend the structure rather than store more compressive energy. This bifurcation is detected by the change in the sign of the second variation of the energy:

$$\delta^2 E(\boldsymbol{\phi}; h, \lambda) = \int_{\Omega_h} (W_{FF}(\nabla \boldsymbol{y}(x; h, \lambda))\nabla\boldsymbol{\phi}, \nabla\boldsymbol{\phi})\, dx, \qquad \boldsymbol{\phi} \in V_h, \tag{3.2.5}$$

where $V_h \subset W^{1,\infty}(\Omega_h)$ is the vector space of admissible variations resulting from the loading program (3.2.1), i.e., from the admissible set $\mathcal{A}_\lambda$ in (3.2.2). Namely, this means that there exists $\lambda_{crit}(h) > 0$ such that the second variation is nonnegative when $0 < \lambda < \lambda_{crit}(h)$ for all test functions $\boldsymbol{\phi} \in V_h$, but for $\lambda > \lambda_{crit}(h)$ it can become negative for some choice of $\boldsymbol{\phi}$. This observation leads to the following mathematical definition of the critical buckling load:

$$\lambda_{\text{crit}}(h) = \inf\left\{\lambda > 0 : \delta^2 E(\boldsymbol{\phi}; h, \lambda) < 0 \text{ for some } \boldsymbol{\phi} \in V_h\right\}. \tag{3.2.6}$$

The body Ω_h is said to undergo a *near-flip buckling*, if $\lim_{h\to 0} \lambda_{\text{crit}}(h) = 0$. A buckling mode is generally understood as a variation $\boldsymbol{\phi}_h^* \in V_h$ different from zero, such that

$$\delta^2 E\left(\boldsymbol{\phi}_h^*; h, \lambda_{crit}(h)\right) = 0.$$

In practice one is only interested in the leading term asymptotics (or sometimes even in the scaling) of the buckling load in h as $h \to 0$. Note that if one perturbs $\lambda_{\text{crit}}(h)$ by a small factor ϵ, i.e., one replaces it by $(1+\epsilon)\lambda_{\text{crit}}(h)$, then the second variation changes approximately by $\epsilon\lambda_{\text{crit}}(h)\frac{\partial\left(\delta^2 E\right)}{\partial\lambda}(\boldsymbol{\phi}_h^*; h, \lambda_{\text{crit}}(h))$.

This means that second variations at two different variations $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ differing by an infinitesimal value compared to $\lambda_{\text{crit}}(h)\frac{\partial(\delta^2 E)}{\partial\lambda}(\boldsymbol{\phi}_h^*; h, \lambda_{\text{crit}}(h))$ should not be distinguished. This observation leads to the following new definition of buckling loads and buckling modes in the broader sense.

Definition 3.2.2. *We say that a function $\lambda(h) \to 0$, as $h \to 0$ is a buckling load if*

$$\lim_{h\to 0} \frac{\lambda(h)}{\lambda_{crit}(h)} = 1.$$

Similarly, a buckling mode is a family of variations $\boldsymbol{\phi}_h \in V_h \backslash \{0\}$, such that

$$\lim_{h\to 0} \frac{\delta^2 E(\boldsymbol{\phi}_h; h, \lambda_{crit}(h))}{\lambda_{crit}(h)\frac{\partial(\delta^2 E)}{\partial\lambda}(\boldsymbol{\phi}_h; h, \lambda_{crit}(h))} = 0$$

It turns out that under the conditions in Definition 3.2.1, the buckling load and buckling modes can be captured by the so called *constitutively linearized second variation* [**bib:Gra.Tru.**]:

$$\delta^2 E_{\text{cl}}(\boldsymbol{\phi}; h, \lambda) = \int_{\Omega_h} \left(\langle \boldsymbol{L}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle + \lambda \left\langle \boldsymbol{\sigma}_h, \nabla\boldsymbol{\phi}^T \nabla\boldsymbol{\phi}\right\rangle\right) d\boldsymbol{x}, \qquad \boldsymbol{\phi} \in V_h, \tag{3.2.7}$$

where $\boldsymbol{\sigma}_h(\boldsymbol{x}) = \boldsymbol{L}_0 e\left(\boldsymbol{u}^h(\boldsymbol{x})\right)$ is the linear elastic stress. From the condition (P4, chapter 2) on $\boldsymbol{L}_0$, the first term in (3.2.7) is always nonnegative, thus the potentially destabilizing variations will be in the set:

$$V_h^d = \left\{\phi \in V_h : \left\langle \boldsymbol{\sigma}_h, \nabla\boldsymbol{\phi}^T \nabla\boldsymbol{\phi}\right\rangle \leq 0\right\}. \tag{3.2.8}$$

Further, the *constitutively linearized critical strain* will be obtained by minimizing the Rayleigh quotient

$$\Re(h, \boldsymbol{\phi}) = -\frac{\int_{\Omega_h} \langle \boldsymbol{L}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle\, dx}{\int_{\Omega_h} \langle \boldsymbol{\sigma}_h, \nabla\boldsymbol{\phi}^T \nabla\boldsymbol{\phi}\rangle\, dx}, \tag{3.2.9}$$

over the set of destabilizing variations V_h^d. One can formalize this in the following definition.

Definition 3.2.3. *The constitutively linearized buckling load $\lambda_{\text{cl}}(h)$ is defined by*

$$\lambda_{\text{cl}}(h) = \inf_{\boldsymbol{\phi}\in V_h^d} \Re(h, \boldsymbol{\phi}). \tag{3.2.10}$$

The family of variations $\left\{\boldsymbol{\phi}_h \in V_h^d : h \in (0, h_0)\right\}$ *is called a constitutively linearized buckling mode if*

$$\lim_{h\to 0} \frac{\Re\,(h, \boldsymbol{\phi}_h)}{\lambda_{\mathrm{cl}}(h)} = 1. \tag{3.2.11}$$

Definition 3.2.4. *The body* Ω_h *is slender if* $\lim_{h\to 0} K\,(\Omega_h) = 0$, *where the Korn constant is defined as*

$$K\,(\Omega_h) = \inf_{\phi \in V_h} \frac{\|e(\boldsymbol{\phi})\|^2_{L^2(\Omega_h)}}{\|\nabla \boldsymbol{\phi}\|^2_{L^2(\Omega_h)}}.$$

The following theorem, proven in [**bib:Gra.Har.2**], provides a formula for the buckling load and buckling modes.

THEOREM **3.2.5.** *Suppose that the body is slender in the sense of Definition 3.2.4. Assume that the constitutively linearized critical load* $\lambda_{\mathrm{cl}}(h)$, *defined in (3.2.10) satisfies* $\lambda_{\mathrm{cl}}(h) > 0$ *for all sufficiently small* h *and*

$$\lim_{h\to 0} \frac{\lambda_{\mathrm{cl}}(h)^2}{K\,(\Omega_h)} = 0.$$

Then $\lambda_{\mathrm{cl}}(h)$ *is the buckling load and any constitutively linearized buckling mode* $\boldsymbol{\phi}_h$ *is a buckling mode in the sense of Definition 3.2.2.*

Now, in order to study the problem of buckling of cylindrical shells under axial compression, we see from Theorem 3.2.5 and from the above formalism, that we need to gain appropriate information about the trivial branch associated with axial compression, the associated stress tensor, the Korn constant, and the load $\lambda_{\mathrm{cl}}(h)$ for our especific problem. This is done in the following sections.

3.3 Problem setting and main results

In this section, we formulate the problems under consideration and the main results. Let $\boldsymbol{\alpha}(\theta)$ be a closed convex curve in the XY-plane that is parameterized by the arc length and has period/length p. Then the cylindrical surface S having horizontal cross section as the curve $\boldsymbol{\alpha}$ and height $L > 0$ will be given by the formula

$$S: \quad \boldsymbol{r}(\theta, z) = \boldsymbol{\alpha}(\theta) + z\boldsymbol{e}_z, \quad \theta \in [0, p],\ z \in [0, L].$$

Denote $\boldsymbol{e}_\theta = \boldsymbol{\alpha}'(\theta)$ and by $\boldsymbol{e}_t$ the tangent and normal vectors to $\boldsymbol{\alpha}$ respectively. This will give rise to the local orthonormal basis $(\boldsymbol{e}_t, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$. We will be dealing with cylindrical shells Ω_h with mid-surface S and thickness h, i.e.,

$$\Omega_h = \{(t, \theta, z) \ : \ t \in [-h/2, h/2],\ \theta \in [0, p],\ z \in [0, L]\}. \tag{3.3.1}$$

Denoting $k_\theta = \|\boldsymbol{\alpha}''\|$ the curvature of the cross-section $\boldsymbol{\alpha}$, we have that the gradient of any vector field $\boldsymbol{\phi} = (\phi_t, \phi_\theta, \phi_z) \in H^1(\Omega_h, \mathbb{R}^3)$ will be given in the local basis $(\boldsymbol{e}_t, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$ by

$$\nabla\boldsymbol{\phi} = \begin{bmatrix} \phi_{t,t} & \frac{\phi_{t,\theta} - k_\theta\,\phi_\theta}{k_\theta\,t+1} & \phi_{t,z} \\ \phi_{\theta,t} & \frac{\phi_{\theta,\theta} + k_\theta\,\phi_t}{k_\theta\,t+1} & \phi_{\theta,z} \\ \phi_{z,t} & \frac{\phi_{z,\theta}}{k_\theta\,t+1} & \phi_{z,z} \end{bmatrix}, \tag{3.3.2}$$

where $f_{,\eta}$ inside the gradient matrix denotes the partial derivative $\partial_\eta f$. Assume that the shell Ω_h is resting on a substrate and is undergoing uniform and homogeneous vertical/axial loading[1]

$$\boldsymbol{t}(\boldsymbol{x}, h, \lambda) = -\lambda\boldsymbol{e}_z \tag{3.3.3}$$

applied at the top of the shell. The main problem that we are concerned with is the determination of the asymptotics of critical buckling load $\lambda(h)$ in the thickness h as $h \to 0$.

As already mentioned in Section 3.1, one has $\lambda(h) \sim h$ as $h \to 0$ in the case when the cross-section $\boldsymbol{\alpha}$ is a circle. We will be considering the cases when $\boldsymbol{\alpha}$ has positive curvature everywhere ($k(\theta) > 0$ for all $\theta \in [0, p]$), and the case when $\boldsymbol{\alpha}$ has positive curvature everywhere except for finitely many points on the curve, at which the curvature has to

[1]The terminology "axial load" will be used even if $\boldsymbol{\alpha}$ does not have point-symmetry, and thus Ω_h will have no axis.

vanish as $\boldsymbol{\alpha}$ is convex and thus $k(\theta) \geq 0$ for all $\theta \in [0, p]$.

Let us mention that in what follows the norm $\|f\|$ will be the L^2 norm $\|f\|_{L^2(\Omega_h)}$. For the problem under consideration, the natural choice for the vector space V_h will be the subspace of all displacements $\boldsymbol{\phi} \in H^1(\Omega_h)$ that vanish at the top and the bottom of the shell, *i.e.*,

$$V_h = \{\boldsymbol{\phi} \in H^1(\Omega_h) \ : \ \boldsymbol{\phi}(t,\theta,0) = \boldsymbol{\phi}(t,\theta,L) = 0, \quad (t,\theta) \in [-h/2, h/2] \times [0,p]\}. \quad (3.3.4)$$

This means that one shall study the stability of the trivial branch within the set of all Lipschitz deformations satisfying the same Dirichlet boundary conditions on the top and bottom of the cylinder. However, we will prove a stronger stability result, i.e., stability within a wider class of deformations. Note that if one only prescribes the values of the vertical component y_z of the trivial branch $\boldsymbol{y}$ at the top and the bottom of the shell, then the cylinder may undergo flip instability through infinitesimal rotations in the cross-section plane ([**bib:Gra.Har.2**],[**bib:Gra.Tru.**]). Therefore one needs to impose some condition on y_t or y_θ to rule this possibility out. Following [**bib:Gra.Har.3**], we choose to impose zero integral condition on the tangential component ϕ_θ in the z direction, which gives the alternative subspace

$$V_h^\theta = \{\boldsymbol{\phi} \in H^1(\Omega_h) \ : \ \phi_z|_{z=0} = \phi_z|_{z=L} = \int_0^L \phi_\theta(t,\theta,z)dz = 0, \quad \forall (t,\theta)\}. \qquad (3.3.5)$$

Let us explain this choice. Observe that we can extend the cylinder to the lower half-space by mirror reflection about the plane $z = 0$, and consequently the components ϕ_t and ϕ_θ as even functions and ϕ_z as an odd function, preserving the structure of the symmetric gradient $e(\boldsymbol{\phi})$. This would allow us to expand the components ϕ_t and ϕ_θ in Fourier space in cosine series and the ϕ_z component in sine series in the z variable; see (3.4.19) and the paragraph between (3.4.18) and (3.4.19). Then the zero integral condition simply means that the independent of z term in the expansion of ϕ_θ is nonexistent. This would clearly prevent the cylinder from undergoing infinitesimal rotations in the cross-section plane, which affects only the independent of z variable terms in $\boldsymbol{\phi}$.

Note also, that unlike perfect circular cylindrical shells, where homogeneous deformations result in no change in the tangential component, (thus in [**bib:Gra.Har.2**], the authors have prescribed the tangential component ϕ_θ at the top and the bottom of the cylinder too), homogeneous deformations in general cylindrical shells with any cross sections result in nonzero displacements in the tangential component too, thus a more relaxed subspace, such as (3.3.5) has to be considered. We will prove the following theorems, where the vector space V is either V_h or V_h^θ.

THEOREM **3.3.1.** *Assume $\boldsymbol{\alpha}$ is a convex C^2 regular curve in the XY plane and assume the cylindrical shell Ω_h is given as in (3.3.1). Assume Ω_h is undergoing vertical loading as in (3.3.3), and the admissible variations $\boldsymbol{\phi}$ belong to the subspace V. The following statements hold:*

(i) If

$$\min_{\theta\in[0,p]} k_\theta(\theta) = k > 0,$$

then one has

$$\lambda(h) \sim h,$$

as $h \to 0$.

(ii) Assume in addition that the cross section $\boldsymbol{\alpha}$ is of class[2] C^5, and assume $k_\theta(\theta) > 0$ except for finitely many points $\theta_1, \theta_2, \ldots, \theta_n \in (0, p)$, where one has $k_\theta(\theta_i) = 0$ and $k''_\theta(\theta_i) \neq 0$[3] for $i = 1, 2, \ldots, n$. This will imply quadratic growth of the curvature at the points θ_i, i.e.,

$$c|\theta-\theta_i|^2 \le k_\theta(\theta) \le \frac{1}{c}|\theta-\theta_i|^2, \qquad \theta \in [0,p],\ i = 1, 2, \ldots, n, \tag{3.3.6}$$

for some constant $c > 0$. Then there exist constants $C_1, C_2 > 0$, depending only on

[2]Note that the higher regularity C^5 is only required for the Ansatz construction in (3.4.50).
[3]Due to the fact $k_\theta \ge 0$, we have $k'_\theta(\theta_i) = 0$ for $i = 1, 2, \ldots, n$.

the cylinder height L and the cross-section $\boldsymbol{\alpha}$, such that one has

$$C_1 h^{8/5} \leq \lambda(h) \leq C_2 h^{3/2},$$

as $h \to 0$.

A remark is in order.

Remark 3.3.2. *Part (i) of Thereom 3.3.1 implies that no matter what the geometry of the cross-section $\boldsymbol{\alpha}$ is, the critical buckling load $\lambda(h)$ will scale like h as $h \to 0$ as long as $\boldsymbol{\alpha}$ has strictly positive curvature everywhere. This in particular means that the buckling load asymptotics in the vanishing thickness is independent of the fact whether $\boldsymbol{\alpha}$ has any kind of symmetry or not. Hence,* ***the critical buckling load asymptotics is not sensitive to the initial symmetry of the undeformed configuration.*** *Part (ii) provides some evidence for the phenomenon that the buckling load may drop to $h^{3/2}$ if the cylinder has some zero longitudinal curvature sections.*

In order to prove Theorem 3.3.1, we will need the following Korn and Korn-like inequalities that can be considered as part of the main results of the paper.

THEOREM **3.3.3.** *Let the cylindrical shell Ω_h be as in Theorem 3.3.1. The following statements hold:*

(i) If

$$\min_{\theta\in[0,p]} k_\theta(\theta) = k > 0$$

then there exist a constant $\tilde{h} > 0$, depending only on $L, \max k_\theta$, and k, such that

$$\inf_{\boldsymbol{u}\in V} \frac{\|e(\boldsymbol{\phi})\|^2}{\|\mathrm{col}_3(\nabla\boldsymbol{\phi})\|^2} \sim h, \tag{3.3.7}$$

for all $h \in (0, \tilde{h})$.

(ii) Assume the conditions in part (ii) of Theorem 3.3.1 are satisfied. Then there exist constants $C_1, C_2, C_3, C_4, \tilde{h} > 0$, depending only on $L, \max k_\theta$, and c, such that the

Korn and Korn-like inequalities hold:

$$C_1 h^{12/7} \leq \inf_{\phi \in V} \frac{\|e(\boldsymbol{\phi})\|^2}{\|\nabla \boldsymbol{\phi}\|^2} \leq C_2 h^{5/3}, \qquad C_3 h^{8/5} \leq \inf_{\phi \in V} \frac{\|e(\boldsymbol{\phi})\|^2}{\|\mathrm{col}_3(\nabla \boldsymbol{\phi})\|^2} \leq C_4 h^{3/2}, \tag{3.3.8}$$

for all $h \in (0, \tilde{h})$. These results are presented in the schematic table 3.2 below.

Remark 3.3.4. *It is proven in [**bib:Gra.Har.4**], that under the assumptions in part (i) of Theorem 3.3.3, Korn's first inequality holds:*

$$\|\nabla \boldsymbol{\phi}\|^2 \leq \frac{C}{h^{3/2}} \|e(\boldsymbol{\phi})\|^2, \tag{3.3.9}$$

for all vector fields $\boldsymbol{\phi} \in V_h$ and all $h \in (0, \tilde{h})$. This will be a useful component in the proof of part (i) of Theorem 3.3.1.

Cross-section (C.-S.) and load	C.-S. with uniformly positive curvature, vertical load $k(\theta) > 0$, $\boldsymbol{t} = -\lambda \boldsymbol{e}_z$	C.-S. with positive curvature, except at finitely many points, vertical load $k(\theta) \geq 0$ and $k(\theta_n) = 0$ for finitely many θ_n, $\boldsymbol{t} = -\lambda \boldsymbol{e}_z$
Buckling load asymptotics	$\lambda(h) = Ch$	$Ch^{8/5} \leq \lambda(h) \leq Ch^{3/2}$

Table 3.2: The dependence of the critical buckling load of convex cylindrical shells on the cross-section curvature. Cross-sections with uniformly positive curvature versus cross-sections with nonnegative curvature, that vanish only at finitely many points on the curve.

3.4 Proof of the main results

3.4.1 The trivial branch and the stress tensor

In this section, we will demonstrate how one can apply the theory presented in Section 3.2 to the problem of axial compression of cylindrical shells with any convex cross-section.

First we will calculate the trivial branch $\boldsymbol{y}(\boldsymbol{x}, \lambda, h)$ resulting from the compression $\boldsymbol{t}(\boldsymbol{x}, \lambda, h) = -\lambda \boldsymbol{e}_z$ at the top of the shell. The vector field $\boldsymbol{u}_h$ then will be obtained, which will yield a formula for the linear elastic stress tensor $\boldsymbol{\sigma}_h$, thus the minimization problem (3.2.10) will be identified, which will be studied in the next section.

The compression problem is equivalent to the boundary-value problem, where one prescribes the displacement at the top and the bottom of the shell and at the same time solves the system of Euler-Lagrange equations (equations of equilibrium). The boundary conditions on the vertical component of $\boldsymbol{y}$ yielding from the axial compression will be

$$y_z(t,\theta,0,\lambda,h)=0 \text{ at the bottom,} y_z(t,\theta,L,\lambda,h)=(1-\lambda)L \text{ at the top.}$$

It is natural to seek the trivial branch among homogeneous deformations, yielding homogeneous thickening of the shell in the cross-section plane which is clearly given by

$$y_t=(1+a(\lambda))(t+\boldsymbol{\alpha}(\theta)\cdot\boldsymbol{e}_t(\theta)), \qquad y_\theta=(1+a(\lambda))\boldsymbol{\alpha}(\theta)\cdot\boldsymbol{e}_\theta(\theta), \qquad y_z=(1-\lambda)z, \tag{3.4.1}$$

where $\boldsymbol{y}(t,\theta,z)=y_t\boldsymbol{e}_t+y_\theta\boldsymbol{e}_\theta+y_z\boldsymbol{e}_z$, and $a(\lambda)$ is a smooth enough function satisfying $a(0)=0$, and will be determined later by the equations of equilibrium and the natural boundary conditions yielding from the energy minimization problem.

For the sake of notation simplicity, let $\boldsymbol{y}(t,\theta,z;h,\lambda)=\boldsymbol{y}(t,\theta,z)=\boldsymbol{y}(\boldsymbol{x})$ while keeping in mind that the trivial branch can depend on h and λ. The plan is to prove the existence of a trivial branch given as in (3.4.1). It is easy to see that when minimizing the elastic energy $\int_{\Omega_h}W(\nabla\bar{\boldsymbol{y}})d\boldsymbol{x}$ subject to boundary conditions $\bar{\boldsymbol{y}}(\boldsymbol{x})=\boldsymbol{y}(\boldsymbol{x})$ at $z=0,L$ for $\boldsymbol{y}$ as in (3.4.1), then any local minimizer $\bar{\boldsymbol{y}}\in H^1(\Omega_h)$ must satisfy the equations of equilibrium

$$\nabla\cdot\boldsymbol{P}(\nabla\bar{\boldsymbol{y}}(\boldsymbol{x}))=0, \qquad \boldsymbol{x}\in\Omega_h, \tag{3.4.2}$$

together with the natural boundary conditions

$$\boldsymbol{P}(\nabla\bar{\boldsymbol{y}})\boldsymbol{e}_t=0 \qquad \text{at} \qquad t=\pm h/2, \tag{3.4.3}$$

where $\boldsymbol{P}(\boldsymbol{F})=W_{\boldsymbol{F}}(\boldsymbol{F})$ is the Piola-Kirchhoff stress tensor. Also, note that if we do not prescribe the t or θ component of the field $\bar{y}$ at $z=0,L$, then we will get additional natural boundary conditions such as

$$\boldsymbol{P}(\nabla\bar{\boldsymbol{y}})\boldsymbol{e}_z\cdot\boldsymbol{e}_t=0 \qquad \text{at} \qquad z=0,L, \tag{3.4.4}$$

$$\boldsymbol{P}(\nabla\bar{\boldsymbol{y}})\boldsymbol{e}_z\cdot\boldsymbol{e}_\theta=0 \qquad \text{at} \qquad z=0,L,$$

respectively. The existence of the trivial branch is proved in the following Lemma:

LEMMA **3.4.1.** *Assume that $W(\boldsymbol{F})$ is three times continuously differentiable in a neighborhood of the identity matrix $\boldsymbol{F}=\boldsymbol{I}$ and satisfies the properties (P1)-(P6) from section 2.2. Then there exists a constant $\lambda_0>0$ and a unique function $a(\lambda)\in C^2([0,\lambda_0],\mathbb{R})$, such that $a(0)=0$ and the family $\boldsymbol{y}(\boldsymbol{x})=(y_t,y_\theta,y_z)$ given by (3.4.1) satisfies the equations of equilibrium (3.4.2) and all of the boundary conditions (3.4.3)-(3.4.4). Moreover, the trivial branch $\boldsymbol{y}(\boldsymbol{x})$ also fulfills all the conditions in Definition 3.2.1 as is required for the general theory to apply.*

Proof. For the proof we adopt the strategy in [**bib:Gra.Har.2**]. First of all, note that as homogeneous deformations always satisfy the equations of equilibrium, we only have to verify the boundary conditions in (3.4.3) and (3.4.4). Letting $\boldsymbol{F}=\nabla\boldsymbol{y}$ and $\boldsymbol{C}=\boldsymbol{F}^T\boldsymbol{F}$, we have by the isotropy (P6) that $W(\boldsymbol{F})=\tilde{W}(\boldsymbol{C})$ for some function $\tilde{W}$ that is three times continuously differentiable in a neighborhood of the identity matrix $\boldsymbol{I}$. We have by simple algebra for the Piola-Kirchhoff stress tensor the formula

$$\boldsymbol{P}(\boldsymbol{F})=W_{\boldsymbol{F}}(\boldsymbol{F})=2\boldsymbol{F}\tilde{W}_{\boldsymbol{C}}(\boldsymbol{C}). \tag{3.4.5}$$

Taking into account the form given in (3.4.1) and Frenet-Serret formulas $\boldsymbol{\alpha}''=-k_\theta\boldsymbol{e}_t$, $\boldsymbol{e}_t'=k_\theta\boldsymbol{\alpha}'$, we have

$$\begin{aligned}
&y_{t,t}=1+a(\lambda),\quad y_{t,\theta}=k_\theta(1+a(\lambda))\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}',\quad y_{t,z}=0,\\
&y_{\theta,t}=0,\quad y_{\theta,\theta}=(1+a(\lambda))(1+\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}''),\quad y_{\theta,z}=0,\\
&y_{z,t}=0,\quad y_{z,\theta}=0,\quad y_{z,z}=1-\lambda.
\end{aligned}$$

Consequently, recalling the formula (3.3.2), we have that the gradient $\nabla\boldsymbol{y}$ in the curvilinear coordinates $\boldsymbol{e}_t, \boldsymbol{e}_\theta, \boldsymbol{e}_z$ is represented as:

$$\nabla \boldsymbol{y} = \begin{bmatrix} y_{t,t} & \frac{y_{t,\theta} - k_\theta\, y_\theta}{k_\theta\, t+1} & y_{t,z} \\ y_{\theta,t} & \frac{y_{\theta,\theta} + k_\theta\, y_t}{k_\theta\, t+1} & y_{\theta,z} \\ y_{z,t} & \frac{y_{z,\theta}}{k_\theta\, t+1} & y_{z,z} \end{bmatrix} = \begin{bmatrix} 1+a(\lambda) & 0 & 0 \\ 0 & 1+a(\lambda) & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}, \tag{3.4.6}$$

and thus we can calculate

$$\boldsymbol{C} = \begin{bmatrix} (1+a(\lambda))^2 & 0 & 0 \\ 0 & (1+a(\lambda))^2 & 0 \\ 0 & 0 & (1-\lambda)^2 \end{bmatrix}. \tag{3.4.7}$$

This clearly shows that the matrices $\boldsymbol{F}$ and $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ are diagonal. Additionally, the frame indifference condition (P3) implies that $\tilde{W}\left(\boldsymbol{R}\boldsymbol{C}\boldsymbol{R}^T\right) = \tilde{W}(\boldsymbol{C})$ for all $\boldsymbol{R} \in SO(3)$. Differentiating this relation in $\boldsymbol{R}$ at $\boldsymbol{R} = \boldsymbol{I}$ one concludes that $\tilde{W}_{\boldsymbol{C}}(\boldsymbol{C})$ commutes with $\boldsymbol{C}$ for any $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$ and $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$; see [**bib:Gur.Fri.Ana.**] for details. Consequently, if $\boldsymbol{C}$ is diagonal with distinct diagonal entries, then $\tilde{W}_{\boldsymbol{C}}(\boldsymbol{C})$ must be diagonal as well, while if $\boldsymbol{C}$ is diagonal with possibly equal diagonal entries, then small diagonal perturbations will reduce the situation to the first case, and thus $\tilde{W}_{\boldsymbol{C}}(\boldsymbol{C})$ will be diagonal too as long as $\boldsymbol{C}$ is so. Hence, as in our case, the matrix $\boldsymbol{C}$ is diagonal, also $\tilde{W}_{\boldsymbol{C}}(\boldsymbol{C})$ and the Piola-Kirchhoff stress tensor $\boldsymbol{P}(\boldsymbol{F})$ given in (3.4.5) have to be diagonal as well. Therefore the boundary conditions in (3.4.4) are automatically satisfied, and the one in (3.4.3) reduces to the single equation

$$\hat{W}_{\boldsymbol{C}}\left((1+a(\lambda))^2\left(\boldsymbol{e}_t\otimes\boldsymbol{e}_t + \boldsymbol{e}_\theta\otimes\boldsymbol{e}_\theta\right) + (1-\lambda)^2\boldsymbol{e}_z\otimes\boldsymbol{e}_z\right)\boldsymbol{e}_t\cdot\boldsymbol{e}_t = 0, \tag{3.4.8}$$

which solvability is guaranteed by the implicit function theorem. Indeed, the absence of prestress condition (P2) and (3.4.5) for $\boldsymbol{F} = \boldsymbol{I}$ imply that equality (3.4.8) is fulfilled at the point $(0, a(0)) = (0, 0)$ and for $\lambda = 0$, from (P5) we get

$$\boldsymbol{L}_0 \boldsymbol{e}_t \cdot \boldsymbol{e}_t \neq 0.$$

Consequently, the implicit function theorem guarantees the existence of a C^2 smooth $a(\lambda)$ function in some neighborhood of $\lambda = 0$, which completes existence part of Lemma 3.4.1. Finally, since

$$\boldsymbol{u}_h(\boldsymbol{x}) = \left.\frac{\partial \boldsymbol{y}(\boldsymbol{x}; h, \lambda)}{\partial \lambda}\right|_{\lambda=0} = a'(0)(t + \boldsymbol{\alpha} \cdot \boldsymbol{e}_t)\boldsymbol{e}_t + (a'(0)\boldsymbol{\alpha} \cdot \boldsymbol{e}_\theta)\boldsymbol{e}_\theta - z\boldsymbol{e}_z,$$

does not depend on h and $a(\lambda) \in C^2$ we can conclude that conditions (3.2.3) and (3.2.4) are fulfilled as well.

$\square$

Now in order to identify the minimization problem (3.2.10) and the Korn constant, we need to calculate the linear elastic stress tensor $\nabla \boldsymbol{u}_h(x)$. We have differentiating (3.4.6) in λ at $\lambda = 0$, that

$$\nabla \boldsymbol{u}_h(x) = \begin{bmatrix} a'(0) & 0 & 0 \\ 0 & a'(0) & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

thus we get

$$e(\boldsymbol{u}_h(x)) = \begin{bmatrix} a'(0) & 0 & 0 \\ 0 & a'(0) & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where $a'(0) = \nu$ is the Poisson's ratio. As the material is isotropic we have for the linear elastic stress tensor $\boldsymbol{\sigma}_h$ the formula $\sigma_h^{ij} = \frac{2\mu\nu}{1-2\nu}\delta_{ij}\mathrm{Tr}(e(\boldsymbol{u}_h)) + 2\mu e_{ij}(\boldsymbol{u}_h)$, where μ is the

Lamé parameter. This leads to the form

$$\boldsymbol{\sigma}_h = \begin{bmatrix} 0 & 0 & 0 \\ \\ 0 & 0 & 0 \\ \\ 0 & 0 & -E \end{bmatrix}, \tag{3.4.9}$$

where E is the Young's modulus. Now we can identify the set V_h^d of destabilizing variations given in (3.2.8), which will be the set all variations $\boldsymbol{\phi} \in V_h$ such that

$$-\int_{\Omega_h} (\boldsymbol{\sigma}_h, \nabla\boldsymbol{\phi}^T \nabla\boldsymbol{\phi}) d\boldsymbol{x} = E(\|\phi_{t,z}\|^2 + \|\phi_{\theta,z}\|^2 + \|\phi_{z,z}\|^2) \geq 0,$$

i.e., it coincides exactly with V_h. Finally, the minimization problem in (3.2.10) to be studied reduces to

$$\lambda_{cl}(h) = \inf_{\boldsymbol{\phi}\in V_h} \frac{\int_{\Omega_h} \langle \boldsymbol{L}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle \, d\boldsymbol{x}}{E \cdot \|\mathrm{col}_3(\nabla\boldsymbol{\phi})\|^2}, \tag{3.4.10}$$

which will be addressed in Section 3.4.4.

3.4.2 Korn's Inequalities: Ansatz free lower bounds

Proof of Theorem 3.3.3. **Part (i).** In what follows all the constants $C > 0$ and $C_i > 0$ will depend only on $L, \max k_\theta, k$, and c. While we will do the proof for both spaces V_h and V_h^θ in parallel, we will provide additional details to address the more specific case $V = V_h^\theta$ when needed. The tools and strategies developed in [**bib:Gra.Har.1**, **bib:Gra.Har.4**, **bib:Harutyunyan.1**] will be adopted. Namely, we introduce the so-called simplified gradient $\boldsymbol{G}$, which is obtained by inserting $t = 0$ in the denominators of the second column of (3.3.2):

$$\boldsymbol{G} = \begin{bmatrix} \phi_{t,t} & \phi_{t,\theta} - k_\theta \phi_\theta & \phi_{t,z} \\ \phi_{\theta,t} & \phi_{\theta,\theta} + k_\theta \phi_t & \phi_{\theta,z} \\ \phi_{z,t} & \phi_{z,\theta} & \phi_{z,z} \end{bmatrix}. \tag{3.4.11}$$

For the sake of simplicity, we will be proving all the estimates in Theorem 3.3.3 with the gradient $\nabla\boldsymbol{\phi}$ replaced by the simplified gradient $\boldsymbol{G}$ first, and then we will make the reverse step by virtue of the obvious bounds

$$\|\boldsymbol{G}^{sym} - e(\boldsymbol{\phi})\| \leq \|\boldsymbol{G} - \nabla\boldsymbol{\phi}\| \leq h\|\nabla\boldsymbol{\phi}\| \quad \text{for all} \quad \boldsymbol{\phi} \in H^1(\Omega_h), \tag{3.4.12}$$

where $\boldsymbol{G}^{sym} = \frac{1}{2}(\boldsymbol{G} + \boldsymbol{G}^T)$ is the symmetric part of $\boldsymbol{G}$. We aim to prove the following bounds

$$\|\phi_{t,z}\|^2 \leq \frac{C}{h}\|e(\boldsymbol{\phi})\|^2, \quad \|\phi_{\theta,z}\|^2 \leq \frac{C}{\sqrt{h}}\|e(\boldsymbol{\phi})\|^2, \quad \|\phi_{z,z}\|^2 \leq C\|e(\boldsymbol{\phi})\|^2, \quad \text{for all } \boldsymbol{\phi} \in V. \tag{3.4.13}$$

First of all, note that by density we can without loss of generality assume that the field $\boldsymbol{\phi}$ is of class C^2. For the zz component we have the obvious estimate

$$\|\phi_{z,z}\| \leq \|e(\boldsymbol{\phi})\|. \tag{3.4.14}$$

For the sake of simplicity, denote for any functions $f, g \in H^1(\Omega_h)$ the inner product $(f, g) = \int_{\Omega_h} fg$. In order to estimate the second component of the third column, we can integrate by parts using the boundary conditions on the component ϕ_z in z, and periodicity in θ :

$$\begin{aligned} \|\phi_{\theta,z}\|^2 &= (\phi_{\theta,z}, 2\boldsymbol{G}_{\theta z}^{sym} - \phi_{z,\theta}) \\ &= 2(\phi_{\theta,z}, \boldsymbol{G}_{\theta z}^{sym}) - (\phi_{\theta,z}, \phi_{z,\theta}) \\ &= 2(\phi_{\theta,z}, \boldsymbol{G}_{\theta z}^{sym}) - (\phi_{\theta,\theta}, \phi_{z,z}) \\ &= 2(\phi_{\theta,z}, \boldsymbol{G}_{\theta z}^{sym}) - (\boldsymbol{G}_{\theta\theta}^{sym} - k_\theta\phi_t, \boldsymbol{G}_{zz}^{sym}), \end{aligned}$$

thus we obtain by the Cauchy-Schwartz inequality the estimate

$$\|\phi_{\theta,z}\|^2 \leq 6\|\boldsymbol{G}^{sym}\|(\|\boldsymbol{G}^{sym}\| + \|k_\theta\phi_t\|) \leq C\|\boldsymbol{G}^{sym}\|(\|\boldsymbol{G}^{sym}\| + \|\phi_t\|). \tag{3.4.15}$$

Next, we recall the following Korn and Korn interpolation inequalities proven in Theorems 3.2 and 3.1 of [**bib:Gra.Har.4**] respectively:

$$\|\nabla\boldsymbol{\phi}\|^2 \leq \frac{C}{h^{3/2}}\|e(\boldsymbol{\phi})\|^2, \quad \|\nabla\boldsymbol{\phi}\|^2 \leq C\|e(\boldsymbol{\phi})\|\left(\frac{\|\phi_t\|}{h} + \|e(\boldsymbol{\phi})\|\right), \quad \text{for all} \quad \boldsymbol{\phi} \in V_h. \tag{3.4.16}$$

Note that the inequalities in (3.4.16) were derived in [**bib:Gra.Har.4**] under the boundary conditions in V_h, thus we will need to prove similar to (3.4.16) estimates for the vector space V_h^θ too. To that end, we invoke the following universal Korn interpolation inequality proven in [**bib:Harutyunyan.1**]. The estimate holds for any shells Ω_h with bounded principal curvatures and for any vector fields $\boldsymbol{\phi} \in H^1(\Omega_h)$ (even without any boundary conditions):

$$\|\nabla\boldsymbol{\phi}\|^2 \leq C\left(\frac{1}{h}\|e(\boldsymbol{\phi})\| \cdot \|\phi_t\| + \|\boldsymbol{\phi}\|^2 + \|e(\boldsymbol{\phi})\|^2\right), \qquad \text{for all} \qquad \boldsymbol{\phi} \in H^1(\Omega_h). \tag{3.4.17}$$

Observe that z and θ components of fields $\boldsymbol{\phi} \in V_h^\theta$, satisfy Poincaré inequality in the z direction, thus keeping in mind (3.4.14) and (3.4.15), we obtain from (3.4.17) the simplified estimate

$$\|\nabla\boldsymbol{\phi}\|^2 \leq C\left(\frac{1}{h}\|e(\boldsymbol{\phi})\| \cdot \|\phi_t\| + \|\phi_t\|^2 + \|e(\boldsymbol{\phi})\|^2\right), \qquad \text{for all} \qquad \boldsymbol{\phi} \in V. \tag{3.4.18}$$

Note next that (3.4.18) combined with (3.4.12) and the Cauchy inequality implies the bound

$$\|e(\boldsymbol{\phi})\| \leq 2\|\boldsymbol{G}^{sym}\| + Ch^{1/2}\|\phi_t\|, \tag{3.4.19}$$

thus we get from (3.4.18) an analogous estimate

$$\|\nabla\boldsymbol{\phi}\|^2 \leq C\left(\frac{1}{h}\|\boldsymbol{G}^{sym}\| \cdot \|\phi_t\| + \frac{\|\phi_t\|^2}{h^{1/2}} + \|\boldsymbol{G}^{sym}\|^2\right), \qquad \text{for all} \quad \boldsymbol{\phi} \in V. \tag{3.4.20}$$

In order now to get a sharp estimate on the z derivative of ϕ_t, we extend the cylinder Ω_h to the lower half-space by mirror reflection, and accordingly the t and θ components of the vector field $\boldsymbol{\phi}$ as even functions, and the z component as an odd function, which is possible due to the imposed zero boundary conditions. The extended version are denoted by $\bar{\Omega}_h$ and $\bar{\boldsymbol{\phi}}$, where we clearly have $\bar{\boldsymbol{\phi}} \in H^1(\bar{\Omega}_h)$. The point is that this extension simply doubles all the norms under consideration, thus we can prove all the inequalities under consideration for the extended fields. The functions $\bar{\phi}_t$, $\bar{\phi}_\theta$ and $\bar{\phi}_z$ can then be written in Fourier space in the z variable in H^1 :

$$\begin{cases} \bar{\phi}_t = \sum_{m=0}^{\infty} \bar{\phi}_t^m(t,\theta)\cos(\frac{\pi m z}{L}), \\ \bar{\phi}_\theta = \sum_{m=0}^{\infty} \bar{\phi}_\theta^m(t,\theta)\cos(\frac{\pi m z}{L}), \\ \bar{\phi}_z = \sum_{m=0}^{\infty} \bar{\phi}_z^m(t,\theta)\sin(\frac{\pi m z}{L}). \end{cases} \tag{3.4.21}$$

Observe that in each norm under consideration the Fourier modes separate, thus we can prove all the inequalities under consideration for a fixed Fourier mode with a wavenumber $m \geq 0$. If $m = 0$, then we have $\phi_{t,z}^0 = 0$ and $\phi_{\theta,z}^0 = 0$, thus (3.4.13) holds. Assuming $m > 0$, we have $\|\bar{\phi}_{\theta,z}\| = m\|\bar{\phi}_\theta\|$, thus (3.4.15) implies the bound

$$m^2\|\bar{\phi}_\theta\|^2 \leq C\|\bar{\boldsymbol{G}}^{sym}\|(\|\bar{\boldsymbol{G}}^{sym}\| + \|\bar{\phi}_t\|). \tag{3.4.22}$$

Next we have $e(\bar{\boldsymbol{\phi}})_{\theta\theta} = \bar{\boldsymbol{G}}_{\theta\theta} = \bar{\phi}_{\theta,\theta} + k(\theta)\bar{\phi}_t$, thus we have integrating by parts in θ and utilizing periodicity:

$$\begin{aligned} \int_{\Omega_h} k(\theta)\bar{\phi}_t^2 &= \int_{\Omega_h} \bar{\boldsymbol{G}}_{\theta\theta}^{sym}\bar{\phi}_t + \int_{\Omega_h} (\bar{\phi}_\theta\bar{\phi}_t)_{,\theta} + \int_{\Omega_h} \bar{\phi}_\theta\bar{\phi}_{t,\theta} \\ &= \int_{\Omega_h} \bar{\boldsymbol{G}}_{\theta\theta}^{sym}\bar{\phi}_t + \int_{\Omega_h} \bar{\phi}_\theta\bar{\phi}_{t,\theta} \end{aligned}$$

We can substitute $\bar{\phi}_{t,\theta} = \bar{\boldsymbol{G}}_{t\theta} + k(\theta)\bar{\phi}_\theta$ to get

$$\left\|\sqrt{k(\theta)}\bar{\phi}_t\right\|^2 \leq C\int_{\Omega_h} |\bar{\phi}_t\bar{\boldsymbol{G}}^{sym}_{\theta\theta}| + |\bar{\phi}_\theta\bar{\boldsymbol{G}}_{t\theta}| + |\bar{\phi}_\theta|^2. \tag{3.4.23}$$

Observe that (3.4.22), (3.4.23) and an application of the Cauchy inequality imply the bound

$$\left\|\bar{\phi}_t\right\|^4 \leq C\left(\left\|\bar{\boldsymbol{G}}^{sym}\right\|^4 + \left\|\bar{\phi}_\theta\right\|^2\left\|\bar{\boldsymbol{G}}\right\|^2\right). \tag{3.4.24}$$

Note that (3.4.12) implies that analogous to (3.4.20) inequalities hold for $\nabla\boldsymbol{\phi}$ replaced by $\boldsymbol{G}$, thus owing to (3.4.24), (3.4.22) and the new version off (3.4.20), we discover

$$\left\|\bar{\phi}_t\right\|^2 \leq C\left(\left\|\bar{\boldsymbol{G}}^{sym}\right\|^2 + \frac{1}{m\sqrt{h}}\left\|\bar{\phi}_t\right\|\left\|\bar{\boldsymbol{G}}^{sym}\right\|\right). \tag{3.4.25}$$

It remains to note that, upon an application of the Cauchy inequality, (3.4.25) implies the desired bound

$$\|\bar{\phi}_{t,z}\|^2 = m^2\left\|\bar{\phi}_t\right\|^2 \leq \frac{\left\|\bar{\boldsymbol{G}}^{sym}\right\|^2}{h}. \tag{3.4.26}$$

Observe that on one hand (3.4.26) implies that $\left\|\bar{\phi}_t\right\| \leq \frac{\left\|\bar{\boldsymbol{G}}^{sym}\right\|}{\sqrt{h}}$, thus we get from (3.4.20) that

$$\|\nabla\boldsymbol{\phi}\|^2 \leq \frac{C}{h\sqrt{h}}\|\boldsymbol{G}^{sym}\|^2 \qquad \text{for all} \quad \boldsymbol{\phi}\in V. \tag{3.4.27}$$

Next, owing back to (3.4.12), we derive from (3.4.27) the analogous estimate

$$\|\nabla\boldsymbol{\phi}\|^2 \leq \frac{C}{h\sqrt{h}}\|e(\boldsymbol{\phi})\|^2 \qquad \text{for all} \quad \boldsymbol{\phi}\in V. \tag{3.4.28}$$

On one hand, it remains to note that (3.4.28) combined with (3.4.12), (3.4.14), (3.4.15), and (3.4.26) imply the bound

$$\|\mathrm{col}_3(\nabla\boldsymbol{\phi})\|^2 \leq \frac{C}{h}\|e(\boldsymbol{\phi})\|^2, \qquad \text{for all} \qquad \boldsymbol{\phi}\in V, \tag{3.4.29}$$

which confirms one direction in (3.3.7). The proof of the other direction is by an Ansatz construction and is postponed until Section 3.4.4.

We now turn to the second part, where the curvature $k(\theta)$ vanishes at finitely many points.

Part (ii). We will prove the Ansatz-free lower bounds

$$\|\mathrm{col}_3(\nabla\boldsymbol{\phi})\|^2 \leq \frac{C}{h^{8/5}}\|e(\boldsymbol{\phi})\|^2, \qquad \|\nabla\boldsymbol{\phi}\|^2 \leq \frac{C}{h^{12/7}}\|e(\boldsymbol{\phi})\|^2, \qquad \text{for all} \qquad \boldsymbol{\phi}\in V. \tag{3.4.30}$$

In the sequel, we will be working with the extended vector field $\bar{\boldsymbol{\phi}}$, but will drop the "bar" to keep the notation simpler. Assume first that the wavenumber m. Observe that if $\|\phi_t\| \leq \|e(\boldsymbol{\phi})\|$ then (3.4.18) would imply (3.4.30) so we assume that

$$\|e(\boldsymbol{\phi})\| \leq \|\phi_t\|\,. \tag{3.4.31}$$

We can assume without loss of generality that the domain of the θ variable is $[-1,1]$. Also, for the simplicity of the presentation, we will assume that there is only one point on the curve $\boldsymbol{\alpha}$ where the curvature vanishes, the general case being analogous. Consequently, assume $n=1$ and $\theta_1=0$ in (3.3.6). Let $\delta>0$ be a small parameter yet to be chosen and let $I_0=[-\delta,\delta], I_1=[-1,1]-I_0$, and $\boldsymbol{\phi}^i=\boldsymbol{\phi}\chi_{I_i}$, $i=0,1$. Recall that (3.4.23) implies

$$\|\sqrt{k(\theta)}\phi_t\|^2 \leq C\left(\|\phi_t\|\cdot\|\boldsymbol{G}^{sym}\| + \frac{\|\phi_\theta\|^2}{\epsilon} + \epsilon\,\|\boldsymbol{G}\|^2\right), \tag{3.4.32}$$

for any $\epsilon\in(0,\infty)$. From the obvious bound $k(\theta)\leq C\sqrt{k(\theta)}$ and inequality (3.4.15) we have

$$\|\phi_\theta\|^2 \leq \frac{C}{m^2}(\|\boldsymbol{G}^{sym}\|^2 + \|\boldsymbol{G}^{sym}\|\cdot\|\sqrt{k(\theta)}\phi_t\|). \tag{3.4.33}$$

Therefore combining (3.4.32) and (3.4.33) we arrive at

$$\|\sqrt{k(\theta)}\phi_t\|^2 \leq C\left(\|\phi_t\|\cdot\|\boldsymbol{G}^{sym}\| + \frac{\|\boldsymbol{G}^{sym}\|^2+\|\boldsymbol{G}^{sym}\|\cdot\|\sqrt{k(\theta)}\phi_t\|}{m^2\epsilon} + \epsilon\,\|\boldsymbol{G})\|^2\right),$$

and applying Young's inequality again we arrive at the key estimate

$$\|\sqrt{k(\theta)}\phi_t\|^2 \leq C\left(\|\phi_t\|\cdot\|\boldsymbol{G}^{sym}\| + \frac{\|\boldsymbol{G}^{sym}\|^2}{m^2\epsilon} + \frac{\|\boldsymbol{G}^{sym}\|^2}{m^4\epsilon^2} + \epsilon\|\boldsymbol{G}\|^2\right). \tag{3.4.34}$$

Next we utilize (3.4.34) to bound $\|\phi_t^1\|^2$. We have by (3.3.6) that $\min_{I_1}k(\theta)=k_\delta\geq c\delta^2$, thus substituting $\epsilon = \delta^2\eta^2$ we get from (3.4.34) the bound

$$\begin{aligned} \left\|\phi_t^1\right\|^2 &\leq \frac{1}{k_\delta}\|\sqrt{k(\theta)}\phi_t\|^2 \\ &\leq \frac{\|\phi_t\|^2}{50} + C\left(\left(\frac{1}{\delta^4} + \frac{1}{m^2\delta^4\eta^2} + \frac{1}{m^4\delta^6\eta^4}\right)\|\boldsymbol{G}^{sym}\|^2 + \eta^2\|\boldsymbol{G}\|^2\right). \end{aligned} \tag{3.4.35}$$

In order to bound $\left\|\phi_t^0\right\|^2$ we choose a smooth cut-off function $\varphi\colon [-1,1] \to \mathbb{R}$ supported in $[-2\delta, 2\delta]$ such that

$$\varphi(\theta) = \begin{cases} 1, & \theta \in I_0, \\ 0, & \theta \in [1,-1] - 2I_0, \\ |\varphi'(\theta)| \leq \frac{2}{\delta}, & \theta \in [1,-1]. \end{cases}$$

By Poincare inequality, we have that

$$\begin{aligned} \left\|\phi_t^0\right\|^2 &\leq \|\phi_t\varphi\|^2 \\ &\leq \delta^2\left\|\partial_\theta(\phi_t\varphi)\right\|^2 \\ &\leq \delta^2\left(\left\|\boldsymbol{G}\chi_{2I_0}\right\|^2 + \left\|k(\theta)\phi_\theta\chi_{2I_0}\right\|^2 + \frac{4}{\delta^2}\left\|\phi_t\chi_{I_1}\right\|^2\right) \\ &\leq C\delta^2\left(\|\boldsymbol{G}\|^2 + \|\phi_\theta\|^2\right) + 4\left\|\phi_t^1\right\|^2 \\ &\leq C\delta^2\|\boldsymbol{G}\|^2 + 4\left\|\phi_t^1\right\|^2, \end{aligned} \tag{3.4.36}$$

where we used the obvious bound $\|\phi_\theta\| \leq C\|\boldsymbol{G}\|$. Putting now (3.4.35) and (3.4.36) together we arrive at

$$\|\phi_t\|^2 \leq C(\delta^2 + \eta^2)\|\boldsymbol{G}\|^2 + C\left(\frac{1}{\delta^4} + \frac{1}{m^2\delta^4\eta^2} + \frac{1}{m^4\delta^6\eta^4}\right)\|\boldsymbol{G}^{sym}\|^2 \tag{3.4.37}$$

Observe that the universal interpolation inequality in (3.4.18) together with the bounds (3.4.14), (3.4.22), and (3.4.31) imply another key estimate:

$$\|\boldsymbol{G}\|^2 \leq \frac{C}{h}\|\boldsymbol{G}^{sym}\|\|\phi_t\|. \tag{3.4.38}$$

In order to estimate $\|\boldsymbol{G}\|$, we first combine (3.4.37) and (3.4.38) to get by an application of the Cauchy inequality:

$$\|\phi_t\|^2 \leq C\left(\frac{\delta^4+\eta^4}{h^2}+\frac{1}{\delta^4}+\frac{1}{m^2\delta^4\eta^2}+\frac{1}{m^4\delta^6\eta^4}\right)\|\boldsymbol{G}^{sym}\|^2 . \tag{3.4.39}$$

Finally keeping in mind that $m \geq 1$, we choose $\eta=\delta=h^{1/7}$ to optimize (3.4.39). This gives

$$\|\boldsymbol{G}\|^2 \leq \frac{C}{h^{12/7}}\|\boldsymbol{G}^{sym}\|^2 , \tag{3.4.40}$$

and consequently also the second inequality in (3.4.30) through (3.4.12). In order to prove the first inequality in (3.4.30), we note that (3.4.38) implies in particular the bound

$$\|\phi_{t,z}\|^2 \leq \frac{C}{h}\|\boldsymbol{G}^{sym}\|\|\phi_t\|,$$

which is equivalent to

$$m^2\|\phi_t\|^2 \leq \frac{C}{m^2h^2}\|\boldsymbol{G}^{sym}\|^2 . \tag{3.4.41}$$

Next we choose $\eta=\delta$ in (3.4.39) to get the simplified variant

$$m^2\|\phi_t\|^2 \leq C\left(\frac{m^2\delta^4}{h^2}+\frac{m^2}{\delta^4}+\frac{1}{\delta^6}+\frac{1}{m^2\delta^{10}}\right)\|\boldsymbol{G}^{sym}\|^2 . \tag{3.4.42}$$

We need to obtain an optimal estimate for $m^2\|\phi_t\|^2$ from (3.4.41) and (3.4.42) regardless of the value of m, by choosing the parameter $\delta>0$ appropriately. We choose δ so that the values of the first and last summands on the right-hand side of (3.4.39) coincide: $\frac{m^2\delta^4}{h^2}=\frac{1}{m^2\delta^{10}}$. This gives $\delta=\frac{h^{1/7}}{m^{2/7}}$ and (3.4.39) reduces to

$$m^2\|\phi_t\|^2 \leq C\left(\frac{m^{6/7}}{h^{10/7}}+\frac{m^{22/7}}{h^{4/7}}+\frac{m^{12/7}}{h^{6/7}}\right)\|\boldsymbol{G}^{sym}\|^2 . \tag{3.4.43}$$

It remains to note that if $m \geq \frac{1}{h^{1/5}}$, then (3.4.38) would give

$$m^2\|\phi_t\|^2 \leq \frac{C}{h^{8/5}}\|\boldsymbol{G}^{sym}\|^2 . \tag{3.4.44}$$

If otherwise $m \geq \frac{1}{h^{1/5}}$, then we would get the same estimate (3.4.41) this time from (3.4.40) instead. Consequently (3.4.41) is fulfilled independently of $m \in \mathbb{N}$. Finally putting together (3.4.30) and (3.4.41) we arrive at the first estimate in (3.4.26) in the case $m \geq 1$. In the case $m = 0$ there is no $z-$variable dependence, thus obviously have $\text{col}_3(\nabla \boldsymbol{\phi}) = 0$ and $\phi_\theta = \phi_z = 0$, hence both lower bounds in (3.3.8) become trivial. This completes the proof of the Ansatz-free lower bound parts of Theorem 3.3.3.

$\square$

3.4.3 The Ansätze

Part (i). An Ansatz realizing the asymptotics in (3.3.7) can be constructed in numerous ways. For instance, one Kirchhoff-like Ansatz which can be as such was constructed in [**bib:Harutyunyan.2**], see also [**bib:Harutyunyan.3**]. We present it here for the convenience of the reader. One chooses

$$\begin{cases} \phi_t = W(\frac{\theta}{\sqrt{h}}, \frac{z-L/2}{\sqrt{h}}), \\ \\ \phi_\theta = -\frac{t}{\sqrt{h}} \cdot W_{,\theta}(\frac{\theta}{\sqrt{h}}, \frac{z-L/2}{\sqrt{h}}), \\ \\ \phi_z = -\frac{t}{\sqrt{h}} \cdot W_{,z}(\frac{\theta}{\sqrt{h}}, \frac{z-L/2}{\sqrt{h}}), \end{cases} \tag{3.4.45}$$

where W is a smooth function compactly supported in $(0,p) \times (0,L)$. Also, W is chosen so that W and all its first and second order derivatives be of order one. It is then easy to see that one gets for this choice $\|e(\boldsymbol{\phi})\| \sim h$ and $\|\text{col}_3(\nabla\boldsymbol{\phi})\| = \sqrt{h}$ as $h \to 0$. This gives the asymptotics

$$\frac{\|e(\boldsymbol{\phi})\|^2}{\|\text{col}_3(\nabla\boldsymbol{\phi})\|^2} \sim h,$$

as $h \to 0$, i.e., (3.3.7).

Part (ii) a): Second estimate in (3.3.8). The idea and the novelty, in this case, is to localize the Ansätze at the zero curvature points and make use of the fact that the curvature vanishes. Namely, assume again the domain of the variable θ is $[-1,1]$ and the point $\theta = 0$ is a zero curvature point, i.e., $\theta(0) = 0$ and $c\theta^2 \leq k(\theta) \leq \frac{1}{c}\theta^2$ for all $\theta \in [-1,1]$. Let $\delta = h^\alpha$ be a small parameter ($\alpha > 0$) yet to be chosen. We adjust (3.4.45) as

$$\begin{cases} \phi_t = W(\frac{\theta}{\delta}, \frac{z-L/2}{\delta}), \\ \\ \phi_\theta = -\frac{t}{\delta} \cdot W_{,\theta}(\frac{\theta}{\delta}, \frac{z-L/2}{\delta}), \\ \\ \phi_z = -\frac{t}{\delta} \cdot W_{,z}(\frac{\theta}{\delta}, \frac{z-L/2}{\delta}), \end{cases} \tag{3.4.46}$$

where $W(\theta, z)$ is again a smooth function, compactly supported on $D = (-1,1)^2$ such that W and all its first and second order derivatives be of order one. Computing the simplified gradient $\boldsymbol{G}$ in (3.4.11) we get:

$$\boldsymbol{G} = \begin{bmatrix} 0 & \frac{W_{,\theta} - k(\theta) t W_{,\theta}}{\delta} & \frac{W_{,z}}{\delta} \\ \\ -\frac{W_{,\theta}}{\delta} & \frac{tW_{,\theta\theta}}{\delta^2} + k(\theta)W & \frac{tW_{,\theta z}}{\delta^2} \\ \\ -\frac{W_{,z}}{\delta} & \frac{tW_{,z\theta}}{\delta^2} & -\frac{tW_{,zz}}{\delta^2} \end{bmatrix}$$

and

$$\boldsymbol{G}^{sym} = \begin{bmatrix} 0 & -\frac{k(\theta) t W_{,\theta}}{2\delta} & 0 \\ \\ -\frac{k(\theta) t W_{,\theta}}{2\delta} & \frac{tW_{,\theta\theta}}{\delta^2} + k(\theta)W & \frac{2tW_{,\theta z}}{\delta^2} \\ \\ 0 & \frac{2tW_{,\theta z}}{\delta^2} & -\frac{tW_{,zz}}{\delta^2} \end{bmatrix}.$$

Now choosing $\delta = h^{1/4}$ and recalling that $k(\theta) \sim \theta^2$, it is easy to see that $\|\mathrm{col}_3(\nabla\boldsymbol{\phi})\|^2 = h$, and $\|e(\boldsymbol{\phi})\|^2 \sim h^{5/2}$, as $h \to 0$. This realizes the asymptotics

$$\frac{\|e(\boldsymbol{\phi})\|^2}{\|\mathrm{col}_3(\nabla\boldsymbol{\phi})\|^2} \sim h^{3/2}$$

as $h \to 0$, i.e., the right-hand side of the second inequality in (3.3.8).

Remark 3.4.2. *It is easy to see that if the curvature $k(\theta)$ has a zero of order $\beta \geq 2$, then by choosing $\delta = h^{\frac{1}{\beta+2}}$, the Ansatz in (3.4.43) would actually give us*

$$\frac{\|e(\boldsymbol{\phi})\|^2}{\|\mathrm{col}_3(\nabla\boldsymbol{\phi})\|^2} \sim h^{\frac{2\beta+2}{\beta+2}},$$

as $h \to 0$.

Part (ii) B): First estimate in (3.3.8). We construct the Ansatz utilizing the idea of linearization in t suggested in [**bib:Gra.Har.1**]. Namely, we seek the Ansatz in the following form

$$\begin{cases} \phi_t = u, \\ \phi_\theta = tv_1 + v_2, \\ \phi_z = tw_1 + w_2, \end{cases} \tag{3.4.47}$$

where the functions u, v_1, v_2, w_1 and w_2 depend only on θ and z. The simplified gradient will then be given by

$$\boldsymbol{G} = \begin{bmatrix} 0 & u_{,\theta} - k(\theta)\,(tv_1 + v_2) & u_{,z} \\ v_1 & tv_{1,\theta} + v_{2,\theta} + k(\theta)u & tv_{1,z} + v_{2,z} \\ w_1 & tw_{1,\theta} + w_{2,\theta} & tw_{1,z} + w_{2,z} \end{bmatrix}.$$

In order to make the symmetric part of the gradient small, we choose the functions u, v_1, v_2, w_1, and w_2 to satisfy the relationships

$$v_1 = -u_{,\theta}, \qquad w_1 = -u_{,z}, \qquad v_{2,\theta} = -k(\theta)u. \tag{3.4.48}$$

This will reduce the simplified gradient to

$$\boldsymbol{G} = \begin{bmatrix} 0 & u_{,\theta} - k(\theta)\,(tv_1 + v_2) & u_{,z} \\ v_1 & tv_{1,\theta} & tv_{1,z} + v_{2,z} \\ w_1 & tw_{1,\theta} + w_{2,\theta} & tw_{1,z} + w_{2,z} \end{bmatrix}, \tag{3.4.49}$$

and the symmetric part will be

$$\boldsymbol{G}^{sym} = \begin{bmatrix} 0 & -\frac{1}{2}k(\theta)\,(tv_1+v_2) & 0 \\ -\frac{1}{2}k(\theta)\,(tv_1+v_2) & tv_{1,\theta} & \frac{1}{2}(t(v_{1,z}+w_{1,\theta})+v_{2,z}+w_{2,\theta}) \\ 0 & \frac{1}{2}(t(v_{1,z}+w_{1,\theta})+v_{2,z}+w_{2,\theta}) & tw_{1,z}+w_{2,z} \end{bmatrix} \tag{3.4.50}$$

Thus to make the θz component small, we need a new relationship:

$$w_{2,\theta} = -v_{2,z}, \tag{3.4.51}$$

which simplifies $\boldsymbol{G}^{sym}$ further to

$$e(\boldsymbol{G}) = \begin{bmatrix} 0 & -\frac{1}{2}k(\theta)\,(tv_1+v_2) & 0 \\ -\frac{1}{2}k(\theta)\,(tv_1+v_2) & tv_{1,\theta} & \frac{t}{2}(v_{1,z}+w_{1,\theta}) \\ 0 & \frac{t}{2}(v_{1,z}+w_{1,\theta}) & tw_{1,z}+w_{2,z} \end{bmatrix}. \tag{3.4.52}$$

Let now $W(\theta, z)$ be a smooth compactly supported function on $D = (-1, 1) \times (0, L)$ such that W and all its first, second, and third order derivatives be of order one. We choose

$$\begin{cases} w_2 = -k^2(\theta)W_{,z}(\frac{\theta}{\delta}, z) \\ v_2 = 2k(\theta)k'(\theta)W(\frac{\theta}{\delta}, z) + \frac{k(\theta)^2}{\delta}W_{,\theta}(\frac{\theta}{\delta}, z) \\ u = -\frac{2k'^2(\theta)}{k(\theta)}W(\frac{\theta}{\delta}, z) - 2k''(\theta)W(\frac{\theta}{\delta}, z) - \frac{4k'(\theta)}{\delta}W_{,\theta}(\frac{\theta}{\delta}, z) - \frac{k(\theta)}{\delta^2}W_{,\theta\theta}(\frac{\theta}{\delta}, z) \\ v_1 = -u_{,\theta} \\ w_1 = -u_{,z}, \end{cases} \tag{3.4.53}$$

where $\delta = h^\beta$ ($\beta > 0$) is a small parameter yet to be chosen. From the fact that $\boldsymbol{\alpha}$ is of class C^5 and $k(0) = k'(0) = 0$, $k''(0) > 0$, we have for small enough h the obvious bounds $|k'''(\theta)| + |k''(\theta)| \leq C$, $|k'(\theta)| \leq C|\theta|$, $c\theta^2 \leq |k(\theta)| \leq \frac{1}{c}\theta^2$, for $\theta \in (-\delta, \delta)$. Consequently, we can easily verify that

$$\frac{\|\boldsymbol{G}^{sym}\|^2}{\|\boldsymbol{G}\|^2} \sim \max\{h^{6\beta}(h^{2-2\beta}+h^{6\beta}), h^{2-2\beta}, h^2+h^{10\beta}\}$$

as $h \to 0$. In order to minimize the left hand side we choose $\beta = 1/6$ that gives the desired result

$$\frac{\|e(\boldsymbol{\phi})\|^2}{\|\nabla\boldsymbol{\phi}\|^2} \sim h^{\frac{5}{3}}.$$

3.4.4 The buckling load

Proof of Theorem 3.3.1. In this section, we will study the stability of the homogeneous trivial branch given by (in (3.4.1))

$$y_t = (1+a(\lambda))(t+\boldsymbol{\alpha}(\theta)\cdot\boldsymbol{e}_t(\theta)), \qquad y_\theta = (1+a(\lambda))\boldsymbol{\alpha}(\theta)\cdot\boldsymbol{e}_\theta(\theta), \qquad y_z = (1-\lambda)z, \tag{3.4.54}$$

within the theory of buckling of slender structures presented in Section 3.2. Let us start by verifying that the conditions for the applicability of the theory are fulfilled. These consist of the conditions in Definition 3.2.1 and Theorem 3.2.5. The conditions in Definition 3.2.1 have already been verified by Lemma 3.4.1, while the conditions in Theorem 3.2.5 immediately follow from Theorem 3.3.3 and Remark 3.3.4. This implies that in fact, we can calculate the buckling load as in (3.2.10), which we have proven reduces to the minimization problem in (3.4.10), i.e.,

$$\lambda(h) = \inf_{\boldsymbol{\phi}\in V} \frac{\int_{\Omega_h} \langle \boldsymbol{L}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle\, d\boldsymbol{x}}{E\cdot\|\mathrm{col}_3(\nabla\boldsymbol{\phi})\|^2}. \tag{3.4.55}$$

By properties (P4) and (P5) the tensor $\boldsymbol{L}_0$ is positive definite on symmetric matrices, i.e., there exists a positive constant $C > 0$ such that

$$\frac{1}{C}\|e(\boldsymbol{\phi})\|^2 \leq \int_{\Omega_h} \langle \boldsymbol{L}_0 e(\boldsymbol{\phi}), e(\boldsymbol{\phi})\rangle\, d\boldsymbol{x} \leq C\|e(\boldsymbol{\phi})\|^2,$$

for all $\boldsymbol{\phi} \in H^1(\Omega_h)$. Hence (3.4.52) yields the asymptotics

$$\lambda(h) \sim \inf_{\boldsymbol{\phi} \in V} \frac{\|e(\boldsymbol{\phi})\|^2}{\|\mathrm{col}_3(\nabla \boldsymbol{\phi})\|^2}, \tag{3.4.56}$$

and Theorem 3.3.1 immediately follows from Theorem 3.3.3.

□

3.5 Explicit trivial branch for Neo-Hookean materials

In this section, we provide an explicit form for the function $a(\lambda)$ in (3.4.1) and (3.4.8) for the special case of Neo-Hookean solids. In that case the Piola-Kirchhoff stress tensor is given by

$$\boldsymbol{P}(\boldsymbol{F}) = \tilde{\mu}(\boldsymbol{F} - \boldsymbol{F}^{-T}) + 2\tilde{\lambda}(J-1)J\boldsymbol{F}^{-T}, \quad J = \det(\boldsymbol{F}),$$

where $\tilde{\mu}$ and $\tilde{\lambda}$ are the Lamé parameters[4]. Letting $b(\lambda) = (1 + a(\lambda))^2$, the system (3.4.3) will reduce to

$$b(\lambda)^2(1-\lambda)^2 - b(\lambda)(1 - \lambda - \frac{\tilde{\mu}}{\tilde{\lambda}}) - \frac{\tilde{\mu}}{\tilde{\lambda}} = 0,$$

which gives the solution

$$b(\lambda) = \frac{1 - \lambda - \frac{\tilde{\mu}}{\tilde{\lambda}} + \sqrt{(1 - \lambda - \frac{\tilde{\mu}}{\tilde{\lambda}})^2 + 4(1-\lambda)^2\frac{\tilde{\mu}}{\tilde{\lambda}}}}{2(1-\lambda)^2},$$

and for a we get

$$a(\lambda) = \frac{\sqrt{1 - \lambda - \frac{\tilde{\mu}}{\tilde{\lambda}} + \sqrt{(1 - \lambda - \frac{\tilde{\mu}}{\tilde{\lambda}})^2 + 4(1-\lambda)^2\frac{\tilde{\mu}}{\tilde{\lambda}}}}}{\sqrt{2}(1-\lambda)} - 1.$$

[4]Note that we are using the symbol "tilde" over the letters to avoid confusion between the Lamé parameter $\tilde{\lambda}$ and the loading parameter λ.

Chapter 4

Weighted Korn and Poincaré inequalities

4.1 Introduction

Comment 1. Add importance of Weighted Korn/ Poincare inequalities Papers: [**wKorn1**, **wKorn2**, **wKorn3**, **surveyQuasilinearSystems**]

Most of the results were inspired by the paper of Sergi Conti and Barbara Zwicknagl, [**conti0**]. In this paper it was proven a simpler weighed Poincaré inequality, which was crucial to a new proof of the Korn Inequality. In this chapter, we will use similar techniques to prove a stronger version of the weighted Poincaré inequality (4.4) that will imply a weighted version of the Korn inequality (4.5). Additionally, we will also introduce a new weighted Korn Inequality for plates in 4.6.

4.2 Main results and Important Definitions

To be able to prove weighted Poincaré and Korn inequalities we need some conditions on the domain. In particular, we will need the domain to be uniformly Lipschitz. We will use the definition used [**conti0**] of uniformly Lipschitz domains:

Definition 4.2.1 (Uniformly Lipschitz domains). *Let $L, R > 0$. An open set $\Omega \subseteq \mathbb{R}^n$ is (L, R)-Lipschitz if there is $\varepsilon > 0$ such that:*

1. $|\boldsymbol{x} - \boldsymbol{y}| < R\varepsilon$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \Omega$*;*
2. *For each* $\boldsymbol{x} \in \partial\Omega$ *there are* $f_{\boldsymbol{x}} \in \operatorname{Lip}(\mathbb{R}^{n-1}; \mathbb{R})$ *with* $\operatorname{Lip}(f_{\boldsymbol{x}}) \leq L$ *and an isometry* $A_{\boldsymbol{x}} : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $B_\varepsilon(\boldsymbol{x}) \cap \Omega = B_\varepsilon(\boldsymbol{x}) \cap V_{\boldsymbol{x}}$*, where*

$$V_{\boldsymbol{x}} := A_{\boldsymbol{x}} \left\{ (\boldsymbol{y}', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : y_n < f_{\boldsymbol{x}}(\boldsymbol{y}') \right\}$$

So for uniform Lipschtiz domains, we can prove the Weighted Poincaré inequality and its respectively non-linear version, uniform rigidity:

Theorem **4.2.2** (Weighted Korn Inequality and Uniform Rigidity). *Let $\Omega \subset \mathbb{R}^n$ be a connected, bounded (L, R)-Lipschitz set, Γ an nonempty close subset of $\partial\Omega$ and $\delta_\Gamma(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, \partial\Gamma)$. Then for any $\boldsymbol{u} \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^k)$, with $p \in [1, \infty)$, and every $\alpha \geq 0$ there is are $S \in \mathbb{R}^{n\times n}_{\mathrm{skw}}$ and $R \in \mathrm{SO}(n)$ such that*

$$\|\delta_\Gamma(\boldsymbol{x})^\alpha (\nabla \boldsymbol{u} - S)\|_{L^p(\Omega)} \leq c(n, p, \alpha, L, R) \left\|\delta_\Gamma(\boldsymbol{x})^\alpha e(\boldsymbol{u})\right\|_{L^p(\Omega)}.$$

and

$$\|\delta_\Gamma(\boldsymbol{x})^\alpha (\nabla \boldsymbol{u} - R)\|_{L^p(\Omega)} \leq c(n, p, \alpha, L, R) \|\delta_\Gamma(\boldsymbol{x})^\alpha \operatorname{dist}(\nabla \boldsymbol{u}, \mathrm{SO}(n))\|_{L^p(\Omega)}.$$

Additionally, we can prove similar results for plates $\Omega_h = \Omega \times I_h$, where the middle surface Ω satisfies the same condition as in the previous theorem.

Theorem **4.2.3** (Weighted Korn Inequality and Uniform Rigidity for Plates). *Let $\Omega_h = \Omega \times I_h \subset \mathbb{R}^3$ be a shell such that Ω is a connected, bounded (L, R)-Lipschitz set and $I_h = [-h, h]$ for small $h > 0$. Additionally, let $\delta'(\boldsymbol{x}) = \delta_{\partial\Omega \times I_h} = \operatorname{dist}(\boldsymbol{x}, \partial\Omega \times I_h)$ and $p \in (1, \infty)$. Then for any $\boldsymbol{u} \in W^{1,p}(\Omega_h; \mathbb{R}^n)$ and any $\alpha \geq 0$, there are $R \in \mathrm{SO}(3)$ and*

$S \in \mathbb{R}^{3\times 3}_{\mathrm{skw}}$ *such that*

$$\|\delta'(\boldsymbol{x})^{\alpha}(\nabla \boldsymbol{u} - S)\|_{L^p(\Omega)} \leq \frac{c(n,p,\alpha,L,R)}{h} \|\delta'(\boldsymbol{x})^{\alpha} e(\boldsymbol{u})\|_{L^p(\Omega)} .$$

and

$$\|\delta'(\boldsymbol{x})^{\alpha}(\nabla \boldsymbol{u} - R)\|_{L^p(\Omega)} \leq \frac{c(n,p,\alpha,L,R)}{h}\|\delta'(\boldsymbol{x})^{\alpha} \operatorname{dist}(\nabla \boldsymbol{u}, \mathrm{SO}(n))\|_{L^p(\Omega)}.$$

In order to prove these results, a new Weighted Poincaré inequality was necessary. Variations of Poincaré inequalities are important tools in proving the existence and uniqueness of several minimization problems, so we believe that this result deserves to be stated on its own.

Theorem **4.2.4** (Weighted Poincaré Inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a connected, bounded (L,R)-Lipschitz set, Γ an nonempty closed subset of $\partial\Omega$ and $\delta_\Gamma(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, \partial\Gamma)$. Then for any $\boldsymbol{u} \in W^{1,p}_{\mathrm{loc}}\left(\Omega; \mathbb{R}^k\right)$, with $p \in [1,\infty)$, and every $\alpha \geq 0$ there is $\boldsymbol{a} \in \mathbb{R}^k$ such that:*

$$\|\delta_\Gamma(\boldsymbol{x})^{\alpha}(\boldsymbol{u} - \boldsymbol{a})\|_{L^p(\Omega)} \leq c(n,p,\alpha,L,R)\|\delta_\Gamma(\boldsymbol{x})^{1+\alpha}\nabla \boldsymbol{u}\|_{L^p(\Omega)}.$$

Remark 4.2.5. *Theorems 4.2.2 and 4.2.4 will be used several times for the particular case where $\Gamma = \partial\Omega$. Additionally, for the last result, if we consider $\alpha = 0$ we achieve the Weighted Poincare inequality [**conti0**], and since we have that $\delta_\Gamma \leq \operatorname{diam}(\Omega)$ the traditional inequality is also a corollary of this theorem. For both Korn inequalities, if we assume $\alpha = 0$ we also get the traditional Korn inequality.*

4.3 Auxiliary Results and Lemmas

The main results of this chapter are inspired by the proofs of [**conti0**], so the first step is to generalize Lemma 5.6 from that paper, which can be considered a Weighted Poincaré in 1D.

Lemma **4.3.1.** *Let $I = (a,b) \subseteq \mathbb{R}$ and $E = [a, a+\epsilon]$ with $0 < \epsilon < b - a$. For all*

$\varphi \in C^1(I), \alpha > 0 \in \mathbb{R}, A \in \mathbb{R}$ *and* $p \in [1, \infty)$ *we have that*

$$\int_I |(b-x)^\alpha(\phi - A)|^p dx \le 2^p \left(\frac{|I|}{|E|}\right)^{\alpha p+1} \int_E |(b-x)^\alpha(\beta - A)|^p dx$$
$$+ 2^p \left(\frac{|I|}{|E|}\right)^{\alpha p+1} \left(\frac{p}{\alpha+1}\right)^p \int_I |(b-x)^{\alpha+1}\phi'|^p dx.$$

Proof. Step 1: Let $\beta = \phi(a)$, then by the fundamental theorem of calculus applied to $(\phi - \beta)^p$ we get the following bound

$$|\phi - \beta|^p(x) \le p \int_a^x |\phi(t) - \beta|^{p-1} |\phi'(t)| \, dt.$$

Now if we multiply last equation by $(b-x)^{\alpha p}$ and integrate over I we get

$$\int_I |(b-x)^\alpha(\phi - \beta)|^p(x) \le p \int_I \int_a^x (b-x)^{\alpha p} |\phi(t) - \beta|^{p-1} |\phi'(t)| \, dt dx.$$

Using Fubini's theorem we can conclude that

$$\int_I |(b-x)^\alpha(\phi - \beta)|^p(x) \le \frac{p}{\alpha+1} \int_I (b-t)^{\alpha p+1} |\phi(t) - \beta|^{p-1} |\phi'(t)| \, dt.$$

Since $\alpha p + 1 = \alpha(p-1) + (1+\alpha)$, using Holder's inequality we have that

$$\int_I |(b-x)^\alpha(\phi - \beta)|^p dx \le \frac{p}{\alpha+1} \left\| ((b-t)^\alpha(\phi(t) - \beta))^{p-1} \right\|_{L^{p'}(I)} \left\| (t-b)^{1+\alpha}\phi'(t) \right\|_{L^p(I)}.$$

where $p' = \frac{p}{p-1}$. Additionally, since

$$\left\| ((b-t)^\alpha(\phi(t) - \beta))^{p-1} \right\|_{L^{p'}(I)} = \| (b-t)^\alpha(\phi(t) - \beta) \|_{L^p(I)}^{p-1},$$

we can conclude the first step with the following inequality

$$\| (b-x)^\alpha(\phi - \beta) \|_{L^p(I)} \le \frac{p}{\alpha+1} \left\| (b-x)^{1+\alpha}\phi' \right\|_{L^p(I)}. \tag{4.3.1}$$

Step 2: Since we need to prove the inequality for any $A \in \mathbb{R}$ and not only β, the goal of this step is to analyze the integral of $|(b-x)^\alpha(\beta - A)|^p$. Since β and A are constants we can compute $\int_I |(b-x)^\alpha(\beta - A)|^p dx$ and $\int_E |(b-x)^\alpha(\beta - A)|^p dx$ exactly, and conclude that

$$\frac{\int_I |(b-x)^\alpha(\beta-A)|^p dx}{\int_E |(b-x)^\alpha(\beta-A)|^p dx} = \frac{|I|^{\alpha+1}}{|E|^{\alpha+1}} \geq 1,$$

and consequently

$$\int_I |(b-x)^\alpha(\beta-A)|^p dx \leq \left(\frac{|I|}{|E|}\right)^{\alpha+1} \int_E |(b-x)^\alpha(\beta-A)|^p dx.$$

Step 3: To conclude the proof we just need to apply triangular inequality two more times and use inequality 4.3.1

$$\begin{aligned}
\int_I |(b-x)^\alpha(\phi-A)|^p dx &\leq 2^{p-1}\int_I |(b-x)^\alpha(\phi-\beta)|^p dx + 2^{p-1}\int_I |(b-x)^\alpha(\beta-A)|^p dx \\
&\leq 2^{p-1}\int_I |(b-x)^\alpha(\phi-\beta)|^p dx + 2^{p-1}\left(\frac{|I|}{|E|}\right)^{\alpha p+1}\int_E |(b-x)^\alpha(\beta-A)|^p dx \\
&\leq 2^{p}\left(\frac{|I|}{|E|}\right)^{\alpha p+1}\left(\int_E |(b-x)^\alpha(\phi-A)|^p dx + \int_I |(b-x)^\alpha(\phi-\beta)|^p dx\right) \\
&\leq 2^{p}\left(\frac{|I|}{|E|}\right)^{\alpha p+1}\left(\int_E |(b-x)^\alpha(\phi-A)|^p dx + \left(\frac{p}{\alpha+1}\right)^p\int_I |(b-x)^{\alpha+1}\phi'|^p dx\right).
\end{aligned}$$

□

The next step is to use the properties of uniformly Lipschitz domains and the previous result to prove something similar in higher dimensions, close to the boundary.

Lemma **4.3.2.** *Let $\Omega \subset \mathbb{R}^n$ be (L,R)-Lipschitz, ε as in Definition 4.2.1, $\boldsymbol{x}_* \in \Gamma \subset \partial\Omega$, $r \in (0, \varepsilon/(4+4L)]$, $p \in [1,\infty)$ and $\delta_\Gamma(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x},\Gamma)$. For any $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ there is $a \in \mathbb{R}$ such that*

$$\|\delta_\Gamma(\boldsymbol{x})^\alpha(u-a)\|_{L^p(\Omega\cap B_r(\boldsymbol{x}_*))} \leq c(n,p,\alpha,L)\|\delta_\Gamma(\boldsymbol{x})^{(\alpha+1)}\nabla u\|_{L^p(\Omega)}.$$

Proof. By the definition of $(L,R)-Lipschitz$ we have that $B(\boldsymbol{x}_*,\varepsilon)\cap\Omega = B(\boldsymbol{x}_*,\varepsilon)\cap V$, where

$$V := A\left\{(\boldsymbol{y}', y_n) \in \mathbb{R}^{n-1}\times\mathbb{R} : y_n < f(\boldsymbol{y}')\right\},$$

for an isometry A and an L-Lipschitz function, f. Since all the results are invariant under rotation and translations we can, *w.l.o.g.* assume that $A = I$, $\boldsymbol{x}_* = 0$ and $f(\mathbf{0}) = 0$ to simplify the notation.

Let $\tau := rL \in (0, \varepsilon/4)$, and consider the cylinder $T := B'_r \times (-3\tau, -2\tau)$ where B'_r is the projection of $B_r(\boldsymbol{x}_*)$ into $x_n = 0$. For any $\boldsymbol{x}' \in B'_r$ we have $f(\boldsymbol{x}') \geq -rL = -\tau$, and therefore $4\tau \geq f(\boldsymbol{x}') - x_n \geq \tau$ for all $(\boldsymbol{x}', x_n) \in T$. Further, since $(3h)^2 + r^2 \leq (9L^2+1)r^2 \leq \varepsilon^2$, we obtain that T is still inside $B_\varepsilon \cap V$ as we can see in the figure 4.1.

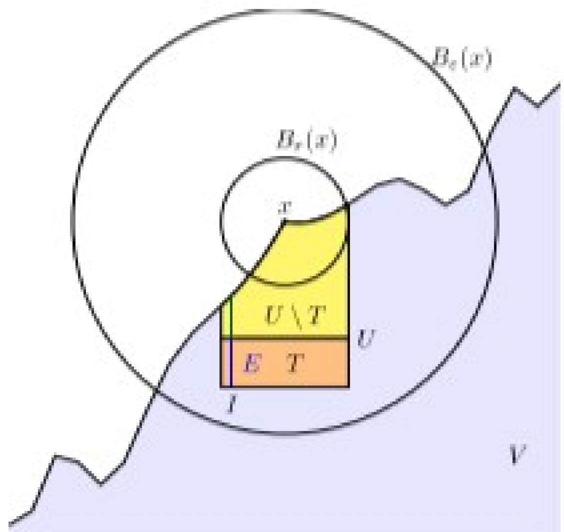


Figure 4.1: Sketch of the geometry in the construction of Lemma 4.3.2

Additionally, since $\operatorname{diam}(T) \leq ch$ by the usual Poincaré inequality there exist $a \in \mathbb{R}$ with

$$\begin{aligned}\int_T |f(\boldsymbol{x}') - x_n|^{\alpha p} |u - a|^p d\boldsymbol{x} &\leq 4^{\alpha p} \tau^{\alpha p} \int_T |u - a|^p d\boldsymbol{x} \\ &\leq 4^{\alpha p} c(n, p, L) \tau^{(1+\alpha)p} \int_T |\nabla u|^p d\boldsymbol{x} \\ &\leq 4^{\alpha p} c(n, p, L) \int_T |f(\boldsymbol{x}') - x_n|^{(1+\alpha)p} |\nabla u|^p d\boldsymbol{x}.\end{aligned}$$

For the next step we will apply Lemma 4.3.1 to $u(\boldsymbol{x}', \cdot)$ for each $\boldsymbol{x}' \in B'_r$, with $I = (-3\tau, f(\boldsymbol{x}'))$ and $E = (-3\tau, -2\tau)$. Since , $\tau = \mathcal{L}^1(E) \leq \mathcal{L}^1(I) \leq 4\tau$, we get

$$\int_I |f(\boldsymbol{x}') - x_n|^{\alpha p} |u(\boldsymbol{x}', x_n) - a|^p dx_n \leq c(n, p, \alpha, L) \int_I |f(\boldsymbol{x}') - x_n|^{(\alpha+1)p} |\nabla u|^p dx_n$$

$$+ c(n,p,\alpha,L)\int_E |f(\boldsymbol{x}')-x_n|^{\alpha p}|u(\boldsymbol{x}',x_n)-\alpha|^p dx_n.$$

Let $U := (B_r' \times (-3h,\infty))\cap V$, so that $B_r(\boldsymbol{x})\cap\Omega = B_r(\boldsymbol{x}_*)\cap U$ and $U \subseteq B_\varepsilon(\boldsymbol{x}_*)\cap V = B_\varepsilon(\boldsymbol{x}_*)\cap\Omega$ as we can see in the figure 4.1. We integrate over $\boldsymbol{x}' \in B_r'$, and use both of the above inequalities to conclude

$$\begin{aligned}\int_U |f(\boldsymbol{x}')-x_n|^{\alpha p}|u-\alpha|^p d\boldsymbol{x} &\leq c\int_{B_r'}\int_I |f(\boldsymbol{x}')-x_n|^{(\alpha+1)p}|\nabla u|^p dx_n d\boldsymbol{x}' \\ &\quad + c\int_T |f(\boldsymbol{x}')-x_n|^{(\alpha+1)p}|\nabla u|^p d\boldsymbol{x} \\ &\leq c(\alpha,p,n,L)\int_U |f(\boldsymbol{x}')-x_n|^{(\alpha+1)p}|\nabla u|^p d\boldsymbol{x}.\end{aligned}$$

Additionally, by construction of U we have that $|f(\boldsymbol{x}')-x_n|$ is comparable with $\delta(\boldsymbol{x})$. In [**conti0**] it is proven that there exists $c(L)>0$, such that

$$|f(\boldsymbol{x}')-x_n| \leq c(L)\delta_\partial\Omega(\boldsymbol{x}) \leq c(L)\delta_\Gamma(\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{x}\in U,$$

and for a lower bound, we use the fact that $\boldsymbol{x}^* \in \Gamma$ to get

$$\delta_\Gamma(\boldsymbol{x}) \leq \operatorname{dist}(\boldsymbol{x},\boldsymbol{x}^*) \leq \sqrt{9\tau^2+r^2} = \sqrt{9L+1}r \leq \frac{\sqrt{9L+1}}{L}|f(\boldsymbol{x}')-x_n|,$$

so in fact there exist $c(L)$ such that

$$\frac{1}{c(L)}\delta_\Gamma(\boldsymbol{x}) \leq |f(\boldsymbol{x}')-x_n| \leq c(L)\delta_\Gamma(\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{x}\in U.$$

which concludes the proof since $B_r(\boldsymbol{x}_*)\cap V \subseteq U$. □

Finally, to put all these pieces together we will need Whitney Covering Lemma. A more general version of the lemma can be seen in [**stein**], but for simplicity, we will introduce the necessary statement for our framework:

LEMMA **4.3.3** (Whitney covering lemma)**.** *Let $\Gamma \in \mathbb{R}^n$ be a closed non-emptyset set. We can cover Γ^c by a collection of closed cubes Q_j that are essentially disjoint, and whose size is comparable to their distance from Γ, i.e*

1. $\cup_j Q_j = \Gamma^c$ *and* Q_j *'s have disjoint interiors,*

2. $\sqrt{n}\ell\,(Q_j) \leq \operatorname{dist}(Q_j, \Gamma) \leq 4\sqrt{n}\ell\,(Q_j)$*,*

3. If the boundaries of two cubes Q_j *and* Q_k *touch then* $\frac{1}{4} \leq \frac{\ell(Q_j)}{\ell(Q_k)} \leq 4$*,*

4. For a given Q_j *there exist at most* $12^n Q_k$ *'s that touch it.*

Where $\ell(Q)$ *denotes the length of a cube* Q*.*

Additionally, we can also conclude that $2Q_j \subset \Gamma^c$ *and that the cover* $\{2Q_j\}_j$ *has the finite overlap property.*

Remark 4.3.4. *This Lemma is often used to cover an open set* $\Omega \in \mathbb{R}^n$ *by choosing* $\Gamma = \partial\Omega$ *and restricting only to the cubes inside* Ω*.*

We will notice that part of our cover will be outside of Ω. Since our functions are just defined in Ω to handle this problem we will consider the following extension from [**evansGa**].

LEMMA **4.3.5.** *Let* Ω *be a* (L, R)*-Lipschitz domain and* $\boldsymbol{u} \in W^{1,p}(\Omega, \mathbb{R}^k)$*, with* $p \in [1, \infty)$*, then there exists a gradient preserving extension* $\tilde{\boldsymbol{u}} \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$*, i.e.,*

$$\|\nabla(\tilde{\boldsymbol{u}})\|_{L^p(\mathbb{R}^n)} \leq c(n, L, R)\|\nabla(\boldsymbol{u})\|_{L^p(\Omega)},$$

and

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = u(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \Omega.$$

4.4 Proof of Weighted Poincare inequality for bulk domains

In this section, we will prove the weighted Poincaré inequality 4.2.4. The proof is harder than the one in [**conti0**] since a Vitali's cover is not sufficient, in our case, we will need to apply a Whitney cover for the interior and a Vitali's cover close to Γ.

Proof of Theorem 4.2.4. First notice that it suffices to consider the scalar case, additionally, using a density argument we can also assume u in $C^1(\Omega)$. For brevity, let $A := \|\delta_\Gamma(\boldsymbol{x})^{1+\alpha}\nabla u\|_{L^p(\Omega)}$. Let ε be as in Definition 4.2.1 and fix $r_B := \varepsilon/(12(L+1))$ (the reason will become clear below).

Step 1: Let's start by using a Vitali's cover $\Gamma_{2r_B} := \{\boldsymbol{x} \in \Omega \ : \ \delta_\Gamma(\boldsymbol{x}) \leq 2r_B\}$ with balls of radius $4Lr_B$ and centers $\boldsymbol{x}^i$, *i.e.*:

$$\mathcal{U}^\Gamma = \{B_i^\Gamma := B_{4Lr_B}(\boldsymbol{x}^i) \cap \Omega,\ \boldsymbol{x}^0, \dots, \boldsymbol{x}^K \in \Gamma,\ |\boldsymbol{x}^i - \boldsymbol{x}^{i-1}| < \frac{r_B}{L}\}.$$

Claim: There exist $c(n, L)$ *such that, for each* $i = 1, \cdots, K$, *we have that*

$$|B_i^\Gamma \cap B_{i-1}^\Gamma \cap \Gamma_{r_B}^c| \geq c(n, L) r_B^n \quad \textit{and} \quad \|\delta_\Gamma(\boldsymbol{x})^\alpha\|^p_{L^p\left(B_i^\Gamma \cap B_{i-1}^\Gamma\right)} \geq c_L r_B^{n+\alpha p}.$$

First of all, we can, *w.l.o.g.*, consider $L \geq 1$ so we have that $B_i^\Gamma \cap B_{i-1}^\Gamma \subset B_\varepsilon(\boldsymbol{x}^i)$. Additionally, to simplify the proof, we can consider $\boldsymbol{x}^i = (\boldsymbol{x}^{i'}, x_n^i)$, $f_{\boldsymbol{x}^i}(\boldsymbol{x}^{i'}) = 0$, $f_{\boldsymbol{x}^i}(\boldsymbol{x}^{i-1'}) \geq 0$ and $A_{\boldsymbol{x}^i} = I$.

Let $m_i = \min_{B_{r_B/L}(\boldsymbol{x}^{i'})} f_{\boldsymbol{x}^i}(\boldsymbol{x}')$, since $f_{\boldsymbol{x}^i}$ is a $L-$lipschitz function we have that

$$m_i \geq -r_B \quad \text{and} \quad f_{\boldsymbol{x}^i}(\boldsymbol{x}^{i-1'}) \leq r_B.$$

So, in fact the intersection is minimized when $\boldsymbol{x}^{i-1} = (\boldsymbol{y}', y_n) = (\boldsymbol{x}^{i'} + \frac{r_B}{L}\frac{\boldsymbol{x}^{i-1'} - \boldsymbol{x}^{i'}}{|\boldsymbol{x}^{i-1} - \boldsymbol{x}^{i'}|}, r_B)$, which implies that

$$\begin{aligned} B_i^\Gamma \cap B_{i-1}^\Gamma \cap \Gamma_{r_B}^c &\supset B_i^\Gamma \cap B_{i-1}^\Gamma \cap \{(\boldsymbol{x}', x_n) : \boldsymbol{x}' \in B_{r_B/L}(\boldsymbol{x}^i) \ \wedge \ x_n \leq m_i - r_B\} \\ &\supset B_i^\Gamma \cap B_{4Lr_B}(\boldsymbol{y}) \cap \{(\boldsymbol{x}', x_n) : \boldsymbol{x}' \in B_{r_B/L}(\boldsymbol{x}^i) \ \wedge \ x_n \leq m_i - r_B\}. \end{aligned}$$

Additionally, the cylinder

$$C_{\boldsymbol{y}} = B(\boldsymbol{x}^{i'}, \frac{r_B}{L}) \times \left[\left(-4\sqrt{1 - \frac{r_B}{4L^2}} + 1\right) r_B, m_i - r_B\right],$$

is also included in the intersection, so

$$
\begin{aligned}
|B_i^\Gamma \cap B_{i-1}^\Gamma \cap \Gamma_{r_B}^c| \geq |C_{\boldsymbol{y}}| & \\
&\geq \omega(n-1)\left(\frac{2r_B}{L}\right)^{n-1}\left[\left(4\sqrt{1-\frac{r_B}{4L^2}}-1\right)r_B + m_i - r_B\right] \\
&\geq \omega(n-1)\left(\frac{2}{L}\right)^{n-1}\left(4\sqrt{1-\frac{1}{4L^2}}-3\right)r_B^n.
\end{aligned}
$$

So the claim is proven, with $c(n,L) = \omega(n-1)\left(\frac{2}{L}\right)^{n-1}\left(4\sqrt{1-\frac{1}{4L^2}}-3\right)$, where $\omega(n)$ is the volume of the unit ball in $\mathbb{R}^n$.

Additionally, we can conclude that

$$
\begin{aligned}
\|\delta_\Gamma(\boldsymbol{x})^\alpha\|^p_{L^p\left(B_i^\Gamma \cap B_{i-1}^\Gamma\right)} &\geq r_B^{\alpha p}|B_i^\Gamma \cap B_{i-1}^\Gamma \cap \Gamma_{r_B}^c| \\
&\geq c_L r_B^{n+\alpha p}.
\end{aligned}
$$

Step 2: To work away from Γ we will consider the Whitney cover, $\{\hat{Q}_j\}_{j\in\mathbb{N}}$ of the open set Γ^c defined in the Lemma 4.3.3. More precisely, we consider the cubes centered in $\boldsymbol{x}^j$, $\hat{Q}_j := \boldsymbol{x}^j + (-r_j, r_j)^n$ from the Whitney cover and the bigger cubes $Q_j := \boldsymbol{x}^j + (-\frac{3}{2}r_j, \frac{3}{2}r_j)^n$. Additionally we will consider only the sub-cover of $\{Q_j\}_{j\in\mathbb{N}}$ such that the respective $\hat{Q}_j$ intersects $\Omega \cap \Gamma_{r_B}^c$, i.e,

$$
\mathcal{U}^{int} := \{Q_j^{int} := Q_j,\ \hat{Q}_j \cap \Omega \cap \Gamma_{r_B}^c \neq \emptyset\}.
$$

For a better understanding of this cover, we can go over its main properties:

1. $2\sqrt{n}r_j \leq \delta_\Gamma(\hat{Q}_j^{int}) \leq 8\sqrt{n}r_j$ and $\frac{3\sqrt{n}}{2}r_j \leq \delta_\Gamma(Q_j^{int}) \leq 6\sqrt{n}r_j$,

2. If the boundaries of two cubes $\hat{Q}_j^{int}$ and $\hat{Q}_i^{int}$ touch then $\frac{1}{4} \leq \frac{r_j}{r_i} \leq 4$,

3. For a given Q_j^{int}, it intersect a finite number of Q_i^{int} 's,

4. For every j, we have that $r_j \geq \frac{r_B}{\sqrt{n}}$,

5. The cover is finite, so, after rearranging we can consider $j = 0, 1, \cdots = J$ for some $J \in \mathbb{N}$.

Similar to the previous claim we would like to be able to control the size of the intersection of 2 cubes in this cover. In fact, we will just need to do it when $\hat{Q}_j^{int}$ and $\hat{Q}_i^{int}$ are adjacent. *W.l.o.g.* we can assume that $\ell(\hat{Q}_j^{int}) \leq \ell(\hat{Q}_i^{int})$. Then

$$|Q_j^{int} \cap Q_i^{int}| \geq \frac{1}{2}|\hat{Q}_j^{int}| = \frac{1}{2}(2r_j)^n.$$

Lastly, is important to remark that this cover actually covers more than Ω, but u is just defined in Ω. To overcome this problem we can use the gradient preserving extension, $\tilde{u}$, defined in Lemma 4.3.5. To keep the notation simple we will still use u instead of $\tilde{u}$, but we will keep in mind that we can consider u to be defined in $\mathbb{R}^n$ when necessary.

Step 3: The next step is to understand how both covers interact, so it will be useful to merge and relabel them. Let

$$\Omega^{ext} = \bigcup_{j=0}^{K} B_j^{\Gamma} \bigcup_{j=0}^{J} Q_j^{int},$$

and it's cover

$$\mathcal{U} = \{B_j := B_j^{\Gamma},\ j = 0, \ldots, K\ B_{K+j} := Q_j^{int},\ j = 0, \ldots, J\}.$$

To move from the inside to the outside cover we need to be able to control the size of the intersection for some interior ball B_{K+j} and boundary ball B_k. This will not be possible for every intersecting ball, so let's assume that the respective smaller cuber, $\hat{Q}_j^{int}$ intersects Γ_{r_B}. Thisthat $\boldsymbol{x}_{K+j} \in \Gamma_{2r_B}$, so there must be a boundary ball containing $\boldsymbol{x}_{K+j}$, let's make it B_k.

By property 4 of the Whitney cover, $r_{K+j} \geq \frac{r_B}{\sqrt{n}}$, we have the desired bounds:

$$|B_{K+j} \cap B_k| \geq c(L,n) r_B^n, \text{ and } \|\delta_\Gamma(\boldsymbol{x})^\alpha\|^p_{L^p\left(B_{K+j} \cap B_k\right)} \geq c(n,L) r_{K+j}^{n+\alpha p}.$$

Step 4: To understand why we built our cover this way we will see that in fact we can prove a weighted Poincare in each set $B_j,\ j = 1, \ldots, K+J$. For $j \le K$ we can use Theorem 4.3.2 to conclude that there exist $a_j \in \mathbb{R}$ such that

$$\|\delta_\Gamma(x)^\alpha (u - a_j)\|_{L^p(B_j \cap \Omega)} \le c(n,p,\alpha,L)A,$$

and for $j > K$ we can apply the standard poincaré inequality and property 1, to obtain a_j:

$$\begin{aligned}
\|\delta_\Gamma(\boldsymbol{x})^\alpha (u - a_j)\|_{L^p(B_j)} &\le (4\sqrt{n} r_j)^\alpha \|(u - a_j)\|_{L^p(B_j)} \\
&\le c(n,p,\alpha,L) r_j^\alpha \|\nabla u\|_{L^p(B_j)} \\
&\le c(n,p,\alpha,L) \|\delta_\Gamma(\boldsymbol{x})^\alpha \nabla u\|_{L^p(B_j)}.
\end{aligned}$$

Step 5: The main problem is that for each different set, we have a different constant a_j, so we need to find a way to control the size of a_j when we move from one set to another. Fix $k \in \{0, 1 \ldots K+J\}$, and consider the sequence $j_0 := 0, j_1, \ldots, j_H := k$ such that $B_{j_h} \cap B_{j_{h+1}} \cap \Omega \ne \emptyset$ for all h. This is possible since $\Omega^{ext} = \cup B_k \cap \Omega \bigcup \cup B_{K+j'}$ is connected, which means that there is a continuous curve in Ω^{ext} which joins a point $\boldsymbol{x}^0$ and $\boldsymbol{x}^k$; as the curve is compact it is covered by finitely many of the balls. We can further assume, by step 1,2,3, that:

$$\|\delta_\Gamma^\alpha(\boldsymbol{x})\|_{L^p\left(B_{j_{h+1}} \cap B_{j_h}\right)} \ge c(n,p,L) r_{j_h}^{n/p+\alpha}.$$

Using the last inequality we can then control the size of $|a_{j_h} - a_{j_{h+1}}|$:

$$\begin{aligned}
r_{j_h}^{n/p+\alpha} |a_{j_h} - a_{j_{h+1}}| &\le c \|\delta_\Gamma^\alpha(\boldsymbol{x})\|_{L^p\left(B_{j_{h+1}} \cap B_{j_h}\right)} |a_{j_h} - a_{j_{h+1}}| \\
&\le c \|\delta(\boldsymbol{x})^\alpha (a_{j_h} - a_{j_{h+1}})\|_{L^p\left(B_{j_{h+1}} \cap B_{j_h}\right)} \\
&\le c \|\delta(\boldsymbol{x})^\alpha (u - a_{j_h})\|_{L^p\left(B_{j_h}\right)} + \|\delta(\boldsymbol{x})^\alpha (u - a_{j_{h+1}})\|_{L^p\left(B_{j_{h+1}}\right)} \\
&\le c(n,p,\alpha,L)A.
\end{aligned}$$

Step 5: To finalize, for each B_k we can always create a finite path that starts at B_0

and finishes at B_k. Besides that the size of all the balls are comparable, i.e., there exist $C = C(L, R)$ such that $\frac{1}{C} \leq \frac{r_{k_1}}{r_{k_2}} \leq C$ for every $k_1, k_2 \in [0, K+J]$. So taking $a = a_0$ we have that

$$\begin{aligned}
\|\delta(\boldsymbol{x})^\alpha(u-a)\|_{L^p(\Omega)} &\leq \sum_{k=0}^{K+J} \|\delta(\boldsymbol{x})^\alpha(u-a_0)\|_{L^p(B_k\cap\Omega)} \\
&\leq \sum_{k=0}^{K+J} \left[\|\delta(\boldsymbol{x})^\alpha(u-a_k)\|_{L^p(B_k)} + \|\delta(\boldsymbol{x})^\alpha(a_k-a)\|_{L^p(B_k)}\right] \\
&\leq C(K+J)A + \sum_{k=0}^{K+J}\sum_{h=0}^{H_k} \left\|\delta(\boldsymbol{x})^\alpha(a_{j_{h+1}} - a_{j_h})\right\|_{L^p(B_k)} \\
&\leq C(K+J)A + \sum_{k=0}^{K+J}\sum_{h=0}^{H_k} c r_k^{n/p+\alpha} \left|a_{j_{h+1}} - a_{j_h}\right| \\
&\leq C(n,p,\alpha,L,R)A,
\end{aligned}$$

as desired. □

Remark 4.4.1. *Even if $u = 0$ in Γ we can't assume that $a = 0$ as usual in other poincaré inequalities. This happens because even to control the function near the boundary we always use information from the interior. To confirm this we can look at the following 1d counter-example, for $\Omega = [0,1]$ and $\Gamma = \{0\}$. Take*

$$u_n = \begin{cases} n \ if \ \delta_\Gamma(x) \geq \frac{1}{n} \\ \delta_\Gamma(x) n^2 \ if \ \delta(x) \leq \frac{1}{n} \end{cases} \qquad |u_n'| = \begin{cases} 0 \ if \ \delta_\Gamma(x) > \frac{1}{n} \\ n^2 \ if \ \delta_G G(x) \leq \frac{1}{n} \end{cases}$$

So for $n > 4$ we have that

$$\int_0^1 \delta_\Gamma(x)^{\alpha p} u_n(x)^p \geq \int_{\delta_\Gamma(x)\geq 1/4} \delta_\Gamma(x)^{\alpha p} n^p \geq \frac{1}{4^{\alpha p}}\frac{1}{2} n^p \to \infty$$

However

$$\int_0^1 \delta_\Gamma(x)^{(\alpha+1)p} |u_n'(x)|^p = \int_0^{\frac{1}{n}} x^{(\alpha+1)p} n^{2p} = \frac{2}{(\alpha+1)p+1} n^{2p-(\alpha+1)p-1} = Cn^{p(1-\alpha)-1}$$

so we conclude that for any $p > 0, \alpha > 0$

$$\frac{\|\delta(x)^{\alpha}|u_n(x)|\|^p}{\|\delta(x)^{\alpha+1}|u_n'(x)|\|^p} \geq C\frac{n^p}{n^{p(1-\alpha)-1}} = n^{1+p\alpha} \to \infty$$

So there is no guarantee that $a = 0$ even if it vanishes on Γ.

4.5 Weighted Korn inequality for a bulk domain

In this section, we will prove the weighted Korn inequality 4.2.2. The idea is very similar to the proof of the weighted Poincaré inequality (Theorem 4.2.4), but we can use a simpler cover. Results like this normally are proved for $\Gamma = \partial\Omega$, but we will prove it for a more general Γ because it will be necessary to prove the result for plates.

Proof of Theorem 4.2.2. When Ω is a cube and $\alpha = 0$ both of these results are well known (Theorem 2.3.1, 2.3.2), and the technique to extend to the general equation is identically in both cases so we will just prove the second inequality.

Contrary to the proof of the weighted Poincaré Inequality, for this theorem, we do not need a different cover near Γ, we will just use the Whitney cover from lemma 4.3.3:

$$\mathcal{U} := \{Q_j,\ \hat{Q}_j \cap \Omega \neq \emptyset,\ j = 0, \ldots, J\}, \qquad \Omega^{ext} = \cup_j \hat{Q}_j.$$

As in the previous theorem, we will use the gradient preserving extension, $\tilde{\boldsymbol{u}}$, defined in Lemma 4.3.5, while still using $\boldsymbol{u}$.

Step 1: The first step is to apply the normal Uniform Rigidity inequality to each cube in the cover. So for each Q_j we have that there exist $R_j \in SO(n)$ such that

$$\begin{aligned}
\|\delta_\Gamma(\boldsymbol{x})^\alpha \nabla(\boldsymbol{u} - R_j)\|_{L^p(Q_j)} &\leq (4\sqrt{n}r_j)^\alpha \|\nabla(\boldsymbol{u} - R_j)\|_{L^p(Q_j)} \\
&\leq C(4\sqrt{n}r_j)^\alpha \|\mathrm{dist}\,(\nabla \boldsymbol{u}, \mathrm{SO}(n))\|_{L^p(Q_j)} \\
&\leq C(n,p,\alpha) \|\delta_\Gamma(\boldsymbol{x})^\alpha\, \mathrm{dist}\,(\nabla \boldsymbol{u}, \mathrm{SO}(n))\|_{L^p(Q_j)} .
\end{aligned}$$

Notice that since all the sets are cubes, the constant $C(n,p,\alpha)$ is the same for every

j.

Step 2: Again, we have a different constant for each cube, but this time to find the right constant we will use a different technique. Let start by constructing a partition of unity subordinated to $\mathcal{U}$, i.e., fix $\varphi^* \in C_c^\infty\left((-1,1)^n;[0,1]\right)$ with $\varphi^* = 1$ on $\left(-\frac{1}{2},\frac{1}{2}\right)^n$, let $\hat{\varphi}_j(\boldsymbol{x}) := \varphi^*\left(\left(\boldsymbol{x}-\boldsymbol{x}^j\right)/2r_j\right)$ and $\varphi_j := \hat{\varphi}_j / \sum_k \hat{\varphi}_k$ which is well define in Ω^{ext} since there is at least one $\hat{\varphi}_k > 0$, so it have the following properties:

1. $\varphi_j \in C_c^\infty\left(Q_j \cap \Omega^{ext}\right)$,
2. $\sum_j \varphi_j = 1$ in Ω^{ext},
3. $|\nabla\varphi_j| \le c/r_j$.

Then we can define a smooth interpolation of all R_j defined in Ω^{ext},

$$\beta := \sum_j \varphi_j R_j.$$

Since the overlap of the sets is finite, by the properties of the partition of unity we have that:

$$\begin{aligned}
\|\delta_\Gamma(\boldsymbol{x})^\alpha(\nabla\boldsymbol{u}-\beta)\|_{L^p(\Omega)} &= \left\|\delta_\Gamma(\boldsymbol{x})^\alpha \sum_j \varphi_j\left(\nabla u - R_j\right)\right\|_{L^p(\Omega)} \\
&\le C\sum_j \|\delta_\Gamma(\boldsymbol{x})^\alpha(\nabla\boldsymbol{u}-R_j)\|_{L^p(Q_j)} \\
&\le C(n,p)\left\|\delta_\Gamma(\boldsymbol{x})^\alpha \operatorname{dist}(\nabla\boldsymbol{u},\mathrm{SO}(n))\right\|_{L^p(\Omega)}.
\end{aligned}$$

Step 3: This looks similar to what we need to prove, however, β is not constant, so we will have to apply weighted Poincaré inequality proved in 4.2.4. To do so, we will need to control the size of the gradient of β. Since, $\sum_j \nabla\varphi_j = 0$ on Ω we have that

$$\nabla\beta = \sum_j \nabla\varphi_j R_j = \sum_j \nabla\varphi_j\left(R_j - \nabla\boldsymbol{u}\right).$$

Additionally, recall that $|\nabla\varphi_j| \le C/r_j$, and that $\delta_\Gamma(Q_j) \le Cr_j$, which implies that

$$\delta_\Gamma(\boldsymbol{x})\,|\nabla\varphi_j|\,(\boldsymbol{x}) \le C\chi_{Q_j}(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x}\in\Omega.$$

Therefore

$$\begin{aligned}\|\delta_\Gamma(\boldsymbol{x})^{(1+\alpha)}\nabla\beta\|_{L^p(\Omega)} &\le C\sum_j \|\delta_\Gamma(\boldsymbol{x})^{(1+\alpha)}\nabla\varphi_j\,(\nabla\boldsymbol{u}-R_j)\,\|_{L^p(Q_j)}\\ &\le C(n,p,\alpha)\sum_j \|\delta_\Gamma(\boldsymbol{x})^{\alpha}\,(\nabla\boldsymbol{u}-R_j)\,\|_{L^p(Q_j)}.\end{aligned}$$

So, by applying the weighted Poincaré inequality to β, we obtain $R_*\in\mathbb{R}^{n\times n}$, such that

$$\begin{aligned}\|\delta_\Gamma(\boldsymbol{x})^{\alpha}(\beta-R_*)\|_{L^p(\Omega)} &\le C\|\delta_\Gamma(\boldsymbol{x})^{(1+\alpha)}\nabla\beta\|_{L^p(\Omega)}\\ &\le C(n,p,\alpha,L,R)\|\delta_\Gamma(\boldsymbol{x})^{\alpha}\,\mathrm{dist}(\nabla\boldsymbol{u},\mathrm{SO}(n))\|_{L^p(\Omega)}.\end{aligned}$$

Step 4: The last problem that needs to be fixed is that R_* is not necessarily in $\mathrm{SO}(n)$.Although, $\mathrm{SO}(n)$ is compact, so we can take R be the closest matrix to R_* in $\mathrm{SO}(n)$, which satisfies:

$$|R-R_*| = \mathrm{dist}\,(R_*,\mathrm{SO}(n)) \le |R_*-\nabla\boldsymbol{u}|\,(\boldsymbol{x}) + \mathrm{dist}(\nabla\boldsymbol{u}(x),\mathrm{SO}(n)),$$

for every $\boldsymbol{x}$. So we can conclude that

$$\begin{aligned}\|\delta_\Gamma(\boldsymbol{x})^{\alpha}(\nabla\boldsymbol{u}-R)\|_{L^p(\Omega)} &\le \|\delta_\Gamma(\boldsymbol{x})^{\alpha}(\nabla\boldsymbol{u}-R_*)\|_{L^p(\Omega)} + \|\delta_\Gamma(\boldsymbol{x})^{\alpha}(R_*-R)\|_{L^p(\Omega)}\\ &\le 2\|\delta_\Gamma(\boldsymbol{x})^{\alpha}(\nabla\boldsymbol{u}-R_*)\|_{L^p(\Omega)} + \|\delta_\Gamma(\boldsymbol{x})^{\alpha}\,\mathrm{dist}(\nabla\boldsymbol{u},\mathrm{SO}(n))\|_{L^p(\Omega)}\\ &\le 2\|\delta_\Gamma(\boldsymbol{x})^{\alpha}(\nabla\boldsymbol{u}-\beta)\|_{L^p(\Omega)} + 2\|\delta_\Gamma(\boldsymbol{x})^{\alpha}(\beta-R_*)\|_{L^p(\Omega)}\\ &\qquad + \|\delta_\Gamma(\boldsymbol{x})^{\alpha}\,\mathrm{dist}(\nabla\boldsymbol{u},\mathrm{SO}(n))\|_{L^p(\Omega)}\\ &\le C(n,p,\alpha,L,R)\|\delta_\Gamma(\boldsymbol{x})^{\alpha}\,\mathrm{dist}(\nabla\boldsymbol{u},\mathrm{SO}(n))\|_{L^p(\Omega)},\end{aligned}$$

as desired.

□

4.6 Weighted Korn Inequality for Plates

Theoretically, the previous result is also true for plates, however, to be able to control the size of the constant when the thickness of the plate goes to zero we need a different approach.

Proof of Theorem 4.2.3. Again we will just need to focus on one of the inequalities, this time we will just prove the first one. Similar to the proof of the weighted poincare inequality, Theorem 4.2.4, we need two different covers.

We will start by covering the 2d plate Ω, using $\Gamma = \partial\Omega$ and $r_B = h$; then we will create cylinders with the previous balls as a base to cover Ω_h. To simplify the notation we will consider $\boldsymbol{x} = (\boldsymbol{x}', z)$, where $\boldsymbol{x}' \in \Omega$ and $z \in I_h$. Additionally, B' and Q' will be a subset of Ω and δ' will be the distance in 2D.

Step 1: Near the boundary, we can cover the set $\partial\Omega'_h = \{\boldsymbol{x}' \in \Omega : \delta'_{\partial\Omega(\boldsymbol{x}')} < h\}$, with $\mathcal{U}^{ext} := \{B'_i = B'_{2h}(\boldsymbol{x}'_i),\ \boldsymbol{x}'_0, \boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_K \in \partial\Omega\}$ with the following properties:

1. $K = \mathcal{O}(1/h)$,
2. $|\boldsymbol{x}'_i - \boldsymbol{x}'_{i-1}| \leq h$,
3. $\sum \chi_{B'(\boldsymbol{x}'_i, 2h)\cap\Omega} \leq C = \mathcal{O}(1)$,
4. for h small enough, we also have that for $\boldsymbol{x}' \in B'_j \cap \Omega$,

$$\delta'_{\partial\Omega}(\boldsymbol{x}') \leq \delta'_{B'_j\cap\partial\Omega}(\boldsymbol{x}') \leq c(L)\delta'_{\partial\Omega}(\boldsymbol{x}').$$

For each $j = 0, \ldots, K$, define $C_j = B'_j \times I_h$ and $\Gamma = B'_j \times I_h \cap \partial\Omega \times I_h$. So, since $\delta'(\boldsymbol{x}) = \delta_{\partial\Omega\times I_h}(\boldsymbol{x}) = \delta'_{\partial\Omega}(\boldsymbol{x}')$ and by property 4 of the cover, for each $\boldsymbol{x} \in C_j$ we have that:

$$\delta'(\boldsymbol{x}) \leq \delta_\Gamma(\boldsymbol{x}) \leq C(L)\delta'(\boldsymbol{x})$$

so we can apply the weighted Korn inequality for a bulk domain, Theorem 4.2.2, to conclude that there exists S_j such that

$$\begin{aligned}\|\delta'(\boldsymbol{x})^\alpha(\nabla\boldsymbol{u}-S_j)\|_{L^p(C_j\cap\Omega_h)} &\leq \|\delta_\Gamma(\boldsymbol{x})^\alpha(\nabla\boldsymbol{u}-S_j)\|_{L^p(C_j\cap\Omega_h)}\\ &\leq C\|\delta_\Gamma(\boldsymbol{x})\alpha e(\boldsymbol{u})\|_{L^p(C_j\cap\Omega_h)}\\ &\leq C(p,\alpha,L,R)\|\delta'(\boldsymbol{x})^\alpha e(\boldsymbol{u})\|_{L^p(C_j\cap\Omega_h)}\end{aligned}$$

Step 2: For the interior, we will consider a Whitney cover of $\partial\Omega^c$, and keep only the cubes that intersect $\Omega\cap\partial\Omega_h'^c$, more precisely,

$$\mathcal{U}^{int} := \{B'_{K+j} := Q'_j,\ \hat{Q}'{}_j\cap\Omega\cap\Omega\cap\partial\Omega_h'^c\neq\emptyset\}.$$

Since $\frac{h}{2}\leq\ell(B'_{K+j})\leq\operatorname{diam}(\Omega)$, for every $j=1,\dots,J$ we can apply the normal Korn inequality for plates, Theorem 2.3.3, to $C_{K+j}=B'_{K+j}\times I_h$, to find S_{K+j} such that:

$$\begin{aligned}\|\delta'(\boldsymbol{x})^\alpha(\nabla\boldsymbol{u}-S_{K+j})\|_{L^p(C_{K+j})} &\leq C\ell(B'_{K+j})^\alpha\|(\nabla\boldsymbol{u}-S_{K+j})\|_{L^p(C_{K+j})}\\ &\leq \frac{C}{h}\ell(B'_{K+j})^\alpha\|e(\boldsymbol{u})\|_{L^p(C_{K+j})}\\ &\leq \frac{C(p,\alpha)}{h}\|\delta'(\boldsymbol{x})^\alpha e(\boldsymbol{u})\|_{L^p(C_{K+j})}.\end{aligned}$$

Step 3: To find a constant that works in the full domain we will use a partition of unity again for both covers together $\mathcal{U}=\mathcal{U}^{ext}\bigcup\mathcal{U}^{int}=\bigcup_{j=0}^{K+J}B'_j$. For the interior cubes, fix $\varphi^*\in C_c^\infty\left((-1,1)^2;[0,1]\right)$ with $\varphi^*=1$ on $\left(-\frac{1}{2},\frac{1}{2}\right)^2$, let $\hat{\varphi}_j(\boldsymbol{x}):=\varphi^*\left((\boldsymbol{x}-\boldsymbol{x}_j)/r_j\right)$. For the exterior balls, we can do something very similar but with a radial function instead. In the end, we can define the final functions as $\varphi'_j:=\hat{\varphi}_j/\sum_k\hat{\varphi}_k$ which is well defined in Ω since there is at least one $\hat{\varphi}_k>0$ even in $\partial\Omega$. Additionally, to cover the full Ω_h we can extend each function φ'_j to the cylinder C_j, i.e., let $\varphi(\boldsymbol{x})=\varphi(\boldsymbol{x}',z)=\varphi'(\boldsymbol{x}')$. So for these functions, we have that:

1. for $j\leq K$, $\varphi'_j\in C_c^\infty\left(B'_j\right)$, $\varphi_j\in C^\infty\left(C_j\right)$ and $|\nabla\varphi_j|\leq C/r_j$,

2. for $j>K$, $\varphi'_j\in C_c^\infty\left(B'_j\cap\Omega\right)$, $\varphi_j\in C^\infty\left(C_j\right)$ and $|\nabla\varphi_j|\leq C/r_j=C/h$,

3. $\sum_{j=0}^{K+J} \varphi_j = 1$ in Ω_h.

Taking in account the previous definitions we can define $\beta : \Omega \to \mathbb{R}^{3\times 3}$ as a smooth interpolation between all $S_j, \beta := \sum_j \varphi_j S_j$. Notice that an interior set just overlaps with $\mathcal{O}(1)$ number of other interior sets, however, it can overlap with $\mathcal{O}(1/h)$ exterior cylinders. That would not create a problem since the Korn constant for the outside cylinders does not depend on h, so we can still conclude that:

$$\begin{aligned}\|\delta(\boldsymbol{x})^\alpha(\nabla \boldsymbol{u} - \beta)\|_{L^p(\Omega_h)} &= \left\|\delta(\boldsymbol{x})^\alpha \sum_j \varphi_j \left(\nabla \boldsymbol{u} - S_j\right)\right\|_{L^p(\Omega_h)} \\ &\leq \frac{C(p,\alpha,L,R)}{h} \|\delta(\boldsymbol{x})^\alpha e(\boldsymbol{u})\|_{L^p(\Omega_h)}.\end{aligned}$$

Step 4: This and the following step are identical to the proof of the weighted Korn inequality for bulk domains, Theorem 4.2.2, we just need to be careful that the cover is more complex. First, we will try to approximate β by a constant using the weighted Poincaré inequality, for that we need to bound $\nabla\beta$. Since $\sum_j \nabla\varphi_j = 0$ on Ω_h we have that

$$\nabla\beta = \sum_j \nabla\varphi_j S_j = \sum_j \nabla\varphi_j \left(S_j - \nabla \boldsymbol{u}\right),$$

and

1. for $\boldsymbol{x} \in C_j,\ j > K$

$$|\nabla\varphi_j|\,(\boldsymbol{x}) \leq c/r_j, \qquad \delta'(\boldsymbol{x}) \leq \delta'_{\partial\Omega}\left(B'_j\right) \leq C r_j,$$

2. for $\boldsymbol{x} \in C_j \cap \Omega_h,\ j \leq K$

$$|\nabla\varphi_j|\,(\boldsymbol{x}) \leq C/h, \qquad \delta'(\boldsymbol{x}) \leq C\delta'_{\partial\Omega}(\boldsymbol{x}') \leq Ch.$$

So, as desired, for every $\boldsymbol{x} \in \Omega_h,\ \boldsymbol{x} \in C_j$ for some j, we have that

$$\delta'(\boldsymbol{x})\,|\nabla\varphi_j|\,(\boldsymbol{x}) \leq C\chi_{C_j}(\boldsymbol{x}),$$

Therefore

$$
\begin{aligned}
\|\delta'(\boldsymbol{x})^{(1+\alpha)}\nabla\beta\|_{L^p(\Omega_h)} &\le C\left\|\sum_j \delta(\boldsymbol{x})^{(1+\alpha)}\nabla\varphi_j(\nabla\boldsymbol{u}-S_j)\right\|_{L^p(C_j)} \\
&\le C\left\|\sum_j \delta(\boldsymbol{x})^{\alpha}(\nabla\boldsymbol{u}-S_j)\right\|_{L^p(C_j)} \\
&\le \frac{C(p,\alpha,L,R)}{h}\|\delta'(\boldsymbol{x})^{\alpha}e(\boldsymbol{u})\|_{L^p(\Omega_h)}.
\end{aligned}
$$

We then apply the weighted Poincaré inequality to β with $\Gamma = \partial\Omega \times I_h$ to obtain that there is $S_* \in \mathbb{R}^{3\times 3}$ with

$$
\|\delta'(\boldsymbol{x})^{\alpha}(\beta - S_*)\|_{L^p(\Omega_h)} \le c\|\delta'(\boldsymbol{x})^{(1+\alpha)}\nabla\beta\|_{L(\Omega_h)} \le \frac{c}{h}\|\delta'(\boldsymbol{x})^{\alpha}e(\boldsymbol{u})\|_{L^p(\Omega_h)}.
$$

Step 5: One more time, S_* might not be skew symmetric. To fix this we can let S be the skew-symmetric matrix closest to S_*. Then, using that $|S - S_*| = \operatorname{dist}\left(S_*, \mathbb{R}^{3\times 3}_{\mathrm{skw}}\right) \le |S_* - \nabla\boldsymbol{u}|(\boldsymbol{x}) + \operatorname{dist}(\nabla\boldsymbol{u}(\boldsymbol{x}), \mathbb{R}^{3\times 3}_{\mathrm{skw}}) = |S_* - \nabla\boldsymbol{u}|(\boldsymbol{x}) + e(\boldsymbol{u}(\boldsymbol{x}))$ for every $\boldsymbol{x} \in \Omega_h$ we obtain that

$$
\begin{aligned}
\|\delta'(\boldsymbol{x})^{\alpha}(\nabla\boldsymbol{u} - S)\|_{L^p(\Omega_h)} &\le \|\delta'(\boldsymbol{x})^{\alpha}(\nabla\boldsymbol{u} - S^*)\|_{L^p(\Omega_h)} + \|\delta'(\boldsymbol{x})^{\alpha}(S^* - S)\|_{L^p(\Omega_h)} \\
&\le 2\|\delta'(\boldsymbol{x})^{\alpha}(\nabla\boldsymbol{u} - S^*)\|_{L^p(\Omega_h)} + \|\delta'(\boldsymbol{x})^{\alpha}e(\boldsymbol{u})\|_{L^p(\Omega_h)} \\
&\le \frac{C(p,\alpha,L,R)}{h}\|\delta'(\boldsymbol{x})^{\alpha}e(\boldsymbol{u})\|,
\end{aligned}
$$

as desired.

$\square$

Chapter 5

Korn inequality in plates for functions of special bounded deformation (SBD)

5.1 Introduction

As we seen, in section 2.4, functions of Bounded Deformation arise naturally in the study of fracture mechanics, They have $\mathcal{H}^{n-1}-$rectifiable a jump set, and away from the jump set the symmetrized gradient is a Radon Measure. To understand this space better in section 5.2 we will go over some of its properties and how it relates to the well-known space of Bounded Variation Functions. Then we will review the recent Korn inequalities in bulk domains in section 5.3. Lastly, we will go over a new proof for Korn inequality in plates for the Sobolev function in section 5.4 and extend it to special functions of bounded deformation in section 5.5.

5.2 Bounded Variation and Bounded Deformation functions

While we care mostly about Bounded Deformation functions (BD), we will also expose some properties of Bounded Variation functions (BV). Both spaces are very similar and it will be useful to understand how they are related, additionally, BV functions have been studied for a longer time so their theory is very well documented, for more details the reader can look into chapter 5 of Lawrence Evans and Ronald Gariepy book [**evansGa**] or [**SBV1**, **SBV2**, **SBV3**]. Besides that bounded variations functions are a bit easier to understand since they can be considered real functions, while bounded deformation functions only make sense as vector-valued functions. Most of the results for bounded deformation functions can be found in [**Ambrosio1997**, **RogerPaper**, **RogerBook**, **SBD1**, **SBD2**], if the reader is interested.

In simple terms, BV and BD functions are functions where their gradient and symmetric gradient, respectively, exist only in the sense of distribution. So to fully understand this chapter a good understanding of Measure Theory and Functional Analysis is necessary. We will try to make it as self-contained as possible but we will assume the reader is familiar with basic concepts and results of the field, like Hausdorff measures, absolute continuity of Measures, Lebesgue Decomposition Theorem, Riesz Representation Theorem, etc. For more details on these topics see [**evansGa**].

5.2.1 Definitions

As mentioned before, we will start with the definition of functions of Bounded Variation (BV). There are two equivalent ways to define this space, the first one is by using the total variation of the function ([**evansGa**]) and the second is to use the Riesz Representation theorem, to actually obtain a distributional derivative ([**evansGa**]).

Definition 5.2.1 (Bounded Variation Function)**.** *Let $U \subset \mathbb{R}^n$ and $\boldsymbol{f} \in L^1(U)$. Then, $\boldsymbol{f}$*

is a function of bounded variation in U if

$$\sup\left\{\int_U \boldsymbol{f}\,\mathrm{div}\,\boldsymbol{\phi}\;d\boldsymbol{x}\;,\;\boldsymbol{\phi}\in C_c^1\left(U;\mathbb{R}^n\right),|\boldsymbol{\phi}|\leq 1\right\}<\infty.$$

We write $BV(U)$ to denote the space of functions of bounded variation in U.

Definition 5.2.2 (BV Structure Theorem)**.** *Let $U\subset\mathbb{R}^n$ and $\boldsymbol{f}\in L^1(U)$. Then, $\boldsymbol{f}$ is a function of bounded variation in U if there exist a signed Radon measure μ on U such that , we have*

$$\int_U \boldsymbol{f}\,\mathrm{div}\,\boldsymbol{\phi} d\boldsymbol{x}=-\int_U \boldsymbol{\phi}\mu,\quad \textit{for all}\quad \boldsymbol{\phi}\in C_c^1\left(U;\mathbb{R}^n\right)$$

So the weak first partial derivatives of a BV function are Randon measures and we can therefore denote

$$D\boldsymbol{f}:=\mu.$$

Remark 5.2.3. *The notation for the derivative changes very often in the literature. Considering signed measures simplifies a lot of the notation, but in some cases, it is not done, [**evansGa**] for example. In this book, our notation is equivalent to:*

$$|\mu|=\|D\boldsymbol{f}\|\qquad \mu=[D\boldsymbol{f}]:=\|D\boldsymbol{f}\|\llcorner\sigma$$

For the case of Bounded Deformation functions, the first definition is more complicated, since we just have a distributional derivative of the symmetric gradient. So we will focus on the second one, which is more intuitive, [**Ambrosio1997**]

Definition 5.2.4 (Bounded Deformation Function)**.** *Let $U\subset\mathbb{R}^n$ and $\boldsymbol{f}=(f_1,\ldots,f_n)\in L^1(U;\mathbb{R}^n)$. Then, $\boldsymbol{f}$ is a function of bounded deformation in U if the symmetric part of the distributional gradiet of $\boldsymbol{f}$,i.e.:*

$$E_{ij}\boldsymbol{f}:=\frac{1}{2}(D_if_j+D_jf_i)$$

is a Randon measure with bounded total variation in U for any $i,j=1,\ldots,n$. We write $BD(U)$ to denote the space of functions of bounded deformation in U.

Additionally, this is a Banach space for the norm

$$\|\boldsymbol{f}\|_{BD(U)} = \|\boldsymbol{f}\|_{L^1(U)} + \sum_{i,j} |E_{ij}\boldsymbol{f}|(U)$$

However, we can look at each direction separately to get real value functions and get a more intuitive definition([**RogerPaper**]):

Definition 5.2.5. *For every $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$, let $D_{\boldsymbol{\xi}}$ be the distributional derivative in the direction $\boldsymbol{\xi}$ defined by $D_{\boldsymbol{\xi}}\boldsymbol{v} = \sum_j \xi_j D_j \boldsymbol{v}$ and for every function $\boldsymbol{f} \in L^1(U)$ define the function $f^{\boldsymbol{\xi}} : U \to \mathbf{R}$ by $f^{\boldsymbol{\xi}}(\boldsymbol{x}) = (\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{\xi})$. Then $\boldsymbol{f} \in BD(U)$ if*

$$D_{\boldsymbol{\xi}} f^{\boldsymbol{\xi}} = (E\boldsymbol{f}\boldsymbol{\xi}, \boldsymbol{\xi})$$

is a bounded Radon measure on U for every $\boldsymbol{\xi}$ of the form $\boldsymbol{e}_i + \boldsymbol{e}_j$, $i, j = 1, \dots, n$.

Conversely, if $\boldsymbol{f} \in BD(U)$, then $D_{\boldsymbol{\xi}} f^{\boldsymbol{\xi}}$ is a bounded Radon measure on U for every $\boldsymbol{\xi} \in \mathbf{R}^n$.

5.2.2 Jump Set characterization

The main reason these functions are used in fracture mechanics is because they can admit jumps to represent cracks in the material. So in this subsection, we will try to understand how to define the jump set and its properties. This is much easier to understand for real functions, so we will focus first on functions of Bounded Variations.

To start, let's recall the definition of the approximate limit for a real-valued function f at a point $\boldsymbol{x}$([**evansGa**]):

Definition 5.2.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$.*

1. *We say $\lambda(\boldsymbol{x})$ is the approximate* $\limsup$ *of f as $\boldsymbol{y} \to \boldsymbol{x}$, written*

$$\operatorname{ap} \limsup_{\boldsymbol{y} \to \boldsymbol{x}} f(\boldsymbol{y}) = \lambda(\boldsymbol{x}),$$

if $\lambda(\boldsymbol{x})$ *is the infimum of the real numbers* t *such that*

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B_r(\boldsymbol{x}) \cap \{f > t\})}{\mathcal{L}^n(B_r(\boldsymbol{x}))} = 0.$$

2. *Similarly,* $\mu(\boldsymbol{x})$ *is the approximate* $\liminf$ *of* f *as* $\boldsymbol{y} \to x$*, written*

$$\text{ap } \liminf_{\boldsymbol{y}\to\boldsymbol{x}} f(\boldsymbol{y}) = \mu(\boldsymbol{x}),$$

if $\mu(\boldsymbol{x})$ *is the supremum of the real numbers* t *such that*

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B_r(\boldsymbol{x}) \cap \{f < t\})}{\mathcal{L}^n(B_r(\boldsymbol{x}))} = 0.$$

Additionally, we say $f : \mathbb{R}^n \to \mathbb{R}$ *is approximately continuous at* $\boldsymbol{x} \in \mathbb{R}^n$ *if*

$$\lambda(\boldsymbol{x}) = \mu(\boldsymbol{x}) = f(\boldsymbol{x}).$$

For real-valued functions, we can consider the jump set to be the points at which the approximate limit does not exist [**evansGa**], i.e.

$$J_f := \{\boldsymbol{x} \in \mathbb{R}^n \mid \lambda(\boldsymbol{x}) < \mu(\boldsymbol{x})\},$$

We can notice that all the Lebesgue points are also points of approximate continuity so, for a $\mathcal{L}^n$-measurable function f, we have that

$$\mathcal{L}^n(J_f) = 0.$$

However, we will see that if we assume more regularity on the function f, we will get more information on the jump set.

In the case of vector-valued functions we need to consider all the directions in which the jump can occur, for that let's introduce one-sided Lebesgue limits [**Ambrosio1997**],

Definition 5.2.7. *Let* $\boldsymbol{\nu}$ *be a unit vector in* $\mathbb{R}^n, \boldsymbol{x} \in \mathbb{R}^n$*. We define the hyperplane*

$$H_{\boldsymbol{\nu}} := \{\boldsymbol{y} \in \mathbb{R}^n \mid \boldsymbol{\nu} \cdot (\boldsymbol{y} - \boldsymbol{x}) = 0\},$$

and the half-spaces

$$H_{\boldsymbol{\nu}}^{+} := \{\boldsymbol{y} \in \mathbb{R}^n \mid \boldsymbol{\nu} \cdot (\boldsymbol{y} - \boldsymbol{x}) \geq 0\},$$

$$H_{\boldsymbol{\nu}}^{-} := \{\boldsymbol{y} \in \mathbb{R}^n \mid \boldsymbol{\nu} \cdot (\boldsymbol{y} - \boldsymbol{x}) \leq 0\}.$$

Definition 5.2.8 (One-sided Lebesgue limits)**.** *Let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$. We say that $\boldsymbol{f}$ has one-sided Lebesgue limits $\boldsymbol{f}^{+}_{\boldsymbol{\nu_f}}(\boldsymbol{x})$ and $\boldsymbol{f}^{-}_{\boldsymbol{\nu_f}}(\boldsymbol{x})$ at $\boldsymbol{x} \in \mathbb{R}^n$, with respect to a suitable direction $\boldsymbol{\nu}_f \in S^{n-1}$, if*

$$\lim_{r \to 0^+} \frac{1}{r^n} \int_{B_r(\boldsymbol{x}) \cap H^{\pm}_{\boldsymbol{\nu}_f}} \left| \boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}^{\pm}_{\boldsymbol{\nu_f}}(\boldsymbol{x}) \right| d\boldsymbol{y} = 0.$$

The Jump set $J_{\boldsymbol{f}}$ is then defined as the non-Lebesgue points such that both one-sided Lebesgue limits exist and are different for some suitable direction $\boldsymbol{\nu}_f \in S^{n-1}$,*i.e.*

$$J_{\boldsymbol{f}} := \{\boldsymbol{x} \in \mathbb{R}^n : \ \boldsymbol{f}^{+}_{\boldsymbol{\nu_f}}(\boldsymbol{x}), \boldsymbol{f}^{-}_{\boldsymbol{\nu_f}}(\boldsymbol{x}) \text{ exist and } \boldsymbol{f}^{+}_{\boldsymbol{\nu_f}}(\boldsymbol{x}) \neq \boldsymbol{f}^{-}_{\boldsymbol{\nu_f}}(\boldsymbol{x}), \text{ for some } \boldsymbol{\nu}_f \in S^{n-1}\}$$

Again we can easily see that $\mathcal{L}^n(J_{\boldsymbol{f}}) = 0$, but in fact for both, BV and BD, functions we have that the jump set is much more regular, it is countably rectifiable([**evansGa**] and [**Ambrosio1997**]). This is a very strong property but general enough to model cracks in fracture mechanics.

THEOREM **5.2.9** ($\mathcal{H}^{n-1}$ a.e. rectifiable)**.** *Let $\boldsymbol{f} \in BV(U)$ or $\boldsymbol{f} \in BD(U)$, then there exist countably many C^1-hypersurfaces, $\{M_k\}_{k=1}^{\infty}$, such that*

$$\mathcal{H}^{n-1}\left(J_{\boldsymbol{f}} - \bigcup_{k=1}^{\infty} M_k \right) = 0.$$

Additionally for each M_k we have a unique C^1 outer vector $\boldsymbol{\nu}^k$ such that

$$\boldsymbol{\nu}_{\boldsymbol{f}}(\boldsymbol{x}) = \boldsymbol{\nu}^k(\boldsymbol{x}), \quad \boldsymbol{x} \in M_k,$$

and $\boldsymbol{\nu}_{\boldsymbol{f}}(\boldsymbol{x})$ is $\mathcal{H}^{n-1}$a.e. continuous in $J_{\boldsymbol{f}}$

5.2.3 Lebesgue decomposition of Df and Ef

Now that we understand the irregular part of BV and BD functions better we can look into the Lebesgue decomposition of the measures associated with these functions. This is a very important result because it will help us understand how to characterize the energy of materials, more details in [**Ambrosio1997**].

Definition 5.2.10. *Let* $\boldsymbol{f} \in BV(U)$. *Then we can decompose the Radon measure* $D\boldsymbol{f}$ *into*

$$D\boldsymbol{f} = D_{ac}\boldsymbol{f} + D_J\boldsymbol{f} + D_c\boldsymbol{f}$$

such that:

1. *The approximate differential of* $\boldsymbol{f}$ *is the density* $\nabla\boldsymbol{f}$, *which is the absolute continuous with respect to* $\mathcal{L}^n$, *i.e.,*

$$D_{ac}\boldsymbol{f} = \nabla\boldsymbol{f}\mathscr{L}^n.$$

2. *The jump part of* $D\boldsymbol{f}$ *is the restriction* $D_J\boldsymbol{f}$ *of the singular part of the measure,* $D_s\boldsymbol{f}$, *to the jump set* $J_{\boldsymbol{f}}$, *i.e.,*

$$D_{\boldsymbol{f}}\boldsymbol{f} := D_s\boldsymbol{f} \llcorner J_{\boldsymbol{f}}$$

3. *The Cantor part of* $D\boldsymbol{f}$ *is then the remaining part of the measure, i.e.,*

$$D_c\boldsymbol{f} := D_s\boldsymbol{f} \llcorner (U \backslash J_{\boldsymbol{f}})$$

This can be simplified a bit more. Since the cantor part D_c vanishes for all Borel sets B with $\mathscr{H}^{n-1}(B) < +\infty$, we know that the jump part is the only $n-1$ dimension part of $D\boldsymbol{f}$. Which, as we saw before, is a countably $\mathcal{H}^{n-1}-$rectifiable set, so we can rewrite it as:

$$D_J\boldsymbol{f} = \left[\left(\boldsymbol{f}^+_{\boldsymbol{\nu_f}} - \boldsymbol{f}^-_{\boldsymbol{\nu_f}}\right) \otimes \boldsymbol{\nu_f}\right] \mathscr{H}^{n-1} \llcorner J_{\boldsymbol{f}}.$$

For BD functions we have similar results and conclusions so we will just present them all together:

Definition 5.2.11. *Let* $\boldsymbol{f} \in BD(U)$. *Then we can decompose the vector value Randon* *measure $E\boldsymbol{f}$ into*

$$E\boldsymbol{f} = E_{ac}\boldsymbol{f} + E_J\boldsymbol{f} + E_c\boldsymbol{f}$$

such that:

1. *The approximate symmetric differential of $\boldsymbol{f}$ is the density $\mathcal{E}\boldsymbol{f}$ of $E\boldsymbol{f}$ with respect to $\mathcal{L}^n$, i.e.,*

$$E_{ac}\boldsymbol{f} = \mathcal{E}\boldsymbol{f}\mathscr{L}^n.$$

2. *The jump part of $E\boldsymbol{f}$ is the restriction $E_J\boldsymbol{f}$ of the singular part, $E_s\boldsymbol{f}$, to the jump set $J_{\boldsymbol{f}}$, i.e.,*

$$E_J\boldsymbol{f} := E_s\boldsymbol{f} \llcorner J_{\boldsymbol{f}} = \left[\left(\boldsymbol{f}^+_{\boldsymbol{\nu}_{\boldsymbol{f}}} - \boldsymbol{f}^-_{\boldsymbol{\nu}_{\boldsymbol{f}}}\right) \odot \boldsymbol{\nu}_{\boldsymbol{f}}\right] \mathscr{H}^{n-1} \llcorner J_{\boldsymbol{f}}\ ,$$

   where $a \odot b = \frac{1}{2}(a \otimes b + b \otimes a)$.

3. *The Cantor part of $E\boldsymbol{f}$ is then the remaining part of the measure, i.e.,*

$$E_c u := E_s\boldsymbol{f} \llcorner (U \backslash J_{\boldsymbol{f}})\ ,$$

   and vanishes for all Borel sets B with $\mathscr{H}^{n-1}(B) < +\infty$.

Working with the cantor part of the measure is in general very difficult and not very realistic, so it is common to work with functions where this part of the measure vanishes:

Definition 5.2.12 (SBV and SBD)**.** *Let $\boldsymbol{f} \in BV(U)$ and $\boldsymbol{g} \in BD(u)$. Then if,*

$$D_c\boldsymbol{f} = 0, \qquad\qquad E_c\boldsymbol{g} = 0$$

they are considered special functions of bounded variation and special functions of bounded deformation respectively. Additionally, the sets of these functions are normally noted as $SBV(U)$ and $SBD(U)$.

5.2.4 Structure Theorem for BD functions

As we have seen before, an important definition for BD functions is based on looking at the real function $f^{\boldsymbol{\xi}}(x) = (\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{\xi})$, for each direction $\boldsymbol{\xi}$. So an important question to ask is how does the jump set of f relate to the jump set of $\boldsymbol{f}^{\boldsymbol{\xi}}$. The answer is given by the following theorem but to understand the theorem completely we need to introduce some new definitions. For any $\boldsymbol{y}, \boldsymbol{\xi} \in \mathbb{R}^n$ and any $B \subset \mathbb{R}^n$ consider the following spaces:

$\pi_{\boldsymbol{\xi}} := \{\boldsymbol{y} \in \mathbf{R}^n : (\boldsymbol{y}, \boldsymbol{\xi}) = 0\}$, hyperplane orthogonal to $\boldsymbol{\xi}$

$B^{\boldsymbol{\xi}}_{\boldsymbol{y}} := \{t \in \mathbf{R} : \boldsymbol{y} + t\boldsymbol{\xi} \in B\}$, one dimensional representation of B in $\boldsymbol{\xi}$-coordinates

$B^{\boldsymbol{\xi}} := \left\{\boldsymbol{y} \in \pi_{\boldsymbol{\xi}} : B^{\boldsymbol{\xi}}_{\boldsymbol{y}} \neq \emptyset\right\}$, projection of B in $\pi_{\boldsymbol{\xi}}$

$J^{\boldsymbol{\xi}}_{\boldsymbol{f}} := \left\{x \in J_{\boldsymbol{f}} : \left(u^+(\boldsymbol{x}) - u^-(\boldsymbol{x}), \boldsymbol{\xi}\right) \neq 0\right\}$, subset of $J_{\boldsymbol{f}}$ with jumps in $\boldsymbol{\xi}$-direction.

and the following functions:

$$f^{\boldsymbol{\xi}}_{\boldsymbol{y}}(t) := f^{\boldsymbol{\xi}}(\boldsymbol{y} + t\boldsymbol{\xi}) = (\boldsymbol{f}(\boldsymbol{y} + t\boldsymbol{\xi}), \boldsymbol{\xi}) \quad \forall t \in U^{\boldsymbol{\xi}}_{\boldsymbol{y}},$$

$$\boldsymbol{f}^*(\boldsymbol{x}) := \begin{cases} \lim_{r\to 0} -\!\!\!\!\!\!\int_{B(\boldsymbol{x},r)} \boldsymbol{f} d\boldsymbol{y} & \text{if this limit exists} \\ 0 & \text{otherwise} \end{cases}$$

Then theorem 4.5 of [**Ambrosio1997**] states that:

THEOREM **5.2.13** (Structure Theorem). *Let $\boldsymbol{f} \in BD(U)$ and let $\boldsymbol{\xi} \in \mathbf{R}^n$ with $\boldsymbol{\xi} \neq 0$. Then*

1. $(\mathcal{E}\boldsymbol{f}\boldsymbol{\xi}, \boldsymbol{\xi}) = \int_{U^{\boldsymbol{\xi}}} \nabla \boldsymbol{f}^{\boldsymbol{\xi}}_{\boldsymbol{y}} d\mathscr{H}^{n-1}(\boldsymbol{y}), \quad |(\mathcal{E}\boldsymbol{f}\boldsymbol{\xi}, \boldsymbol{\xi})| = \int_{U^{\boldsymbol{\xi}}} \left|\nabla \boldsymbol{f}^{\boldsymbol{\xi}}_{\boldsymbol{y}}\right| d\mathscr{H}^{n-1}(\boldsymbol{y}).$

2. *For $\mathscr{H}^{n-1}$-almost every $\boldsymbol{y} \in U^{\boldsymbol{\xi}}$, the functions $\boldsymbol{f}^{\boldsymbol{\xi}}_{\boldsymbol{y}}$ and $\boldsymbol{f}^{*\boldsymbol{\xi}}_{\boldsymbol{y}}$ belong to $BV\left(U^{\boldsymbol{\xi}}_{\boldsymbol{y}}\right)$ and coincide $\mathscr{L}^1$-almost everywhere on $U^{\boldsymbol{\xi}}_{\boldsymbol{y}}$, the measures $\left|D\boldsymbol{f}^{\boldsymbol{\xi}}_{\boldsymbol{y}}\right|$ and $V\boldsymbol{f}^{*\boldsymbol{\xi}}_{\boldsymbol{y}}$ coincide on $U^{\boldsymbol{\xi}}_{\boldsymbol{y}}$, and*

$$(\mathcal{E}\boldsymbol{f}(\boldsymbol{y} + t\boldsymbol{\xi})\boldsymbol{\xi}, \boldsymbol{\xi}) = \nabla \boldsymbol{f}^{\boldsymbol{\xi}}_{\boldsymbol{y}}(t) = \left(\boldsymbol{f}^{*\boldsymbol{\xi}}_{\boldsymbol{y}}\right)'(t)$$

for $\mathscr{L}^1$-almost every $t \in U^{\boldsymbol{\xi}}_{\boldsymbol{y}}$.

3. $(E_J\boldsymbol{f}\boldsymbol{\xi},\boldsymbol{\xi}) = \int_{U\boldsymbol{\xi}} D_J f_{\boldsymbol{y}}^{\boldsymbol{\xi}} d\mathscr{H}^{n-1}(\boldsymbol{y}), \quad |(E_J\boldsymbol{f}\boldsymbol{\xi},\boldsymbol{\xi})| = \int_{U\boldsymbol{\xi}} \left|D_J f_{\boldsymbol{y}}^{\boldsymbol{\xi}}\right| d\mathscr{H}^{n-1}(\boldsymbol{y})$.

4. $\left(J_{\boldsymbol{f}}^{\boldsymbol{\xi}}\right)_{\boldsymbol{y}}^{\boldsymbol{\xi}} = J_{f_{\boldsymbol{y}}^{\boldsymbol{\xi}}}$ *for* $\mathscr{H}^{n-1}$*-almost every* $\boldsymbol{y} \in U^{\boldsymbol{\xi}}$

$$\left(\boldsymbol{f}^{+}(\boldsymbol{y}+t\boldsymbol{\xi}),\boldsymbol{\xi}\right) = \left(f_{\boldsymbol{y}}^{\boldsymbol{\xi}}\right)^{+}(t) = \lim_{s\to t^{+}} \boldsymbol{f}_{\boldsymbol{y}}^{*\boldsymbol{\xi}}(s),$$
$$\left(\boldsymbol{f}^{-}(\boldsymbol{y}+t\boldsymbol{\xi}),\boldsymbol{\xi}\right) = \left(f_{\boldsymbol{y}}^{\boldsymbol{\xi}}\right)^{-}(t) = \lim_{s\to t^{-}} \boldsymbol{f}_{\boldsymbol{y}}^{*\boldsymbol{\xi}}(s),$$

and for every $t \in \left(J_{\boldsymbol{f}}^{\boldsymbol{\xi}}\right)_{\boldsymbol{y}}^{\boldsymbol{\xi}}$ *where the normals to* $J_{\boldsymbol{f}}$ *and* $J_{f_{\boldsymbol{y}}^{\boldsymbol{\xi}}}$ *are oriented so that* $(\boldsymbol{\nu}_{\boldsymbol{f}},\boldsymbol{\xi}) \geq 0$.

5. $(E_c\boldsymbol{f}\boldsymbol{\xi},\boldsymbol{\xi}) = \int_{U\boldsymbol{\xi}} D_c f_{\boldsymbol{y}}^{\boldsymbol{\xi}} d\mathscr{H}^{n-1}(\boldsymbol{y}), |(E_c\boldsymbol{f}\boldsymbol{\xi},\boldsymbol{\xi})| = \int_{U\boldsymbol{\xi}} \left|D_c f_{\boldsymbol{y}}^{\boldsymbol{\xi}}\right| d\mathscr{H}^{n-1}(\boldsymbol{y})$.

There are a lot of things to understand in the previous Theorem, however the most important for this Thesis is to understand part 4. In simpler terms, it means that when we fix a direction, and look at the function $f_{\boldsymbol{y}}^{\boldsymbol{\xi}}$ we get a function of bounded variation, and the jump set of this function is the projection of the jump set of $\boldsymbol{f}$ in the direction $\boldsymbol{\xi}$. This is very important because it means that we can study the jump set of $\boldsymbol{f}$ by looking at the jump set of $f_{\boldsymbol{y}}^{\boldsymbol{\xi}}$ for almost every $\boldsymbol{y}$ and $\boldsymbol{\xi}$.

To conclude we will add a simple Corolary about SBD functions and a final remark about the inclusion of all the spaces introduced in this section from [**SBD3**].

Corollary 5.2.14. *Let* $\boldsymbol{f} \in BD(U)$ *and let* $\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^n$ *be a basis of* $\mathbb{R}^n$*. Then the following two conditions are equivalent:*

1. $\boldsymbol{f} \in \mathrm{SBD}(U)$.

2. *For every* $\boldsymbol{\xi} = \boldsymbol{\xi}^i + \boldsymbol{\xi}^j$ *with* $1 \leqq i, j \leq n$*, we have* $f_{\boldsymbol{y}}^{\boldsymbol{\xi}} \in SBV\left(U_{\boldsymbol{y}}^{\boldsymbol{\xi}}\right)$ *for* $\mathscr{H}^{n-1}$ *almost every* $\boldsymbol{y} \in U\boldsymbol{\xi}$.

Theorem **5.2.15.** *We have* $BV(U;\mathbf{R}^n) \cap SBD(U) = SBV(U;\mathbf{R}^n)$*. Moreover the inclusions* $SBV(U;\mathbf{R}^n) \subseteq SBD(U) \subseteq BD(U)$ *are strict.*

5.3 Known Korn's Inequalities for SBD functions

As we have seen in Chapter 2, Korn inequalities are very important for the development of the theory of elasticity, so in the last decade, many authors have been working to extend this result and improve Fracture Mechanics theory. Several function spaces derived from BD were considered, which makes this problem particularly challenging since the gradient of this type of function might not even be a bounded measure, in fact, the problem of whether the analog of Korn's inequality for SBD functions $\boldsymbol{u}$ in $\Omega \backslash J_{\boldsymbol{u}}$ is true, is still open. In order to overcome this issue, several research groups remove from Ω a small neighborhood ω of the jump set and prove the inequality in $\Omega \backslash \omega$. This was first established by Chambolle, Conti, and Francfort in 2014 [**kornBD1**]. However since this topic has been so active the same group of authors have been able to improve the result several times [**kornBD2**, **kornBD3**]. Simultaneously, Manuel Friedrich was able to also prove the similar Korn-Poincaré version of the results in a series of papers [**kornBD4**, **kornBD5**].

All the results are very similar, however, they can use different spaces or have different assumptions on the jump and different bounds on the size of ω. The main goal of this chapter is to extend this type of inequality for plates, and we believe that the original technique used will work for most of the cases. To improve the readability of the results we chose to work with the Theorem 1.1 from [**kornBD3**]:

THEOREM **5.3.1.** *Let $n \in \mathbb{N}$ with $n \geq 2, p \in (1, \infty)$, and let $\Omega \subset \mathbb{R}^n$ be a bounded connected and open Lipschitz set. There exists $c = c(n, p, \Omega) > 0$ such that, for any $\boldsymbol{u} \in SBD^p(\Omega)$, there is a set of finite perimeter $\omega \subset \Omega$ with $\mathcal{H}^{n-1}(\partial^* \omega) \leq c\mathcal{H}^{n-1}(J_u)$ and a skew-symmetric matrix A such that*

$$\int_{\Omega \backslash \omega} |\nabla \boldsymbol{u} - A|^p dx \leq c(n, p, \Omega) \int_{\Omega} |e(\boldsymbol{u})|^p dx$$

The constant c is invariant under uniform scalings of the domain.

Remark 5.3.2. *In fact, the Theorem is valid for generalized functions of special bounded deformations, $GSBD^p$, however, to not introduce more complexity to this work we will just consider the case of SBD^p functions, that is, functions in SBD such that $e(\boldsymbol{u}) \in L^p(\Omega)$.*

5.4 Korn inequality in plates for Sobolev functions, revisited

Having established the Korn inequality for functions in cubes, we can hope that extending the result for plates would follow the same steps as in the case of Sobolev functions, however is not that simple. The traditional proof of Korn's inequality with the desirable asymptotics in a plate [**bib:Fri.Jam.Mue.1**, **bib:Fri.Jam.Mue.2**, **conti0**] normally follows the following steps (Very similar to what we did for the weighted Korn inequality 4.6):

1. Split the plate in N^2 overlapping cubes of size h^3.
2. Apply the Korn inequality for bulk domains, 2.3.1, in each cube to obtain skew-symmetric matrices $A_{i,j}$.
3. Interpolate the matrices $A_{i,j}$ using a partition of unity to obtain a piecewise function R_h.
4. Apply Poincaré inequality to R_h to obtain constant matrix R.
5. Choose the closest skew-symmetric matrix to R to obtain the desired matrix A.

Almost all the steps are valid for SBD^p functions, except for the Poincaré inequality. Even if we are just applying the inequality in a piecewise linear function, as we can see from the Theorem 5.3.1, the inequality is not valid for the full domain, we need to exclude a set ω around the jump. While we can still apply the Poincaré Inequality in sets with small holes, there are no known bounds for the Poincaré constant, and it can be arbitrarily large. This was the main obstacle while extending the Korn inequality for SBD^p functions in plates.

To overcome this problem we will start by giving an alternate proof of the Korn inequality for Sobolev functions in plates that do not need the Poincaré inequality but still give the desired asymptotics, $\mathcal{O}(\frac{1}{h})$. After understanding better this new technique we will be able to prove the final result easily.

THEOREM **5.4.1** (Korn's first inequality in plates for Sobolev functions)**.** *Let $D = [0,1]^2$ and $\Omega_h = D \times [-h,h]$ for some small h. Then for any $\boldsymbol{u} \in W^{1,p}(\Omega_h : \mathbb{R}^3)$ there is a skew-symmetric matrix A such that*

$$\|\nabla \boldsymbol{u} - A\|_{L^p(\Omega_h)} \leq \frac{C(p)}{h} \|e(\boldsymbol{u})\|_{L^p(\Omega_h)}.$$

Proof. Since we are just interest on the asymptotics when $h \to 0$, we can assume, *w.l.g.*, that $N = \frac{1}{h} \in \mathbb{N}$.

Step 1: The first step is to split Ω_h in overlapping cubes and apply the traditional Korn's first inequality in each cube. Consider the $(N-1)^2$ overlapping cubes of size $(2h)^3$:

$$D_{i,j} = \left[\frac{i-1}{N}, \frac{i+1}{N}\right] \times \left[\frac{j-1}{N}, \frac{j+1}{N}\right] \times [-h,h], \qquad i,j = 1, \dots, N-1,$$

and apply Korn inequality (2.3.1) to each cube to obtain obtain a skew-symmetric matrix $A_{i,j}$ such that

$$\|\nabla \boldsymbol{u} - A_{i,j}\|_{L^p(D_{i,j})} \leq C_0(p) \|e(\boldsymbol{u})\|_{L^p(D_{i,j})} \tag{5.4.1}$$

where C_0 is independent of the cube and h.

Step 2: To be able to find the matrix that works in the full domain Ω_h we need to see how the matrices $A_{i,j}$ change between two intersecting cubes. Let $D_{i,j}$ and $D_{i',j'}$ be

2 intersecting cubes and the respective matrices $A_{i,j}$ and $A_{i',j'}$, then

$$
\begin{aligned}
(2h)^3|A_{i,j} - A_{i',j'}|^p &= \|A_{i,j} - A_{i',j'}\|^p_{L^p(D_{i,j}\cap D_{i',j'})} \\
&\leq 2^{p-1}\|\nabla \boldsymbol{u} - A_{i,j}\|^p_{L^p(D_{i,j}\cap D_{i',j'})}p + \|\nabla \boldsymbol{u} - A_{i',j'}\|^p_{L^p(D_{i,j}\cap D_{i',j'})} \\
&\leq 2^{p-1}\|\nabla \boldsymbol{u} - A_{i,j}\|^p_{L^p(D_{i,j})}p + \|\nabla \boldsymbol{u} - A_{i',j'}\|^p_{L^p(D_{i',j'})} \\
&\leq 2^{p-1}C_0\left(\|e(\boldsymbol{u})\|^p_{L^p(D_{i,j})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{i',j'})}\right).
\end{aligned}
$$

Additionally, since all A's are constant and all the cubes have the same size, for any (k,l) we have that

$$
\|A_{i,j} - A_{i',j'}\|^p_{L^p(D_{k,l})} \leq 2^pC_0\left(\|e(\boldsymbol{u})\|^p_{L^p(D_{i,j})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{i',j'})}\right). \tag{5.4.2}
$$

Step 3: For non-intersecting cubes we need to create a path between them, in this step we will see how that works and how can it help. First let's fix a matrix in the first collum, $j = 1$, and fix some row i, then:

$$
\|\nabla \boldsymbol{u} - A_{i,1}\|^p_{L^p(\Omega_h)} \leq \sum_{k=1}^{N-1}\sum_{l=1}^{N-1}\|\nabla \boldsymbol{u} - A_{i,1}\|^p_{L^p(D_{k,l})}.
$$

To bound each term in the double sum, we need to consider a path between $D_{i,1}$ and $D_{k,l}$, so for $k \geq i$, we can consider a path of size less than $2N$ of the form

$$
D_{i,1} \to D_{i,2} \to \cdots \to D_{i,l} \to D_{i+1,l} \to \cdots \to D_{k,l},
$$

as we can see in the figure 5.1.

For each term in the double sum, we can split it again through all the cubes in the path and apply inequality 5.4.2:

$$
\begin{aligned}
\|\nabla u - A_{i,1}\|^p_{L^p(D_{k,l})} &= \left\|\nabla \boldsymbol{u} + \sum_{l'=2}^{l}(A_{i,l'} - A_{i,l'-1}) + \sum_{k'=i+1}^{k}(A_{k',l} - A_{k'-1,l}) - A_{k,l}\right\|^p_{L^p(D_{k,l})} \\
&\leq (2N)^{p-1}\Bigg(\|\nabla \boldsymbol{u} - A_{k,l}\|^p_{L^p(D_{k,l})} + \sum_{l'=2}^{l}\|A_{i,l'} - A_{i,l'-1}\|^p_{L^p(D_{k,l})}
\end{aligned}
$$

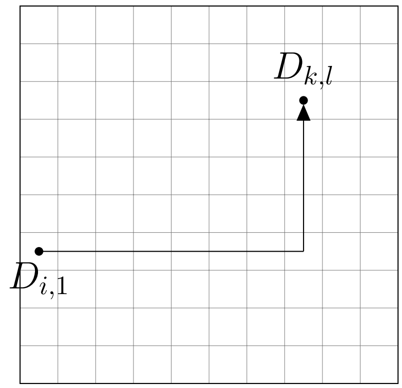


Figure 5.1: In this grid we consider that the cubes do not intersect each other for better visualization. The arrow represents one of the possible path we can take.

$$
\begin{aligned}
&\qquad\qquad + \sum_{k'=i+1}^{k} \|A_{k',l} - A_{k'-1,l}\|^p_{L^p(D_{k,l})}\Bigg) \\
&\leq C_0(2N)^{p-1}\Bigg(\|e(\boldsymbol{u})\|^p_{L^p(D_{k,l})} + 2^p \sum_{l'=2}^{l}\left(\|e(\boldsymbol{u})\|^p_{L^p(D_{i,l'})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{i,l'-1})}\right) \\
&\qquad\qquad + 2^p \sum_{k'=i+1}^{k}\left(\|e(\boldsymbol{u})\|^p_{L^p(D_{k',l})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{k'-1,l})}|\right)\Bigg) \\
&\leq C_0 2^{2p+1} N^{p-1}\left(\|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{-,l})}\right),
\end{aligned}
$$

where $D_{i,-} = \cup_{l=1}^{N-1} D_{i,l}$ is the row i and $D_{-,l} = \cup_{i=1}^{N-1} D_{i,l}$ is the column l.

For $k < i$ a very similar argument holds and we get the same bound:

$$
\|\nabla \boldsymbol{u} - A_{i,1}\|^p_{L^p(D_{k,l})} \leq C_0 2^{2p+1} N^{p-1}\left(\|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{-,l})}\right).
$$

Therefore, by summing over all k, l we obtain:

$$
\begin{aligned}
\sum_{k=1}^{N-1}\sum_{l=1}^{N-1} \|\nabla \boldsymbol{u} - A_{i,1}\|^p_{L^p(D_{k,l})} &\leq C_0 2^{2p+1} N^{p-1} \sum_{k=1}^{N-1}\sum_{l=1}^{N-1}\left(\|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{-,l})}\right) \\
&\leq C_0 2^{2p+1} N^{p-1}\left(N^2\|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + 2\sum_{k=1}^{N-1}\|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)}\right) \\
&\leq C_0 2^{2p+1} N^{p-1}\left(N^2\|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + 2N\|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)}\right).
\end{aligned}
$$

Step 4: At first glance it looks like we finished the proof, however, we don't have the desired power of N. To get the desired power notice that we still have freedom over i.

Summing over all i we have that:

$$\sum_{i=1}^{N-1} \|\nabla \boldsymbol{u} - A_{i,0}\|^p_{L^p(\Omega_h)} \leq C_0 2^{2p+1} N^{p-1} \left(N^2 \sum_{i=1}^{N-1} \|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + 2N^2 \|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)} \right)$$
$$\leq C_0(p) 2^{2p+3} N^{p+1} \|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)}.$$

So in fact the power of N doesn't change, but we are summing over $N-1$ elements, so there must exist some i' that will satisfy the inequality we are aiming at:

$$\|\nabla \boldsymbol{u} - A_{i',0}\|^p_{L^p(\Omega_h)} \leq C(p) N^p \|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)} = \frac{C(p)}{h^p} \|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)}.$$

□

5.5 Korn inequality for plates in the space SBD

In this section, we will prove the main result of this chapter, the Korn Inequality in plates for SBD^p functions. More challenges will show up because not every path will be admissible due to the existence of jumps/cracks, however, the general idea will be very similar to the previous proof.

THEOREM **5.5.1.** *Let $D = [-1,1]^2$ and $\Omega_h = D \times [-h,h]$ for some small h. Then for $\delta > 0$ small enough, there exists $C = C(p) > 0$, such that for any $\boldsymbol{u} \in SBD^p(\Omega_h)$ with $\mathcal{H}^2(J_u) < \delta h$ there is a set $\omega \in \Omega_h$ with $\mathcal{L}^3(\omega) \leq C\mathcal{H}^2(J_u)$ and a skew-symmetric matrix A such that*

$$\|\nabla \boldsymbol{u} - A\|^p_{L^p(\Omega_h \setminus \omega)} dx \leq \frac{C(p)}{h^p} \|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)}$$

Proof. The beginning of the proof is exactly the same, we start by considering $N = \frac{1}{h} \in \mathbb{N}$ and apply the Korn inequality 5.3.1 at each overlapping cube:

$$D_{i,j} = \left[\frac{i-1}{N}, \frac{i+1}{N}\right] \times \left[\frac{j-1}{N}, \frac{j+1}{N}\right] \times [-h,h], \qquad i,j = 1,\dots,N-1.$$

to obtain the skew-symmetric matrixes $A_{i,j}$ and the sets of finite perimeter $\omega_{i,j} \subset D_{i,j}$

with $\mathcal{H}^2(\partial^*\omega_{i,j}) \le C\mathcal{H}^2(J_u \cap D_{i,j})$ such that

$$\|\nabla \boldsymbol{u} - A_{i,j}\|^p_{L^p(D_{i,j}\setminus\omega_{i,j})} \le C_0(p)\|e(\boldsymbol{u})\|^p_{L^p(D_{i,j})}. \tag{5.5.3}$$

However, when working with plates, bounding the perimeter is not sufficient. So using the size of each cube we can also bound the volume of each $\omega_{i,j}$. Applying the isoperimetric inequality, we get that:

$$\mathcal{L}(\omega_{i,j}) = (\mathcal{L}(\omega_{i,j}))^{2/3}(\mathcal{L}(\omega_{i,j}))^{1/3} \le 2Ch\mathcal{H}^2(\partial^*\omega_{i,j}) \le 2C_p h\mathcal{H}^2(J_u \cap D_{i,j}).$$

Step 2: Here is when things start getting a bit different, because of $\omega_{i,j}$, we can't always control the difference between $A_{i,j}$ and $A_{i',j'}$ for two intersecting cubes $D_{i,j}$ and $D_{i',j'}$. In fact, to do that, we also need to control the size of $\omega_{i,j}$ and $\omega_{i',j'}$, so let's label the cubes according to the portion of the jump that lands on them, i.e, let $D_{k,j}$ be a good cube if

$$\mathcal{H}^2(J_u \cap D_{k,j}) \le \frac{1}{4C_pC_0}h^2,\ \mathcal{L}(\omega_{k,j}) \le h^3,$$

and a bad cube otherwise, we can define

$$G_{idx} = \{(k,j) : D_{k,j} \text{ is a good cube}\},$$

$$B_{idx} = \{(k,j) : D_{k,j} \text{ is a bad cube}\}.$$

Now for this definition to be useful we need to make sure that we have enough good cubes to represent the plate, but since we control the size of the jump and since every point in Ω_h can be at most at 4 cubes, then

$$\begin{aligned}\mathcal{H}^2(J_u) &= \mathcal{H}^2\left(\bigcup_{i,j}(J_u \cap D_{i,j})\right)\\ &\ge \frac{1}{4}\sum_{k,j}\mathcal{H}^2(J_u \cap D_{i,j})\\ &\ge Ch^2\mathcal{H}^0(B_{idx}).\end{aligned}$$

and consequently, we can bound the number of bad cubes two in different ways:

$$\mathcal{H}^0(B_{idx}) \leq \frac{C\mathcal{H}^2(J_u)}{h^2}, \tag{5.5.4}$$

$$\mathcal{H}^0(B_{idx}) \leq C\delta N. \tag{5.5.5}$$

Step 3: Consider 2 intersecting good cubes, $D_{i,j}, D_{i',j'}$, then we can actually bound the difference of $A_{i,j}$ and $A_{i',j'}$. First notice that by the definition of a good cube, we can find a lower bound of the good part of their intersection, *i.e.*, $\mathcal{L}((D_{i,j} \cap D_{i',j'}) \backslash (\omega_{i,j} \cup \omega_{i',j'})) \geq 2h^3$ and then we can repeat the same argument as before

$$\begin{aligned}
(2h)^3 |A_{i,j} - A_{i',j'}|^p &\leq \|A_{i,j} - A_{i',j'}\|^p_{L^p((D_{i,j} \cap D_{i',j'}) \backslash (\omega_{i,j} \cup \omega_{i',j'}))} \\
&\leq 2^{p-1} \left(\|\nabla \boldsymbol{u} - A_{i,j}\|^p_{L^p(D_{i,j} \backslash \omega_{i,j})} + \|\nabla \boldsymbol{u} - A_{i',j'}\|^p_{L^p(D_{i',j'} \backslash \omega_{i',j'})} \right) \\
&\leq C(p) \left(\|e(\boldsymbol{u})\|^p_{L^p(D_{i,j})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{i',j'})} \right).
\end{aligned}$$

Furthermore, for any target good cube, $D_{i'',j''}$, that we will need to integrate over, we also have that

$$\begin{aligned}
\|A_{k,j} - A_{k',j'}\|^p_{L^p(D_{i'',j''} \backslash \omega_{i'',j''})} &\leq 8h^3 |A_{k,j} - A_{k',j'}|^p \\
&\leq C \left(\|e(\boldsymbol{u})\|^p_{L^p(D_{i,j} \backslash \omega_{i,j})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{i',j'} \backslash \omega_{i',j'})} dx \right).
\end{aligned}$$

Step 4: In this next step we will define the set ω. At least we know that it has to contain all $\omega_{i,j}$, but from the previous step. it would also make sense to add all the bad cubes. Maybe that would be enough, but to simplify the proof we will consider a bigger ω that is still small enough.

Let $D_{i,-} = \cup_{j'} D_{i,j'}$, $D_{-,j} = \cup_{i'} D_{i',j}$ be the row i and column j respectively. We say that $D_{i,-}$ is a good row if $D_{i,j'}$ is a good cube for all j' and a bad row otherwise, and similarly for $D_{-,j}$. To simplify the notation let's define:

$$B^r_{idx} = \{i : D_{i,-} \text{ is a bad row}\}, \qquad \mathcal{H}^0(B^r_{idx}) \leq \mathcal{H}^0(B_{idx}),$$

$$B^c_{idx} = \{j : D_{-,j} \text{ is a bad column}\}, \qquad \mathcal{H}^0(B^c_{idx}) \leq \mathcal{H}^0(B_{idx}).$$

To make it easy to construct paths between good cubes, we can exclude all the cubes that belong to both a bad row and a bad collum, *i.e.* (check figure 5.2 for a better understanding):

$$\omega = \left(\bigcup_{i \in B^r_{idx}, j \in B^c_{idx}} D_{i,j} \right) \bigcup \left(\bigcup_{i,j} \omega_{i,j} \right).$$

To bound the size of ω we will use the fact that we can't have that many bad rows and columns, so

$$\begin{aligned}
\mathcal{L}(\omega) &\leq (2h)^3 \mathcal{H}^0(B^r_{idx}) \mathcal{H}^0(B^c_{idx}) + \sum_{(i,j) \notin B_{idx}} \mathcal{L}(\omega_{i,j}) \\
&\leq \delta 2^6 C^2 c^2 \mathcal{H}(J_u)^2 + 8Cch\mathcal{H}^2(J_u) \\
&\leq C\delta \mathcal{H}^2(J_u).
\end{aligned}$$

Step 5: Now that we finally define all the good cubes, we need to find paths of $\mathcal{O}(N)$ cubes to connect them. For the rest of the proof to follow as before we need to make sure that the path only passes through good cubes and additionally, no more than $\mathcal{O}(N)$ paths can pass in the same cube outside row i.

For this let's start by considering the sequences containing the indexes of the good and bad columns, g_l and b_l, respectively:

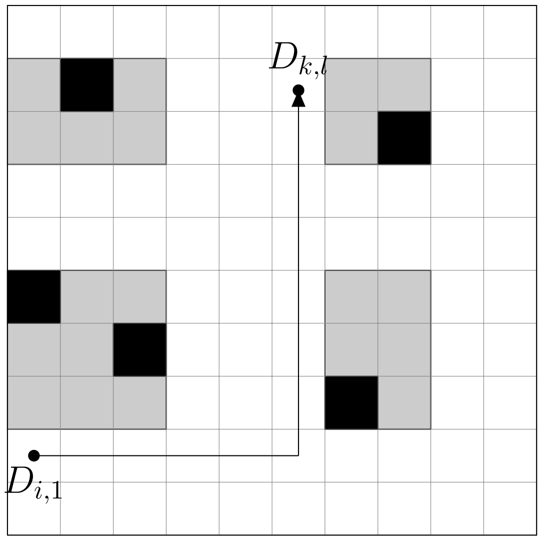
$$g_l = l\text{-th smallest index not in } B^c_{idx} \qquad l = 1, \cdots G = (N-1) - \mathcal{H}^0(B^c_{idx}),$$

$$b_l = l\text{-th smallest index in } B^c_{idx} \qquad l = 1, \cdots B = (N-1) - \mathcal{H}^0(B^c_{idx}).$$

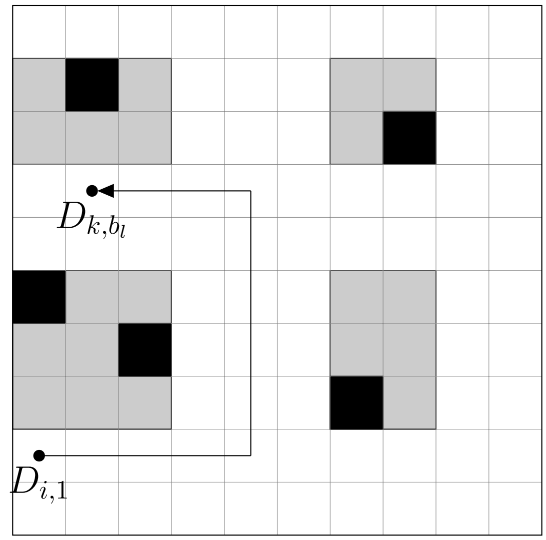
Consider $j = 1$ and fix $i \notin B^r_{idx}$, so

$$\begin{aligned}
\int_{\Omega \setminus \omega} \|\nabla \boldsymbol{u} - A_{i,1}\|^p_{L^p(\Omega_h \setminus \omega)} \leq & \sum_{l=1}^{G} \sum_{k=0}^{N-1} \|\nabla \boldsymbol{u} - A_{i,1}\|^p_{L^p(D_{k,g_l} \setminus \omega_{h,g_l})} \\
&+ \sum_{l=1}^{B} \sum_{k \notin B^r_{idx}} \|\nabla \boldsymbol{u} - A_{i,j_1}\|^p_{L^p(D_{k,b_l} \setminus \omega_{k,b_l})}.
\end{aligned}$$

W.l.o.g. we can consider $k \geq i$ and approach the two terms on the right-hand side separately. For the first term, the target cube, D_{k,g_l}, is in a good column, and for the second the target cube will not be in a good column but it will be in a good row, so as we can see in the figure 5.2 there are two different types of paths we can use.



(a) For the case that the target cube $D_{k,l}$ is in a good column we can take a similar path as before.



(b) When the target cube is not in a good collum but it's in a good row we can take a path like shown in figure.

Figure 5.2: In this grid we consider that the cubes do not intersect each other for better visualization. The black cubes represent bad cubes because $\omega_{i,j}$ is too big, while gray ones are cubes that are part of a bad row and a bad collumn.

For the first case(5.2a)) the path is simillar to the previous proof:

$$D_{i,1} \to D_{i,2} \to \cdots \to D_{i,g_l} \to D_{i+1,g_l} \to \cdots \to D_{k,g_l}.$$

So as in the case of $\mathcal{L}^p$ functions, we have that

$$\|\nabla \boldsymbol{u} - A_{i,1}\|^p_{L^p(D_{k,g_l} \setminus \omega_{k,g_l})} \leq CN^{p-1} \left(\int_{D_{i,-}} \|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{-,g_l})} \right).$$

However, for the second term, we need to find a path that goes around the islands created by ω, but we need to be careful to not have too many paths passing in the same column. For δ small enough, there will be fewer bad columns than good ones so we can consider the following path represented in the figure 5.2b):

$$D_{i,1} \to \cdots \to D_{i,g_l} \to \cdots \to D_{k,g_l} \to \cdots \to D_{k,b_l}.$$

And repeating the same steps as in the Sobolev case we get:

$$\|\nabla \boldsymbol{u} - A_{i,1}\|^p_{L^p(D_{k,b_l}\setminus\omega_{k,b_l})} \leq CN^{p-1}\left(\|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{k,-})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{-,g_l})}\right).$$

So substituting in both terms of (**??**) we get:

$$\begin{aligned}
\sum_{l=1}^{G}\sum_{k=0}^{N-1}\|\nabla \boldsymbol{u} - A_{i,1}\|^p_{L^p(D_{k,g_l}\setminus\omega_{k,g_l})} &\leq CN^{p-1}\sum_{l=1}^{G}\sum_{k=0}^{N-1}\left(\|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{-,g_l})}\right)\\
&\leq CN^{p-1}\left(N^2\|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + 2\sum_{k=1}^{N-1}\|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)}\right)\\
&= CN^{p+1}\|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + CN^{p}\|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)},
\end{aligned}$$

$$\begin{aligned}
&\sum_{l=1}^{B}\sum_{k\notin b^r_{idx}}\|\nabla \boldsymbol{u} - A_{i,1}\|^p_{L^p(D_{k,b_l}\setminus\omega_{k,b_l})}\\
&\qquad\leq CN^{p-1}\sum_{l=1}^{B}\sum_{k\notin b^r_{idx}}\left(\|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{k,-})} + \|e(\boldsymbol{u})\|^p_{L^p(D_{-,b_l})}\right)\\
&\qquad\leq CN^{p-1}\left(N^2\|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + 2\sum_{l=1}^{B}\|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)} + 2\sum_{k\notin b^r_{idx}}\|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)}\right)\\
&\qquad\leq CN^{p+1}\|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + CN^{p}\|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)}.
\end{aligned}$$

Step 6: Now we can conclude the proof by summing over all $i \notin B^r_{idx}$ to get the desired asymptotics.

$$\begin{aligned}\sum_{i \notin B^r_{idx}} \|\nabla \boldsymbol{u} - A_{i,0}\|^p_{L^p(\Omega_h \setminus \omega)} &\leq CN^{p+1} \sum_{i \notin B^r_{idx}} \|e(\boldsymbol{u})\|^p_{L^p(D_{i,-})} + CN^p \sum_{i \notin B^r_{idx}} \|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)} \\ &\leq CN^{p+1} \|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)},\end{aligned}$$

and since we are summing over at least $N/2$ elements, there must exist i' such that

$$\|\nabla \boldsymbol{u} - A_{i',0}\|^p_{L^p(\Omega_h \setminus \omega)} \leq C(p) N^p \|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)} = \frac{C(p)}{h^p} \|e(\boldsymbol{u})\|^p_{L^p(\Omega_h)},$$

as desired.

$\square$