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Introduction to Numerical Methods

Exercise no. 12

Hand in before the beginning of the exercise class on 19.01.2023

Exercise 12.1 (4 points) Let $Q \in \mathbb{R}^{n\times n}$ be an orthonormal matrix, that is $Q^TQ = I$. Show that $\|Qx\|_2 = \|x\|_2$ for all $x \in \mathbb{R}^n$, $\|Q\|_2 = 1$ and $\kappa_2(Q) = 1$.

Exercise 12.2 (4 points) For every vector $x \in \mathbb{R}^n$ let the Taxicab norm be given by

$$\|x\|_1 := |x_1| + |x_2| + \ldots + |x_n|.$$

(Similarly for $\mathbb{R}^m$). Show that the Taxicab norm induces the following matrix norm:

$$\|A\|_1 := \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}| \quad \forall A \in \mathbb{R}^{m\times n} \quad \text{(Column sum norm).}$$

Exercise 12.3 (2 points) (Bonus) For every matrix $A \in \mathbb{R}^{m\times n}$ the Frobenius norm is defined by

$$\|A\|_F := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}.$$

Now, let $m = n$ and $n > 1$. Show that the Frobenius norm is not induced by any vector norm, i.e. there exist no vector norm $\|\cdot\|\colon \mathbb{R}^n \to \mathbb{R}$, such that for every matrix $A \in \mathbb{R}^{n\times n}$ it holds

$$\|A\|_F = \max_{0\neq x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|}$$

Exercise 12.4 (4 points)(Bonus) Let $A \in \mathbb{R}^{n\times n}$ be a regular matrix.

a) Show that $x \in \mathbb{R}^n$ is an eigenvector of A for the eigenvalue $\lambda \in \mathbb{R}$ if and only if $x \in \mathbb{R}^n$ is an eigenvector of A^{-1} for the eigenvalue $\lambda^{-1} \in \mathbb{R}$.

b) Additionally, let A be symmetric. Show that

$$\kappa_2(A) = \frac{|\lambda_n|}{|\lambda_1|}$$

where $\lambda_n \in \mathbb{R}$ denotes the eigenvalue of A with the largest absolute value and λ_1 denotes the eigenvalue of A with the smallest absolute value.