Universität Duisburg-Essen – Winter term 22/23

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Mock Exam Introduction to Numerical Methods

XX.XX.2023, XX:00-XX:00 Uhr (120 minutes)

- Place your student ID card in front of you on your desk so that your identity can be checked.
- The exam consists of **6 tasks**. For every task there are two additional pages provided. If the available space is not sufficient, ask the supervisor for additional paper. Your own paper is not allowed. The exam must remain stapled.
- If you wish to leave the room, please sign out with the supervisor first. An early submission of the exam is not possible during the last 30 minutes.
- You should only take part in this exam if you feel that you are in good health. This is assumed at the beginning of the examination. It is not possible to submit a medical attestation after the examination.
- It is allowed to use a calculator. **No** other auxiliary means are allowed. Electronic devices such as cell phones will be immediately considered as an attempt to cheat without further examination of their use.
- All steps of the solutions must be described in a **comprehensible** and **readable** way! Results from the lecture, however, may be used without a proof.

Student - Id - Number:

Last name:

First name:

Degree program:

I have read the above points. My personal data is correct:

Signature: ____

Exercise	1	2	3	4	5	6	Sum
Achieved points							
Max. points	8	8	8	8	8	8	48

Grade	Signature supervisor

To be filled in at the post-exam review:

Day of post-exam review	Signature

1. Let the interval $I = [-1, 1]$ and the function $f : I \to \mathbb{R}, x \mapsto 2x^3 + x^2 - x - 2$ be given.
   (a) (4 points) Compute the interpolation polynomial p of degree 2 of f with respect to the nodes $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.
   (b) (2 points) Estimate the interpolation error $\|f - p\|_\infty$ based on the result from the lecture.
   (c) (2 points) We consider the decompositions

$$\mathcal{T} = \{-3, -1, 1, 3\}, \quad \widetilde{\mathcal{T}} = \{-3, -1, 0, -1, 3\}.$$

   Decide if the following statements are true or false:
   (i) The function $f : [-3, 3] \to \mathbb{R}, x \mapsto |x|$ is an element of $S_{1,\mathcal{T}}$. True ☐ False ☐
   (ii) The function $f : [-3, 3] \to \mathbb{R}, x \mapsto |x|$ is an element of $S_{1,\widetilde{\mathcal{T}}}$. True ☐ False ☐
   (iii) For $k \in \mathbb{N}$, it holds $S_{k,\mathcal{T}} \subset S_{k+1,\mathcal{T}}$. True ☐ False ☐
   (iv) For $k \in \mathbb{N}$, it holds $S_{k+1,\mathcal{T}} \subset S_{k,\mathcal{T}}$. True ☐ False ☐

2. (a) (4 points) Using the trapezoidal rule, approximate the integral

$$\int_0^4 x^3 + 2x^2 dx.$$

Give an estimate for the approximation error, based on the results from the lecture.

(b) (4 points) Let $x_0 = 0$, $x_1 = \pi$, $x_2 = 2\pi$ and

$$I(f) := \int_0^{2\pi} f(x)dx, \quad Q(f) := 2\pi \sum_{i=0}^{2} \lambda_i f(x_i)$$

with $\lambda_i \in \mathbb{R}$ for $i = 0, 1, 2$.

(i) Compute $\lambda_i \in \mathbb{R}$ for $i = 0, 1, 2$, such that the quadrature rule Q is exact for the functions $f, g, h \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 1, \quad g(x) = x^2, \quad h(x) = \cos(x) \quad \forall x \in \mathbb{R}.$$

(ii) With this choice of λ_i for $i = 0, 1, 2$, is the quadrature rule Q also exact for $f \colon \mathbb{R} \to \mathbb{R}, x \mapsto \sin(x)$?

3. Consider the Trapezoidal rule given by

$$Q_T(f) \approx \frac{b-a}{2}\Big(f(a)+f(b)\Big),$$

with error estimated by

$$\left| \int_a^b f(x)\,dx - Q_T(f) \right| \leq \frac{(b-a)^3}{12} \max_{x\in[a,b]} \|f''(x)\|,$$

(a) (2 points) Formulate the composite trapezoidal rule for approximating the integral of f on $[a,b]$ by using the nodes $\{x_0, x_1, \ldots, x_n\} \subset [a,b], n \in \mathbb{N}$.

(b) (3 points) Let $f : [0,2] \to \mathbb{R}$, given by $f(x) = e^{x^2}$. Estimate the error of the approximation given by the Composite Trapezoidal rule.

(c) (3 points) Consider the function $f\colon [0,3] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 2, & x \in [0,1] \\ x+1, & x \in [1,2] \\ (x-2)^3+3, & x \in [2,3]. \end{cases}$$

Find a Composite rule that gives the exact value of the integral of f on $[0,3]$ using the least amount of evaluations of the function f. Justify your answer.

4. (a) (3 points) Give the three decomposition methods from the lecture and sufficient conditions for their existence.

   (b) (7 points) Let
$$A = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & -1 \\ 2 & -1 & 1 & 1 \end{pmatrix}.$$
   Apply a decomposition method from (a) to the matrix A, and explain how you would solve the linear system $Ax = b$ for a vector $b \in \mathbb{R}^4$, using this decomposition.

5. (a) (4 points) Let an arbitrary matrix A be given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$, such that $ad - bc \neq 0$. Show that $\kappa_\infty(A) = \kappa_1(A)$.
Hint: The inverse of A is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(b) (2 points) Find a matrix $A \in \mathbb{R}^{3\times 3}$ such that $\kappa_\infty(A) \neq \kappa_1(A)$.

6. Let

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

(a) (3 points) Compute two steps of the Gauss–Seidel method with respect to the initial vector $x^{(0)} = (1, 1, 1, 1)^T$.

(b) (3 points) Show that the Jacobi method does not converge for all initial guesses $x^{(0)} \in \mathbb{R}^4$.