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Introduction to Numerical Methods

Algorithms no. 1

Algorithm 1 LU decomposition

Input: A square matrix A of order $n > 0$

Output: A lower triangular unitary matrix L and a upper triangular matrix U, both of order n, satisfying $A = LU$

Set $A^{(1)} \leftarrow A$
for $i = 1$ to $n - 1$ **do**
    Define the Frobenius Matrix

$$L_i = \begin{pmatrix} 1 & & & & & \\ & \dots & & & & \\ & & 1 & & & \\ & & -l_{i+1,i} & 1 & & \\ & & \dots & & \dots & \\ & & -l_{n,i} & \dots & & 1 \end{pmatrix}$$

where $l_{j,i} = a_{j,i}^{(i)} / a_{i,i}^{(i)}$ for all $j = i + 1, \dots, n$.
    Set $A^{(i+1)} = L_i A^{(i)}$.
end for
Set $U \coloneqq A^{(n)}$ and $L \coloneqq L_1^{-1} L_2^{-1} \cdot \dots \cdot L_{n-1}^{-1}$

Algorithm 2 LU decomposition with partial pivoting

Input: A square matrix A of order $n > 0$

Output: A lower triangular unitary matrix L and a upper triangular matrix U, both of order n, satisfying $A = LU$

Set $A^{(1)} = A$
for $i = 1$ to $n - 1$ **do**
  Choose an index $j \geq i$ with $|a_{ji}^{(i)}| \geq |a_{ki}^{(i)}|$
  for all $k \geq i$ **do**
    Set $\tilde{A}^{(i)} = P_{ij} A^{(i)}$
    Define the Frobenius matrix

$$L_i = \begin{pmatrix} 1 & & & & & \\ & \dots & & & & \\ & & 1 & & & \\ & & -l_{i+1,i} & 1 & & \\ & & \dots & & \dots & \\ & & -l_{n,i} & \dots & & 1 \end{pmatrix}$$

  where $l_{j,i} = \tilde{a}_{j,i}^{(i)} / \tilde{a}_{i,i}^{(i)}$ for all $j = i + 1, \dots, n$
    Set $A^{(i+1)} = L_i \tilde{A}^{(i)}$ and $P_i = P_{ij}$
  end for
end for
Set $U := A^{(n)}$, $P = P_{n-1} P_{n-2} \cdot \dots \cdot P_1$ and $L := \hat{L}_1^{-1} \hat{L}_2^{-1} \cdot \dots \cdot \hat{L}_{n-1}^{-1}$
  where $\hat{L}_k = P_{n-1} \cdot \dots \cdot P_{k+1} L_k P_{k+1}^T \cdot \dots \cdot P_{n-1}^T$ for all $k = 1, \dots, n - 1$.

Algorithm 3 Cholesky decomposition

Input: A symmetric and positive definite matrix A of order n

Output: A lower triangular matrix L of order n, satisfying $A = LL^T$

for $j = 1$ to n **do**
  $l_{jj} := \sqrt{a_{jj} - \sum_{m=1}^{j-1} l_{jm}^2}$ ▷ undefined sums are regarded as 0
  for $i = (j + 1)$ to n **do**
    $l_{ij} := (a_{ij} - \sum_{m=1}^{j-1} l_{im} l_{jm}) / l_{jj}$
  end for
end for
Set $L = (l_{ij})_{1 \leq i,j \leq n}$