Maurice Hensel
Andre Souza
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Introduction to Numerical Methods

Exercise no. 13

Hand in before the beginning of the exercise class on 26.01.2023

Exercise 13.1 (2 points) Let

$$A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$$

with $a \in \mathbb{R}\backslash\{-1, 1\}$. Determine $\|A\|_\infty$, $\|A\|_2$, $\kappa_\infty(A)$ and $\kappa_2(A)$.

Exercise 13.2 (4 points) We consider the matrix A, the vector b and the initial value $x^{(0)}$, given by

$$A = \begin{pmatrix} 5 & 3 & 1 \\ 3 & 12 & 0 \\ 1 & 0 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad x^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

i) Give the Jacobi iteration matrix G and compute one step of the Jacobi method.

ii) Compute $\|G\|_\infty$. Can you guarantee the convergence of the Jacobi method?

iii) Compute the *a posteriori* error bound for $e^{(2)} = x_e - x^{(2)}$ regarding the norm $\|\cdot\|_\infty$.

iv) Compute the *a priori* error bound for $e^{(2)} = x_e - x^{(2)}$ regarding the norm $\|\cdot\|_\infty$. How many iterations are necessary to guarantee that the error is $\leq 10^{-8}$.

Exercise 13.3 (2 points) We consider the matrix A, the vector b and the initial value $x^{(0)}$, given by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

i) Compute x_e, the exact solution of $Ax = b$.

ii) Compute one step of the symmetric Gauss-Seidel method.

Exercise 13.4 (2 points)(Bonus) Show that the error of the Gauss Seidel method propagates as follows

$$e^{(k+1)} = -(D + L)^{-1} U e^{(k)}.$$

Exercise 13.5 (2 points)(Bonus) We consider the matrix A and the vector b, given by

$$A = \begin{pmatrix} 0.5 & 0.25 \\ 0.25 & 0.5 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

a) Give the iteration rule for the Jacobi method.

b) Consider the initial value

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and compute two iteration steps of the Jacobi method.