2. Autovalores y Autovectores

Definición

$T(x)=\lambda x$

Sea A una matriz de orden n , una matriz X no nula de orden $n\times 1$ es un autovector de la matriz A si existe un escalar $\lambda\in\mathbb{R}$ tal que se verifica

$$AX=\lambda X$$

$$Ax-\lambda x=\theta$$

$\lambda=?$

$$\underbrace{(A-\lambda I)}X=\theta$$

$$\det(A-\lambda I)=0$$

Para la determinación de los autovalores se tiene que hallar las raíces del polinomio característico asociado el cual se define como

$$p(\lambda)=|A-\lambda I|$$

$$|A-\lambda I|=0$$

$$p(\lambda)=0$$

Ejemplo: Hallar los autovalores de la matriz

$$A=\begin{bmatrix}2 & -5\\ -2 & -1\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}2 & -5\\ -2 & -1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2-\lambda & -5\\ -2 & -1-\lambda\end{bmatrix}$$

$$p(\lambda)=|A-\lambda I|=(2-\lambda)(-1-\lambda)-10$$

$$=(\lambda+1)(\lambda-2)-10$$

$$p(\lambda)=\lambda^2-\lambda-12$$

$$p(\lambda)=0\Rightarrow\lambda^2-\lambda-12=0$$

$$(\lambda-4)(\lambda+3)=0\Rightarrow\lambda=4$$

$$\lambda=-3$$

Los autovalores de la matriz A son $\{4,-3\}$

Ejemplo: Hallar los autovalores de la matriz

$$A=\begin{bmatrix}4 & -4\\ -4 & -2\end{bmatrix}$$

Autovalores $\{-4,6\}$

Para hallar los autovectores se tiene que determinar las solucione son nulas del sistema

$$(A-\lambda I)X=\theta$$

Ejemplo: Hallar los autovectores de la matriz

$$A=\begin{bmatrix}2 & -5\\ -2 & -1\end{bmatrix}$$

Dado que los autovalores de la matriz A, son $\{4,-3\}$ tenemos:

$$(A - \lambda I)X = \theta$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & -5 \\ -2 & -1-\lambda \end{bmatrix}$$

Para $\lambda = 4$ $\quad (A-4I)X = \theta$

$$[A-4I \,\vdots\, \theta]$$

$$\left[\begin{array}{cc|c} -2 & -5 & 0 \\ -2 & -5 & 0 \end{array}\right] \rightarrow \begin{array}{ccc} x & y & \\ \end{array}\left[\begin{array}{cc|c} -2 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right] \iff -2x-5y=0$$

$$x = -\frac{5}{2}y$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{5}{2}y \\ y \end{bmatrix} = y\begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix} \qquad V_1 = \begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix}$$

autovector asociado a $\lambda = 4$

Para $\lambda = -3$. $\quad (A-(-3)I)X = \theta$

$$[A+3I \,\vdots\, \theta]$$

$$\left[\begin{array}{cc|c} 5 & -5 & 0 \\ -2 & 2 & 0 \end{array}\right] \rightarrow \begin{array}{ccc} x & y & \\ \end{array}\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right] \iff x-y=0$$

$$x = y$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

autovector asociado a $\lambda = -3$

Ejemplo: Hallar los autovectores de la matriz

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

Autovalores:

$$A - \lambda I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix}$$

$$p(\lambda) = |A - \lambda I| = (1-\lambda)(1-\lambda) - 9 = \lambda^2 - 2\lambda + 1 - 9$$

$$p(\lambda) = \lambda^2 - 2\lambda - 8$$

$$p(\lambda) = 0 \implies \lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda - 4)(\lambda + 2) = 0 \implies \lambda = 4 \lor \lambda = -2$$

$$\text{Autovalores} = \{-2, 4\}$$

Autovectores

$$(A - \lambda I)X = \theta \longrightarrow [A - \lambda I \,\vdots\, \theta]$$

$$A-\lambda I=\begin{bmatrix}1-\lambda & 3\\ 3 & 1-\lambda\end{bmatrix}$$

Para $\lambda=-2$: $(A+2I)X=\theta \longrightarrow [A+2I\,\vdots\,\theta]$

$$\begin{bmatrix}3 & 3 & \vdots & 0\\ 3 & 3 & \vdots & 0\end{bmatrix}\longrightarrow \begin{array}{c}\begin{array}{cccc}x & y & & \end{array}\\ \begin{bmatrix}1 & 1 & \vdots & 0\\ 0 & 0 & \vdots & 0\end{bmatrix}\end{array} \qquad \begin{aligned}x+y&=0\\ y&=-x\end{aligned}$$

$$X=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x\\ -x\end{bmatrix}=x\begin{bmatrix}1\\ -1\end{bmatrix} \qquad v_1=\begin{bmatrix}1\\ -1\end{bmatrix}$$ autovector asociado a $\lambda=-2$

Para $\lambda=4$: $(A-4I)X=\theta \rightarrow [A-4I\ \ 0]$

$$\begin{bmatrix}-3 & 3 & \vdots & 0\\ 3 & -3 & \vdots & 0\end{bmatrix}\rightarrow \begin{array}{c}\begin{array}{cccc}x & y & & \end{array}\\ \begin{bmatrix}1 & -1 & \vdots & 0\\ 0 & 0 & \vdots & 0\end{bmatrix}\end{array} \qquad \begin{aligned}x-y&=0\\ x&=y\end{aligned}$$

$$X=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x\\ x\end{bmatrix}=x\begin{bmatrix}1\\ 1\end{bmatrix} \qquad v_2=\begin{bmatrix}1\\ 1\end{bmatrix}$$ autovector asociado a $\lambda=4$

Ejemplo: Hallar los autovectores de la matriz

$$A=\begin{bmatrix}1 & 5\\ 5 & 1\end{bmatrix}$$

Autovalores: $\lambda=-4$
$\lambda=6$

Autovectores

Para $\lambda=-4$ $\quad v_1=\begin{bmatrix}1\\ -1\end{bmatrix}$

Para $\lambda=6$ $\quad v_2=\begin{bmatrix}1\\ 1\end{bmatrix}$

$$Ax=\lambda x$$

$$Av_1=-4v_1 = -4v_1+0v_2 \qquad (Tv_1)_\beta=\begin{bmatrix}-4\\ 0\end{bmatrix}$$

$$Av_2=6v_2 = 0v_1+6v_2 \qquad (Tv_2)_\beta=\begin{bmatrix}0\\ 6\end{bmatrix}$$

$$\beta=\{v_1,v_2\}$$

$$A=[T]_\beta^\beta=\begin{bmatrix}-4 & 0\\ 0 & 6\end{bmatrix}$$

$$T:\underset{\beta}{\mathbb{R}^2}\xrightarrow{A}\underset{\beta}{\mathbb{R}^2}$$

$$A = [T]_{\beta}^{\beta}$$

$$
\begin{tikzcd}
\mathbb{V}_{B_1} \arrow[r, "T"] \arrow[d, dash] & \mathbb{V}_{B_1} \arrow[d, dash] \\
\mathbb{V}_{B_2} \arrow[r, "T"] & \mathbb{V}_{B_2}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{V}_{B_1} \arrow[r, "\overset{A}{\shortparallel}\,{[T]_{B_1}}"] \arrow[d, "M"'] & \mathbb{V}_{B_1} \\
\mathbb{V}_{B_2} \arrow[r, "\overset{D}{\shortparallel}\,{[T]_{B_2}}"'] & \mathbb{V}_{B_2} \arrow[u, "M^{-1}"']
\end{tikzcd}
\qquad A = M^{-1}DM
$$

$$A = M^{-1}DM$$

$$A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

$$D = \begin{array}{c} \begin{array}{cc} \lambda_1 & \lambda_2 \end{array} \\ \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix} \end{array}$$

$$M^{-1} = \begin{array}{c} \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$A = M^{-1} D M$$

$$\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

Ejemplo: Hallar los autovectores de la matriz

$$A = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Autovalores

$$A - \lambda I = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} \boxed{3-\lambda \quad -1 \quad -1} \\ 1 \quad 1-\lambda \quad -1 \\ 1 \quad -1 \quad 1-\lambda \end{bmatrix} \end{array}$$

$p(\lambda)=|A-\lambda I|$

$$= (+)(3-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} + (-)(-1)\begin{vmatrix} 1 & -1 \\ 1 & 1-\lambda \end{vmatrix} + (+)(-1)\begin{vmatrix} 1 & 1-\lambda \\ 1 & -1 \end{vmatrix}$$

$$p(\lambda)= \lambda^3-5\lambda^2+8\lambda-4$$

$$\left.\begin{array}{l} \lambda=1 \\ \lambda=2 \\ \lambda=2 \end{array}\right\}$$

- Autovectores

Para $\lambda=1$: $(A-1\cdot I)X=\theta$

$$[A-I \,\vdots\, \theta]$$

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \to \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{l} x-z=0 \\ -y+z=0 \end{array} \uparrow \quad \begin{array}{l} x=z \\ y=z \end{array}$$

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} z \\ z \\ z \end{bmatrix}=z\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Para $\lambda=2$: $(A-2I)X=\theta$

$$[A-2I \,|\, \theta]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right] \to \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$x-y-z=0$$
$$x=y+z$$

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} y+z \\ y \\ z \end{bmatrix}=\begin{bmatrix} y \\ y \\ 0 \end{bmatrix}+\begin{bmatrix} z \\ 0 \\ z \end{bmatrix}=y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}+z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Ejemplo: Hallar los autovalores y autovectores de la matriz

$$A=\begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

Autovectores

$$p(\lambda)=|A-\lambda I|=-\lambda(\lambda-5)^2$$

$$p(\lambda)=0 \Rightarrow \lambda=0$$

$$p(\lambda)=0 \Rightarrow \begin{array}{l}\lambda=0\\ \lambda=5\end{array}$$

Autovectores $\quad (A-\lambda I)X=0 \qquad A-\lambda I=\begin{bmatrix}5-\lambda & -4 & 0\\ 1 & -\lambda & 2\\ 0 & 2 & 5-\lambda\end{bmatrix}$

para $\lambda=0$

$$\left[\begin{array}{ccc|c}5 & -4 & 0 & 0\\ 1 & 0 & 2 & 0\\ 0 & 2 & 5 & 0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1 & 0 & 2 & 0\\ 5 & -4 & 0 & 0\\ 0 & 2 & 5 & 0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1 & 0 & 2 & 0\\ 0 & -4 & -10 & 0\\ 0 & 2 & 5 & 0\end{array}\right] \rightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c}1 & 0 & 2 & 0\\ 0 & 2 & 5 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \end{array}$$

$$\begin{array}{r} x+2z=0\\ 2y+5z=0\end{array} \uparrow \quad \begin{array}{l} x=-2z\\ y=-\frac{5}{2}z\end{array}$$

$$X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-2z\\ -\frac{5}{2}z\\ z\end{bmatrix}=z\begin{bmatrix}-2\\ -\frac{5}{2}\\ 1\end{bmatrix}$$

Para $\lambda=5$

$$\left[\begin{array}{ccc|c}0 & -4 & 0 & 0\\ 1 & -5 & 2 & 0\\ 0 & 2 & 0 & 0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1 & -5 & 2 & 0\\ 0 & -4 & 0 & 0\\ 0 & 2 & 0 & 0\end{array}\right] \rightarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c}1 & -5 & 2 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \end{array}$$

$$\begin{array}{r} x-5y+2z=0\\ 2y=0\end{array} \uparrow \quad \begin{array}{l} x=2z\\ y=0\end{array}$$

$$X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2z\\ 0\\ z\end{bmatrix}=z\begin{bmatrix}2\\ 0\\ 1\end{bmatrix}$$