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Computational Finance and FinTech
Financial Time Series

Contents

4 Financial Time Series

- Further reading: **Py4Fi, Chapter 8**
- This session also covers material not in **Py4Fi**.
- Time series are ubiquitous in finance.
- `pandas` is the main library in Python to deal with time series.

4.1 Financial Data

Financial data

- For the time being we work with locally stored data files.
- These are in `.csv`-files (comma-separated values), where the data entries in each row are separated by commas.
- Some initialisation:

[1]:
```
import numpy as np
import pandas as pd
from pylab import mpl, plt
plt.style.use('seaborn')
mpl.rcParams['font.family'] = 'serif'
%matplotlib inline
```

Data import

- `pandas` provides a numer of different functions and `DataFrame` methods for importing and exporting data.
- Here we use `pd.read_csv()`.
- The file that we load contains end-of-day data for different financial instruments retrieved from Thomson Reuters.

[2]:
```
filename = './data/tr_eikon_eod_data.csv' # path and filename
f = open(filename, 'r')
f.readlines()[:5]  # show first five lines
```

[2]:
```
['Date,AAPL.O,MSFT.O,INTC.O,AMZN.O,GS.N,SPY,.SPX,.VIX,EUR=,XAU=,GDX,GLD\n',
 '2010-01-01,,,,,,,,,1.4323,1096.35,,\n',
 '2010-01-04,30.57282657,30.95,20.88,133.9,173.08,113.33,1132.99,20.04,1.4411,11
20.0,47.71,109.8\n',
 '2010-01-05,30.625683660000004,30.96,20.87,134.69,176.14,113.63,1136.52,19.35,1
.4368,1118.65,48.17,109.7\n',
```

```
'2010-01-06,30.138541290000003,30.77,20.8,132.25,174.26,113.71,1137.14,19.16,1.
4412,1138.5,49.34,111.51\n']
```

Data import

[3]:
```python
data = pd.read_csv(filename,  # import csv-data into DataFrame
                   index_col=0, # take first column as index
                   parse_dates=True)  # index values are datetime
```

[4]:
```python
data.info()  # information about the DataFrame object
```

```
<class 'pandas.core.frame.DataFrame'>
DatetimeIndex: 2216 entries, 2010-01-01 to 2018-06-29
Data columns (total 12 columns):
 #   Column  Non-Null Count  Dtype
---  ------  --------------  -----
 0   AAPL.O  2138 non-null   float64
 1   MSFT.O  2138 non-null   float64
 2   INTC.O  2138 non-null   float64
 3   AMZN.O  2138 non-null   float64
 4   GS.N    2138 non-null   float64
 5   SPY     2138 non-null   float64
 6   .SPX    2138 non-null   float64
 7   .VIX    2138 non-null   float64
 8   EUR=    2216 non-null   float64
 9   XAU=    2211 non-null   float64
 10  GDX     2138 non-null   float64
 11  GLD     2138 non-null   float64
dtypes: float64(12)
memory usage: 225.1 KB
```

Data import

[5]:
```python
data.head()
```

[5]:
```
               AAPL.O  MSFT.O  INTC.O  AMZN.O    GS.N     SPY     .SPX   .VIX  \
Date
2010-01-01        NaN     NaN     NaN     NaN     NaN     NaN      NaN    NaN
2010-01-04  30.572827  30.950   20.88  133.90  173.08  113.33  1132.99  20.04
2010-01-05  30.625684  30.960   20.87  134.69  176.14  113.63  1136.52  19.35
2010-01-06  30.138541  30.770   20.80  132.25  174.26  113.71  1137.14  19.16
2010-01-07  30.082827  30.452   20.60  130.00  177.67  114.19  1141.69  19.06

              EUR=     XAU=    GDX     GLD
Date
2010-01-01  1.4323  1096.35    NaN     NaN
2010-01-04  1.4411  1120.00  47.71  109.80
2010-01-05  1.4368  1118.65  48.17  109.70
2010-01-06  1.4412  1138.50  49.34  111.51
2010-01-07  1.4318  1131.90  49.10  110.82
```

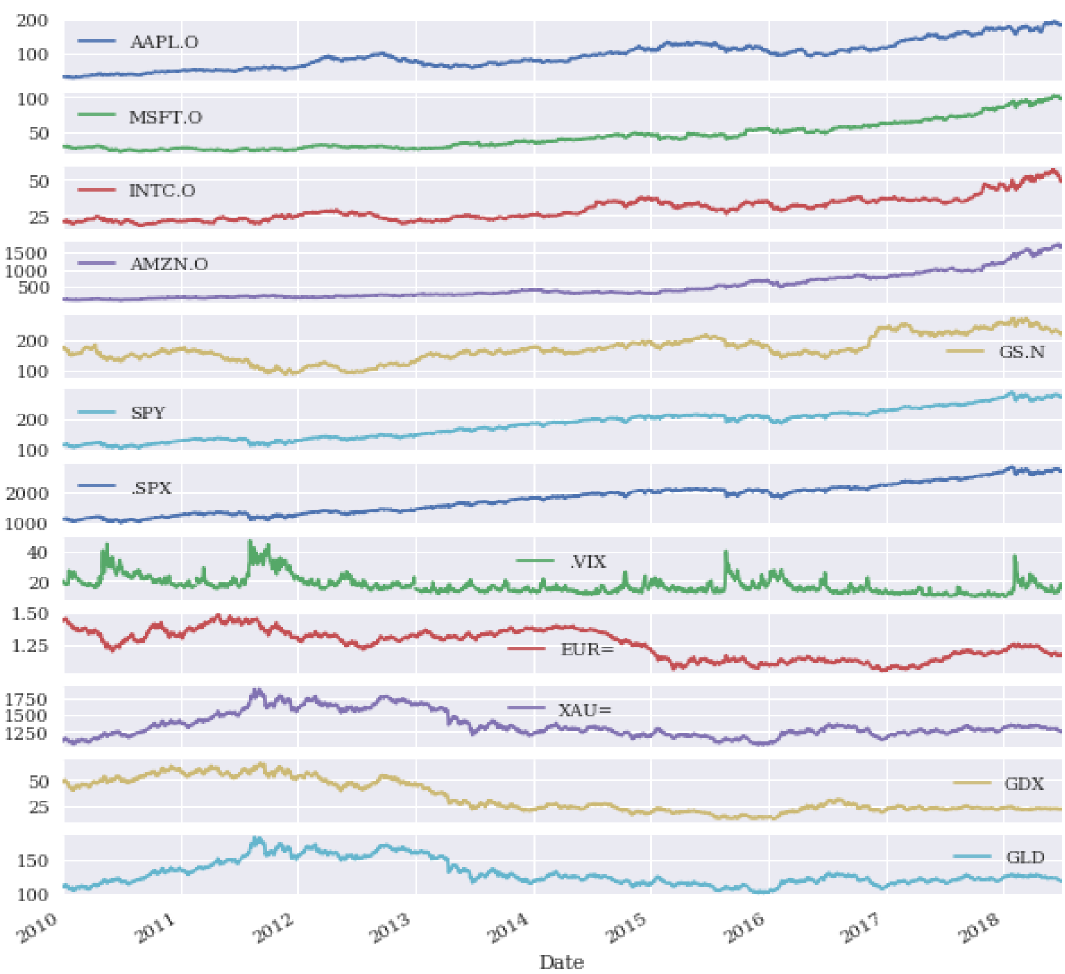
Data import

[6]:
```python
data.tail()
```

[6]:
```
            AAPL.O  MSFT.O  INTC.O   AMZN.O    GS.N     SPY     .SPX   .VIX  \
Date
2018-06-25  182.17   98.39   50.71  1663.15  221.54  271.00  2717.07  17.33
2018-06-26  184.43   99.08   49.67  1691.09  221.58  271.60  2723.06  15.92
2018-06-27  184.16   97.54   48.76  1660.51  220.18  269.35  2699.63  17.91
2018-06-28  185.50   98.63   49.25  1701.45  223.42  270.89  2716.31  16.85
2018-06-29  185.11   98.61   49.71  1699.80  220.57  271.28  2718.37  16.09

              EUR=     XAU=    GDX     GLD
Date
2018-06-25  1.1702  1265.00  22.01  119.89
2018-06-26  1.1645  1258.64  21.95  119.26
2018-06-27  1.1552  1251.62  21.81  118.58
2018-06-28  1.1567  1247.88  21.93  118.22
2018-06-29  1.1683  1252.25  22.31  118.65
```

Data import

[7]:
```python
data.plot(figsize=(10, 10), subplots=True);
```



Data import

- The identifiers used by Thomson Reuters are so-called RIC's.

- The financial instruments in the data set are:

[8]:
```
instruments = ['Apple Stock', 'Microsoft Stock',
               'Intel Stock', 'Amazon Stock', 'Goldman Sachs Stock',
               'SPDR S&P 500 ETF Trust', 'S&P 500 Index',
               'VIX Volatility Index', 'EUR/USD Exchange Rate',
               'Gold Price', 'VanEck Vectors Gold Miners ETF',
               'SPDR Gold Trust']
```

Data import

[9]:
```
for ric, name in zip(data.columns, instruments):
    print('{:8s} | {}'.format(ric, name))
```

```
AAPL.O   | Apple Stock
MSFT.O   | Microsoft Stock
INTC.O   | Intel Stock
AMZN.O   | Amazon Stock
GS.N     | Goldman Sachs Stock
SPY      | SPDR S&P 500 ETF Trust
.SPX     | S&P 500 Index
.VIX     | VIX Volatility Index
EUR=     | EUR/USD Exchange Rate
XAU=     | Gold Price
GDX      | VanEck Vectors Gold Miners ETF
GLD      | SPDR Gold Trust
```

Summary statistics

[10]:
```
data.describe().round(2)
```

[10]:

	AAPL.O	MSFT.O	INTC.O	AMZN.O	GS.N	SPY	.SPX	.VIX \
count	2138.00	2138.00	2138.00	2138.00	2138.00	2138.00	2138.00	2138.00
mean	93.46	44.56	29.36	480.46	170.22	180.32	1802.71	17.03
std	40.55	19.53	8.17	372.31	42.48	48.19	483.34	5.88
min	27.44	23.01	17.66	108.61	87.70	102.20	1022.58	9.14
25%	60.29	28.57	22.51	213.60	146.61	133.99	1338.57	13.07
50%	90.55	39.66	27.33	322.06	164.43	186.32	1863.08	15.58
75%	117.24	54.37	34.71	698.85	192.13	210.99	2108.94	19.07
max	193.98	102.49	57.08	1750.08	273.38	286.58	2872.87	48.00

	EUR=	XAU=	GDX	GLD
count	2216.00	2211.00	2138.00	2138.00
mean	1.25	1349.01	33.57	130.09
std	0.11	188.75	15.17	18.78
min	1.04	1051.36	12.47	100.50
25%	1.13	1221.53	22.14	117.40
50%	1.27	1292.61	25.62	124.00
75%	1.35	1428.24	48.34	139.00
max	1.48	1898.99	66.63	184.59

Summary statistics

- The `aggregate()`-function allows to customise the statistics viewed:

[11]:
```python
data.aggregate([min,
                np.mean,
                np.std,
                np.median,
                max]
).round(2)
```

[11]:

	AAPL.O	MSFT.O	INTC.O	AMZN.O	GS.N	SPY	.SPX	.VIX	EUR=	\
min	27.44	23.01	17.66	108.61	87.70	102.20	1022.58	9.14	1.04	
mean	93.46	44.56	29.36	480.46	170.22	180.32	1802.71	17.03	1.25	
std	40.55	19.53	8.17	372.31	42.48	48.19	483.34	5.88	0.11	
median	90.55	39.66	27.33	322.06	164.43	186.32	1863.08	15.58	1.27	
max	193.98	102.49	57.08	1750.08	273.38	286.58	2872.87	48.00	1.48	

	XAU=	GDX	GLD
min	1051.36	12.47	100.50
mean	1349.01	33.57	130.09
std	188.75	15.17	18.78
median	1292.61	25.62	124.00
max	1898.99	66.63	184.59

Returns

- When working with financial data we typically (=always - you must have good reasons to deviate from this) work with performance data, i.e., **returns**.
- Reasoning:
  - Historical data are mainly used to make forecasts one or several time periods forward.
  - The daily average stock price over the last eight years is meaningless to make a forecast for tomorrow's stock price.
  - However, the daily returns are possible scenarios for the next time period(s).
- The function `pct_change()` calculates discrete returns:

$$r_t^{\mathrm{d}} = \frac{S_t - S_{t-1}}{S_{t-1}},$$

where S_t denotes the stock price at time t.

Returns

[12]:
```python
data.pct_change().round(3).head()
```

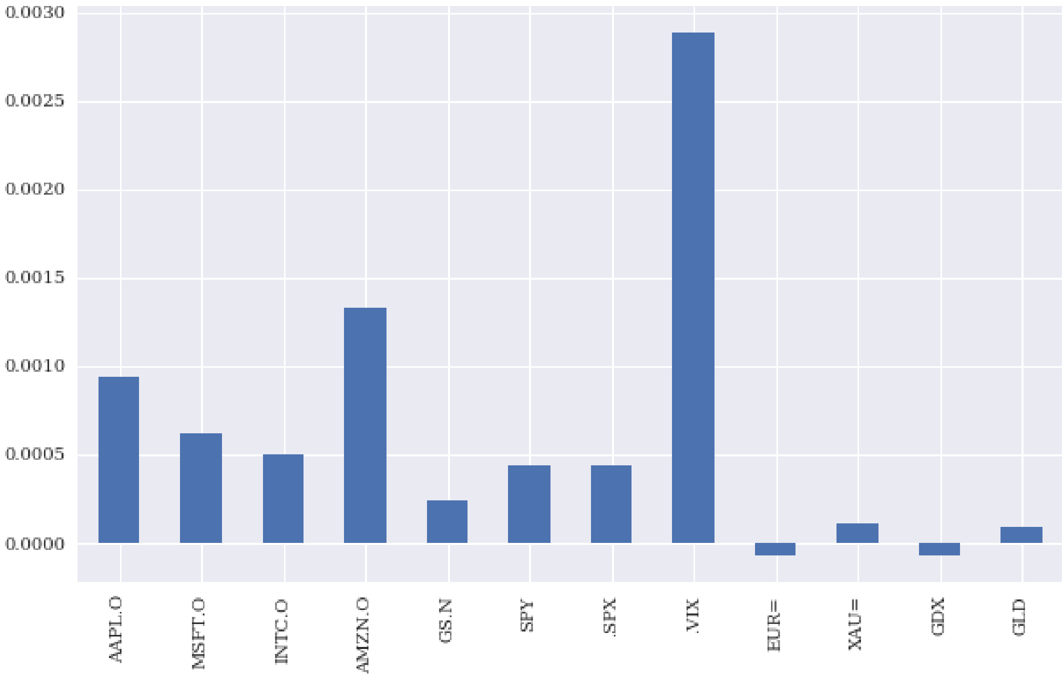
[12]:

	AAPL.O	MSFT.O	INTC.O	AMZN.O	GS.N	SPY	.SPX	.VIX	EUR=	\
Date										
2010-01-01	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	
2010-01-04	NaN	NaN	NaN	NaN	NaN	NaN	NaN	NaN	0.006	
2010-01-05	0.002	0.000	-0.000	0.006	0.018	0.003	0.003	-0.034	-0.003	
2010-01-06	-0.016	-0.006	-0.003	-0.018	-0.011	0.001	0.001	-0.010	0.003	
2010-01-07	-0.002	-0.010	-0.010	-0.017	0.020	0.004	0.004	-0.005	-0.007	

	XAU=	GDX	GLD
Date			
2010-01-01	NaN	NaN	NaN
2010-01-04	0.022	NaN	NaN
2010-01-05	-0.001	0.010	-0.001
2010-01-06	0.018	0.024	0.016
2010-01-07	-0.006	-0.005	-0.006

Returns

[13]:
```
data.pct_change().mean().plot(kind='bar', figsize=(10, 6));
```



Returns

- In finance, **log-returns**, also called **continuous returns**, are often preferred over discrete returns:

$$r_t^c = \ln\left(\frac{S_t}{S_{t-1}}\right).$$

- The main reason is that log-return are additive over time.
- For example, the log-return from $t-1$ to $t+1$ is the sum of the single-period log-returns:

$$r_{t-1,t+1}^c = \ln\left(\frac{S_{t+1}}{S_t}\right) + \ln\left(\frac{S_t}{S_{t-1}}\right) = \ln\left(\frac{S_{t+1}}{S_t}\cdot\frac{S_t}{S_{t-1}}\right) = \ln\left(\frac{S_{t+1}}{S_{t-1}}\right).$$

- Note: If the sampling (time) intervall is small (e.g. one day or one week), then the difference between discrete returns and log-returns is negligible.

Returns

[14]:
```
rets = np.log(data / data.shift(1))  # calculates log-returns in a vectorised way
```

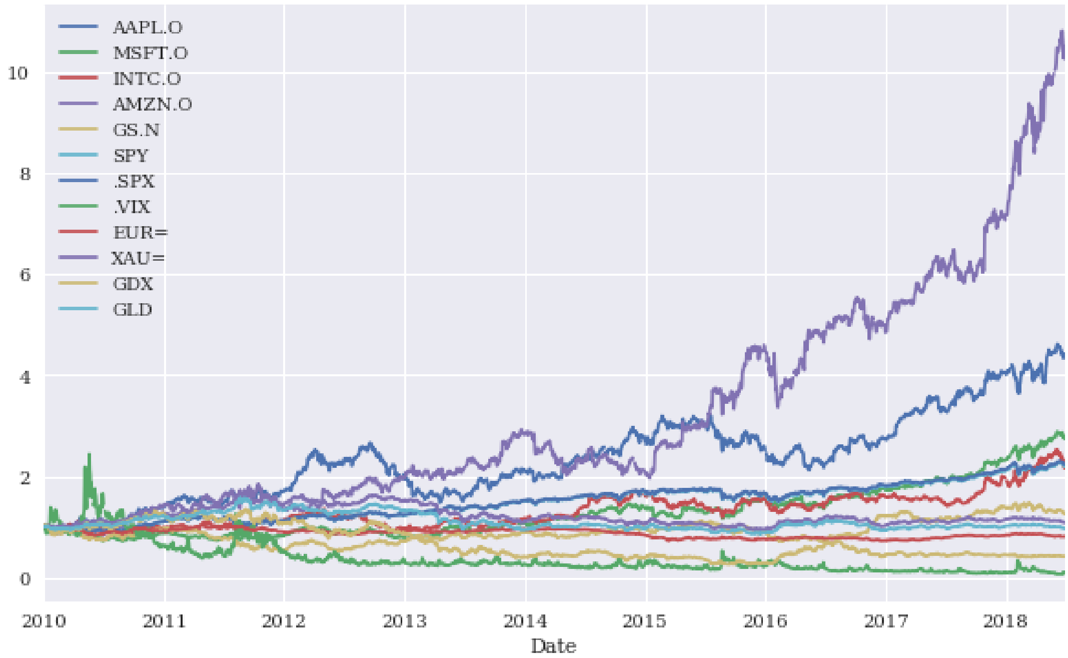
[15]:
```
rets.head().round(3)
```

[15]:
```
            AAPL.O  MSFT.O  INTC.O  AMZN.O   GS.N    SPY   .SPX   .VIX   EUR=  \
Date
2010-01-01     NaN     NaN     NaN     NaN    NaN    NaN    NaN    NaN    NaN
2010-01-04     NaN     NaN     NaN     NaN    NaN    NaN    NaN    NaN  0.006
2010-01-05   0.002   0.000  -0.000   0.006  0.018  0.003  0.003 -0.035 -0.003
2010-01-06  -0.016  -0.006  -0.003  -0.018 -0.011  0.001  0.001 -0.010  0.003
2010-01-07  -0.002  -0.010  -0.010  -0.017  0.019  0.004  0.004 -0.005 -0.007
```

```
            XAU=    GDX    GLD
Date
2010-01-01    NaN    NaN    NaN
2010-01-04  0.021    NaN    NaN
2010-01-05 -0.001  0.010 -0.001
2010-01-06  0.018  0.024  0.016
2010-01-07 -0.006 -0.005 -0.006
```

Returns

[16]:
```
rets.cumsum().apply(np.exp).plot(figsize=(10, 6));  # recover price paths from␣
↪log-returns
```



Resampling

- Down-sampling is achieved by `resample()`:

[17]:
```
data.resample('1w', label='right').last().head()  # down-sample to weekly time␣
↪intervals
```

[17]:
```
               AAPL.O  MSFT.O  INTC.O  AMZN.O    GS.N     SPY     .SPX   .VIX  \
Date
2010-01-03        NaN     NaN     NaN     NaN     NaN     NaN      NaN    NaN
2010-01-10  30.282827   30.66   20.83  133.52  174.31  114.57  1144.98  18.13
2010-01-17  29.418542   30.86   20.80  127.14  165.21  113.64  1136.03  17.91
2010-01-24  28.249972   28.96   19.91  121.43  154.12  109.21  1091.76  27.31
2010-01-31  27.437544   28.18   19.40  125.41  148.72  107.39  1073.87  24.62

              EUR=     XAU=    GDX     GLD
Date
2010-01-03  1.4323  1096.35    NaN     NaN
2010-01-10  1.4412  1136.10  49.84  111.37
```

```
2010-01-17  1.4382  1129.90  47.42  110.86
2010-01-24  1.4137  1092.60  43.79  107.17
2010-01-31  1.3862  1081.05  40.72  105.96
```

4.2 Correlation analysis and linear regression

- To further illustrate how to work with financial time series we consider the S&P 500 stock index and the VIX volatility index.
- Empirical stylised fact: As the S&P 500 rises, the VIX falls, and vice versa.
- Note: This is about **correlation** not **causation**.

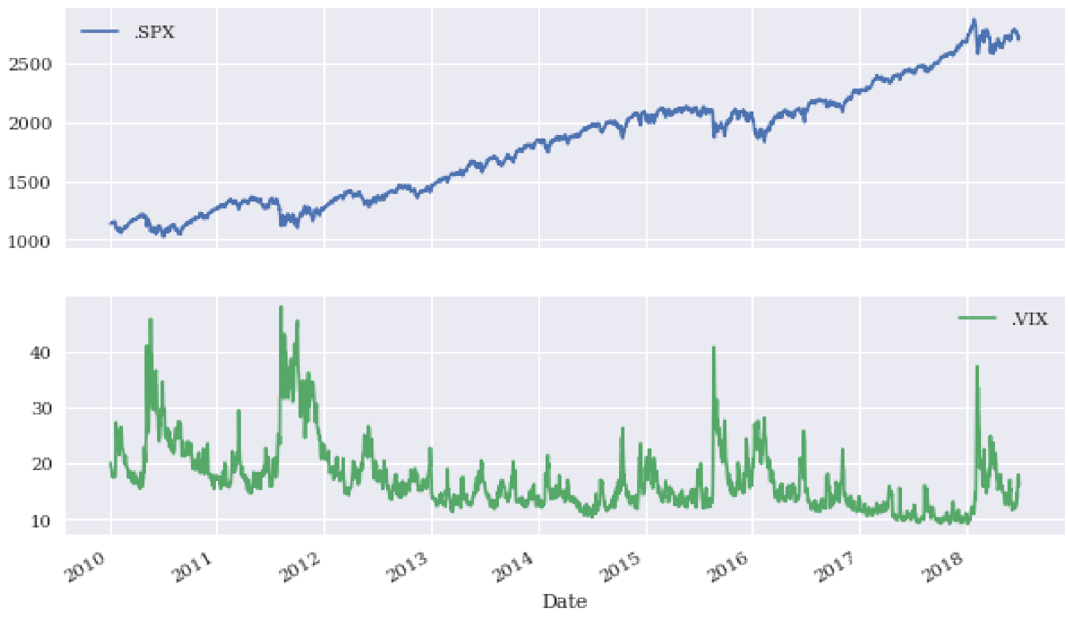
Correlation analysis

[18]:
```
# EOD data from Thomson Reuters Eikon Data API
raw = pd.read_csv('./data/tr_eikon_eod_data.csv', index_col=0, parse_dates=True)
data = raw[['.SPX', '.VIX']].dropna()
data.tail()
```

[18]:
```
               .SPX   .VIX
Date
2018-06-25  2717.07  17.33
2018-06-26  2723.06  15.92
2018-06-27  2699.63  17.91
2018-06-28  2716.31  16.85
2018-06-29  2718.37  16.09
```

Correlation analysis

[19]:
```
data.plot(subplots=True, figsize=(10, 6));
```



Correlation analysis

- Transform both data series into log-returns:

[20]:
```
rets = np.log(data / data.shift(1))
rets.head()
```

[20]:
```
                .SPX      .VIX
Date
2010-01-04       NaN       NaN
2010-01-05  0.003111 -0.035038
2010-01-06  0.000545 -0.009868
2010-01-07  0.003993 -0.005233
2010-01-08  0.002878 -0.050024
```

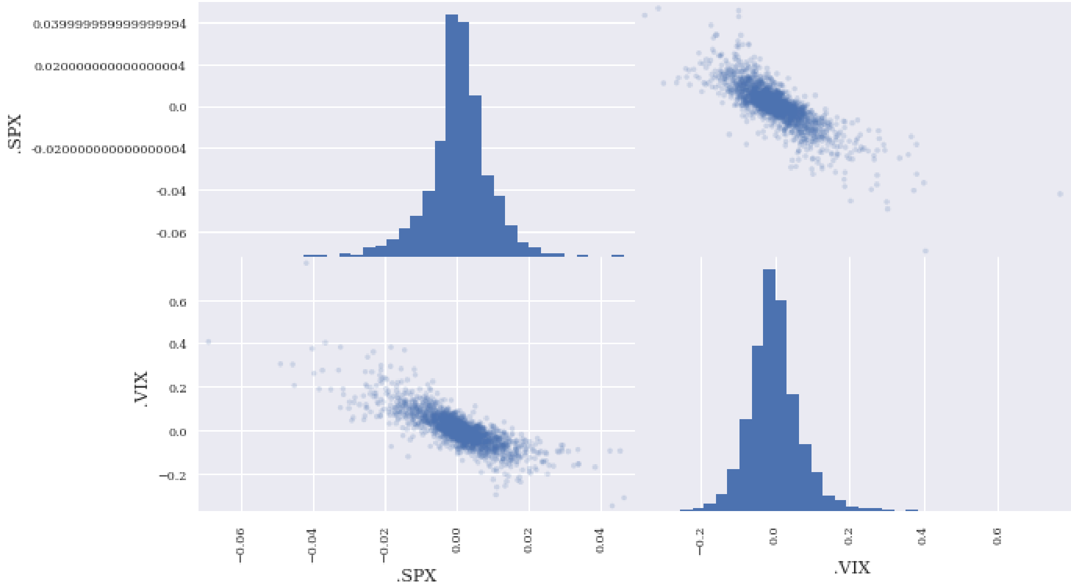
[21]:
```
rets.dropna(inplace=True) # drop NaN (not-a-number) entries
```

Correlation analysis

[22]:
```
rets.plot(subplots=True, figsize=(10, 6));
```

Correlation analysis

[23]:
```
pd.plotting.scatter_matrix(rets,
                           alpha=0.2,
                           diagonal='hist',
                           hist_kwds={'bins': 35},
                           figsize=(10, 6));
```



Correlation analysis

```
[24]: rets.corr()
```

```
[24]:            .SPX      .VIX
      .SPX  1.000000 -0.804382
      .VIX -0.804382  1.000000
```

OLS regression

- **Linear regression** captures the linear relationship between two variables.
- For two variables x, y, we postulate a linear relationship:

$$y = \alpha + \beta x + \varepsilon, \quad \alpha, \beta \in \mathbb{R}.$$

- Here, α is the **intercept**, β is the **slope (coefficient)** and ε is the **error term**.
- Given data sample of joint observations $(x_1, y_1), \ldots, (x_n, y_n)$, we set

$$y_i = \hat{\alpha} + \hat{\beta} x_i + \hat{\varepsilon}_i,$$

where $\hat{\alpha}$ and $\hat{\beta}$ are estimates of α, β and $\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_n$ are the so-called **residuals**.
- The **ordinary least squares (OLS)** estimator $\hat{\alpha}, \hat{\beta}$ corresponds to those values of α, β that minimise the sum of squared residuals:

$$\min_{\alpha,\beta} \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2.$$

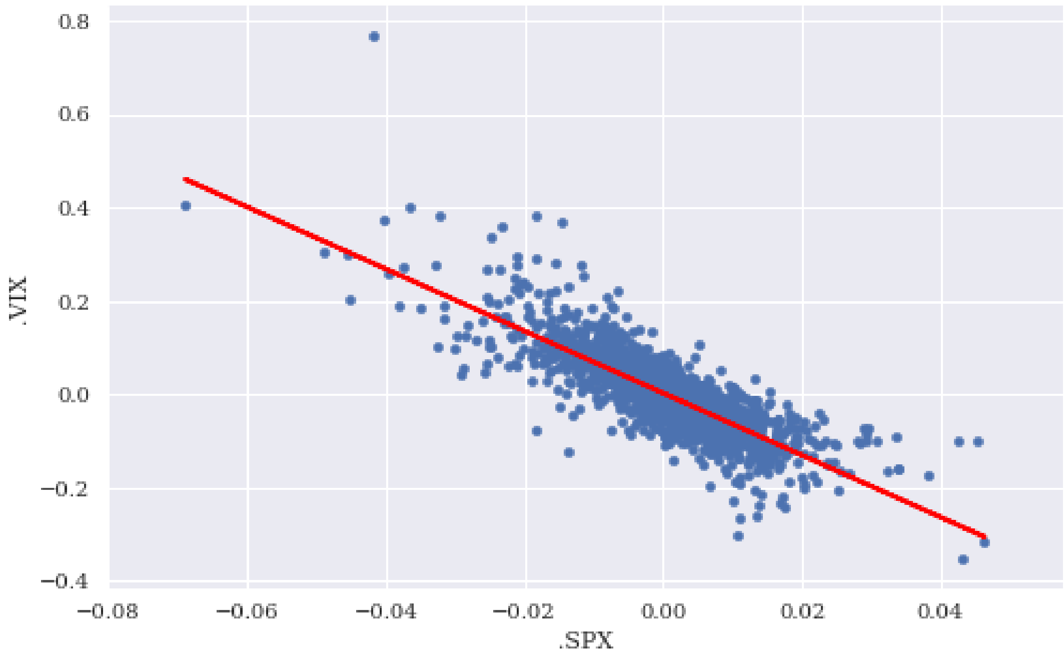
OLS regressions

- Simplest form of OLS regression:

```
[25]: reg = np.polyfit(rets['.SPX'], rets['.VIX'], deg=1)  # fit a linear equation (a␣
      ↪polynomial of degree 1)
      reg.view() # the fitted paramters
```

[25]: array([-6.65160028e+00, 2.62132142e-03])

[26]:
```
ax = rets.plot(kind='scatter', x='.SPX', y='.VIX', figsize=(8, 5))
ax.plot(rets['.SPX'], np.polyval(reg, rets['.SPX']), 'r', lw=2);
```



OLS regression

- To do a more refined OLS regression with a proper analysis, use the package `statsmodels`.

[27]:
```
import statsmodels.api as sm

Y=rets['.VIX']
X=rets['.SPX']
X = sm.add_constant(X)
```

[28]:
```
model = sm.OLS(Y,X)
results = model.fit()
```

[29]:
```
results.params
```

[29]:
```
const     0.002621
.SPX     -6.651600
dtype: float64
```

[30]:
```
results.predict()[0:10]
```

[30]:
```
array([-0.01807052, -0.0010063 , -0.0239404 , -0.01651898, -0.00898726,
        0.06531557, -0.05252965, -0.01349928,  0.07500527, -0.08000615])
```

OLS regression

[31]:
```
print(results.summary())
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                   .VIX   R-squared:                       0.647
Model:                            OLS   Adj. R-squared:                  0.647
Method:                 Least Squares   F-statistic:                     3914.
Date:                Thu, 02 Apr 2020   Prob (F-statistic):               0.00
Time:                        22:31:16   Log-Likelihood:                 3550.1
No. Observations:                2137   AIC:                            -7096.
Df Residuals:                    2135   BIC:                            -7085.
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0026      0.001      2.633      0.009       0.001       0.005
.SPX          -6.6516      0.106    -62.559      0.000      -6.860      -6.443
==============================================================================
Omnibus:                      518.582   Durbin-Watson:                   2.094
Prob(Omnibus):                  0.000   Jarque-Bera (JB):             6789.425
Skew:                           0.766   Prob(JB):                         0.00
Kurtosis:                      11.597   Cond. No.                         107.
==============================================================================

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.
```

OLS regression: Interpretation of output and forecasting

- The column `coef` lists the coefficients of the regression: the coefficient in the row labelled `const` corresponds to $\hat{\alpha}$ $(= 0.0026)$ and the coefficient in the row `.SPX` denotes $\hat{\beta}$ $(= -6.6515)$.
- The estimated model in the example is thus:

$$.\text{VIX} = 0.0026 - 6.6516.\text{SPX}.$$

- The best forecast of the VIX return when observing an S&P return of 2% is therefore $0.0026 - 6.6516 \cdot 0.02 = -0.130432 = -13.0432\%$.

OLS regression: Validation (R^2)

- To **validate** the model, i.e., to determine, if the model in itself and the explanatory variable(s) make sense, we look R^2 and various p-values (or confidence intervals or t-statistics).
- R^2 measures the fraction of variance in the dependent variable Y that is captured by the regression line; $1 - R^2$ is the fraction of Y-variance that remaines in the residuals ε_i^2, $i = 1, \ldots, n$.
- In the output above R^2 is given as 0.647. In other words, 64.7% of the variance in VIX returns are "explained" by SPX returns.
- A high R^2 (and this one is high) is necessary for making forecasts.

OLS regression: Validation (confidence interval)

- An important hypothesis to test in any regression model is whether the explanatory variable(s) have an effect on the independent variable.
- This can be translated into testing whether $\beta \neq 0$. ($\beta = 0$ is the same as saying that the X variable can be removed from the model.)
- Formally, we test the null hypothesis $H_0 : \beta = 0$ against the alternative hypothesis $H_1 : \beta \neq 0$.

- There are several statistics to come to the same conclusion: confidence intervals, t-statistics and p-values.
- The **confidence interval** is an interval around the estimate $\hat{\beta}$ that we are confident contains the true parameter β. A typial **confidence level** is 95%.
- If the 95% confidence interval does **not** contain 0, then we say β is **statistically significant** at the 5% (=1-95%) level, and we conclude that $\beta \neq 0$.

OLS regression: Validation (t-statistic)

- The t-statistic corresponds to the **number of standard deviations** that the estimated coefficient $\hat{\beta}$ is away from 0 (the mean under H_0).
- For a normal distribution, we have the following rules of thumb:
  - 66% of observations lie within one standard deviation of the mean
  - 95% of observations lie within two standard deviations of the mean
  - 99.7% of observations lie within three standard deviations of the mean

- If the sample size is large enough, then the t-statistic is approximately normally distributed, and if it is large (in absolute terms), then this is an indication against $\beta = 0$.
- In the example above, the t-statistics is -62.559, i.e., $\hat{\beta}$ is approx. 63 standard deviations away from zero, which is practically impossible.

OLS regression: Validation (p-value)

- The p-value expresses the probability of observing a coefficient estimate as extreme (away from zero) as $\hat{\beta}$ under H_0, i.e., when $\beta = 0$.
- In other words, it measures the probability of observing a t-statistic as extreme as the one observed if $\beta = 0$.
- If the p-value (column `P>|t|`) is smaller than the desired level of significance (typically 5%), then the H_0 can be rejected and we conclude that $\beta \neq 0$.
- In the example above, the p-value is given as 0.000, i.e., it is so small, that we can conclude the estimated coefficient $\hat{\beta}$ is so extreme (= away from zero) that is virtually impossible to obtain such an estimated if $\beta = 0$.
- Finally, the F-test tests the hypotheses $H_0 : R^2 = 0$ versus $H_1 : R^2 \neq 0$. In a multiple regression with k independent variables, this is equivalent to $H_0 : \beta_1 = \cdots = \beta_k = 0$.
- In the example above, the p-value of the F-test is 0, so we conclude that the model overall has explanatory power.

4.3 Time series models: Empirical stylised facts

- We discuss empirical stylised facts of financial time series.
- The GARCH model is the standard workhorse in financial time series analysis.

Time series models

- Load data set containing of daily DAX closing prices (1990-2019):

[32]:
```
dax = pd.read_csv('./_src/yahoo_GDAXI.csv',index_col = 0,na_values = 'null')
dax.head()
```

[32]:
```
                   Open         High          Low        Close    Adj Close  \
Date
1990-01-02  1788.890015  1788.890015  1788.890015  1788.890015  1788.890015
1990-01-03  1867.290039  1867.290039  1867.290039  1867.290039  1867.290039
1990-01-04  1830.920044  1830.920044  1830.920044  1830.920044  1830.920044
1990-01-05  1812.900024  1812.900024  1812.900024  1812.900024  1812.900024
```

```
1990-01-08  1841.469971  1841.469971  1841.469971  1841.469971  1841.469971

            Volume
Date
1990-01-02     0.0
1990-01-03     0.0
1990-01-04     0.0
1990-01-05     0.0
1990-01-08     0.0
```

Time series models

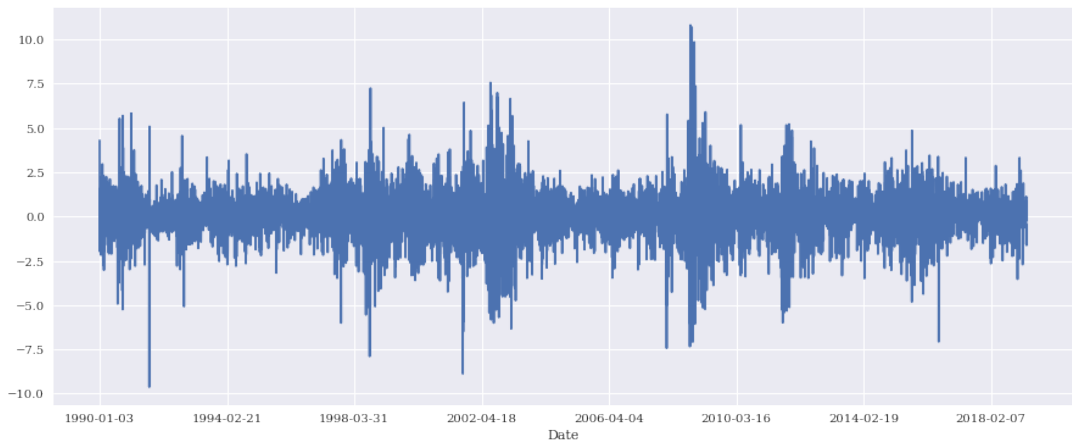
- Transform closing prices to log returns:

[33]:
```
data=dax['Close']
returns = 100*np.log(data / data.shift(1))
returns.dropna(inplace=True)
returns.head()
```

[33]:
```
Date
1990-01-03    4.289288
1990-01-04   -1.966961
1990-01-05   -0.989081
1990-01-08    1.563636
1990-01-09    1.297570
Name: Close, dtype: float64
```

Time series models

[34]:
```
returns.plot(figsize=(15,6));
```



Time series models

- When working with a data sample, we often assumes that the data are independent and identically distributed ("iid").
- The previous plot shows that the "iid" assumption is violated.
- The "iid" assumption is in general not justified for financial data, and more sophisticated models for time series are more appropriate for capturing phenomena such as volatility clustering.

Time series models

- An **empirical stylised fact** of a financial time series is an empirical observations that applies to the majority of (daily) series of asset returns, such as log-returns of equities, indexes, FX rates and commodity prices (see Mcneil, 2005, and Cont, 2001).
- Generally accepted stylised facts of asset returns are:

  1. Return series are not iid although they show little serial correlation.
  2. Series of absolute or squared returns show profound serial correlation.
  3. Conditional expected returns are close to zero.
  4. Volatility appears to vary over time.
  5. Return series are leptokurtic or heavy-tailed.
  6. Extreme returns appear in clusters.
     A.J. McNeil, R. Frey, and P. Embrechts. Quantitative Risk Management. Princeton University Press, Princeton, NJ, 2005. R. Cont. Empirical properties of asset returns: stylized facts and statistical issues. Quantitative Finance, 1(2):223–236, 2001.

Time series models

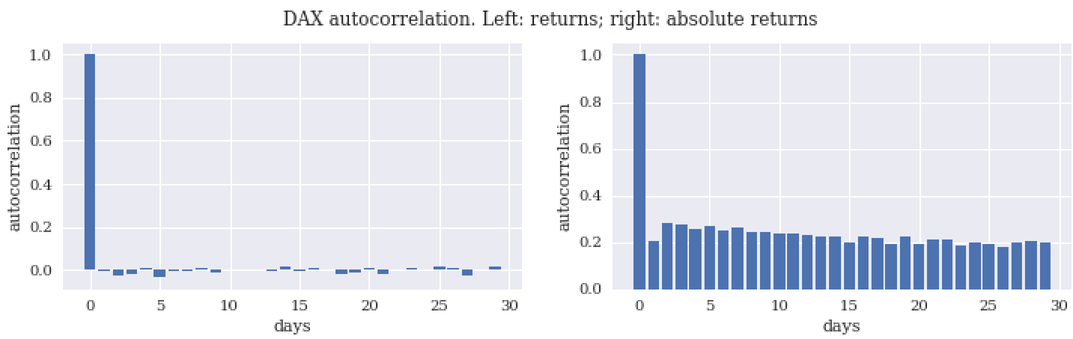
- The figure below illustrates the first three stylised facts (serial correlation = autocorrelation):

[35]:
```
ac = [];
acabs=[];
for i in range(0,30):
    ac.append(returns.autocorr(i))
    acabs.append(abs(returns).autocorr(i))
```

Time series models

- The figure below illustrates the first three stylised facts (serial correlation = autocorrelation):

[36]:
```
fig = plt.figure(figsize=(12,3))
fig.suptitle("DAX autocorrelation. Left: returns; right: absolute returns")
plt.subplot(121)
plt.bar(range(0,30), ac);
plt.xlabel('days');
plt.ylabel('autocorrelation');
plt.subplot(122)
plt.bar(range(0,30), acabs);
plt.xlabel('days');
plt.ylabel('autocorrelation');
```



Time series models

- The excess kurtosis of the DAX returns suggests that more extreme events occurs than a normal distribution would suggest.

[37]:
```
returns.kurtosis()
```

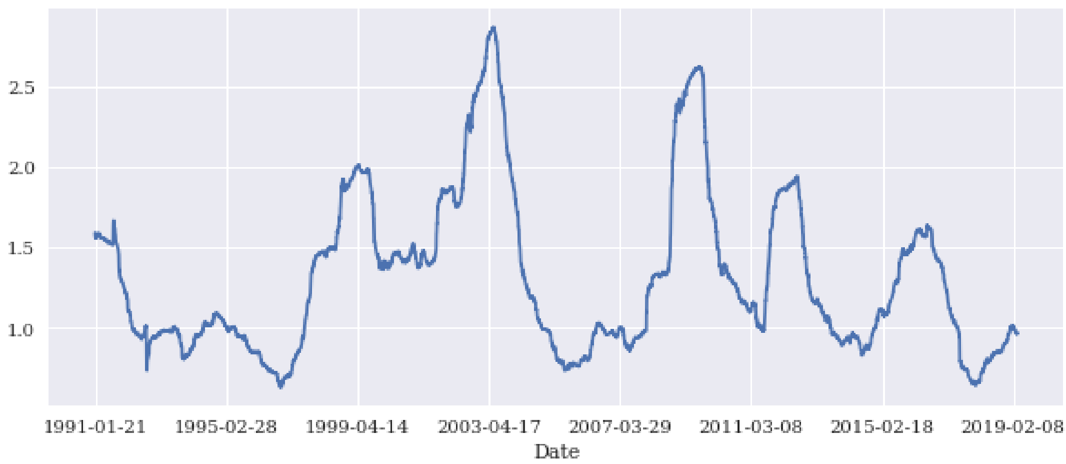
[37]: 4.7289880389545775

Time series models

- The following figures shows the DAX volatility based on a rolling time windows of 252 trading days (approx. one year).
- This illustrates that volatility varies over time.

[38]:
```
vol=returns.rolling(window=252).std()
vol.dropna(inplace=True)
```

[39]:
```
vol.plot(figsize=(10,4));
```



Time series models

- The following figure illustrates the 100 most extreme DAX returns over the time period 1990-2019.
- These are not evenly spaced, but appear in clusters.

[40]:
```
m = abs(returns).sort_values()[-100] # the top 100 returns are greater than this
m
```

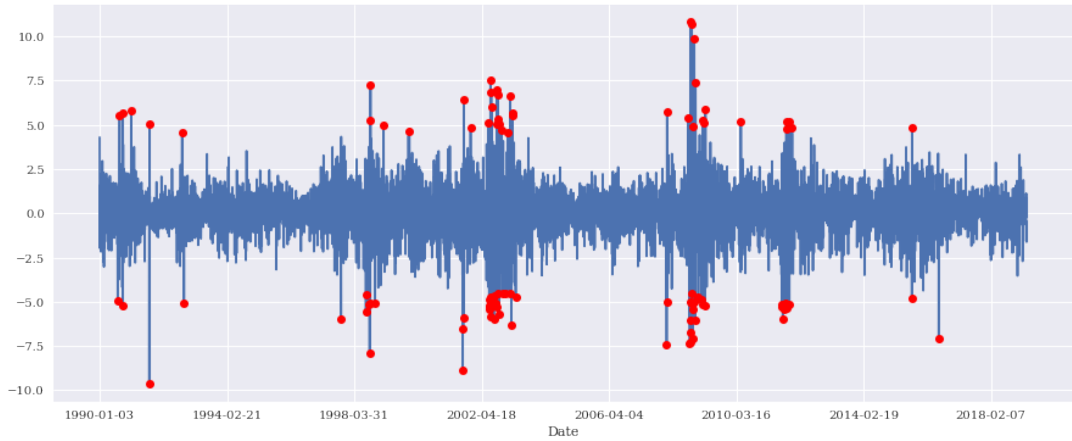
[40]: 4.49364555843897

[41]:
```
mreturns = returns.loc[abs(returns) > m]
```

[42]:
```
ret = pd.DataFrame(returns, index=returns.index)
mret = pd.DataFrame(mreturns, index=mreturns.index)
all = ret.join(mret, lsuffix='_caller', rsuffix='_other') # merge the data into one␣
↪DataFrame
```

Time series models

```
[43]: all.plot(figsize=(15,6), style=['', 'ro'], legend=None);
```



Time series models

- These phenomena typically become less pronounced as the time period between successive returns is increased.
- For daily or weekly data, however, it is clear that a model needs to capture the time series variations, most importantly the time-varying volatility.

4.4 Time Series Models: GARCH

The GARCH model

- The class of **GARCH (generalised autoregressive conditional heteroskedastic) models** incorporate time-varying volatility, autocorrelation in absolute / squared returns and fatter tails than suggested by the normal distribution (see Bollerlslev, 1986).
- The GARCH(1,1) is the simplest and most widely used of the family of GARCH-type models.
- A process $X = (X_t)_{t\in\mathbb{Z}}$ is a **GARCH(1,1) process** if it is satisfies

$$\begin{aligned} X_t &= \sigma_t Z_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2 + \beta\sigma_{t-1}^2, \end{aligned}$$

  where the **innovations** Z_t, $t = 1, 2, \ldots$ are iid standard normally distributed, and $\alpha_0 > 0$, $\alpha_1 \geq 0$ and $\beta \geq 0$.
- In this model periods of high volatility tend to be **persistent**, that is, if either $|X_{t-1}|$ or σ_{t-1} are large, then $|X_t|$ has a tendency to be large as well, which in turn causes a high volatility.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics, pp. 31 (3), 307–327.

Properties of the GARCH model

Proposition

Let X be a a GARCH(1,1) process satisfying $\alpha_1 + \beta < 1$. Then, for all $s, t \in \mathbb{Z}$, 1. $\mathbb{E}(X_t) = 0$; 2. $\mathrm{Var}(X_t) = \dfrac{\alpha_0}{1 - \alpha_1 - \beta}$; 3. the **autocorrelation** $\mathbb{E}(X_t X_s)/\sqrt{\mathrm{Var}(X_t)\,\mathrm{Var}(X_s)}$ is 0 whenever $s \neq t$; 4. the variance of X_t conditional on the information up to time $t-1$ is σ_t^2; 5. the kurtosis of X_t is

$$\frac{\mathbb{E}(X_t^4)}{\mathbb{E}(X_t^2)^2} = \frac{3\,(1 - (\alpha_1 + \beta)^2)}{1 - (\alpha_1 + \beta)^2 - 2\,\alpha_1^2},$$

In particular, X_t has a positive excess kurtosis.

Variants of the GARCH process

- The more general GARCH(p, q) model is defined by setting the variance to

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2.$$

- There are many extensions of GARCH processes (Integrated GARCH, GARCH with leverage, Threshold GARCH, ...).

Fitting a GARCH model

- Given a time series, such as the DAX returns, and postulating a GARCH model, we find the parameters that provide the "best" fit for the data.
- The best fit is generally obtained via the method of **maximum likelihood**.
- The `arch` library in Python will do this for us.

Fitting a GARCH model

[44]:
```
from arch import arch_model
ret_demeaned=returns-returns.mean(); # de-mean process, i.e., adjust so that mean␣
↪is zero
am = arch_model(ret_demeaned, mean = 'Zero')
res = am.fit()
```

```
Iteration:      1,   Func. Count:      5,   Neg. LLF: 11626.026847183497
Iteration:      2,   Func. Count:     13,   Neg. LLF: 11625.965550453562
Iteration:      3,   Func. Count:     19,   Neg. LLF: 11623.43241756365
Iteration:      4,   Func. Count:     25,   Neg. LLF: 11622.72549915859
Iteration:      5,   Func. Count:     31,   Neg. LLF: 11622.613321349227
Iteration:      6,   Func. Count:     37,   Neg. LLF: 11622.038129157363
Iteration:      7,   Func. Count:     42,   Neg. LLF: 11621.980562427689
Iteration:      8,   Func. Count:     47,   Neg. LLF: 11621.978840084295
Iteration:      9,   Func. Count:     52,   Neg. LLF: 11621.978820104492
Optimization terminated successfully.    (Exit mode 0)
            Current function value: 11621.97881942417
            Iterations: 9
            Function evaluations: 53
            Gradient evaluations: 9
```

Fitting a GARCH model

[45]:
```
res
```

[45]:
```
                       Zero Mean - GARCH Model Results
==============================================================================
Dep. Variable:                  Close   R-squared:                       0.000
Mean Model:                 Zero Mean   Adj. R-squared:                  0.000
Vol Model:                      GARCH   Log-Likelihood:               -11622.0
Distribution:                  Normal   AIC:                           23250.0
Method:            Maximum Likelihood   BIC:                           23270.6
                                        No. Observations:                 7282
Date:                Thu, Apr 02 2020   Df Residuals:                     7279
Time:                        22:31:17   Df Model:                            3
                              Volatility Model
============================================================================
                 coef    std err          t      P>|t|      95.0% Conf. Int.
----------------------------------------------------------------------------
```

```
omega          0.0283  8.755e-03      3.228  1.248e-03 [1.110e-02,4.542e-02]
alpha[1]       0.0823  8.774e-03      9.384  6.344e-21 [6.514e-02,9.953e-02]
beta[1]        0.9019  1.016e-02     88.765      0.000     [  0.882,   0.922]
============================================================================

Covariance estimator: robust
ARCHModelResult, id: 0x1a24aca450
```

Fitting a GARCH model

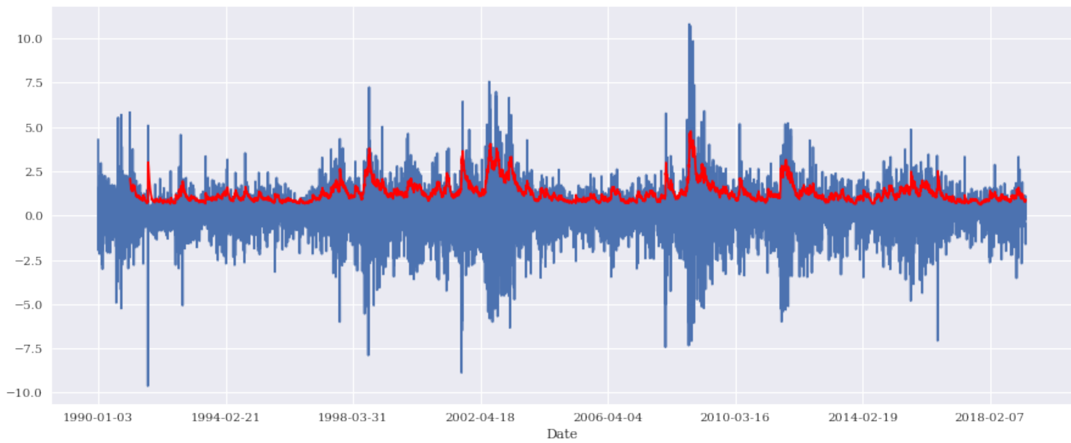
- The following parameters are obtained: $\alpha_0 = 0.0283$, $\alpha_1 = 0.0823$ and $\beta = 0.9019$.
- All estimates are statistically significant (p-values<0.01).

[46]:
```
ret = pd.DataFrame(returns, index=returns.index)
vol = pd.DataFrame(res.conditional_volatility, index=vol.index)
all = ret.join(vol, lsuffix='_caller', rsuffix='_other') # merge the data into one
 ↪DataFrame
```

Fitting a GARCH model

- The plot shows the DAX returns together with the fitted GARCH volatility.
- The initial volatility is typically chosen as the time series' unconditional volatility.

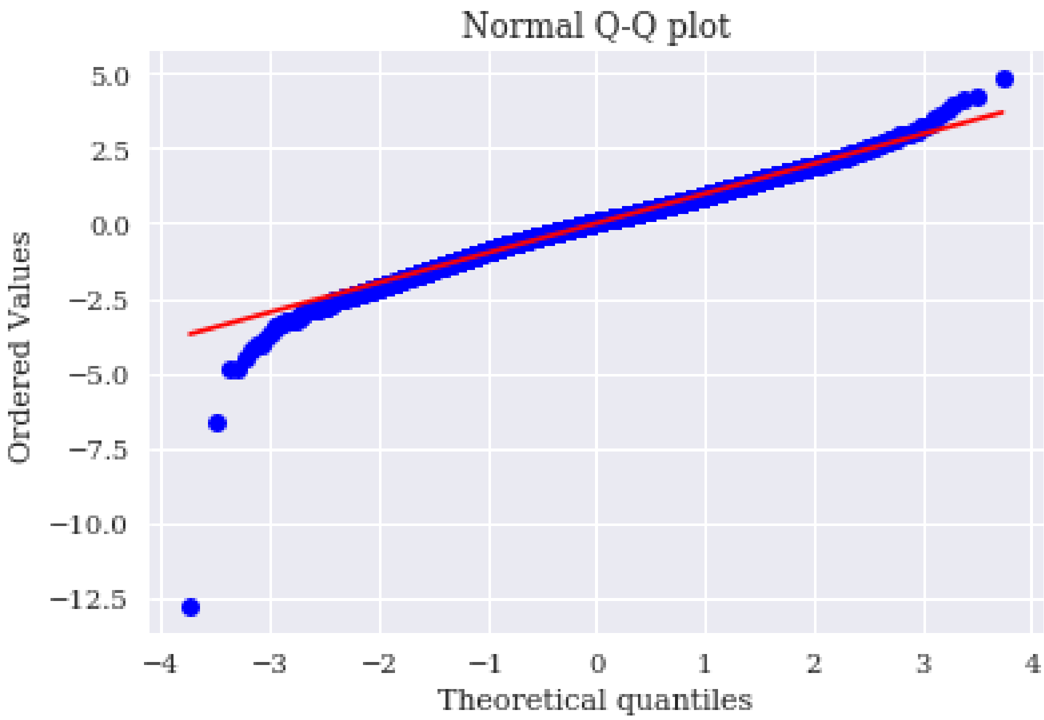
[47]:
```
all.plot(figsize=(15,6), style=['', 'r'],legend=None);
```



Validating a GARCH model

- To check the quality of the fit, one can compare the "residuals" $Z_t = X_t/\sigma_t$ with a standard normal distribution via a QQ-plot, see figure below.

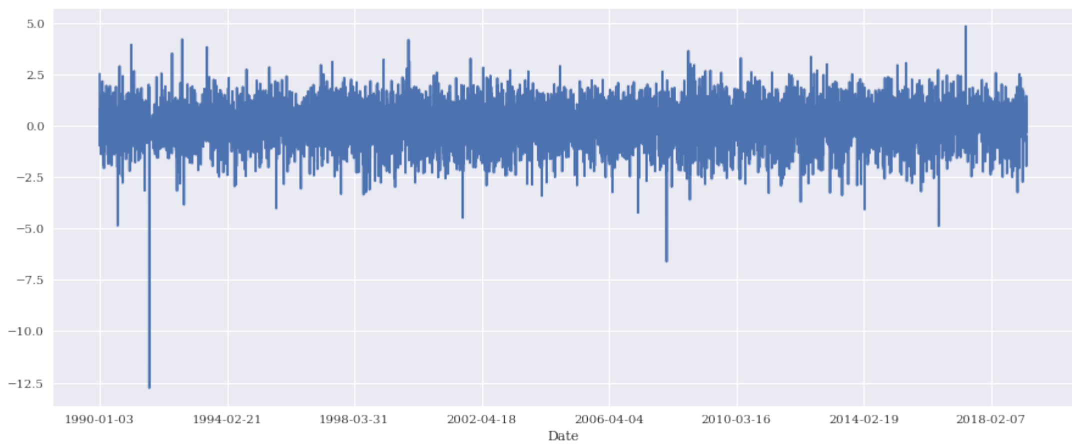
[48]:
```
import scipy.stats as stats
residuals=res.resid/res.conditional_volatility; # the residuals
stats.probplot(residuals, dist="norm", plot=plt)
plt.title("Normal Q-Q plot")
plt.show()
```



Validating a GARCH model

- This is what the residuals look like:

[49]:
```
residuals.plot(figsize=(15,6));
```



Validating a GARCH model

- In case the residuals do not fit the normal distribution, one may - in a second step - fit the residuals to a more appropriate distribution, such as the more heavy-tailed Student t.

Volatility forecasting

- One use of GARCH models is to forecast future volatility.
- Given asset returns $x_0, \ldots, x_t$ assume that a GARCH model has been fitted and that the condition $\alpha_1 + \beta < 1$ (see Proposition above) is fulfilled.
- A prediction of σ^2_{t+1} is given by

$$\hat{\sigma}^2_{t+1} = \mathbb{E}(X^2_{t+1}|X_t, \sigma_t) = \alpha_0 + \alpha_1 X_t^2 + \beta\sigma_t^2,$$

  and, more generally for one time period h periods forward,

$$\hat{\sigma}^2_{t+h} = \mathbb{E}(X^2_{t+h}|X_t, \sigma_t) = \alpha_0 \sum_{i=0}^{h-1} (\alpha_1 + \beta)^i + (\alpha_1 + \beta)^{h-1} (\alpha_1 X_t^2 + \beta\sigma_t^2).$$

- A derivation of this formula is beyond the scope of the course.

Volatility forecasting

[50]:
```
res.params
```

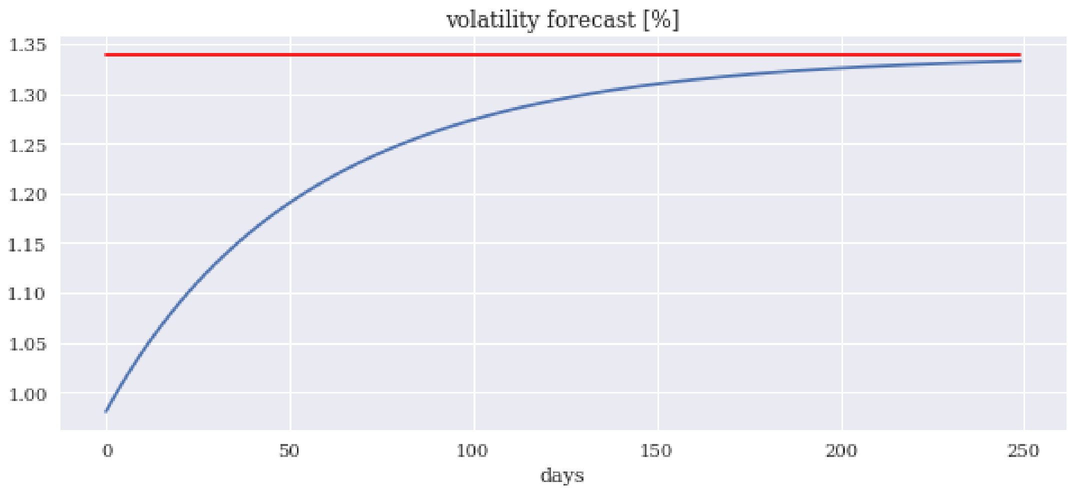
[50]:
```
omega       0.028257
alpha[1]    0.082337
beta[1]     0.901897
Name: params, dtype: float64
```

[51]:
```
sigmasq_f=[]
tmp=[]
alpha0=res.params[0]
alpha1=res.params[1]
beta=res.params[2]

for i in range(0,251):
    tmp.append((alpha1 + beta)**i)

for h in range(1,251):
     sigmasq_f.append(alpha0 * np.sum(tmp[0:h]) + tmp[h-1] * (alpha1 *␣
↪returns[-1]**2 \
                                                                  + beta * res.
↪conditional_volatility[-1]**2))
```

Volatility forecasting

- The figure below shows the volatility forecast for the DAX return data.
- The red line shows the unconditional standard deviation $\sqrt{\dfrac{\alpha_0}{1 - \alpha_1 - \beta}}$.

[52]:
```
unconditional_vol=np.sqrt(res.params[0]/(1-res.params[1]-res.params[2]))*np.
↪ones(len(sigmasq_f))
plt.figure(figsize=(10, 4))
plt.plot(np.sqrt(sigmasq_f))
plt.plot(unconditional_vol, 'r')
plt.title('volatility forecast [%]')
plt.xlabel('days');
```


volatility forecast [%]
1.35
1.30
1.25
1.20
1.15
1.10
1.05
1.00
0
50
100
150
200
250
days