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Computational Finance and FinTech – Problem Set 6 with solutions

Exercise 1. Write a function that calculates the replicating portfolio (x, y) for an arbitrary claim $\Phi(Z)$ in a one-period model. Use `np.linalg.solve`.
Test your function on the example from class, where $r = 5\%$, $S_0 = 100$, $u = 1.2$, $d = 0.8$, and the claim under consideration is a call option with strike price $K = 105$.

Solution 1.

```
import numpy as np

def replication(B0, r, S0, u, d, Phiu, Phid):
    return np.linalg.solve(np.array([[B0*(1+r), S0*u], [B0*(1+r), S0*d]]),\
                           np.array([Phiu,Phid]))

[x,y]=replication(1, 0.05, 100, 1.2, 0.8, 15, 0)
```

The output is: $x = -28.571428571428557$ and $y = 0.375$.

Exercise 2. Write a function that calculates the risk-neutral probabilities and test your function on the example from class, where $r = 5\%$, $S_0 = 100$, $u = 1.2$, $d = 0.8$.

Solution 2.

```
import numpy as np

def q_probabilities(r, S0, u, d):
    return np.linalg.solve(np.array([[1, 1], [d, u]]), np.array([1,1+r]))

[qd, qu] = q_probabilities(0.05, 100, 1.2, 0.8)
```

The output is $q_d = 0.37499999999999999$ and $q_u = 0.62500000000000001$.

Exercise 3. A stock price is currently trading at €40. It is known that at the end of one month it will be either €42 or €38. The risk-free interest rate is 8% per annum with continuous compounding (verify first that this is the same as a monthly interest rate of 0.6689%).

(a) What is the value of a one-month European call option with a strike price €39?

(b) Explain how the seller of the option can hedge their exposure.

(c) Explain how one could make an arbitrage profit if the option traded in the market for €2.

(d) Calculate the option price via the risk-neutral expectation.

Solution 3.

(a) We solve

$$e^{0.08\cdot 1/12}x + 42y = 3$$
$$e^{0.08\cdot 1/12}x + 38y = 0$$

to give $x = -28.3106$ and $y = 0.75$. The value of this replicating portfolio is $-28.3106+30 = 1.6894$.

In Python: `[x,y] =replication(1, 0.00689, 40, 42/40, 38/40, 3, 0).`

Result: `[-28.30497869677919, 0.75].`

Portfolio value: `x + y * 40` gives `1.695021303220809`.

(b) The seller of the option can eliminate the risk from selling the option by buying the replicating portfolio as this creates a synthetic long position in the option.

(c) If the option can be traded for €2, then one would sell the option and buy the replicating portfolio, thus locking in a profit of $2 - 1.6894 = 0.3106$.

In Python:
Risk-neutral probabiltiies: `[qd,qu] = q_probabilities(0.00689, 40, 42/40, 38/40)` gives `[0.4310999999999995, 0.5689000000000005].`
Discount expected payoff:
`1/(1.00689)* (qd * 0 + qu * 3)` gives `1.69502130322081`.

Exercise 4. Consider a two-step binomial tree model where, in each step, the stock price increases by 20% or decreases by 10%. The interest rate is 10% (simple compounding). The current stock price is 100.

(a) Consider a call option with strike 110 that expires after two time periods.

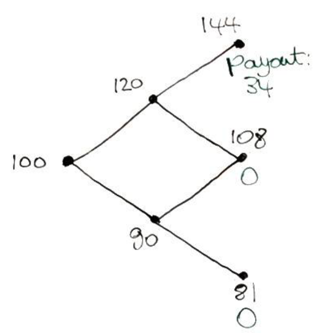
(i) Draw a tree showing the evolution of the stock price and the payout of the option.
(ii) Determine the option price at every node in the tree.
(iii) Calculate the option price using the method of risk-neutral pricing.

(b) Determine the price of an American put option with strike 110. Explain why the result differs from the price of a European put option.

Solution 4.

(a)

(i)



(ii)
- Option price at $S_1 = 120$: replicating portfolio: $x = -84.2975$, $y = 0.9444$, so $V_1 = 20.6061$. (To calculate x note that the bond price at time 1 is 1.1.)
- Option price at $S = 90$: replicating portfolio: $x = 0$, $y = 0$, so $V_1 = 0$.
- Option price at $S_0 = 100$: replicating portfolio $x = -56.1985$, $y = 0.6869$ so $V_0 = 12.4915$. (Note: use $\phi(u) = 20.6061$ and $\phi(d) = 0$, i.e., the prices of the option at time 1.)

(iii) Risk-neutral probabilities: $q_u = \dfrac{2}{3}$, $q_d = \dfrac{1}{3}$.

Risk-neutral pricing:

$$\frac{1}{(1+R)^2}\mathbb{E}[(S_2 - 110)^+] = \frac{1}{(1+R)^2}\left(34 \cdot q_u^2 + 0 \cdot 2 \cdot q_u \cdot q_d + 0 \cdot q_d^2\right)$$
$$= 0.8264 \cdot 15.1111 = 12.49.$$

(b) If the stock price goes down in the first period, then the value of continuing to hold the put option (the continuation value) is

$$\frac{1}{1.1}\left[q_d \cdot (110 - 81) + q_u \cdot (110 - 108)\right] = 10.$$

In this state, a better choice is to exercise the option, which gives a payoff of $110 - 90 = 20$. Hence, the *value* of the option in this state is $\max(10, 20) = 20$.

If the stock price goes up, then the time-1 value is 0.6061.

The value at time 0 is thus

$$\frac{1}{1.1}\left[q_d \cdot 20 + q_u \cdot 0.6061\right] = 6.4279.$$

European-style options cannot be exercised early, hence their price is lower. In this case, the option could not be exercised at time 1, so the payoff of 20 could not be realised.