TALLER DE PROBLEMAS (Clase #5)
Sección 2.1.8, ejercicios 3 y 10

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6. En el caso 3D tenemos que si $\{\mathbf{e}_i\}$ define un sistema de coordenadas (dextrógiro) no necesariamente ortogonal, entonces, demuestre que:

   a)
$$\mathbf{e}^i = \frac{\mathbf{e}_j \times \mathbf{e}_k}{\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k)}, \quad i,j,k = 1,2,3 \text{ y sus permutaciones cíclicas}$$

   b) si los volumenes: $V = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)$ y $\tilde{V} = \mathbf{e}^1 \cdot (\mathbf{e}^2 \times \mathbf{e}^3)$, entonces $V\tilde{V} = 1$.

   c) ¿Qué vector satisface $\mathbf{a} \cdot \mathbf{e}^i = 1$? Demuestre que $\mathbf{a}$ es único.

d) Encuentre el producto vectorial de dos vectores $\mathbf{a}$ y $\mathbf{b}$ que están representados en un sistema de coordenadas oblicuo: Dada la base: $\mathbf{w}_1 = 4\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\mathbf{w}_2 = 3\mathbf{i}+3\mathbf{j}$, $\mathbf{w}_3 = 2\mathbf{k}$. Entonces encuentre:

1) Las bases recíprocas $\{\mathbf{e}^i\}$.
2) Las componentes covariantes y contravariantes del vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

d

1 $\quad w_1 = (4,2,1) \quad W_2 = (3,3,0) \quad W_3 = (0,0,2)$

$$\langle e^1 | w_1 \rangle = (a_1, b_1, c_1)\begin{pmatrix}4\\2\\1\end{pmatrix} = 4a_1 + 2b_1 + c_1 = 1$$

$$\langle e^1 | w_2 \rangle = (a_1, b_1, c_1)\begin{pmatrix}3\\3\\0\end{pmatrix} = 3a_1 + 3b_1 = 0$$

$$\langle e^1 | w_3 \rangle = (a_1, b_1, c_1)\begin{pmatrix}0\\0\\2\end{pmatrix} = 2c_1 = 0$$

$c_1 = 0 \quad a_1 = -b_1 \quad 4a_1 - 2a_1 = 1$

$2a_1 = 1 \quad a_1 = \frac{1}{2} \quad b_1 = -\frac{1}{2}$

$$\langle e^2 | w_1 \rangle = (a_2, b_2, c_2)\begin{pmatrix}4\\2\\1\end{pmatrix} = 4a_2 + 2b_2 + C_2 = 0$$

$C_2 = 0 \quad 4a_2 = -2b_2$

$-2a_2 = b_2$

$\langle e^2 | w_2 \rangle = 3a_2 + 3b_2 = 1$

$\langle e^2 | w_3 \rangle = 2c_2 = 0$

$3a_2 - 6a_2 = 1$

$-3a_2 = 1 \quad a_2 = -\frac{1}{3}$

$3b_2 = 2$

$b_2 = \frac{2}{3}$

$$\langle e^3 | w_1 \rangle = 4a_3 + 2b_3 + c_3 = 0$$

$$\langle e^3 | w_2 \rangle = 3a_3 + 3b_3 = 0$$

$$\langle e^3 | w_3 \rangle = 2c_3 = 1$$

$c_3 = \frac{1}{2}$ $\qquad a_3 = -b_3 \qquad 4a_3 - 2a_3 = -\frac{1}{2}$

$b_3 = \frac{1}{4} \qquad 2a_3 = -\frac{1}{2}$

$a_3 = -\frac{1}{4}$

$$\langle e^1 | = \left(\frac{1}{2}, -\frac{1}{2}, 0 \right)$$

$$\langle e^2 | = \left(-\frac{1}{3}, \frac{2}{3}, 0 \right)$$

$$\langle e^3 | = \left(-\frac{1}{4}, \frac{1}{4}, \frac{1}{2} \right)$$

(2)

$$a = i + 2j + 3k = (1\ 2\ 3)$$

Contravariantes

$$a^1 = \langle e^1 | a \rangle = \frac{1}{2} - 1 + 0 = -\frac{1}{2}$$

$$a^2 = \langle e^2 | a \rangle = -\frac{1}{3} + \frac{4}{3} + 0 = 1$$

$$a^3 = \langle e^3 | a \rangle = -\frac{1}{4} + \frac{2}{4} + \frac{3}{2} = \frac{7}{4}$$

$$a^i = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{7}{4} \end{pmatrix}$$

Covariantes

$$b_1 = \langle b | w_1 \rangle = 4 + 4 + 3 = 11$$

$$b_2 = \langle b | w_2 \rangle = 3 + 6 + 0 = 9$$

$$b_3 = \langle b | w_3 \rangle = 2 + 0 + 0 = 2$$

$$b_i = (11\ \ 9\ \ 2)$$

7. Considere una vez más el espacio vectorial de matrices hermíticas 2×2 y la definición de producto interno $\langle a | b \rangle \rightleftharpoons \mathrm{Tr}(\mathbb{A}^\dagger \mathbb{B})$ que introdujimos en los ejercicios de la sección 2.2.6. Hemos comprobado que las matrices de Pauli $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ –presentadas tambien en los ejercicios de la sección 2.2.6– forman base para ese espacio (ver ejercicios sección 2.3.8). Encuentre entonces la base dual asociada a las base de Pauli y, adicionalmente dado un vector genérico en este espacio vectorial, encuentre también su 1-forma asociada.

(7)

$$e^{i+} = \begin{pmatrix} a_i & c_i^* \\ b_i^* & d_i \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad e^i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e^{i+} = \begin{pmatrix} a_i & c_i^* \\ b_i^* & d_i \end{pmatrix}$$

$$c_i^* = d_i$$

$$\langle e^1 | \sigma_1 \rangle = Tr\left(A^+ B\right) = Tr\begin{bmatrix} c_1^* & a_1 \\ b_1 & b_1^* \end{bmatrix} = b_1^* + c_1^* = 1$$

$$\langle e^1 | \sigma_2 \rangle = Tr\begin{bmatrix} -i c_1^* & \\ & -i b_1^* \end{bmatrix} = i c_1^* - i b_1^* = 0$$

$$\langle e^1 | \sigma_3 \rangle = Tr\begin{bmatrix} a_1 & \\ & -d_1 \end{bmatrix} = a_1 - d_1 = 0$$

$$\langle e^1 | \sigma_0 \rangle = Tr\begin{bmatrix} a_1 & \\ & d_1 \end{bmatrix} = a_1 + d_1 = 0$$

$a_1 = 0 \quad d_1 = 0 \qquad c_1^* = b_1^* \qquad c_1, d_1 \in \mathbb{R}$

$c_1^* = b_1 \qquad c_1 = b_1$

$b_1 = \frac{1}{2} \quad c_1 = \frac{1}{2}$

$$e^1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

$$\langle e^2 | \sigma_1 \rangle = c_2^* + b_2^* = 0$$

$$\langle e^2 | \sigma_2 \rangle = i c_2^* - i b_2^* = 1$$

$$c_2 = b_2^* = x + y i$$

$$b_2 = c_2^* = x - y i$$

$$\langle e^2 | \sigma_3 \rangle = a_2 - d_2 = 0$$

$$\langle e^2 | \sigma_0 \rangle = a_2 + d_2 = 0$$

$a_2 = 0 \quad d_2 = 0$

$$x - yi = -x - yi$$

$$\Rightarrow x = 0$$

$$y \in \mathbb{R}$$

$$c_2 = yi$$
$$b_2 = -yi$$

$$\boxed{y = 1}$$

$$\langle e^2 | = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

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$$\langle e^3 | \sigma_1 \rangle = c_3^* + b_3^* = 0$$

$$\langle e^3 | \sigma_2 \rangle = i c_3^* - i b_3^* = 0$$

$$\langle e^3 | \sigma_3 \rangle = a_3 - d_3 = 1$$

$$\langle e^3 | \sigma_0 \rangle = d_3 + d_3 = 0$$

$$\langle e^3 | = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$a_3 = -d_3$

$$-2 d_3 = 1$$

$$d_3 = -\frac{1}{2}$$

$$a_3 = \frac{1}{2}$$

$c_3 = 0$

$b_3 = 0$

$$\langle e^0 | \sigma_1 \rangle = c_0^* + b_0^* = 0$$

$$\langle e^0 | \sigma_2 \rangle = i c_0^* - i b_0^* = 0$$

$$\langle e^0 | \sigma_3 \rangle = a_0 - d_0 = 0$$

$$\langle e^0 | \sigma_0 \rangle = a_0 + d_0 = 1$$

$c_0 = 0$

$b_0 = 0$

$$a_0 = d_0$$

$$2a_0 = 1$$

$$a_0 = \frac{1}{2} \qquad d_0 = \frac{1}{2}$$

$$\langle e^0 | = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$a = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$\tilde{a}^1 = \langle \tilde{e}^1 | a \rangle = Tr\left[\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \cdot \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \right] = -\frac{1}{2}i + \frac{1}{2}i = 0$$

$$\tilde{a}^2 = \langle \tilde{e}^2 | a \rangle = Tr\left[\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \right] = 1 + 1 = 2$$

$$\tilde{a}^3 = \langle \tilde{e}^3 | a \rangle = Tr\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \right] = \frac{1}{2} - \frac{1}{2} = 0$$

$$\tilde{a}^0 = \langle \tilde{e}^0 | a \rangle = Tr\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \right] = \frac{1}{2} + \frac{1}{2} = 1$$

$$\langle a | = 2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + 1 \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$\langle a | = \begin{pmatrix} \frac{1}{2} & -2i \\ 2i & \frac{1}{2} \end{pmatrix}$$