Improved Classical and Quantum Algorithms for Subset-Sum

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Outline

Introduction

The Subset-Sum Problem

Problem

Given: $\mathbf{a} = (a_1, \ldots, a_{\mathbf{n}})$ *a vector of* ℓ*-bit integers, and an* ℓ*-bit target* S*, find* $\mathbf{e} = (e_1, \ldots, e_{\mathbf{n}}) \in \{0,1\}^{\mathbf{n}}$ *such that* $\mathbf{e} \cdot \mathbf{a} = \sum_i e_i a_i = S \mod 2^{\ell}$.

- The decision version is NP-complete
- The low-density case ($\ell \gg \mathbf{n}$) is related to lattice SVP
- The high-density case ($\ell \ll \mathbf{n}$) is solvable efficiently
- The density-1 case ($\ell \simeq \mathbf{n}$) is hard

Subset-sums in (post-quantum) cryptography

- Repeatedly used as a hard problem for post-quantum cryptography [a]
- Similar techniques that we will see in this presentation apply to other problems (generic decoding algorithms) [b]
- Solving subset-sums is also useful in quantum hidden shift algorithms [c]

[a]Lyubashevsky, Palacio, and Segev, *"Public-Key Cryptographic Primitives Provably as Secure as Subset Sum"*, TCC 10

[b]Kachigar and Tillich, *"Quantum Information Set Decoding Algorithms"*, PQCrypto 17

[c]Bonnetain, *Improved Low-qubit Hidden Shift Algorithms*, 2019

The random Subset-Sum Problem

Problem

Given: $\mathbf{a} = (a_1, \ldots, a_{\mathbf{n}})$ *a vector of* **n***-bit integers, and an* **n***-bit target* S*, find* $\mathbf{e} = (e_1, \ldots, e_{\mathbf{n}}) \in \{0,1\}^{\mathbf{n}}$ *such that* $\mathbf{e} \cdot \mathbf{a} = \sum_i e_i a_i = S \mod 2^{\mathbf{n}}$; ***where*** $\mathbf{a}, S$ ***are selected uniformly at random.***

- Classical and quantum algorithms run in time $\widetilde{\mathcal{O}}\left(2^{\beta \mathbf{n}}\right)$: we are interested in the value of β
- In this talk, we optimize the **time exponent** (not the memory)
- We consider w.l.o.g. that $\mathbf{e}$ has Hamming weight $\frac{\mathbf{n}}{2}$

Classical algorithms

The time is $\widetilde{\mathcal{O}}\left(2^{\beta \mathbf{n}}\right)$.

Technique	β	Ref.
MIM	0.5	HS74 (Slide 8)
4-list merge	0.5	SS81 (Slide 9)
$\{0, 1\}$	0.3370	HGJ10 (Slide 18)
$\{-1, 0, 1\}$	0.2909	BCJ11 (Slide 23)
$\{-1, 0, 1\}$ + NN	0.287	Ilya Ozerov's PhD thesis
$\{-1, 0, 1, 2\}$	0.283	Ours

Classical algorithm: meet-in-the-middle

Cut the solution $\mathbf{e} = \underbrace{(\underbrace{0 \dots 0}_{\mathbf{n}/2 \text{ bits}} \mid \underbrace{* \dots *}_{\mathbf{n}/2 \text{ bits}})}_{2^{\mathbf{n}/2} \text{ choices: } L_l} + \underbrace{(\underbrace{* \dots *}_{\mathbf{n}/2 \text{ bits}} \mid \underbrace{0 \dots 0}_{\mathbf{n}/2 \text{ bits}})}_{2^{\mathbf{n}/2} \text{ choices: } L_r}$

Then find $\mathbf{e_l} \in L_l, \mathbf{e_r} \in L_r$ s.t. $\mathbf{e_l} \cdot \mathbf{a} = -\mathbf{e_r} \cdot \mathbf{a} + S \mod 2^{\mathbf{n}}$.

Complexities

Time: $\mathcal{O}\left(2^{\mathbf{n}/2}\right)$ (best worst-case time); **Memory:** $\mathcal{O}\left(2^{\mathbf{n}/2}\right)$

Horowitz and Sahni, *"Computing Partitions with Applications to the Knapsack Problem"*, J. ACM

Schroeppel and Shamir's 4-list merging

By cutting in 4 instead of 2, we can decrease the memory to $2^{\mathbf{n}/4}$.

$$\mathbf{e} = \underbrace{(*|0|0|0)}_{\mathbf{e_0} \in L_0} + \underbrace{(0|*|0|0)}_{\mathbf{e_1} \in L_1} + \underbrace{(0|0|*|0)}_{\mathbf{e_2} \in L_2} + \underbrace{(0|0|0|*)}_{\mathbf{e_3} \in L_3}$$

We now look for $(\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}) \in L_0 \times L_1 \times L_2 \times L_3$ s.t.

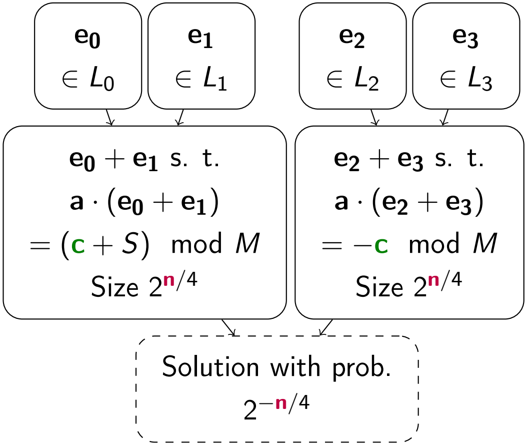
$$\mathbf{a} \cdot (\mathbf{e_0} + \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}) = S \mod 2^{\mathbf{n}}$$

$\Rightarrow$ there is still one solution among $2^{\mathbf{n}/4} \times 2^{\mathbf{n}/4} \times 2^{\mathbf{n}/4} \times 2^{\mathbf{n}/4}$.

Schroeppel and Shamir, *"A $T = O(2^{n/2})$, $S = O(2^{n/4})$ Algorithm for Certain NP-Complete Problems"*, SIAM 81

Schroeppel and Shamir's 4-list merging (ctd.)

- Choose an $n/4$-bit number $c \mod M$
- Repeat for every value of c:


$e_0 \in L_0$
$e_1 \in L_1$
$e_2 \in L_2$
$e_3 \in L_3$
$e_0 + e_1$ s. t. $a \cdot (e_0 + e_1) = (c + S) \mod M$
Size $2^{n/4}$
$e_2 + e_3$ s. t. $a \cdot (e_2 + e_3) = -c \mod M$
Size $2^{n/4}$
Solution with prob. $2^{-n/4}$

Complexities

Time: $\mathcal{O}\left(2^{n/4} \times 2^{n/4}\right)$

Memory: $\mathcal{O}\left(2^{n/4}\right)$

Representations

Breaking the $2^{n/2}$ bound

- When we have **one** solution among 2^n tuples, we don't know of any better time than $2^{n/2}$
- The idea of Howgrave-Graham and Joux (HGJ): cut $\mathbf{e}$ with respect to its Hamming weight

Suppose that $\mathbf{e}$ is of weight $n/2$ (worst case). Write for example:

$$\underbrace{\mathbf{e}}_{\substack{\text{Weight}\\ n/2}} = \underbrace{\mathbf{e_0}}_{\substack{\text{Weight}\\ n/8}} + \underbrace{\mathbf{e_1}}_{\substack{\text{Weight}\\ n/8}} + \underbrace{\mathbf{e_2}}_{\substack{\text{Weight}\\ n/8}} + \underbrace{\mathbf{e_3}}_{\substack{\text{Weight}\\ n/8}}$$

notice that: $\binom{n}{n/8}^4 \simeq 2^{2.174n} \ggg \binom{n}{n/2} \simeq 2^n$: many solution tuples!

Howgrave-Graham and Joux, *"New Generic Algorithms for Hard Knapsacks"*, EC 10

Notations

We introduce **distributions** and **weight constraints**.

Distributions

$\mathbf{e} \in D^{\mathbf{n}}[\alpha]$ if $\mathbf{e}$ contains $\alpha\mathbf{n}$ “1” and $(1-\alpha)\mathbf{n}$ “0”.

Then if $\mathbf{e_1} \in D^{\mathbf{n}}[\alpha_1], \mathbf{e_2} \in D^{\mathbf{n}}[\alpha_2]$ we have $\mathbf{e_1} + \mathbf{e_2} \in D^{\mathbf{n}}[\alpha_1 + \alpha_2]$ **with some probability** (to be continued).

Weight constraints

$\mathbf{e}$ has “a $\mathbf{cn}$-bit weight constraint” if we are able to constrain the (knapsack) weight of $\mathbf{e}$ as $\mathbf{e} \cdot \mathbf{a} = s \mod M$ for (previously) chosen $\mathbf{cn}$-bit integers M and s.

If $\mathbf{e_1}$ has a $\mathbf{cn}$-bit-weight cons. and $\mathbf{e_2}$ has a $\mathbf{cn}$-bit-weight cons., then $\mathbf{e_1} + \mathbf{e_2}$ as well by linearity (but the precise moduli don't matter!)

Example: Schroeppel-Shamir

$$\begin{tikzcd}
 & \begin{array}{c} L_1, \mathbf{0} \\ \{0\}^{\mathbf{n}/4} \times D^{\mathbf{n}/4}[\frac{1}{2}] \times \{0\}^{\mathbf{n}/2} \end{array} \arrow[dd] & & \begin{array}{c} L_2, \mathbf{0} \\ \{0\}^{\mathbf{n}/2} \times D^{\mathbf{n}/4}[\frac{1}{2}] \times \{0\}^{\mathbf{n}/4} \end{array} \arrow[dd] \\
\begin{array}{c} L_0, \mathbf{0} \\ D^{\mathbf{n}/4}[\frac{1}{2}] \times \{0\}^{3\mathbf{n}/4} \end{array} \arrow[dr] & & \begin{array}{c} L_3, \mathbf{0} \\ \{0\}^{3\mathbf{n}/4} \times D^{\mathbf{n}/4}[\frac{1}{2}] \end{array} \arrow[dr] & \\
 & \begin{array}{c} L_0^1, \mathbf{c} = \mathbf{\frac{1}{4}} \\ D^{\mathbf{n}/2}[\frac{1}{2}] \times \{0\}^{\mathbf{n}/2} \end{array} \arrow[dr] & & \begin{array}{c} L_1^1, \mathbf{c} = \mathbf{\frac{1}{4}} \\ \{0\}^{\mathbf{n}/2} \times D^{\mathbf{n}/2}[\frac{1}{2}] \end{array} \arrow[dl] \\
 & & L^0, \mathbf{1} &
\end{tikzcd}$$

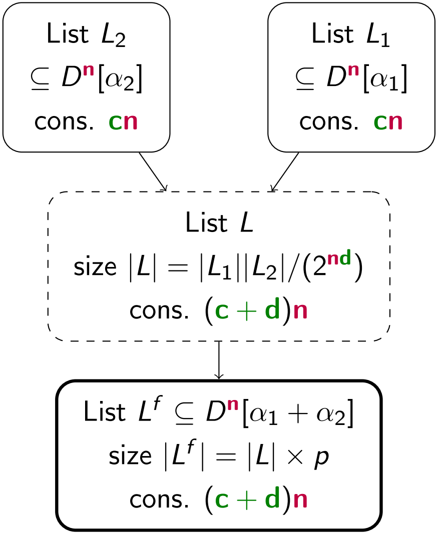
Generic layout

Solution

The solution $\mathbf{e}$ is the only vector with an $\mathbf{n}$-bit weight constraint and $\mathbf{e} \in D^{\mathbf{n}}[1/2]$.

- We start from vectors $\mathbf{e}$ with a 0-bit weight constraint and distributions $D^{\mathbf{n}}[\alpha]$
- We sum them, trying to increase the weight constraint ("merging")
- Eventually we get to a $\mathbf{n}$-bit weight constraint and a distribution $D^{\mathbf{n}}[1/2]$

Merging and filtering


List L_2
$\subseteq D^{\mathbf{n}}[\alpha_2]$
cons. $\mathbf{cn}$
List L_1
$\subseteq D^{\mathbf{n}}[\alpha_1]$
cons. $\mathbf{cn}$
List L
size $|L| = |L_1||L_2|/(2^{\mathbf{nd}})$
cons. $(\mathbf{c} + \mathbf{d})\mathbf{n}$
List $L^f \subseteq D^{\mathbf{n}}[\alpha_1 + \alpha_2]$
size $|L^f| = |L| \times p$
cons. $(\mathbf{c} + \mathbf{d})\mathbf{n}$

Step 1: merging

Find pairs with more constrained weights.

We produce L in time $\max(\min(|L_1|, |L_2|), |L|)$.

Step 2: filtering

Keep only the $\mathbf{e_1} + \mathbf{e_2}$ that conform to the expected distribution.

p is the "filtering probability" for: $D^{\mathbf{n}}[\alpha_1] \times D^{\mathbf{n}}[\alpha_2] \rightarrow D^{\mathbf{n}}[\alpha_1 + \alpha_2]$

Merging and filtering (ctd.)

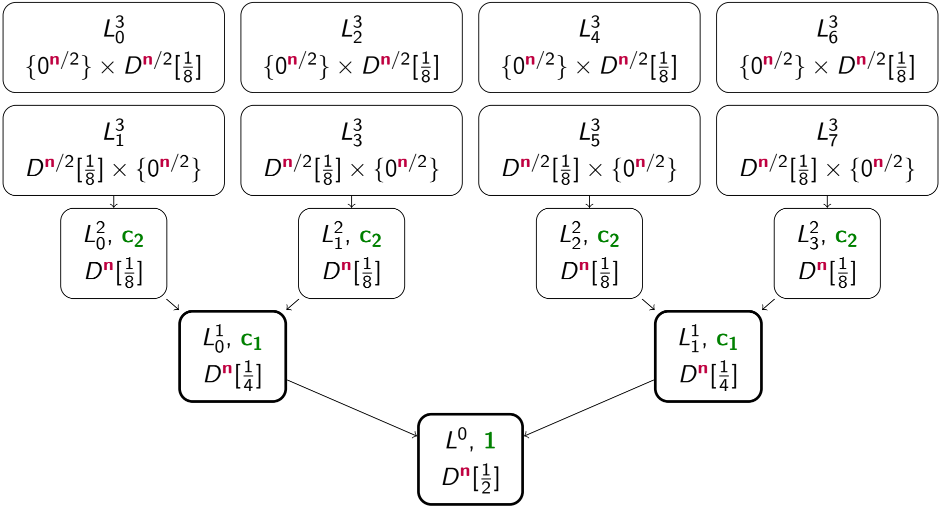
Heuristic

The vectors in L^f are uniformly distributed in $D^{\mathbf{n}}[\alpha_1 + \alpha_2]$.

Approximations

- Representation sets have size: $D^{\mathbf{n}}[\alpha] = \binom{\mathbf{n}}{\alpha \mathbf{n}} \simeq 2^{h(\alpha)\mathbf{n}}$
- $h(x) = -x \log x - (1-x)\log(1-x)$
- In general we have a filtering probability $D^{\mathbf{n}}[\alpha_1] \times D^{\mathbf{n}}[\alpha_2] \to D^{\mathbf{n}}[\alpha_1 + \alpha_2]$ of: $\simeq 2^{(h(\alpha_1/(1-\alpha_2)) - h(\alpha_1))\mathbf{n}}$

The HGJ algorithm

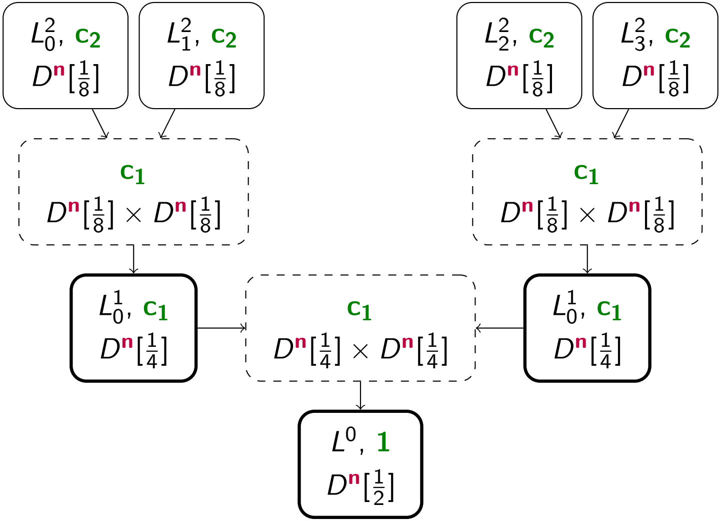

L_0^3 $\{0^{n/2}\} \times D^{n/2}[\frac{1}{8}]$
L_2^3 $\{0^{n/2}\} \times D^{n/2}[\frac{1}{8}]$
L_4^3 $\{0^{n/2}\} \times D^{n/2}[\frac{1}{8}]$
L_6^3 $\{0^{n/2}\} \times D^{n/2}[\frac{1}{8}]$
L_1^3 $D^{n/2}[\frac{1}{8}] \times \{0^{n/2}\}$
L_3^3 $D^{n/2}[\frac{1}{8}] \times \{0^{n/2}\}$
L_5^3 $D^{n/2}[\frac{1}{8}] \times \{0^{n/2}\}$
L_7^3 $D^{n/2}[\frac{1}{8}] \times \{0^{n/2}\}$
L_0^2, c_2 $D^n[\frac{1}{8}]$
L_1^2, c_2 $D^n[\frac{1}{8}]$
L_2^2, c_2 $D^n[\frac{1}{8}]$
L_3^2, c_2 $D^n[\frac{1}{8}]$
L_0^1, c_1 $D^n[\frac{1}{4}]$
L_1^1, c_1 $D^n[\frac{1}{4}]$
L^0, 1 $D^n[\frac{1}{2}]$

HGJ step 3: left-right split with a modulus

$$\begin{tikzcd}
\boxed{\begin{array}{c} L_6^3 \\ \{0^{\mathbf{n}/2}\} \times D^{\mathbf{n}/2}[\frac{1}{8}] \end{array}} \arrow[dr] & & \boxed{\begin{array}{c} L_7^3 \\ D^{\mathbf{n}/2}[\frac{1}{8}] \times \{0^{\mathbf{n}/2}\} \end{array}} \arrow[dl] \\
& \boxed{\begin{array}{c} L_3^2, \mathbf{c_2} \\ D^{\mathbf{n}}[\frac{1}{8}] \end{array}} &
\end{tikzcd}$$

At this level we can afford to **merge without filtering**: find vectors $\mathbf{e} \in D^{\mathbf{n}}[\frac{1}{8}]$ with a $\mathbf{c_2}$-weight constraint (on $\mathbf{e} \cdot \mathbf{a}$).

HGJ step 2 and 1: merge and filter


L^2_0, c_2 $D^n[\frac{1}{8}]$
L^2_1, c_2 $D^n[\frac{1}{8}]$
L^2_2, c_2 $D^n[\frac{1}{8}]$
L^2_3, c_2 $D^n[\frac{1}{8}]$
c_1 $D^n[\frac{1}{8}] \times D^n[\frac{1}{8}]$
c_1 $D^n[\frac{1}{8}] \times D^n[\frac{1}{8}]$
L^1_0, c_1 $D^n[\frac{1}{4}]$
c_1 $D^n[\frac{1}{4}] \times D^n[\frac{1}{4}]$
L^1_0, c_1 $D^n[\frac{1}{4}]$
L^0, 1 $D^n[\frac{1}{2}]$

An optimization problem

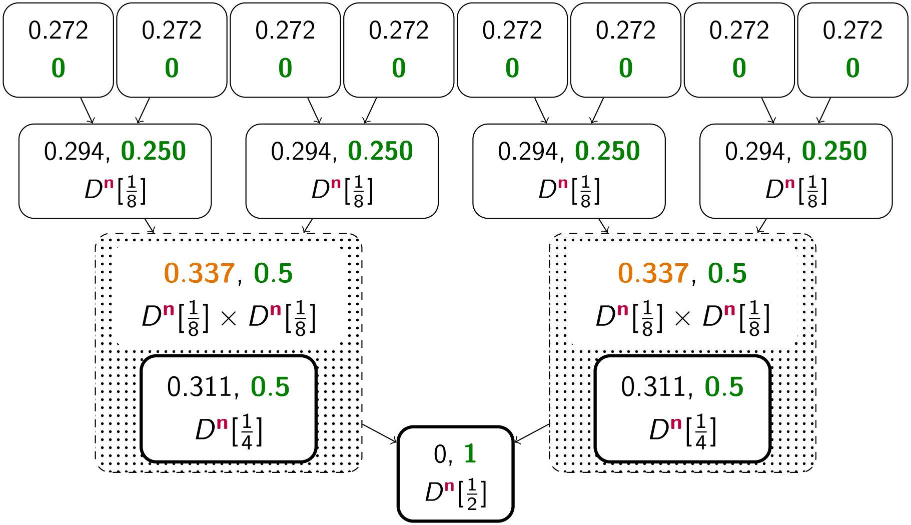
We write all parameters in $\log_2$, relative to $\mathbf{n}$:

$$\begin{array}{c||c} |D^{\mathbf{n}/2}[\frac{1}{8}]| & h(1/8)/2 \simeq 0.2718 \\ |L_i^j| & \ell_i^j \\ |L_1||L_2|/(2^{\mathbf{nd}}) & \ell_1 + \ell_2 - d \\ |L| \times p & \ell + \mathrm{pf} \end{array}$$

We compute the filtering probabilities:

- $D^{\mathbf{n}}[0, \frac{1}{8}] \times D^{\mathbf{n}}[0, \frac{1}{8}] \rightarrow D^{\mathbf{n}}[0, \frac{1}{4}]$: $2^{-0.02585\mathbf{n}}$
- $D^{\mathbf{n}}[0, \frac{1}{4}] \times D^{\mathbf{n}}[0, \frac{1}{4}] \rightarrow D^{\mathbf{n}}[0, \frac{1}{2}]$: $2^{-0.12256\mathbf{n}}$

Optimized parameters for HGJ


0.272
0
0.294, 0.250
$D^{\mathbf{n}}[\frac{1}{8}]$
0.337, 0.5
$D^{\mathbf{n}}[\frac{1}{8}] \times D^{\mathbf{n}}[\frac{1}{8}]$
0.311, 0.5
$D^{\mathbf{n}}[\frac{1}{4}]$
0, 1
$D^{\mathbf{n}}[\frac{1}{2}]$

Better representations: BCJ

Sample distributions with $\{-1, 0, 1\}$.

- Of course we still need to obtain $D^{\mathbf{n}}[\frac{1}{2}]$ in the end
- The "-1" need to be canceled out by "1"
- The "-1" shouldn't sum up to "-2"!
- More parameters, new filtering probabilities
- Improvement on the time exponent: $0.291 < 0.337$

Becker, Coron, and Joux, *"Improved Generic Algorithms for Hard Knapsacks"*, EC 11

New results

Idea 1: do not saturate the lists

Starting lists are **not equal** to $D^{\mathbf{n}/2}[*]$ but **sampled** u.a.r. from it.

BCJ without saturation: $0.289 < 0.291$

Idea 2: still better representations

Why stop at "-1"? Add some "2".

- Of course we still need to obtain $D^{\mathbf{n}}[\frac{1}{2}]$ in the end
- Some "1" can sum up to "2" (but not too much)
- A "-1" and a "2" give a "1"

BCJ without saturation and with "2": $0.283 < 0.289$

Bonnetain et al., *Improved Classical and Quantum Algorithms for Subset-Sum*, ePrint 2020/168

Going Quantum

Classical search

Let $\underbrace{X}_{\text{Search space, size } N} = \underbrace{G}_{\text{Good ones, size } T} \cup \underbrace{B}_{\text{Bad ones, size } N-T}$

Let `Sample` and `Test` be functions to *sample* x *from* X and *test if* $x \in G$, in time $t_{\texttt{Sample}}$ and $t_{\texttt{Test}}$.

There exists a function $\texttt{Sample}_G$ that samples from G in time:

$$\frac{N}{T}\left(t_{\texttt{Sample}} + t_{\texttt{Test}}\right)$$

$\Rightarrow$ we transform a sampling procedure for the "search space" into a sampling procedure for the "solution space".

Quantum search (amplitude amplification)

$$\underbrace{X}_{\text{Search space, size } N} = \underbrace{G}_{\text{Good ones, size } T} \cup \underbrace{B}_{\text{Bad ones, size } N-T}$$

Let `QSample` and `QTest` be quantum algorithms to **quantumly sample X** and **quantumly test if $x \in G$**, in time $t_{\texttt{Sample}}$ and $t_{\texttt{Test}}$.

There exists an algorithm $\texttt{QSample}_G$ that samples G in time:

$$\sqrt{\frac{N}{T}}\left(t_{\texttt{QSample}} + t_{\texttt{QTest}}\right)$$

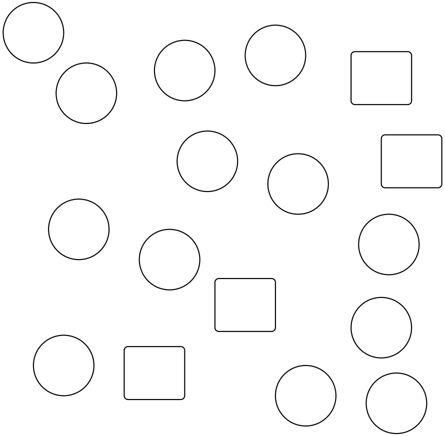
Quantum test: means testing **any $x \in X$ in superposition**

Quantum sample: means producing a uniform superposition of X

Brassard et al., *"Quantum amplitude amplification and estimation"*, 2002

Classical search vs. quantum search

In the classical realm, we test elements x at random until we have found (a random) $x \in G$.



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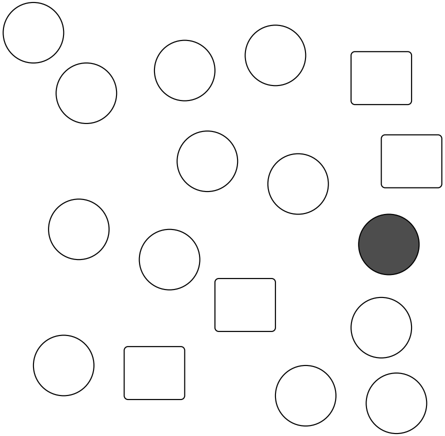

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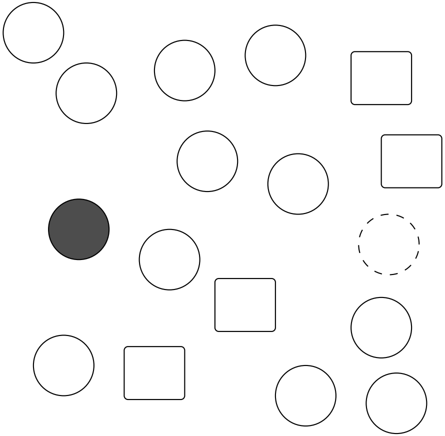

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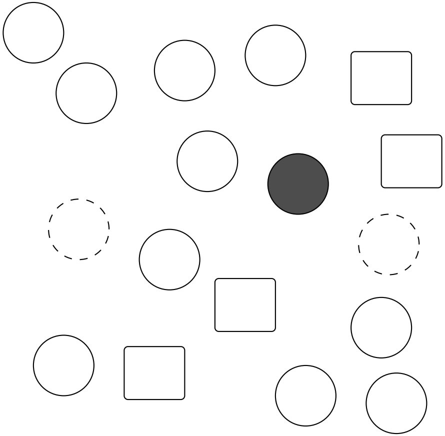

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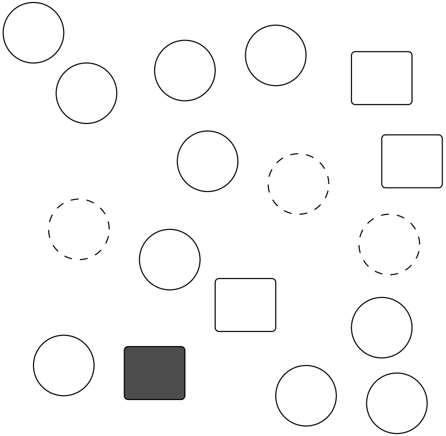
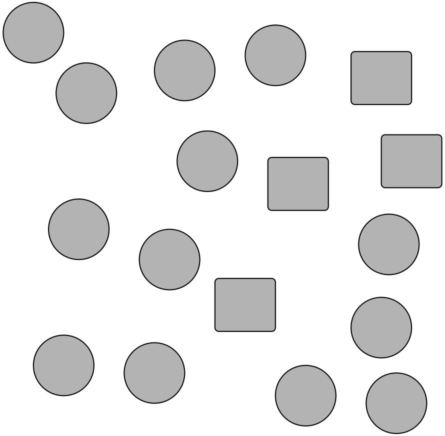


Classical search vs. quantum search

In the quantum realm, we move globally from X to G.



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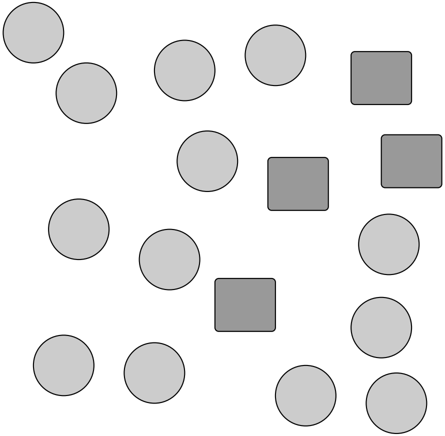

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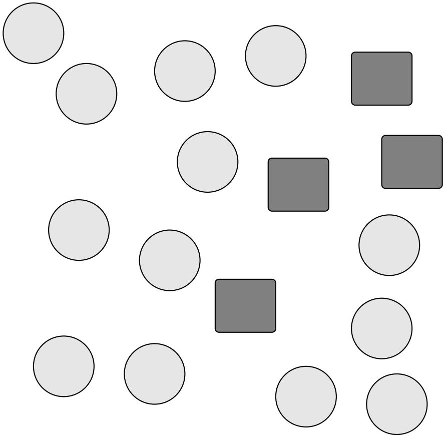

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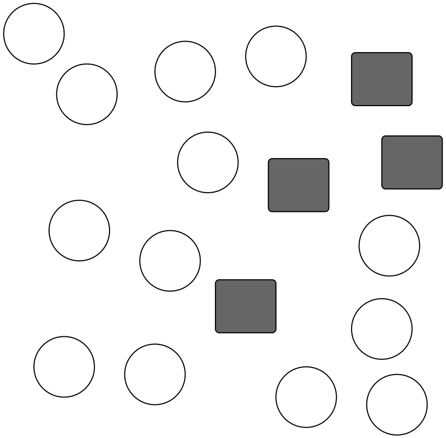


Interlude: quantum memory models

What happens if X is in memory?

Classical sample: only reads a single $x \in X$ (easy)

Quantum sample: must read all of X in superposition (maybe not easy): this is quantum random access

	Quantum random access	
Classical write	Classical memory quantum random access **QRACM**	$\Rightarrow$ This section
Quantum write	Quantum memory quantum random access **QRAQM**	Previous quantum subset-sum algos

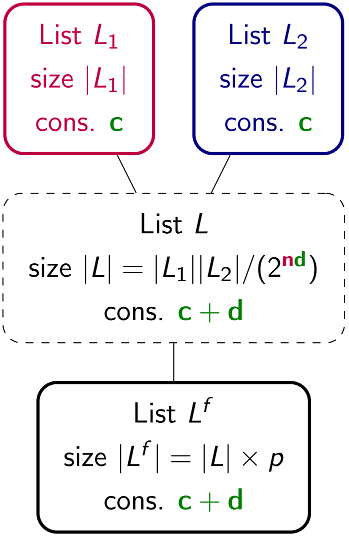
Quantum algorithms for subset-sum

The time is $\widetilde{\mathcal{O}}\left(2^{\beta \mathbf{n}}\right)$.

Technique	β	Ref.	Classical version	Model
MIM	0.3334	BHT98	HS74	QRACM
4-list merge	0.3	BJLM13	SS81	QRAQM
$\{0,1\}$	0.241	BJLM13	HGJ10	QRAQM + **conj.**
$\{0,1\}$	**0.2356**	**Ours**	**HGJ10**	**QRACM**
$\{-1,0,1\}$	0.226	HM18	BCJ11	QRAQM + **conj.**
$\{-1,0,1,2\}$	0.2156	Ours		QRAQM + **conj.**
$\{-1,0,1,2\}$	0.2182	Ours		QRAQM

Subset-Sum with Quantum Search

"Sampling-and-filtering"



Let's separate:

- **The sampled list**
- **The intermediate list**

We turn samples from L_1 into:

- Samples from L:

$$t_{\texttt{Sample}(L)} = \max\left(\frac{2^{\mathbf{nd}}}{|L_2|}, 1\right) t_{\texttt{Sample}(L_1)}$$

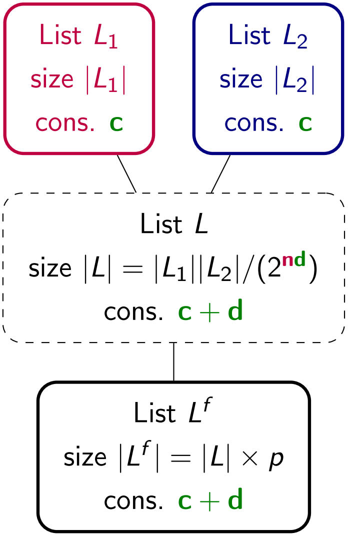
(find an element with same modulus)

- Samples from L^f:

$$t_{\texttt{Sample}(L_f)} = \frac{1}{p} t_{\texttt{Sample}(L)}$$

(wait until the filter is passed)

Quantum "sampling-and-filtering"


List L_1
size $|L_1|$
cons. $\mathbf{c}$
List L_2
size $|L_2|$
cons. $\mathbf{c}$
List L
size $|L| = |L_1||L_2|/(2^{\mathbf{nd}})$
cons. $\mathbf{c} + \mathbf{d}$
List L^f
size $|L^f| = |L| \times p$
cons. $\mathbf{c} + \mathbf{d}$

Assume that we have quantum samples from L_1.

Then we have:

- Quantum samples from L:

$$t_{\mathtt{QSample}(L)} = \max\left(\sqrt{\frac{2^{\mathbf{nd}}}{|L_2|}}, 1\right) t_{\mathtt{QSample}(L_1)}$$

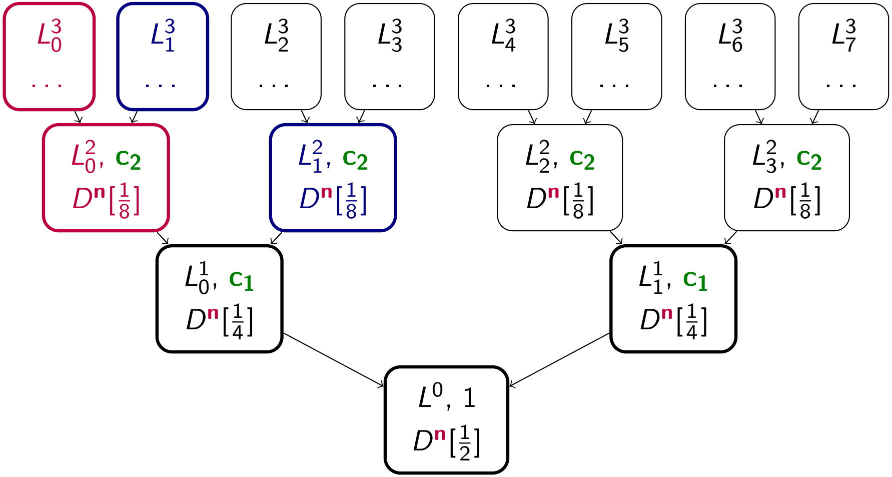
- Quantum samples from L^f:

$$t_{\mathtt{Sample}(L_f)} = \sqrt{\frac{1}{p}}\, t_{\mathtt{Sample}(L)}$$

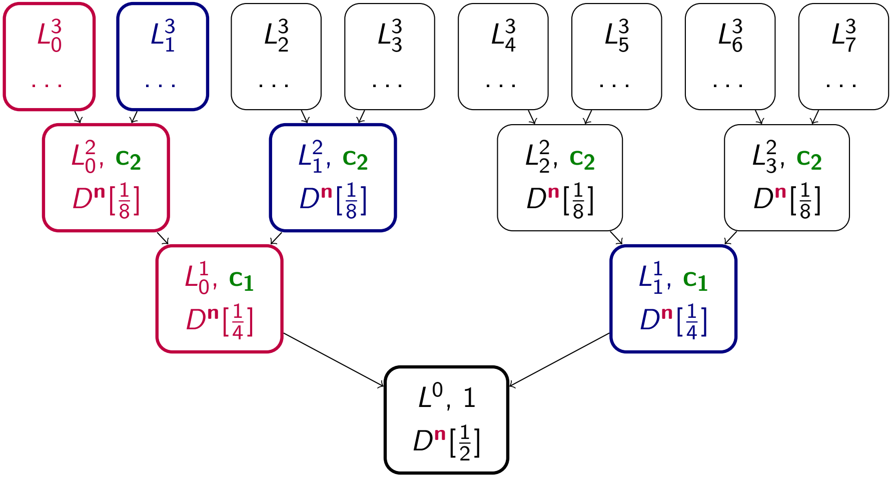
HGJ in the “sampling” framework

$$\begin{tikzcd}
\boxed{\begin{array}{c} L_0^3 \\ \dots \end{array}} \arrow[dr] & & \boxed{\begin{array}{c} L_1^3 \\ \dots \end{array}} \arrow[dl] & & \boxed{\begin{array}{c} L_2^3 \\ \dots \end{array}} \arrow[dr] & & \boxed{\begin{array}{c} L_3^3 \\ \dots \end{array}} \arrow[dl] & & \boxed{\begin{array}{c} L_4^3 \\ \dots \end{array}} \arrow[dr] & & \boxed{\begin{array}{c} L_5^3 \\ \dots \end{array}} \arrow[dl] & & \boxed{\begin{array}{c} L_6^3 \\ \dots \end{array}} \arrow[dr] & & \boxed{\begin{array}{c} L_7^3 \\ \dots \end{array}} \arrow[dl] \\
& \boxed{\begin{array}{c} L_0^2, \mathbf{c_2} \\ D^{\mathbf{n}}[\frac{1}{8}] \end{array}} \arrow[drr] & & & & \boxed{\begin{array}{c} L_1^2, \mathbf{c_2} \\ D^{\mathbf{n}}[\frac{1}{8}] \end{array}} \arrow[dll] & & & & \boxed{\begin{array}{c} L_2^2, \mathbf{c_2} \\ D^{\mathbf{n}}[\frac{1}{8}] \end{array}} \arrow[drr] & & & & \boxed{\begin{array}{c} L_3^2, \mathbf{c_2} \\ D^{\mathbf{n}}[\frac{1}{8}] \end{array}} \arrow[dll] & \\
& & & \boxed{\begin{array}{c} L_0^1, \mathbf{c_1} \\ D^{\mathbf{n}}[\frac{1}{4}] \end{array}} \arrow[drrrr] & & & & & & & & \boxed{\begin{array}{c} L_1^1, \mathbf{c_1} \\ D^{\mathbf{n}}[\frac{1}{4}] \end{array}} \arrow[dllll] & & & \\
& & & & & & & \boxed{\begin{array}{c} L^0, 1 \\ D^{\mathbf{n}}[\frac{1}{2}] \end{array}} & & & & & & &
\end{tikzcd}$$

HGJ in the “sampling” framework


L_0^3 ...
L_1^3 ...
L_2^3 ...
L_3^3 ...
L_4^3 ...
L_5^3 ...
L_6^3 ...
L_7^3 ...
L_0^2, $\mathbf{c}_2$ $D^{\mathbf{n}}[\frac{1}{8}]$
L_1^2, $\mathbf{c}_2$ $D^{\mathbf{n}}[\frac{1}{8}]$
L_2^2, $\mathbf{c}_2$ $D^{\mathbf{n}}[\frac{1}{8}]$
L_3^2, $\mathbf{c}_2$ $D^{\mathbf{n}}[\frac{1}{8}]$
L_0^1, $\mathbf{c}_1$ $D^{\mathbf{n}}[\frac{1}{4}]$
L_1^1, $\mathbf{c}_1$ $D^{\mathbf{n}}[\frac{1}{4}]$
L^0, 1 $D^{\mathbf{n}}[\frac{1}{2}]$

HGJ in the “sampling” framework


L_0^3 ...
L_1^3 ...
L_2^3 ...
L_3^3 ...
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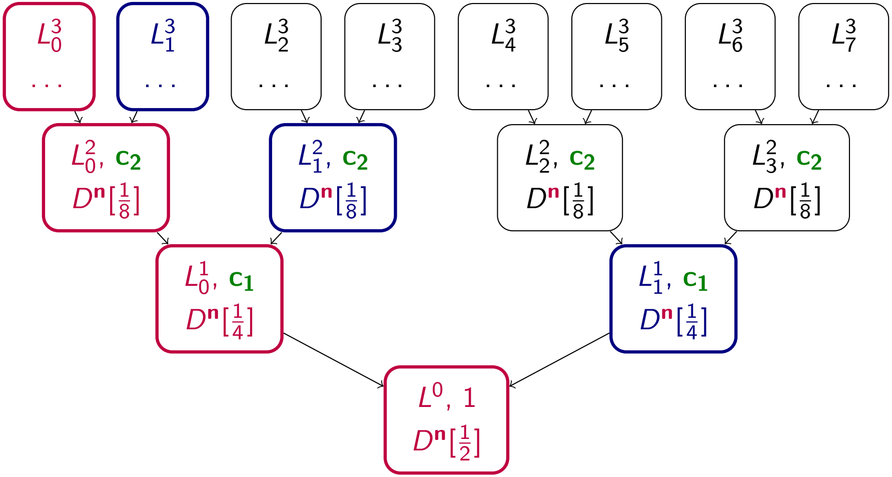
L_0^1, $\mathbf{c_1}$ $D^{\mathbf{n}}[\frac{1}{4}]$
L_1^1, $\mathbf{c_1}$ $D^{\mathbf{n}}[\frac{1}{4}]$
L^0, 1 $D^{\mathbf{n}}[\frac{1}{2}]$

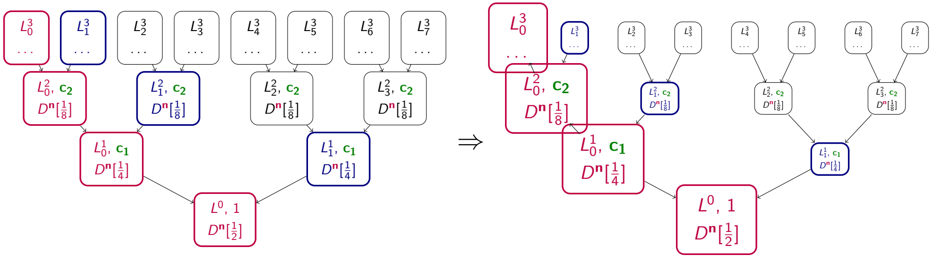
HGJ in the “sampling” framework


$$\begin{tikzcd}
L_0^3 \ \dots \arrow[dr] & L_1^3 \ \dots \arrow[d] & L_2^3 \ \dots \arrow[d] & L_3^3 \ \dots \arrow[dl] & L_4^3 \ \dots \arrow[d] & L_5^3 \ \dots \arrow[dl] & L_6^3 \ \dots \arrow[d] & L_7^3 \ \dots \arrow[dl] \\
& L_0^2,\ \mathbf{c_2}\ D^{\mathbf{n}}[\tfrac{1}{8}] \arrow[dr] & L_1^2,\ \mathbf{c_2}\ D^{\mathbf{n}}[\tfrac{1}{8}] \arrow[d] & & L_2^2,\ \mathbf{c_2}\ D^{\mathbf{n}}[\tfrac{1}{8}] \arrow[dr] & & L_3^2,\ \mathbf{c_2}\ D^{\mathbf{n}}[\tfrac{1}{8}] \arrow[dl] & \\
& & L_0^1,\ \mathbf{c_1}\ D^{\mathbf{n}}[\tfrac{1}{4}] \arrow[drr] & & & L_1^1,\ \mathbf{c_1}\ D^{\mathbf{n}}[\tfrac{1}{4}] \arrow[dll] & & \\
& & & & L^0,\ 1\ D^{\mathbf{n}}[\tfrac{1}{2}] & & &
\end{tikzcd}$$

Using quantum search

Quantum search will square-root the sampling time of L^0. But it's useless if the intermediate lists cost the same as before.

$\Rightarrow$ we make the tree unbalanced.



L_0^3 ...
L_1^3 ...
L_2^3 ...
L_3^3 ...
L_4^3 ...
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L_0^1, $\mathbf{c}_1$ $D^{\mathbf{n}}[\frac{1}{4}]$
L_1^1, $\mathbf{c}_1$ $D^{\mathbf{n}}[\frac{1}{4}]$
L^0, 1 $D^{\mathbf{n}}[\frac{1}{2}]$
$\Rightarrow$

Details and result

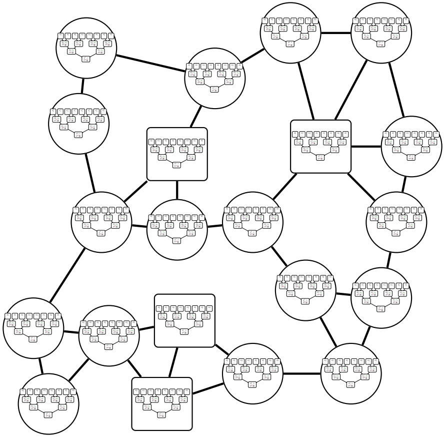
- Unbalanced left-right split of L_0^3 and L_1^3, unbalanced weights for the lists
- L_1^3, L_1^2, L_1^1 are intermediate lists stored in QRACM (classical data with quantum random access)
- **only** poly(**n**) **quantum storage needed**

$$\underbrace{0.226 \text{ (HM18)} <}_{\substack{\text{We use only } \{0,1\} \\ \text{representations}}} \quad \mathbf{0.2356} \quad \underbrace{< 0.241 \text{ (BJLM13)}}_{\text{We filter more efficiently}}$$

Subset-Sum with Quantum Walks

A classical walk for HGJ

Reduce the HGJ merging tree to a smaller tree, with smaller starting lists. Now L^0 does not always contain a solution.



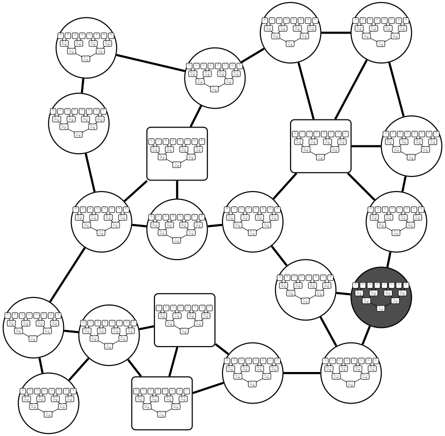
Walking on the graph

We use a (regular, undirected) Johnson graph $J(D, L)$.

- A vertex contains a product of 8 small lists $L_i^3, 0 \leq i \leq 7$, of size $|L_i^3| = L$, chosen among distributions $|D^i| = D$, **and** the whole tree built from these lists.
- There are $\binom{D}{L}^8$ vertices.
- Some vertices are **marked**: they contain the knapsack solution.
- We go from one to another by changing an element in a list L_i^3 **and** updating the tree.

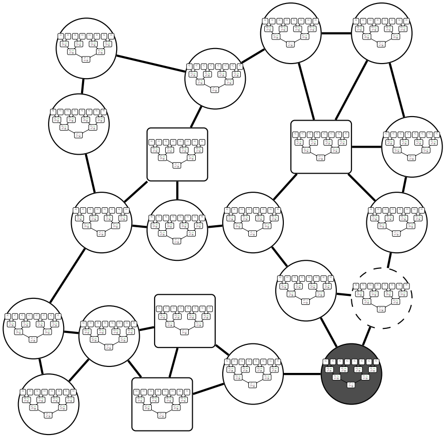
Classical random walk

We move to random neighbors until we find a marked vertex.



Classical random walk

We move to random neighbors until we find a marked vertex.



Classical random walk

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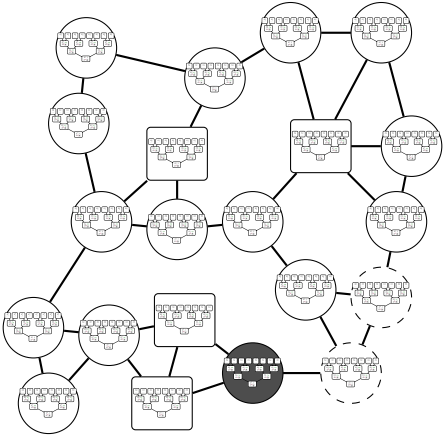

Classical random walk

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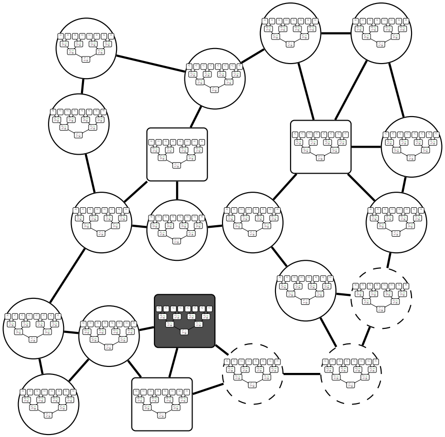


Cost of a classical random walk

We need procedures:

- To **setup** a starting arbitrary vertex (S)
- To **move** from one vertex to one of its neighbors (U)
- To **check** if a vertex is marked (trivial) (C)

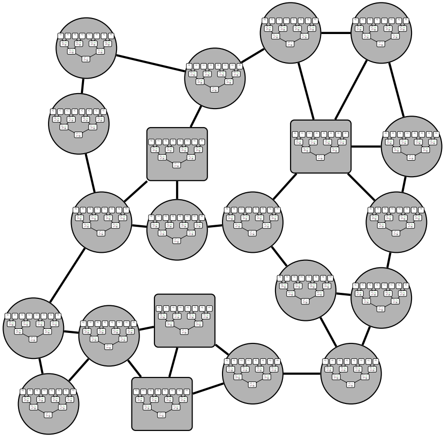
We will find a marked vertex in time:

$$\mathsf{S} + \underbrace{\frac{1}{\epsilon}}_{\substack{\epsilon \text{ proportion of} \\ \text{marked vertices}}} \left(\underbrace{\frac{1}{\delta}}_{\substack{\delta \text{ spectral gap} \\ \text{of the graph}}} \mathsf{U} + \mathsf{C} \right)$$

where $\frac{1}{\delta}$ is the number of updates before we reach a new uniformly random vertex. In a Johnson graph $J(D, L)$, $\frac{1}{\delta} \simeq L$. (We need to replace all elements.)

Quantum walk

As in quantum search, the walk transforms a uniform superposition over the whole graph into a superposition over marked vertices.



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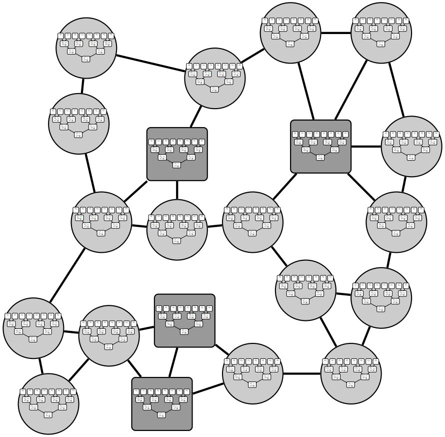

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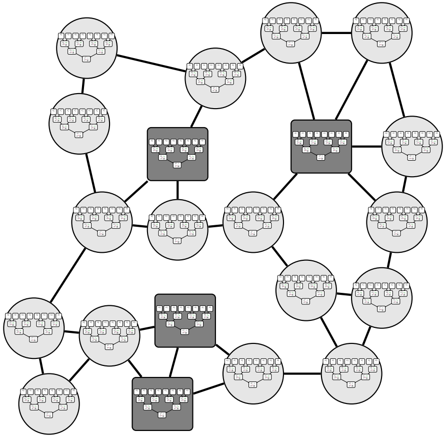

Quantum walk

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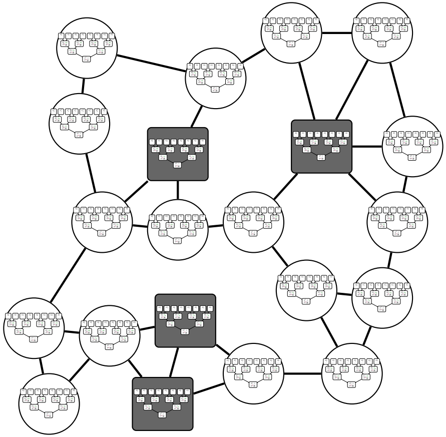


Time of a quantum walk (MNRS framework)

- The **setup** now requires to create a superposition over **all** vertices
- As in quantum search, we perform $\sqrt{\frac{1}{\epsilon}}$ steps instead of $\frac{1}{\epsilon}$
- But the mixing is also accelerated!

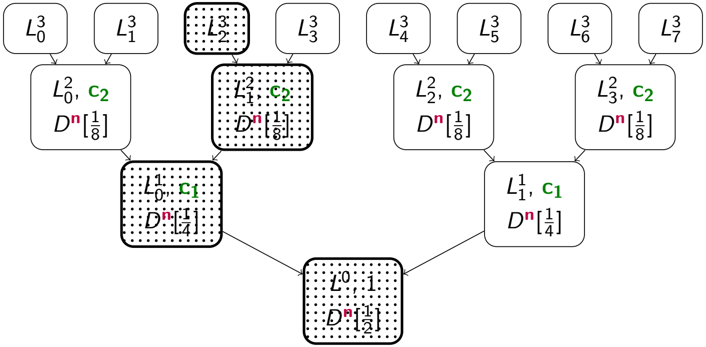
$$\mathsf{S} + \underbrace{\sqrt{\frac{1}{\epsilon}}}_{\text{Walk steps}} \left(\underbrace{\sqrt{\frac{1}{\delta}}\mathsf{U}}_{\text{Mixing time}} + \mathsf{C} \right)$$

- The **Update** handles all vertices and all edges in superposition

Magniez et al., *"Search via quantum walk"*, SIAM 11

Tracking the updates

- The update U must replace one element in a lower-level list **and** update the merging tree data structure.


L_0^3
L_1^3
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L_0^2, c_2 $D^{\mathbf{n}}[\frac{1}{8}]$
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L_3^2, c_2 $D^{\mathbf{n}}[\frac{1}{8}]$
L_0^1, c_1 $D^{\mathbf{n}}[\frac{1}{4}]$
L_1^1, c_1 $D^{\mathbf{n}}[\frac{1}{4}]$
$L^0, 1$ $D^{\mathbf{n}}[\frac{1}{2}]$

- **On average,** there is a single replacement to make at each level (no problem classically).

Superposition updates

- The update needs to take a fixed time.
- Since we are handling all vertices **and all edges** in superposition, there are cases when updating the tree would cost an exponential time.

Can we abort the bad cases?

Not in the MNRS framework: the data structure (the tree) must depend only on the vertex (the initial lists).

The quantum walk conjecture of Helm and May (TQC18)

With an update of expected time $\mathcal{O}(1)$, we can still do the runtime analysis as if it had an exact time $\mathcal{O}(1)$.

Helm and May, *"Subset Sum Quantumly in 1.17^n"*, TQC 18

New results

- We have modified the **data structure** to guarantee the update time
- This reduces (marginally) the number of marked vertices

Fact

Previous quantum walks for subset-sum do not require the update conjecture.

- However, this data structure is not enough for our best algorithms . . .

Summary of quantum walk results

Technique	Time	Ref.	Classical version	Model
$\{0,1\}$	0.241	BJLM13	HGJ10	QRAQM + **conj.**
$\{-1,0,1\}$	0.226	HM18	BCJ11	QRAQM + **conj.**
$\{-1,0,1,2\}$	0.2156	Ours		QRAQM + **conj.**
$\{-1,0,1,2\}$	0.2182	Ours		QRAQM

Conclusion and open questions

On classical algorithms

More symbols and nearest-neighbor techniques should improve the exponent $\Rightarrow$ how far could we go?

On quantum algorithms with quantum search

Better representations should improve the exponent . . . if we manage to make the optimization converge.

Conclusion and open questions (ctd.)

On quantum walks

- The update conjecture can be removed from previous works . . . but not completely from ours
- It seems that we still need to adapt the MNRS Quantum Walk framework

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Thank you!