ECE 316 – Lecture Notes Chapters 4-7

# Chapter 4: Discrete Random Variables

Random variables are real-valued functions of outcomes defined within the same sample space.

Example: Let denote the number of heads of 3 tossed coins. Then, is a random variable with values:

Example: Independent trials of flipping a coin with probability of getting heads, are performed until heads occurs. Let denote the number of flips.

So

## Discrete Random Variables

A random variable that can take on at most a countable number of possible values

## Probability Mass Function

The **probability mass function (PMF)** of a discrete random variable is the probability that is equal to that value:

Example: The PMF of a random variable X is given by , where and is a positive number. Find:

## Cumulative Distribution Function

## Indicator Variable

An **indicator variable** for an event is 1 if the event occurs, 0 otherwise:

The **cumulative distribution function (CDF)** of a discrete random variable is the sum of all the PMFs for less than or equal to a value :

## Expectation

The **expectation/expected value** of a discrete random variable with PMF is the sum of each value of multiplied with its corresponding PMF:

If is a random variable, then so is any function of it, . The expectation of is:

Example:

Example: A product sold seasonally yields a net profit of dollars for each unit sold and a net loss of dollars for each unit left unsold when the season ends. The number of units of the product that are ordered by customers is a random variable having PMF .

Determine the number of units the store should stock, to maximize its expected profit.

Let denote the number of units ordered by customers.

Let denote the number of units to be stocked.

The profit

## Discrete Integer Optimization

Want to check if

We want to know when

Hence, the optimal profit occurs when

**Corollary:**

For a random variable , if and are constants then:

Proof:

The expectation of , is known as the **first moment**.

is the **nth moment**.

## Variance

For a random variable , the **variance** is how far away the results are from the expectation:

For constants and :

– doesn’t affect variance since it is just a shift

Example: Calculate if is the outcome of a die.

## Common Discrete Random Variables

**Poisson Random Variable**

**Binomial Random Variable**   
Number of successes in independent trials, each with success probability

**Bernoulli Random Variable**   
Success or failure

## Geometric Random Variables

Recall: Geometric series

**Geometric Random Variable**   
Independent trials until a success occurs, where the probability of success is . The number of trials that it takes to get a success is a geometric random variable .

Note that

# Chapter 5: Continuous Random Variables

Continuous random variables take on an uncountable number of values

## Probability Density Function

The **probability density function (PDF)** of a continuous random variable is the probability that belongs to a region :

Common cases:

Example: For a random variable with PDF

1. What is the value of ?
2. Find .

## Cumulative Distribution Function

The **cumulative distribution function (CDF)** of a continuous random variable is the probability that is less than or equal to a value :

Example: If is a continuous random variable with CDF and PDF , find the probability density function of

In general, by chain rule:

## Expectation

The **expectation/expected value** of a continuous random variable is:

Example: Find if

Example: Find for

## Uniform Random Variables

A random variable is **uniformly distributed** over if:

Example: A bus arrives at a stop at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits:

1. Less than 5 minutes for a bus

Let denote the time that the passenger arrives at the stop.

In order to wait less than 5 minutes for a bus, the passenger needs to arrive between 7:10 and 7:15 or between 7:25 and 7:30.

So, we want to find .

1. More than 10 minutes for a bus

Let denote the time that the passenger arrives at the stop.

In order to wait more than 10 minutes for a bus, the passenger needs to arrive between 7:00 and 7:05 or between 7:15 and 7:20.

So, we want to find .

## Normal/Gaussian Random Variable

is a **normal/Gaussian random variable** if its PDF is:

If is normally distributed with , then is normally distributed with .

For the **standard normal distribution**, :

For a standard normal distribution, , since

For the **general case**, :

Given a random variable , where ,

Proof:

Example: Gaussian noise in a communicator

Let be a random variable such that for bit 1, is transmitted and for bit 0, is transmitted.

Let be the received value: . The received value is decoded into bit 1 if and bit 0 if .

## Exponential Random Variables

An **exponential random variable** with parameter has:

## The Gamma Distribution

A **gamma distribution** with parameters has:

Where

Example: In queuing systems,

* The total number of customers arriving is
* The number of customers arriving at a time is
* The number of customers in an inter-arrival interval is

# Chapter 6: Jointly Distributed Random Variables

## Joint Distribution Functions

The **joint cumulative distribution function (JCDF)** of two random variables and is the probability that the two random variables are less than certain values:

In the case where one value is infinite:

is the **marginal distribution function** of

## Joint Probability Mass Function (Discrete)

The **joint probability mass function (JPMF)** of two discrete random variables and is the probability that the two random variables take on certain values:

and are the **marginal PMFs** of and

Example: 3 balls are selected from 3 red, 4 white, and 5 blue. Let X and Y denote, respectively, the number of red and white balls chosen. Then the JPMF of X and Y is

Two continuous random variables and are jointly continuous if there exists a **joint probability density function (JPDF)**  such that, for every set :

In the case where (a rectangle),

The **marginal PDF** is equivalent to if A or B were :

## Joint Probability Density Function (Continuous)

Similarly,

## Joint Cumulative Density Function

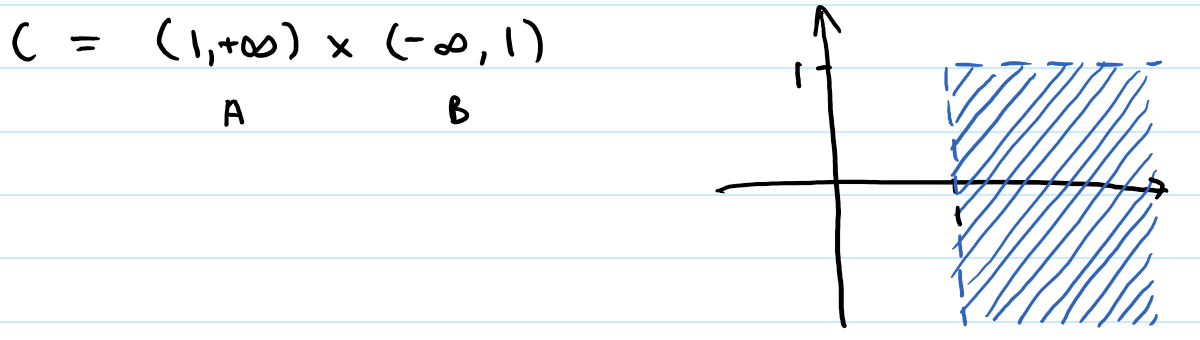
The **joint cumulative density function (JCDF)** of two continuous random variables and is the probability that the two random variables are less than certain values:

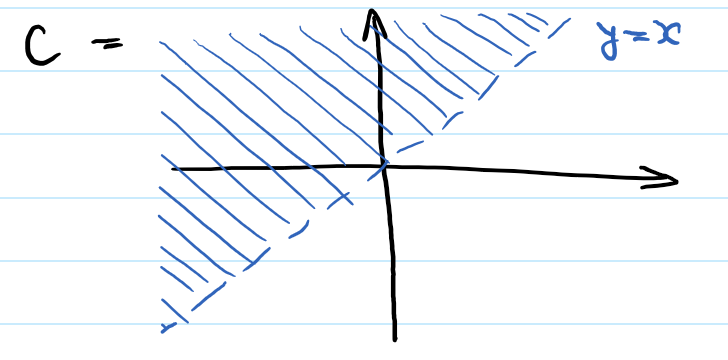
For example, if and :

Conversely,

𝑓𝑡𝑦 re less than certain valuesss random variable

Example: The JPDF of X and Y is



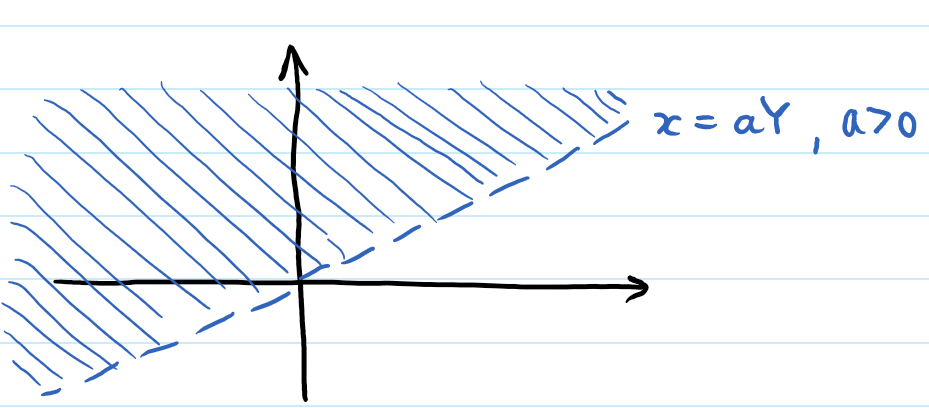


Ranges: or

Example: The joint density of X and Y is

Find the density function of X/Y.

Let . To solve this problem, first find then derive .



Ranges: or

## Independent Random Variables

Two random variables and are independent if for any two sets of real numbers and ,

Because this is true for any and , we only need to consider :

Then, two random variables X and Y are **independent** if:

For discrete random variables, this means:

For continuous random variables, this means:

𝑓𝑡𝑦 re less than certain valuesss random variable

𝑓𝑡𝑦 re less than certain valuesss random variable𝑓𝑡𝑦 re less than certain valuesss random variable

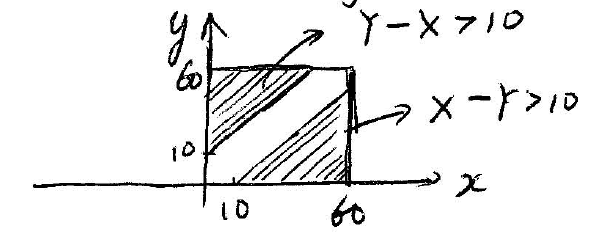
Example: independent trials with a common success probability . Let be the number of successes in the first trials, be the number of successes in the last trials, and be the total number of successes in the trials.

1. Are and independent?   
   Yes
2. Are and independent?

No, since the value of affects the distribution of (e.g. can take on the value with some probability, but if then

Example: A man and a woman decide to meet at a certain location. If each person independently arrives at a time uniformly distributed between 12pm and 1pm, find the probability that the first to arrive has to wait longer than 10 minutes.

Let X be the arrival time of the man, and Y be the arrival time of the woman. Then they are both uniformly distributed variables (0, 60).



Ranges: or

## Sums of Independent Random Variables

For two independent random variables and , with PDFs and , the PDF of the sum of and is the convolution of the individual density functions:

𝑓𝑡𝑦 re less than certain valuesss random variable𝑓𝑡𝑦 re less than certain valuesss random variable𝑓𝑡𝑦 re less than certain valuesss random variable

Example: If X and Y are independent and both uniformly distributed (0,1), calculate the probability density of X+Y.

Since for ,

And for ,

## Gaussian Distributions

If and they are all independent, then the parameters of the Gaussian distribution can be summed:

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## Poisson Distributions

If and are independent, then the parameters of the Poisson distribution can be summed:

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## Binomial Distributions

If and are independent, then the first parameter of the Binomial distribution can be summed:

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## Conditional Distributions: Discrete Case

If and are discrete random variables, then the **conditional CDF** of given is:

If and are discrete random variables, then the **conditional PMF** of given is:

If and are independent, then

Example: The JPMF of X and Y is . Calculate the conditional PMF of X given that Y=1.

## Conditional Distributions: Continuous Case

If and are continuous random variables, then the **conditional CDF** of given is:

If and are continuous random variables, then the **conditional PDF** of given is:

Example: The JPDF of X and Y is . Find .

# Chapter 7: Properties of Expectations

## Expectations of Sums of Random Variables

Let be **independent and identically distributed (iid)** random variables with the same distribution function .

The **sample mean** is used to estimate :

Example: Expectation of binomial random variable

Let . Find .

## Covariance, Variance of Sums, and Correlations

If and are independent, then for any functions and :

**Properties**:

The **covariance** between and is:

Example:

## Conditional Expectations

The **conditional expectation** of and is:

**Discrete**:

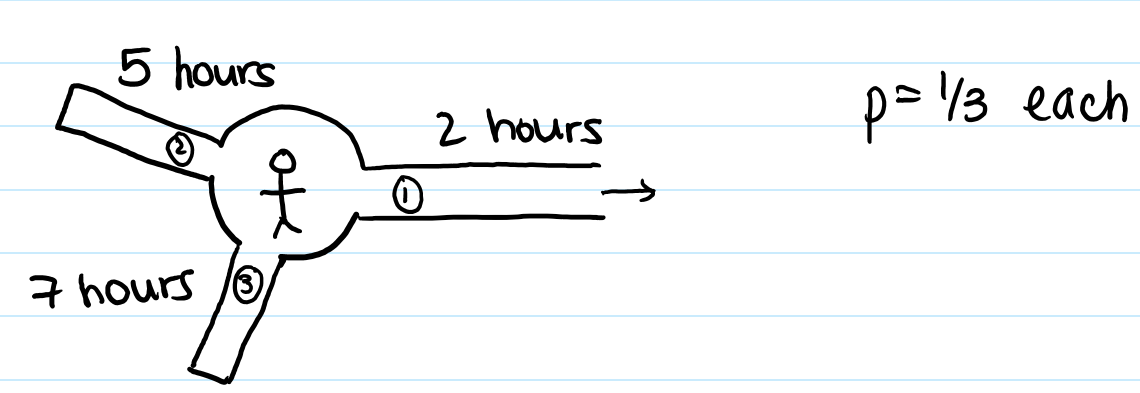
**Continuous**:

Example: Suppose . Compute .

## Computing Expectations by Conditioning

Sometimes it might be easier to compute the expectation of a random variable through its conditional expectation:

Example: A miner is trapped. What is the expected time to get out?



Let be the expected time to get out. Let be the path picked; then .

## Moment-Generating Functions

The **moment-generating function (MGF)** is the expectation of :

**Discrete**:

**Continuous**:

The MGF generates moments easily:

Example:

Example:

Example: Gaussian

For

If X and Y are independent, then the **MGF of X+Y** is the product of their individual MGFs:

Example:

s

Thus,