Foundations of Operations Research - Notes - v0.2.0-dev

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Preface

Every theory section in these notes has been taken from the sources:

- Course slides. [1]

About:

GitHub repository

These notes are an unofficial resource and shouldn't replace the course material or any other book on foundations of operations research. It is not made for commercial purposes. I've made the following notes to help me improve my knowledge and maybe it can be helpful for everyone.

As I have highlighted, a student should choose the teacher's material or a book on the topic. These notes can only be a helpful material.

Contents

1 Introduction

Definition 1

Operations Research (OR), often shortened to the initialism `OR`, is the branch of mathematics in which **mathematical models** and **quantitative methods** (e.g. optimization, game theory, simulation) are **used to analyze complex decision-making problems** and **find (near-)optimal solutions**.

The overall and primary *goal* is to *help make better decisions*.

OR can be seen as an interdisciplinary field at the intersection of applied mathematics, computer science, economics, and industrial engineering.

Operations research is often concerned with **determining the extreme values of some real-world objective**: the *maximum* (of profit, performance, or yield) or *minimum* (of loss, risk, or cost). Originating in military efforts before World War II, its techniques have grown to concern problems in a variety of industries. [2]

Its origins date back to World War II: teams of scientists were asked to research the most efficient way to conduct operations (e.g., to optimize the allocation of scarce resources).

In the decades after the war, the techniques became public and were applied more widely to problems in business, industry, and society.

During the industrial boom, the substantial increase in the size of firms and organizations led to more complex decision problems.

There are favorable circumstances: rapid progress in OR and in numerical analysis methods, and the advent and spread of computers (available computing power and widespread software).

1.1 Decision-making problems

Decision-making problems are analyzed using mathematical models and quantitative methods.

Definition 2

Decision-making problems are problems in which we must **choose** a (feasible) **solution among a large number of alternatives based on one or several criteria**.

In other words, they are complex decision-making problems that are **addressed through a mathematical modeling approach** (mathematical models, algorithms, and computer implementations).

Some practical **examples** include assignment problem, network design, shortest paths, personnel scheduling, service management, multicriteria problem, and maximum clique.

1.1.1 Assignment problem

A mathematical definition of an **assignment problem** is as follows. Given m jobs and m machines, suppose that each job can be executed by any machine and that t_{ij} is the execution time of job J_i on machine M_j.

	M_1	M_2	M_3
J_1	2	6	3
J_2	8	4	9
J_3	5	7	8

Table 1: Example of an assignment problem table.

The **main goal** is to **decide which job to assign to each machine in order to minimize the total execution time**. Also, (constraints) each job must be assigned to exactly one machine, and each machine must be assigned to exactly one job.

The **number** of feasible **solutions** is the permutations, then the **factorial of** m: $m!$.

1.1.2 Network design

The **network design problem** is characterized as **how to connect n cities** (offices) **via a collection of possible links** so as (*main goal*) **to minimize the total link cost**.

Using mathematical terms, given a graph $G = (N, E)$ with a node $i \in N$ for each city and an edge $\{i, j\} \in E$ of cost c_{ij}, select a subset of edges of minimum total cost, guaranteeing that all pairs of nodes are connected.

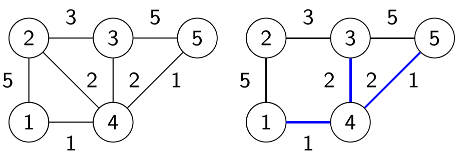


Figure 1: Examples of network design graphs.

The **number** of alternative **solutions** is at most 2^m, where $m = |E|$.

1.1.3 Shortest path

The **shortest path problem** is similar to network design. Given a direct path that represents a road network with distances (traveling times) for each arc, determine the shortest (fastest) path between two points (nodes).

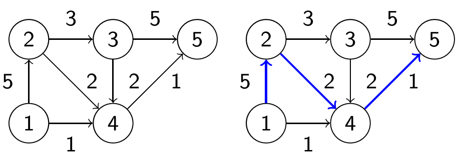


Figure 2: Examples of shortest path graphs.

1.1.4 Other problems

Other decision-making problems are:

- **Personnel scheduling problem**: determine the week schedule for the hospital personnel, so as to minimize the number of people involved (physicians, nurses, ...) while meeting the daily requirements.
- **Service management problem**: determine how many counters/desks to open at a given time of the day so that the average customer waiting time does not exceed a certain value (guarantee a given service quality).
- **Multicriteria problem**: decide which laptop to buy considering the price, the weight and the performance.
- **Maximum clique problem**: determine the complete subgraph of a graph, with maximum number of vertices.

1.2 Scheme of an OR study

The most important and common **steps** in operational research are:

1. **Problem**. Define the problem;
2. **Model**. Build the model;
3. **Algorithm**. Select or develop an appropriate algorithm;
4. **Implementation**. Implementing or using an efficient computer program;
5. **Results**. Analyze the results.

Definition 3

A mathematical **model** is a **simplified representation of a real-world problem**.

To define a mathematical model, it is necessary to identify the fundamental elements of the problem and the main relationships between them. But **how can we decide** the *number of decision makers*, the *number of objectives* and the *level of uncertainty in the parameters*? It depends on the environment. If we have:

- One decision maker, one object, then we will use **mathematical programming**.
- One decision maker, multiple objectives, then we will use **multi-objective programming**.
- Uncertainty greater than zero, then we will use **stochastic programming**.
- Multiple decision makers, then we will use **game theory**.

Example 1: production planning

A company produces 3 types of electronic devices: D_1, D_2, D_3; going through 3 main phases of the production process: assembly, refinement and quality control.
Time (in minutes) required for each phase and product:

	D_1	D_2	D_3
Assembly	80	70	120
Refinement	70	90	20
Quality control	40	30	20

Available resources within the planning horizon (depend on the workforce) in minutes:

Assembly	Refinement	Quality control
30′000	25′000	18′000

Unitary profit (in KEuro):

D_1	D_2	D_3
1.6	1	2

Assumption: the company can sell whatever it produces.
Give a mathematical model for determining a production plan which maximizes the total profit.

- **Decision variables**, x_j is equal to the number of devices D_j produced, for $j = 1, 2, 3$.
- **Objective function**: max $z = 1.6x_1 + 1x_2 + 2x_3$.
- **Constraints**: the production capacity limit for each phase:

$$\begin{aligned} 80x_1 + 70x_2 + 120x_3 &\leq 30'000 && \text{(assembly)} \\ 70x_1 + 90x_2 + 20x_3 &\leq 25'000 && \text{(refinement)} \\ 40x_1 + 30x_2 + 20x_3 &\leq 18'000 && \text{(quality control)} \end{aligned}$$

- **Non-negative variables**: $x_1, x_2, x_3 \geq 0$ may be fractional (real) values.

Example 2: portfolio selection problem

An insurance company must decide which investments to select out of a given set of possible assets (stocks, bonds, options, gold certificates, real estate, ...).

Investments	area	capital (c_j K)	expected return (r_j)
A	Germany	150	11%
B	Italy	150	9%
C	U.S.A.	60	13%
D	Italy	100	10%
E	Italy	125	8%
F	France	100	7%
G	Italy	50	3%
H	UK	80	5%

Legend:

- A and B: automotive
- C and D: ICT
- E and F: real estate
- G: short term treasury bounds
- H: long term treasury bounds

The available capital is: 600 KEuro.
At most 5 investments to avoid excessive fragmentation.
Geographic diversification to limit risk: ≤ 3 investments in Italy and ≤ 3 abroad.
Give a mathematical model for deciding which investments to select so as to maximize the expected return while satisfying the constraints.

- **Decision variables**, x_j is equal to 1 if j-th investment is selected and $x_j = 0$ otherwise, for $j = 1, \ldots, 8$.

- **Objective function**: $\max\ z = \sum_{j=1}^{8} c_j\, r_j\, x_j$.

- **Constraints**:

$$\begin{array}{rcll} \sum_{j=1}^{8} c_j\, x_j & \leq & 600 & \text{(capital)} \\ \sum_{j=1}^{8} x_j & \leq & 5 & \text{(max 5 investments)} \\ x_2 + x_4 + x_5 + x_7 & \leq & 3 & \text{(max 3 in Italy)} \\ x_1 + x_3 + x_6 + x_8 & \leq & 3 & \text{(max 3 abroad)} \end{array}$$

- **Binary (integer) variables**: $x_j \in \{0, 1\}$ and $1 \leq j \leq 8$.

<u>**Possible variant**</u>. In order to limit the risk, if any of the ICT investment is selected then at least one of the treasury bond must be selected.

- **Objective function**: $\max z = \sum_{j=1}^{8} c_j\, r_j\, x_j$.

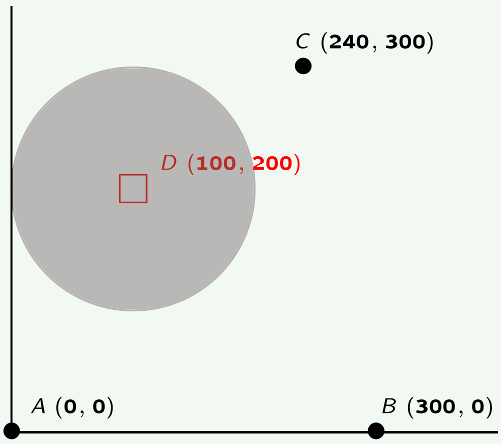
- **Constraints**:

$$\begin{array}{rcll} \sum_{j=1}^{8} c_j\, x_j & \leq & 600 & \text{(capital)} \\ \sum_{j=1}^{8} x_j & \leq & 5 & \text{(max 5 investments)} \\ x_2 + x_4 + x_5 + x_7 & \leq & 3 & \text{(max 3 in Italy)} \\ x_1 + x_3 + x_6 + x_8 & \leq & 3 & \text{(max 3 abroad)} \\ \frac{x_3 + x_4}{2} & \leq & x_7 + x_8 & \text{(investment in treasury bonds)} \end{array}$$

- **Binary (integer) variables**: $x_j \in \{0, 1\}$ and $1 \leq j \leq 8$.

Example 3: facility location

Consider 3 oil pits, located in positions A, B and C, from which oil is extracted.



Connect them to a refinery with pipelines whose cost is proportional to the square of their length.
The refinery must be at least 100 km away from point $D = (100, 200)$, but the oil pipelines can cross the corresponding forbidden zone.
Give a mathematical model to decide where to locate the refinery so as to minimize the total pipeline cost.

- **Decision variables**, x_1, x_2 cartesian coordinates of the refinery.
- **Objective function**:

$$\begin{aligned}\min z = \; & \left[(x_1 - 0)^2 + (x_2 - 0)^2\right] + \\ & \left[(x_1 - 300)^2 + (x_2 - 0)^2\right] + \\ & \left[(x_1 - 240)^2 + (x_2 - 300)^2\right]\end{aligned}$$

- **Constraints**:

$$\sqrt{(x_1 - 100)^2 + (x_2 - 200)^2} \geq 100$$

- **Variables**: $x_1, x_2 \in \mathbb{R}$.

1.3 Mathematical programming/optimization

Mathematical Optimization or **Mathematical Programming** is the **selection of a best element**, with regard to some criteria, **from some set of available alternatives**.

In the more general approach, an optimization problem consists of **maximizing** or **minimizing a real function** by systematically choosing input values from within an allowed set and computing the value of the function. The generalization of optimization theory and techniques to other formulations constitutes a large area of applied mathematics.

❓ Okay, how is it defined mathematically?

Mathematical Operation problems belong to the category of decision-making problems. They are characterized by a **single decision maker**, a **single objective**, and **reliable parameter estimates**. In mathematical language, we can say:

$$\text{opt } f(\mathbf{x}) \quad \text{with} \quad \mathbf{x} \in X \quad \text{and} \quad \text{opt } = \begin{Bmatrix} \min \\ \max \end{Bmatrix}$$

Where:

- $\mathbf{x} \in \mathbb{R}^n$ **decision variables**. They are numerical variables whose values identify a solution of the problem.
- $X \subseteq \mathbb{R}^n$ **feasible region**. Distinguishes between feasible and infeasible solutions (via constraints):

$$X = \left\{ \mathbf{x} \in \mathbb{R}^n \ : \ g_i(\mathbf{x}) \begin{Bmatrix} = \\ \leq \\ \geq \end{Bmatrix} 0, i = 1, \dots, m \right\}$$

- $f : X \to \mathbb{R}$ **objective function**.Expresses in quantitative terms the value or cost of each feasible solution.

Note an interesting observation:

$$\max \{ f(\mathbf{x}) : \ \mathbf{x} \in X \} = -\min \{ -f(\mathbf{x}) : \ \mathbf{x} \in X \}$$

❓ More specifically, how can we solve these problems?

It depends on how hard the given problem is to solve.

- The problem has an **easy/medium level** of complexity. It makes sense to use the **Global Optima** technique. It consists in finding a feasible solution that is **globally optimum**, then a vector $\mathbf{x}^* \in X$ such that:

$$\begin{aligned} f(\mathbf{x}^*) \leq f(\mathbf{x}) \quad \forall \mathbf{x} \in X \quad \text{if opt } = \min \\ f(\mathbf{x}^*) \geq f(\mathbf{x}) \quad \forall \mathbf{x} \in X \quad \text{if opt } = \max \end{aligned}$$

  Unfortunately, this method is not perfect and it may happen that the given problem occurs:

  - Is **infeasible**, so the feasible region is empty: $X = \emptyset$.
  - Is **unbounded**: $\forall c \in \mathbb{R}$, $\exists \mathbf{x}_c \in X$ such that $f(\mathbf{x}_c) \le c$ or $f(\mathbf{x}_c) \ge c$.
  - Has a **single optimal solution**.
  - Has a **large number** (even an infinite number) **of optimal solutions** (with the same optimal value!).

- The problem has a **difficult/hard level** of complexity. Then the **Local Optima** is the best choice. It consists in finding a feasible solution that is **local optimum** (main different against global optima technique), then a vector $\widehat{\mathbf{x}} \in X$ such that:

  $$\begin{array}{lll} f(\widehat{\mathbf{x}}) \le f(\mathbf{x}) & \forall \mathbf{x} \text{ with } \mathbf{x} \in X \text{ and } \|\mathbf{x} - \widehat{\mathbf{x}}\| \le \varepsilon & \text{if opt } = \min \\ f(\widehat{\mathbf{x}}) \ge f(\mathbf{x}) & \forall \mathbf{x} \text{ with } \mathbf{x} \in X \text{ and } \|\mathbf{x} - \widehat{\mathbf{x}}\| \le \varepsilon & \text{if opt } = \max \end{array}$$

  For an appropriate value $\varepsilon > 0$.

  In this case, it may happen that the given **problem has many local optima**.

Categories

A Mathematical Programming can be **categorized** depending on the feasible region:

- **Linear Programming (LP)**. The function f is linear:

  $$X = \left\{ \mathbf{x} \in \mathbb{R}^n \ : \ g_i(\mathbf{x}) \left\{ \begin{array}{c} = \\ \le \\ \ge \end{array} \right\} 0, \ i = 1, \dots, m \right\} \text{ with } g_i \text{ linear } \forall i$$

  An **example** is the *production planning*.

- **Integer Linear Programming (ILP)**. The function f is linear:

  $$X = \left\{ \mathbf{x} \in \mathbb{R}^n \ : \ g_i(\mathbf{x}) \left\{ \begin{array}{c} = \\ \le \\ \ge \end{array} \right\} 0, \ i = 1, \dots, m \right\} \cap \mathbb{Z}^n \text{ with } g_i \text{ linear } \forall i$$

  An **example** is the *portfolio selection* (finance). As we can see, the ILP technique is identical to LP with additional integrality constraints on the variables.

- **Nonlinear Programming (NLP)**. The function f is convex/regular or non convex/regular:

  $$X = \left\{ \mathbf{x} \in \mathbb{R}^n \ : \ g_i(\mathbf{x}) \left\{ \begin{array}{c} = \\ \le \\ \ge \end{array} \right\} 0, \ i = 1, \dots, m \right\}$$

  With g_i convex/regular or not convex/regular $\forall i$.

  An **example** is the *facility location* (with g_i convex).

History of Mathematical Programming

It is correct to report the history of mathematical programming:

1826/27 Joseph Fourier presents a method to solve systems of linear inequalities (Fourier-Motzkin) and discusses some LPs with 2-3 variables.

1939 Leonid Kantorovitch lays the bases of LP (Nobel prize 1975).

1947 George Dantzig proposes independently LP and invents the simplex algorithm.

1958 Ralph Gomory proposes a cutting plane method for ILP problems.

References

[1] Braz Pascoal Marta Margarida. Foundations of Operations Research. Slides from the HPC-E master's degree course on Politecnico di Milano, 2024.

[2] Wikipedia. Operations research - Wikipedia. https://en.wikipedia.org/wiki/Operations_research. [Accessed 08-09-2024].

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