Numerical Linear Algebra - Notes - v0.1.0-dev

260236

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Preface

Every theory section in these notes has been taken from the sources:

- Course slides. [1]

About:

GitHub repository

These notes are an unofficial resource and shouldn't replace the course material or any other book on numerical linear algebra. It is not made for commercial purposes. I've made the following notes to help me improve my knowledge and maybe it can be helpful for everyone.

As I have highlighted, a student should choose the teacher's material or a book on the topic. These notes can only be a helpful material.

Contents

1 Preliminaries

This section introduces some of the basic topics used throughout the course.

1.1 Notation

We try to use the same notation for anything.

- **Vectors**. With $\mathbb{R}$ is a set of real numbers (scalars) and $\mathbb{R}^n$ is a space of column vectors with n real elements.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Vectors with all zeros and all ones:

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

- **Matrices**. With $\mathbb{R}^{m \times n}$ is a space of $m \times n$ matrices with real elements:

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & & \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

Identity matrix $\mathbf{I} \in \mathbb{R}^{n \times n}$:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_n \end{bmatrix}$$

Where $\mathbf{e}_i$, $i = 1, 2, \ldots, n$ are the canonical vectors.

$$\mathbf{e}_i = \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \end{bmatrix}^T$$

Where 1 is the i-th entry.

1.2 Matrix Operations

Some basic matrix operations:

- **Inner products**. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ then:

$$\mathbf{x}^T \mathbf{y} = \sum_{i=1,\dots,n} x_i y_i$$

For real vectors, the commutative property is true:

$$\mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x}$$

Furthermore, the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are **orthogonal** if:

$$\mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} = \mathbf{0}$$

And finally, some useful properties of matrix multiplication:

1. Multiplication by the *identity* changes nothing.

$$A \in \mathbb{R}^{n \times m} \;\Rightarrow\; \mathbf{I}_n A = A = A \mathbf{I}_m$$

2. Associativity:

$$A\,(BC) = (AB)\,C$$

3. Distributive:

$$A\,(B + D) = AB + AD$$

4. No commutativity:

$$AB \neq BA$$

5. Transpose of product:

$$(AB)^T = B^T A^T$$

- **Matrix powers**. For $A \in \mathbb{R}^{n \times n}$ with $A \neq \mathbf{0}$:

$$A^0 = \mathbf{I}_n \qquad A^k = \underbrace{A \cdots A}_{k \text{ times}} = AA^{k-1} \qquad k \geq 1$$

Furthermore, $A \in \mathbb{R}^{n \times n}$ is:

  - **Idempotent** (projector) $A^2 = A$
  - **Nilpotent** $A^k = \mathbf{0}$ for some integer $k \geq 1$

- **Inverse**. For $A \in \mathbb{R}^{n \times n}$ is **nonsingular** (**invertible**), if exists A^{-1} with:

$$AA^{-1} = \mathbf{I}_n = A^{-1} A \tag{1}$$

Inverse and transposition are interchangeable:

$$A^{-T} \triangleq \left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Furthermore, an inverse of a product for a matrix $A \in \mathbb{R}^{n \times n}$ can be expressed as:

$$(AB)^{-1} = B^{-1} A^{-1}$$

Finally, remark that if $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$ and $A\mathbf{x} = 0$, then A is **singular**.

- **Orthogonal matrices**. Given a matrix $A \in \mathbb{R}^{n\times n}$ that is *invertible*, the matrix A is said to be **orthogonal** if:

$$A^{-1} = A^T \;\Rightarrow\; A^T A = \mathbf{I}_n = AA^T$$

- **Triangular matrices**. There are two types of triangular matrices:

  1. **Upper triangular matrix**:

$$\mathbf{U} = \begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n} \\ 0 & u_{2,2} & \cdots & u_{2,n} \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{n,n} \end{bmatrix}$$

  $\mathbf{U}$ is **nonsingular** if and only if $u_{ii} \neq 0$ for $i = 1, \dots, n$.

  2. **Lower triangular matrix**:

$$\mathbf{L} = \begin{bmatrix} l_{1,1} & 0 & \cdots & 0 \\ l_{2,1} & l_{2,2} & \cdots & 0 \\ \vdots & \dots & \ddots & \vdots \\ l_{n,1} & l_{n,2} & \cdots & l_{n,n} \end{bmatrix}$$

  $\mathbf{L}$ is **nonsingular** if and only if $l_{ii} \neq 0$ for $i = 1, \dots, n$.

- **Unitary triangular matrices**. Are matrices similar to the lower and upper matrices, but they have the main diagonal composed of ones.

  1. **Unitary upper triangular matrix**:

$$\mathbf{U} = \begin{bmatrix} 1 & u_{1,2} & \cdots & u_{1,n} \\ 0 & 1 & \cdots & u_{2,n} \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

  2. **Unitary lower triangular matrix**:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{2,1} & 1 & \cdots & 0 \\ \vdots & \dots & \ddots & \vdots \\ l_{n,1} & l_{n,2} & \cdots & 1 \end{bmatrix}$$

1.3 Basic matrix decomposition

In the Numerical Linear Algebra course, we will use three main decomposition:

- **LU factorization with (partial) pivoting**. If $A \in \mathbb{R}^{n \times n}$ is a nonsingular matrix, then:
$$PA = LU$$
Where:
  - P is a permutation matrix
  - L is an unit lower triangular matrix
  - U is an upper triangular matrix

  Note that the linear system solution:
$$A\mathbf{x} = \mathbf{b}$$
Can be solved directly by calculation:
$$PA = LU$$
This way the complexity is equal to $O\left(n^3\right)$. So a smarter way to reduce complexity is to use the *divide et impera* (or *divide and conquer*) technique. Then solve the system:
$$\begin{cases} L\mathbf{y} = P\mathbf{b} & \rightarrow \text{ unit lower triangular system, complexity } O\left(n^2\right) \\ U\mathbf{x} = \mathbf{y} & \rightarrow \text{ upper triangular system, complexity } O\left(n^2\right) \end{cases}$$

- **Cholesky decomposition**. If $A \in \mathbb{R}^{n \times n}$ is a symmetric[1] and positive definite[2], then:
$$A = L^T L$$
Where L is a lower triangular matrix (with positive entries on the diagonal). Also note that the linear system solution:
$$A\mathbf{x} = \mathbf{b}$$
Can be solved directly by calculation:
$$A = L^T L$$
This way the complexity is equal to $O\left(n^3\right)$. So a smarter way to reduce complexity is to use the *divide et impera* (or *divide and conquer*) technique. Then solve the system:
$$\begin{cases} L^T\mathbf{y} = \mathbf{b} & \rightarrow \text{ lower triangular system, complexity } O\left(n^2\right) \\ L\mathbf{x} = \mathbf{y} & \rightarrow \text{ upper triangular system, complexity } O\left(n^2\right) \end{cases}$$

[1] $A^T = A$
[2] $\mathbf{z}^T A \mathbf{z} > 0 \qquad \forall \mathbf{z} \neq 0$

- **QR decomposition**. If $A \in \mathbb{R}^{n \times n}$ is a nonsingular matrix, then:

$$A = QR$$

Where:

  - Q is an orthogonal matrix
  - R is an upper triangular

Note that the linear system solution:

$$A\mathbf{x} = \mathbf{b}$$

Can be solved directly by calculation:

$$A = QR$$

This way the complexity is equal to $O\left(n^3\right)$. So a smarter way to reduce complexity is to use the *divide et impera* (or *divide and conquer*) technique. Then:

  1. Multiply $\mathbf{c} = Q^T \mathbf{b}$, complexity $O\left(n^2\right)$
  2. Solve the lower triangular system $R\mathbf{x} = \mathbf{c}$, complexity $O\left(n^2\right)$

References

[1] Antonietti Paola Francesca. Numerical Linear Algebra. Slides from the HPC-E master's degree course on Politecnico di Milano, 2024.

Index