

Exponential Integral $E_1(x)$

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1. Pre-stuff

1.1. Informational

- Github: <https://github.com/AndreWaehlich/Exponential-Integral>
- Some links about the exponential integral:
 - wolfram.com:
 - * E_1 and Ei :
 - <http://mathworld.wolfram.com/ExponentialIntegral.html>
 - <http://functions.wolfram.com/GammaBetaErf/ExpIntegralEi/>
 - [http://www.wolframalpha.com/input/?i=ExpIntegralE\[1,x\]](http://www.wolframalpha.com/input/?i=ExpIntegralE[1,x])
 - [http://www.wolframalpha.com/input/?i=ExpIntegralEi\[x\]](http://www.wolframalpha.com/input/?i=ExpIntegralEi[x])
 - * E_n :
 - <http://mathworld.wolfram.com/En-Function.html>
 - <http://functions.wolfram.com/GammaBetaErf/ExpIntegralE/>

1.2. Mathematical notes

1. All variables and parameters are considered real, if not stated otherwise.
2. The notation $[f(x)]_{x=a}^b$ means $f(b) - f(a)$. In the same sense, if for example $b = \infty$, interpret as $[f(x)]_{x=a}^{\infty} = \lim_{x \rightarrow \infty} (f(x)) - f(a)$.

1.3. Definitions

1.3.1. Definition of Exponential Integral $E_1(x)$

$$E_1(x) = \int_x^{\infty} \frac{e^{-w}}{w} dw, \quad (x > 0) \quad (1)$$

1.3.2. Definition of $D(x)$

$$D(x) = e^x E_1(x), \quad (x > 0) \quad (2)$$

1.3.3. Definition of $Y(a, b, c, d)$

$$Y(a, b, c, d) = \int_0^d e^{ax} E_1[b(x+c)] dx \quad (3)$$

2. Exponential Integral $E_1(x)$

2.1. Integral Representations

2.1.1. $E_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt$, source is eq. (2b) of [De 90]

Use substitution $w = xt$:

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw, \quad (x > 0) \quad (4)$$

$$= \int_1^\infty \frac{e^{-xt}}{t} dt \quad (5)$$

2.1.2. $E_1(x) = e^{-x} \int_0^1 \frac{1}{x - \ln t} dt$, source is eq. (4) in sec. 3.3 of [GN69]

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw, \quad (x > 0) \quad (6)$$

$$= e^{-x} \int_x^\infty \frac{e^{x-w}}{w} dw \quad (7)$$

First use substitution $(x - w) = -y$, then use substitution $y = -\ln t$:

$$e^{-x} \int_x^\infty \frac{e^{x-w}}{w} dw = e^{-x} \int_0^\infty \frac{e^{-y}}{x + y} dy \quad (8)$$

$$= e^{-x} \int_1^0 \frac{t}{x - \ln t} \left(\frac{-1}{t} \right) dt \quad (9)$$

$$= e^{-x} \int_0^1 \frac{1}{x - \ln t} dt \quad (10)$$

2.1.3. $E_1(x) = e^{-x} \int_0^\infty \frac{e^{-xt}}{t+1} dt$

Use substitution $\frac{w}{x} - 1 = t$:

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw \quad (11)$$

$$= \int_0^\infty \frac{e^{-x(t+1)}}{x(t+1)} (x dt) \quad (12)$$

$$= e^{-x} \int_0^\infty \frac{e^{-xt}}{t+1} dt \quad (13)$$

2.2. Special Values

2.2.1. $\frac{d}{dx} E_1 [b(x+c)] = -\frac{e^{-b(x+c)}}{x+c}$

Start with integral representation of $E_1(x)$ found in subsection 2.1.1:

$$\frac{d}{dx} E_1 [b(x+c)] = \frac{d}{dx} \int_1^\infty \frac{e^{-b(x+c)t}}{t} dt \quad (14)$$

$$= \int_1^\infty (-bt) \frac{e^{-b(x+c)t}}{t} dt \quad (15)$$

$$= -b \int_1^\infty e^{-b(x+c)t} dt \quad (16)$$

$$= \frac{-b}{-b(x+c)} \left[e^{-b(x+c)t} \right]_{t=1}^\infty \quad (17)$$

$$= -\frac{e^{-b(x+c)}}{x+c} \quad (b(x+c) > 0) \quad (18)$$

2.3. Integrals involving $E_1(x)$

Integral	Y-representation	Location	Location in [De 90]
$\int_0^d e^{ax} E_1 [b(x+c)] dx$	$Y(a, b, c, d)$	(31), p. 6	eq. (A2a)
$\int_0^d e^{ax} E_1 [bd] dx$	$Y(a, b, 0, d)$	(65), p. 9	eq. (A2b)
$\int_0^\infty e^{ax} E_1 [b(x+c)] dx$	$Y(a, b, c, \infty)$	(42), p. 7	eq. (A2c)
$\int_0^\infty e^{ax} E_1 [bx] dx$	$Y(a, b, 0, \infty)$	(52), p. 8	eq. (A2d)

Table 1: Table of integrals encountered in [De 90]

2.3.1. $\int_a^b \frac{e^{-x}}{x} dx = E_1(a) - E_1(b)$

Start with the negative derivative from subsection 2.2.1, while setting $b = 1$ and $c = 0$:

$$-\frac{d}{dx} E_1(x) = \frac{e^{-x}}{x} \quad (19)$$

Now, just apply the *fundamental theorem of calculus* (in German: *Hauptsatz der Differential- und Integralrechnung*): For certain conditions to $f(x)$, a, b and if $\frac{d}{dx} F(x) = f(x)$, then $\int_a^b f(x) dx = F(b) - F(a)$. Applying this to equation (19) leads to the desired result. We assume $a \neq b$, so that the integral is not trivially zero. Also required due to the infinite discontinuity of the integrand at $x = 0$ is that a and b are *both* positive or *both* negative, furthermore both must be non-zero:

$$\int_a^b \frac{e^{-x}}{x} dx = E_1(a) - E_1(b) \quad (a \neq b, ab > 0) \quad (20)$$

$$\begin{aligned} \text{2.3.2. } \int_0^d e^{ax} E_1[b(x+c)] dx = \\ \frac{e^{-bc}}{a} (e^{(a-b)d} \{D[b(c+d)] - D[(-a+b)(c+d)]\} - D[bc] + D[(-a+b)c]) \end{aligned}$$

This is the general case of $Y(a, b, c, d) = \int_0^d e^{ax} E_1[b(x+c)] dx$, found in [De 90]. Special cases, e.g. $c = 0$ are considered in the following subsections. Now, first integrate by parts, using the derivative $\frac{d}{dx} E_1[b(x+c)] = -\frac{e^{-b(x+c)}}{x+c}$ from subsection 2.2.1:

$$\int_0^d e^{ax} E_1[b(x+c)] = \left\{ \frac{e^{ax}}{a} E_1[b(x+c)] \right\}_{x=0}^d - \int_0^d \frac{e^{ax}}{a} \frac{d}{dx} E_1[b(x+c)] dx \quad (21)$$

$$= \frac{e^{ad}}{a} E_1[b(d+c)] - \frac{E_1[bc]}{a} - \int_0^d \frac{e^{ax}}{a} \left\{ -\frac{e^{-b(x+c)}}{x+c} \right\} dx \quad (22)$$

$$= \frac{e^{ad}}{a} E_1[b(d+c)] - \frac{E_1[bc]}{a} + \frac{e^{-bc}}{a} \int_0^d \frac{e^{-x(b-a)}}{x+c} dx \quad (23)$$

To simplify the integral in the last summand, first use the abbreviation $\phi = b-a$ and the substitution $x+c = y$, then use the substitution $(b-a)y = \phi y = t$. In the final step, to get to the intermediate result of equation (28), use the integral $\int_a^b \frac{e^{-x}}{x} dx = E_1(a) - E_1(b)$, found in subsection 2.3.1:

$$\int_0^d \frac{e^{-x(b-a)}}{x+c} dx = \int_0^d \frac{e^{-x\phi}}{x+c} dx \quad (24)$$

$$= \int_c^{d+c} \frac{e^{-(y-c)\phi}}{y} dy \quad (25)$$

$$= e^{c\phi} \int_c^{d+c} \frac{e^{-y\phi}}{y} dy \quad (26)$$

$$= e^{c\phi} \int_{\phi c}^{\phi(d+c)} \frac{e^{-t}}{t\phi^{-1}} \phi^{-1} dt \quad (27)$$

$$= e^{c\phi} \{E_1[\phi c] - E_1[\phi(d+c)]\} \quad (28)$$

To arrive at the desired result we just put the intermediate step (28) into equation (23), and restore the abbreviation $\phi = b-a$. Then rearrange the terms to get to equation (A2a) from the appendix of [De 90]. In the last steps use the definition of $D(x)$ from eq. (2), which can be found in subsection 1.3.2 on page 2:

$$\begin{aligned} \int_0^d e^{ax} E_1[b(x+c)] dx &= \frac{e^{ad}}{a} E_1[b(d+c)] - \frac{E_1[bc]}{a} \\ &\quad + \frac{e^{-bc} e^{c(b-a)}}{a} \{E_1[(b-a)c] - E_1[(b-a)(d+c)]\} \end{aligned} \quad (29)$$

$$\begin{aligned} &= \frac{1}{a} \left\{ e^{ad} E_1[b(c+d)] - e^{-ac} E_1[(-a+b)(c+d)] \right. \\ &\quad \left. - E_1[bc] + e^{-ac} E_1[(-a+b)c] \right\} \end{aligned} \quad (30)$$

$$\begin{aligned} &= \frac{e^{-bc}}{a} \left(e^{(a-b)d} \{D[b(c+d)] - D[(-a+b)(c+d)]\} \right. \\ &\quad \left. - D[bc] + D[(-a+b)c] \right) \end{aligned} \quad (31)$$

Note that – as always when using the definition of E_1 from equation (1) – the arguments of the exponential integrals $E_1(x)$ and $D(x)$ must be positive:

$$b(c+d) > 0 \quad (32a)$$

$$(-a+b)(c+d) > 0 \quad (32b)$$

$$bc > 0 \quad (32c)$$

$$(-a+b)c > 0 \quad (32d)$$

We can see, that for the interesting special case $c = 0$, the result is not defined: It violates the conditions (32c) and (32d).

$$\mathbf{2.3.3.} \quad \int_0^\infty e^{ax} E_1[b(x+c)] dx = \frac{1}{a} e^{-bc} \{-D[bc] + D[c(-a+b)]\}$$

Start with the integral representation $E_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt$ from subsection 2.1.1:

$$\int_0^\infty e^{ax} E_1[b(x+c)] dx = \int_0^\infty e^{ax} \int_1^\infty \frac{e^{-b(x+c)t}}{t} dt dx \quad (33)$$

$$= \int_1^\infty dt \frac{e^{-bct}}{t} \int_0^\infty dx e^{x(a-bt)} \quad (34)$$

$$= \int_1^\infty dt \frac{e^{-bct}}{t} \left(-\frac{1}{a-bt} \right), \quad (a-bt < 0) \quad (35)$$

$$= - \int_1^\infty dt \frac{e^{-bct}}{a} \frac{a-bt+bt}{t(a-bt)} \quad (36)$$

$$= - \int_1^\infty dt \frac{e^{-bct}}{a} \left(\frac{1}{t} + \frac{b}{a-bt} \right) \quad (37)$$

The first term in the last equation we recognize as $E_1(bc)$. For the second summand we first apply the substitution $-(a-bt) = y$ and then $\frac{y}{b-a} = x$ to also get an exponential integral:

$$\int_0^\infty e^{ax} E_1 [b(x+c)] dx = -\frac{1}{a} \left\{ E_1 [bc] - \int_{b-a}^\infty \frac{be^{-bc\frac{a+y}{b}}}{y} \frac{dy}{b} \right\} \quad (38)$$

$$= -\frac{1}{a} \left\{ E_1 [bc] - e^{-ac} \int_{b-a}^\infty \frac{e^{-cy}}{y} dy \right\} \quad (39)$$

$$= -\frac{1}{a} \left\{ E_1 [bc] - e^{-ac} \int_1^\infty \frac{e^{-c(b-a)x}}{(b-a)x} (b-a) dx \right\} \quad (40)$$

$$= -\frac{1}{a} \{ E_1 [bc] - e^{-ac} E_1 [c(-a+b)] \} \quad (41)$$

$$= \frac{1}{a} e^{-bc} \{ -D [bc] + D [c(-a+b)] \} \quad (42)$$

In the end we arrive at the equation (A2c) found in the appendix of [De 90]. We have to comply to the following restrictions:

$$a - bt < 0 \stackrel{(35)}{\Rightarrow} a - b < 0 \Rightarrow a < b \quad (43a)$$

$$bc > 0 \quad (43b)$$

2.3.4. $\int_0^\infty e^{ax} E_1 (bx) dx = -\frac{1}{a} \ln \left(1 - \frac{a}{b} \right)$

Start with the integral representation $E_1 (bx) = e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt$ from subsection 2.1.3, which is only valid for $bx > 0$. Then multiply the equation with $e^{ax} dx$, while integrating both sides from $x = 0$ to infinity. After that just change the integration order:

$$\int_0^\infty e^{ax} E_1 (bx) dx = \int_0^\infty e^{ax} \left[e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt \right] dx, \quad (b > 0) \quad (44)$$

$$= \int_0^\infty \frac{dt}{1+t} \int_0^\infty e^{-(a+b+bt)x} dx \quad (45)$$

$$= \int_0^\infty \frac{1}{1+t} \frac{1}{(-a+b+bt)} dt, \quad (-a+b+bt > 0) \quad (46)$$

We can see the requirement $-a+b+bt > 0$ yields $-a+b > 0$ for the integration limit $t = 0$, or equivalently $a < b$. On this result we want to use the substitution $\frac{1}{1+t} = z$. Then, in the next step, substitute $b - az = x$:

$$\int_0^\infty e^{ax} E_1(bx) dx = \int_1^0 z \frac{1}{(-a + \frac{b}{z})} \left(-\frac{1}{z^2} dz \right) \quad (47)$$

$$= \int_0^1 \frac{1}{(b - az)} dz \quad (48)$$

$$= \int_b^{b-a} \frac{1}{x} \left(-\frac{1}{a} dx \right) \quad (49)$$

$$= -\frac{1}{a} [\ln x]_{x=b}^{b-a} \quad (50)$$

$$= -\frac{1}{a} [\ln(b-a) - \ln(b)] \quad (51)$$

$$= -\frac{1}{a} \ln\left(1 - \frac{a}{b}\right), \quad (a < b, 0 < b) \quad (52)$$

This end result is equation (A2d) from [De 90]. In terms of the Y -function this is $Y(a, b, 0, \infty)$. For a more general derivation based on that of [Sch43, pp. 73 sq.] see section A.3 in the appendix. The equation can also be found as equation (R10) in [She55].

2.3.5. $\int_0^d e^{ax} E_1(bx) dx = \frac{1}{a} (e^{(a-b)d} \{D[bd] - D[(-a+b)d]\} - \ln[1 - \frac{a}{b}])$

To start we plug in the integral representation $E_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt$ from subsection 2.1.1:

$$\int_0^d e^{ax} E_1(bx) dx = \int_0^d e^{ax} \left(\int_1^\infty \frac{e^{-bxt}}{t} dt \right) dx \quad (53)$$

$$= \int_1^\infty \frac{dt}{t} \int_0^d dx e^{(a-bt)x} \quad (54)$$

$$= \int_1^\infty \frac{dt}{t} \left(\frac{1}{a-bt} e^{(a-bt)d} - \frac{1}{a-bt} \right), \quad (a < b) \quad (55)$$

$$= \int_1^\infty \frac{dt}{a} \frac{a}{t(a-bt)} e^{(a-bt)d} - \int_1^\infty dt \frac{1}{t(a-bt)} \quad (56)$$

$$= \int_1^\infty \frac{dt}{a} \frac{a-bt+bt}{t(a-bt)} e^{(a-bt)d} - \int_1^\infty dt \frac{1}{t(a-bt)} \quad (57)$$

$$= \int_1^\infty \frac{dt}{a} \left(\frac{1}{t} + \frac{b}{a-bt} \right) e^{(a-bt)d} - \int_1^\infty dt \frac{1}{t(a-bt)} \quad (58)$$

$$= \frac{e^{ad}}{a} \int_1^\infty dt \frac{e^{-dbt}}{t} + \frac{1}{a} \int_1^\infty dt \frac{be^{(a-bt)d}}{a-bt} - \int_1^\infty dt \frac{1}{t(a-bt)} \quad (59)$$

Looking at the first term in the last equation we see the integral representation of $E_1(bd)$. To simplify to second summand we first apply the substitution $-(a-bt) = y$ and after applying $\frac{y}{b-a} = x$ see the integral representation of $E_1[d(b-a)]$. And finally for the last term in the equation we use the substitution $\frac{1}{a-bt} = z$ and then apply $1-az = w$:

$$\int_0^d e^{ax} E_1(bx) dx = \frac{e^{ad}}{a} E_1[bd] + \frac{1}{a} \int_{b-a}^{\infty} dy \frac{e^{-yd}}{-y} - \int_{(a-b)^{-1}}^0 \frac{b}{a - \frac{1}{z}} \frac{z}{b} \frac{dz}{z^2} \quad (60)$$

$$= \frac{e^{ad}}{a} E_1[bd] - \frac{1}{a} \int_1^{\infty} dx \frac{e^{-d(b-a)x}}{(b-a)x} (b-a) - \int_1^{1-\frac{a}{a-b}} \frac{1}{w} \left(-\frac{dw}{a} \right) \quad (61)$$

$$= \frac{e^{ad}}{a} E_1[bd] - \frac{1}{a} E_1[(b-a)d] + \frac{1}{a} \ln \left[1 - \frac{a}{a-b} \right] \quad (62)$$

$$= \frac{e^{ad}}{a} E_1[bd] - \frac{1}{a} E_1[(-a+b)d] + \frac{1}{a} \ln \left[\frac{a-b-a}{a-b} \right] \quad (63)$$

$$= \frac{1}{a} \left\{ e^{ad} E_1[bd] - E_1[(-a+b)d] - \ln \left[-\frac{a-b}{b} \right] \right\} \quad (64)$$

$$= \frac{1}{a} \left(e^{(a-b)d} \{ D[bd] - D[(-a+b)d] \} - \ln \left[1 - \frac{a}{b} \right] \right) \quad (65)$$

We have to apply the following restrictions:

$$b > 0 \quad (66a)$$

$$b > a \quad (66b)$$

$$bd > 0 \xrightarrow{(66b)} d > 0 \quad (66c)$$

Equation (65) is equation (A2b) from the appendix of [De 90]. It is the special case $Y(a, b, 0, d) := W(a, b, d)$. It can also be found as equation (R11) and (R22) of [She55].

3. References

- [De 90] D. K. G. De Boer. ‘Calculation of x-ray fluorescence intensities from bulk and multilayer samples’. In: *X-Ray Spectrometry* 19.3 (1990), pp. 145–154. URL: <http://onlinelibrary.wiley.com/doi/10.1002/xrs.1300190312/abstract>.
- [DLMF] *NIST Digital Library of Mathematical Functions*. <http://dlmf.nist.gov/>, Release 1.0.10 of 2015-08-07. Online companion to [Olv+10]. URL: <http://dlmf.nist.gov/>.
- [GN69] Murray Geller and Edward W. Ng. ‘A table of integrals of the exponential integral’. In: *Journal of research of the National Bureau of Standards B* 73 (1969), pp. 191–210. URL: http://nvlpubs.nist.gov/nistpubs/jres/73B/jresv73Bn3p191_A1b.pdf (visited on 23/12/2015).
- [Olv+10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, eds. *NIST Handbook of Mathematical Functions*. Print companion to [DLMF]. New York, NY: Cambridge University Press, 2010.
- [Sch43] Oskar Schlömilch. *Beiträge zur Theorie bestimmter Integrale*. Jena: Friedrich Frommann, 1843. URL: <http://www.mdz-nbn-resolving.de/urn/resolver.pl?urn=urn:nbn:de:bvb:12-bsb10053982-5>.
- [She55] Jacob Sherman. ‘The theoretical derivation of fluorescent X-ray intensities from mixtures’. In: *Spectrochimica Acta* 7 (1955), pp. 283–306. ISSN: 0371-1951. DOI: [http://dx.doi.org/10.1016/0371-1951\(55\)80041-0](http://dx.doi.org/10.1016/0371-1951(55)80041-0). URL: <http://www.sciencedirect.com/science/article/pii/0371195155800410>.

A. Unused stuff

A.1. Definition of Logarithmic Integral $li(x)$

$$li(x) = \int_0^x \frac{1}{\ln t} dt, \quad (x > 0) \quad (67)$$

Note that for $x = 1$ the integral diverges (because of the singularity of the integrand $\frac{1}{\ln t}$) and for $x > 1$ the Cauchy principle value has to be employed to interpret the integral. From the definition of the exponential integral $E_1(x)$ in equation (1) one can see the following relation, when utilizing the substitution $t = e^{-w}$:

$$li(e^{-x}) = \int_0^{e^{-x}} \frac{1}{\ln t} dt \quad (68)$$

$$= \int_\infty^x \left(-\frac{1}{w} \right) (-e^{-w} dw) \quad (69)$$

$$= - \int_x^\infty \frac{e^{-w}}{w} dw \quad (70)$$

$$= -E_1(x) \quad (71)$$

A.2. Definition of Gamma function $\Gamma(x)$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad (x > 0) \quad (72)$$

A.3. More general derivation of section 2.3.4

This more general derivation of subsection 2.3.4 is based on that of [Sch43, pp. 73 sq.]. Start with the integral representation $E_1(bx) = e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt$ from subsection 2.1.3, which is only valid for $bx > 0$. Then multiply the equation with $x^{\mu-1} e^{ax} dx$, while integrating both sides from $x = 0$ to infinity:

$$\int_0^\infty x^{\mu-1} e^{ax} E_1(bx) dx = \int_0^\infty x^{\mu-1} e^{ax} \left[e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt \right] dx, \quad (b > 0) \quad (73)$$

$$= \int_0^\infty \frac{dt}{1+t} \int_0^\infty x^{\mu-1} e^{-(a-bt)x} dx \quad (74)$$

In the last step we just changed the integration order. Now we take a look at the integral over x . Using the substitution $kx = u$, and also taking advantage of the abbreviation $k = -a + b + bt$, we get:

$$\int_0^\infty x^{\mu-1} e^{-kx} dx = \int_0^\infty \left(\frac{u}{k}\right)^{\mu-1} e^{-u} \left(\frac{1}{k} du\right), \quad (k > 0) \quad (75)$$

$$= k^{-\mu} \int_0^\infty u^{\mu-1} e^{-u} du \quad (76)$$

$$= \frac{\Gamma(\mu)}{k^\mu}, \quad (k > 0, \mu > 0) \quad (77)$$

Here we used the definition of the Gamma function $\Gamma(\mu)$ from equation (72), which is only valid for $\mu > 0$. When applying the requirement $k > 0$ to the abbreviation $k = -a + b + bt > 0$, we can see this yields $-a + b > 0$ for the integration limit $t = 0$, or equivalently $a < b$. On this result we want to use the substitution $\frac{1}{1+t} = z$, i.e. $t = \frac{1}{z} - 1$:

$$\int_0^\infty x^{\mu-1} e^{ax} E_1(bx) dx = \Gamma(\mu) \int_0^\infty \frac{1}{(1+t)} \frac{1}{(-a + b(1+t))^\mu} dt \quad (78)$$

$$= \Gamma(\mu) \int_1^0 z \frac{1}{(-a + \frac{b}{z})^\mu} \left(-\frac{1}{z^2} dz\right) \quad (79)$$

$$= \Gamma(\mu) \int_0^1 \frac{z^{-1}}{(\frac{b}{z} - a)^\mu} \frac{z^\mu}{z^\mu} dz \quad (80)$$

$$= \Gamma(\mu) \int_0^1 \frac{z^{\mu-1}}{(b - az)^\mu} dz, \quad (a < b, \mu > 0) \quad (81)$$

Now, we can just set $\mu = 1$, use $\Gamma(1) = 1$ and employ the substitution $b - az = x$ to get the desired end result of equation (52):

$$\int_0^\infty e^{ax} E_1(bx) dx = \Gamma(1) \int_0^1 \frac{1}{b - az} dz \quad (82)$$

$$= \int_b^{b-a} \frac{1}{x} \left(-\frac{1}{a} dx\right) \quad (83)$$

$$= -\frac{1}{a} [\ln x]_{x=b}^{b-a} \quad (84)$$

$$= -\frac{1}{a} [\ln(b-a) - \ln(b)] \quad (85)$$

$$= -\frac{1}{a} \ln\left(1 - \frac{a}{b}\right), \quad (a < b, 0 < b) \quad (86)$$