Exponential Integral $E_1(x)$

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1. Introduction

1.1. Informational

- Github: `https://github.com/AndreWaehlisch/Exponential-Integral`
- Some links about the exponential integral:
  - wolfram.com:
    - E_1 and Ei:
      - `http://mathworld.wolfram.com/ExponentialIntegral.html`
      - `http://functions.wolfram.com/GammaBetaErf/ExpIntegralEi/`
      - `http://www.wolframalpha.com/input/?i=ExpIntegralE[1,x]`
      - `http://www.wolframalpha.com/input/?i=ExpIntegralEi[x]`
    - E_n:
      - `http://mathworld.wolfram.com/En-Function.html`
      - `http://functions.wolfram.com/GammaBetaErf/ExpIntegralE/`

1.2. Mathematical notes

1. All variables and parameters are considered real, if not stated otherwise.
2. The notation $[f(x)]_{x=a}^{b}$ means $f(b) - (a)$. In the same sense, if for example $b = \infty$, interpret as $[f(x)]_{x=a}^{\infty} = \lim_{x\to\infty}(f(x)) - f(a)$.

1.3. Definitions

1.3.1. Definition of Exponential Integral $E_1(x)$

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw, \qquad (x > 0) \tag{1}$$

1.3.2. Definition of $D(x)$

$$D(x) = e^x E_1(x), \qquad (x > 0) \tag{2}$$

1.3.3. Definition of $Y(a, b, c, d)$

$$Y(a, b, c, d) = \int_0^d e^{ax} E_1[b(x + c)] dx, \qquad (d > 0, b > 0, c > 0) \tag{3}$$

2. Exponential Integral $E_1(x)$

2.1. Integral Representations

2.1.1. $E_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt$, source is eq. (2b) of [De 90]

Use substitution $w = xt$:

$$
\begin{aligned}
E_1(x) &= \int_x^\infty \frac{e^{-w}}{w} dw, \qquad (x > 0) && (4)\\
&= \int_1^\infty \frac{e^{-xt}}{t} dt && (5)
\end{aligned}
$$

2.1.2. $E_1(x) = e^{-x} \int_0^1 \frac{1}{x - \ln t} dt$, source is eq. (4) in sec. 3.3 of [GN69]

$$
\begin{aligned}
E_1(x) &= \int_x^\infty \frac{e^{-w}}{w} dw, \qquad (x > 0) && (6)\\
&= e^{-x} \int_x^\infty \frac{e^{x-w}}{w} dw && (7)
\end{aligned}
$$

First use substitution $(x - w) = -y$, then use substitution $y = -\ln t$:

$$
\begin{aligned}
e^{-x} \int_x^\infty \frac{e^{x-w}}{w} dw &= e^{-x} \int_0^\infty \frac{e^{-y}}{x + y} dy && (8)\\
&= e^{-x} \int_1^0 \frac{t}{x - \ln t} \left(\frac{-1}{t} \right) dt && (9)\\
&= e^{-x} \int_0^1 \frac{1}{x - \ln t} dt && (10)
\end{aligned}
$$

2.1.3. $E_1(x) = e^{-x} \int_0^\infty \frac{e^{-xt}}{t+1} dt$

Use substitution $\frac{w}{x} - 1 = t$:

$$
\begin{aligned}
E_1(x) &= \int_x^\infty \frac{e^{-w}}{w} dw && (11)\\
&= \int_0^\infty \frac{e^{-x(t+1)}}{x(t+1)} (x dt) && (12)\\
&= e^{-x} \int_0^\infty \frac{e^{-xt}}{t+1} dt && (13)
\end{aligned}
$$

2.1.4. $E_1(x) = \int_0^{\frac{\pi}{2}} \tan(\alpha) e^{-\frac{x}{\cos\alpha}} d\alpha$, this is eq. (2a) from [De 90]

Starting with the representation of equation (5) and using the substitution $t = \frac{1}{\cos\alpha}$ we immediately arrive at the result:

$$\begin{aligned} E_1(x) &= \int_1^\infty \frac{e^{-xt}}{t} dt && (14) \\ &= \int_0^{\frac{\pi}{2}} \cos(\alpha) e^{-\frac{x}{\cos\alpha}} \frac{\tan\alpha}{\cos\alpha} d\alpha && (15) \\ &= \int_0^{\frac{\pi}{2}} \tan(\alpha) e^{-\frac{x}{\cos\alpha}} d\alpha && (16) \end{aligned}$$

2.2. Special Values

2.2.1. $\frac{d}{dx} E_1[b(x+c)] = -\frac{e^{-b(x+c)}}{x+c}$

Start with integral representation of $E_1(x)$ found in subsection 2.1.1:

$$\begin{aligned} \frac{d}{dx} E_1[b(x+c)] &= \frac{d}{dx} \int_1^\infty \frac{e^{-b(x+c)t}}{t} dt && (17) \\ &= \int_1^\infty (-bt) \frac{e^{-b(x+c)t}}{t} dt && (18) \\ &= -b \int_1^\infty e^{-b(x+c)t} dt && (19) \\ &= \frac{-b}{-b(x+c)} \left[e^{-b(x+c)t} \right]_{t=1}^\infty && (20) \\ &= -\frac{e^{-b(x+c)}}{x+c} \qquad (b(x+c) > 0) && (21) \end{aligned}$$

2.2.2. A shifted integration range

Starting with the integral representation of $E_1(x)$ from subsection 2.1.4 we don't want to integrate from 0 to $\frac{\pi}{2}$, but instead only up to $\frac{\pi}{2} - \psi$, with the condition $\psi \in \left]0, \frac{\pi}{2}\right[$. The first step is to undo the substitution $t = \frac{1}{\cos\alpha}$ and use $\cos\left(\frac{\pi}{2} - \psi\right) = \sin\psi$:

$$\int_0^{\frac{\pi}{2}-\psi} \tan(\alpha) e^{-\frac{x}{\cos\alpha}} d\alpha = \int_1^{\frac{1}{\sin\psi}} \frac{e^{-xt}}{t} dt, \qquad \left(0 < \psi < \frac{\pi}{2}\right) \tag{22}$$

Now, notice that if $0 < \psi < \frac{\pi}{2}$ then $\infty > \frac{1}{\sin\psi} > 1$. Split the first integral representation of $E_1(x)$ from equation (5) up into two summands:

$$E_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt = \int_1^{\frac{1}{\sin\psi}} \frac{e^{-xt}}{t} dt + \int_{\frac{1}{\sin\psi}}^\infty \frac{e^{-xt}}{t} dt \tag{23}$$

The first term on the right-hand side is equation (22) from above. Using the substitution $xt = r$ notice that the second term is the exponential integral again, but with a different argument:

$$\int_0^{\frac{\pi}{2}-\psi} \tan(\alpha)\, e^{-\frac{x}{\cos\alpha}} d\alpha = \int_1^{\infty} \frac{e^{-xt}}{t} dt - \int_{\frac{1}{\sin\psi}}^{\infty} \frac{e^{-xt}}{t} dt \tag{24}$$

$$= E_1(x) - \int_{\frac{x}{\sin\psi}}^{\infty} \frac{e^{-r}}{r} dr \tag{25}$$

$$= E_1(x) - E_1\left(\frac{x}{\sin\psi}\right) \tag{26}$$

$$= E_1(x) - E_1(x \csc\psi) \tag{27}$$

2.2.3. Limits with exponential function

First, see that the following inequality holds true, for a derivation see [Gau59].

$$\frac{1}{2}\ln\left(1+\frac{2}{x}\right) \leq e^x E_1(x) \leq \ln\left(1+\frac{1}{x}\right), \qquad (x > 0) \tag{28}$$

When taking a look at the definition in equation 1, it is clear that the monotonically decreasing integrand forces the limit of the exponential integral to zero:

$$\lim_{x\to\infty} E_1(x) = 0 \tag{29}$$

$$\lim_{x\to\infty} e^{-x} E_1(x) = 0 \tag{30}$$

Where the second limit follows trivially. When looking at inequality (28) above we notice that both the function on the right-hand side and left-hand side tend to zero. Which means that the middle part tends to zero as well:

$$\lim_{x\to\infty} e^x E_1(x) = \lim_{x\to\infty} D(x) = 0 \tag{31}$$

Multiplying with the inverse exponential function leads again to

$$\lim_{x\to\infty} e^{-x} D(x) = \lim_{x\to\infty} E_1(x) = 0 \tag{32}$$

after cancelling the exponential function terms. However, $\lim_{x\to\infty} e^x D(x) = e^{2x} E_1(x)$ diverges.

Integral	Y-representation	Location this work	Location in [De 90]
$\int_0^d e^{ax} E_1 [b(x+c)]\, dx$	$Y(a,b,c,d)$	eq. (45), p. 8	eq. (A2a)
$\int_0^d e^{ax} E_1 [bd]\, dx$	$Y(a,b,0,d)$	eq. (79), p. 11	eq. (A2b)
$\int_0^\infty e^{ax} E_1 [b(x+c)]\, dx$	$Y(a,b,c,\infty)$	eq. (56), p. 9	eq. (A2c)
$\int_0^\infty e^{ax} E_1 [bx]\, dx$	$Y(a,b,0,\infty)$	eq. (66), p. 10	eq. (A2d)

Table 1: Integrals encountered in [De 90]

2.3. Integrals involving $E_1(x)$

2.3.1. $\int_a^b \frac{e^{-x}}{x} dx = E_1(a) - E_1(b)$

Start with the negative derivative from subsection 2.2.1, while setting $b = 1$ and $c = 0$:

$$-\frac{d}{dx} E_1(x) = \frac{e^{-x}}{x} \tag{33}$$

Now, just apply the *fundamental theorem of calculus* (in German: *Hauptsatz der Differential- und Integralrechnung*): For certain conditions to $f(x), a, b$ and if $\frac{d}{dx} F(x) = f(x)$, then $\int_a^b f(x)\, dx = F(b) - F(a)$. Applying this to equation (33) leads to the desired result. We assume $a \neq b$, so that the integral is not trivially zero. Also required due to the infinite discontinuity of the integrand at $x = 0$ is that a and b are *both* positive or *both* negative, furthermore both must be non-zero:

$$\int_a^b \frac{e^{-x}}{x} dx = E_1(a) - E_1(b) \qquad (a \neq b, ab > 0) \tag{34}$$

2.3.2. $\int_0^d e^{ax} E_1 [b(x+c)]\, dx =$ $\frac{e^{-bc}}{a} \left(e^{(a-b)d} \{ D[b(c+d)] - D[(-a+b)(c+d)] \} - D[bc] + D[(-a+b)c] \right)$

This is the general case of $Y(a,b,c,d) = \int_0^d e^{ax} E_1 [b(x+c)]\, dx$, found in [De 90]. Special cases, e.g. $c = 0$ are considered in the following subsections. Now, first integrate by parts, using the derivative $\frac{d}{dx} E_1 [b(x+c)] = -\frac{e^{-b(x+c)}}{x+c}$ from subsection 2.2.1:

$$\int_0^d e^{ax} E_1 [b(x+c)] = \left\{ \frac{e^{ax}}{a} E_1 [b(x+c)] \right\}_{x=0}^{d} - \int_0^d \frac{e^{ax}}{a} \frac{d}{dx} E_1 [b(x+c)]\, dx \tag{35}$$

$$= \frac{e^{ad}}{a} E_1 [b(d+c)] - \frac{E_1 [bc]}{a} - \int_0^d \frac{e^{ax}}{a} \left\{ -\frac{e^{-b(x+c)}}{x+c} \right\} dx \tag{36}$$

$$= \frac{e^{ad}}{a} E_1 [b(d+c)] - \frac{E_1 [bc]}{a} + \frac{e^{-bc}}{a} \int_0^d \frac{e^{-x(b-a)}}{x+c} dx \tag{37}$$

To simplify the integral in the last summand, first use the abbreviation $\phi = b - a$ and the substitution $x + c = y$, then use the substitution $(b - a)\, y = \phi y = t$. In the final step, to get to the intermediate result of equation (42), use the integral $\int_a^b \frac{e^{-x}}{x} dx = E_1(a) - E_1(b)$, found in subsection 2.3.1:

$$\int_0^d \frac{e^{-x(b-a)}}{x+c} dx = \int_0^d \frac{e^{-x\phi}}{x+c} dx \tag{38}$$

$$= \int_c^{d+c} \frac{e^{-(y-c)\phi}}{y} dy \tag{39}$$

$$= e^{c\phi} \int_c^{d+c} \frac{e^{-y\phi}}{y} dy \tag{40}$$

$$= e^{c\phi} \int_{\phi c}^{\phi(d+c)} \frac{e^{-t}}{t\phi^{-1}} \phi^{-1} dt \tag{41}$$

$$= e^{c\phi} \left\{ E_1[\phi c] - E_1[\phi(d+c)] \right\} \tag{42}$$

To arrive at the desired result we just put the intermediate step (42) into equation (37), and restore the abbreviation $\phi = b - a$. Then rearrange the terms to get to equation (A2a) from the appendix of [De 90]. In the last steps use the definition of $D(x)$ from eq. (2), which can be found in subsection 1.3.2 on page 3:

$$\begin{aligned}\int_0^d e^{ax} E_1[b(x+c)] =& \frac{e^{ad}}{a} E_1[b(d+c)] - \frac{E_1[bc]}{a} \\ &+ \frac{e^{-bc} e^{c(b-a)}}{a} \left\{ E_1[(b-a)c] - E_1[(b-a)(d+c)] \right\}\end{aligned} \tag{43}$$

$$\begin{aligned}=& \frac{1}{a} \Big\{ e^{ad} E_1[b(c+d)] - e^{-ac} E_1[(-a+b)(c+d)] \\ &- E_1[bc] + e^{-ac} E_1[(-a+b)c] \Big\}\end{aligned} \tag{44}$$

$$\begin{aligned}=& \frac{e^{-bc}}{a} \Big(e^{(a-b)d} \left\{ D[b(c+d)] - D[(-a+b)(c+d)] \right\} \\ &- D[bc] + D[(-a+b)c] \Big)\end{aligned} \tag{45}$$

Note that – as always when using the definition of $E_1(x)$ from equation (1) – the arguments of the exponential integrals $E_1(x)$ and $D(x)$ must be positive:

$$b(c+d) > 0 \tag{46a}$$

$$(-a+b)(c+d) > 0 \tag{46b}$$

$$bc > 0 \tag{46c}$$

$$(-a+b)c > 0 \tag{46d}$$

We can see, that for the interesting special case $c = 0$, the result is not defined: It violates the conditions (46c) and (46d).

2.3.3. $\int_0^\infty e^{ax} E_1 [b(x+c)]\, dx = \frac{1}{a} e^{-bc} \{-D[bc] + D[c(-a+b)]\}$

Start with the integral representation $E_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt$ from subsection 2.1.1:

$$\int_0^\infty e^{ax} E_1 [b(x+c)]\, dx = \int_0^\infty e^{ax} \int_1^\infty \frac{e^{-b(x+c)t}}{t} dt x \tag{47}$$

$$= \int_1^\infty dt \frac{e^{-bct}}{t} \int_0^\infty dx e^{x(a-bt)} \tag{48}$$

$$= \int_1^\infty dt \frac{e^{-bct}}{t} \left(-\frac{1}{a-bt}\right), \qquad (a-bt<0) \tag{49}$$

$$= -\int_1^\infty dt \frac{e^{-bct}}{a} \frac{a-bt+bt}{t(a-bt)} \tag{50}$$

$$= -\int_1^\infty dt \frac{e^{-bct}}{a} \left(\frac{1}{t} + \frac{b}{a-bt}\right) \tag{51}$$

The first term in the last equation we recognize as $E_1(bc)$. For the second summand we first apply the substitution $-(a-bt) = y$ and then $\frac{y}{b-a} = x$ to also get an exponential integral:

$$\int_0^\infty e^{ax} E_1 [b(x+c)]\, dx = -\frac{1}{a} \left\{ E_1[bc] - \int_{b-a}^\infty \frac{b e^{-bc\frac{a+y}{b}}}{y} \frac{dy}{b} \right\} \tag{52}$$

$$= -\frac{1}{a} \left\{ E_1[bc] - e^{-ac} \int_{b-a}^\infty \frac{e^{-cy}}{y} dy \right\} \tag{53}$$

$$= -\frac{1}{a} \left\{ E_1[bc] - e^{-ac} \int_1^\infty \frac{e^{-c(b-a)x}}{(b-a)x} (b-a)\, dx \right\} \tag{54}$$

$$= -\frac{1}{a} \left\{ E_1[bc] - e^{-ac} E_1[c(-a+b)] \right\} \tag{55}$$

$$= \frac{1}{a} e^{-bc} \{-D[bc] + D[c(-a+b)]\} \tag{56}$$

In the end we arrive at the equation (A2c) found in the appendix of [De 90]. We have to comply to the following restrictions:

$$a - bt < 0 \overset{(49)}{\Rightarrow} a - b < 0 \Rightarrow a < b \tag{57a}$$

$$bc > 0 \tag{57b}$$

2.3.4. $\int_0^\infty e^{ax} E_1(bx)\, dx = -\frac{1}{a} \ln\left(1 - \frac{a}{b}\right)$

Start with the integral representation $E_1(bx) = e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt$ from subsection 2.1.3, which is only valid for $bx > 0$. Then multiply the equation with $e^{ax} dx$, while integrating both sides from $x = 0$ to ∞. After that just change the integration order:

$$\int_0^\infty e^{ax} E_1(bx)\,dx = \int_0^\infty e^{ax}\left[e^{-bx}\int_0^\infty \frac{e^{-bxt}}{t+1}dt\right]dx, \qquad (b>0) \tag{58}$$

$$= \int_0^\infty \frac{dt}{1+t}\int_0^\infty e^{-(-a+b+bt)x}dx \tag{59}$$

$$= \int_0^\infty \frac{1}{1+t}\frac{1}{(-a+b+bt)}dt, \qquad (-a+b+bt>0) \tag{60}$$

We can see the requirement $-a+b+bt>0$ yields $-a+b>0$ for the integration limit $t=0$, or equivalently $a<b$. On this result we want to use the substitution $\frac{1}{1+t}=z$. Then, in the next step, substitute $b-az=x$:

$$\int_0^\infty e^{ax} E_1(bx)\,dx = \int_1^0 z\frac{1}{\left(-a+\frac{b}{z}\right)}\left(-\frac{1}{z^2}dz\right) \tag{61}$$

$$= \int_0^1 \frac{1}{(b-az)}dz \tag{62}$$

$$= \int_b^{b-a} \frac{1}{x}\left(-\frac{1}{a}dx\right) \tag{63}$$

$$= -\frac{1}{a}\left[\ln x\right]_{x=b}^{b-a} \tag{64}$$

$$= -\frac{1}{a}\left[\ln(b-a)-\ln(b)\right] \tag{65}$$

$$= -\frac{1}{a}\ln\left(1-\frac{a}{b}\right), \qquad (a<b, 0<b) \tag{66}$$

This end result is equation (A2d) from [De 90]. In terms of the Y-function this is $Y(a,b,0,\infty)$. For a more general derivation based on that of [Sch43, pp. 73 sq.] see section A.3 in the appendix. The equation can also be found as equation (R10) in [She55].

2.3.5. $\int_0^d e^{ax} E_1(bx)\,dx = \frac{1}{a}\left(e^{(a-b)d}\left\{D[bd]-D[(-a+b)d]\right\}-\ln\left[1-\frac{a}{b}\right]\right)$

To start we plug in the integral representation $E_1(x)=\int_1^\infty \frac{e^{-xt}}{t}dt$ from subsection 2.1.1:

$$\begin{aligned}
\int_0^d e^{ax} E_1(bx)\, dx &= \int_0^d e^{ax} \left(\int_1^\infty \frac{e^{-bxt}}{t} dt \right) dx && (67)\\
&= \int_1^\infty \frac{dt}{t} \int_0^d dx e^{(a-bt)x} && (68)\\
&= \int_1^\infty \frac{dt}{t} \left(\frac{1}{a-bt} e^{(a-bt)d} - \frac{1}{a-bt} \right), \qquad (a<b) && (69)\\
&= \int_1^\infty \frac{dt}{a} \frac{a}{t(a-bt)} e^{(a-bt)d} - \int_1^\infty dt \frac{1}{t(a-bt)} && (70)\\
&= \int_1^\infty \frac{dt}{a} \frac{a-bt+bt}{t(a-bt)} e^{(a-bt)d} - \int_1^\infty dt \frac{1}{t(a-bt)} && (71)\\
&= \int_1^\infty \frac{dt}{a} \left(\frac{1}{t} + \frac{b}{a-bt} \right) e^{(a-bt)d} - \int_1^\infty dt \frac{1}{t(a-bt)} && (72)\\
&= \frac{e^{ad}}{a} \int_1^\infty dt \frac{e^{-dbt}}{t} + \frac{1}{a} \int_1^\infty dt \frac{be^{(a-bt)d}}{a-bt} - \int_1^\infty dt \frac{1}{t(a-bt)} && (73)
\end{aligned}$$

Looking at the first term in the last equation we see the integral representation of $E_1(bd)$. To simplify the second summand we first apply the substitution $-(a-bt)=y$ and after applying $\frac{y}{b-a}=x$ see the integral representation of $E_1[d(b-a)]$. And finally for the last term in the equation we use the substitution $\frac{1}{a-bt}=z$ and then apply $1-az=w$:

$$\begin{aligned}
\int_0^d e^{ax} E_1(bx)\, dx &= \frac{e^{ad}}{a} E_1[bd] + \frac{1}{a} \int_{b-a}^\infty dy \frac{e^{-yd}}{-y} - \int_{(a-b)^{-1}}^0 \frac{b}{a-\frac{1}{z}} \frac{z}{b} \frac{dz}{z^2} && (74)\\
&= \frac{e^{ad}}{a} E_1[bd] - \frac{1}{a} \int_1^\infty dx \frac{e^{-d(b-a)x}}{(b-a)x} (b-a) - \int_1^{1-\frac{a}{a-b}} \frac{1}{w} \left(-\frac{dw}{a} \right) && (75)\\
&= \frac{e^{ad}}{a} E_1[bd] - \frac{1}{a} E_1[(b-a)d] + \frac{1}{a} \ln \left[1 - \frac{a}{a-b} \right] && (76)\\
&= \frac{e^{ad}}{a} E_1[bd] - \frac{1}{a} E_1[(-a+b)d] + \frac{1}{a} \ln \left[\frac{a-b-a}{a-b} \right] && (77)\\
&= \frac{1}{a} \left\{ e^{ad} E_1[bd] - E_1[(-a+b)d] - \ln \left[-\frac{a-b}{b} \right] \right\} && (78)\\
&= \frac{1}{a} \left(e^{(a-b)d} \left\{ D[bd] - D[(-a+b)d] \right\} - \ln \left[1 - \frac{a}{b} \right] \right) && (79)
\end{aligned}$$

We have to apply the following restrictions:

$$b > 0 \tag{80a}$$
$$b > a \tag{80b}$$
$$bd > 0 \overset{(80b)}{\Rightarrow} d > 0 \tag{80c}$$

Equation (79) is equation (A2b) from the appendix of [De 90]. It is the special case $Y(a, b, 0, d) \coloneqq W(a, b, d)$. It can also be found as equation (R11) and (R22) of [She55].

2.4. Negative arguments

In the previous derivations we always assumed the argument of the exponential integral to be positive. But one might want to evaluate the exponential integral for negative arguments too. That's possible if we take the *Cauchy principal value* $\mathcal{PV}$ of the otherwise diverging integral, which is defined as (e.g. [Bro+08]):

$$\mathcal{PV}\left[\int_a^b f(x)\,dx\right] = \lim_{\epsilon\to 0^+}\left[\int_a^{x_0-\epsilon} f(x)\,dx + \int_{x_0+\epsilon}^b f(x)\,dx\right] \tag{81}$$

for a function $f(x)$ with a singularity at $x = x_0$. The integration bounds $a < x_0 < b$ can be $\pm\infty$ as well. We will use this for $x_0 = 0$ on the definition of the exponential integral:

$$\mathcal{PV}\left[\int_x^\infty \frac{e^{-w}}{w}dw\right] = \lim_{\epsilon\to 0^+}\left[\int_x^{-\epsilon} \frac{e^{-w}}{w}dw + \int_\epsilon^\infty \frac{e^{-w}}{w}dw\right] \tag{82}$$

Let us now assume a negative argument $x < 0$. Then we can define $a = -x > 0$ and split up the integral. For positive values of a the Cauchy principal value converges and the limit is just the exponential integral $E_1(a)$:

$$\begin{aligned} E_1(x=-a) = \mathcal{PV}\left[\int_{-a}^\infty \frac{e^{-w}}{w}dw\right] &= \mathcal{PV}\left[\int_{-a}^a \frac{e^{-w}}{w}dw + \int_a^\infty \frac{e^{-w}}{w}dw\right] && (83)\\ &= \mathcal{PV}\left[\int_{-a}^a \frac{e^{-w}}{w}dw\right] + E_1(a) && (84)\end{aligned}$$

The remaining Cauchy principal value can be readily evaluated (see e.g. [DR84]), when we split the integrand into an odd and an even part:

$$\begin{aligned} op(w) &= \frac{1}{2}\left[f(w) - f(-w)\right] && (85)\\ ep(w) &= \frac{1}{2}\left[f(w) + f(-w)\right] && (86)\\ f(w) &= \frac{e^{-w}}{w} = op(w) + ep(w) && (87)\end{aligned}$$

Then the odd part will vanish inside the integral, because we integrate over an symmetric interval $\int_{-a}^a op(w)\,dw = 0$. We can also exploit the behaviour of even functions in the same symmetric interval $\int_{-a}^a ep(w)\,dw = 2\cdot\int_0^a ep(w)\,dw$:

$$\mathcal{PV}\left[\int_{-a}^{a} \frac{e^{-w}}{w} dw\right] = \mathcal{PV}\left[\int_{-a}^{a} op\left(w\right) + ep\left(w\right) dw\right] \tag{88}$$

$$= 2 \cdot \mathcal{PV}\left[\int_{0}^{a} ep\left(w\right) dw\right] \tag{89}$$

$$= \mathcal{PV}\left[\int_{0}^{a} \left(\frac{e^{-w}}{w} - \frac{e^{w}}{w}\right) dw\right] \tag{90}$$

The last integral converges (even without the Cauchy principal value), and can easily be numerically integrated. For example:

$$E_1\left(-1\right) = \int_{0}^{1} \left(\frac{e^{-w}}{w} - \frac{e^{w}}{w}\right) dw + E_1\left(1\right) \tag{91}$$

$$\approx -1.89512 \tag{92}$$

3. References

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A. Collection of unused definitions and derivations

A.1. Definition of Logarithmic Integral $li\left(x\right)$

$$li\left(x\right) = \int_0^x \frac{1}{\ln t} dt, \qquad (x > 0) \tag{93}$$

Note that for $x = 1$ the integral diverges (because of the singularity of the integrand $\frac{1}{\ln t}$) and for $x > 1$ the Cauchy principle value has to be employed to interpret the integral. From the definition of the exponential integral $E_1\left(x\right)$ in equation (1) one can see the following relation, when utilizing the substitution $t = e^{-w}$:

$$li\left(e^{-x}\right) = \int_0^{e^{-x}} \frac{1}{\ln t} dt \tag{94}$$

$$= \int_\infty^x \left(-\frac{1}{w}\right)\left(-e^{-w} dw\right) \tag{95}$$

$$= -\int_x^\infty \frac{e^{-w}}{w} dw \tag{96}$$

$$= -E_1\left(x\right) \tag{97}$$

A.2. Definition of Gamma function $\Gamma\left(x\right)$

$$\Gamma\left(x\right) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad (x > 0) \tag{98}$$

A.3. More general derivation of section 2.3.4

This more general derivation of subsection 2.3.4 is based on that of [Sch43, pp. 73 sq.]. Start with the integral representation $E_1\left(bx\right) = e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt$ from subsection 2.1.3, which is only valid for $bx > 0$. Then multiply the equation with $x^{\mu-1} e^{ax} dx$, while integrating both sides from $x = 0$ to infinity:

$$\int_0^\infty x^{\mu-1} e^{ax} E_1\left(bx\right) dx = \int_0^\infty x^{\mu-1} e^{ax} \left[e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt\right] dx, \qquad (b > 0) \tag{99}$$

$$= \int_0^\infty \frac{dt}{1+t} \int_0^\infty x^{\mu-1} e^{-(-a+b+bt)x} dx \tag{100}$$

In the last step we just changed the integration order. Now we take a look at the integral over x. Using the substitution $kx = u$, and also taking advantage of the abbreviation $k = -a + b + bt$, we get:

$$
\begin{aligned}
\int_0^\infty x^{\mu-1} e^{-kx} dx &= \int_0^\infty \left(\frac{u}{k}\right)^{\mu-1} e^{-u} \left(\frac{1}{k} du\right), \qquad (k>0) &\quad (101)\\
&= k^{-\mu} \int_0^\infty u^{\mu-1} e^{-u} du &\quad (102)\\
&= \frac{\Gamma(\mu)}{k^\mu}, \qquad (k>0, \mu>0) &\quad (103)
\end{aligned}
$$

Here we used the definition of the Gamma function $\Gamma(\mu)$ from equation (98), which is only valid for $\mu > 0$. When applying the requirement $k > 0$ to the abbreviation $k = -a + b + bt > 0$, we can see this yields $-a + b > 0$ for the integration limit $t = 0$, or equivalently $a < b$. On this result we want to use the substitution $\frac{1}{1+t} = z$, i.e. $t = \frac{1}{z} - 1$:

$$
\begin{aligned}
\int_0^\infty x^{\mu-1} e^{ax} E_1(bx)\, dx &= \Gamma(\mu) \int_0^\infty \frac{1}{(1+t)} \frac{1}{(-a + b(1+t))^\mu} dt &\quad (104)\\
&= \Gamma(\mu) \int_1^0 z \frac{1}{\left(-a + \frac{b}{z}\right)^\mu} \left(-\frac{1}{z^2} dz\right) &\quad (105)\\
&= \Gamma(\mu) \int_0^1 \frac{z^{-1}}{\left(\frac{b}{z} - a\right)^\mu} \frac{z^\mu}{z^\mu} dz &\quad (106)\\
&= \Gamma(\mu) \int_0^1 \frac{z^{\mu-1}}{(b - az)^\mu} dz, \qquad (a<b, \mu>0) &\quad (107)
\end{aligned}
$$

Now, we can just set $\mu = 1$, use $\Gamma(1) = 1$ and employ the substitution $b - az = x$ to get the desired end result of equation (66):

$$
\begin{aligned}
\int_0^\infty e^{ax} E_1(bx) &= \Gamma(1) \int_0^1 \frac{1}{b - az} dz &\quad (108)\\
&= \int_b^{b-a} \frac{1}{x} \left(-\frac{1}{a} dx\right) &\quad (109)\\
&= -\frac{1}{a} \left[\ln x\right]_{x=b}^{b-a} &\quad (110)\\
&= -\frac{1}{a} \left[\ln(b-a) - \ln(b)\right] &\quad (111)\\
&= -\frac{1}{a} \ln\left(1 - \frac{a}{b}\right), \qquad (a<b, 0<b) &\quad (112)
\end{aligned}
$$