Exponential Integral $E_1(x)$

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1. Pre-stuff

1.1. Informational

- Github: `https://github.com/AndreWaehlisch/Exponential-Integral`
- Some links about the exponential integral:
  - wolfram.com:
    * E_1 and Ei:
      · `http://mathworld.wolfram.com/ExponentialIntegral.html`
      · `http://functions.wolfram.com/GammaBetaErf/ExpIntegralEi/`
      · `http://www.wolframalpha.com/input/?i=ExpIntegralE[1,x]`
      · `http://www.wolframalpha.com/input/?i=ExpIntegralEi[x]`
    * E_n:
      · `http://mathworld.wolfram.com/En-Function.html`
      · `http://functions.wolfram.com/GammaBetaErf/ExpIntegralE/`

1.2. Mathematical notes

1. All variables and parameters are considered real, if not stated otherwise.

1.3. Definitions

1.3.1. Definition of Exponential Integral $E_1(x)$

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw, \qquad (x > 0) \tag{1}$$

1.3.2. Definition of $D(x)$

$$D(x) = e^x E_1(x), \qquad (x > 0) \tag{2}$$

1.3.3. Definition of $Y(a, b, c, d)$

$$Y(a, b, c, d) = \int_0^d e^{ax} E_1[b(x + c)]\, dx \tag{3}$$

2. Exponential Integral $E_1(x)$

2.1. Integral Representations

2.1.1. $E_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt$, source is eq. (2b) of [De 90]

Use substitution $w = xt$:

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw, \qquad (x > 0) \tag{4}$$

$$= \int_1^\infty \frac{e^{-xt}}{t} dt \tag{5}$$

2.1.2. $E_1(x) = e^{-x} \int_0^1 \frac{1}{x - \ln t} dt$, source is eq. (4) in sec. 3.3 of [GN69]

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw, \qquad (x > 0) \tag{6}$$

$$= e^{-x} \int_x^\infty \frac{e^{x-w}}{w} dw \tag{7}$$

First use substitution $(x - w) = -y$, then use substitution $y = -\ln t$:

$$e^{-x} \int_x^\infty \frac{e^{x-w}}{w} dw = e^{-x} \int_0^\infty \frac{e^{-y}}{x+y} dy \tag{8}$$

$$= e^{-x} \int_1^0 \frac{t}{x - \ln t} \left(\frac{-1}{t} \right) dt \tag{9}$$

$$= e^{-x} \int_0^1 \frac{1}{x - \ln t} dt \tag{10}$$

2.1.3. $E_1(x) = e^{-x} \int_0^\infty \frac{e^{-xt}}{t+1} dt$

Use substitution $\frac{w}{x} - 1 = t$:

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw \tag{11}$$

$$= \int_0^\infty \frac{e^{-x(t+1)}}{x(t+1)} (x dt) \tag{12}$$

$$= e^{-x} \int_0^\infty \frac{e^{-xt}}{t+1} dt \tag{13}$$

2.2. Special Values

2.2.1. $\frac{d}{dx}E_1\left[b\left(x+c\right)\right]=-\frac{e^{-b(x+c)}}{x+c}$

Start with integral representation of $E_1\left(x\right)$ found in subsection 2.1.1:

$$\begin{aligned}\frac{d}{dx}E_1\left[b\left(x+c\right)\right] &= \frac{d}{dx}\int_1^\infty \frac{e^{-b(x+c)t}}{t}dt && (14)\\ &= \int_1^\infty \left(-bt\right)\frac{e^{-b(x+c)t}}{t}dt && (15)\\ &= -b\int_1^\infty e^{-b(x+c)t}dt && (16)\\ &= \frac{-b}{-b\left(x+c\right)}\left[e^{-b(x+c)t}\right]_{t=1}^\infty && (17)\\ &= -\frac{e^{-b(x+c)}}{x+c} \qquad \left(b\left(x+c\right)>0\right) && (18)\end{aligned}$$

2.2.2. $\lim_{x\to\infty}E_1\left[b\left(x+c\right)\right]=0$

Start with integral representation of $E_1\left(x\right)$ found in subsection 2.1.2:

$$\begin{aligned}\lim_{x\to\infty}E_1\left[b\left(x+c\right)\right] &= \lim_{x\to\infty}e^{-b(x+c)}\int_0^1\frac{1}{b\left(x+c\right)-\ln t}dt && (19)\\ &= e^{-bc}\int_0^1\left[\lim_{x\to\infty}\frac{e^{-bx}}{bx+bc-\ln t}\right]dt && (20)\end{aligned}$$

The perhaps suspected problem value of $\ln t \xrightarrow{t\to 0} -\infty$ in the limit is in fact no problem, when you rewrite it like the following:

$$\begin{aligned}\lim_{x\to\infty}\frac{e^{-bc}}{bx+bc-\ln\frac{1}{x}} &= \left(\lim_{x\to\infty}e^{-bc}\right)\cdot\left(\lim_{x\to\infty}\frac{1}{bx+bc-\ln\frac{1}{x}}\right) && (21)\\ &= 0\cdot 0 && (22)\\ &= 0 \qquad \left(b\geq 0\right) && (23)\end{aligned}$$

The desired limit follows. We have to take into account, that for $\ln t \xrightarrow{t\to 1} 0$ the limit only exists for positive b:

$$\lim_{x\to\infty}E_1\left[b\left(x+c\right)\right]=0 \qquad \left(b>0\right) \tag{24}$$

2.2.3. $\lim_{x\to\infty} e^{ax} E_1 [b (x + c)] = 0$

Use subsection 2.2.2 to evaluate the limit in the first step, $\lim_{x\to\infty} E_1 [b (x + c)] = 0$ (which is only valid for $b > 0$), then use the rule of de L'Hospital:

$$\lim_{x\to\infty} e^{ax} E_1 [b (x + c)] = \lim_{x\to\infty} \frac{E_1 [b (x + c)]}{e^{-ax}} \tag{25}$$

$$= \frac{0“}{„0} \qquad (a > 0, b > 0) \tag{26}$$

$$\lim_{x\to\infty} \frac{\frac{d}{dx} E_1 [b (x + c)]}{\frac{d}{dx} e^{-ax}} = \lim_{x\to\infty} \frac{-\left(\frac{e^{-b(x+c)}}{x+c}\right)}{-a e^{-ax}} \tag{27}$$

$$= \frac{1}{a} \lim_{x\to\infty} \frac{e^{-b(x+c)+ax}}{x + c} \tag{28}$$

$$= \frac{e^{-bc}}{a} \lim_{x\to\infty} \frac{e^{x(a-b)}}{x + c} \tag{29}$$

$$= 0 \qquad (a > 0, b > 0, a - b < 0) \tag{30}$$

2.3. Integrals involving $E_1 (x)$

Integral	Y-representation	Location	Location in [De 90]
$\int_0^d e^{ax} E_1 [b (x + c)] \, dx$	$Y (a, b, c, d)$	(43), p. 8	eq. (A2a)
$\int_0^d e^{ax} E_1 [bd] \, dx$	$Y (a, b, 0, d)$	(69), p. 11	eq. (A2b)
$\int_0^\infty e^{ax} E_1 [b (x + c)] \, dx$	$Y (a, b, c, \infty)$	(52), p. 9	eq. (A2c)
$\int_0^\infty e^{ax} E_1 [bx] \, dx$	$Y (a, b, 0, \infty)$	(61), p. 10	eq. (A2d)

Table 1: Table of integrals encountered in [De 90]

2.3.1. $\int_a^b \frac{e^{-x}}{x} dx = E_1 (a) - E_1 (b)$

Start with the negative derivative from subsection 2.2.1, while setting $b = 1$ and $c = 0$:

$$-\frac{d}{dx} E_1 (x) = \frac{e^{-x}}{x} \tag{31}$$

Now, just apply the *fundamental theorem of calculus* (in German: *Hauptsatz der Differential- und Integralrechnung*): For certain conditions to $f (x), a, b$ and if $\frac{d}{dx} F (x) = f (x)$, then $\int_a^b f (x) \, dx = F (b) - F (a)$. Applying this to equation (31) leads to the desired result. We assume $a \neq b$, so that the integral is not trivially zero. Also required due to the infinite discontinuity of the integrand at $x = 0$ is that a and b are *both* positive or *both* negative, furthermore both must be non-zero:

$$\int_a^b \frac{e^{-x}}{x}dx = E_1(a) - E_1(b) \qquad (a \neq b, ab > 0) \tag{32}$$

2.3.2. $\int_0^d e^{ax} E_1[b(x+c)]\,dx =$
$\frac{e^{-bc}}{a}\left(e^{(a-b)d}\{D[b(c+d)] - D[(-a+b)(c+d)]\} - D[bc] + D[(-a+b)c]\right)$

This is the general case of $Y(a,b,c,d) = \int_0^d e^{ax} E_1[b(x+c)]\,dx$, found in [De 90]. Special cases, e.g. $c = 0$ are considered in the following subsections. Now, first integrate by parts, using the derivative $\frac{d}{dx}E_1[b(x+c)] = -\frac{e^{-b(x+c)}}{x+c}$ from subsection 2.2.1:

$$\int_0^d e^{ax} E_1[b(x+c)] = \left\{\frac{e^{ax}}{a}E_1[b(x+c)]\right\}_{x=0}^{d} - \int_0^d \frac{e^{ax}}{a}\frac{d}{dx}E_1[b(x+c)]\,dx \tag{33}$$

$$= \frac{e^{ad}}{a}E_1[b(d+c)] - \frac{E_1[bc]}{a} - \int_0^d \frac{e^{ax}}{a}\left\{-\frac{e^{-b(x+c)}}{x+c}\right\}dx \tag{34}$$

$$= \frac{e^{ad}}{a}E_1[b(d+c)] - \frac{E_1[bc]}{a} + \frac{e^{-bc}}{a}\int_0^d \frac{e^{-x(b-a)}}{x+c}dx \tag{35}$$

To simplify the integral in the last summand, first use the abbreviation $\phi = b-a$ and the substitution $x + c = y$, then use the substitution $(b-a)y = \phi y = t$. In the final step, to get to the intermediate result of equation (40), use the integral $\int_a^b \frac{e^{-x}}{x}dx = E_1(a) - E_1(b)$, found in subsection 2.3.1:

$$\int_0^d \frac{e^{-x(b-a)}}{x+c}dx = \int_0^d \frac{e^{-x\phi}}{x+c}dx \tag{36}$$

$$= \int_c^{d+c} \frac{e^{-(y-c)\phi}}{y}dy \tag{37}$$

$$= e^{c\phi}\int_c^{d+c} \frac{e^{-y\phi}}{y}dy \tag{38}$$

$$= e^{c\phi}\int_{\phi c}^{\phi(d+c)} \frac{e^{-t}}{t\phi^{-1}}\phi^{-1}dt \tag{39}$$

$$= e^{c\phi}\{E_1[\phi c] - E_1[\phi(d+c)]\} \tag{40}$$

To arrive at the desired result we just put the intermediate step (40) into equation (35), and restore the abbreviation $\phi = b-a$. Then rearrange the terms to get to equation (A2a) from the appendix of [De 90]. In the last steps use the definition of $D(x)$ from eq. (2), which can be found in subsection 1.3.2 on page 3:

$$
\begin{aligned}
\int_0^d e^{ax} E_1\left[b\left(x+c\right)\right] =& \frac{e^{ad}}{a} E_1\left[b\left(d+c\right)\right] - \frac{E_1\left[bc\right]}{a} \\
&+ \frac{e^{-bc}e^{c(b-a)}}{a}\left\{E_1\left[\left(b-a\right)c\right] - E_1\left[\left(b-a\right)\left(d+c\right)\right]\right\}
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
=& \frac{1}{a}\Big\{e^{ad} E_1\left[b\left(c+d\right)\right] - e^{-ac} E_1\left[\left(-a+b\right)\left(c+d\right)\right] \\
&- E_1\left[bc\right] + e^{-ac} E_1\left[\left(-a+b\right)c\right]\Big\}
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
=& \frac{e^{-bc}}{a}\Big(e^{(a-b)d}\left\{D\left[b\left(c+d\right)\right] - D\left[\left(-a+b\right)\left(c+d\right)\right]\right\} \\
&- D\left[bc\right] + D\left[\left(-a+b\right)c\right]\Big)
\end{aligned} \tag{43}
$$

Note that – as always when using the definition of E_1 from equation (1) – the arguments of the exponential integrals $E_1\left(x\right)$ and $D\left(x\right)$ must be positive:

$$b\left(c+d\right) > 0 \tag{44a}$$
$$\left(-a+b\right)\left(c+d\right) > 0 \tag{44b}$$
$$bc > 0 \tag{44c}$$
$$\left(-a+b\right)c > 0 \tag{44d}$$

We can see, that for the interesting special case $c = 0$, the result is not defined: It violates the conditions (44c) and (44d).

2.3.3. $\int_0^\infty e^{ax} E_1\left[b\left(x+c\right)\right] dx = \frac{1}{a} e^{-bc}\left\{-D\left[bc\right] + D\left[c\left(b-a\right)\right]\right\}$

The result can be obtained from the general case $Y\left(a,b,c,d\right)$ in the preceding subsection 2.3.2, when setting $d=\infty$ and using the limit of subsection 2.2.3. But we can repeat the equivalent derivation of this special case $Y\left(a,b,c,\infty\right) = \int_0^\infty e^{ax} E_1\left[b\left(x+c\right)\right] dx$: First integrate by parts using the derivative $\frac{d}{dx} E_1\left[b\left(x+c\right)\right] = -\frac{e^{-b(x+c)}}{x+c}$ from subsection 2.2.1, then make use of the limit $\lim_{x\to\infty} e^{ax} E_1\left[b\left(x+c\right)\right] = 0$, which can be found in subsection 2.2.3:

$$\int_0^\infty e^{ax} E_1\left[b\left(x+c\right)\right] dx = \left[\frac{e^{ax}}{a} E_1\left[b\left(x+c\right)\right]\right]_{x=0}^\infty - \int_0^\infty \frac{e^{ax}}{a}\left(-\frac{e^{-b(x+c)}}{x+c}\right) dx \tag{45}$$

$$= -\frac{E_1\left(bc\right)}{a} + \frac{e^{-bc}}{a}\int_0^\infty \frac{e^{-x(b-a)}}{x+c} dx \tag{46}$$

Now, first use substitution $x\left(b-a\right) = y$ with $\frac{dx}{dy} = \left(b-a\right)^{-1}$, then use substitution $y + c\left(b-a\right) = t$. In the last steps use the definition of $D\left(x\right)$ from eq. (2), which can be found in subsection 1.3.2 on page 3. In the end we arrive at the equation (A2c) found in the appendix of [De 90]. Provided positive c the condition $a-b<0$ results from eq. (50):

$$\int_0^\infty e^{ax} E_1\left[b(x+c)\right]dx = -\frac{E_1(bc)}{a} + \frac{e^{-bc}}{a}\int_0^\infty \frac{e^{-y}}{\frac{y}{b-a}+c}\frac{dy}{b-a} \tag{47}$$

$$= -\frac{E_1(bc)}{a} + \frac{e^{-bc}}{a}\int_0^\infty \frac{e^{-y}}{y+c(b-a)}dy \tag{48}$$

$$= -\frac{E_1(bc)}{a} + \frac{e^{-bc}}{a}\int_{c(b-a)}^\infty \frac{e^{-t}e^{c(b-a)}}{t}dt \tag{49}$$

$$= -\frac{E_1(bc)}{a} + \frac{e^{-ac}}{a}E_1\left[c(b-a)\right] \qquad (c(b-a)>0) \tag{50}$$

$$= \frac{1}{a}e^{-bc}\left\{-D\left[bc\right] + e^{-ac}e^{bc}e^{-c(b-a)}D\left[c(b-a)\right]\right\} \tag{51}$$

$$= \frac{1}{a}e^{-bc}\left\{-D\left[bc\right] + D\left[c(b-a)\right]\right\} \tag{52}$$

2.3.4. $\int_0^\infty e^{ax}E_1(bx)\,dx = -\frac{1}{a}\ln\left(1-\frac{a}{b}\right)$

Start with the integral representation $E_1(bx) = e^{-bx}\int_0^\infty \frac{e^{-bxt}}{t+1}dt$ from subsection 2.1.3, which is only valid for $bx>0$. Then multiply the equation with $e^{ax}dx$, while integrating both sides from $x=0$ to infinity. After that just change the integration order:

$$\int_0^\infty e^{ax}E_1(bx)\,dx = \int_0^\infty e^{ax}\left[e^{-bx}\int_0^\infty \frac{e^{-bxt}}{t+1}dt\right]dx, \qquad (b>0) \tag{53}$$

$$= \int_0^\infty \frac{dt}{1+t}\int_0^\infty e^{-(-a+b+bt)x}dx \tag{54}$$

$$= \int_0^\infty \frac{1}{1+t}\frac{1}{(-a+b+bt)}dt, \qquad (-a+b+bt>0) \tag{55}$$

We can see the requirement $-a+b+bt>0$ yields $-a+b>0$ for the integration limit $t=0$, or equivalently $a<b$. On this result we want to use the substitution $\frac{1}{1+t}=z$. Then, in the next step, substitute $b-az=x$:

$$\int_0^{\infty} e^{ax} E_1(bx)\, dx = \int_1^0 z \frac{1}{\left(-a+\frac{b}{z}\right)} \left(-\frac{1}{z^2} dz\right) \tag{56}$$

$$= \int_0^1 \frac{1}{(b-az)} dz \tag{57}$$

$$= \int_b^{b-a} \frac{1}{x} \left(-\frac{1}{a} dx\right) \tag{58}$$

$$= -\frac{1}{a} [\ln x]_{x=b}^{b-a} \tag{59}$$

$$= -\frac{1}{a} [\ln(b-a) - \ln(b)] \tag{60}$$

$$= -\frac{1}{a} \ln\left(1-\frac{a}{b}\right), \qquad (a<b, 0<b) \tag{61}$$

This end result is equation (A2d) from [De 90]. In terms of the Y-function this is $Y(a,b,0,\infty)$. For a more general derivation based on that of [Sch43, pp. 73 sq.] see section A.3 in the appendix.

2.3.5. $\int_0^d e^{ax} E_1(bx)\, dx = \frac{1}{a}\left(e^{(a-b)d}\{D[bd] - D[(-a+b)d]\} - \ln\left(a-\frac{a}{b}\right)\right)$

With a look at the preceding subsection we write:

$$\int_0^d e^{ax} E_1(bx)\, dx = \int_0^{\infty} e^{ax} E_1(bx)\, dx - \int_d^{\infty} e^{ax} E_1(bx)\, dx \tag{62}$$

The solution to the first integral on the right side can be found in the preceding subsection 2.3.4. We just have to solve the second integral, starting with integrating by parts. We utilize the derivative $\frac{d}{dx} E_1(bx) = -\frac{e^{-bx}}{x}$ from subsection 2.2.1 (for $bx > 0$) and the limit $\lim_{x\to\infty} e^{ax} E_1(bx) = 0$ from subsection 2.2.3 (for $a>0, b>0, a<b$) again:

$$\int_d^{\infty} e^{ax} E_1(bx)\, dx = \left[\frac{e^{ax}}{a} E_1(bx)\right]_{x=d}^{\infty} - \int_d^{\infty} \frac{e^{ax}}{a} \left(-\frac{e^{-bx}}{x}\right) dx \tag{63}$$

$$= -\frac{e^{ad}}{a} E_1(bd) + \frac{1}{a} \int_d^{\infty} \frac{e^{-x(b-a)}}{x} dx \tag{64}$$

Now we just need to substitute $\frac{x}{d} = t$ and apply the integral representation $E_1(x) = \int_1^{\infty} \frac{e^{-xt}}{t} dt$ from subsection 2.1.1:

$$\int_d^{\infty} e^{ax} E_1(bx)\, dx = -\frac{e^{ad}}{a} E_1(bd) + \frac{1}{a} \int_1^{\infty} \frac{e^{-d(b-a)t}}{t} dt \tag{65}$$

$$= -\frac{e^{ad}}{a} E_1(bd) + \frac{1}{a} E_1[d(b-a)] \tag{66}$$

Getting back to the entire integral in equation (62), we can now use (61) from the previous subsection ($\int_0^\infty e^{ax} E_1(bx)\, dx = -\frac{1}{a}\ln\left(1-\frac{a}{b}\right)$) and get the final result. In the lest step we use the definition of the D-function from subsection 1.3.2:

$$\int_0^d e^{ax} E_1(bx)\, dx = -\frac{1}{a}\ln\left(1-\frac{a}{b}\right) + \frac{e^{ad}}{a} E_1[bd] - \frac{1}{a} E_1[d(b-a)] \tag{67}$$

$$= \frac{1}{a}\left\{ e^{ad} E_1[bd] - E_1[(-a+b)d] - \ln\left(1-\frac{a}{b}\right)\right\} \tag{68}$$

$$= \frac{1}{a}\left(e^{(a-b)d}\left\{ D[bd] - D[(-a+b)d]\right\} - \ln\left(a-\frac{a}{b}\right)\right) \tag{69}$$

We have to apply the following restrictions:

$$a > 0 \tag{70a}$$

$$b > 0 \tag{70b}$$

$$a < b \tag{70c}$$

$$bd > 0 \overset{(70b)}{\Rightarrow} d > 0 \tag{70d}$$

$$d(b-a) > 0 \qquad \text{(redundant)} \tag{70e}$$

Equation (69) is equation (A2b) from the appendix of [De 90]. It is the special case $Y(a, b, 0, d) \coloneqq W(a, b, d)$.

3. References

[De 90] D. K. G. De Boer. 'Calculation of x-ray fluorescence intensities from bulk and multilayer samples'. In: *X-Ray Spectrometry* 19.3 (1990), pp. 145–154. URL: http://onlinelibrary.wiley.com/doi/10.1002/xrs.1300190312/abstract.

[GN69] Murray Geller and Edward W. Ng. 'A table of integrals of the exponential integral'. In: *Journal of research of the National Bureau of Standards B* 73 (1969), pp. 191–210. URL: http://nvlpubs.nist.gov/nistpubs/jres/73B/jresv73Bn3p191_A1b.pdf (visited on 23/12/2015).

[Sch43] Oskar Schlömilch. *Beiträge zur Theorie bestimmter Integrale*. Jena: Friedrich Frommann, 1843. URL: http://www.mdz-nbn-resolving.de/urn/resolver.pl?urn=urn:nbn:de:bvb:12-bsb10053982-5.

A. Unused stuff

A.1. Definition of Logarithmic Integral $li\left(x\right)$

$$li\left(x\right) = \int_0^x \frac{1}{\ln t} dt, \qquad (x > 0) \tag{71}$$

Note that for $x = 1$ the integral diverges (because of the singularity of the integrand $\frac{1}{\ln t}$) and for $x > 1$ the Cauchy principle value has to be employed to interpret the integral. From the definition of the exponential integral $E_1\left(x\right)$ in equation (1) one can see the following relation, when utilizing the substitution $t = e^{-w}$:

$$\begin{aligned} li\left(e^{-x}\right) &= \int_0^{e^{-x}} \frac{1}{\ln t} dt & (72) \\ &= \int_\infty^x \left(-\frac{1}{w}\right)\left(-e^{-w} dw\right) & (73) \\ &= -\int_x^\infty \frac{e^{-w}}{w} dw & (74) \\ &= -E_1\left(x\right) & (75) \end{aligned}$$

A.2. Definition of Gamma function $\Gamma\left(x\right)$

$$\Gamma\left(x\right) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad (x > 0) \tag{76}$$

A.3. More general derivation of section 2.3.4

This more general derivation of subsection 2.3.4 is based on that of [Sch43, pp. 73 sq.]. Start with the integral representation $E_1\left(bx\right) = e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt$ from subsection 2.1.3, which is only valid for $bx > 0$. Then multiply the equation with $x^{\mu-1} e^{ax} dx$, while integrating both sides from $x = 0$ to infinity:

$$\begin{aligned} \int_0^\infty x^{\mu-1} e^{ax} E_1\left(bx\right) dx &= \int_0^\infty x^{\mu-1} e^{ax} \left[e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt\right] dx, \qquad (b > 0) & (77) \\ &= \int_0^\infty \frac{dt}{1+t} \int_0^\infty x^{\mu-1} e^{-(-a+b+bt)x} dx & (78) \end{aligned}$$

In the last step we just changed the integration order. Now we take a look at the integral over x. Using the substitution $kx = u$, and also taking advantage of the abbreviation $k = -a + b + bt$, we get:

$$\int_0^\infty x^{\mu-1}e^{-kx}dx = \int_0^\infty \left(\frac{u}{k}\right)^{\mu-1} e^{-u}\left(\frac{1}{k}du\right), \qquad (k>0) \tag{79}$$

$$= k^{-\mu}\int_0^\infty u^{\mu-1}e^{-u}du \tag{80}$$

$$= \frac{\Gamma(\mu)}{k^\mu}, \qquad (k>0, \mu>0) \tag{81}$$

Here we used the definition of the Gamma function $\Gamma(\mu)$ from equation (76), which is only valid for $\mu > 0$. When applying the requirement $k > 0$ to the abbreviation $k = -a + b + bt > 0$, we can see this yields $-a + b > 0$ for the integration limit $t = 0$, or equivalently $a < b$. On this result we want to use the substitution $\frac{1}{1+t} = z$, i.e. $t = \frac{1}{z} - 1$:

$$\int_0^\infty x^{\mu-1}e^{ax}E_1(bx)\,dx = \Gamma(\mu)\int_0^\infty \frac{1}{(1+t)}\frac{1}{(-a+b(1+t))^\mu}dt \tag{82}$$

$$= \Gamma(\mu)\int_1^0 z\frac{1}{\left(-a+\frac{b}{z}\right)^\mu}\left(-\frac{1}{z^2}dz\right) \tag{83}$$

$$= \Gamma(\mu)\int_0^1 \frac{z^{-1}}{\left(\frac{b}{z}-a\right)^\mu}\frac{z^\mu}{z^\mu}dz \tag{84}$$

$$= \Gamma(\mu)\int_0^1 \frac{z^{\mu-1}}{(b-az)^\mu}dz, \qquad (a<b, \mu>0) \tag{85}$$

Now, we can just set $\mu = 1$, use $\Gamma(1) = 1$ and employ the substitution $b - az = x$ to get the desired end result of equation (61):

$$\int_0^\infty e^{ax}E_1(bx) = \Gamma(1)\int_0^1 \frac{1}{b-az}dz \tag{86}$$

$$= \int_b^{b-a}\frac{1}{x}\left(-\frac{1}{a}dx\right) \tag{87}$$

$$= -\frac{1}{a}\left[\ln x\right]_{x=b}^{b-a} \tag{88}$$

$$= -\frac{1}{a}\left[\ln(b-a) - \ln(b)\right] \tag{89}$$

$$= -\frac{1}{a}\ln\left(1-\frac{a}{b}\right), \qquad (a<b, 0<b) \tag{90}$$