Exponential Integral $E_1(x)$

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Contents

1. Pre-stuff

1.1. Informational

- Github: `https://github.com/AndreWaehlisch/Exponential-Integral`
- Some links about the exponential integral:
  - wolfram.com:
    - E_1 and Ei:
      - `http://mathworld.wolfram.com/ExponentialIntegral.html`
      - `http://functions.wolfram.com/GammaBetaErf/ExpIntegralEi/`
      - `http://www.wolframalpha.com/input/?i=ExpIntegralE[1,x]`
      - `http://www.wolframalpha.com/input/?i=ExpIntegralEi[x]`
    - E_n:
      - `http://mathworld.wolfram.com/En-Function.html`
      - `http://functions.wolfram.com/GammaBetaErf/ExpIntegralE/`

1.2. Mathematical notes

1. All variables and parameters are considered real, if not stated otherwise.
2. The notation $[f(x)]_{x=a}^{b}$ means $f(b) - (a)$. In the same sense, if for example $b = \infty$, interpret as $[f(x)]_{x=a}^{\infty} = \lim_{x\to\infty}(f(x)) - f(a)$.

1.3. Definitions

1.3.1. Definition of Exponential Integral $E_1(x)$

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw, \qquad (x > 0) \tag{1}$$

1.3.2. Definition of $D(x)$

$$D(x) = e^x E_1(x), \qquad (x > 0) \tag{2}$$

1.3.3. Definition of $Y(a, b, c, d)$

$$Y(a, b, c, d) = \int_0^d e^{ax} E_1[b(x + c)]\, dx, \qquad (d > 0, b > 0, c > 0) \tag{3}$$

2. Exponential Integral $E_1(x)$

2.1. Integral Representations

2.1.1. $E_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt$, source is eq. (2b) of [De 90]

Use substitution $w = xt$:

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw, \qquad (x > 0) \tag{4}$$

$$= \int_1^\infty \frac{e^{-xt}}{t} dt \tag{5}$$

2.1.2. $E_1(x) = e^{-x} \int_0^1 \frac{1}{x - \ln t} dt$, source is eq. (4) in sec. 3.3 of [GN69]

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw, \qquad (x > 0) \tag{6}$$

$$= e^{-x} \int_x^\infty \frac{e^{x-w}}{w} dw \tag{7}$$

First use substitution $(x - w) = -y$, then use substitution $y = -\ln t$:

$$e^{-x} \int_x^\infty \frac{e^{x-w}}{w} dw = e^{-x} \int_0^\infty \frac{e^{-y}}{x + y} dy \tag{8}$$

$$= e^{-x} \int_1^0 \frac{t}{x - \ln t} \left(\frac{-1}{t} \right) dt \tag{9}$$

$$= e^{-x} \int_0^1 \frac{1}{x - \ln t} dt \tag{10}$$

2.1.3. $E_1(x) = e^{-x} \int_0^\infty \frac{e^{-xt}}{t+1} dt$

Use substitution $\frac{w}{x} - 1 = t$:

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw \tag{11}$$

$$= \int_0^\infty \frac{e^{-x(t+1)}}{x(t+1)} (x dt) \tag{12}$$

$$= e^{-x} \int_0^\infty \frac{e^{-xt}}{t + 1} dt \tag{13}$$

2.1.4. $E_1(x) = \int_0^{\frac{\pi}{2}} \tan(\alpha) e^{-\frac{x}{\cos\alpha}} d\alpha$, this is eq. (2a) from [De 90]

Starting with the representation of equation (5) and using the substitution $t = \frac{1}{\cos\alpha}$ we immediately arrive at the result:

$$E_1(x) = \int_1^{\infty} \frac{e^{-xt}}{t} dt \tag{14}$$

$$= \int_0^{\frac{\pi}{2}} \cos(\alpha) e^{-\frac{x}{\cos\alpha}} \frac{\tan\alpha}{\cos\alpha} d\alpha \tag{15}$$

$$= \int_0^{\frac{\pi}{2}} \tan(\alpha) e^{-\frac{x}{\cos\alpha}} d\alpha \tag{16}$$

2.2. Special Values

2.2.1. $\frac{d}{dx} E_1[b(x+c)] = -\frac{e^{-b(x+c)}}{x+c}$

Start with integral representation of $E_1(x)$ found in subsection 2.1.1:

$$\frac{d}{dx} E_1[b(x+c)] = \frac{d}{dx} \int_1^{\infty} \frac{e^{-b(x+c)t}}{t} dt \tag{17}$$

$$= \int_1^{\infty} (-bt) \frac{e^{-b(x+c)t}}{t} dt \tag{18}$$

$$= -b \int_1^{\infty} e^{-b(x+c)t} dt \tag{19}$$

$$= \frac{-b}{-b(x+c)} \left[e^{-b(x+c)t} \right]_{t=1}^{\infty} \tag{20}$$

$$= -\frac{e^{-b(x+c)}}{x+c} \qquad (b(x+c) > 0) \tag{21}$$

2.3. Integrals involving $E_1(x)$

Integral	Y-representation	Location	Location in [De 90]
$\int_0^d e^{ax} E_1[b(x+c)]\, dx$	$Y(a,b,c,d)$	(34), p. 6	eq. (A2a)
$\int_0^d e^{ax} E_1[bd]\, dx$	$Y(a,b,0,d)$	(68), p. 9	eq. (A2b)
$\int_0^{\infty} e^{ax} E_1[b(x+c)]\, dx$	$Y(a,b,c,\infty)$	(45), p. 7	eq. (A2c)
$\int_0^{\infty} e^{ax} E_1[bx]\, dx$	$Y(a,b,0,\infty)$	(55), p. 8	eq. (A2d)

Table 1: Table of integrals encountered in [De 90]

2.3.1. $\int_a^b \frac{e^{-x}}{x} dx = E_1(a) - E_1(b)$

Start with the negative derivative from subsection 2.2.1, while setting $b = 1$ and $c = 0$:

$$-\frac{d}{dx} E_1(x) = \frac{e^{-x}}{x} \tag{22}$$

Now, just apply the *fundamental theorem of calculus* (in German: *Hauptsatz der Differential- und Integralrechnung*): For certain conditions to $f(x), a, b$ and if $\frac{d}{dx} F(x) = f(x)$, then $\int_a^b f(x)\, dx = F(b) - F(a)$. Applying this to equation (22) leads to the desired result. We assume $a \neq b$, so that the integral is not trivially zero. Also required due to the infinite discontinuity of the integrand at $x = 0$ is that a and b are *both* positive or *both* negative, furthermore both must be non-zero:

$$\int_a^b \frac{e^{-x}}{x} dx = E_1(a) - E_1(b) \qquad (a \neq b, ab > 0) \tag{23}$$

2.3.2. $\int_0^d e^{ax} E_1[b(x+c)]\, dx = \frac{e^{-bc}}{a} \left(e^{(a-b)d} \{D[b(c+d)] - D[(-a+b)(c+d)]\} - D[bc] + D[(-a+b)c]\right)$

This is the general case of $Y(a,b,c,d) = \int_0^d e^{ax} E_1[b(x+c)]\, dx$, found in [De 90]. Special cases, e.g. $c = 0$ are considered in the following subsections. Now, first integrate by parts, using the derivative $\frac{d}{dx} E_1[b(x+c)] = -\frac{e^{-b(x+c)}}{x+c}$ from subsection 2.2.1:

$$\int_0^d e^{ax} E_1[b(x+c)] = \left\{\frac{e^{ax}}{a} E_1[b(x+c)]\right\}_{x=0}^{d} - \int_0^d \frac{e^{ax}}{a} \frac{d}{dx} E_1[b(x+c)]\, dx \tag{24}$$

$$= \frac{e^{ad}}{a} E_1[b(d+c)] - \frac{E_1[bc]}{a} - \int_0^d \frac{e^{ax}}{a} \left\{-\frac{e^{-b(x+c)}}{x+c}\right\} dx \tag{25}$$

$$= \frac{e^{ad}}{a} E_1[b(d+c)] - \frac{E_1[bc]}{a} + \frac{e^{-bc}}{a} \int_0^d \frac{e^{-x(b-a)}}{x+c} dx \tag{26}$$

To simplify the integral in the last summand, first use the abbreviation $\phi = b - a$ and the substitution $x + c = y$, then use the substitution $(b-a)y = \phi y = t$. In the final step, to get to the intermediate result of equation (31), use the integral $\int_a^b \frac{e^{-x}}{x} dx = E_1(a) - E_1(b)$, found in subsection 2.3.1:

$$
\begin{aligned}
\int_0^d \frac{e^{-x(b-a)}}{x+c} dx &= \int_0^d \frac{e^{-x\phi}}{x+c} dx && (27)\\
&= \int_c^{d+c} \frac{e^{-(y-c)\phi}}{y} dy && (28)\\
&= e^{c\phi} \int_c^{d+c} \frac{e^{-y\phi}}{y} dy && (29)\\
&= e^{c\phi} \int_{\phi c}^{\phi(d+c)} \frac{e^{-t}}{t\phi^{-1}} \phi^{-1} dt && (30)\\
&= e^{c\phi} \left\{E_1 [\phi c] - E_1 [\phi (d+c)]\right\} && (31)
\end{aligned}
$$

To arrive at the desired result we just put the intermediate step (31) into equation (26), and restore the abbreviation $\phi = b - a$. Then rearrange the terms to get to equation (A2a) from the appendix of [De 90]. In the last steps use the definition of $D(x)$ from eq. (2), which can be found in subsection 1.3.2 on page 2:

$$
\begin{aligned}
\int_0^d e^{ax} E_1 [b(x+c)] =& \frac{e^{ad}}{a} E_1 [b(d+c)] - \frac{E_1 [bc]}{a} \\
&+ \frac{e^{-bc} e^{c(b-a)}}{a} \left\{E_1 [(b-a)c] - E_1 [(b-a)(d+c)]\right\} && (32)\\
=& \frac{1}{a} \Big\{ e^{ad} E_1 [b(c+d)] - e^{-ac} E_1 [(-a+b)(c+d)] \\
&- E_1 [bc] + e^{-ac} E_1 [(-a+b)c] \Big\} && (33)\\
=& \frac{e^{-bc}}{a} \Big(e^{(a-b)d} \left\{D [b(c+d)] - D [(-a+b)(c+d)]\right\} \\
&- D [bc] + D [(-a+b)c] \Big) && (34)
\end{aligned}
$$

Note that – as always when using the definition of E_1 from equation (1) – the arguments of the exponential integrals $E_1(x)$ and $D(x)$ must be positive:

$$
\begin{aligned}
b(c+d) &> 0 && (35a)\\
(-a+b)(c+d) &> 0 && (35b)\\
bc &> 0 && (35c)\\
(-a+b)c &> 0 && (35d)
\end{aligned}
$$

We can see, that for the interesting special case $c = 0$, the result is not defined: It violates the conditions (35c) and (35d).

2.3.3. $\int_0^\infty e^{ax} E_1 [b(x+c)] \, dx = \frac{1}{a} e^{-bc} \{-D[bc] + D[c(-a+b)]\}$

Start with the integral representation $E_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt$ from subsection 2.1.1:

$$\int_0^\infty e^{ax} E_1 [b(x+c)] \, dx = \int_0^\infty e^{ax} \int_1^\infty \frac{e^{-b(x+c)t}}{t} dt x \tag{36}$$

$$= \int_1^\infty dt \frac{e^{-bct}}{t} \int_0^\infty dx e^{x(a-bt)} \tag{37}$$

$$= \int_1^\infty dt \frac{e^{-bct}}{t} \left(-\frac{1}{a-bt}\right), \qquad (a - bt < 0) \tag{38}$$

$$= -\int_1^\infty dt \frac{e^{-bct}}{a} \frac{a - bt + bt}{t(a-bt)} \tag{39}$$

$$= -\int_1^\infty dt \frac{e^{-bct}}{a} \left(\frac{1}{t} + \frac{b}{a-bt}\right) \tag{40}$$

The first term in the last equation we recognize as $E_1(bc)$. For the second summand we first apply the substitution $-(a-bt) = y$ and then $\frac{y}{b-a} = x$ to also get an exponential integral:

$$\int_0^\infty e^{ax} E_1 [b(x+c)] \, dx = -\frac{1}{a} \left\{ E_1 [bc] - \int_{b-a}^\infty \frac{b e^{-bc \frac{a+y}{b}}}{y} \frac{dy}{b} \right\} \tag{41}$$

$$= -\frac{1}{a} \left\{ E_1 [bc] - e^{-ac} \int_{b-a}^\infty \frac{e^{-cy}}{y} dy \right\} \tag{42}$$

$$= -\frac{1}{a} \left\{ E_1 [bc] - e^{-ac} \int_1^\infty \frac{e^{-c(b-a)x}}{(b-a)x} (b-a) \, dx \right\} \tag{43}$$

$$= -\frac{1}{a} \{ E_1 [bc] - e^{-ac} E_1 [c(-a+b)] \} \tag{44}$$

$$= \frac{1}{a} e^{-bc} \{-D[bc] + D[c(-a+b)]\} \tag{45}$$

In the end we arrive at the equation (A2c) found in the appendix of [De 90]. We have to comply to the following restrictions:

$$a - bt < 0 \overset{(38)}{\Rightarrow} a - b < 0 \Rightarrow a < b \tag{46a}$$

$$bc > 0 \tag{46b}$$

2.3.4. $\int_0^\infty e^{ax} E_1 (bx) \, dx = -\frac{1}{a} \ln\left(1 - \frac{a}{b}\right)$

Start with the integral representation $E_1(bx) = e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt$ from subsection 2.1.3, which is only valid for $bx > 0$. Then multiply the equation with $e^{ax} dx$, while integrating both sides from $x = 0$ to infinity. After that just change the integration order:

$$\int_0^\infty e^{ax} E_1(bx)\, dx = \int_0^\infty e^{ax} \left[e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt \right] dx, \qquad (b > 0) \tag{47}$$

$$= \int_0^\infty \frac{dt}{1+t} \int_0^\infty e^{-(-a+b+bt)x} dx \tag{48}$$

$$= \int_0^\infty \frac{1}{1+t} \frac{1}{(-a+b+bt)} dt, \qquad (-a+b+bt > 0) \tag{49}$$

We can see the requirement $-a+b+bt > 0$ yields $-a+b > 0$ for the integration limit $t = 0$, or equivalently $a < b$. On this result we want to use the substitution $\frac{1}{1+t} = z$. Then, in the next step, substitute $b - az = x$:

$$\int_0^\infty e^{ax} E_1(bx)\, dx = \int_1^0 z \frac{1}{\left(-a + \frac{b}{z}\right)} \left(-\frac{1}{z^2} dz \right) \tag{50}$$

$$= \int_0^1 \frac{1}{(b - az)} dz \tag{51}$$

$$= \int_b^{b-a} \frac{1}{x} \left(-\frac{1}{a} dx \right) \tag{52}$$

$$= -\frac{1}{a} \left[\ln x \right]_{x=b}^{b-a} \tag{53}$$

$$= -\frac{1}{a} \left[\ln(b-a) - \ln(b) \right] \tag{54}$$

$$= -\frac{1}{a} \ln\left(1 - \frac{a}{b}\right), \qquad (a < b, 0 < b) \tag{55}$$

This end result is equation (A2d) from [De 90]. In terms of the Y-function this is $Y(a, b, 0, \infty)$. For a more general derivation based on that of [Sch43, pp. 73 sq.] see section A.3 in the appendix. The equation can also be found as equation (R10) in [She55].

2.3.5. $\int_0^d e^{ax} E_1(bx)\, dx = \frac{1}{a} \left(e^{(a-b)d} \left\{ D[bd] - D[(-a+b)d] \right\} - \ln\left[1 - \frac{a}{b}\right] \right)$

To start we plug in the integral representation $E_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt$ from subsection 2.1.1:

$$\int_0^d e^{ax} E_1(bx)\,dx = \int_0^d e^{ax}\left(\int_1^\infty \frac{e^{-bxt}}{t}dt\right)dx \tag{56}$$
$$= \int_1^\infty \frac{dt}{t}\int_0^d dx e^{(a-bt)x} \tag{57}$$
$$= \int_1^\infty \frac{dt}{t}\left(\frac{1}{a-bt}e^{(a-bt)d} - \frac{1}{a-bt}\right), \qquad (a<b) \tag{58}$$
$$= \int_1^\infty \frac{dt}{a}\frac{a}{t(a-bt)}e^{(a-bt)d} - \int_1^\infty dt\frac{1}{t(a-bt)} \tag{59}$$
$$= \int_1^\infty \frac{dt}{a}\frac{a-bt+bt}{t(a-bt)}e^{(a-bt)d} - \int_1^\infty dt\frac{1}{t(a-bt)} \tag{60}$$
$$= \int_1^\infty \frac{dt}{a}\left(\frac{1}{t}+\frac{b}{a-bt}\right)e^{(a-bt)d} - \int_1^\infty dt\frac{1}{t(a-bt)} \tag{61}$$
$$= \frac{e^{ad}}{a}\int_1^\infty dt\frac{e^{-dbt}}{t} + \frac{1}{a}\int_1^\infty dt\frac{be^{(a-bt)d}}{a-bt} - \int_1^\infty dt\frac{1}{t(a-bt)} \tag{62}$$

Looking at the first term in the last equation we see the integral representation of $E_1(bd)$. To simplify to second summand we first apply the substitution $-(a-bt)=y$ and after applying $\frac{y}{b-a}=x$ see the integral representation of $E_1[d(b-a)]$. And finally for the last term in the equation we use the substitution $\frac{1}{a-bt}=z$ and then apply $1-az=w$:

$$\int_0^d e^{ax}E_1(bx)\,dx = \frac{e^{ad}}{a}E_1[bd] + \frac{1}{a}\int_{b-a}^\infty dy\frac{e^{-yd}}{-y} - \int_{(a-b)^{-1}}^0 \frac{b}{a-\frac{1}{z}}\frac{z}{b}\frac{dz}{z^2} \tag{63}$$
$$= \frac{e^{ad}}{a}E_1[bd] - \frac{1}{a}\int_1^\infty dx\frac{e^{-d(b-a)x}}{(b-a)x}(b-a) - \int_1^{1-\frac{a}{a-b}}\frac{1}{w}\left(-\frac{dw}{a}\right) \tag{64}$$
$$= \frac{e^{ad}}{a}E_1[bd] - \frac{1}{a}E_1[(b-a)d] + \frac{1}{a}\ln\left[1-\frac{a}{a-b}\right] \tag{65}$$
$$= \frac{e^{ad}}{a}E_1[bd] - \frac{1}{a}E_1[(-a+b)d] + \frac{1}{a}\ln\left[\frac{a-b-a}{a-b}\right] \tag{66}$$
$$= \frac{1}{a}\left\{e^{ad}E_1[bd] - E_1[(-a+b)d] - \ln\left[-\frac{a-b}{b}\right]\right\} \tag{67}$$
$$= \frac{1}{a}\left(e^{(a-b)d}\left\{D[bd] - D[(-a+b)d]\right\} - \ln\left[1-\frac{a}{b}\right]\right) \tag{68}$$

We have to apply the following restrictions:

$$b > 0 \tag{69a}$$
$$b > a \tag{69b}$$
$$bd > 0 \overset{(69b)}{\Rightarrow} d > 0 \tag{69c}$$

Equation (68) is equation (A2b) from the appendix of [De 90]. It is the special case $Y(a, b, 0, d) \coloneqq W(a, b, d)$. It can also be found as equation (R11) and (R22) of [She55].

3. References

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A. Unused stuff

A.1. Definition of Logarithmic Integral $li\left(x\right)$

$$li\left(x\right) = \int_0^x \frac{1}{\ln t} dt, \qquad (x > 0) \tag{70}$$

Note that for $x = 1$ the integral diverges (because of the singularity of the integrand $\frac{1}{\ln t}$) and for $x > 1$ the Cauchy principle value has to be employed to interpret the integral. From the definition of the exponential integral $E_1\left(x\right)$ in equation (1) one can see the following relation, when utilizing the substitution $t = e^{-w}$:

$$li\left(e^{-x}\right) = \int_0^{e^{-x}} \frac{1}{\ln t} dt \tag{71}$$

$$= \int_\infty^x \left(-\frac{1}{w}\right)\left(-e^{-w} dw\right) \tag{72}$$

$$= -\int_x^\infty \frac{e^{-w}}{w} dw \tag{73}$$

$$= -E_1\left(x\right) \tag{74}$$

A.2. Definition of Gamma function $\Gamma\left(x\right)$

$$\Gamma\left(x\right) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad (x > 0) \tag{75}$$

A.3. More general derivation of section 2.3.4

This more general derivation of subsection 2.3.4 is based on that of [Sch43, pp. 73 sq.]. Start with the integral representation $E_1\left(bx\right) = e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt$ from subsection 2.1.3, which is only valid for $bx > 0$. Then multiply the equation with $x^{\mu-1} e^{ax} dx$, while integrating both sides from $x = 0$ to infinity:

$$\int_0^\infty x^{\mu-1} e^{ax} E_1\left(bx\right) dx = \int_0^\infty x^{\mu-1} e^{ax} \left[e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt\right] dx, \qquad (b > 0) \tag{76}$$

$$= \int_0^\infty \frac{dt}{1+t} \int_0^\infty x^{\mu-1} e^{-(-a+b+bt)x} dx \tag{77}$$

In the last step we just changed the integration order. Now we take a look at the integral over x. Using the substitution $kx = u$, and also taking advantage of the abbreviation $k = -a + b + bt$, we get:

$$\int_0^\infty x^{\mu-1}e^{-kx}dx = \int_0^\infty \left(\frac{u}{k}\right)^{\mu-1} e^{-u}\left(\frac{1}{k}du\right), \qquad (k>0) \tag{78}$$

$$= k^{-\mu}\int_0^\infty u^{\mu-1}e^{-u}du \tag{79}$$

$$= \frac{\Gamma(\mu)}{k^\mu}, \qquad (k>0, \mu>0) \tag{80}$$

Here we used the definition of the Gamma function $\Gamma(\mu)$ from equation (75), which is only valid for $\mu > 0$. When applying the requirement $k > 0$ to the abbreviation $k = -a + b + bt > 0$, we can see this yields $-a + b > 0$ for the integration limit $t = 0$, or equivalently $a < b$. On this result we want to use the substitution $\frac{1}{1+t} = z$, i.e. $t = \frac{1}{z} - 1$:

$$\int_0^\infty x^{\mu-1}e^{ax}E_1(bx)\,dx = \Gamma(\mu)\int_0^\infty \frac{1}{(1+t)}\frac{1}{(-a+b(1+t))^\mu}dt \tag{81}$$

$$= \Gamma(\mu)\int_1^0 z\frac{1}{\left(-a+\frac{b}{z}\right)^\mu}\left(-\frac{1}{z^2}dz\right) \tag{82}$$

$$= \Gamma(\mu)\int_0^1 \frac{z^{-1}}{\left(\frac{b}{z}-a\right)^\mu}\frac{z^\mu}{z^\mu}dz \tag{83}$$

$$= \Gamma(\mu)\int_0^1 \frac{z^{\mu-1}}{(b-az)^\mu}dz, \qquad (a<b, \mu>0) \tag{84}$$

Now, we can just set $\mu = 1$, use $\Gamma(1) = 1$ and employ the substitution $b - az = x$ to get the desired end result of equation (55):

$$\int_0^\infty e^{ax}E_1(bx) = \Gamma(1)\int_0^1 \frac{1}{b-az}dz \tag{85}$$

$$= \int_b^{b-a}\frac{1}{x}\left(-\frac{1}{a}dx\right) \tag{86}$$

$$= -\frac{1}{a}\left[\ln x\right]_{x=b}^{b-a} \tag{87}$$

$$= -\frac{1}{a}\left[\ln(b-a) - \ln(b)\right] \tag{88}$$

$$= -\frac{1}{a}\ln\left(1-\frac{a}{b}\right), \qquad (a<b, 0<b) \tag{89}$$