Exponential Integral $E_1(x)$

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Contents

1. Pre-stuff

1.1. Informational

- Github: `https://github.com/AndreWaehlisch/Exponential-Integral`
- Some links about the exponential integral:
  - wolfram.com:
    * E_1 and Ei:
      · `http://mathworld.wolfram.com/ExponentialIntegral.html`
      · `http://functions.wolfram.com/GammaBetaErf/ExpIntegralEi/`
      · `http://www.wolframalpha.com/input/?i=ExpIntegralE[1,x]`
      · `http://www.wolframalpha.com/input/?i=ExpIntegralEi[x]`
    * E_n:
      · `http://mathworld.wolfram.com/En-Function.html`
      · `http://functions.wolfram.com/GammaBetaErf/ExpIntegralE/`

1.2. Mathematical notes

1. All variables and parameters are considered real, if not stated otherwise.
2. The notation $[f(x)]_{x=a}^{b}$ means $f(b) - (a)$. In the same sense, if for example $b = \infty$, interpret as $[f(x)]_{x=a}^{\infty} = \lim_{x\to\infty}(f(x)) - f(a)$.

1.3. Definitions

1.3.1. Definition of Exponential Integral $E_1(x)$

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw, \qquad (x > 0) \tag{1}$$

1.3.2. Definition of $D(x)$

$$D(x) = e^x E_1(x), \qquad (x > 0) \tag{2}$$

1.3.3. Definition of $Y(a,b,c,d)$

$$Y(a,b,c,d) = \int_0^d e^{ax} E_1[b(x+c)]\, dx, \qquad (d > 0, b > 0, c > 0) \tag{3}$$

2. Exponential Integral $E_1(x)$

2.1. Integral Representations

2.1.1. $E_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt$, source is eq. (2b) of [De 90]

Use substitution $w = xt$:

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw, \qquad (x > 0) \tag{4}$$

$$= \int_1^\infty \frac{e^{-xt}}{t} dt \tag{5}$$

2.1.2. $E_1(x) = e^{-x} \int_0^1 \frac{1}{x - \ln t} dt$, source is eq. (4) in sec. 3.3 of [GN69]

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw, \qquad (x > 0) \tag{6}$$

$$= e^{-x} \int_x^\infty \frac{e^{x-w}}{w} dw \tag{7}$$

First use substitution $(x - w) = -y$, then use substitution $y = -\ln t$:

$$e^{-x} \int_x^\infty \frac{e^{x-w}}{w} dw = e^{-x} \int_0^\infty \frac{e^{-y}}{x + y} dy \tag{8}$$

$$= e^{-x} \int_1^0 \frac{t}{x - \ln t} \left(\frac{-1}{t} \right) dt \tag{9}$$

$$= e^{-x} \int_0^1 \frac{1}{x - \ln t} dt \tag{10}$$

2.1.3. $E_1(x) = e^{-x} \int_0^\infty \frac{e^{-xt}}{t+1} dt$

Use substitution $\frac{w}{x} - 1 = t$:

$$E_1(x) = \int_x^\infty \frac{e^{-w}}{w} dw \tag{11}$$

$$= \int_0^\infty \frac{e^{-x(t+1)}}{x(t+1)} (x dt) \tag{12}$$

$$= e^{-x} \int_0^\infty \frac{e^{-xt}}{t+1} dt \tag{13}$$

2.2. Special Values

2.2.1. $\frac{d}{dx}E_1\left[b\left(x+c\right)\right]=-\frac{e^{-b(x+c)}}{x+c}$

Start with integral representation of $E_1\left(x\right)$ found in subsection 2.1.1:

$$\begin{aligned}
\frac{d}{dx}E_1\left[b\left(x+c\right)\right] &= \frac{d}{dx}\int_1^\infty \frac{e^{-b(x+c)t}}{t}dt && (14)\\
&= \int_1^\infty \left(-bt\right)\frac{e^{-b(x+c)t}}{t}dt && (15)\\
&= -b\int_1^\infty e^{-b(x+c)t}dt && (16)\\
&= \frac{-b}{-b\left(x+c\right)}\left[e^{-b(x+c)t}\right]_{t=1}^{\infty} && (17)\\
&= -\frac{e^{-b(x+c)}}{x+c} \qquad \left(b\left(x+c\right)>0\right) && (18)
\end{aligned}$$

2.3. Integrals involving $E_1\left(x\right)$

Integral	Y-representation	Location	Location in [De 90]
$\int_0^d e^{ax}E_1\left[b\left(x+c\right)\right]dx$	$Y\left(a,b,c,d\right)$	(31), p. 6	eq. (A2a)
$\int_0^d e^{ax}E_1\left[bd\right]dx$	$Y\left(a,b,0,d\right)$	(65), p. 9	eq. (A2b)
$\int_0^\infty e^{ax}E_1\left[b\left(x+c\right)\right]dx$	$Y\left(a,b,c,\infty\right)$	(42), p. 7	eq. (A2c)
$\int_0^\infty e^{ax}E_1\left[bx\right]dx$	$Y\left(a,b,0,\infty\right)$	(52), p. 8	eq. (A2d)

Table 1: Table of integrals encountered in [De 90]

2.3.1. $\int_a^b \frac{e^{-x}}{x}dx=E_1\left(a\right)-E_1\left(b\right)$

Start with the negative derivative from subsection 2.2.1, while setting $b=1$ and $c=0$:

$$-\frac{d}{dx}E_1\left(x\right)=\frac{e^{-x}}{x} \tag{19}$$

Now, just apply the *fundamental theorem of calculus* (in German: *Hauptsatz der Differential- und Integralrechnung*): For certain conditions to $f\left(x\right),a,b$ and if $\frac{d}{dx}F\left(x\right)=f\left(x\right)$, then $\int_a^b f\left(x\right)dx=F\left(b\right)-F\left(a\right)$. Applying this to equation (19) leads to the desired result. We assume $a\neq b$, so that the integral is not trivially zero. Also required due to the infinite discontinuity of the integrand at $x=0$ is that a and b are *both* positive or *both* negative, furthermore both must be non-zero:

$$\int_a^b \frac{e^{-x}}{x} dx = E_1(a) - E_1(b) \qquad (a \neq b, ab > 0) \tag{20}$$

2.3.2. $\int_0^d e^{ax} E_1[b(x+c)]\, dx =$
$\frac{e^{-bc}}{a} \left(e^{(a-b)d} \{D[b(c+d)] - D[(-a+b)(c+d)]\} - D[bc] + D[(-a+b)c]\right)$

This is the general case of $Y(a,b,c,d) = \int_0^d e^{ax} E_1[b(x+c)]\, dx$, found in [De 90]. Special cases, e.g. $c = 0$ are considered in the following subsections. Now, first integrate by parts, using the derivative $\frac{d}{dx} E_1[b(x+c)] = -\frac{e^{-b(x+c)}}{x+c}$ from subsection 2.2.1:

$$\int_0^d e^{ax} E_1[b(x+c)] = \left\{\frac{e^{ax}}{a} E_1[b(x+c)]\right\}_{x=0}^{d} - \int_0^d \frac{e^{ax}}{a} \frac{d}{dx} E_1[b(x+c)]\, dx \tag{21}$$

$$= \frac{e^{ad}}{a} E_1[b(d+c)] - \frac{E_1[bc]}{a} - \int_0^d \frac{e^{ax}}{a} \left\{-\frac{e^{-b(x+c)}}{x+c}\right\} dx \tag{22}$$

$$= \frac{e^{ad}}{a} E_1[b(d+c)] - \frac{E_1[bc]}{a} + \frac{e^{-bc}}{a} \int_0^d \frac{e^{-x(b-a)}}{x+c} dx \tag{23}$$

To simplify the integral in the last summand, first use the abbreviation $\phi = b - a$ and the substitution $x + c = y$, then use the substitution $(b-a)\, y = \phi y = t$. In the final step, to get to the intermediate result of equation (28), use the integral $\int_a^b \frac{e^{-x}}{x} dx = E_1(a) - E_1(b)$, found in subsection 2.3.1:

$$\int_0^d \frac{e^{-x(b-a)}}{x+c} dx = \int_0^d \frac{e^{-x\phi}}{x+c} dx \tag{24}$$

$$= \int_c^{d+c} \frac{e^{-(y-c)\phi}}{y} dy \tag{25}$$

$$= e^{c\phi} \int_c^{d+c} \frac{e^{-y\phi}}{y} dy \tag{26}$$

$$= e^{c\phi} \int_{\phi c}^{\phi(d+c)} \frac{e^{-t}}{t\phi^{-1}} \phi^{-1} dt \tag{27}$$

$$= e^{c\phi} \{E_1[\phi c] - E_1[\phi(d+c)]\} \tag{28}$$

To arrive at the desired result we just put the intermediate step (28) into equation (23), and restore the abbreviation $\phi = b - a$. Then rearrange the terms to get to equation (A2a) from the appendix of [De 90]. In the last steps use the definition of $D(x)$ from eq. (2), which can be found in subsection 1.3.2 on page 2:

$$
\begin{aligned}
\int_0^d e^{ax} E_1\left[b\left(x+c\right)\right] =& \frac{e^{ad}}{a} E_1\left[b\left(d+c\right)\right] - \frac{E_1\left[bc\right]}{a} \\
&+ \frac{e^{-bc}e^{c(b-a)}}{a}\left\{E_1\left[\left(b-a\right)c\right] - E_1\left[\left(b-a\right)\left(d+c\right)\right]\right\}
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
=& \frac{1}{a}\Big\{e^{ad} E_1\left[b\left(c+d\right)\right] - e^{-ac} E_1\left[\left(-a+b\right)\left(c+d\right)\right] \\
&- E_1\left[bc\right] + e^{-ac} E_1\left[\left(-a+b\right)c\right]\Big\}
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
=& \frac{e^{-bc}}{a}\Big(e^{(a-b)d}\left\{D\left[b\left(c+d\right)\right] - D\left[\left(-a+b\right)\left(c+d\right)\right]\right\} \\
&- D\left[bc\right] + D\left[\left(-a+b\right)c\right]\Big)
\end{aligned} \tag{31}
$$

Note that – as always when using the definition of E_1 from equation (1) – the arguments of the exponential integrals $E_1\left(x\right)$ and $D\left(x\right)$ must be positive:

$$
b\left(c+d\right) > 0 \tag{32a}
$$

$$
\left(-a+b\right)\left(c+d\right) > 0 \tag{32b}
$$

$$
bc > 0 \tag{32c}
$$

$$
\left(-a+b\right)c > 0 \tag{32d}
$$

We can see, that for the interesting special case $c = 0$, the result is not defined: It violates the conditions (32c) and (32d).

2.3.3. $\int_0^\infty e^{ax} E_1\left[b\left(x+c\right)\right]dx = \frac{1}{a}e^{-bc}\left\{-D\left[bc\right] + D\left[c\left(-a+b\right)\right]\right\}$

Start with the integral representation $E_1\left(x\right) = \int_1^\infty \frac{e^{-xt}}{t}dt$ from subsection 2.1.1:

$$
\int_0^\infty e^{ax} E_1\left[b\left(x+c\right)\right]dx = \int_0^\infty e^{ax}\int_1^\infty \frac{e^{-b(x+c)t}}{t} dtx \tag{33}
$$

$$
= \int_1^\infty dt \frac{e^{-bct}}{t}\int_0^\infty dx e^{x(a-bt)} \tag{34}
$$

$$
= \int_1^\infty dt \frac{e^{-bct}}{t}\left(-\frac{1}{a-bt}\right), \qquad \left(a-bt<0\right) \tag{35}
$$

$$
= -\int_1^\infty dt \frac{e^{-bct}}{a}\frac{a-bt+bt}{t\left(a-bt\right)} \tag{36}
$$

$$
= -\int_1^\infty dt \frac{e^{-bct}}{a}\left(\frac{1}{t} + \frac{b}{a-bt}\right) \tag{37}
$$

The first term in the last equation we recognize as $E_1\left(bc\right)$. For the second summand we first apply the substitution $-\left(a-bt\right) = y$ and then $\frac{y}{b-a} = x$ to also get an exponential integral:

$$\int_0^\infty e^{ax} E_1\left[b\left(x+c\right)\right] dx = -\frac{1}{a}\left\{E_1\left[bc\right] - \int_{b-a}^\infty \frac{be^{-bc\frac{a+y}{b}}}{y}\frac{dy}{b}\right\} \tag{38}$$

$$= -\frac{1}{a}\left\{E_1\left[bc\right] - e^{-ac}\int_{b-a}^\infty \frac{e^{-cy}}{y}dy\right\} \tag{39}$$

$$= -\frac{1}{a}\left\{E_1\left[bc\right] - e^{-ac}\int_1^\infty \frac{e^{-c(b-a)x}}{\left(b-a\right)x}\left(b-a\right)dx\right\} \tag{40}$$

$$= -\frac{1}{a}\left\{E_1\left[bc\right] - e^{-ac}E_1\left[c\left(-a+b\right)\right]\right\} \tag{41}$$

$$= \frac{1}{a}e^{-bc}\left\{-D\left[bc\right] + D\left[c\left(-a+b\right)\right]\right\} \tag{42}$$

In the end we arrive at the equation (A2c) found in the appendix of [De 90]. We have to comply to the following restrictions:

$$a - bt < 0 \overset{(35)}{\Rightarrow} a - b < 0 \Rightarrow a < b \tag{43a}$$

$$bc > 0 \tag{43b}$$

2.3.4. $\int_0^\infty e^{ax} E_1\left(bx\right) dx = -\frac{1}{a}\ln\left(1-\frac{a}{b}\right)$

Start with the integral representation $E_1\left(bx\right) = e^{-bx}\int_0^\infty \frac{e^{-bxt}}{t+1}dt$ from subsection 2.1.3, which is only valid for $bx > 0$. Then multiply the equation with $e^{ax}dx$, while integrating both sides from $x = 0$ to infinity. After that just change the integration order:

$$\int_0^\infty e^{ax} E_1\left(bx\right) dx = \int_0^\infty e^{ax}\left[e^{-bx}\int_0^\infty \frac{e^{-bxt}}{t+1}dt\right]dx, \qquad \left(b > 0\right) \tag{44}$$

$$= \int_0^\infty \frac{dt}{1+t}\int_0^\infty e^{-(-a+b+bt)x}dx \tag{45}$$

$$= \int_0^\infty \frac{1}{1+t}\frac{1}{\left(-a+b+bt\right)}dt, \qquad \left(-a+b+bt > 0\right) \tag{46}$$

We can see the requirement $-a+b+bt > 0$ yields $-a+b > 0$ for the integration limit $t = 0$, or equivalently $a < b$. On this result we want to use the substitution $\frac{1}{1+t} = z$. Then, in the next step, substitute $b - az = x$:

$$
\begin{aligned}
\int_0^\infty e^{ax} E_1(bx)\,dx &= \int_1^0 z \frac{1}{\left(-a+\frac{b}{z}\right)} \left(-\frac{1}{z^2} dz\right) && (47)\\
&= \int_0^1 \frac{1}{(b-az)} dz && (48)\\
&= \int_b^{b-a} \frac{1}{x} \left(-\frac{1}{a} dx\right) && (49)\\
&= -\frac{1}{a} \left[\ln x\right]_{x=b}^{b-a} && (50)\\
&= -\frac{1}{a} \left[\ln(b-a) - \ln(b)\right] && (51)\\
&= -\frac{1}{a} \ln\left(1-\frac{a}{b}\right), \qquad (a<b, 0<b) && (52)
\end{aligned}
$$

This end result is equation (A2d) from [De 90]. In terms of the Y-function this is $Y(a, b, 0, \infty)$. For a more general derivation based on that of [Sch43, pp. 73 sq.] see section A.3 in the appendix. The equation can also be found as equation (R10) in [She55].

2.3.5. $\int_0^d e^{ax} E_1(bx)\,dx = \frac{1}{a}\left(e^{(a-b)d}\left\{D[bd] - D[(-a+b)d]\right\} - \ln\left[1-\frac{a}{b}\right]\right)$

To start we plug in the integral representation $E_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt$ from subsection 2.1.1:

$$
\begin{aligned}
\int_0^d e^{ax} E_1(bx)\,dx &= \int_0^d e^{ax} \left(\int_1^\infty \frac{e^{-bxt}}{t} dt\right) dx && (53)\\
&= \int_1^\infty \frac{dt}{t} \int_0^d dx e^{(a-bt)x} && (54)\\
&= \int_1^\infty \frac{dt}{t} \left(\frac{1}{a-bt} e^{(a-bt)d} - \frac{1}{a-bt}\right), \qquad (a<b) && (55)\\
&= \int_1^\infty \frac{dt}{a} \frac{a}{t(a-bt)} e^{(a-bt)d} - \int_1^\infty dt \frac{1}{t(a-bt)} && (56)\\
&= \int_1^\infty \frac{dt}{a} \frac{a-bt+bt}{t(a-bt)} e^{(a-bt)d} - \int_1^\infty dt \frac{1}{t(a-bt)} && (57)\\
&= \int_1^\infty \frac{dt}{a} \left(\frac{1}{t} + \frac{b}{a-bt}\right) e^{(a-bt)d} - \int_1^\infty dt \frac{1}{t(a-bt)} && (58)\\
&= \frac{e^{ad}}{a} \int_1^\infty dt \frac{e^{-dbt}}{t} + \frac{1}{a} \int_1^\infty dt \frac{be^{(a-bt)d}}{a-bt} - \int_1^\infty dt \frac{1}{t(a-bt)} && (59)
\end{aligned}
$$

Looking at the first term in the last equation we see the integral representation of $E_1(bd)$. To simplify to second summand we first apply the substitution $-(a-bt) = y$ and after applying $\frac{y}{b-a} = x$ see the integral representation of $E_1[d(b-a)]$. And finally for the last term in the equation we use the substitution $\frac{1}{a-bt} = z$ and then apply $1-az = w$:

$$
\begin{aligned}
\int_0^d e^{ax} E_1(bx)\,dx &= \frac{e^{ad}}{a} E_1[bd] + \frac{1}{a}\int_{b-a}^{\infty} dy \frac{e^{-yd}}{-y} - \int_{(a-b)^{-1}}^{0} \frac{b}{a-\frac{1}{z}} \frac{z}{b} \frac{dz}{z^2} && (60)\\
&= \frac{e^{ad}}{a} E_1[bd] - \frac{1}{a}\int_{1}^{\infty} dx \frac{e^{-d(b-a)x}}{(b-a)\,x}(b-a) - \int_{1}^{1-\frac{a}{a-b}} \frac{1}{w}\left(-\frac{dw}{a}\right) && (61)\\
&= \frac{e^{ad}}{a} E_1[bd] - \frac{1}{a} E_1[(b-a)\,d] + \frac{1}{a}\ln\left[1-\frac{a}{a-b}\right] && (62)\\
&= \frac{e^{ad}}{a} E_1[bd] - \frac{1}{a} E_1[(-a+b)\,d] + \frac{1}{a}\ln\left[\frac{a-b-a}{a-b}\right] && (63)\\
&= \frac{1}{a}\left\{e^{ad} E_1[bd] - E_1[(-a+b)\,d] - \ln\left[-\frac{a-b}{b}\right]\right\} && (64)\\
&= \frac{1}{a}\left(e^{(a-b)d}\left\{D[bd] - D[(-a+b)\,d]\right\} - \ln\left[1-\frac{a}{b}\right]\right) && (65)
\end{aligned}
$$

We have to apply the following restrictions:

$$
\begin{aligned}
b &> 0 && \text{(66a)}\\
b &> a && \text{(66b)}\\
bd > 0 &\overset{(66b)}{\Rightarrow} d > 0 && \text{(66c)}
\end{aligned}
$$

Equation (65) is equation (A2b) from the appendix of [De 90]. It is the special case $Y(a,b,0,d) \coloneqq W(a,b,d)$. It can also be found as equation (R11) and (R22) of [She55].

3. References

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A. Unused stuff

A.1. Definition of Logarithmic Integral $li\left(x\right)$

$$li\left(x\right) = \int_0^x \frac{1}{\ln t} dt, \qquad (x > 0) \tag{67}$$

Note that for $x = 1$ the integral diverges (because of the singularity of the integrand $\frac{1}{\ln t}$) and for $x > 1$ the Cauchy principle value has to be employed to interpret the integral. From the definition of the exponential integral $E_1\left(x\right)$ in equation (1) one can see the following relation, when utilizing the substitution $t = e^{-w}$:

$$\begin{aligned} li\left(e^{-x}\right) &= \int_0^{e^{-x}} \frac{1}{\ln t} dt && (68) \\ &= \int_\infty^x \left(-\frac{1}{w}\right)\left(-e^{-w} dw\right) && (69) \\ &= -\int_x^\infty \frac{e^{-w}}{w} dw && (70) \\ &= -E_1\left(x\right) && (71) \end{aligned}$$

A.2. Definition of Gamma function $\Gamma\left(x\right)$

$$\Gamma\left(x\right) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad (x > 0) \tag{72}$$

A.3. More general derivation of section 2.3.4

This more general derivation of subsection 2.3.4 is based on that of [Sch43, pp. 73 sq.]. Start with the integral representation $E_1\left(bx\right) = e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt$ from subsection 2.1.3, which is only valid for $bx > 0$. Then multiply the equation with $x^{\mu-1} e^{ax} dx$, while integrating both sides from $x = 0$ to infinity:

$$\begin{aligned} \int_0^\infty x^{\mu-1} e^{ax} E_1\left(bx\right) dx &= \int_0^\infty x^{\mu-1} e^{ax} \left[e^{-bx} \int_0^\infty \frac{e^{-bxt}}{t+1} dt\right] dx, \qquad (b > 0) && (73) \\ &= \int_0^\infty \frac{dt}{1+t} \int_0^\infty x^{\mu-1} e^{-(-a+b+bt)x} dx && (74) \end{aligned}$$

In the last step we just changed the integration order. Now we take a look at the integral over x. Using the substitution $kx = u$, and also taking advantage of the abbreviation $k = -a + b + bt$, we get:

$$\int_0^\infty x^{\mu-1}e^{-kx}dx = \int_0^\infty \left(\frac{u}{k}\right)^{\mu-1} e^{-u}\left(\frac{1}{k}du\right), \qquad (k>0) \tag{75}$$

$$= k^{-\mu}\int_0^\infty u^{\mu-1}e^{-u}du \tag{76}$$

$$= \frac{\Gamma(\mu)}{k^\mu}, \qquad (k>0, \mu>0) \tag{77}$$

Here we used the definition of the Gamma function $\Gamma(\mu)$ from equation (72), which is only valid for $\mu > 0$. When applying the requirement $k > 0$ to the abbreviation $k = -a + b + bt > 0$, we can see this yields $-a + b > 0$ for the integration limit $t = 0$, or equivalently $a < b$. On this result we want to use the substitution $\frac{1}{1+t} = z$, i.e. $t = \frac{1}{z} - 1$:

$$\int_0^\infty x^{\mu-1}e^{ax}E_1(bx)\,dx = \Gamma(\mu)\int_0^\infty \frac{1}{(1+t)}\frac{1}{(-a+b(1+t))^\mu}dt \tag{78}$$

$$= \Gamma(\mu)\int_1^0 z\frac{1}{\left(-a+\frac{b}{z}\right)^\mu}\left(-\frac{1}{z^2}dz\right) \tag{79}$$

$$= \Gamma(\mu)\int_0^1 \frac{z^{-1}}{\left(\frac{b}{z}-a\right)^\mu}\frac{z^\mu}{z^\mu}dz \tag{80}$$

$$= \Gamma(\mu)\int_0^1 \frac{z^{\mu-1}}{(b-az)^\mu}dz, \qquad (a<b, \mu>0) \tag{81}$$

Now, we can just set $\mu = 1$, use $\Gamma(1) = 1$ and employ the substitution $b - az = x$ to get the desired end result of equation (52):

$$\int_0^\infty e^{ax}E_1(bx) = \Gamma(1)\int_0^1 \frac{1}{b-az}dz \tag{82}$$

$$= \int_b^{b-a} \frac{1}{x}\left(-\frac{1}{a}dx\right) \tag{83}$$

$$= -\frac{1}{a}\left[\ln x\right]_{x=b}^{b-a} \tag{84}$$

$$= -\frac{1}{a}\left[\ln(b-a) - \ln(b)\right] \tag{85}$$

$$= -\frac{1}{a}\ln\left(1-\frac{a}{b}\right), \qquad (a<b, 0<b) \tag{86}$$