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Infinite Symmetry in the Quantum Hall Effect

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ABSTRACT

Free planar electrons in a uniform magnetic field are shown to possess the symmetry of area-preserving diffeomorphisms (W-infinity algebra). Intuitively, this is a consequence of gauge invariance, which forces dynamics to depend only on the flux. The infinity of generators of this symmetry act within each Landau level, which is infinite-dimensional in the thermodynamical limit. The incompressible ground states corresponding to completely filled Landau levels (integer quantum Hall effect) are shown to be infinitely symmetric, since they are annihilated by an infinite subset of generators. This geometrical characterization of incompressibility also holds for fractional fillings of the lowest level (simplest fractional Hall effect) in the presence of Haldane's effective two-body interactions. Although these modify the symmetry algebra, the corresponding incompressible ground states proposed by Laughlin are again symmetric with respect to the modified infinite algebra.

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1. Introduction

The quantum Hall system [1] is a fascinating example of a *quantum fluid.* Two-dimensional electrons in an external uniform magnetic field occupy highly degenerate Landau levels. At low temperatures and large fields, they have strong quantum correlations which lead to collective motion and macroscopical quantum effects. The macroscopical quantum states are few, "simple" and universal, and they find their experimental evidence in a discrete series of quantized values for the Hall conductivity:

$$\begin{aligned} &\sigma_{xy} = \nu e^2/h, \\ &\nu = 1\ (\pm 10^{-8}),\ \frac{1}{3}\ (\pm 10^{-5}),\ \frac{1}{5},\ \frac{1}{7},\ \frac{2}{5},\frac{2}{7},\dots,\ \frac{6}{13},\dots,\ \frac{2}{3},\ 2,\ \frac{5}{2},\dots, \end{aligned} \tag{1.1}$$

where ν is interpreted as the *filling fraction*, the ratio between the number of electrons and the degeneracy of the Landau levels.

The observed universality clearly calls for an approach based on an effective field theory and renormalization group ideas [2] [3]. However, the most striking feature is the extraordinary precision of these rational values for the conductivity. Exact numbers usually indicate that dynamics is dominated by symmetry and topology, as is familiar from infinite conformal symmetry in two dimensions [4], and from topological quantization conditions of gauge fields [5]. Due to these expectations, the quantum Hall effect has attracted the interest of theoretical physicists beyond the border of the solid-state community.

In this paper, we show that an infinite-dimensional symmetry is indeed present in the Hall system at integer and (the simplest) fractional fillings. This yields a strong geometrical characterization of *incompressibility* of the ground state and it may be a key idea to understand the exactness of the collective behavior of electrons. Before specifying the content of this paper, let us briefly review the phenomenology of the problem and previous approaches based on symmetry principles.

The established phenomenology is based on the seminal work of Laughlin, Haldane, Halperin and others [6] [7] [8] [1]. It comprises a body of theoretical arguments substantiated by numerical simulations, which account for the exactness and stability of the ground state at the plateaus of the Hall conductivity. The main idea is the existence of an *incompressible quantum fluid* at specific rational values of the filling. Repelling electrons assume the most "symmetric" configuration compatible with their density, such that the ground state has a gap and such that there are no phonons. This picture clearly applies to the case of completely filled Landau levels (integer Hall effect). Compressions would excite electrons to higher levels, but these are suppressed by the large cyclotron energy. The Hall conduction is due to an overall rigid motion of the droplet of quantum fluid, which yields (1.1) for integer ν. In this case, Coulomb repulsion can be neglected since its typical scale is much smaller than the gap. Moreover, topological arguments have been developed to show that the free-electron picture is robust in the presence of impurities [9].

At filling fractions $\nu = 1/m$, with m an odd integer, Laughlin's wave function

$$\Psi(\mathbf{x_i}) = \prod_{i<j} (z_i - z_j)^m \ \mathrm{e}^{-\frac{1}{2\ell^2}\sum_i |z_i|^2} , \tag{1.2}$$

($z_j = x_j + iy_j$ is the position of j-th electron and $\ell = \sqrt{2\hbar c/eB}$ is the magnetic length) exhibits a similar incompressibility [10]. It correctly approximates the numerical ground state for a large class of repulsive interactions, and excitations have a finite gap [11]. Laughlin developed the incompressibility picture, as well as the properties of excitations, by interpreting $|\Psi|^2$ as a classical probability distribution for a two-dimensional Coulomb gas of charges, the *plasma analogy* [10]. For $\nu = 1/m$, this plasma is a liquid, the charges are completely screened and stability amounts to local electrical neutrality.

A very interesting consequence of this theory is that the charged excitations are *anyons*, *i.e.*, particles with fractional statistics $\theta/\pi = 1/m$ [12], and fractional charge e/m [13]. Moreover, Haldane [7] and Halperin [8] constructed a whole hierarchy of new incompressible quantum fluids for other filling fractions $\nu = \frac{2}{5}, \frac{2}{7}, \ldots$, by extending Laughlin's arguments on electrons to anyons. The hierarchical structure of plateaus and excitations has been observed experimentally and strongly supports the entire picture.

As we already mentioned, we believe that this picture should be supplemented with exact statements coming from symmetry. We have in mind the exact fractional critical exponents that follow from the infinite conformal symmetry of two-dimensional critical phenomena [4]. Actually, a formal analogy between the quantum Hall effect and two-dimensional conformal field theory was observed by Fubini [14], and Stone [15], and further developed in references [16]. This analogy is based on the similarity between the (holomorphic part of the) Laughlin wave function (1.2) and the correlators of vertex operators in a conformal theory with central charge $c = 1$.

Further extensions of this analogy have produced new results [17]. A new type of incompressible fluid was suggested at $\nu = 1/2$, which has been recently observed experimentally in double layer samples [18]. Moreover, the possibility that the excitations in this system possess non-abelian statistics is currently being debated [17][18].

These interesting developments call for an explanation of this analogy from a direct analysis of the microscopic physics of the quantum Hall effect. At first sight, this looks rather odd, given that conformal symmetry acts in $(1+1)$-dimensional relativistic field theory, whereas the problem at hand is $(2+1)$-dimensional and non-relativistic. A first positive indication is that the Hilbert space of electrons restricted to a given Landau level is the Bargmann space of analytic functions [19] [20]. This projection produces a dimensional reduction of the particle phase space from four to two dimensions.

In Section 2, we shall indeed show that there exists an infinite-dimensional symmetry in the free electron theory of Landau levels. However, the relevant algebra is not the Virasoro algebra but a quantum version of the algebra of area-preserving diffeomorphisms. These types of quantum algebras are called W_∞ and are currently the subject of active investigations in two-dimensional gravity [21].

In Section 3, the incompressibility of completely filled Landau levels is related to the infinite symmetry. Namely, the ground state is shown to be annihilated by infinitely many (in the thermodynamical limit) symmetry generators. This is analogous to the invariance of the vacuum in conformal field theory, *i.e.* the highest-weight conditions.

In Section 4, the symmetry characterization of incompressibility is extended to the Laughlin ground state (1.2) at $\nu = 1/m$. It is convenient and customary in this case to limit the theory to the lowest Landau level, and to introduce an effective short-range interaction for which the Laughlin wave function is an exact incompressible eigenstate [7][11]. This interaction deforms the infinite symmetry algebra of the free case. While the full properties of this theory, and of the deformed algebra, are not discussed in this paper, we prove that the Laughlin wave function for $\nu = 1/m$ satisfies again the highest weight conditions as in the $\nu = n$ case.

Finally, in the conclusions we discuss excitations and a geometrical description of the fluid.

2. Infinite conserved charges in the non-interacting theory

2.1 Landau levels

Let us begin by reviewing some basic facts about non-interacting electrons in an external uniform magnetic field B orthogonal to the plane [19][20]. We first discuss the one-body problem and shall deal later with the many-body situation and second quantization. The action S, and the Hamiltonian H, of an electron of charge e and mass m are:

$$\begin{aligned} S &= \int dt \left(\frac{1}{2}m\mathbf{v}^2 + \frac{e}{c}\mathbf{v}\cdot\mathbf{A}\right) , \\ H &= \frac{1}{2}m\mathbf{v}^2 = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 , \end{aligned} \tag{2.1}$$

where we choose * the symmetric gauge $\mathbf{A} = \frac{B}{2}(-y, x)$.

At the semiclassical level (exact in this case), the classical circular motion of cyclotron frequency $\omega = eB/mc$ gives rise to quantization of the (kinetic) energy $E = \hbar\omega n$ and thus to quantized radii r_n. The corresponding orbits enclose a flux multiple of the quantum unit of flux $\Phi_0 = hc/e$, *i.e.* $\pi r_n^2 B = n\Phi_0$. The center of the orbit is instead free, thus excitations of arbitrary angular momentum cost no energy and are degenerate. The flux quantization implies that the magnetic field has an associated unit of length, the *magnetic length* $\ell = \sqrt{2\hbar c/eB}$. From now on we choose units $\hbar = c = \ell = 1$, so that $eB = 2$ and $\Phi_0 = 2\pi/e$. Moreover, for convenience we take $m = e = 1$ in these units, thus $\omega = 2$.

At the quantum level, the canonical momentum is realized as $\mathbf{p} = -i\nabla$ acting on functions of the coordinate $\mathbf{x} = (x, y)$. The Hamiltonian and (canonical) angular momentum can be written in terms of a pair of independent harmonic oscillators:

$$\begin{aligned} H &= 2a^\dagger a + 1 , \\ J &= b^\dagger b - a^\dagger a , \end{aligned} \tag{2.2}$$

In complex notation $z = x + iy$, $\bar{z} = x - iy$, $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial\bar{z}$, these operators are

$$\begin{aligned} a &= \frac{z}{2} + \bar{\partial} , \qquad a^\dagger = \frac{\bar{z}}{2} - \partial , \\ b &= \frac{\bar{z}}{2} + \partial , \qquad b^\dagger = \frac{z}{2} - \bar{\partial} , \end{aligned} \tag{2.3}$$

* Our results will naturally be gauge-independent. We shall comment later on the formulas for other gauges.

and satisfy the commutation relations

$$[a, a^{\dagger}] = 1 , \qquad [b, b^{\dagger}] = 1 , \tag{2.4}$$

with all other commutators vanishing. Starting from the vacuum $\Psi_0 = \frac{1}{\sqrt{\pi}} \exp\left(-|z|^2/2\right)$, satisfying $a\Psi_0 = b\Psi_0 = 0$, one indeed finds energy $(a^{\dagger})$-excitations, the Landau levels, which are infinitely degenerate with respect to the angular momentum $(b^{\dagger})$-excitations. The general wave function of energy n and angular momentum l is given by

$$\Psi_{n,l}(\mathbf{x}) = \frac{(b^{\dagger})^{l+n}}{\sqrt{(l+n)!}} \frac{(a^{\dagger})^n}{\sqrt{n!}} \Psi_0(\mathbf{x}) = \sqrt{\frac{n!}{\pi(l+n)!}}\; z^l L_n^l(|z|^2)\, \mathrm{e}^{-\frac{|z|^2}{2}} , \tag{2.5}$$

where $L_n^l(x)$ are the generalized Laguerre polynomials and $n \geq 0, l+n \geq 0$. For example, the wave functions of the lowest Landau level satisfy $a\Psi_{0,l} = 0$, and read

$$\Psi_{0,l}(\mathbf{x}) = \frac{z^l}{\sqrt{l!}} \frac{1}{\sqrt{\pi}}\, \mathrm{e}^{-\frac{|z|^2}{2}} . \tag{2.6}$$

These functions are peaked at circular orbits, such that the degenerate levels have an onion-like structure. The degeneracy $N_{\mathcal{A}}$ of levels occupying an area $\mathcal{A}$ is equal to the flux through it in quantum units, $N_{\mathcal{A}} = \Phi/\Phi_0 = B\mathcal{A}/2\pi$, thus the degeneracy is finite in a finite sample.

The many-body problem of N electrons will clearly be described in first quantization by taking N copies of the previous operators labelled by an index i: $a, b \rightarrow a_i, b_i$. In this case, the parameter B acts as an external pressure, because it controls the number of states and thus the density of electrons per state. Actually, the latter is the correct quantum measure of electron density, the *filling fraction* ν

$$\nu = \frac{N}{N_{\mathcal{A}}} \propto \frac{1}{B} . \qquad \text{(finite sample)} \tag{2.7}$$

In the infinite plane, and within the lowest Landau level, the commonly used definition is:

$$\nu = \frac{N(N-1)}{2J} . \qquad \text{(infinite plane)} \tag{2.8}$$

Indeed, the area occupied by the electrons is proportional to the total angular momentum $J = \sum_{i=1}^{N} J_i$, whose minimal value is $N(N-1)/2$, by Fermi statistics.

To be more precise, the filling fraction ν has to be understood as a thermodynamical expectation value, and in this sense it appears in the conductivity, eq. (1.1). Therefore, eq.

(2.8) indicates that $1/\nu$ is proportional to the average value of J. In order to control J in the infinite plane with an external parameter, we introduce a central harmonic potential which confines the particles

$$H \to \mathcal{G} = H + \frac{\lambda}{2} \sum_{i=1}^{N} |z_i|^2 , \tag{2.9}$$

where $\mathcal{G}$ is the Gibbs free energy. Indeed, this modification is physical since it corresponds precisely to the effect of the neutralizing positive charged background [22]. The harmonic potential can be written in terms of a and b's and therefore of J and H

$$\mathcal{G} = (1+\alpha)H + \alpha J , \tag{2.10}$$

where $\lambda = \alpha(\alpha+2)$. Thus, α is the external parameter conjugate to $H+J$, *i.e.*, to $1/\nu$. Since $(J+H) \geq 0$ measures the area occupied by the electrons, α is a confining pressure (infinite plane) and it is equivalent to $1/B$ (finite sample). Finally, note that the value of ν will be dominated by the angular momentum of the ground state and fluctuations in J of $O(1,N)$ will describe the characteristic excitations.

In second quantization, the field operator is written as

$$\hat{\Psi}(\mathbf{x},t) = \sum_{n,k=0}^{\infty} F_k^{(n)} \Psi_{n,k-n}(\mathbf{x})\ \mathrm{e}^{-i(2n+1)t} , \tag{2.11}$$

in terms of the wave functions (2.5), the fermionic Fock annihilators $F_k^{(n)}$, and creators $F_k^{(n)\dagger}$, with

$$\{F_k^{(n)}, F_l^{(m)\dagger}\} = \delta_{n,m}\delta_{k,l} . \tag{2.12}$$

The N-body wave function is given by the Fock average

$$\Psi_E(\mathbf{x_1},\ldots,\mathbf{x_N}) = \langle 0|\hat{\Psi}(\mathbf{x_1},0)\ldots\hat{\Psi}(\mathbf{x_N},0)|N,E\rangle, \tag{2.13}$$

where $|0\rangle$ is the Fock vacuum, and $|N,E\rangle$ the corresponding Fock state. Second quantization corresponds to the canonical quantization of the non-relativistic Schrödinger field theory whose action is:

$$S = \int dt\ d^2\mathbf{x} \left(i\Psi^\dagger \frac{\partial}{\partial t}\Psi - \frac{1}{2}(\mathbf{D}\Psi)^\dagger \cdot (\mathbf{D}\Psi) \right) , \tag{2.14}$$

where $\mathbf{D} = \nabla - i\mathbf{A}$. The commutation relations (2.12) correspond to the canonical commutator

$$\{\hat{\Psi}(\mathbf{x},t), \hat{\Psi}(\mathbf{y},t)^{\dagger}\} = \delta^{(2)}(\mathbf{x}-\mathbf{y}) \quad , \tag{2.15}$$

and the conserved Hamiltonian H and number operator N are given by

$$H = \int d^2\mathbf{x}\ \frac{1}{2}\left(\mathbf{D}\hat{\Psi}\right)^{\dagger} \cdot \left(\mathbf{D}\hat{\Psi}\right), \qquad N = \int d^2\mathbf{x}\ \hat{\Psi}^{\dagger}\hat{\Psi}. \tag{2.16}$$

2.2 Infinite quasi-local conserved charges

We now return to discuss the meaning of the a and b oscillators. The operators a,$a^{\dagger}$ are covariant derivatives, as can be seen by restoring their B dependence

$$a = \bar{\partial} + \frac{B}{4}z \quad , \quad a^{\dagger} = -\partial + \frac{B}{4}\bar{z} \quad , \quad [a, a^{\dagger}] = \frac{B}{2} \ . \tag{2.17}$$

The operators b and $b^{\dagger}$ are generators of *magnetic translations*, *i.e.*, of translations combined with gauge transformations. In a uniform magnetic field, there is clearly translation invariance. Therefore, there should be two derivative operators commuting with the Hamiltonian, $[b, H] = [b^{\dagger}, H] = 0$. However, the choice of gauge apparently breaks translation invariance. Therefore, a translation should be accompanied by a compensating gauge transformation*. Actually, by exponentiation, the finite magnetic translations $T_{\epsilon,\bar{\epsilon}}$ give rise to a projective representation of the translation group

$$\begin{aligned} T_{\epsilon,\bar{\epsilon}}\psi(z,\bar{z}) &\equiv \mathrm{e}^{\epsilon b - \bar{\epsilon}\bar{b}}\Psi = \mathrm{e}^{\frac{1}{2}(\epsilon\bar{z}-\bar{\epsilon}z)}\Psi(z+\epsilon, \bar{z}+\bar{\epsilon}), \\ T_{\epsilon,\bar{\epsilon}}\ T_{\lambda,\bar{\lambda}} &= \mathrm{e}^{\frac{1}{2}(\epsilon\bar{\lambda}-\bar{\epsilon}\lambda)}\ T_{\epsilon+\lambda,\bar{\epsilon}+\bar{\lambda}} \ . \end{aligned} \tag{2.18}$$

The fact that the two correct translation operators commute with H, but not among themselves $([b, b^{\dagger}] = 1)$, accounts for the infinite degeneracy of Landau levels on the unbounded plane. It also gives other interesting effects in finite geometries [16].

It is worth stressing that the magnetic translation algebra is gauge invariant, even if the explicit form of b and $b^{\dagger}$ in terms of ∂_i and x_i depends on the gauge. As discussed in [14][24], in a general gauge they are defined by $b_i = \partial_i + i\Lambda_i$, where $\Lambda_i(\mathbf{x})$ are at most linear in the coordinates and are determined by the conditions $[b_i, p_j - A_j] = 0, \quad i,j = 1,2$. These

* A general discussion of symmetry in the presence of a background is given in [23].

imply $[b_1, b_2] = [p_1 - A_1, p_2 - A_2] = -iB$, which establishes manifest gauge invariance. Thus far, we discussed well-established matters.

We would now like to make a new, yet simple, remark: *there are more operators which commute with the Hamiltonian.* They are given by polynomials of the non-commuting operators b and $b^\dagger$, defined by:

$$\begin{aligned} &\mathcal{L}_{n,m} \equiv (b^\dagger)^{n+1}\, b^{m+1}, \qquad n,m \geq -1, \\ &[\mathcal{L}_{n,m}, H] = 0. \end{aligned} \tag{2.19}$$

These generate *quantum area-preserving diffeomorphisms* (up to gauge transformations). Indeed, from the definitions (2.3) and (2.4) we deduce that

$$[\mathcal{L}_{n,m}, \mathcal{L}_{k,l}] = \hbar\left((m+1)(k+1) - (n+1)(l+1)\right)\mathcal{L}_{n+k,m+l} + O(\hbar^2), \tag{2.20}$$

where the missing terms correspond to contractions of more derivatives, and hence have higher powers of $\hbar$. The $O(\hbar)$ term, the classical algebra, identifies our algebra as that of area-preserving diffeomorphisms or isometries. This is called w_∞ in the context of two-dimensional gravity [21]. Quantum deformations of this algebra are called, collectively, W_∞-algebras. There are many such deformations, which differ in the higher order terms in $\hbar$ *.

The $\mathcal{L}_{n,m}$ generically involve powers of derivatives higher than one, and therefore are not generators of local coordinate transformations on the wave function; they are *quasi-local* operators. The generating function of $\mathcal{L}_{n,m}$ is actually the finite magnetic translation discussed before (2.18). The only local coordinate transformations are the translations $\mathcal{L}_{0,-1}, \mathcal{L}_{-1,0}$, and rotations $(\mathcal{L}_{0,0} - a^\dagger a)$ with which we started.

For N particles, the first-quantized form of the generators $\mathcal{L}_{n,m}$ is

$$\mathcal{L}_{n,m} \equiv \sum_{i=1}^{N} (b_i^\dagger)^{n+1} b_i^{m+1}, \qquad n,m \geq -1. \tag{2.21}$$

Note that in the thermodynamical limit they become an infinite set of independent charges.

The second-quantized formulae might help to convince the reader that the $\mathcal{L}_{n,m}$ are bona-fide conserved charges arising from conserved (quasi)-local currents. Their second quantized expression is

$$\mathcal{L}_{n,m} = \int d^2\mathbf{x}\ \hat{\Psi}^\dagger\ (b^\dagger)^{n+1}(b)^{m+1}\ \hat{\Psi}, \tag{2.22}$$

* They can be changed by modifying the ordering of b, $b^\dagger$ in the definition (2.19).

where $\hat{\Psi}$ are the field operators introduced in eq. (2.11). Since $[(b^\dagger)^{n+1}(b)^{m+1}, H] = 0$, the $\mathcal{L}_{n,m}$ are conserved charges. This can be cast in the usual form of current conservation

$$\partial_0 J^0_{(n,m)} + \partial_i J^i_{(n,m)} = 0, \tag{2.23}$$

where

$$\begin{aligned} J^0_{(n,m)}(\mathbf{x}) &\equiv \hat{\Psi}^\dagger (b^\dagger)^{n+1}(b)^{m+1}\hat{\Psi}, \\ J^j_{(n,m)}(\mathbf{x}) &\equiv \frac{i}{2}\left[\left(D_j\hat{\Psi}\right)^\dagger (b^\dagger)^{n+1} b^{m+1}\hat{\Psi} - \hat{\Psi}^\dagger (b^\dagger)^{n+1} b^{m+1} D_j\hat{\Psi}\right], \end{aligned} \tag{2.24}$$

Current conservation follows from the operator equation of motion of the Hamiltonian (2.16), and the mentioned fact that b, $b^\dagger$ commute with the covariant derivatives. It is worthwhile to point out that strictly speaking, the observables have to be real, and they correspond to the eigenvalues of $Re(\mathcal{L}_{n,m})$ and $Im(\mathcal{L}_{n,m})$.

2.3 Classical symmetry

In order to gain some geometrical intuition, let us clarify the classical origin of the symmetry in our problem. In relativistic field theory we usually look for symmetries as those coordinate transformations that leave the action invariant. However, from a Hamiltonian point of view, symmetries are canonical transformations that leave the Hamiltonian invariant. Besides coordinate transformations, there are more general transformations which change momenta independently of the coordinates. These are usually disregarded in field theory, since they act non-locally in the Hilbert space, and therefore are difficult to implement at the quantum level*. As we now show, the W_∞-symmetry in our problem is the quantum version of a canonical symmetry of the classical Hamiltonian (2.1).

To begin, we briefly recall some basic facts about canonical transformations in the simplest example of a two-dimensional phase space (q,p). Canonical transformations are diffeomorphisms of the phase space which preserve the symplectic two-form $\omega = dq \wedge dp$, *i.e.* the area form [26]. The infinitesimal transformations have a generating function $F = F(p,q)$,

$$\begin{aligned} Q &= q + \delta q, \qquad \delta q = \{q, F\}_{PB} = \frac{\partial F}{\partial p}, \\ P &= p + \delta p, \qquad \delta p = \{p, F\}_{PB} = -\frac{\partial F}{\partial q}, \end{aligned} \tag{2.25}$$

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* Nevertheless, we quote the example of the Bäcklund transformation in the canonical quantization of Liouville theory [25].

where $\{\ ,\ \}_{PB}$ are the Poisson brackets, $\{q,p\}=1$. The "point" transformations, $\delta q = f(q)$, are given by $F = pf(q)$, under which p transforms as a derivative. A complete basis of generators is given by the full power series expansion of $F(p,q)$, $F_{n,m} = -q^{n+1}p^{m+1}$. The Poisson brackets of two such generators give a representation of the classical w_∞ algebra [21].

Let us now explain how this algebra can arise in our problem, which has a *four-dimensional* phase space $\{x, p_x, y, p_y\}$. The classical Hamiltonian (2.1) can be written as

$$H = \frac{1}{2}\left((p_x+y)^2 + (p_y - x)^2\right) = \frac{1}{2}\left(v_x^2 + v_y^2\right), \tag{2.26}$$

The most general canonical transformations which leave it invariant are generated by

$$\begin{aligned} \mathcal{L}^{(cl)}_{n,m}\,(\mathbf{x},\mathbf{p}) &= (b^\dagger)^{n+1} b^{m+1} \quad , \qquad b = \frac{1}{2}\left((p_y + x) + i(p_x - y)\right), \\ \delta H &= \{H, \mathcal{L}^{(cl)}_{n,m}\}_{PB} = 0 \ , \end{aligned} \tag{2.27}$$

as is clear from the classical limit of the discussion in section 2.2, $\mathcal{L}_{n,m} \to \mathcal{L}^{(cl)}_{n,m}$, and $[\ ,\ ] \to i\hbar\{\ ,\ \}_{PB} + O(\hbar^2)$. This also implies a crucial property of these transformations, namely, that they leave invariant the two combinations $v_x = \mathrm{Im}\ a$, $v_y = -\mathrm{Re}\ a$, where $a = \frac{1}{2}\left(-(p_y - x) + i(p_x + y)\right)$, *i.e.*,

$$\delta(p_x + y) = 0, \qquad \delta(p_y - x) = 0\ . \tag{2.28}$$

Therefore, the $\mathcal{L}^{(cl)}_{n,m}$ act non-trivially only on a two-dimensional subspace of the four-dimensional phase space, characterized by constant energy. It is a peculiar property of this problem that a two-dimensional subspace admits a symplectic structure in terms of b and $b^\dagger$. Thus, the infinite canonical transformations on this subspace are infinite symmetries and satisfy the w_∞ algebra

$$\{\mathcal{L}^{(cl)}_{n,m}, \mathcal{L}^{(cl)}_{k,l}\}_{PB} = \left((m+1)(k+1) - (n+1)(l+1)\right)\ \mathcal{L}^{(cl)}_{n+k,m+l}\ , \tag{2.29}$$

which is the classical limit of (2.20).

Upon quantization, something peculiar happens: within each Landau level, the previous two-dimensional subspace is identified with the actual coordinate space of the electrons. As emphasized in [27], this *phase-space reduction* can be obtained by taking the limit $m \to 0$ of the action (2.1):

$$\tilde{S} = \lim_{m\to 0} S = \oint_{orbit} dt\ \dot{\mathbf{x}}\cdot\mathbf{A} = \Phi = \int dt\ \left(\dot{y}x - \dot{x}y\right)\ , \tag{2.30}$$

where Φ denotes the flux piercing the area encircled by the orbit. The action $\tilde{S}$ ("Chern-Simons mechanics") describes classically the residual degree of freedom within each Landau level (the constant energy of the level is renormalized to zero). Note that $\tilde{S}$ in (2.30) is of first order in time derivatives, thus the symplectic structure is immediately evident and implies that only one of the original coordinates remains a coordinate in the Hamiltonian sense - the other becomes the conjugate momentum. This is the advertised phase-space reduction induced by the external magnetic field.

Since $\tilde{H} = 0$, the symmetries of the Hamiltonian are all canonical transformations of the reduced phase space. Equivalently, the symmetries of the action $\tilde{S}$ are the area preserving diffeomorphisms of the coordinate space, which leave Φ invariant.

2.4 The full W_∞ algebra

We now present some additional properties of the full quantum algebra and discuss its relation to the standard nomenclature [21]. The full commutator (2.20) follows from the Leibnitz differentiation rule ($\hbar = 1$)

$$[\ \mathcal{L}_{n,m}\ ,\ \mathcal{L}_{k,l}\ ] = \sum_{s=0}^{Min(m,k)} \frac{(m+1)!(k+1)!}{(m-s)!(k-s)!(s+1)!}\ \mathcal{L}_{n+k-s,\ m+l-s} - (m \leftrightarrow l, n \leftrightarrow k) \tag{2.31}$$

Special cases are given by:

$$\begin{array}{lll} i) & [\ \mathcal{L}_{n,0}, \mathcal{L}_{k,0}\ ] = (k-n)\mathcal{L}_{n+k,0}\ ; & \\ ii) & [\ \mathcal{L}_{0,n}, \mathcal{L}_{0,k}\ ] = (n-k)\mathcal{L}_{0,n+m}\ ; & \\ iii) & [\ \mathcal{L}_{n,n}, \mathcal{L}_{k,k}\ ] = 0 & \text{(Cartan subalgebra)}\ ; \\ iv) & [\ \mathcal{L}_{0,0}, \mathcal{L}_{n,m}\ ] = (n-m)\mathcal{L}_{n,m}\ . & \end{array} \tag{2.32}$$

The first two are Virasoro algebras, but the negative ($n < -1$) modes are missing. Later, we shall show that $\mathcal{L}_{n,0} = \sum_i z_i^{n+1}\partial_i$, when acting on analytic functions of the lowest level (2.6): these look like the conformal transformations which were assumed in references [16]; however, there are important differences. The missing negative modes correspond to singular transformations at the origin, which are not allowed on our two-dimensional Hilbert space ($(b^\dagger)^{-k}$ is not defined). On the other hand, Virasoro transformations are reparametrizations of the circle and can be extended to singular transformations of the plane. Moreover, the incompleteness of the algebra prevents us from considering possible central extensions. In the Virasoro algebra $L_k^\dagger = L_{-k}$, but in our case $\mathcal{L}_{n,0}^\dagger = \mathcal{L}_{0,n}$, and these do not satisfy a Virasoro algebra with the $\mathcal{L}_{k,0}$.

The standard notation of W_∞ algebra stresses its relation to the Virasoro algebra. Indeed, we can express our operators as

$$V_n^i = -\,\mathcal{L}_{n+i,i}, \qquad i \geq -1, \quad n \geq -i-1 \quad , \tag{2.33}$$

where the index $i+2$ corresponds to the conformal spin. For $i = 0$, the Virasoro subalgebra is satisfied by the Fourier modes of the spin-two stress-tensor. For $i = 1$, one has the Zamolodchikov spin-three current, and similarly for higher spins. The index n is the conformal dimension, namely the eigenvalue of $\mathcal{L}_{0,0}$. In our case, the Fourier modes V_n^i are bounded from below ($n \geq -i-1$), while in conformal field theory they are unbounded $\{V_n^i, \quad n \in \mathbf{Z}, \quad i \geq 0 \quad \text{in CFT}\}$. Therefore, our algebra corresponds to the so-called "wedge" $W_\Lambda = \{V_n^i, |n| \leq i+1\}$, plus the positive modes $n > i+1$. In the standard notation of W_∞, it reads (restoring $\hbar$ for comparison with reference [21]):

$$[V_n^i, V_m^j] = \hbar\lambda V_{n+m}^{i+j} + \hbar^2\eta V_{n+m}^{i+j-1} + \hbar^3\gamma V_{n+m}^{i+j-2} + \ldots + \hbar^{i+j+1-q}\ \rho V_{n+m}^q \quad , \tag{2.34}$$

where $q = Min(Max(i-m, j), Max(i, j-n))$. In eq. (2.20), we omitted the structure constants, which differ from those of reference [21], apart from the classical one. As anticipated, their form depends on the choice of quantum ordering and it is largely arbitrary, provided the Jacobi identity is satisfied. Therefore, the notion of W_∞ really stands for a large class of quantum algebras. As in the case of the Virasoro subalgebra, W_∞ admits in general a central term $\delta^{ij}\delta_{n+m,0}c_i(n)$. In its minimal form [21], $c_i(n)$ vanish inside the wedge, thus it cannot be added to our algebra.

3. Infinite symmetry and incompressibility of the ground state at integer fillings

In the previous section, we revealed an infinite-dimensional algebra in the non-interacting theory; we now investigate its relevance for the integer Hall effect. Up to now, we have been concerned primarily with the commutators of the generators with the Hamiltonian. Before we can assert that there is a symmetry in the quantum theory, we should discuss the action of the generators on the states, in particular the ground state $\mid \Omega \,\rangle$. We shall find an implementation of the symmetry analogous to conformal field theory. The ground state satisfies the highest weight conditions, *i.e.*, it is annihilated by an infinite subset of "lowering" operators, while the others create excitations. In conformal field theory, this condition is

$$L_n \mid \Omega \,\rangle = 0, \qquad n \geq -1 \quad . \tag{3.1}$$

Let us first understand the action of the $\mathcal{L}_{n,m}$ in Fock space. Their second-quantized expression is obtained by inserting the field operator (2.11) into definition (2.22)

$$\begin{aligned}
\mathcal{L}_{n,m} &= \sum_{k,j=0}^{\infty} \sum_{l,i=0}^{\infty} F_j^{(k)\dagger} F_i^{(l)} \int d^2\mathbf{x}\ \overline{\Psi_0}(\mathbf{x}) \frac{a^k}{\sqrt{k!}} \frac{b^j}{\sqrt{j!}} (b^\dagger)^{n+1} b^{m+1} \frac{\left(a^\dagger\right)^l}{\sqrt{l!}} \frac{(b^\dagger)^i}{\sqrt{i!}} \Psi_0(\mathbf{x}) \\
&= \sum_{r=0}^{\infty} \mathcal{L}^{(r)}_{n,m}, \\
\mathcal{L}^{(r)}_{n,m} &= \sum_{k\geq m+1} \frac{\sqrt{k!(k+n-m)!}}{(k-m-1)!} F^{(r)\dagger}_{k+n-m} F^{(r)}_k .
\end{aligned} \tag{3.2}$$

The $\mathcal{L}_{n,m}$ split into identical copies $\mathcal{L}^{(r)}_{n,m}$ acting independently within each level (r). The bilinears $F^\dagger_{k+s} F_k$ show that $\mathcal{L}_{n+s,n}$, for fixed s, displaces electrons by increasing $(s > 0)$ or lowering $(s < 0)$ their angular momentum. They act as *raising* and *lowering* operators within each level. Clearly, the bilinears $F^\dagger_{k+s} F_k$ give the simplest bosonic description of the Hilbert space at each level, because any fermionic state can be connected to any other by their repeated action. Therefore they achieve a "bosonization" of the non-relativistic fermion theory in each level. Moreover, the $\mathcal{L}^{(r)}_{n,m}$ similarly span the single-level Hilbert space, given that $\{F^{(r)\dagger}_{k+s} F^{(r)}_k\}$ and $\{\mathcal{L}^{(r)}_{n,m}\}$ are equivalent sets. Actually, eq. (2.21) can be inverted within the r-th level as follows:

$$F^{(r)\dagger}_a F^{(r)}_b = \frac{1}{\sqrt{a!b!}} \sum_{k=-1}^{\infty} \frac{(-1)^{k+1}}{(k+1)!} \mathcal{L}^{(r)}_{a+k,b+k} \quad . \tag{3.3}$$

As a consequence, the algebra of the $\mathcal{L}_{n,m}$ is a "spectrum-generating algebra" for the angular momentum.

Next, we discuss some physical issues of the integer Hall effect. The non-interacting electron theory considered so far is generically unphysical. The Coulomb interaction lifts the degeneracy of the Landau levels such that the properties of the ground state depend on the value of the filling fraction ν. However, for integer filling $\nu = 1, 2, \ldots$, the non-interacting theory makes physical sense * [9]. Under the experimental conditions of large magnetic fields, the energy gap between Landau levels is larger than the typical energy scale of the Coulomb interaction:

$$\hbar \frac{eB}{mc} > \frac{e^2}{\kappa \ell}, \tag{3.4}$$

where κ is the dielectric constant. Therefore, for integer ν, the Landau levels retain their character, the ground state is unique and has a gap: this corresponds to an *incompressible* quantum fluid [10]. Transitions of one or more electrons to higher levels would reduce the angular momentum, *i.e.*, would compress the fluid; however, these are forbidden by the large gap.

Moreover, angular momentum excitations, *i.e.*, decompressions, are controlled in our geometry by the confining potential (2.10) discussed in section 2.1. Their energy is given by the Gibbs energy $\mathcal{G} = \alpha J = \alpha \mathcal{L}_{0,0}$ (within each Landau level), where the value of $\alpha \propto 1/B$ allows for completely filled Landau levels, say the first one. Angular momentum excitations become energy excitations, and we achieve a very close analogy with conformal field theory, where similarly $H = L_0$. Indeed, the incompressible ground state at $\nu = 1$ satisfies highest weight conditions analogous to (3.1), as we now show.

The ground state is given by

$$|\,\Omega\,\rangle \;=\; |N, \nu = 1\rangle = F_0^{(1)\dagger} F_1^{(1)\dagger} \ldots F_{N-1}^{(1)\dagger} |\,0\,\rangle \tag{3.5}$$

in second quantization, and by

$$\Psi_{\nu=1}(\mathbf{x_1}, \ldots, \mathbf{x_N}) = \prod_{i<j}^{N} (z_i - z_j)\; e^{-\frac{1}{2}\sum_{i=1}^{N} |z_i|^2} \tag{3.6}$$

in first quantization. The highest weight conditions read

$$\mathcal{L}_{n,m} \Psi_{\nu=1} = 0, \qquad \text{for} \quad -1 \le n < m, \quad m \ge 0 \quad , \tag{3.7}$$

* A more precise statement based on the renormalization group will be made in section 4.

and follow from the nature of $\mathcal{L}_{n,m}$. Indeed, $\mathcal{L}_{n,m} \to \mathcal{L}^{(1)}_{n,m}$ when acting on the lowest level, where they lower the angular momentum for $n < m$ and therefore compress the fluid. Thus, they vanish on a completely filled state. Moreover, they cannot generate compression transitions to higher levels, energetically forbidden, because they commute with the Hamiltonian.

Eqs. (3.7) precisely state *the incompressibility conditions of the filled Landau level.* Geometrically, they express the symmetry, *i.e.*, stability of this fluid *. Similarly, in conformal field theory, the highest weight conditions (3.1) express the invariance of the vacuum under conformal transformations regular at the origin.

More precisely, let us compare both conditions in the customary W_∞ notation (see Section 2.4):

$$V^i_n |\, \Omega \,\rangle = 0, \qquad i = 0, 1, \ldots, \ -i-1 \le n < 0 \quad . \quad \text{(Hall effect)} \tag{3.8}$$

These conditions have to be compared with those corresponding to the W_∞-symmetric vacuum in conformal theory:

$$V^i_n |\, \Omega \,\rangle = 0, \qquad i = 0, 1, \ldots, \ -i-1 \le n < \infty \quad . \quad \text{(CFT)} \tag{3.9}$$

Note that all conditions in (3.8) are contained in (3.9). In the Hall case, one finds by inspection that there are $O(N^2)$ non-trivial conditions for N electrons, such that the ground state is infinitely symmetric in the thermodynamical limit.

The same incompressibility conditions (3.7) are easily extended to k-completely filled Landau levels. They read:

$$\mathcal{L}_{n,m} |N, \nu = k\rangle = 0, \qquad \text{for} \quad -1 \le n < m, \quad m \ge 0, \quad k = 1, 2, \ldots \quad . \tag{3.10}$$

Indeed, using second quantization, for $N = kN_L$ electrons:

$$|N, \nu = k\rangle = \prod_{r=1}^{k} \left(F_0^{(r)\dagger} F_1^{(r)\dagger} \ldots F_{N_L-1}^{(r)\dagger} \right) |0\rangle, \tag{3.11}$$

thus,

$$\mathcal{L}_{n,m} |N, \nu = k\rangle = \sum_{r=1}^{k} (\ldots)\, \mathcal{L}^{(r)}_{n,m} \left(F_0^{(r)\dagger} F_1^{(r)\dagger} \ldots F_{N_L-1}^{(r)\dagger} \right) |0\rangle = 0, \tag{3.12}$$

by commuting operators for different levels.

In summary, we have shown that in the case of the integer Hall effect, the incompressibility of the ground state corresponds to an infinite symmetry under the W_∞ algebra.

* Note that the condition of magnetic translation invariance, $\mathcal{L}_{-1,0}\Psi = 0$, has been previously recognized as a necessary condition for incompressibility of the ground state [22][20].

4. Infinite symmetry and incompressibility at fractional fillings

In this section, we show that the incompressibility of the Laughlin wave function at fractional fillings $\nu = 1/m$, m odd, is again characterized by a set of highest weight conditions, similar to the integer case.

4.1 Projection onto the lowest Landau level

For fractional fillings, non-interacting electrons have access to a large reservoir of degenerate states. One has to understand how the repulsive interaction manages to arrange the electrons in a collective "symmetric" state, such that the ground state is unique and has a gap in the thermodynamical limit. We already remarked that under experimental conditions, the Landau level gap is larger than the typical scale of the Coulomb interaction, eq. (3.4). A better estimate of the interaction scale is given by the size of the gap for the low-lying excitations above the $\nu = 1/m$ ground state, which numerical simulations and Laughlin's theory predict to be a few percent of the Landau level gap [10][11]. Therefore, it is probably sufficient to limit the theory to the lowest level. The Laughlin wave functions (1.2) belong indeed to this level, *i.e.*, they are of the general form

$$\Psi(\mathbf{x_1}, \dots, \mathbf{x_N}) = e^{-\frac{1}{2}\sum_i |z_i|^2} \varphi(z_1, \dots, z_N), \tag{4.1}$$

where φ is an entire analytic function of $z_1, \dots, z_N$ (see section 2.1). The projection onto the lowest level is largely accepted in the literature for $\nu = 1/m$, while it is being debated in the case of hierarchical constructions at fillings $\nu = p/q$ [28].

The projection leads us to the Bargmann space of coherent states for the b, $b^\dagger$ oscillators. Let us recall some of its basic features [19][20]. The exponential factor in (4.1) is absorbed into the measure and operators act only on the analytic part of the wave function $\varphi(z)$

$$\Psi = e^{-\frac{1}{2}z\bar{z}} \varphi(z), \qquad \langle \phi \mid \varphi \rangle = \int \frac{dz d\bar{z}}{2\pi i} e^{-z\bar{z}} \overline{\phi(z)} \varphi(z). \tag{4.2}$$

From eqs. (2.3), b and $b^\dagger$ are indeed represented by z and ∂, respectively. The orthogonality and completeness conditions for coherent states are (see section 2.1)

$$\mid z \rangle = e^{\bar{z} b^\dagger} \mid \Psi_0 \rangle, \qquad \varphi(z) = \langle z \mid \varphi \rangle, \tag{4.3}$$

$$\langle z \mid \eta \rangle = e^{z\bar{\eta}} \ , \qquad \text{(identity kernel)} \tag{4.4}$$

$$\mathbf{1} \ = \ \int \frac{dz d\bar{z}}{2\pi i} e^{-z\bar{z}} \mid z\rangle \ \langle z \mid . \tag{4.5}$$

Notice that $\bar{z}$ acts on the left and is the adjoint of ∂, because one finds by partial integration that $\langle\phi|\partial\varphi\rangle \ = \ \langle z\phi \ |\varphi\rangle$. This confirms that the two-dimensional plane $(z,\bar{z})$ is a phase space for the one-level theory, in agreement with the classical analysis of section 2.3. Any operator A is represented by an integral kernel $A(z,\bar{\eta})$ and acts as follows

$$(A\varphi)(z) \ = \ \langle z|A\varphi\rangle \ = \ \int \frac{d\eta d\bar{\eta}}{2\pi i} e^{-\eta\bar{\eta}} A(z,\bar{\eta}) \ \varphi(\eta). \tag{4.6}$$

In particular, the kernel $A(z,\bar{\eta})$ of a (quasi)-local operator A is the same operator applied to the identity kernel

$$A \ \varphi(z) \ = \ z^k \partial^r \varphi(z) \quad \leftrightarrow \quad A(z,\bar{\eta}) \ = \ z^k \left(\frac{\partial}{\partial z}\right)^r e^{z\bar{\eta}} = \ z^k \bar{\eta}^r e^{z\bar{\eta}}. \tag{4.7}$$

The projector P_0 onto the lowest level acts on a generic wave function Ψ, setting to zero its higher level components. This is achieved by using the identity kernel

$$\Psi(z,\bar{z}) \ = \ e^{-\frac{1}{2}z\bar{z}} \Phi(z,\bar{z}), \qquad (P_0\Phi)(z) \ = \ \int \frac{d\eta d\bar{\eta}}{2\pi i} e^{-\eta\bar{\eta}} \ e^{z\bar{\eta}} \ \Phi(\eta,\bar{\eta}). \tag{4.8}$$

Indeed, if $\Phi = (a^\dagger) \ \phi(\eta,\bar{\eta}) \ = \ (\bar{\eta} - \partial_\eta)\,\phi$, it vanishes inside the integral by partial integration. Similarly, the projection of local operators $V(\bar{z},z)$ is obtained by moving all $\bar{z}$ powers to the left of the z's (normal ordering) and by replacing $\bar{z} \ \rightarrow \ \partial$ [19] :

$$P_0 \ V(\bar{z},z) \ P_0 \ = : V(\partial, z) : \quad . \tag{4.9}$$

For example, $P_0 \ |z|^2 \ P_0 \ = \ \partial z \ = \ 1 + z\partial$. The same result is obtained by applying P_0 in the integral form (4.8) on both sides [20].

The generators of area-preserving diffeomorphisms (2.21) take the form

$$\mathcal{L}_{n,m} \ = \ \sum_{i=1}^{N} z_i^{n+1} \left(\frac{\partial}{\partial z_i}\right)^{m+1} \quad . \tag{4.10}$$

As anticipated in section 2.4, the $\mathcal{L}_{n,0}$ actually generates conformal transformations on analytic wave functions. Note that, as a result of the projection, the subset of local transformations of our algebra is enlarged to include this infinite subset.

4.2 Effective interaction for $\nu = 1/m$

A convenient parametrization of projected two-body interactions is given by a "partial-wave" analysis or, equivalently, by a short-range expansion * [19][20]. The previous rule of projection (4.8) tells us that the kernel of a projected two-body operator, $V(|z_1 - z_2|)$, is given by

$$
\begin{aligned}
(P_0\ V\ P_0)(z_1, z_2; \bar{\eta}_1, \bar{\eta}_2) &= \int d^2\rho\ d^2\xi\ e^{-\rho\bar{\rho} - \xi\bar{\xi}}\ e^{z_1\bar{\rho} + z_2\bar{\xi}}\ V(|\rho - \xi|)\ e^{\rho\bar{\eta}_1 + \xi\bar{\eta}_2} \\
&= e^{\frac{z_1 + z_2}{\sqrt{2}}}\ e^{\frac{\bar{\eta}_1 + \bar{\eta}_2}{\sqrt{2}}} \sum_{n=0}^{\infty} \frac{e_n}{n!} \left(\frac{z_1 - z_2}{\sqrt{2}}\right)^n \left(\frac{\bar{\eta}_1 - \bar{\eta}_2}{\sqrt{2}}\right)^n,
\end{aligned}
\tag{4.11}
$$

where the amplitudes e_n of the partial waves are

$$
e_n = \frac{1}{n!} \int_0^{\infty} dr^2 e^{-r^2} r^{2n} V(\sqrt{2}r). \tag{4.12}
$$

The terms in the expansion of the kernel can be rewritten as

$$
P_0 V(1,2) P_0 = \sum_{n=0}^{\infty} e_n V_n(1,2), \qquad V_n = |J_{12} = n\ \rangle\ \langle\ J_{12} = n\ |. \tag{4.13}
$$

Here, V_n is the projection on the relative angular momentum $J_{12} = n$. Indeed, when acting on a two-particle wave function, it gives

$$
\varphi(z_1, z_2) = \sum_{n,m} C_{n,m} \left(\frac{z_1 - z_2}{\sqrt{2}}\right)^n \left(\frac{z_1 + z_2}{\sqrt{2}}\right)^m, \tag{4.14}
$$

$$
(V_n\varphi)(z_1, z_2) = \left(\frac{z_1 - z_2}{\sqrt{2}}\right)^n \sum_m C_{n,m} \left(\frac{z_1 + z_2}{\sqrt{2}}\right)^m, \tag{4.15}
$$

i.e., it selects the term of $O\left((z_1 - z_2)^n\right)$ in the expansion of $\varphi(z_1, z_2)$ around $\dfrac{z_1 + z_2}{2}$. By antisymmetry, even powers never appear; thus, we should only consider V_{2k+1}, $k = 0, 1, 2, \ldots$. An equivalent notation [22], valid only within expectation values, is

$$
V_n(1,2) \sim \Delta^n\ \delta^{(2)}(\mathbf{x_1} - \mathbf{x_2}), \qquad < \Psi|V_n|\Psi > = \int d^2\mathbf{x}\ \overline{\Psi}(\mathbf{x})\ \Delta^n \Psi(\mathbf{x}), \tag{4.16}
$$

* This is also reminiscent of the operator product expansion in conformal field theory [29].

where Δ is the Laplacian. For example, the partial waves of the Coulomb interaction are

$$H^{(c)}(1,2) \;=\; \frac{1}{|\mathbf{x_1}-\mathbf{x_2}|} \qquad \leftrightarrow \qquad e_n^{(c)} = \frac{\Gamma(n+\frac{1}{2})}{n!\sqrt{2}} \overset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2n}} \quad . \tag{4.17}$$

Note that the $e_n^{(c)}$ are positive and monotonically decreasing. Their overall dimensionful factor e^2/ℓ is set to one in our units.

Let us now discuss an *effective* short-range interaction which has been introduced for the Hall systems at filling fractions $\nu = 1/m$, say $1/3$ (Haldane's "pseudo-potentials" [11][7]). We would like to motivate it by using a renormalization group picture. Indeed, the phenomenological and numerical results in the literature on the Laughlin wave function strongly suggest such a picture. We have in mind the renormalization-group flow in the space of effective interactions (4.13) and we would like to associate a fixed point to each $\nu = 1/m$ plateau, with a corresponding effective interaction. The main supporting facts are:

i) There is *universality*, namely the Laughlin wave function

$$\phi_\nu(z_1,\ldots,z_N) \;=\; \prod_{i<j}(z_i-z_j)^m, \qquad \nu = \frac{1}{m}, \tag{4.18}$$

is an extremely good approximation to the numerical ground state for a large class of repulsive interactions at $\nu = 1/m$ [10]. The approximation improves as one approaches the thermodynamical limit [30] *.

ii) The Laughlin wave function is the *exact* incompressible ground state for the short-range potential [7][22],

$$H^{(m)} \;=\; \sum_{\substack{k=1 \\ k \text{ odd}}}^{m-2} e_k V_k, \tag{4.19}$$

where $e_1,\ldots,e_{m-2}$ are positive couplings (this statement will be proven later).

iii) There is a tunable mass scale Λ in the field-theoretical sense

$$\Lambda^2 \sim \alpha \qquad \left(\sim \frac{1}{B} \qquad \text{finite sample}\right), \tag{4.20}$$

* Compare with finite-size effects at the critical point in field theory.

which controls the density $\nu \propto 1/J \sim \alpha$, by eqs. (2.8)(2.10), and allows one to move among plateaus. Indeed, the typical inter-electron distance (in correct quantum units) is $1/\sqrt{\alpha}$, which plays the role of correlation length.

iv) The incompressibility of the $\nu = 1/m$ ground-states should be of the same nature as in the $\nu = 1$ case.

These facts suggest the following renormalization group scenario. The Coulomb interaction (including possible lattice impurities) corresponds to the "bare", or classical, interaction, whose couplings are given by (4.17). At a given value of α, the electrons test the Coulomb interaction at all scales smaller than $1/\sqrt{\alpha}$. The bare couplings e_k, originally monotonically decreasing for $k \to \infty$, have a renormalization group flow.

For (α such that) $\nu \simeq 1/m$, we assume that this flow is attracted to the fixed point,

$$\{e_k^{(eff)}(\frac{1}{\nu} = m)\} = \{e_1(m), \ldots, e_{m-2}(m), 0, 0, \ldots\}, \tag{4.21}$$

characterized by non-vanishing short-range effective couplings (e_k, $k < m$) and by vanishing long-range ones ($k \geq m$). Intuitively, the latter become irrelevant because they are not dynamically important at the length scales allowed by the density. Therefore, we are going to consider the short-range interaction (4.19), as an exact description of the interaction at the Hall plateaus $\nu = 1/m$, in the sense of fixed-points of the renormalization group.

Whilst this picture needs to be clarified by a computation of beta functions, it is substantiated by the numerical results reported in ref. [11]. Haldane observed that the ground state and the lowest lying states behave smoothly when the long-range couplings of the Coulomb interaction $e_k^{(c)}$, $k \geq m$, are turned off. Here we only presented his arguments in a more appealing language, by stating that the Coulomb and short-range $H^{(m)}$ interaction belong to the same universality class for $\nu = 1/m$.

Our characterization of the plateaus as fixed-points is found for integer ν in the phase diagram of the effective field theory approach of ref. [31]. Recently, an extension of this phase diagram for fractional filling has been proposed [3]. The $\nu = 1/m$ plateaus appear as fixed points on the boundary of the diagram, corresponding to accumulation points of a purely massive phase, like the high-temperature fixed point of the Ising model, where $\Lambda^2 = T - T_c \to \infty$.

4.3 Exactness and infinite symmetry of Laughlin's wave function

Let us now study the properties of the fixed-point theory at $\nu = 1/m$. It describes electrons constrained to the lowest Landau level and interacting via the two-body repulsive potential

$$H^{(m)} \;=\; \sum_{\substack{k=1 \\ k \text{ odd}}}^{m-2} e_k \sum_{i<j} V_k(i,j)\ , \qquad V_k(i,j) \;=\; \mid J_{ij}=k\ \rangle\ \langle J_{ij}=k \mid, \tag{4.22}$$

where V_k is the short-range potential defined previously. A more explicit form of its action on the holomorphic wave function is given by

$$\begin{aligned} V_k(i,j)\ \varphi(z_1,\ldots,z_i,\ldots,z_j,\ldots,z_N) \;&=\; \\ (z_i-z_j)^k\, \frac{1}{k!}\left(\frac{\partial}{\partial x}\right)^k \varphi&\left(z_1,\ldots,\frac{z_i+z_j+x}{2},\ldots,\frac{z_i+z_j-x}{2},\ldots,z_N\right)\Bigg|_{x=0} \end{aligned} \tag{4.23}$$

which generalizes the two-body case (4.15).

By antisymmetry, the analytic wave-function φ in (4.1) has the general form

$$\varphi = \prod_{i<j}(z_i-z_j)\ P(z_1,\ldots,z_N), \tag{4.24}$$

where $\prod(z_i-z_j)$ is the Vandermonde determinant and P is a completely symmetric homogeneous polynomial.

For the first non-trivial case, $\nu = 1/3$, the Hamiltonian is

$$H^{(3)} \;=\; e_1 \sum_{i<j} V_1(i,j) \qquad , \qquad e_1 > 0\ . \tag{4.25}$$

We now discuss its spectrum. The $E=0$ eigenspace is highly degenerate and contains wave-functions which go to zero at coincidence points faster than (z_i-z_j) for any pair $i,j=1,\ldots,N$. The $E>0$ eigenstates contain those which vanish as $O(z_i-z_j)$, *i.e.* minimally. For example, $H^{(3)}$ can be easily diagonalized for $N=2$, by using eq. (4.23),

$$\begin{aligned} E=0,&\quad \phi_0 = (z_1-z_2)^3\ P(z_1,z_2), &\quad \Theta_0 = 1 - V_1, \\ E=e_1,&\quad \phi_1 = (z_1-z_2)\ Q(z_1+z_2), &\quad \Theta_1 = e_1\ V_1, \end{aligned} \tag{4.26}$$

where Θ_0, Θ_1 are the projectors on the corresponding eigenspaces,

$$\mathbf{1} = \Theta_0 + \Theta_1, \quad \Theta_0\Theta_1 = \Theta_1\Theta_0 = 0, \quad H^{(3)}\Theta_0 = 0, \quad H^{(3)}\Theta_1 = e_1\ \Theta_1 \quad . \tag{4.27}$$

We now find the ground state of the interacting theory and later discuss its incompressibility. We prove the following statement: *the $E = 0$ eigenspace of the Hamiltonian (4.25) consists of wave functions which contain the third power of the Vandermonde as a factor.*

$$\varphi_0 = \prod_{i<j} (z_i - z_j)^3 P(z_1, \ldots, z_N), \qquad \leftrightarrow \qquad H^{(3)} \varphi_0 = 0, \tag{4.28}$$

where P is a symmetric homogeneous polynomial. Clearly wave-functions of this form have vanishing energy, by eq. (4.23). Conversely, let us find the solutions to $H^{(3)}\phi = 0$. Note that projectors have positive expectation value, thus we can write

$$0 \;=\; \langle \phi | H^{(3)} | \phi \rangle = e_1 \sum_{i<j} \| \, V_1(i,j) | \phi \rangle \|^2 \;\rightarrow\; V_1(i,j)\phi = 0 \;, \quad \forall \, i,j. \tag{4.29}$$

There are as many independent conditions as the number of pairs. Again, by eq. (4.23), ϕ should vanish faster than $(z_i - z_j)$ for any pair; thus, by antisymmetry, it should have at least a third-order zero for every pair. Moreover, ϕ is an analytic polynomial (*i.e.*, entire) function of $z_1, \ldots, z_N$, which is completely determined by its zeroes (whose number is given by J). Therefore, ϕ should have a third power of the Vandermonde determinant as a factor, *q.e.d.**

A corollary to this statement is that the Laughlin wave function (4.18) is the *exact non-degenerate ground state* at $\nu = 1/3$, as firstly observed by Haldane [11][7]. On the plane, the value $\nu = 1/3$ corresponds to angular momentum $J = 3N(N-1)/2$. Actually, $J = \sum_i z_i \partial_i$ gives the degree of homogeneity, so that P has degree zero in this case, $P = 1$; thus, φ_0 at $\nu = 1/3$ is the Laughlin wave function (4.18). This ground state is unique and incompressible, because it has the minimal value of J in the $E = 0$ eigenspace. Indeed, in eq. (4.28) we proved that $E = 0$ implies $J \geq 3N(N-1)/2$. Conversely, any state of smaller J should necessarily have $E > 0$, *i.e.*, it has an energy gap.

On the other hand, decompressions, $J > 3N(N-1)/2$, also develop a gap after inclusion of the confining potential (2.10)

$$H^{(3)} \rightarrow \mathcal{G}^{(3)} = H^{(3)} + \alpha \mathcal{L}_{0,0}, \tag{4.30}$$

in complete analogy with the integer filling case discussed in section 3.

---

* The structure of inclusion of the Hilbert space for $\nu = 1, \frac{1}{3}, \frac{1}{5}, \ldots$ resembles the projections in rational conformal field theories, like the projection of the free bosonic Fock space carried on by the Coulomb gas technique [32].

We shall now discuss the geometrical conditions expressing the incompressibility of the Laughlin wave function. To this end, we introduce the projector Θ_0 on the $H^{(3)} = 0$ subspace

$$\Theta_0 \ = \ \sum_i \mid \varphi_{0,i} \rangle \ \langle \varphi_{0,i} \mid, \qquad H^{(3)}\Theta_0 = 0, \tag{4.31}$$

where $\{\varphi_{0,i}\}$ is an orthogonal set of φ_0's of eq. (4.26).

Next, we modify the generators $\mathcal{L}_{n,m}$ of the non-interacting symmetry algebra (2.19) as follows

$$\mathcal{L}_{n,m} = \sum_i z_i^{n+1} \ \partial_i^{m+1}, \qquad \to \qquad \hat{\mathcal{L}}_{n,m} \ \equiv \ \Theta_0 \ \mathcal{L}_{n,m} \ \Theta_0. \tag{4.32}$$

The projected generators $\hat{\mathcal{L}}_{n,m}$ commute with $H^{(3)}$ and produce compressions and decompressions in the $E = 0$ subspace described before (4.28). They satisfy a modified algebra, due to the projection Θ_0. Nevertheless, their angular momentum eigenvalue is unchanged because $\mathcal{L}_{0,0} = \hat{\mathcal{L}}_{0,0}$, and $[\hat{\mathcal{L}}_{0,0}, \hat{\mathcal{L}}_{n,m}] = (n-m)\hat{\mathcal{L}}_{n,m}$, in the $E = 0$ subspace.

Acting on the Laughlin wave function (4.18), we find

$$\hat{\mathcal{L}}_{n,m} \ \phi_{\frac{1}{3}} \ = \ \Theta_0 \ \mathcal{L}_{n,m} \ \Theta_0 \ \prod_{i<j} (z_i - z_j)^3 \ = \ 0, \quad \text{for} \quad n < m, \ m \geq 0. \tag{4.33}$$

Proof: consider $\Phi = \mathcal{L}_{n,m}\Theta_0 \prod (z_i - z_j)^3$. $\mathcal{L}_{n,m}$ lowers the angular momentum, thus Φ has $J = 3N(N-1)/2 \ -(m-n) \ < \ 3N(N-1)/2$. We already proved that functions with such values of J are necessarily a superposition of $E > 0$ eigenstates only. Therefore, $\Theta_0\Phi = 0$, by eq.(4.31), *q.e.d.*.

Therefore, *eqs. (4.33) are the infinite incompressibility conditions satisfied by the Laughlin wave function at* $\nu = 1/m$.

We achieve a description of incompressibility of the quantum fluid analogous to the integer ν case. Among the transitions generated by the $\mathcal{L}_{n,m}$, we neglected compressions to $E > 0$ states, which are forbidden by the macroscopic gap. The remaining ones (the $\hat{\mathcal{L}}_{n,m}$, $n < m, m \geq 0$), which are energetically allowed, were used to establish the incompressibility of the ground state, *i.e.*, its infinite symmetry. Notice that this exact characterization is possible due to the peculiar properties of the two-body effective interaction, which renders the interacting theory much like the free one.

For the sake of completeness, a remark may be useful at this point. We treated differently compressions and decompressions both in the integer and fractional cases. Following Haldane [11], we gave a special role to compressions, because they characterize the effective interaction which develops at the plateaus. Indeed, being a short-range interaction, it allows for any decompression. Therefore, compressions are characteristic, and determine the relevant symmetry algebra, while decompressions are generic.

Unfortunately, at present we lack a complete understanding of the projection Θ_0 (4.27), and thus, of the $\hat{\mathcal{L}}_{n,m}$ algebra in this more interesting fractional case. Clearly, a second-quantized formulation will make manifest the action of the $\hat{\mathcal{L}}_{n,n}$ in the $E = 0$ subspace.

Nevertheless, it is interesting to remark that the conformal subalgebra is preserved by the projection, $\hat{\mathcal{L}}_{n,0} = \mathcal{L}_{n,0}$. Indeed, these transformations are locally a rotation and a dilatation, which keep the order of the zeroes of the wave function. This may be an indication that this subalgebra plays a distinguished role at each one of the plateaus.

5. Summary and perspectives

In summary, we have exposed the existence of an infinite symmetry in the many-body quantum problem of planar electrons in a uniform magnetic field. In the absence of two-body interactions, the symmetry generators satisfy an algebra of (quantum) area-preserving diffeomorphisms. Its classical origin are canonical transformations of the four-dimensional phase space that leave invariant the Hamiltonian. Upon quantization, the external magnetic field quenches the kinetic energy and the phase space is consequently reduced from four to two dimensions.

At the quantum level, these generators have been used to describe excitations within any Landau level. The incompressibility of completely filled levels was expressed by a set of highest weight conditions (3.7),(3.12), of the type which appear in conformal field theory.

In the presence of Haldane's effective two-body interactions, the basic picture does not change: there is still an infinite algebra and the Laughlin ground states (1.2) are annihlilated by an infinite subset of generators. In this case, the explicit expressions of the generators and their algebra are not fully understood at the moment. The exact nature of the latter and the eventual emergence of a full Virasoro algebra constitute important unsettled issues which are under current investigation.

Another interesting problem is to further elucidate the geometrical properties of excited states. For example, let us consider the (second-quantized) Fourier-transformed density operator $\rho(\mathbf{k})$:

$$\rho(\mathbf{k}) = \int d^2\mathbf{k}\ e^{i\mathbf{k}\cdot\mathbf{x}}\ \rho(\mathbf{x}). \tag{5.1}$$

When projected onto the first Landau level this operator can be written,

$$\rho(\mathbf{k}) = e^{-\frac{1}{4}k\bar{k}} \sum_{n,m} \frac{1}{n!m!} \left(\frac{i\bar{k}}{2}\right)^n \left(\frac{ik}{2}\right)^m \mathcal{L}_{n-1,m-1}. \tag{5.2}$$

Due to the character of the $\mathcal{L}_{n,m}$, it creates intra-level bosonic excitations labelled by a wave vector $\mathbf{k}$: indeed, these are the *magneto-phonons* and *magneto-rotons* neutral excitations studied in [33].

More relevant for Laughlin's theory are the *quasi-hole* and *quasi-particle* charged excitations. We expect these to fall into representations of the projected algebra of the fractional case. Hopefully, this will provide the correct characterization of anyons as "vertex operators".

As a last point, we mention the geometrical treatment of the Fermi fluid in terms of *distribution functions*. The expectation value of the density operator $\rho(\mathbf{x},t)$ in any quantum state gives a corresponding quantum distribution function $w(\mathbf{x},t)$ of the fluid. This expectation value is also expressed in terms of $\mathcal{L}_{n,m}$

$$w(\mathbf{x},t) = \frac{1}{\pi}\, e^{-z\bar{z}} \sum_{n,m} \frac{1}{n!m!} z^n \bar{z}^m \langle \mathcal{L}_{n-1,m-1} \rangle. \tag{5.3}$$

Thus, the space of all distribution functions w carries a representation of W_∞. The semi-classical approximation consists in considering distribution functions which carry a representation of w_∞, which is a more tractable algebra. Using this framework, we hope to understand better its geometrical meaning in terms of deformations of the droplet of fluid and the edge states [34]. This semiclassical construction was recently discussed in a "string inspired" geometrical treatment of $(1+1)$-dimensional Fermi fluids [35]. Note, however, that the Fermi fluid considered here is $(2+1)$-dimensional and that the reduction to a two-dimensional phase space is produced by the external magnetic field.

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