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SUM OVER LOOPS WITH AREA-DEPENDENT WEIGHT AND QCD$_2$ SPECTRUM

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An exact method is given for summing over two-dimensional random loops with an "algebraic area" weight of exponential type. The approach applies also to the partition function of the Ising model in an imaginary magnetic field and to a restricted class of contours with a weight of QCD$_2$ type. In the latter case the two-point Green function is calculated and the rôle of loop self-intersections in determining 't Hooft's spectrum is discussed on the basis of an approximate no-intersection condition.

1. Introduction

Much effort has been devoted in the past few years to the computation of the QCD spectrum on the lattice (see e.g. [1]). Such an analysis is usually made by relating the hadron propagator to a statistical sum over Wilson loop averages, and involves in principle two main steps (see e.g. [2]). One is to find the dependence of the loop average on the global geometrical features of the loop, like the perimeter and surface. The other is to evaluate the statistical weight of the loops with the same geometrical features.

Both problems are usually approached simultaneously by approximate numerical methods. On the other hand, it is believed that in QCD the loop average is exponentially damped with the area of some (minimal) surface connected with it. This is confirmed by exact calculations for QCD$_2$ in the $N\to\infty$ limit, where the form of the area dependence is well known [3]. Therefore, we shall take the first step for granted, and we shall attempt in this paper to tackle by analytical methods the second, purely statistical problem, of summing over two-dimensional loops with a given perimeter and surface.

More precisely, we shall concentrate on statistical sums of the type

$$\sum(\tau,\eta)=\sum_{\{C\}} e^{-\tau S(C)}\,\eta^{P(C)}\,, \tag{1.1}$$

where the C's are closed contours in a two-dimensional square lattice, $P(C)$ is the loop perimeter, $S(C)$ is a determination of its area that we shall discuss, and we may also wish to keep fixed one or two points on the contour, in order to analyze particle propagators. Sums of the type (1.1) also occur in statistical models, like the two-dimensional Ising model in an external magnetic field.

Restricting our analysis to $d=2$ considerably simplifies the problem, because it avoids surface fluctuations in the transverse directions, which have been analyzed by Polyakov and by other authors [4]. However, this feature is present in an indirect way, because in two dimensions the probability of the loop intersecting itself is large, and self-intersections are the main difficulty that will prevent us from finding an exact expression for $\sum$ in the most interesting cases.

Let us first note that when the closed loop C has self-intersections (fig. 1) the notion of loop area becomes ambiguous. In fact, one can divide C into simple subcontours C_i in various ways and then combine their areas. Typically, for a given subdivision, we may define an "absolute" area

$$S(\mathrm{C})=\sum_i |A(\mathrm{C}_i)|\,, \tag{1.2}$$

or an "algebraic" area

$$A(\mathrm{C})=\sum_i A(\mathrm{C}_i)\,, \tag{1.3}$$

where now C and the C_i are thought of as oriented loops. The definition (1.2) is most interesting because it is the relevant one for QCD_2 (sect. 4) and, with some modifications, for the Ising model (sect. 3). The algebraic area (1.3) is simpler because it can be easily expressed in terms of random step variables on the contour (sect. 2). In the continuum limit one has in fact

$$A(\mathrm{C})=\int_{\Sigma(\mathrm{C})} \mathrm{d}\sigma(x^\mu)\omega(x^\mu)=\oint \tfrac{1}{2}(x^0\,\mathrm{d}x^1-x^1\,\mathrm{d}x^0) \qquad (\mu=0,1)\,, \tag{1.4}$$

where $\omega(x^\mu)$ is the winding number, i.e. the number of times (with sign) that the contour winds up around x^μ, and $\sum(\mathrm{C})$ is the largest closed surface whose contour belongs to C.

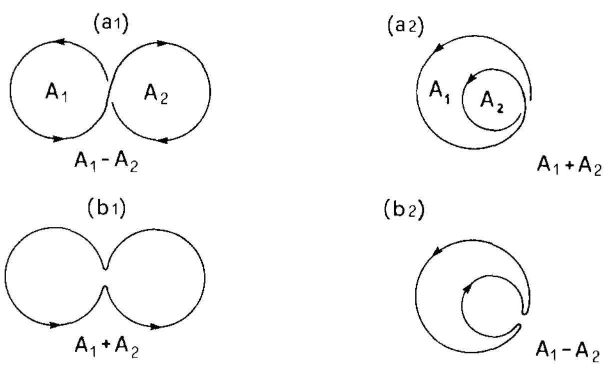


Fig. 1. Examples of (a) self-intersecting and (b) simple contours. The algebraic determination A of the area is as shown. The absolute value determination is $S=|A|$ for simple contours and $S=|A_1|+|A_2|$ for self-intersecting ones.

The first result of this paper is to give an exact expression for the sums of type (1.1) with an oscillating weight $\exp(-i\lambda A)$ ($\tau = i\lambda$) and with the algebraic determination of the area (1.3). The method is presented in sect. 2 and consists in obtaining the weight above by repeated application of a transfer matrix along the *contour* C.

The trick which allows the transfer matrix approach is a sort of Fourier transform performed on the lattice analogue of the loop integral (1.4) and is reminiscent of Kaç's approach to the Curie-Weiss problem (see e.g. [5]). It turns out, unfortunately, that the two-point Green function of this model has no discrete spectrum due to cancellations among various subcontours which are allowed by (1.3) and reduce the damping effect of the oscillating weight.

On the other hand, we have been unable to treat the case (1.2) exactly. This is presumably because the expression (1.2), or its continuum version

$$S(\mathrm{C}) = \int_{\Sigma(\mathrm{C})} \mathrm{d}\sigma(x^\mu)|\omega(x^\mu)|\,, \tag{1.5}$$

does not admit a simple loop integral representation.

The partial results we shall obtain refer to some particular cases where a modification of the transfer matrix method applies to the absolute value determination of the area. In sect. 3 we compute the partition function of the Ising model in an imaginary magnetic field $h = H/kT = \frac{1}{2}i\pi$. In this case the weight is the sign function $(-)^S$ and the distinction between S and A is irrelevant. The result agrees with previous authors' [6] and the calculation turns out to be simpler.

In sect. 4 we analyze QCD_2 by defining a class of contours for which self-intersections (with some proviso) can be counted. By this we mean that we can set up a no-intersection condition so as to treat each simple subcontour separately, by turning eq. (1.3) into eq. (1.2). We are thus able to study the rôle of self-intersections for the propagator spectrum, which is given in sect. 5.

Two features of the results are worth mentioning. First, in the continuum limit this spectrum has the same behaviour at 't Hooft's [7, 8] even if the equivalence of the two eigenvalue problems seems difficult to prove. Secondly, the full Green function and that without self-intersections are rather different. In particular, the latter has no continuum limit except at the spectral points. Since our lattice model based on eqs. (1.1) and (1.2) can also be seen as a form of string quantization, this shows that self-intersections are essential in order to get the correct propagation of the string longitudinal modes [9, 10].

A final point we want to mention. Due to the rôle of self-intersections (sect. 6), evaluating the sum (1.1) with the weight (1.2) is at least as difficult as solving the chain with excluded volume, or the self-avoiding random walk (see e.g. [11]). However, any analytical progress made for the latter problem is likely to be extended to the first by applying the transfer matrix method for the algebraic area weight. This exact method may therefore prove quite useful for future investigations.

2. Random walk with algebraic area weight

Here we present the method for evaluating the sum over closed loops:

$$\sum(\lambda, \eta) = \sum_{\{C(\sigma_i)\}} e^{-i\lambda A(C)} \eta^{P(C)} = \sum_{N=0}^{\infty} \eta^N \Sigma_N(\lambda) , \tag{2.1}$$

where $A(C)$ is the "algebraic area" discussed in the introduction.

The loop C will be parametrized in terms of two-dimensional random step variables $\boldsymbol{\sigma}_i$ as

$$C = \{\boldsymbol{\sigma}_1, \cdots, \boldsymbol{\sigma}_N\} , \qquad \sum_{i=1}^{N} \boldsymbol{\sigma}_i = 0 , \tag{2.2a}$$

where

$$\sigma_i^{\mu} = \begin{pmatrix} \sigma_i^0 \\ \sigma_i^1 \end{pmatrix} = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} , \tag{2.2b}$$

and $N = P(C)$ is the contour length. In terms of these variables the algebraic area in eq. (1.4) is given by

$$\begin{aligned} A(C) &= \tfrac{1}{2} \sum_{i,j=1}^{N} \Theta(i-j)(\sigma_i^0 \sigma_j^1 - \sigma_i^1 \sigma_j^0) \\ &= \sum_{i \geq j} \sigma_i^0 \sigma_j^1 . \end{aligned} \tag{2.3}$$

As mentioned in the introduction, the area so defined changes sign with the contour orientation, and adds up algebraically when the contour has self-intersections (fig. 1). Therefore, the sum (2.1) is only loosely connected with the partition function of the Ising model in a magnetic field (sect. 3) and with the Wilson loop weight of QCD_2 (sect. 4). For this reason we have adopted an imaginary coefficient $+i\lambda$ in the exponent instead of the customary (positive) string tension τ.

The advantage of dealing with the algebraic area (2.3) is that the sum (2.1) and the corresponding correlation functions can be reduced, by a transfer matrix method, to a one-dimensional potential problem where the contour length plays the rôle of the evolution time.

Since the exponent in (2.1) corresponds, by (2.3), to a long-range interaction in the variables $\sigma_j^{\mu}(\mu = 0, 1)$, it is not obvious that the transfer matrix method is applicable. Therefore, we shall generalize Kaç's method for the Curie–Weiss model [5] by introducing the gaussian transformation

$$\begin{aligned} \exp\left(-i \sum_{i,j=1}^{N} \sigma_i^0 A_{ij} \sigma_j^1\right) = \frac{1}{(2\pi)^N \det A} \int \prod_{i,j=1}^{N} \mathrm{d}x_i \, \mathrm{d}y_j \, e^{iy_i (A^{-1})_{ij} x_j} \\ \times \exp\left[i(x_i \sigma_i^0 + y_i \sigma_j^1)\right] \end{aligned}$$

for the matrix $A_{ij} = \lambda \Theta(i-j)$ occurring in eq. (2.3) (where, for definiteness, $\Theta(0) = 1$).

Then, one can note that A^{-1} has the short-range form

$$A_{ij}^{-1}=\begin{cases}\lambda^{-1}(\delta_{ij}-\delta_{ij+1}) & (j=1,\ldots,N-1)\\ \lambda^{-1}\delta_{ij} & (j=N)\,,\end{cases}\tag{2.4}$$

in order to rewrite Σ_N as a quantum mechanical path integral for the variables x_i, y_j. In fact, by summing over the σ_i^μ (which occur linearly in the exponent) and by introducing the closed contour constraint

$$\delta^2\left(\sum_{i=1}^{N}\boldsymbol{\sigma}_i\right)=\int_{-\pi}^{\pi}\frac{\mathrm{d}q_1\,\mathrm{d}q_0}{(2\pi)^2}\exp\left[i\sum_i(\sigma_i^0q_0+\sigma_i^1q_1)\right],\tag{2.5}$$

we obtain

$$\Sigma_N(\lambda)=\int_{-\pi}^{\pi}\frac{\mathrm{d}q_0\,\mathrm{d}q_1}{(2\pi)^2}\int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(\frac{\mathrm{d}x_i\,\mathrm{d}y_i}{2\pi}\right)\exp\left\{i\left[\sum_{i=2}^{N}(x_i-x_{i-1})y_i+x_1y_1\right]\right\}$$

$$\times\prod_{i=1}^{N}\{\tfrac{1}{2}[\cos(\lambda x_i+q_0)+\cos(y_i+q_1)]\}\,,\tag{2.6}$$

where we have used the normalized sum

$$\sum_{\sigma_i^\mu}\tfrac{1}{4}\exp\left(i(\sigma_i^1y_i+\sigma_i^0x_i)\right)=\tfrac{1}{2}(\cos y_i+\cos x_i)\,.\tag{2.7}$$

We can see that eq. (2.6) looks like a hamiltonian path integral in the conjugate variables x, y with a momentum-dependent transfer matrix roughly given by eq. (2.7). In fact, by performing the y-integrations, and introducing the x-representation $y\to p=-i\partial/\partial x$, we obtain

$$\Sigma_N(\lambda)=\int_{-\pi}^{\pi}\frac{\mathrm{d}q_0\,\mathrm{d}q_1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}x\,\langle x|\frac{1}{2^N}[\cos(\lambda x+q_0)+\cos(p+q_1)]^N|0\rangle$$

$$=\int_{-\pi}^{\pi}\frac{\mathrm{d}q_0\,\mathrm{d}q_1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{-iq_1x}\left\langle x+\frac{q_0}{\lambda}\right|\frac{1}{2^N}(\cos(\lambda x)+\cos p)^N\left|\frac{q_0}{\lambda}\right\rangle,\tag{2.8}$$

where the last form is obtained by using translational invariance and is meaningful only for $\lambda\neq 0$.

Eq. (2.8) is the main result of this section. The sum (2.1) has been reduced to a random walk problem with the transfer matrix

$$K=\tfrac{1}{2}(\cos(\lambda x)+\cos p)\,.\tag{2.9}$$

By acting along the contour N times, this matrix builds up the correct area weight, with $\Sigma_N\sim$ "Tr" (K^N), where the so-called trace actually means the integral operation in (2.8).

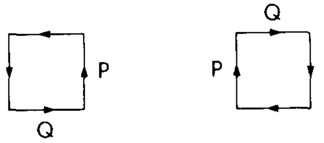


Fig. 2. Plaquette contours for the P- and Q-operators.

The rôle of the x and p variables is better understood by introducing the step operators in the x, y directions:

$$P = \mathrm{e}^{ip}, \qquad Q = \mathrm{e}^{i\lambda x}, \tag{2.10}$$

which satisfy the algebraic relations

$$P^+Q^+PQ = \mathrm{e}^{i\lambda}, \qquad Q^+P^+QP = \mathrm{e}^{-i\lambda}. \tag{2.11}$$

This means that, acting along a plaquette, such operators build up a unit of area which is dependent on the contour orientation (fig. 2).

Once the main formula (2.8) is given, all subsequent computations involving the algebraic area weight are trivially obtained. Firstly, by performing the summation over N we obtain a closed expression for $\Sigma(\lambda, \eta)$, eq. (2.1):

$$\Sigma(\lambda, \eta) = \text{“Tr”}\,[1 - \tfrac{1}{2}\eta(\cos(\lambda x + q_0) + \cos(p + q_1))]^{-1}, \tag{2.12}$$

where

$$\text{“Tr”}\, O = \int_{-\pi}^{\pi} \frac{\mathrm{d}q_0\,\mathrm{d}q_1}{(2\pi)^2} \int_{-\infty}^{\infty} \mathrm{d}x\, \langle x|O(\boldsymbol{q})|0\rangle .$$

It is easy to check on (2.12) that the $\lambda \to 0$ limit yields the “free” random loop sum

$$\Sigma(0, \eta) = \int_{-\pi}^{\pi} \frac{\mathrm{d}q_0\,\mathrm{d}q_1}{(2\pi)^2} [1 - \tfrac{1}{2}\eta(\cos q_0 + \cos q_1)]^{-1}. \tag{2.13}$$

If the random walk is interpreted as the propagator of some scalar “quark” field $\phi(x)$, this sum evaluates the loop contributions to $\langle 0|\phi^+(0)\phi(0)|0\rangle$ [2, 12].

Secondly, one may compute the two-point correlation function which should be interpreted as the composite particle propagator $\Delta(x^\mu, \eta, \lambda)$ of a $q\bar{q}$ state (sect. 4). This is obtained by fixing two points, $x^\mu = 0$ and $x^\mu = (x^0, x^1)$, along the closed loop, and performing the random sum with the weight (2.3).

By defining the Fourier transform

$$\Delta(P^\mu; \eta, \lambda) = \int \mathrm{d}^2x\, \Delta(x^\mu; \eta, \lambda)\, \mathrm{e}^{i\boldsymbol{P}\cdot\boldsymbol{x}}, \tag{2.14}$$

we can see that Δ only differs from Σ by the introduction of the factors $\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}$, $\boldsymbol{x} = \sum_{i=1}^{M} \boldsymbol{\sigma}_i$, along the first M steps (fig. 3). Therefore, we obtain the closed expression

$$\Delta(P^\mu; \eta, \lambda) = \text{“Tr”}\,\{[1 - \tfrac{1}{2}\eta K(\boldsymbol{q} + \boldsymbol{P})]^{-1}[1 - \tfrac{1}{2}\eta K(\boldsymbol{q})]^{-1}\}, \tag{2.15}$$

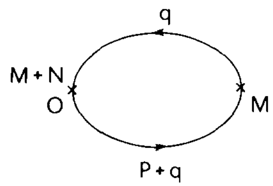


Fig. 3. Loops for the q$\bar{q}$ propagator. The q and $\bar{q}$ lines walk M and N steps and have momenta $P+q$ and $-q$ respectively.

which, in the $\lambda=0$ limit, reduces to the convolution of two free particle propagators:

$$\Delta(P^{\mu};\eta,0)=\int_{-\pi}^{\pi}\frac{\mathrm{d}^2q}{(2\pi)^2}\{1-\tfrac{1}{2}\eta\,[\cos(q_1+P_1)+\cos(q_0+P_0)]\}^{-1}$$
$$\times\{1-\tfrac{1}{2}\eta[\cos q_0+\cos q_1]\}^{-1}\,. \tag{2.16}$$

It is easy to check that eq. (2.16) exhibits the free two-particle spectrum when continued to the Minkowski region $P_0=-iE$ with $P_1=0$. In fact by performing the q_1 integration, we get

$$\Delta(P_0,0;\eta,0)=\left(\frac{2}{\eta}\right)^2\frac{1}{2\pi}\int_{\chi_0}^{\chi_1}\mathrm{d}\chi\cosh\chi(\cosh2\chi-\cos P_0)^{-1}$$
$$\times[(\cosh\chi-\cosh\chi_0)(\cosh\chi_1-\cosh\chi)]^{-1/2}\,, \tag{2.17}$$

where $\cosh\chi=2/\eta-\cos q_0$, $\cosh\chi_0=2/\eta-1$, $\cosh\chi_1=2/\eta+1$. This shows a cut for $2\chi_0\leqslant E\leqslant 2\chi_1$ which, in the continuum limit $a\to0$ with $1/\eta=1+\frac{1}{4}m^2a^2$ becomes the two-particle cut $E>2\chi_0=2ma$, where m is the "quark" mass.

The above conclusions can be extended to the case of the random walk of a spinning quark by modifying à la Wilson the perimeter weight as a function of the hopping parameter η:

$$\eta^N\to\eta^N\,\mathrm{Tr}\prod_{i=1}^{N}\tfrac{1}{2}(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\gamma})\,. \tag{2.18}$$

The effect of such an inclusion is to yield a new transfer matrix

$$K_{(1/2)}=\tfrac{1}{2}[\exp(i\gamma_0\lambda x)+\exp(i\gamma_1 p)]\,, \tag{2.19}$$

which replaces K in eq. (2.9), where now a trace over spin indices is understood. The expression for the two-point function is similar to eq. (2.15), where however one should introduce at points 1 and M the initial and final γ-matrices giving the structure of the composite state operator.

With this identification of the random walks with quark propagators we can now discuss the most direct interpretation of the algebraic area weight (2.3) in the sum (2.1). It corresponds to the quantum motion of the (scalar) quark in a uniform external magnetic field $H=i\lambda$ [13]. In fact, the lagrangian contains in such a case the term $\mathscr{L}_1 = e\oint dx^\mu A_\mu$, which for $eA_\mu = \frac{1}{2}i\lambda\varepsilon_{\mu\nu}x^\nu$ is precisely the algebraic area (2.3). (The imaginary value of the field $H = i\lambda$ gives rise to the oscillatory area weight.) This observation makes it clear that the composite Green function (2.15) cannot possibly give rise to bound states because the external field tends to drive them to (imaginary) infinity.

In order to better analyze this point, let us consider the continuum limit of eq. (2.15), defined by letting the unit of area λ and the lattice spacing a go both to zero, with λ/a^2 fixed. This defines a physical string tension $\sigma = \lambda/a^2$, in terms of which all dimensional quantities should be expressed.

It is therefore convenient to introduce the rescaled variables

$$\hat{x} = \sqrt{\lambda}\,x\,, \qquad \hat{p} = \frac{p}{\sqrt{\lambda}}\,, \qquad \hat{q}^\mu = \frac{q^\mu}{\sqrt{\lambda}}\,, \qquad \hat{P}^\mu = \frac{P^\mu}{\sqrt{\lambda}}\,, \qquad \hat{m}^2 = \frac{m^2}{\lambda}\,, \tag{2.20}$$

which have finite continuum values. Then the transfer matrix (2.9) has the quadratic limit

$$K \underset{\lambda\to 0}{\simeq} 1 - \tfrac{1}{4}\lambda(\hat{x}^2 + \hat{p}^2)\,, \tag{2.21}$$

and the expression (2.15) becomes

$$\Delta(P_0, 0; \eta, \lambda) = \lambda \int_{-\pi/\sqrt{\lambda}}^{\pi/\sqrt{\lambda}} \frac{dq_0\, dq_1}{(2\pi)^2} \int_{-\infty}^{\infty} d\hat{x}\, e^{-iq_0\hat{x}}$$
$$\times \langle \hat{x} + \hat{q}_1 | (1-\eta K)^{-1}\, e^{-i\hat{P}_0\hat{x}} (1-\eta K)^{-1}\, e^{i\hat{P}_0\hat{x}} | \hat{q}_1 \rangle\,. \tag{2.22}$$

This formula is evaluated in appendix A in terms of the evolution kernel of the one-dimensional harmonic oscillator. The result is

$$\Delta(P_0, 0; \eta, \lambda) = \frac{2}{\pi\lambda} \int_0^\infty \frac{d\Sigma\,\Sigma}{\sinh \Sigma} \int_{-1}^{+1} d\chi\, \exp\left[-\left(\hat{m}^2\Sigma + \tfrac{1}{2}\hat{P}_0^2 \frac{\cosh \Sigma - \cosh(\Sigma\chi)}{\sinh \Sigma} \right) \right], \tag{2.23}$$

where $\Sigma(1\pm\chi)$ are proper-time variables defined in appendix A.

The singularity of the expression (2.23) in the rotated variable $E^2 = -P_0^2$ is obtained in appendix A by the stationary phase method, and turns out to be a branch point at $E^2 = 4m^2$ as in the free case.

In conclusion, we have seen that the algebraic area weight cannot directly describe a confining interaction. The alternating sign of self-intersecting loops allows cancellations which give no net area weight for a class of diagrams of the type in fig. 4, irrespective of how large the quark oscillations are.

Nevertheless, the method explained in this section can be applied, with the correct area weight, to some particular systems that we shall study in the following. In sect. 3 we shall discuss an application to a statistical problem for which the sign of the area is unimportant, while in sect. 4 we shall analyze some (restricted) loop configurations of QCD_2 which give rise to string quantization with a discrete hadron spectrum.

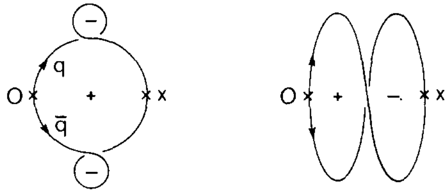


Fig. 4. Loops with (a) q(q̄) self-intersections and (b) qq̄ intersections for which cancellations of the algebraic area occur.

3. Application to the Ising model in a magnetic field

Here we want to use the loop summation method of the previous section for solving an old [6] problem in statistical mechanics: the evaluation of the partition function of the 2-dimensional Ising model in an external magnetic field $H/(kT) = h = \frac{1}{2}i\pi$. This is a case in which the area weight $\exp(-2HS/kT) = (-)^S$ is just a sign function and therefore, as we shall see, no distinction between algebraic and absolute area is needed.

Let us first recall how the area weight emerges, and how the partition function can be related, à la Kaç and Ward [14], to a random-walk problem. Starting from the standard partition function in a square two-dimensional lattice of N sites:

$$Z_N(\beta, h) = \sum_{\{\sigma_i\}} \exp\left[\beta \sum_{\langle i,j\rangle} \sigma_i\sigma_j + h\sum_i \sigma_i\right],$$

$$\beta = J/kT, \qquad h = H/kT, \tag{3.1}$$

we can classify its low-temperature expansion around an ordered configuration (all spins up) in terms of "graphs" which are the boundaries of closed regions with flipped spins. Each graph $\mathscr{G}$ is a non-oriented closed loop of links in the dual lattice and contributes to the exponent of (3.1) the quantity $2\beta(N-P) + h(N-2S)$, where P and S denote its perimeter and (absolute) area (fig. 5). Therefore eq. (3.1) can be rewritten as

$$Z_N(\beta, h) = e^{(2\beta+h)N} \sum_{\{\mathscr{G}\}} e^{-2hS}\eta^P, \qquad \eta = e^{-2\beta}. \tag{3.2}$$

This sum, for $h = 0$, can be related (see e.g. [15]) to a closed random-walk problem where the sum over graphs is replaced by a sum over oriented loops. Since several

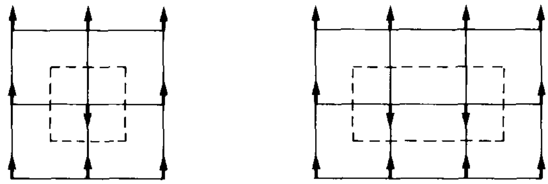


Fig. 5. Graphs on the dual lattice corresponding to one and two flipped spins.

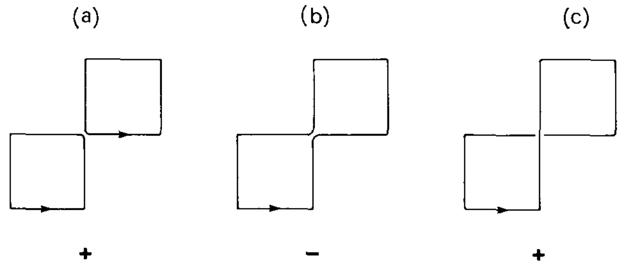


Fig. 6. Example of Kaç-Ward cancellation. Loops (a)-(c) contribute to the same Ising graph with total multiplicity one.

loops correspond to the same graph, multiple counting is avoided by introducing in the sum the phase factor $(-)^{\nu}$, where ν is the number of self-intersections of the loop. In this way each graph is counted only once, due to a pairwise cancellation of the extra loops which have opposite ν-parity (fig. 6).

Therefore, by introducing the link variables σ_i^{μ} in eq. (2.2b) and the closed loop constraint eq. (2.5), we can rewrite eq. (3.2) in the form

$$\begin{aligned} Z_N(\beta, 0) &= \mathrm{e}^{2\beta N} \sum_{\{C(\boldsymbol{\sigma}_i)\}} \eta^P (-1)^{\nu(C)} \int_{-\pi}^{\pi} \frac{\mathrm{d}^2 q}{(2\pi)^2} \exp\left(i\boldsymbol{q} \cdot \sum_{i=1}^{P} \boldsymbol{\sigma}_i \right) \\ &= e^{2\beta N} \exp\left\{ -\frac{1}{2} \sum_{P=0}^{\infty} \int_{-\pi}^{\pi} \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{1}{P} \operatorname{Tr}\left[\eta T_0(\boldsymbol{q})\right]^P \right\} \\ &= \exp(2\beta N)[\det(1-\eta T_0)]^{1/2} , \end{aligned} \tag{3.3}$$

which involves the transfer matrix T_0.

According to the preceding discussion and to eq. (3.3), T_0 is a product of two factors:

$$T_0(\boldsymbol{\sigma}_{i+1}, \boldsymbol{\sigma}_i) = \phi(\boldsymbol{\sigma}_{i+1}, \boldsymbol{\sigma}_i) K_0(\boldsymbol{\sigma}_i) , \tag{3.4}$$

where (i) $K_0(\boldsymbol{\sigma}_i)$ is diagonal with the link and yields the constraint eq. (2.5), i.e.

$$K_0(\sigma_i^0, \sigma_i^1) = \operatorname{diag}(\mathrm{e}^{iq_0}, \mathrm{e}^{-iq_0}, \mathrm{e}^{iq_1}, \mathrm{e}^{-iq_1}) , \tag{3.5}$$

and (ii) ϕ connects links i and $i+1$ by giving a phase factor $\varepsilon = \exp(\frac{1}{4}i\pi)$ ($\varepsilon^* = \exp(-\frac{1}{4}i\pi)$) according to whether there is a $\frac{1}{2}\pi$ ($-\frac{1}{2}\pi$) link rotation in the anticlockwise direction, and is unity (zero) for direct (reverse) steps. In this way ϕ^P builds the phase factor $\exp(\frac{1}{2}i\vartheta) = (-)^{\nu+1}$, where ϑ is the total angle rotated by the tangent along the closed loop. The explicit representation of ϕ with respect to the basis σ_i^{μ}, σ_{i+1}^{μ} is therefore

$$\phi = \begin{vmatrix} 1 & 0 & \varepsilon & \varepsilon^* \\ 0 & 1 & \varepsilon^* & \varepsilon \\ \varepsilon^* & \varepsilon & 1 & 0 \\ \varepsilon & \varepsilon^* & 0 & 1 \end{vmatrix} . \tag{3.6}$$

The evaluation of the 4×4 determinant of eq. (3.3) by using eqs. (3.4)-(3.6) is straightforward and yields the celebrated Onsager [15, 16] solution for the free energy:

$$-\frac{F}{kT}=\lim_{N\to\infty}\frac{1}{N}\log Z_N=-\log\eta+\frac{1}{2}\int_{-\pi}^{\pi}\frac{d^2q}{(2\pi)^2}$$

$$\times\log\left[(1+\eta^2)^2-2\eta(1-\eta^2)(\cos q_0+\cos q_1)\right]$$

$$=\log 2+\frac{1}{2}\int_{-\pi}^{\pi}\frac{d^2q}{(2\pi)^2}\log\left[\cosh^2 2\beta-\sinh 2\beta(\cos q_0+\cos q_1)\right]. \qquad (3.7)$$

It is generally not possible to extend this transfer matrix method to the case $h\neq0$, because the approach of sect. 2 does not apply to the area S occurring in eq. (3.2), which is roughly additive in absolute value when self-intersections occur (fig. 7).

More precisely, since S is the number of flipped spins inside the contour, it adds up in absolute value, except when overlappings occur, in which case two flips may give back no flip. In other words, in the continuum limit, S is related to the winding number $\omega(x)$ of eq. (1.4) by

$$S_{\text{Ising}}=\int d\sigma(x)\langle\omega(x)\rangle, \qquad (3.8a)$$

where

$$\langle\omega(x)\rangle=\tfrac{1}{2}\left[1-(-1)^{\sum_i\omega_i(x)}\right] \qquad (3.8b)$$

is 0 or 1 according to whether the number of flips is even or odd.

However, for the (unphysical) value $h=\frac{1}{2}i\pi$ the area weight is $(-)^S$ and since $(-)^{S_1+S_2}=(-)^{S_1-S_2}$ for integer valued S_i, one can as well replace S with the algebraic area A in eq. (2.3). Furthermore the Kaç-Ward connection with the random loop problem goes through in the same way, because the cancellations occur between contours with the same area weight.

After this discussion, it is clear that the method of sect. 2 can be applied to the case $h=\frac{1}{2}i\pi$. According to it the transfer matrix K_0 in eq. (3.4) is changed into another $K_{\pi/2}$ which yields not only the constraint (2.5) but also the weight $(-)^A$ corresponding to $\lambda=\pi$ in sect. 2. For $\lambda=\pi$ a further simplification occurs, because the step operators of eq. (2.10):

$$P=e^{ip}, \qquad Q=e^{i\pi x} \qquad (\lambda=\pi), \qquad (3.9)$$

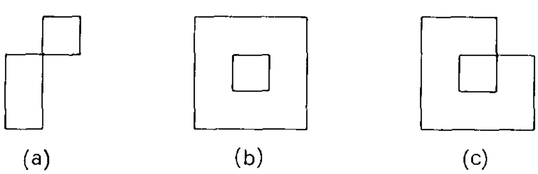


Fig. 7. Example of Ising area weight: (a) $S=3$, (b) $S=8$, (c) $S=7$.

satisfy the simple algebraic relations

$$\{P, Q\} = 0\,, \qquad P^2 = Q^2 = 1\,, \tag{3.10}$$

and therefore admit, for example, the simple representation

$$P = \sigma_1\,, \qquad Q = \sigma_3\,,$$

where σ_i are the Pauli matrices*.

Therefore, the transfer matrix $K_{\pi/2}$ takes the form

$$K_{\pi/2}(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{i+1}) = \mathrm{diag}\,(\mathrm{e}^{-iq_0}Q^+, \mathrm{e}^{iq_0}Q, \mathrm{e}^{-iq_1}P^+, \mathrm{e}^{iq_1}P)\,, \tag{3.11}$$

and the partition function becomes

$$\begin{aligned} Z_N(\beta, \tfrac{1}{2}i\pi) &= \exp\left\{(2\beta + \tfrac{1}{2}i\pi)N - \tfrac{1}{4}\int_{-\pi}^{\pi} \frac{\mathrm{d}^2 q}{(2\pi)^2} \sum_P \frac{1}{P} \mathrm{Tr}\,[\eta T_{\pi/2}(q)]^P\right\} \\ &= \exp\,[(2\beta + \tfrac{1}{2}i\pi)N][\det\,(1 - \eta T_{\pi/2})]^{1/4}\,, \end{aligned} \tag{3.12}$$

where $T_{\pi/2} = \phi K_{\pi/2}$ and the additional factor $\frac{1}{2}$ in the exponent of eq. (3.12) normalizes the trace of the σ-matrices.

Due to (3.11) $T_{\pi/2}$ $(\boldsymbol{q})$ is an 8×8 matrix and the straightforward evaluation of the related determinant in eq. (3.12)** yields the result

$$\begin{aligned} -\frac{F}{kT} = -\log\,\eta + \tfrac{1}{2}i\pi + \tfrac{1}{4}\int_{-\pi}^{\pi} \frac{\mathrm{d}^2 q}{(2\pi)^2} \log\,\{(1-\eta^2)^2 \\ \times[(1+\eta^2)^2 - 2\eta^2(\cos 2q_0 + \cos 2q_1)]\}\,, \end{aligned} \tag{3.13}$$

in agreement with previous, more complicated, calculations [6].

Unlike the $h = 0$ result (3.9), which shows the well-known phase transition at $\mathrm{sh}\,2\beta = 1$, the free energy (3.13) is regular for all temperatures (except $\beta = 0$) because the argument of the logarithm is positive definite. This is because the magnetic field (even if imaginary) tends to depress the disordered phase.

4. QCD$_2$ loops and the role of self-intersections

Sums over loops are of peculiar interest in QCD for the evaluation of composite particle propagators and therefore of the hadronic spectrum. In this section we shall try to extend the method of sect. 2 to the case of QCD$_2$, where however the area of a self-intersecting contour is additive in absolute value. Since this makes it very difficult to perform the sum exactly, we shall consider a restricted class of contours for which the self-intersections can be approximately taken into account, and which is wide enough to yield a discrete spectrum with the correct continuum limit (sect. 5).

* Starting from $x = 0$, P can only build states with $x = n$ (integer) so that Q has only eigenvalues ± 1, and similarly for P. An analogous simplified N-dimensional representation can be built for $\lambda = 2\pi/N$.

** This calculation can be simplified by expressing the matrix in terms of its 2×2 blocks.

Let us first recall that the composite particle* Green function $\Delta(x)$ is related to the Wilson loop by the expression [2, 12]

$$\Delta(x, \eta) = \sum_{\{C\}} \eta^{P(C)} \langle W[C(x)] \rangle , \tag{4.1}$$

where η is the hopping parameter (see e.g. [17]), $P(C)$ is the loop perimeter, $W = \prod_i U_{\sigma_i}$ is the product of field variables U_σ along the closed contour $C(x)$ passing through points 0 and x and $\langle W \rangle$ denotes its average with the Yang–Mills action.

The Wilson loop weight $\langle W(C) \rangle$ is known for QCD_2 in the $N \to \infty$ limit [3] and turns out to be a function of the absolute areas $S_i = |A_i|$ of all simple subcontours C_i into which the (self-intersecting) loop is divided.

More precisely, denoting by I a possible subdivision of C, one has (fig. 1)

$$\langle W(C) \rangle = \sum_{I} \exp\left(-\tau \sum_{i \in I} S_i \right) P_q(S_i) , \tag{4.2}$$

where the sum runs over all the subdivisions of C and P_q is a known polynomial of degree q (the number of overlapping contours minus one) that we shall drop for simplicity from now on.

Therefore, we shall concentrate in the following on the Green function

$$\Delta(0, x; \eta) = \sum_{C, I(C)} \eta^{P(C)} \exp\left[-\tau \sum_{i \in I} S_i(C_i) \right] , \tag{4.3}$$

where the sum runs over loops $C(x)$ passing through the fixed points 0, x.

4.1. SUM OVER RESTRICTED LOOPS

Since the transfer matrix method of sect. 2 works for an algebraic area, we shall attempt to: (i) count the number of self-intersections; (ii) include the algebraic area weight for each simple contour; (iii) take the absolute value of the area by the following projection integral:

$$\begin{aligned} e^{-\tau |A|} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\lambda \, \frac{\sinh \tau}{\cosh \tau - \cos \lambda} \, e^{-i\lambda A} \\ &\underset{\tau \simeq 0}{\simeq} \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \, \frac{\tau}{\tau^2 + \lambda^2} \, e^{-i\lambda A} . \end{aligned} \tag{4.4}$$

* We only consider here the case of spinless quarks. Therefore, we make no effort to avoid "backtracks" (i.e. "spikes") on the loop contour, which in the spinning case are forbidden by the factor (2.18). In any case, summing over backtracks only leads to a finite renormalization of the hopping parameters η_0, η_1 in eq. (4.7), which does not influence the spectrum in the continuum limit when physical masses are introduced.

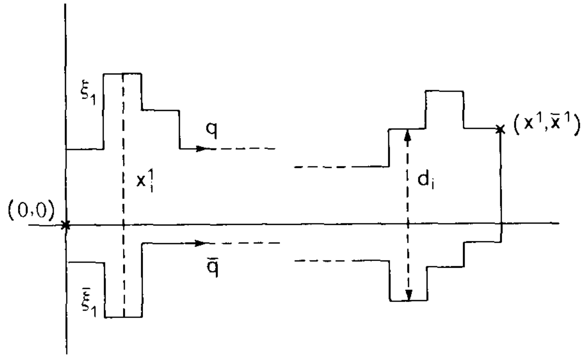


Fig. 8. Summation variables for q$\bar{q}$ loops evolving together in time.

The first step is the most difficult. For instance the requirement that no intersection takes place involves $\frac{1}{2}N(N-1)$ conditions for N links, which are not likely to be tractable in general by a transfer matrix method. Therefore we shall restrict ourselves in the following to a class of contours for which the no-intersection condition is approximately obeyed.

To be more definite, we label a loop C by its q($\bar{q}$) link variables $(\sigma_i^0, \sigma_{i,r}^1)((\bar{\sigma}_j^0, \bar{\sigma}_{j,s}^1))$, where we denote by $i=1,\dots,N_0$ ($j=1,\dots,\bar{N}_0$) the steps in the x^0 direction (that we want to single out) and by $r=1,\dots,n_i$ the steps in the x^1 direction for a given i ($i=1,\dots,N_0+1$). We shall also define the cumulated variables (fig. 8)

$$\begin{cases} x_o = \sum\limits_{i=1}^{N_0} \sigma_i^0 \\ x_i^1 = \sum\limits_{j=1}^{i} \xi_j\,, \qquad \xi_i = \sum\limits_{r=1}^{n_i} \sigma_{i,r}^1\,, \end{cases} \tag{4.5}$$

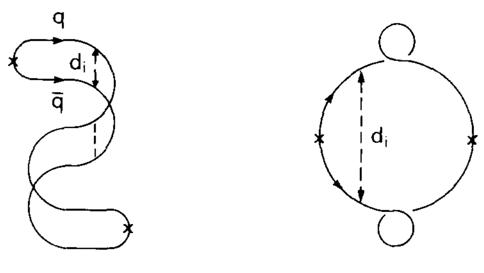
which specify the (x^0, x^1) positions of q, and similarly for $\bar{q}$.

Then the restricted class of contours that we consider is the one for which q and $\bar{q}$ evolve together in x^0, i.e.

$$j=i\,, \qquad \sigma_i^0 = \bar{\sigma}_i^0\,, \qquad N_0 = \bar{N}_0\,. \tag{4.6}$$

This allows to define a "distance" $x_i^1 - \bar{x}_i^1 = d_i$ between q and $\bar{q}$ lines. We shall say that an equal-time intersection of such lines occurs when d_i changes sign for some i. Therefore, the no-intersection condition that we shall consider is $d_i \geq 0$ ($i=1,\dots,N_0$). Notice that this constraint still allows (i) simultaneous self-intersections of both the q and $\bar{q}$ lines and (ii) intersections of the q line with the $\bar{q}$ line at different times (figs. 9a, b). We shall discuss their relevance later on.

Let us now define the Green function Δ_+ for which the sum in (4.3) is restricted to loops without the equal-time self-intersections just defined. First we shall assign to them the algebraic area weight and later on we shall integrate on the parameter



(a) (b)

Fig. 9. (a) $q\bar{q}$ -type and (b) $qq(\bar{q}\bar{q})$-type self-intersections which are still allowed by the constraint $d_i > 0$.

λ as in eq. (4.4) in order to get the weight $e^{-\tau|A|}$. Moreover, since the x^0 and x^1 directions play an asymmetrical rôle in our contours we shall also consider distinct hopping parameters η_0, η_1. By further performing a Fourier transform to the momentum $P^\mu = (P^0, P^1)$ we have therefore

$$\Delta_+(P^\mu;\eta_0,\eta_1,\lambda) = \sum_C (\tfrac{1}{2}\eta_0)^{2N_0}(\tfrac{1}{2}\eta_1)^{N_1(C)} \exp[-i\lambda A(C)] \times \exp[i(P^0x^0 + P^1x^1)]\Theta(C)\,, \tag{4.7}$$

where (x^0, x^1) are related to $(\sigma_i^0, \sigma_{i,r}^1)$ by eq. (4.5); the area A is defined as in eq. (2.3), i.e.

$$A(C) = \sum_{i=1}^{N_0} \sigma_i^0(x_i - \bar{x}_i) = \sum_{i,j} \sigma_i^0(\xi_j - \bar{\xi}_j)\Theta(i-j)$$
$$\Theta(n) = 1\,,\quad n \geq 0\,;\qquad \Theta(n) = 0\,,\quad n<0\,, \tag{4.8}$$

the perimeter in the 1-direction is $N_1 = \sum_{i=1}^{N_0} n_i$, and we have introduced the no-intersection and closure constraint

$$\Theta(C) = \prod_{i=1}^{N_0} \Theta(x_i^1 - \bar{x}_i^1)\delta(x_{N_0+1}^1 - \bar{x}_{N_0+1}^1)$$
$$= \prod_{i=1}^{N_0} \int_{-\pi}^{\pi} \frac{d\delta_i}{2\pi} \frac{e^{i\delta_i(x_i^1 - \bar{x}_i^1)}}{1 - e^{-\varepsilon - i\delta_i}} \int_{-\pi}^{\pi} \frac{d\delta_{N_0+1}}{2\pi} e^{i\delta_{N_0+1}(x_{N_0+1}^1 - \bar{x}_{N_0+1}^1)}\,. \tag{4.9}$$

It is convenient to express (4.7) in terms of the variables σ_i^0 and $\xi_i = x_i^1 - x_{i-1}^1$ defined in (4.5) because they occur directly in the expression (4.8) for the area. Therefore, by introducing the "q-propagator" in the 1-direction

$$g_1(\eta_1, \xi) = \sum_{\{\sigma_k^1, n_1\}} (\tfrac{1}{2}\eta_1)^{n_1}\delta\left(\xi - \sum_{k=1}^{n_1} \sigma_k^1\right)\,, \tag{4.10}$$

and by using (4.3), (4.8) and (4.9), we can rewrite (4.7) in the form

$$\Delta_+(P^\mu;\eta_0,\eta_1,\lambda)=\sum_{\{\xi_i,\sigma_i^0\}}\int\prod_{i=1}^{N_0+1}\left(\frac{\mathrm{d}\delta_i}{2\pi}\right)\mathrm{e}^{i\phi}\,(\tfrac{1}{2}\eta_0)^{2N_0}$$
$$\times\prod_{i=1}^{N_0}\left[\frac{g_1(\eta_1,\xi_i)g_1(\eta_1,\bar{\xi}_i)}{1-\mathrm{e}^{-\varepsilon-i\delta_i}}\right],$$

$$\phi=\sum_{i=1}^{N_0}(\lambda\sigma_i^0+\delta_i)\sum_{j=1}^{i}(\xi_j-\bar{\xi}_j)+P^0\sum_{i=1}^{N_0}\sigma_i^0+(P^1+\delta_{N_0+1})\sum_{j=1}^{N_0+1}\xi_j-\delta_{N_0+1}\sum_{j=1}^{N_0+1}\bar{\xi}_j\,. \quad (4.11)$$

4.2. OPERATOR FORMALISM

In order to perform the sums in (4.11) let us change variables to "momenta" p_j defined as $p_{N_0+1}=\delta_{N_0+1}$, and

$$p_j=\sum_{i=j}^{N_0}(\lambda\sigma_i^0+\delta_i)+\delta_{N_0+1}\qquad(j=1,\dots,N_0)\,,$$
$$\delta_j=p_j-p_{j+1}-\lambda\sigma_j^0\,. \quad (4.12)$$

so that the exponent in (4.11) becomes

$$\phi=\sum_{j=1}^{N_0+1}[\xi_j(\tfrac{1}{2}P^1+p_j)+\bar{\xi}_j(\tfrac{1}{2}P^1-p_j)]+P^0\sum_{j=1}^{N_0}\sigma_j^0\,. \quad (4.13)$$

We can now perform the sums over ξ_i, $\bar{\xi}_i$ and σ_i^0 (which are factorized) by introducing the Fourier transforms

$$\tilde{g}_1(\eta_1,p)=\sum_{\{\sigma_k^1,n_1\}}(\tfrac{1}{2}\eta_1)^{n_1}\exp\left(ip\sum_{k=1}^{n_1}\sigma_k^1\right)=(1-\eta_1\cos p)^{-1}\,,$$
$$\tilde{V}_{P_0}(p)=\tfrac{1}{4}\eta_0^2\sum_{\sigma^0}\frac{\mathrm{e}^{i\sigma^0P_0}}{1-\mathrm{e}^{-\varepsilon-ip+i\lambda\sigma^0}}\,, \quad (4.14)$$

in terms of which (4.11) reads as follows:

$$\Delta_+(P^\mu;\eta_0,\eta_1,\lambda)=\sum_{N_0=0}^{\infty}\int_{-\pi}^{\pi}\prod_{j=1}^{N_0+1}\left(\frac{\mathrm{d}p_j}{2\pi}\right)\prod_{j=1}^{N_0}[\tilde{g}_1(\eta_1;\tfrac{1}{2}P^1+p_j)$$
$$\times\tilde{g}_1(\eta_1;\tfrac{1}{2}P^1-p_j)\tilde{V}_{P^0}(p_j-p_{j+1})]$$
$$\times\tilde{g}_1(\eta_1;\tfrac{1}{2}P^1+p_{N_0+1})\tilde{g}_1(\eta_1;\tfrac{1}{2}P^1-p_{N_0+1})\,. \quad (4.15)$$

It is apparent from the final expression (4.15) that

$$G_{P_1}^0(q)=\tilde{g}_1(\eta_1;\tfrac{1}{2}P_1+p)\tilde{g}_1(\eta_1;\tfrac{1}{2}P_1-p) \quad (4.16)$$

plays the rôle of the free "q$\bar{q}$ propagator", while

$$V^{+}_{P_0}(x) = \int_{-\pi}^{\pi} \frac{dp}{2\pi} e^{ipx} \tilde{V}(p) = \tfrac{1}{4}\eta_0^2 \cos(P_0 - \lambda x)\Theta(x) \tag{4.17}$$

plays the rôle of the potential in an analogue quantum mechanical problem.

By then introducing an operator notation in the conjugate variables (x, p) we obtain from (4.15)

$$\begin{aligned} \Delta_+(P^\mu;\eta_i,\lambda) &= \int \frac{dp_{N_0+1}\,dp_1}{(2\pi)^2} \langle p_{N_0+1}|G^0_{P^1}(1 - V^+_{P^0}G^0_{P^1}]^{-1}|p_1\rangle \\ &= \langle x=0|G^0_{P^1}(1 - V^+_{P^0}G^0_{P^1})^{-1}|x=0\rangle \equiv (G_+)_{00}\,, \end{aligned} \tag{4.18}$$

from which, by (4.4), the final form of the no-intersection propagator follows:

$$\begin{aligned} \Delta_+(P^\mu;\eta_i,\tau) &= \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} \frac{\sinh\tau}{\cosh\tau - \cos\lambda} \langle 0|G^0_{P^1}(1 - V^+_{P^0}G^0_{P^1})^{-1}|0\rangle \\ &\equiv \langle G_+\rangle_{00}\,. \end{aligned} \tag{4.19}$$

We can see from (4.19) that the original problem of the QCD_2 spectrum reduces, for the non-intersecting loops, to the one of finding the poles of the Green function G_+ of the potential system defined by (4.16) and (4.17). The analytic form of the spectrum will be analyzed in sect. 5.

4.2. FORWARD AND BACKWARD EVOLUTION

The expression (4.18) sums over loops without equal-time intersections with the weight $\exp(-\tau|A|)$, where A is the algebraic area. This may give a wrong assignment for intersections of the type in fig. 9, which are not avoided by the constraint $d_i \geq 0$. Such possibilities occur because q and $\bar{q}$ evolve both forward and backward in "time" (i.e. $\sigma_0^i = \pm 1 = \bar{\sigma}_0^i$) and turn out to be depressed when large perimeters are damped, i.e. for massive quarks (cf. sect. 5).

One may think of avoiding the spurious self-intersections altogether by taking only forward evolution*. However this turns out to be equivalent to a nonrelativistic limit and badly violated rotational invariance. In fact, by inserting $\sigma_i^0 = 1$ in eq. (4.11) we obtain a result like (4.19) where V^+ is replaced by

$$V^+_{NR}(x) = \tfrac{1}{8}\eta_0^2 \exp[i(P_0 - \lambda x)]\Theta(x)\,. \tag{4.20}$$

Since this potential is one-sided in the parameter λ, the projection (4.19) is equivalent to the replacement $\lambda = -i\tau$. Therefore, in the Minkowski region $P_0 = -iE$ we get the result

$$\Delta_+^{NR}(-iE, P_1;\eta_i,\tau) = \langle 0|[(G^0_{P_1})^{-1} - \tfrac{1}{8}\eta_0^2 \exp(E - \tau x)\Theta(x)]^{-1}|0\rangle\,. \tag{4.21}$$

* This gives rise to "directed" self-avoiding walks. See e.g. [18].

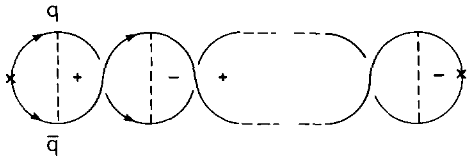


Fig. 10. A q$\bar{\mathrm{q}}$ loops with many equal-time intersections.

The spectrum of (4.21) is easily obtained in the continuum limit of $\tau \to 0$ with E^2/τ fixed in terms of the hamiltonian problem

$$\left[\frac{1}{2\mu}p^2 + \tau x\Theta(x)\right]\psi = (E - E_0)\psi\,, \tag{4.22}$$

where E_0 and μ are mass parameters which are η_i dependent.

Eq. (4.22) yields a spectrum which is typical of a nonrelativistic string ($E - E_0 \sim (n\tau)^{2/3}$) and will be shown in sect. 5 to arise in the nonrelativistic limit of eq. (4.19). In other words in our euclidean lattice we are forced to consider both forward and backward evolution by rotational invariance. This seems to be not the case for path integrals in Minkowski space [10], for which the forward propagation condition has a Lorentz-invariant meaning.

4.4. SUM OVER INTERSECTIONS

It remains to be discussed how the result (4.19) is modified for a general loop with q and $\bar{\mathrm{q}}$ evolving together ($\sigma_i^0 = \bar{\sigma}_i^0$, eq. (4.6)). From fig. 10 it is clear that such loops will consist of successive portions with $d_i \geq O$ (C_+) and $d_i \leq O$ (C_-), where any of the two can be the first, or the last. Correspondingly, the sum (4.3) with the algebraic area weight will be given by operator products of the type*

$$G_+V_+(G_-V_-G_+V_+)^n G_-\,, \tag{4.23}$$

where

$$G_+(P_0, \lambda) = (G_0^{-1} - V_+)^{-1} \tag{4.24}$$

and the first G_+V_+ (last V_+G_-) may also be replaced by 1.

We have now to perform on each factor $(GV)_\pm$ occurring in (4.23) the average (4.19) needed in order to obtain the weight (4.3). This amounts to the replacements

$$(G_+)_{xy} \to \langle G_+\rangle_{xy}\,, \qquad (G_-)_{xy} \to \langle G_-\rangle_{xy} = \langle G_+\rangle_{-x,-y}\,. \tag{4.25}$$

* Notice that, in order to avoid double counting of $d_i = 0$, i.e. of $x = 0$, in (4.23), one should define $\Theta(0) = \frac{1}{2}$ in the expression (4.17) for the potential.

Therefore, by summing over all the possibilities we obtain the final result

$$\Delta(P^\mu;\eta_i,\tau)=\sum_{\langle+,-\rangle}\langle G_+V_+\rangle\langle G_-V_-\rangle\cdots\langle G_-V_-\rangle G_0$$
$$=(1+\langle G_+V_+\rangle)(1-\langle G_-V_-\rangle\langle G_+V_+\rangle)^{-1}(1+\langle G_-V_-\rangle)G_0\,. \tag{4.26}$$

Let us first remark that the operator series (4.26) would simply reduce to a geometric series, were it not for the fact that G_0 allows some evolution between positive and negative x's. However, since the series is alternating, the matrix elements of $G_+V_+(G_-V_-)$ are always between x-values of opposite sign, which only overlap at $x=0$. It is then easy to make this evolution explicit by using the factorized form

$$(G_+)_{yx}=(G_+)_{00}\psi_+(x)\psi_-(y)\qquad(y\leqslant 0\leqslant x)\,, \tag{4.27}$$

where $\psi_\pm$ are the solutions of the homogeneous equation

$$(G_0^{-1}-V_+)\psi_\pm=0\,, \tag{4.28}$$

which are regular for $x\to\pm\infty$ and are normalized to $\psi_\pm(0)=1$.

Note now that ψ_- is evaluated in (4.27) only for $x\leqslant 0$, where it satisfies the free equation ($V_+=0$) and takes the λ-independent form

$$\psi_-(x)=(1+\beta x)\,e^{\alpha x}\,,\qquad G_0^{-1}(i\alpha)=0\quad(x=-n\leqslant 0)\,, \tag{4.29}$$

in which $\cosh\alpha=\frac{3}{2}$ and the parameter β is to be determined by the matching conditions at $x=0$ (cf. sect. 5).

Due to the fact that (4.27) is separable and (4.29) is λ-independent, the averaged operator products (4.26) are easily evaluated in terms of products of the ψ and we get the result

$$\Delta=\langle G_+\rangle_{00}\left[1-\left\langle (G_+)_{00}\sum_{n=0}^{\infty}\psi_-(-n)\psi_+(n)V_+(n)\right\rangle\right]^{-1}$$
$$=\Delta_+\left[1-\sum_{n=0}^{\infty}\langle (G_+)_{-n,n}V_+(n)\rangle\right]^{-1}\,, \tag{4.30}$$

which relates in a simple way the two spectral problems at hand.

5. Spectrum and continuum limit

We shall discuss Δ and Δ_+ in detail only in the continuum limit, where we set $\tau\sim a^2\to 0$ with $x\sqrt{\tau}$ and $p/\sqrt{\tau}$ finite. It is then convenient to define, as in sect. 2, the rescaled variables

$$\hat{x}=\sqrt{\tau}x\,,\qquad \hat{p}=p/\sqrt{\tau}\,,\quad \hat{\lambda}=\lambda/\tau\,,\qquad \hat{P}_0=P_0/\sqrt{\tau}\,, \tag{5.1}$$

in terms of which G_+ should scale as $(P_0^2)^{-1}\sim\tau^{-1}$.

In order to get $G_+^{-1} = G_0^{-1} - V_+ = O(\tau)$ for $x>0$, we have to choose the continuum values of η_0 and η_1 such as to have*

$$(1-\eta_1)^2 = \tfrac{1}{4}\eta_0^2 = \tfrac{1}{2}\eta_1(1-\eta_1)\,, \tag{5.2}$$

modulo terms of the order of the mass parameter $m^2 = O(\tau)$. More precisely, we find, by setting $P_1 = 0$ from now on,

$$\frac{4}{\eta_0^2}(G_0^{-1} - V_+) \simeq \tfrac{1}{2}[4(\hat{p}^2 + m^2) + (\hat{P}_0 - \hat{\lambda}\hat{x})^2]\tau \qquad (x>0) \tag{5.3}$$

for $\eta_1 = \eta_0 = \frac{2}{3}(1 - \frac{1}{3}m^2)$, where $m^2 = \hat{m}^2\tau$.

On the other hand, the naïve continuum limit does not apply for $x \leq 0$, because in this region G_+ takes the values

$$\hat{G}_+^{-1} = \frac{4}{\eta_0^2} G_+^{-1} = \begin{cases} (3-2\cos p)^2 = \dfrac{4}{\eta_0^2} G_0^{-1} = \hat{G}_0^{-1} & (x<0) \\ (3-2\cos p)^2 - \frac{1}{2} & (x=0)\,, \end{cases} \tag{5.4}$$

which are $O(1)$ for $p = \hat{p}\sqrt{\tau} \to 0$. This is natural because the potential problem for G_+ is one-sided ($x>0$), with a potential barrier for $x<0$, which becomes infinite in the continuum limit (fig. 11a). It is the matching between the two regions which provides the value of the $x=0$ matrix element $(G_+)_{00}$ occurring in the expression (4.19) of Δ_+ because of the loop closure condition.

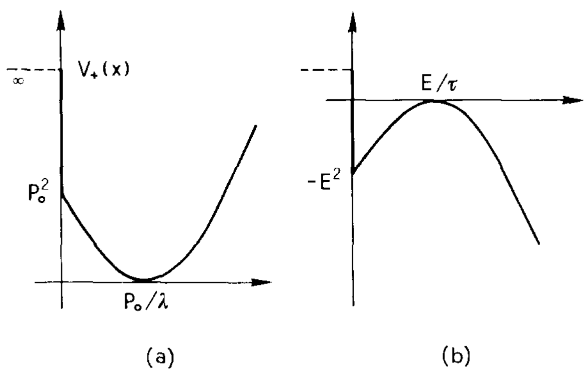


Fig. 11. Shape of the potential in the continuum limit for (a) the euclidean case and (b) the Minkowski case.

* The second condition in (5.2) fixes the relative normalization of kinetic and potential terms and is here chosen so as to recover Lorentz invariance for small values of the spatial momentum P_1, whose effect is to replace $4m^2$ with $4m^2 + P_1^2$ in eq. (5.3).

5.1. EXPLICIT FORM OF Δ_+ AND Δ

If we limit ourselves to $x>0$, the continuum limit of $\hat{G}_+(\hat{x},\hat{y})=\frac{1}{4}\eta_0^2(G_+)_{x,y}$ satisfies, by (5.3), the equation

$$[\hat{p}^2+\hat{m}^2+\tfrac{1}{4}(\hat{P}_0-\hat{\lambda}\hat{x})^2]\hat{G}_+=\frac{1}{2\tau}\delta_{x,y}\simeq\frac{1}{2\sqrt{\tau}}\delta(\hat{x}-\hat{y})\,, \tag{5.5}$$

and is therefore given by the expression

$$\hat{G}_+(\hat{x},\hat{y})=\frac{1}{2\sqrt{\tau}}\frac{\psi_+(\hat{x})\psi_-(\hat{y})\Theta(\hat{x}-\hat{y})+\hat{x}\leftrightarrow\hat{y}}{\psi_+\psi_-'-\psi_+'\psi_-}\,, \tag{5.6}$$

where $\psi_\pm$ are the regular solutions already defined. It follows that

$$\tfrac{1}{4}\eta_0^2\Delta_+=(\hat{G}_+)_{00}=\tfrac{1}{2}[l(\alpha)-\sqrt{\tau}L_+(\hat{P}_0,\hat{\lambda})]^{-1}\,, \tag{5.7}$$

where

$$L_+(\hat{P}_0,\hat{\lambda})=\frac{\psi_+'(0)}{\psi_+(0)}\,,\qquad l(\alpha)=\sqrt{\tau}L_-=\sqrt{\tau}\frac{\psi_-'(0)}{\psi_-(0)}\,. \tag{5.8}$$

Notice that while ψ_+ (and therefore L_+) is determined by its regular behaviour for $x\to\infty$, the value for $l(\alpha)$ is provided by matching the expression

$$\psi_-(x)=1+l(\alpha)x\qquad(x=n>0)\,, \tag{5.9}$$

valid for small $\hat{x}=\sqrt{\tau}n=O(\sqrt{\tau})$, with the expression (4.29) for $x<0$.

In fact, by substituting (5.9) and (4.29) into (5.3) and (5.4) for small values of $p=\hat{p}\sqrt{\tau}$, we obtain 3 matching equations which yield

$$l(\alpha)=\tfrac{7}{4}\sqrt{\tfrac{1}{5}}\,,\qquad\beta=-\sqrt{\tfrac{1}{5}}\,, \tag{5.10}$$

and determine the first two moments of ψ_-, i.e.

$$\sum_{n=0}^{\infty}\psi_-(-n)=\tfrac{1}{2}(1+4l(\alpha))\,,\qquad\sum_{n=0}^{\infty}n\psi_-(-n)=2\,. \tag{5.11}$$

While $l(\alpha)$ determines Δ_+ in (5.7), the moments (5.11) determine Δ in (4.30), where the expression (5.9) can be replaced for small $\hat{x}=n\sqrt{\tau}$. We then obtain

$$\begin{aligned}\tfrac{1}{4}\eta_0^2\Delta(\hat{P}_0,\hat{\tau})&=\tfrac{1}{4}\eta_0^2\Delta_+[1-2\langle(\hat{G}_+)_{00}(l+\sqrt{\tau}L_+)\rangle]^{-1}\\&=\frac{\langle(l-\sqrt{\tau}L_+)^{-1}\rangle}{\langle-4\sqrt{\tau}L_+(l-\sqrt{\tau}L_+)^{-1}\rangle}\,.\end{aligned} \tag{5.12}$$

Eqs. (5.7) and (5.12) give the final expression of Δ_+ and Δ in terms of the logarithmic derivative L_+. An interesting point to note is that for $\tau\to0$ Δ scales as $1/\sqrt{\tau}$, i.e.

$$\tfrac{1}{4}\eta_0^2\Delta(\hat{P}_0,\hat{\tau})\underset{\tau\to0}{\Longrightarrow}-(4\sqrt{\tau}\langle L_+\rangle)^{-1} \tag{5.13}$$

and becomes independent of the quantity $l(\alpha)$, which parametrizes the $x=0$ boundary condition on the lattice.

On the other hand, Δ_+ becomes for $\tau\to 0$ the finite quantity $(2l(\alpha))^{-1}$, which is dependent on the form of G_0 and therefore on the lattice regularization. This finite limit is due to the fact that due to the potential barrier in $x<0$, $\psi_-(0)/\psi_-'(0)=O(\sqrt{\tau})$ and implies that the probability of no intersections is small in two dimensions. In fact it can be checked from the m^2 dependence of (5.12) or (5.17) (which is conjugate to the perimeter dependence) that the difference in the scaling properties of Δ_+ and Δ amounts to a relative suppression factor $1/P(\mathrm{C})$ in their statistical weight.

As a consequence, the propagator Δ_+ does not have a continuum limit except at the spectral points, for which, roughly, $L_+\simeq l/\sqrt{\tau}\to\infty$. In contrast, the propagator Δ has singularities for $\langle L_+\rangle=0$ which roughly corresponds to the vanishing of $\psi_+'(0)$ instead of $\psi_+(0)$*. This difference appears as an energy shift in the two spectra, which turn out however to have the same qualitative features for large quantum numbers.

5.2. SPECTRAL PROPERTIES

The actual evaluation of the spectrum in the two cases requires, by (5.7) and (5.13), the knowledge of $L_+=\psi_+'(0)/\psi_+(0)$, where

$$[4p^2+4m^2+(P_0-\lambda x)^2\Theta(x)]\psi_+=0\,,\qquad \psi_+\underset{x\to\infty}{\simeq}\mathrm{e}^{-\lambda x^2/4}\quad(x\geqslant 0)\,,\tag{5.14}$$

and we shall omit all carets from now on, by understanding that P_0, λ, m are measured in proper units of τ. The solution of (5.14), which corresponds to the potential problem in figs. 11a, b, can be related to the solution of the harmonic oscillator

$$\left(-\frac{\partial^2}{\partial z^2}+z^2+4a^2\right)\varphi(z,a^2)=0\tag{5.15}$$

by the change of variables

$$z=\sqrt{\tfrac{1}{2}}(x\sqrt{|\lambda|}\mp P_0/\sqrt{|\lambda|})\,,\tag{5.16}$$

where the $\pm$ sign holds according to whether $\lambda\lessgtr 0$.

By then using (5.16) in the projection integral (4.19) and by rescaling the τ-variable in the continuum limit, we obtain

$$\langle L_+\rangle=\frac{1}{\pi\sqrt{2}}\int_0^\infty\frac{\mathrm{d}\lambda\sqrt{\lambda}}{1+\lambda^2}\left[l\left(\frac{P_0}{\sqrt{2\lambda}},\frac{m^2}{2\lambda}\right)+P_0\leftrightarrow -P_0\right],\tag{5.17a}$$

$$l(z_0,a^2)=\varphi'(z_0,a^2)/\varphi(z_0,a^2)\,.\tag{5.17b}$$

* This observation is particularly transparent in the nonrelativitic limit of subsect. 4.3 where the analytic continuation $\lambda\to -i\tau$ is straightforward.

The integral (5.17) is evaluated in appendix B. In the (semiclassical) limit $P_0^2 \gg m^2 \gg \tau$ we obtain the result

$$\langle L_+\rangle \simeq -\sqrt{P_0^2+4m^2} \sum_{n=0}^{\infty} \left[\frac{\rho(P_0)}{2\pi(n+\frac{3}{4})}\right]^{1/2} \frac{1}{\rho(P_0)+2\pi(n+\frac{3}{4})}, \tag{5.18}$$

where

$$\rho(P_0) \equiv \tfrac{1}{2}P_0\sqrt{P_0^2+4m^2}+m^2 \log \frac{(P_0+\sqrt{P_0^2+4m^2})^2}{-4m^2} \tag{5.19}$$

has the physical meaning of classical action for the potential problem (5.14), i.e. $\rho(P_0) = -\oint p \, \mathrm{d}x$.

From (5.18) it follows that $\langle L_+\rangle$ does not have poles or zeros for real P_0, while it acquires pole singularities in the Minkowski region of positive $E = iP_0$ for

$$2\pi\tau(n+\tfrac{3}{4}) = -\rho(-iE) = \tfrac{1}{2}E\sqrt{E^2-4m^2} - m^2 \log \frac{(E+\sqrt{E^2-4m^2})^2}{4m^2}. \tag{5.20}$$

These poles give directly the spectrum of Δ_+.

The spectrum of Δ is instead obtained from the zeros of $\langle L_+\rangle$ in (5.18) and yields values for E_n which are intertwined with those in (5.20) and are therefore of the form

$$\tfrac{1}{2}E\sqrt{E^2-4m^2} - m^2 \log \frac{(E+\sqrt{E^2-4m^2})^2}{4m^2} = 2\pi\tau(n+\tfrac{3}{4}) - \pi\tau, \tag{5.21}$$

where the energy shift $(-\pi\tau)$ follows from the nonrelativistic limit (5.26).

The final results (5.20) and (5.21) are very similar to 't Hooft's spectrum, which in turn agrees with string quantization in Minkowski space [9, 10]. However it seems not possible, in our case, to recast the eigenvalue condition $\langle L_+\rangle = 0$ in (5.17) and (5.18) in the form of the hamiltonian problem for the string, which in the temporal gauge reads

$$E\psi = (2\sqrt{p^2+m^2}+\tau|x|)\psi = H_{\mathrm{st}}\psi. \tag{5.22}$$

This is due to the rôle of self-intersections, which make the analytic continuation $(\lambda, P_0) \to (-i\tau, -iE)$ a difficult task, involving the projection integral (5.17).

The expression of $\langle L_+\rangle$ is also evaluated in appendix B for intermediate values of $m^2 = O(\tau)$ and for $m^2/\tau \to 0$. In such a case the q$\bar{\text{q}}$ states become unstable, in the sense that

$$E_n^2 = E_n^{0^2} - i\Gamma, \qquad \Gamma = \log(1+\mathrm{e}^{-\pi m^2/\tau}) = \mathrm{O}(\mathrm{e}^{-\pi m^2/\tau}), \tag{5.23}$$

where the $E_n^{0^2}$ are given by eq. (5.21). The explanation of such a phenomenon is roughly obtained by naïve* continuation of the potential in fig. 11a from real P_0 to

* This naïve picture gives however more states than actually found in (5.20) and (5.21). This is because the correct analytic continuation should take carefully into account the pinch singularities in the contour of eq. (5.17a) (appendix B).

$P_0 = -iE$ (fig. 11b). We can see that the width is expected to be given by the penetration coefficient of the potential barrier, i.e. $\sim\exp(-\pi m^2/\tau)$.

This instability is not found in QCD_2 in the planar approximation which corresponds to a zero-width limit [7, 8]. Its occurrence in our model for small m^2 is to be traced back to the different account of q and $\bar{q}$ contours which have separate or unequal time self-intersections (fig. 9). Presumably, the algebraic assignment of the area in those cases leads to cancellations among subcontours which, if large perimeters are not damped, give rise to unstable states. It is instructive in this respect to discuss the nonrelativistic limit of eq. (5.14), which yields (for $\lambda = -i\tau$, $P_0 = -iE$) the eigenvalue equation

$$(E-2m)\psi_+ = \left[\frac{p^2}{m} + \tau x\Theta(x)\right]\psi_+ . \tag{5.24}$$

This is valid for $E - 2m \ll 2m$, and should be compared with eq. (4.22) derived for $q\bar{q}$ propagation which is only forward in time.

In this limit, which corresponds to a linear approximation of the potential in fig. 11b, both the instability and the wrong self-intersections have disappeared. Furthermore the spectrum of Δ corresponds to the bona fide hamiltonian eigenvalue problem

$$(E-2m)\psi = \left[\frac{p^2}{m} + \tau|x|\right]\psi = H\psi , \tag{5.25}$$

which is obtained by reflection of (5.24) around $x = 0$ and is also identical to the NR limit of the string equation (5.22).

The spectra of (5.24) and (5.25) in the semiclassical limit are given by

$$\oint p\,\mathrm{d}x = \tfrac{4}{3}\sqrt{m}(E-2m)^{3/2} = 2\pi\tau(n+\delta) , \tag{5.26}$$

where $\delta = \frac{1}{4}$ for (5.25) and $\delta = \frac{3}{4}$ for (5.24). This agrees with (5.20) and (5.21) with an energy shift $(-\pi\tau)$.

Note, however, that in the case of the complete potential (5.25) only even eigenfunctions (corresponding to values $2n$ of the quantum number and to $L_+(0) = 0$) are kept in (5.26), due to Bose symmetry.

6. Summary and discussion

In this paper we have attempted to solve by one-dimensional transfer matrix methods some loop resummation problems in two dimensions with area-dependent weight.

Exact results have been obtained for the Green function with the algebraic determination of the area (sect. 2) and for the partition function with a weight of sign type (sect. 3). However, the most interesting cases (like QCD_2 and the Ising model in a magnetic field) basically correspond to the absolute value determination of the area. This one can be treated by our method only if self-intersections are counted and each subcontour is separately considered.

For the case of QCD_2 we have set up in sect. 4 an approximate no-intersection condition for contours which evolve together in time and we have computed in sect. 5 the corresponding Green functions with or without self-intersections. The no-intersection propagator does not have the correct continuum limit, meaning that the no-intersection probability is small in two dimensions. The spectrum of the complete propagator differs from the preceding one by an energy shift, and agrees with 't Hooft's in the continuum limit. Therefore, our sum-over-loops approach reproduces the main features of QCD_2 and can be considered as a good alternative to string quantization in Minkowski space. In particular, it illustrates the important rôle of self-intersecting loops in two dimensions.

There are, however, some unsatisfactory features in our model treatment of self-intersections. One is the fact that the time direction is singled out in the definition of $q\bar{q}$ contours evolving together in time. This generally violates rotational invariance, which is recovered in the continuum limit only up to order P_1^2/M^2, where P_1 is the spatial momentum. The other is that we are able to count equal-time intersections but we still have no control over more complicated ones, as in fig. 9. The fact that the area weight assigned to the latter may be sometimes inconsistent with QCD_2 in the planar approximation is presumably the source of the instability (sect. 4) that we find in the spectrum for small values of the quark mass (i.e. when large perimeters are not damped).

Despite these drawbacks, we believe that the more general approach of (i) counting self-intersections and (ii) constructing the area weight by the exact transfer matrix method of sect. 2 is a good analytical way of learning more about QCD_2 on the lattice. One may envisage doing it systematically starting from the problem of self-avoiding walks with algebraic-area weight. In view of recent progress (see e.g. [9]) in the field, this may be soon a tractable problem and a starting point for the study of self-intersecting loops.

It is a pleasure to thank Enrico Onofri, Giuseppe Marchesini and Giorgio Parisi for interesting discussion and comments. Two of us (M.A. and M.C.) are also grateful to the CERN Theory Division for hospitality while part of this work was being done.

Appendix A

CONTINUUM LIMIT OF PROPAGATOR WITH ALGEBRAIC-AREA WEIGHT

Here we want to prove expression (2.23) of the composite particle propagator Δ with naïve algebraic-area weight and to study its analyticity properties in the energy variable.

In the continuum limit of λ, $a \to 0$ with λ/a^2 fixed, the transfer matrix (2.9) reduces to the form (2.21)

$$1-\eta K \simeq \tfrac{1}{4}\lambda(\hat{m}^2+\hat{x}^2+\hat{p}^2)=\tfrac{1}{2}\lambda H(\hat{m})\,, \tag{A.1}$$

where we have introduced the rescaled variables

$$\hat{m}=m/\sqrt{\lambda}\,,\qquad \hat{x}=x\sqrt{\lambda}\,,\qquad \hat{p}=p/\sqrt{\lambda}\,, \tag{A.2}$$

and the mass parameter m by choosing the value $\eta \simeq 1-\frac{1}{4}m^2$ of the hopping parameter.

Therefore expression (2.22) of Δ can be evaluated in terms of the harmonic oscillator kernel

$$\langle\hat{x}|H^{-1}(\hat{m})|\hat{y}\rangle=\int_0^\infty \mathrm{d}s(2\pi\sinh s)^{-1/2}$$
$$\times\exp\left\{-\tfrac{1}{2}m^2 s-\frac{(\hat{x}^2+\hat{y}^2)\cosh s-2\hat{x}\hat{y}}{2\sinh s}\right\}, \tag{A.3}$$

where s is Schwinger's proper time (which in the continuum corresponds to the number N of steps in the random walk), and the $|\hat{x}\rangle$ states have the continuum normalization $\langle\hat{x}|\hat{y}\rangle=\delta(\hat{x}-\hat{y})$.

Let us not that in eq. (2.22) the $\hat{q}_0$ integration can be performed to yield a factor $(\sin(\pi\hat{x}/\sqrt{\lambda}))(\pi\hat{x})^{-1}$ which in the continuum limit is equivalent to $\delta(\hat{x})$. Therefore we obtain

$$\Delta(P_0;\eta,\lambda)=\frac{4}{\lambda}\int_{-\pi/\sqrt{\lambda}}^{\pi/\sqrt{\lambda}}\frac{\mathrm{d}\hat{q}_1}{2\pi}\langle\hat{q}_1|H^{-1}(\hat{m})\,\mathrm{e}^{-i\hat{x}\hat{P}_0}H^{-1}(\hat{m})\,\mathrm{e}^{i\hat{x}\hat{P}_0}|\hat{q}_1\rangle\,. \tag{A.4}$$

By using the representation (A.3) in eq. (A.4), we obtain

$$\Delta(P_0;\eta,\lambda)=\frac{1}{\pi^2\lambda}\int_0^\infty \mathrm{d}s_1\mathrm{d}s_1\,\mathrm{e}^{-(\hat{m}^2/2)(s_1+s_2)}\int_{-\infty}^{+\infty}\mathrm{d}y\int_{-\pi/\sqrt{\lambda}}^{\pi/\sqrt{\lambda}}\mathrm{d}q_1$$
$$\times(\sinh s_1\sinh s_2)^{-1/2}\,\mathrm{e}^{-\phi}\,,$$
$$\phi=\tfrac{1}{2}\left[(q_1^2+y^2)\left(\frac{\cosh s_1}{\sinh s_1}+\frac{\cosh s_2}{\sinh s_2}\right)-2q_1y\left(\frac{1}{\sinh s_1}+\frac{1}{\sinh s_2}\right)\right]+i\hat{P}_0(y-q_1)\,. \tag{A.5}$$

Here the y-integration can be performed to yield a new gaussian integral in q_1, whose phase is stationary for

$$q_1^0=-\tfrac{1}{2}i\hat{P}_0\frac{\cosh(\chi\Sigma)-\cosh\Sigma}{\sinh\Sigma}\,,\qquad \Sigma=\tfrac{1}{2}(s_1+s_2)\,,\qquad \Delta=\tfrac{1}{2}(s_1-s_2)=\chi\Sigma\,, \tag{A.6}$$

By displacing the q_1 contour along the imaginary axis and letting the integration boundary $|q_{1\,\max}| = \pi/\sqrt{\lambda} \to \infty$, this integral can also be performed to yield

$$\Delta(P_0;\,\eta,\lambda) = \frac{2}{\pi\lambda}\int_0^\infty \frac{\mathrm{d}\Sigma\,\Sigma}{\sinh\Sigma}\int_{-1}^{1}\mathrm{d}\chi\,\exp(-\phi_0(\Sigma,\chi))\,,$$

$$\phi_0(\Sigma,\chi) = \hat{m}^2\Sigma + \tfrac{1}{2}\hat{P}_0^2\frac{\cosh\Sigma - \cosh(\chi\Sigma)}{\sinh\Sigma}\,. \tag{A.7}$$

We can check that (A.7) admits a perturbative expansion in λ, whose first term is the free propagator. In fact, by rescaling the variable $\Sigma \to \lambda\Sigma$ and by expanding in λ the phase ϕ_0, we obtain

$$\phi_0 \simeq [\hat{m}^2 + \tfrac{1}{4}\hat{P}_0^2(1-\chi^2)]\Sigma\lambda = [m^2 + \tfrac{1}{4}P_0^2(1-\chi^2)]\Sigma\,,$$

$$\Delta(P_0;\,\eta,0) = \frac{2}{\pi}\int_{-1}^{1}\mathrm{d}\chi\,[m^2 + \tfrac{1}{4}P_0^2(1-\chi^2)]^{-1}$$

$$= \frac{8}{\pi}[P_0^2(P_0^2+4m^2)]^{-1}\log\frac{P_0+\sqrt{P_0^2+4m^2}}{4m^2}\,,\qquad (P_0^2(P_0^2+4m^2)>0)\,. \tag{A.8}$$

This expression exhibits a normal threshold branch cut in the P_0^2 plane for $-P_0^2 = E^2 \geqslant 4m^2$, due to the pinch (end-point) singularity of the $\chi(\Sigma)$ integral at $\chi = 0$ ($\Sigma = 0$).

The same singularity pattern holds for $\lambda \neq 0$ also, because the integral representation (A.7) can be rewritten as

$$\Delta(P_0;\,\eta,\lambda) = \frac{2}{\pi\lambda}\int_0^\infty \frac{\mathrm{d}\Sigma\,\Sigma}{\sinh\Sigma}\int \frac{\mathrm{d}\phi_0\,\mathrm{e}^{-\phi_0}}{(\partial\phi_0/\partial\chi)(\Sigma,\phi_0)}\,, \tag{A.9}$$

where $\partial\phi_0/\partial\chi = -\frac{1}{2}P_0^2\Sigma(\sinh(\chi\Sigma)/\sinh\Sigma)$ only vanishes for $\chi\Sigma = 0$. Close to $\Sigma = 0$ the χ-integration shows the pinch singularity at $\chi = \pm(1+4m^2/P_0^2)^{1/2}$ as in (A.8) for $-P_0^2 \geqslant 4m^2$.

Appendix B

WAVE FUNCTION IN THE CONTINUUM LIMIT

Here we shall study the detailed properties of the wave function ψ_+ and of its logarithmic derivative, and we shall evaluate the averaging integral (5.17).

The defining equation for ψ_+ (where all carets are omitted)

$$[p^2 + m^2 + \tfrac{1}{4}(P_0 - \lambda x)^2]\psi_+(x) = 0\,,\qquad \psi_+(x) \underset{x\to\infty}{\simeq} \exp(-\tfrac{1}{4}\lambda x^2) \tag{B.1}$$

is reduced to that of an harmonic oscillator by the change of variables (5.16), i.e. $z(x) = \sqrt{\frac{1}{2}}(x\sqrt{|\lambda|} \mp P_0/\sqrt{|\lambda|})$ according to whether $\lambda \gtrless 0$. We then obtain

$$\psi_+ = \varphi(z(x), m^2/\lambda) = \mathrm{e}^{-z^2/2}\,\Psi(\tfrac{1}{4}+a^2, \tfrac{1}{2}, z^2)\,,\qquad a^2 = \frac{m^2}{2\lambda}\,, \tag{B.2}$$

where Ψ is the solution of the hypergeometric equation which is regular for $z \to +\infty$ (see e.g. [20]), i.e.

$$\Psi(\tfrac{1}{4}+a^2, \tfrac{1}{2}, z^2) = \phi(\tfrac{1}{4}+a^2, \tfrac{1}{2}, z^2) - z\frac{\Gamma(\tfrac{3}{4}+a^2)}{\Gamma(\tfrac{1}{4}+a^2)}\phi(\tfrac{3}{4}+a^2, \tfrac{3}{2}, z^2) \tag{B.3a}$$

$$= \frac{\Gamma(\tfrac{3}{4}+a^2)}{\Gamma(\tfrac{1}{4}+a^2)}\int_0^\infty e^{-z^2 s} s^{-\frac{3}{4}+a^2}(1+s)^{-3/4-a^2}\, ds \tag{B.3b}$$

where the last representation holds for $a^2 > -\frac{1}{4}$ and $\operatorname{Re} z > 0$.

We shall now evaluate the integral (5.17a) for the case where both P_0^2 and m^2 are large. By using the asymptotic behaviour of (B.3b) (and of its analytic continuations) for a^2, $z^2 \to \infty$, we find ($\eta = 1/\sqrt{\lambda}$)

$$\begin{aligned} &l(\sqrt{\tfrac{1}{2}}P_0\eta, \tfrac{1}{2}m^2\eta^2) + (P_0 \leftrightarrow -P_0) \\ &= \sqrt{2}P_0\eta \left.\frac{(d/dz)\Psi(\tfrac{1}{4}+\tfrac{1}{2}m^2\eta^2, \tfrac{1}{2}, z^2)}{\Psi(\tfrac{1}{4}+\tfrac{1}{2}m^2\eta^2, \tfrac{1}{2}, z^2)}\right|_{z=P_0\eta/\sqrt{2}} + (P_0 \leftrightarrow -P_0) \\ &= -\sqrt{\tfrac{1}{2}}\eta\sqrt{P_0^2+4m^2}(f(\eta) - f(-\eta)), \end{aligned} \tag{B.4}$$

where $f(\eta)$ is real analytic with the asymptotic behaviour

$$f(\eta) = \begin{cases} \dfrac{\exp(-\frac{1}{2}\eta^2\rho) + i\exp(\frac{1}{2}\eta^2\rho)}{\exp(-\frac{1}{2}\eta^2\rho) - i\exp(\frac{1}{2}\eta^2\rho)}, \quad -2\pi < \arg(\eta^2\rho) < -\pi & \text{(B.5a)} \\ 1, \quad -\pi < \arg(\eta^2\rho) < \pi, & \text{(B.5b)} \end{cases}$$

with ρ given by eq. (B.6).

The result given in (B.4) and (B.5) can also be obtained by semiclassical methods from the action variable

$$\begin{aligned} \rho(P_0, m) &= \tfrac{1}{2}P_0\sqrt{P_0^2+4m^2} + m^2 \log\frac{(P_0+\sqrt{P_0^2+4m^2})^2}{-4m^2} \\ &= -i\tfrac{1}{2}\oint p\, dx \end{aligned} \tag{B.6}$$

for the original potential problem (B.1).

Notice that even if the expression (B.5) is only piecewise analytic, the original function (B.4) is meromorphic in the whole η-plane. Therefore we can use the symmetry $\eta \leftrightarrow -\eta$ of (B.4) in order to rewrite eq. (5.17a) in the form

$$\langle L_+\rangle = -\frac{1}{\pi}\sqrt{P_0^2+4m^2}\int_{-\infty}^{+\infty}\frac{d\eta\, \eta}{1+\eta^4}f(\eta). \tag{B.7}$$

The integrand of eq. (B.7) has the pole structure illustrated in fig. 12 and is well behaved for $|\eta| \to \infty$. Therefore it can be evaluated by the Cauchy theorem. For real

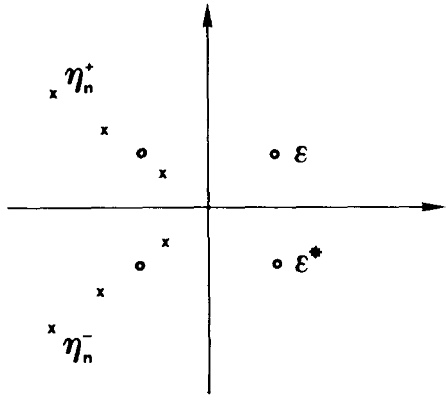


Fig. 12. Poles in the η-plane of the integrand in eq. (C.7). The integration contour runs along the real axis.

P_0 the poles are at $\eta=\pm\varepsilon$, $\pm\varepsilon^*$ ($\varepsilon=\exp(\frac{1}{4}i\pi)$) and at

$$\eta_n^+=-\varepsilon^*\left(\frac{2\pi(n+\frac{3}{4})}{\rho}\right)^{1/2}=-\varepsilon^*\eta_n\,,\qquad \eta_n^-=(\eta_n^+)^*\,,\qquad n=0,1,\ldots,\tag{B.8}$$

where the latter lie in the second and third quadrant respectively and come from the singularities of the asymptotic expression (B.5a). A straightforward calculation then gives

$$\langle L_+\rangle=\begin{cases}-\sqrt{P_0^2+4m^2}\,\dfrac{1}{2i}\left(\displaystyle\sum_n\frac{1+i}{(1+\eta_n)(\eta_n-i)\rho\eta_n}-\text{c.c.}\right) & \text{(B.9a)}\\[2ex] -\sqrt{P_0^2+4m^2}\left(\displaystyle\sum_n\frac{1}{\rho\eta_n(\eta_n^2+1)}+\mathrm{O}\left(\frac{m^2}{E^2}\right)\right), & \text{(B.9b)}\end{cases}$$

where the last expression is obtained by neglecting the ρ-phase (which is small for $m^2\ll E^2$) and coincides, by eq. (B.8), with eq. (5.18) of the text.

For the case when m^2 is small (i.e. $m^2\to 0$) but still $P_0^2\gg 1$, the expression for $\langle L_+\rangle$ can still be obtained from (B.4) and (B.5) except for the replacement

$$\eta^2\rho\to\tfrac{1}{2}\eta^2P_0^2+\tfrac{1}{2}\log 2\,,$$

which follows from the relevant asymptotic behaviour of (B.3) for large z^2 and $a^2\simeq 0$, namely

$$\Psi\simeq\frac{\sqrt{\pi}}{\Gamma(\frac{1}{4})}\begin{cases}\sqrt{2}z^{-1/2}\,, & z\to+\infty\\[1ex] (\sqrt{2}+2i\,\mathrm{e}^{z^2})(-z)^{-1/2}\,, & z\to-\infty\,.\end{cases}\tag{B.10}$$

The poles of $f(\eta)$ in the η-plane are therefore at

$$\eta_n^+=-\frac{\varepsilon^*\sqrt{2}}{P_0}(2\pi(n+\tfrac{3}{4})-\tfrac{1}{2}i\log 2)^{1/2}=-\frac{\varepsilon^*\sqrt{2}}{P_0}\chi_n\,,\qquad \eta_n^-=(\eta_n^+)^*\,,\tag{B.11}$$

and we obtain for $\langle L_+\rangle$ expression (B.9a), but with η_n replaced by χ_n/P_0.

It should be noted that in the case $m \to 0$ the phase in (B.10) should be taken into account in the Wick rotation $P_0 \to -iE$. Since (B.9a) is real analytic (i.e. of the form $f(P_0)+f^*(P_0^*)$), its analytic continuation takes the form $(m^2 \ll E^2)$

$$\langle L_+ \rangle_{P_0=-iE} = \tfrac{1}{2}E\sqrt{E^2-4m^2}\sum_n \left[\frac{1+i}{\chi_n(\chi_n - E)(E+i\chi_n)} - \frac{1-i}{\chi_n^*(\chi_n^*+E)(E+i\chi_n^*)}\right]. \quad \text{(B.12)}$$

The poles of $\langle L_+ \rangle$ in the E-plane come from the first term (from the pinch of $\eta = \eta_n^+$ and $\eta = -\varepsilon$, fig. 12) and are given by

$$E^2 = \chi_n^2 = 4\pi(n+\tfrac{3}{4}) - i\log 2 = E_n^2 - i\Gamma. \quad \text{(B.13)}$$

Therefore these states are not stable, and have a width

$$\gamma_n \simeq \log 2/2E_n .$$

The transition region between $m=0$ and $m \gg \sqrt{\tau}$ can also be treated by analogous methods, with the result that

$$\Gamma = 2E_n\gamma_n = \log(1+\mathrm{e}^{-\pi m^2/\tau}) = \mathrm{O}(\mathrm{e}^{-\pi m^2/\tau}). \quad \text{(B.14)}$$

This value of the width is in agreement with semiclassical arguments on the potential barrier occurring in the naïvely continued potential $V = -\tfrac{1}{2}(E-\tau x)^2$ (fig. 11b).

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