Nuclear Physics B265 [FS15] (1986) 339-363

MEAN FIELD FOR TOPOLOGICAL EXCITATIONS OF Z_2 SPIN AND GAUGE MODELS

A CAPPELLI[1], R LIVI[1], A MARITAN[2] and S RUFFO[1]

Received 19 July 1985

A mean field approach for the disorder parameters is obtained by applying the duality transformation to a Bethe-Peierls type approximation which involves finite clusters Applications to Z_2 models with global and local symmetries are described and systematical improvements are obtained by considering clusters of increasing size For gauge theories a manifestly gauge-invariant mean field is obtained the phenomenology of the spontaneous symmetry breaking is reproduced by considering the suitable order parameters and the standard mean field for gauge theories in four dimensions is reinterpreted

1. Introduction

Mean Field methods (MF) for lattice gauge theories have been extensively studied in the recent years [1] in the functional-variational formulation, especially after it was suggested how to restore gauge invariance [2], broken at an intermediate step of the calculations. The results obtained by this method are quite reliable in four dimensions.

However, the spontaneous symmetry breaking mechanism for gauge theories is not clearly described, while it is manifest in the MF for globally symmetric theories.

In the context of Kramers-Wannier duality transformation [3], a good phenomenology for gauge theories has been derived in terms of condensation of topological excitations [4] ("frustrations", in the Ising model). They are singular configurations of the field which disorder the system non-locally; their condensation in the high-temperature phase is signalled by a suitable "disorder" parameter.

In this paper we present a MF approach based on this disorder parameter, which reproduces the phenomenology of the condensation mechanism.

The most direct way to identify the correct procedure is to consider the three-dimensional case, where duality transformation relates gauge theories to spin theories. More precisely, we shall consider the Ising model on a finite cluster with a border term, where a MF of the Bethe-Peierls (BP) type [5] is defined for the magnetization. Then, by applying the duality transformation, we obtain a MF for the disorder parameter of the Z_2 gauge theory on the dual cluster.

[1] Dipartimento di Fisica, Università degli Studi di Firenze and INFN, Sezione di Firenze, Firenze, Italy
[2] Dipartimento di Fisica, Università degli Studi di Padova and INFN, Sezione di Padova, Padova, Italy

For example, on the smallest clusters defined in sect. 3.1 the following duality relation holds

$$\langle S_0\rangle_{\beta,b} = \langle e^{-2(\beta^*+b^*)\sigma_p}\rangle_{\beta^*,b^*},$$

$$e^{-2\beta} = \tanh\beta^*, \qquad e^{-2b} = \tanh b^*, \tag{1.1}$$

between the magnetization on the spin cluster and the disorder parameter on the gauge cluster. This disorder parameter represents the probability of frustrated configurations on a cube C. In fact it is the ratio of two partition functions, whose actions differ in the sign of the couplings of a plaquette σ_p belonging to C (in the numerator $\prod_C \operatorname{sign}(b^*) = \prod_C \operatorname{sign}(\beta^*) = -1$).

Let us observe that this method: (i) can be extended to the 2d spin and 4d Z_2 gauge theories; (ii) is manifestly gauge invariant and (iii) can be systematically improved by considering larger clusters.

It is quite remarkable that the results obtained in the different framework of Cayley trees [6] are reproduced by our method on the simplest clusters, allowing a more physical interpretation.

Even if this method applies to all abelian groups, here we shall limit ourselves to the case of Z_2 models. The extension of the core of the method to other symmetry groups is straightforward in many aspects and it will be considered elsewhere.

For the sake of clarity, we have chosen to introduce the formalism in the simplest case of the 2d Ising model (sect. 2). Since this theory is selfdual, two "complementary" MF schemes exist, which are based on the magnetization and on the 2d frustration order parameters, respectively Both methods describe the same phase diagram, each with better accuracy in one phase.

The three-dimensional gauge case is discussed in sect. 3, where we also show the global (discrete) symmetry of the dual variables, which forbids isolated frustrations in the ordered phase, and spontaneously breaks when the condensation takes place

A relevant property which characterizes our MF approach is that the space-time dimensionality enters in a non-trivial way through the form of the order parameter. The four-dimensional gauge theory has again, due to selfduality, two complementary MF schemes, which are introduced after the identification of the proper order parameters (sect. 4). They are respectively the product of link variables on a line (Wilson line) and its dual, the product of frustrations on a (dual) line of cubes; both lines extend through the cluster from border to border.

In sect. 5 the MF for the Wilson line will be related to the standard MF method for gauge theories [1]. While in four dimensions our Bethe-Peierls approach fully clarifies the physical basis of the standard MF, in three dimensions it provides the proper description in terms of frustrations, and the standard approach is doubtful in this case. The reliability of the large-volume limit for these line operators in 4d and the relation with finite-temperature lattice gauge theories is also discussed

In appendix A our approach will be compared with the mean plaquette method [7] and in appendix B we shall provide a simple method for evaluating expectation values.

2. Duality and consistency in $d=2$

2.1 DUAL FORM OF THE BETHE-PEIERLS APPROXIMATION

Let us consider the simplest case of the two-dimensional Ising model to exemplify the role of duality in defining a BP approach for the condensation of frustrations.

The strategy of the BP approximation consists in simulating the spontaneous symmetry breaking mechanism by determining selfconsistently the order parameter on a finite cluster with symmetry breaking boundary conditions.

Let us introduce this approach for the cluster in fig. 1a, made of two sites surrounded by their neighbours [5]. The corresponding reduced action is

$$A=\beta \sum_{i=1,2,3} (S_0S_i+S_0'S_i')+\beta S_0S_0'+h(S_0+S_0')+b \sum_{i=1,2,3} (S_i+S_i'), \qquad S=\pm 1, \tag{2.1}$$

where β is the inverse temperature, h is a magnetic field and the last term is responsible for breaking the symmetry $S_i \to -S_i$ when $h \to 0$.

The value of the parameter b is determined as a function of β and h by imposing a consistency equation between internal and external magnetizations

$$\langle S_0\rangle_{\beta,h,b}=\langle S_i\rangle_{\beta,h,b}, \qquad \langle(\cdot)\rangle=\frac{\sum_{\{S\}}(\cdot)\,e^A}{\sum_{\{S\}}e^A}, \tag{2.2}$$

which amounts to requiring translation invariance. For $h=0$, eq. (2.2) can be rewritten in the form

$$\rho=\left(\frac{t+\rho}{1+t\rho}\right)^3, \tag{2.3}$$

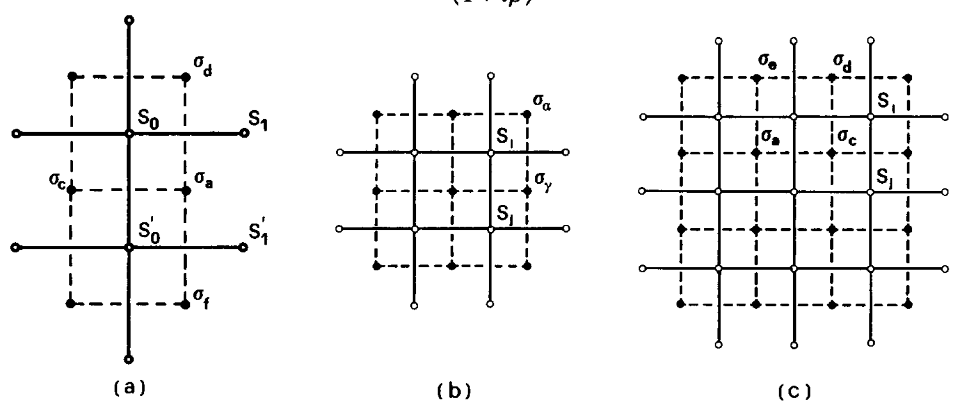


Fig 1 The 2d Ising model clusters the open clusters Λ have white dots and full lines, the closed clusters Λ^* have black dots and dashed lines

where $\rho = e^{-2b}$ and $t = e^{-2\beta}$. Below the critical temperature ($t < t_c = \frac{1}{2}$), besides the solution $\rho = 1$ ($b = 0$) valid for any t, a pair of symmetric solutions appears with $b \neq 0$, giving $\langle S\rangle \neq 0$. These last solutions correspond to the minima of the free energy.

The duality transformation on this cluster will be first performed on the partition function

$$Z_\Lambda = \sum_{\{S\}} e^{A\{S\}} . \tag{2.4}$$

Following the standard procedure (see ref. [4]) one gets for $h = 0$:

$$\begin{aligned} Z_\Lambda &= \sum_{\{S\}} \left\{ \sum_{k=0,1} C_k(\beta)(S_0 S_0')^k \left[\sum_{k_i=0,1} \sum_{r_i=0,1} \prod_{i=1}^{3} C_{k_i}(\beta)(S_0 S_i)^{k_i} C_{r_i}(b) \right.\right. \\ &\quad \left.\left. \times (S_i)^{r_i} \right] \left[\sum_{k_i'=0,1} \sum_{r_i'=0,1} \prod_{i=1}^{3} C_{k_i'}(\beta)(S_0' S_i')^{k_i'} C_{r_i'}(b)(S_i')^{r_i'} \right] \right\} \\ &= \sum_k \sum_{k_i, r_i} \sum_{k_i', r_i'} \hat\delta\left(k + \sum_{i=1}^{3} k_i\right) \hat\delta\left(k + \sum_{i=1}^{3} k_i'\right) \prod_{i=1}^{3} [\hat\delta(k_i + r_i) \\ &\quad \times C_{k_i}(\beta) C_{r_i}(b) \hat\delta(k_i' + r_i') C_{k_i'}(\beta) C_{r_i'}(b)] C_k(\beta) , \end{aligned} \tag{2.5}$$

where

$$C_\alpha(x) = \begin{cases} \cosh x, & \alpha = 0 \\ \sinh x, & \alpha = 1 \end{cases}, \qquad \hat\delta(k) = \begin{cases} 0, & k_{\text{odd}} \\ 1, & k_{\text{even}} \end{cases} . \tag{2.6}$$

Rather than summing explicitly over the $\{k_i, k_i'\}$ and $\{r_i, r_i'\}$ in eq. (2.5) it is convenient, for our purposes, to introduce a representation in terms of dual variables $\{\sigma\}$, as in the infinite lattice transformation, which satisfies the $\hat\delta$ functions:

$$\begin{aligned} &k = \tfrac{1}{2}(1 - \sigma_a \sigma_c) , \\ &k_1 = \tfrac{1}{2}(1 - \sigma_a \sigma_d) , \qquad r_1 = \tfrac{1}{2}(1 - \sigma_a \sigma_d), \ldots , \\ &k_1' = \tfrac{1}{2}(1 - \sigma_a \sigma_f) , \qquad r_1' = \tfrac{1}{2}(1 - \sigma_a \sigma_f), \ldots , \end{aligned} \tag{2.7}$$

where, for instance, σ_a and σ_d belong to the bond orthogonal to $S_0 S_1$ on the dual cluster Λ^* in fig. 1a. Substituting the representation (2.7) in eq. (2 6) one obtains

$$Z_{\Lambda^*} = N \sum_{\{\sigma\}} e^{A^*} = N \sum_{\{\sigma\}} \exp\left[\beta^* \sigma_a \sigma_c + (\beta^* + b^*) \sum_{\langle\alpha,\gamma\rangle} \sigma_\alpha \sigma_\gamma \right] , \tag{2.8}$$

where,

$$N^{-1} = 2^{3/2} (\sinh 2\beta^*)^{7/2} (\sinh 2b^*)^3 , \qquad e^{-2\beta} = \tanh \beta^* , \qquad e^{-2b} = \tanh b^* \tag{2.9}$$

and $\sum_{\langle\alpha,\gamma\rangle}$ runs over the bonds of $\partial\Lambda^*$, the border of Λ^*.

We want to observe that the dual action A^* has given rise to a different boundary term; in what follows we are going to explain how this boundary term* can provide

* This kind of "effective temperature" on the boundary has already been introduced in a self-consistent framework in ref [8]

the dual symmetry breaking mechanism. We need the expression of the dual order parameter. By following a similar procedure as for the partition function we derive the relations:

$$\langle S_0 \rangle_\beta = \langle e^{-2(\beta^*+b^*)\sigma_\alpha\sigma_\gamma} \rangle_{\beta^*}, \tag{2.10a}$$

$$\langle S_i \rangle_\beta = \langle e^{-2b^*\sigma_\alpha\sigma_\gamma} \rangle_{\beta^*}, \tag{2.10b}$$

where $\langle \alpha, \gamma \rangle$ is one of the bonds of $\partial \Lambda^*$ and $\langle (\cdot) \rangle_{\beta^*} = \sum_{\{\sigma\}} (\cdot)\, e^{A^*} / \sum_{\{\sigma\}} e^{A^*}$.

From eq. (2.2) the dual consistency relation is obtained by equating the r.h.s. of eqs. (2.10a) and (2.10b). One obtains again eq (2.3), but

$$\rho = \tanh b^*,$$

$$t = \tanh \beta^*. \tag{2.11}$$

In this dual MF treatment the ordered phase $\beta^* > \beta_c^* = -\frac{1}{2} \ln \tanh \beta_c$ corresponds to the unique solution of eq. (2.3) $b^* = \infty$, while in the disordered phase, $\beta^* < \beta_c^*$, there are two additional solutions for finite b^*. From eqs. (2.10a) and (2.10b) one can see that the disorder parameter is identically zero in the ordered phase.

In the infinite lattice the same behaviour is obtained by volume effects. In fact, the isolated frustration on a square lattice with free boundary conditions is described by the following order parameter [9]

$$\langle S_0 \rangle_\beta = \left\langle \exp\left[-2\beta^* \sum_{\langle \alpha,\gamma \rangle \perp \ell} \sigma_\alpha \sigma_\gamma \right] \right\rangle_{\beta^*}, \tag{2.12}$$

where the line ℓ starts at the site on which S_0 sits and reaches infinity. (The couplings β^* are modified to have $\prod \operatorname{sign}(\beta^*) = -1$ for the bonds on the corresponding plaquette and $\prod \operatorname{sign}(\beta^*) = 1$ otherwise.) One can observe that the extended operator in eq. (2.12), has a local form in our finite cluster approximation, since the infinite line outside the cluster is simulated in eq. (2.10a) by the boundary parameter b^*.

A satisfactory description would require the discussion of the symmetry broken by this condensation. Here we limit ourselves to observe that it can be expressed in terms of a dual parity $\mathcal{P}^* = (-1)^\nu$, where ν is the number of frustrations; therefore, the symmetric vacuum state does not contain isolated frustrations. We postpone a detailed analysis to the more interesting case of Z_2 gauge theory in 3d.

2.2 RESULTS AND FURTHER IMPROVEMENTS

It is more interesting to add some comments on the results of this approach and the possible improvements.

The disorder parameter in the r.h.s. of eqs. (2.10) clearly behaves like a magnetization with a reversed behaviour in temperature and it has the classical critical exponent. The value of the critical temperature, $\beta_c^* = 0.549$, is an upper bound to the true value $\beta_c \simeq 0.441$, because the method overestimates fluctuations, at variance with the standard BP method, which gives the lower bound $\beta_c = 0.347$.

The internal ($\langle\sigma_a\sigma_c\rangle$) and the external (e.g. $\langle\sigma_a\sigma_d\rangle$) link correlation functions, are equal when the consistency equation (2.3) is satisfied. One can check that their behaviour reproduces the strong coupling expansion (SC) up to order $O(t^7)$ while for $\beta^* > \beta^*_c$ only the trivial term in the weak coupling expansion (WC) is present, i.e. $\langle\sigma_a\sigma_c\rangle \equiv 1$.

More generally, this dual BP approximation gives better results for the SC and the standard BP approximation works better for the WC.

In our scheme, we have the possibility to improve the results simply by considering larger clusters like those in figs. 1b and 1c. On the "open" cluster Λ, which has isolated spins on the boundary, the action is

$$A_\Lambda = \beta \sum_{\langle i,j\rangle\in\Lambda} S_i S_j + b \sum_{i\in\partial\Lambda} S_i \,. \tag{2.13}$$

Duality transformation can be performed as in the smaller cluster and it leads to the following action on the "closed" cluster Λ^*,

$$A_{\Lambda^*} = \beta^* \sum_{\langle\alpha,\gamma\rangle\in\Lambda^*} \sigma_\alpha\sigma_\gamma + b^* \sum_{\langle\alpha,\gamma\rangle\in\partial\Lambda^*} \sigma_\alpha\sigma_\gamma \,, \tag{2.14}$$

where β^* and b^* are given in eq. (2.9). For the cluster in fig. 1b the dual consistency equation is obtained again by equating the r.h.s. of eqs. (2.10a, b) (with $\langle\alpha, \gamma\rangle \in \partial\Lambda^*$).

With respect to the previous case, better results are obtained both in the WC and in the SC limits. The SC is better because there are more graphs inside the cluster geometry or, in other words, we are taking into account subleading terms in the Cayley tree resummation of the SC expansion [1].

Moreover, internal spin flips in the cluster Λ^* of fig. 1b is allowed even when the boundary parameter b^* goes to infinity and then the correlation functions are not trivial in the WC limit; however the correct behaviour $\langle\sigma_\alpha\sigma_\gamma\rangle \sim 1 - 4\,e^{-8\beta^*}$ is not yet obtained. This is recovered with the clusters in fig 1c, where we choose the consistency equation*

$$\langle e^{-2\beta^*\sigma_a\sigma_c - 2(\beta^*+b^*)\sigma_e\sigma_d}\rangle_{\beta^*} = \langle e^{-2b^*\sigma_e\sigma_d}\rangle_{\beta^*} \tag{2.15}$$

The specific heat $C_v = 2\beta^{2*}\, d\langle\sigma_a\sigma_c\rangle / d\beta^*$ for this cluster is reported in fig. 2 and compared with the standard BP and the exact results. It shows a finite jump at the critical point. The same kind of singularity is obtained for the cluster of fig. 1b and for larger ones. The critical temperature slowly improves as the cluster size increases ($\beta^*_c = 0.541$, fig. 1b; $\beta^*_c = 0.5245$, fig. 2b).

Finally, one should verify that in this dual MF the magnetization $m^2 = \lim_{|x|\to\infty} \langle\sigma_0\sigma_x\rangle_{\beta^*}$ is different from zero in the ordered phase. We have checked this result by a rough estimate on a strip.

Although we have stressed the pedagogical aspects of the 2d case, we want to remark that the extension of the frustration MF to other interesting open problems in 2d can give completely new results.

* This is the most symmetric among the choices which are present due to the lack of translation invariance, other choices give slight differences which reduce on larger clusters

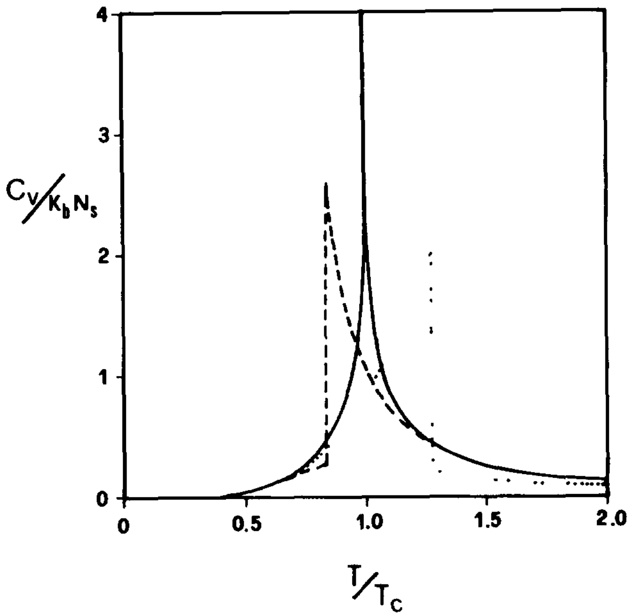


Fig 2 The specific heat of the 2d Ising model exact result (full line), standard Bethe-Peierls (dotted) and our result (dashed) for the cluster in fig 1c

This is the case of the antiferromagnetic triangular lattice, which has not an ordered phase at zero temperature; therefore the standard MF for the magnetization is not defined. Some preliminary results, based on the method discussed in this section, show the absence of a phase transition and a reasonable value for the entropy at zero temperature $\mathcal{S}_0 = 0,2877$ (exact [10], $\mathcal{S}_0 = 0,3231$). A detailed analysis of general frustrated systems will be reported elsewhere

3. Three-dimensional Z_2 gauge theory

3 1 MEAN FIELD FOR THE DISORDER VARIABLES

In the previous section we have introduced a MF method for the frustrations of the 2d Ising model by applying the duality transformation on the standard BP formulation.

In 3d duality relates the Ising model to the Z_2 gauge theory; therefore the extension of our procedure to the 3d Ising model will provide a MF for the frustration of the 3d Z_2 gauge theory [4]. This approach has the remarkable property of being a manifestly gauge invariant description of the phase diagram in terms of local quantities.

Let us proceed as in sect. 2. Starting from the simple Ising cluster Λ in fig. 3a, whose action reads

$$A_\Lambda = \beta\left[S_0S_0' + \sum_{i=1}^{5} S_0S_i + S_0'S_i'\right] + b\sum_{i=1}^{5}(S_i + S_i'), \tag{3.1}$$

we obtain the action of the gauge model on the dual two-cubes cluster Λ^* in the form

$$A_{\Lambda^*} = \beta^*\sum_{p}\sigma_p + b^*\sum_{p}{}'\,\sigma_p, \tag{3.2}$$

where β^* and b^* are defined as in eq. (2.9) and the summations $\sum_p$ and $\sum_p'$ run over all the plaquettes $\sigma_p = \sigma_\mu\sigma_\nu\sigma_\mu\sigma_\nu$ in Λ^* and over the external ones only, respectively. The consistency equation for magnetizations on the Ising cluster, like eq (2.2), transforms under duality into a consistency equation between the frustrations of the Z_2 gauge theory, as follows:

$$\langle e^{-2(\beta^*+b^*)\sigma_{\bar{p}}}\rangle_{\beta^*} = \langle e^{-2b^*\sigma_{\bar{p}}}\rangle_{\beta^*}. \tag{3.3}$$

The l.h.s. of eq. (3.3) is the frustration on an internal cube of Λ^*, while the r.h.s. is interpreted as a frustration on an external cube, sharing the plaquette $\bar{p} \in \partial\Lambda^*$ with the previous one. Eq. (3.3) can be reduced to the following form:

$$t = \frac{s(1+s^2)}{1+s^2+s^4}, \qquad s \equiv 1\ \forall t, \tag{3.4}$$

where $t = \tanh\beta^*$ and $s = \tanh(\beta^* + b^*)$, or equivalently

$$\rho = \left(\frac{t+\rho}{1+t\rho}\right)^5, \tag{3.5}$$

where $\rho = \tanh b^*$.

This last equation was obtained in ref. [6] as a recursive equation which solves the model on a Cayley tree of cubes; it shows a second-order transition at $t_c = \frac{2}{3}$ (i.e. $\beta_c^* = 0.8047$; Monte Carlo, $\beta_c^* = 0.7613$) between an ordered phase with vanishing

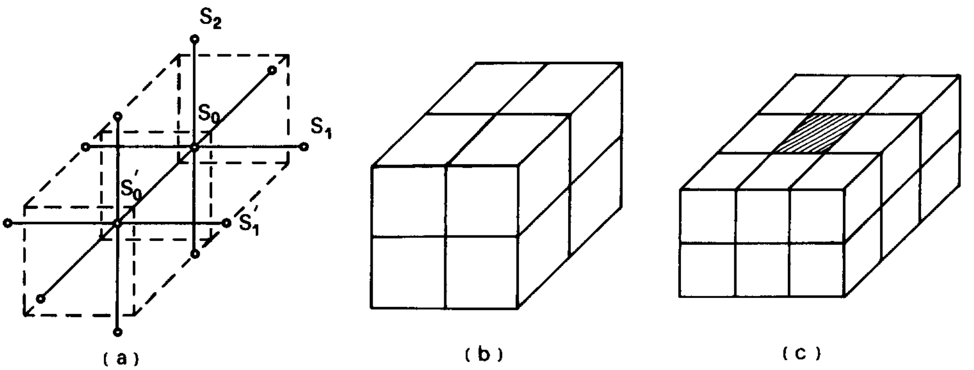


Fig 3 The 3d open cluster (Ising model) Λ and closed clusters Λ^* (Z(2) gauge theory), notations as in fig 1

expectation values in eq. (3.3) ($s \equiv 1$), and a disordered one where isolated frustrations on cubes condensate.

Let us now observe that eqs. (3.4) and (3.5) show, in the dual case, the well-known equivalence between the BP approximation and the translation-invariant solution of the model defined on a Cayley tree [6, 11]. Our method, however, has the advantage of identifying the order parameter and the ansätz action, thus providing an unambiguous recipe to improve this approximation on larger clusters, while the Cayley tree approach by itself cannot be improved straightforwardly.

Let us now show the analogy between the Cayley tree and our method in the case of the simplest cluster of fig. 3a. A frustration on a tree of cubes can be introduced by changing $\beta^* \to -\beta^*$ in the action for the plaquettes orthogonal to a dual line which starts in a cube and goes through a branch of the tree. The recursive equation for the modified branch is nothing else than the consistency equation (3.3), where $-b^*$ is interpreted as the effective parameter.

The improvement of our method for the 2d gauge theory can be obtained, as in sect. 2, by studying the larger clusters in figs. 3b and 3c, whose actions have the general form

$$A_{\Lambda^*} = \beta^* \sum_{p \in \Lambda^*} \sigma_p + b^* \sum_{p \in \partial\Lambda^*} \sigma_p \,. \tag{3.6}$$

The consistency equation for the frustration on a cube is again eq. (3.3), where in the cluster of fig. 3b $\bar{p}$ is one of the equivalent external plaquettes and in the cluster of fig. 3c we choose the shaded plaquette*. On these clusters, the expectation values are ratios of polynomials in t and s, whose coefficients are obtained by a novel method of summation over configurations (see appendix B).

The analysis of these clusters shows the same critical behaviour ($\beta_c^* = 0.800$, fig. 3b; $\beta_c^* = 0.795$. fig. 3c) and the SC and WC expansions of the plaquette are recovered. In particular, the larger cluster in fig. 3c provides the correct second-order WC term. The plaquette is reported in fig. 4 and compared with Monte Carlo data [12] and WC series from ref. [13]. Our good results in the SC region are confirmed by a comparison with Monte Carlo data (the SC series [13] is not significant because it diverges before β_c^*).

These results show the typical features of a MF treatment: a good description of both the phases is obtained and a singularity detects the critical point (at variance with Monte Carlo methods). However, the expected critical behaviour is only approximately reproduced by our method, which gives a cusp singularity.

On the contrary, the standard MF approach for gauge theories [1] gives a first-order transition in 3d, even including corrections; we postpone the comparison of our method with this approach to the next section.

The relation to other MF approaches based on a consistency equation for the plaquette [7] is discussed in appendix A.

* See footnote on page 344

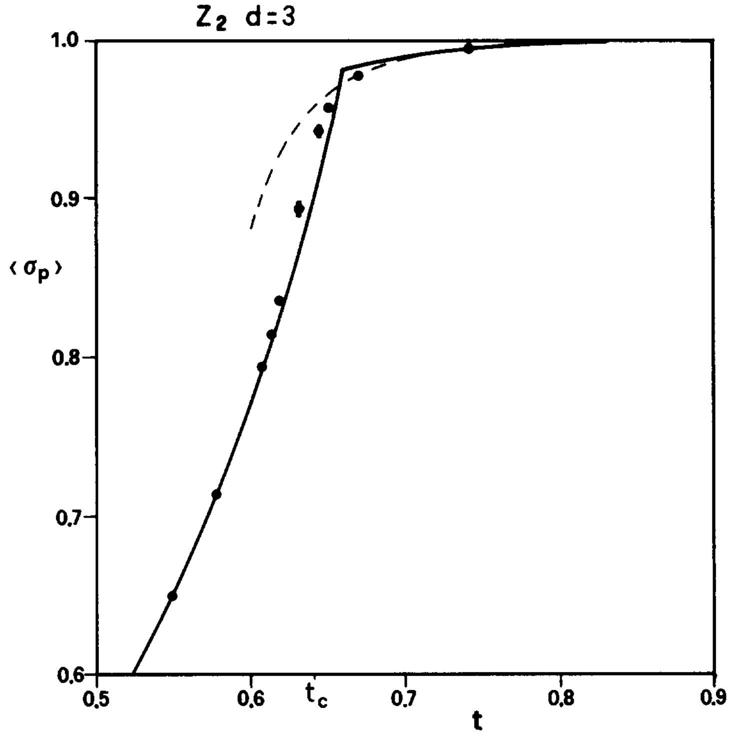


Fig 4 The plaquette in the 3d Z(2) gauge theory our result on the cluster in fig 3c (full line) compared with the WC expansion (dashed) and the Monte Carlo data [12]

3 2 DUAL PARITY SYMMETRY

In the final part of this section we shall explain in more detail the symmetry broken by the condensation of frustrations in three dimensions.

We begin the discussion by writing the partition function of the Ising model on a finite open cluster (see, e.g., fig. 3a), in the presence of a magnetic field h on internal sites and a border term b:

$$Z_\Lambda(\beta, b, h) = \sum_{\{S\}} \exp\left[\beta \sum_{\langle i,j\rangle} S_i S_j + h \sum_{i\in\Lambda\setminus\partial\Lambda} S_i + b \sum_{j\in\partial\Lambda} S_j\right]. \tag{3.7}$$

Let us recall that the parity transformation $\mathscr{P}: S_i \to -S_i$, which is a symmetry of the spin theory for $b, h \to 0$ on a finite cluster Λ, can be identified by checking the relation

$$Z_\Lambda(\beta, -b, -h) \underset{\mathscr{P}}{=} Z_\Lambda(\beta, b, h). \tag{3.8}$$

Furthermore, the relation

$$Z_\Lambda(\beta, -b, h) \underset{\mathscr{P}}{=} Z_\Lambda(\beta, b, -h) \tag{3.9}$$

shows that the external source h and the border term b are equivalent in breaking the degeneracy of the vacuum states before performing the thermodynamic limit* ($\Lambda \to \infty$):

$$\lim_{h\to 0}\lim_{\Lambda\to\infty}\langle S_i\rangle_{\Lambda,h} \neq 0 \qquad (b=0)\,, \tag{3.10}$$

$$\lim_{\Lambda\to\infty}\langle S_i\rangle_{\Lambda,b} \neq 0 \qquad (h=0)\,, \tag{3.11}$$

where

$$\langle S_i\rangle_\Lambda = \Omega_\Lambda^{-1}\frac{\partial}{\partial h}\log Z_\Lambda\,.$$

The dual parity symmetry will be identified by writing down explicitly, in terms of the dual variables $\{\sigma_\mu, A_p\}$, the partition function of the closed dual cluster Λ^* (see, e.g., fig. 3a):

$$\begin{aligned} Z_{\Lambda^*}(\beta^*, h^*, b^*) &= N Z_\Lambda(\beta, h, b) \\ &= \sum_{\{\sigma_\mu, A_p\}}\prod_{C\in\Lambda^*}\left\{\tfrac{1}{2}\left(e^h + e^{-h}\prod_{p\in\partial C}A_p\right)\right\}\prod_{p\in\partial\Lambda^*}\left\{\tfrac{1}{2}\left(e^b + A_p\sigma_p e^{-b}\right)\right\} \\ &\quad\times\exp\left(\beta^*\sum_{p\in\Lambda^*}\sigma_p A_p\right), \end{aligned} \tag{3.12}$$

where N is a normalization factor, $\prod_{C\in\Lambda^*}$ stands for the product over the cubes of $\Lambda^*, \prod_{p\in\partial\Lambda^*}$ over the plaquettes on $\partial\Lambda^*$ and the A_p's are auxiliary spin variables defined on plaquettes (these variables disappear in the limit $h\to 0$, by using the constraints $\prod_{p\in\partial C}A_p = 1\ \forall$ cube).

Performing the symmetry transformation $(b, h)\to(-b, -h)$ in eq. (3.12), the analogous of eq. (3.8) in the dual case holds, because

$$\prod_{C\in\Lambda^*}\left\{\prod_{p\in\partial C}A_p\right\} = \prod_{p\in\partial\Lambda^*}A_p \equiv \mathscr{P}^*\,. \tag{3.13}$$

This transformation acts on the frustration $\langle\mathscr{F}_i\rangle$:

$$\begin{aligned} \langle\mathscr{F}_i\rangle_{\beta^*,h,b} &= \frac{1}{Z_{\Lambda^*}}\sum_{\{A,\sigma\}}\tfrac{1}{2}\left(e^h - e^{-h}\prod_{\partial C_i}A_p\right)\prod_{C\neq C_i}\left\{\tfrac{1}{2}\left(e^h + e^{-h}\prod_{\partial C}A_p\right)\right\} \\ &\quad\times\prod_{p\in\partial\Lambda^*}\left\{\tfrac{1}{2}\left(e^b + e^{-b}A_p\sigma_p\right)\right\}\exp\left(\beta^*\sum_p\sigma_p A_p\right), \end{aligned} \tag{3.14}$$

as follows

$$\begin{aligned} \langle\mathscr{F}_i\rangle_{\beta^*,h,b} &= -\langle\mathscr{F}_i\rangle_{\beta^*,-h,-b} \\ &= \langle\mathscr{P}^*\mathscr{F}_i\mathscr{P}^*\rangle_{\beta^*,h,b} \end{aligned} \tag{3.15}$$

* Analogously with the procedure of the standard Peierls argument [14], we have emphasized the rôle of the parameter b in breaking the symmetry, this rôle is also relevant in the MF approach, where the thermodynamic limit is accomplished by imposing the consistency equation.

This procedure has led to the identification of $\mathcal{P}^*$ in eq. (3.13) as the symmetry operator acting on classical fields: the operator $\mathcal{P}^*$ counts the number ν of frustrated cubes ($\prod_{\partial C} A_p = -1$), modulo two, that is

$$\left\langle \mathcal{P}^* \left\{ \prod_{k=1}^{\nu} \mathcal{F}_{i_k} \right\} \mathcal{P}^* \right\rangle_{h,b=0} = (-1)^{\nu} \left\langle \prod_{k=1}^{\nu} \mathcal{F}_{i_k} \right\rangle_{h,b=0} . \tag{3.16}$$

The conservation (modulo two) of the topological charge ν is violated by the condensation mechanism. Formally, this spontaneous symmetry breaking is defined by eq. (3.10), (3.11) replacing S_i with $\mathcal{F}_i$, and is singled out by adding to the action the external source coupling $h^* \sum_C \{\prod_{\partial C} A_p\}$ or the boundary term $b^* \sum_p A_p \sigma_p$.

4. Frustration mean field in 4d Z_2 gauge theory

In order to extend our procedure to the 4d Z_2 gauge theory, we shall assume that the disorder parameter is again the properly chosen frustration operator: a line of frustrated cubes.

Indeed we shall obtain for this operator a consistency equation on finite clusters, which reduces to the Cayley tree approach [6] in the simplest case. The standard MF [1] will be reinterpreted as a particular limit of our approach (see sect. 5). Moreover this method will provide a gauge-invariant description of the phase diagram, which agrees with the existing phenomenology and improves the results of previous MF methods.

Instead of introducing directly the frustration operator we prefer to start with the more familiar Wilson loop. Then we shall exploit the selfduality of the theory.

The Wilson loop on a finite cluster Λ is given by

$$\begin{aligned} \langle W[\gamma] \rangle &= \left\langle \prod_{\mu \in \gamma} S_\mu \right\rangle_\beta = \left\langle \prod_{p \in \Sigma \;\; \partial\Sigma = \gamma} S_p \right\rangle_\beta \\ &= \frac{1}{Z} \sum_{\{\sigma\}} \prod_{\mu \in \gamma} \{S_\mu\} \exp\left(\beta \sum_{p \in \Lambda} S_p \right), \end{aligned} \tag{4.1}$$

where the S_p's are the plaquettes lying on the surface Σ whose border is the loop γ. We consider a rectangular loop in a space–time plane and we let the space sides coincide with the boundary. The introduction of the 4d order parameter, which we shall denote as "Wilson line", requires to fix the variables on the boundary (e.g. $S_\mu = 1$, $\mu \in \partial\Lambda$). The Wilson loop then reduces to the correlation of two Wilson lines, as follows

$$\langle W[\gamma] \rangle = \langle L(\boldsymbol{x}) L(\boldsymbol{0}) \rangle_\beta , \tag{4.2}$$

where

$$L(\boldsymbol{\xi})_N \equiv \prod_{k=1}^{N} S_{\hat{t}}(\boldsymbol{\xi}, k\hat{t}) \tag{4.3}$$

is the path operator, whose length N is the length of Λ along $\hat{t}$. The order parameter can be identified, as usual in spin models, from the factorization of the correlation (4.2) in the limit $|x| \to \infty$, which is supposed to hold,

$$\langle W[\gamma] \rangle \xrightarrow[|x|\to\infty]{} \langle L(\mathbf{0}) \rangle^2_{\beta,N} = e^{-2NF_q} . \tag{4.4}$$

In eq. (4.4) F_q represents the "quark free energy"; therefore, for finite N, $\langle L(\mathbf{0}) \rangle$ can be considered as an order parameter of the deconfinement transition.

Let us observe that the choice of the boundary conditions in the space directions is necessary for the definition [15] of the expectation value in eq. (4.4) and moreover one has to fix the boundary in the space-time directions for the definition of the expectation value of a single line $\langle L(\mathbf{0}) \rangle$ (see also subsect. 5.3)

We proceed analogously for the dual operators. The dual of the Wilson loop is the 't Hooft loop [4, 16]

$$\begin{aligned} \langle H[\gamma] \rangle &= \left\langle \exp\left[-2\beta^* \sum_{p^* \in \Sigma^*} \sigma_{p^*} \right] \right\rangle_{\beta^*} \\ &= \frac{1}{Z} \sum_{\{\sigma\}} \exp\left[\beta^* \sum_{p^* \in \Lambda^* \setminus \Sigma^*} \sigma_{p^*} - \beta^* \sum_{p^* \in \Sigma^*} \sigma_{p^*} \right] \end{aligned} \tag{4.5}$$

defined on the dual cluster Λ^*, where Σ^* is the collection of plaquettes p^* dual to the plaquettes in Σ and the duality relation is $\langle W[\gamma] \rangle_\beta = \langle H[\gamma] \rangle_{\beta^*}$.

By stretching the loop γ and factorizing the correlation, one obtains the dual order parameter, which is a line of frustrated cubes of the dual lattice*:

$$\langle L^*(x) \rangle_{\beta^*,N} = \left\langle \exp\left[-2\beta^* \sum_{p^* \in \Sigma^*} \sigma_{p^*} \right] \right\rangle_{\beta^*,N} , \tag{4.6}$$

where now Σ^* is dual to a half-plane $\Sigma \subset \Lambda$ and the duality relation is $\langle L(x) \rangle_{\beta,N} = \langle L^*(x) \rangle_{\beta^*,N}$.

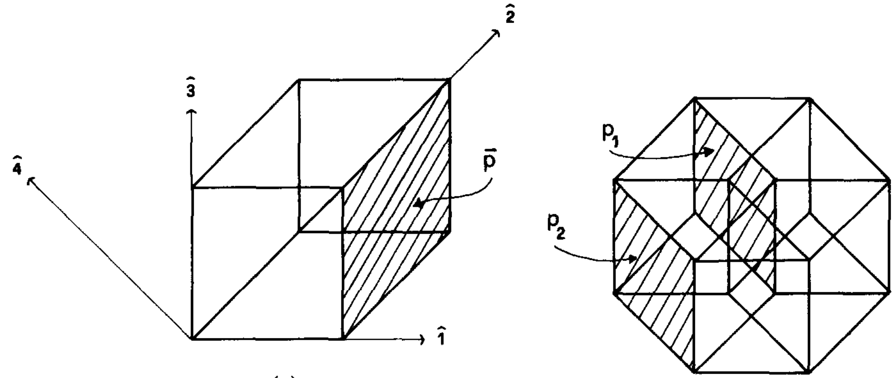
We shall now introduce a BP approach for the frustration operator in eq. (4.6). On a cluster with given length N, the spontaneous symmetry breaking mechanism will be simulated by the consistency equation for the order parameter of N frustrated cubes, which extends from border to border along a dual line.

Let us consider a closed cluster, i.e. a cluster without border plaquettes sharing only one link with the bulk. The smallest one is the cube in fig. 5a and contains only one frustration. In analogy with the 3d case, we consider the following action on this cluster

$$A_{\Lambda^*} = \beta^* \sum_{p^* \in \Lambda^*} \sigma_{p^*} + 3b^* \sum_{p^* \in \partial\Lambda^* \equiv \Lambda^*} \sigma_{p^*} , \tag{4.7}$$

where the factor 3 counts the number of effective external cubes.

* Duality transforms fixed boundary conditions in Λ into free boundary conditions in Λ^*



(a)

Fig 5 Four-dimensional gauge theory clusters (b)

The consistency equation for the frustration order parameter is*

$$\langle e^{-2(\beta^*+3b^*)\sigma_{\bar{p}}}\rangle_{\beta^*}=\langle e^{-2b^*\sigma_{\bar{p}}}\rangle_{\beta^*}\equiv\langle L^*\rangle_{\beta^*,N=1}\,, \tag{4.8}$$

where $\bar{p}$ is a plaquette of the cube. The border term b^* is determined by the implicit equation

$$\rho=\left(\frac{t+3\rho^2+3t\rho+\rho^3}{1+3t\rho^2+3\rho+t\rho^3}\right)^5\,, \tag{4.9}$$

which can be solved in the form

$$t=\frac{z(1-3z^4+3z^{10}-z^{14})}{1-3z^6+3z^{10}-z^{16}}\,,\qquad z\equiv 1\;\forall t\,, \tag{4.10}$$

where $z=\tanh(\beta^*+3b^*)$ and ρ, t are defined in eq. (2.11).

Eq. (4.10) shows a first order transition from an ordered phase ($\langle L^*\rangle_{\beta^*}\equiv 0$, $z\equiv 1$) for $\beta^*>\beta_c^*$ to a disordered one ($\langle L^*\rangle_{\beta^*}\neq 0$, $z<1$). The transition point $\beta_c^*=0.450$ (exact, $\beta_c^*=\frac{1}{2}\ln(\sqrt{2}+1)\cong 0.4407$) is found by equating the free energies of the two phases. These are defined as follows

$$f(\beta^*)=\int_0^{\beta^*}\langle\sigma_{p^*}\rangle_{x,b^*(x)}\,dx+f_0\,, \tag{4.11}$$

where the normalization constants f_0 match the correct SC and WC limits as follows:

$$f_0^{(SC)}=0\,,\qquad f_0^{(WC)}=-\tfrac{1}{2}\ln 2+\int_0^{\infty}(1-\langle\sigma_{p^*}\rangle_{x,b^*=\infty})\,dx\,. \tag{4.12}$$

Eq. (4.9) was already obtained in ref. [6] by solving the Cayley tree of cubes in 4d In our approach we give a more physical interpretation of this result, by observing that the operator defined in eq. (4.8) has the form of the frustration operator on a

* As in the previous section, one can check that the simple one-cube cluster gives the same results as the cluster of 4 cubes with one common plaquette, due to the underlying tree structure

4d tree of cubes, $\langle \exp(-2\beta^* \sum_{p^*} \sigma_{p^*}) \rangle$, where the sum $\sum_{p^*}$ extends to plaquettes orthogonal to a 2d branch of the tree (see also subsect. 3.1). This is an independent check of our strategy and justifies the correctness of this frustration operator as the order parameter in 4d.

Let us now consider the closed cluster of size 2, that is the hypercube in fig. 5b. We introduce a consistency equation for operators of two frustrated cubes, as follows

$$\langle L^* \rangle_{\beta^*,2} \equiv \langle e^{-2(\beta^*+2b^*)(\sigma_{p_1}+\sigma_{p_2})} \rangle_{\beta^*} = \langle e^{-2b^*(\sigma_{p_1}+\sigma_{p_2})} \rangle_{\beta^*} , \tag{4.13}$$

where p_1 and p_2 are the shaded plaquettes in fig. 5b. Evaluating the mean values in eq (4.13) on the cluster action

$$A_{\Lambda^*} = (\beta^* + 2b^*) \sum_{p^* \in \Lambda^*} \sigma_{p^*} , \tag{4.14}$$

as shown in appendix B, we get:

$$t = \frac{u(1-u^2+u^4+4u^6-12u^8-u^{10})}{1-u^2+3u^4+6u^8-u^{10}} , \qquad u = 1 \ \forall t , \tag{4.15}$$

where

$$u = \tanh(\beta^* + 2b^*) .$$

This equation shows indeed the same kind of transition as eq. (4 10), at a slightly different value of the coupling.

Therefore, these finite-cluster approximations can provide a reliable description of the phase transition also in the case of non-local order parameters, defined on an infinite path. We shall discuss in more detail in sect. 5.3 how to perform the large volume limit for path operators, starting from their finite cluster representations.

Now we shall focus on the quantitative results which are obtained from finite clusters. The cube and the hypercube already introduced give a good behaviour in the SC region for the plaquette expectation value; the comparison in fig. 6 with the SC series obtained by Wilson [17] and its Padé analysis by Falcioni et al. [18] shows that our results are better than the series and tends to reproduce part of the expected metastable and unphysical parts of the curve, up to the selfdual point $\langle \sigma_p \rangle = \frac{1}{2}\sqrt{2}$ at $\beta_c^* = \beta_c$.

However the WC behaviour of our results is rather trivial ($\langle \sigma_p \rangle \equiv 1$), since these clusters have no internal links. The singular point β_1^*, i e. the extremum of the WC metastable phase, is mapped to $-\infty$; on the contrary, let us recall that in the standard MF [1] the other singular point of the SC phase β_2^* is mapped to $+\infty$ and β_1^* to a finite value.

In order to obtain, within the same approximation, a reliable result for both in the SC and in the WC, we have considered a larger cluster, which contains an internal plaquette. A closed 4d volume with this property contains too many variables

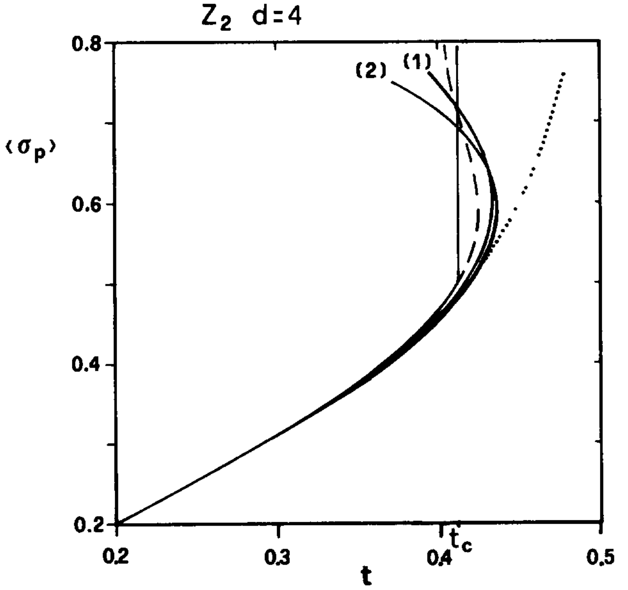


Fig 6 Strong coupling behaviour for the plaquette in 4d frustration MF on the cube (line (1)) and the hypercube (line (2)) of fig 5, the series expansion (dotted) and its Padé analysis (full and dashed line) from ref [18]

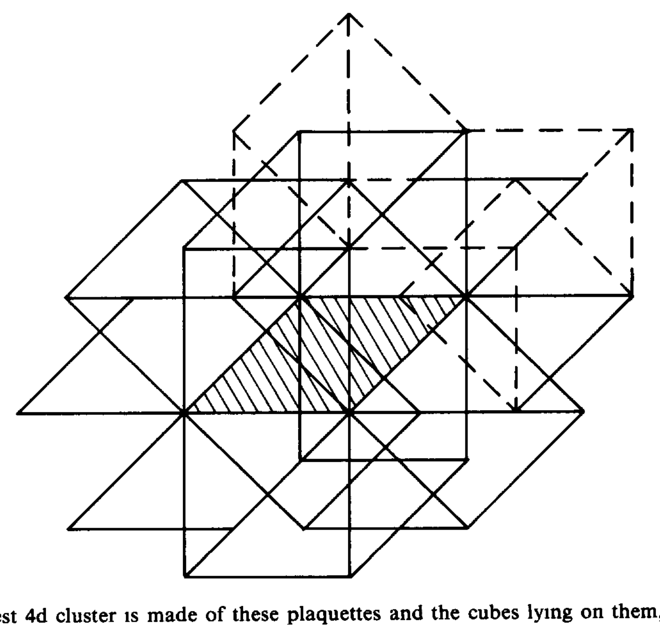


Fig 7 The largest 4d cluster is made of these plaquettes and the cubes lying on them, as the dashed ones The internal plaquette is shaded

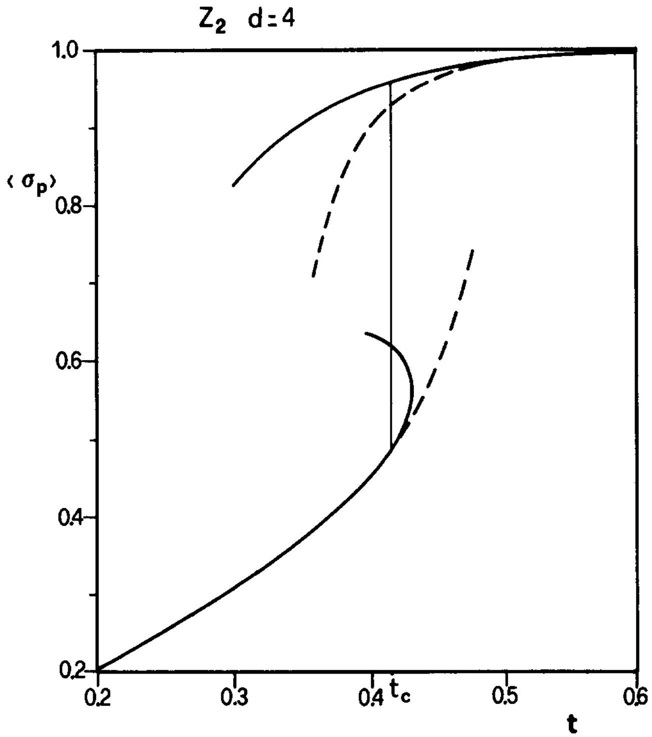


Fig 8 The frustration MF result for the plaquette on the cluster of fig 7 (full line) compared to the series (dashed)

to be handled easily; therefore, for practical reasons, we have considered an appropriate section (fig. 7). For this cluster the consistency equation is again eq. (4.8).

The result for the plaquette is reported in fig. 8 and shows the correct WC limit up to second order. The latent entropy is $\Delta\mathcal{S} = 6\beta^* \Delta(\langle\sigma_p\rangle) \simeq 1.25$ at $\beta^*_c = 0.4434$ (series, $\Delta S = 1.1$). The improvement in the WC region is also manifested by the change of β^*_1 from $-\infty$ to zero. It is natural to expect that the description of both metastable phases becomes better and better on larger clusters.

5. Mean field for the Wilson line

In the first part of this section we shall introduce a MF description à la BP based on the Wilson line (see eq. (4.3)), which is complementary to the approach of the previous section. The standard MF method in four dimensions [1] will be embodied in this description.

In subsect. 5.2 the analogous 3d method will be compared to the frustration MF discussed in sect. 3.

Finally, we shall clarify the symmetry breaking which is singled out by the Wilson line and by the line of frustrations in the dual case (subsect. 5.3).

5 1 THE FOUR-DIMENSIONAL CASE

Let us start from the simplest cluster. The dual of the cube in fig. 5a is the open cluster made of one link, $\hat{\mu}=\hat{4}$, and the surrounding six plaquettes $S_{\hat{\mu}}(0)S_{\hat{\nu}}(0)S_{\hat{\mu}}(\hat{\nu})S_{\hat{\nu}}(\hat{\mu})$, $\hat{\nu}=\pm\hat{1},\pm\hat{2},\pm\hat{3}$. The corresponding action is

$$A_\Lambda=\sum_{\hat{\nu}}\{\beta S_{\hat{\mu}}(0)S_{\hat{\nu}}(0)S_{\hat{\mu}}(\hat{\nu})S_{\hat{\nu}}(\hat{\mu})+b(S_{\hat{\nu}}(0)+S_{\hat{\nu}}(\hat{\mu})+S_{\hat{\mu}}(\hat{\nu}))\}\,. \tag{5.1}$$

Duality transformation on both sides of eq. (4.8) gives the consistency equation

$$\langle L\rangle_{\beta,1}\equiv\langle S_{\hat{\mu}}(0)\rangle_\beta=\langle S_{\hat{\mu}}(\hat{\nu})\rangle_\beta \tag{5.2}$$

between internal and external "one link" Wilson lines. Analogously, the dual of the hypercube (fig 5b) is the cluster made of the 8 links $S_{\hat{\mu}}(0)$, $\hat{\mu}=\pm\hat{1},\dots,\pm\hat{4}$, sharing one site and of the 24 plaquettes lying on them. The cluster action is now

$$A_\Lambda=\beta\sum_{\hat{\mu}\neq\hat{\nu}}S_{\hat{\mu}}(0)S_{\hat{\nu}}(\hat{\mu})S_{\hat{\mu}}(\hat{\nu})S_{\hat{\nu}}(0)+b\sum_{\hat{\mu}\neq\hat{\nu}}(S_{\hat{\mu}}(\hat{\nu})+S_{\hat{\nu}}(\hat{\mu}))\,, \tag{5.3}$$

where $\hat{\mu}\neq\hat{\nu}$ labels pairs of orthogonal links, the consistency equation

$$\langle S_{\hat{\mu}}(0)S_{-\hat{\mu}}(0)\rangle_\beta=\langle S_{\hat{\mu}}(\hat{\nu})S_{-\hat{\mu}}(\hat{\nu})\rangle_\beta\qquad(\hat{\nu}\neq\hat{\mu})\,, \tag{5 4}$$

relates internal and external "two-links" Wilson lines.

One can verify that eq. (5 2) reduces to eq. (4.9) and eq. (5 4) to eq. (4 15) after the identification of variables $\rho=e^{-2b}$, $t=e^{-2\beta}$. Therefore, for these two clusters we obtain again a first-order singularity at the value $\beta_c=0.431$. Without going into further details, we already know that the results of this formulation are better in the WC phase.

Eqs (5.1)-(5 4) suggest the following relevant remarks about the standard MF for 4d gauge theories.

The BP method, based on the consistency eq (5.2), reduces to the standard MF approach up to $1/d$ corrections in the large-d limit (with $2(d-1)\beta=O(1)$). In fact, the standard consistency $m=\tanh[2(d-1)\beta m^3]$ for $m=\langle S_0\rangle$ can be obtained in this limit from the extension of eqs. (5.2) and (4.9) to generic d $(5\to 2d-3)$.

Therefore we conclude that our analysis in sect. 4 on the condensation of frustrations supports, through the selfduality of the 4d theory, the reliability of the standard MF approach in gauge theories.

The standard MF consistency hypothesis, which was guessed in analogy with the spin case, can now be interpreted more physically as representing, on the smaller cluster, the consistency between Wilson lines; or, in other words between the quark free energies. This interpretation is confirmed by the result on larger clusters, e.g. eq. (5 4), which clearly shows the differences between the spin case and the gauge case.

The gauge invariance of this approach is recovered on a large cluster*. As we have already observed, a breaking term $bS_{\hat{\mu}}$ is indeed necessary on the border

* See ref [2] for the standard MF approach

($\hat{\mu} \in \partial\Lambda$) in order to avoid gauge transformations at the ends of the Wilson line [15] and to obtain a non-vanishing mean value; however, this breaking term does not affect gauge invariance in the bulk of the cluster. This choice of the boundary is quite natural, since it generalizes the procedure of spin theories for a correct definition of the magnetization $\langle S_0 \rangle$ [14]

5 2 THE THREE-DIMENSIONAL CASE

Our previous results in 3d and 4d gauge theories have shown the validity of a manifestly gauge-invariant MF approach in terms of frustration operators Moreover the standard MF in the special case of 4d has been reinterpreted through the selfduality of the theory.

In the 3d case, duality relates spin to gauge models and thus it does not allow us to justify the standard MF method for gauge theories. It is well known that this yields a first order phase transition instead of the expected second order one. The approach based on the Wilson line order parameter in 3d also reproduces this result*

Let us discuss more precisely the two independent relations between the Ising model and the Z_2 gauge theory. In subsect. 3.1 the relation between gauge frustration and spin magnetization has provided a MF for the disorder parameter in the gauge theory, which gives the correct order of the phase transition

Now we are concerned with the second relation, which connects the Wilson line with the line of spin frustrations. The condensation of these excitations leads to singular configurations on surfaces.

On the other hand, MF approaches are better suited for describing the thermodynamic properties of bulk quantities; therefore one can guess that they overestimate the condensation of these interface excitations, in such a way to give a first-order transition.

5 3 SPONTANEOUS SYMMETRY BREAKING

The Wilson line has been considered as the order parameter of the deconfinement transition at finite (physical) temperature T [19]. In this case, the line closes by periodic boundary conditions of period $N = 1/T$ and it is gauge invariant (Polyakov loop)

In the 4d Z_2 gauge theory we expect that the deconfined phase extends to $T=0$ for $\beta > \beta_c(0)$ (defined as $\lim_{T\to 0} \beta_c(T) \to \beta_c(0) = \frac{1}{2}\ln(1+\sqrt{2})$). If this is the case, the Polyakov loop (periodic boundary conditions) is the order parameter of the transition at zero temperature, and it should be equal to the Wilson line (fixed boundary) in the limit $N \to \infty$, provided that both limits are not singular**

* In 2d it yields the exact results, at variance with the standard MF approach which gives a first-order phase transition

** Let us remark that our finite-cluster calculations are not concerned with finite temperature, since our boundary conditions simulate an isotropic space–time infinite lattice

The symmetry breaking involved in the $T=0$ and in the $T>0$ cases is the same. Let us consider a line operator pointing along the time direction $\hat{t}$; in this case the symmetry transformation is the following global parity $\mathcal{P}$ of the $S_{\hat{t}}(x)$ variables on a 3d space slice of the lattice [19]:

$$S_{\hat{t}}(x, t_0) \xrightarrow[\mathcal{P}]{} -S_{\hat{t}}(x, t_0)\,, \tag{5.5}$$

where t_0 is a given value in the interval $(1, N)$. Under this transformation the line operator changes as:

$$\langle L(x)\rangle_{\beta,N} \xrightarrow[\mathcal{P}]{} -\langle L(x)\rangle_{\beta,N}\,. \tag{5.6}$$

The corresponding symmetry related to the condensation of lines of frustrations can be identified as in subsect. 3.2. Let us consider the action containing the proper source coupling and boundary term

$$A_\Lambda = \beta \sum_{p\in\Lambda} S_p + h \sum_{x\in\Lambda} \prod_{t=1}^{N} \{S_{\hat{t}}(x, t)\} + \sum_{\hat{\mu}\in\partial\Lambda} b(t) S_{\hat{\mu}}(x, t)\,. \tag{5.7}$$

The global parity symmetry of the partition function reads

$$Z_\Lambda(\beta; b(t); h) \underset{\mathcal{P}}{=} Z_\Lambda(\beta; -b(t_0), b(t\neq t_0); h)\,. \tag{5.8}$$

By checking this relation on the dual form of the partition function we obtain the (classical) parity operator in 4d:

$$\mathcal{P}^*_{t_0} \equiv \prod_{C\in\Omega_{t_0}} \left\{ \prod_{\partial C} A_p \right\} = \mathcal{P}^*\,, \tag{5.9}$$

where Ω_{t_0} is a collection of cubes dual to the links in eq. (5.5). Again, $\mathcal{P}^*$ acting on a field configuration gives $(-1)^\nu$, where ν is the number of lines of frustrated cubes.

One can repeat step by step the discussion of the spontaneous symmetry breaking as in the spin case (subsect. 3.2). The Wilson line shows the perimeter law in the WC

$$\langle L(x)\rangle_{\beta,N} \underset{\substack{\beta\to\infty\\ N\to\infty}}{\sim} \exp\left(-2N\,e^{-12\beta}\right). \tag{5.10}$$

Therefore, we expect that the large-volume limit should be defined for the quark free energy:

$$F_q(\beta) = \lim_{N\to\infty} -\frac{1}{N} \log\langle L(x)\rangle_{\beta,N} < \infty \qquad (b\neq 0)\,. \tag{5.11}$$

According to our previous discussion, e.q. (5.11) eventually characterizes the spontaneous breaking of the parity symmetry in the 4d Z_2 gauge theory.

In order to complete the analysis of the phase structure of the theory, the limit in eq. (5.11) should be tackled by computational methods or by a Peierls-type argument [14]. To our knowledge, this problem has not been considered in the literature. Computational methods [20, 21] and rigorous bounds [22] exist in the context of finite-temperature theories, but they are limited to the region of small values of N or β.

Let us only mention that the most interesting results concerning this problem have been obtained in the framework of MF methods by Alessandrini et al. [21]. A consistency equation for the Polyakov loop (together with the standard MF for the spatial link variables) has been applied successfully to reproduce the whole phase plane (T, β).

However, due to the choice of the axial gauge, one gets $\langle L\rangle \sim 1-2\,e^{-12\beta}$ for $\beta \to \infty$, and the limit $N \to \infty$ in eq. (5.11) is trivially satisfied. We are currently improving these results by a cluster approximation without gauge fixing.

Moreover, Peierls argument for the spin magnetization in 3d can be generalized straightforwardly to the Wilson line, yielding for β large enough ($\beta > \frac{1}{2}\ln 3$) the bound

$$\langle L\rangle_{\beta,N} \geqslant 1-2Nk\,e^{-12\beta}\,, \qquad k>1\,, \tag{5.12}$$

on a finite cluster Λ with fixed boundary ($S_{\hat{\mu}}=1, \hat{\mu} \in \partial\Lambda$). Eq. (5.12) like other bounds in the literature, is ineffective in the limit $N \to \infty$; a better bound is under investigation.

Our results on these subjects will be presented in detail elsewhere [23].

6. Conclusions

In this paper, the BP approximation the disorder parameters of the Ising spin and gauge models has been defined and analyzed in two, three and four dimensions. The description of the phase transition in terms of condensation of frustrations is the relevant and general result of the method, which provides a unified view over both gauge and spin models.

The dimensions of the space-time lattice and of the simplex, on which the interaction is defined, change with the considered model: they enter the MF procedure through the form of the disorder parameter.

This parameter has a local form in 2d and 3d and an extended form in 4d In the latter case (Wilson line) the large-volume limit remains to be carefully analyzed, in connection with finite temperature approaches.

In 3d its behaviour is also worth studying in order to complete the comprehension of these MF methods. In 2d gauge theory the Wilson line MF gives correctly the absence of a phase transition, while the dual formulation does not even exist since the transformation leads to a non-interacting theory.

Furthermore, our MF approach has to be tested in other models, where it should reproduce the existing phenomenology based on the condensation of topological excitations.

$Z(N)$ 2d spin and 4d gauge models, which display three phases [4], would probably require two consistency equations for the order and the disorder parameters at the same time.

Abelian topological excitations of the center $Z(N)$ have been applied in phenomenological descriptions of non-abelian $SU(N)$ gauge models (e.g. the $SU(2)$-$SO(3)$ system). We expect that our method applies to such cases; it could provide non-trivial answers, especially when Monte Carlo calculations become lengthy.

Furthermore our new MF approximation could possibly implement a phenomenological renormalization group approach, analogously to the one developed for the more familiar spin models [24].

It is a pleasure to thank Victor Alessandrini, Giorgio Parisi and Adam Schwimmer for interesting discussions and comments. We are also indebted to Marcello Ciafaloni for his constant support in all the stages of this work. Two of us (R.L. and S.R.) also thank Jean Bernard Zuber and Hendrik Flyvberg for useful discussions.

Appendix A: Mean plaquette consistency equations

Following the authors of ref. [7], we consider in our MF scheme a gauge-invariant consistency equation for the plaquette. As an example, we discuss the 3d gauge theory on the simplest cluster of fig 3a, whose action is given in eq. (3.2) This consistency reads

$$\langle \sigma_{p_{int}} \rangle_{\beta^*} = \langle \sigma_{p_{ext}} \rangle_{\beta^*} , \tag{A.1}$$

where p_{ext} is one of the equivalent external plaquettes and p_{int} is the internal one.

This equation turns out to be identical to eq. (3.4), obtained by a consistency on frustrations. It shows the solution $\langle \sigma_p \rangle \equiv 1 \ \forall \beta^*$ and a non-trivial one which starts at $\beta^* = \beta^*_c$ with a square-root singularity of the MF type.

This could suggest that the plaquette supports a MF approach for gauge theories. However, we shall argue that this conclusion is incorrect.

By imposing the same consistency equation (A.1) on the larger cluster of fig. 3b, we obtain a unique solution $\rho = \rho(\beta^*)$ for every β^*, since the other one ($\langle \sigma_p \rangle \equiv 1$) is not present. The plaquette shows a smooth behaviour, interpolating the correct WC and SC limits, and we have no signals of a phase transition.

In order to explain this result, let us recall that in the MF the consistency equation for the order parameter always has two solutions, corresponding to the symmetric and broken cases respectively.

Unfortunately, the plaquette is not an order parameter, since it is invariant under the dual parity symmetry discussed in subsect. 3.2. In such a case we conclude that the consistency hypothesis cannot single out the degeneration of the vacuum states, then yielding a unique solution, as observed in the larger cluster. Furthermore, we expect a smooth solution, because in the MF the singularity only arises from the matching of two solutions.

According to these remarks, the equivalence between the plaquette and the frustration consistency methods on the smaller cluster has to be considered accidental; in subsect. 3.1 we have seen that the solution $\langle\sigma_p\rangle \equiv 1$ comes just from the absence of link variables completely internal to the cluster.

Moreover, a unique smooth solution is also expected when the plaquette consistency is applied to other gauge models and in other (gauge invariant) MF approaches. Indeed this is the result of our analysis on both gauge (unpublished) and spin [8] models at their lower critical dimension and above.

We have applied the BP method previously discussed and a consistency approach similar to that of ref. [7]. The anomalous results found from the crudest approximations (the solution $\langle\sigma_p\rangle \equiv 1$, the loss of the SC solution or its unexpected divergence) disappear when corrections are included, leading to a unique smooth solution in β^* of the consistency equation. This fact has been also realized in the last paper of ref. [7]

Appendix B: Evaluation of cluster mean values

In this appendix we discuss the algebraic method for the evaluation of mean values on a cluster.

Let us consider, for example, the hypercube in fig. 5b. A mean value is the ratio of two polynomials in $u = \tanh(\beta^* + 2b^*)$ of the 24th order (the number of plaquettes):

$$\langle\mathcal{O}\rangle = \frac{\sum_{\{\sigma\}} \mathcal{O}(\{\sigma_\mu\})\, e^{(\beta^*+2b^*)\sum_p \sigma_p}}{\sum_{\{\sigma\}} e^{(\beta^*+2b^*)\sum_p \sigma_p}} = \frac{N}{Z(u)} \sum_{i=0}^{24} n(i) u^i, \tag{B.1}$$

where $Z(u) = N \sum_i m(i) u^i$ is the partition function of the cluster apart from the normalization function N, which is fixed by the condition $m(0) = 1$.

Our aim is to evaluate the $n(i)$ and $m(i)$ coefficients. First, let us consider the partition function $Z(u)$. The method consists in performing the statistical sum after a reduction of variables, which is possible since the cluster has free boundaries. A simple SC expansion shows that the number of configuration $\sum_i m(i)$ is, at most, equal to 2^k where $k = 8$ is the number of cubes; therefore, there must exist only eight relevant variables.

Duality transformation turns out to be useful in this reduction· on the dual cluster, which is an open star of plaquette, border variables are not interacting and can be summed explicitly. One gets

$$\begin{aligned} Z(u) &= \sum_{\{S_\mu(0)\}} \sum_{\{S_\mu(\hat\nu)\}} \exp\left\{\beta \sum_{\mu\neq\nu} S_{\hat\mu}(0)S_{\hat\nu}(0)S_{\hat\mu}(\hat\nu)S_{\hat\nu}(\hat\mu) + b \sum_{\mu\neq\nu} \left(S_{\hat\mu}(\hat\nu)+S_{\hat\nu}(\hat\mu)\right)\right\} \\ &= M \sum_{\{S_{\hat\mu}(0)\}} \exp\left[\log u\left(\sum_{\mu\neq\nu} \tfrac{1}{2}\left(1 - S_{\hat\mu}(0)S_{\hat\nu}(0)\right)\right)\right] \\ &= M \sum_{i=1}^{24} m(i) u^i , \end{aligned} \tag{B.2}$$

where $\{S_{\hat\mu}(0)\}$ are the internal links, $\hat\mu = \pm\hat 1, \ldots, \pm\hat 4$, $\{S_{\hat\mu}(\hat\nu)\}$ are the border ones and $M = [(1+\rho^2+2t\rho)/(4\rho t^{1/2})]^{24}$.

The evaluation of the numerator in eq. (B.1) is analogous, with minor changes in the form of the exponent in eq (B.2), according to the dual form of the operator $\mathcal{O}$.

On larger clusters the number of cubes increases rapidly: it can be reduced by applying the tricks of finite-lattice calculations [25], e.g. the use of symmetry.

References

[1] J-M Drouffe and J-B Zuber, Phys Reports 102 (1983) 1 and refs therein

[2] E Brèzin and J-M Drouffe, Nucl Phys B200 [FS2] (1982) 93,
B Bunk et al, Z Phys C18 (1983) 239

[3] H A Kramers and G H. Wannier, Phys Rev 60 (1947) 252

[4] R Savit, Rev Mod Phys 52 (1980) 453 and refs therein,
E Fradkin and L Susskind, Phys Rev D17 (1978) 2637,
S Elitzur, R B Pearson and J Shigemitzu, Phys Rev D19 (1979) 3698,
A Ukawa, P Windey and A H Guth, Phys Rev D21 (1980) 1013,
A H Guth, Phys Rev D21 (1980) 2291

[5] H Bethe, Proc Roy Soc (London) 150 (1935) 552,
R Peierls, ibid 151 (1936) 207,
R J Baxter, Exactly solved models in statistical mechanics (Academic Press, London, 1982)
R Livi, A Maritan and S Ruffo, J Phys A16 (1983) 2225,

[6] C Itzykson, R B Pearson and J-B Zuber, Nucl Phys B220 [FS2] (1983) 415,
J-B Zuber, Nucl Phys B235 [FS11] (1984) 435

[7] G G Batrouni, Nucl Phys B208 (1982) 12, 467,
E Dagotto and A Moreo, Phys Rev D30 (1984) 1271,
G G Batrouni, E Dagotto and A Moreo, CLNS preprint 85/638

[8] R Livi, A Maritan and S Ruffo, J Phys A 18 (1985) 1733

[9] L P Kadanoff and H Ceva, Phys Rev B3 (1971) 3918

[10] G H Wannier, Phys Rev 79 (1950) 357,
S Caracciolo, G Parisi and N Sourlas, Nucl Phys B205 [FS5] (1982) 345

[11] E Muller-Hartmann, Z Phys B27 (1977) 161,
T P Eggarter, Phys Rev B9 (1974) 2989,
E Muller-Hartmann and J Zittartz, Phys Rev Lett 33 (1974) 893,
J von Heimburg and H Thomas, J Phys C7 (1974) 3433

[12] M Creutz, L Jacobs and C Rebbi, Phys Reports 95 (1983) 201

[13] C Domb, Ising model, *in* Phase transitions and critical phenomena, eds C Domb and M S Green (Academic Press, London), vol III

[14] R B Griffiths, Rigorous results and theorems, *in* Phase transitions and critical phenomena, vol I
[15] S Elitzur, Phys Rev D12 (1975) 3978
[16] G 't Hooft, Nucl Phys B138 (1978) 1, B153 (1979) 141
[17] Unpublished, see e g ref [1]
[18] M Falcioni et al, Phys Lett 105B (1981) 51
[19] A M Polyakov, Phys Lett 72B (1978) 477,
L Susskind, Phys Rev D20 (1979) 2610,
B Svetitsky and L G Yaffe, Nucl Phys B210 [FS6] (1982) 423,
L D McLerran and B Svetitsky, Phys Rev D24 (1981) 450
[20] M Gross and J F Wheater, Nucl Phys B240 [FS12] (1984) 253 and refs therein
[21] V Alessandrini and Ph Boucaud, Nucl Phys B235 [FS11] (1984) 599
[22] C Borgs and E Seiler, Nucl Phys B215 [FS7] (1983) 125,
E Tomboulis and L Yaffe, Phys Rev D29 (1984) 780
[23] A Cappelli, R Livi and S Ruffo, in preparation
[24] I O Indekeu, A Maritan and A L Stella, J Phys A15 (1982) L291
[25] R B Pearson, Phys Rev B26 (1982) 6285