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$W_{1+\infty}$ Dynamics of Edge Excitations in the Quantum Hall Effect

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Quantum Hall universality classes can be classified by $W_{1+\infty}$ symmetry. We show that this symmetry also governs the dynamics of quantum edge excitations. The Hamiltonian of interacting electrons in the fully-filled first Landau level is expressed in terms of $W_{1+\infty}$ generators. The spectra for both the Coulomb and generic short-range interactions are thus found algebraically. We prove the one-dimensional bosonization of edge excitations in the limit of large number of particles. Moreover, the subleading corrections are given by the higher-spin $W_{1+\infty}$ generators, which measure the radial fluctuations of the electron density. The resulting spectrum for the Coulomb interaction contains a logarithmic enhancement, in agreement with experimental observations. Generic short-range interactions yield a subleading contribution to the spectrum, which can be expressed in terms of the classical capillary frequencies. These results are also extended to the Laughlin fractional fillings $\nu = 1/3, 1/5, \ldots$ by using symmetry arguments.

1. INTRODUCTION

In a recent series of papers [1–5], we have studied the many-body states at the *plateaus* of the quantum Hall effect [6] by means of an *effective field theory* with $W_{1+\infty}$ *dynamical symmetry*.

The effective field theory approach, developed by Landau, Ginsburg, Wilson and others [7], does not attempt to solve the microscopic many-body dynamics, but rather it guesses the macroscopic physics generated by this dynamics. The variables

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of the effective field theory are the relevant low-energy (long-distance) degrees of freedom, which are characterized by a specific symmetry. They describe *universal* properties, which are independent of the microscopic details. This approach is well suited for the quantum Hall effect, where experiments yield *very precise* and *universal* values of the Hall conductivity.

The main input for our effective field theory is the Laughlin theory of *incompressible quantum fluids* [8]; this predicts that the many-body ground states at the plateaus have uniform density and a gap for density waves. We have shown that planar incompressible fluids are completely characterized by a *dynamical symmetry* [9]. This means two basic kinematical properties:

(i) each state of the system can be mapped into any other one by a symmetry transformation, such that the classical configuration space (or the quantum Hilbert space) carries a representation of the classical (quantum) symmetry algebra;

(ii) the conserved quantities (quantum numbers) characterizing the excitations are given by a set of symmetry generators, which are simultaneously diagonal (the Cartan subalgebra); their spectrum is found by algebraic means.

The standard example of effective field theory with dynamical symmetry is *conformal field theory* in $(1+1)$ dimensions [10], which describes the long-distance physics of critical phenomena in two space dimensions, and also non-Fermi liquid behaviour of interacting fermions in $(1+1)$ dimensions [11].

In the quantum Hall effect, the study of representations of the $W_{1+\infty}$ dynamical symmetry has given both the relevant degrees of freedom and their quantum numbers [1]. We have thus obtained an *algebraic* description [2] of *anyons*, collective excitations of *fractional charge*, *spin and statistics* [12], and a classification of quantum Hall universality classes [4, 5], which corresponds to the Jain *hierarchy* [13].[1] This exact algebraic description accounts for the precision of the measured Hall conductivities at the plateaus.

These results on the kinematical data of incompressible fluids are reviewed in Section 2, setting the language for the new results discussed afterwards. Let us briefly summarize them. At the classical level, which is relevant for large samples, the incompressible fluid is characterized by the dynamical symmetry under *area-preserving diffeomorphisms*, obeying the w_∞ algebra [15]. This is because all possible configurations of a *droplet* of uniform density can be obtained by deformations which preserve the area. Furthermore, the relevant degrees of freedom are given by the small fluctuations of a large droplet, say of the shape of a disk, which are localized at the edge—the so-called *edge* waves [16]. The quantization of these waves yields a $(1+1)$-dimensional effective field theory, whose Hilbert space consists of a set of unitary, irreducible, highest-weight representations of $W_{1+\infty}$, the quantum version of the w_∞ algebra [15].

The $W_{1+\infty}$ algebra is linear and infinite dimensional, like the Virasoro conformal algebra [10]. It involves an infinite number of conserved currents V^i_n, having any

[1] See Ref. [14] for related works on the classification of quantum Hall universality classes.

positive (integer) conformal spin $(i+1)$. Its representations are characterized by the quantum numbers corresponding to the eigenvalues of V_0^i [17–19]. The basic quantum numbers are the electric charge V_0^0 and the spin V_0^1 of the excitations. The corresponding currents V_n^0 and V_n^1 generate the Abelian Kac-Moody and Virasoro subalgebras of $W_{1+\infty}$, respectively [10]. The basic operator is not the Hamiltonian, but the charge and spin operators, out of which all the Hilbert space of edge excitations can be constructed. Therefore, we can identify effective theories of edge excitations without specifying completely the dynamics of the incompressible fluid. Moreover, the kinematical data are sufficient to explain the Hall conduction experiments [2, 4, 14].

In this paper, we incorporate the dynamics of quantum edge excitations into the framework of $W_{1+\infty}$ symmetry. Dynamics is important in other types of experiments, where excitations are induced by temperature or other probes. Here, the effective field theory is still useful to predict the thermodynamics and the dynamical response.

In Section 3, we discuss the Hamiltonian of edge excitations. We first show how to use the w_∞ algebra to describe the dynamics of the classical, chiral, incompressible fluid, and rederive the *capillary frequencies* of its eigenoscillations [20]. Next, we study the general microscopic quantum Hamiltonian for interacting electrons in the fully filled Landau level (filling fraction $\nu=1$). It contains two competing contributions: the two-body repulsive interaction and the one-body confining potential, which holds the electrons together in the droplet. For large droplets of radius $R\to\infty$, this microscopic dynamics produces an effective dynamics for the edge excitations, which can be entirely expressed in terms of the generators of the $W_{1+\infty}$ algebra, owing to the dynamical symmetry. The confining potential expands linearly in the generators V_0^i of the Cartan subalgebra, while the two-body interaction is represented *quadratically* in the V_k^i. Both pieces in the effective Hamiltonian are infinite series, which can be organized according to a natural expansion in powers of $1/R$. Only a few leading terms are retained in the thermodynamic limit $R\to\infty$. Specifically, the leading contribution to the interaction term is shown to be quadratic in the Kac-Moody generators V_n^0, for a large variety of microscopic interactions. This result is the *proof* of the one-dimensional *bosonization* of edge excitations, which has been often used in the literature [21, 16], but has only been confirmed by numerical studies so far [22]. Moreover, the $W_{1+\infty}$ algebra describes the subleading corrections and the associated physical picture.

In Section 4, we use the $W_{1+\infty}$ algebra to diagonalize the Hamiltonian derived in Section 3, and compute the spectrum of quantum edge excitations. The confining potential always contributes to the leading order $O(1/R)$, yielding a scale invariant spectrum $\mathscr{E}(q)=vq$ (with $q\equiv \Delta J/R$ the edge momentum) [2]. The interactions can contribute to the same order and modify the spectrum, or, otherwise, be negligible. The *long-range* interactions are of the first type. The *short-range* interactions, with range of the order of the magnetic length ℓ (the ultra-violet cutoff), belong to the second class.

We prove that the three-dimensional Coulomb interaction $V(|\mathbf{x}-\mathbf{y}|)=e^2/|\mathbf{x}-\mathbf{y}|$ modifies the spectrum into $\mathscr{E}(q)\propto q\log(q)$, due to the appearance of an infra-red logarithmic singularity. This result was anticipated by several analytic and numerical approximations [23–25, 22]. Although this spectrum is no more scale invariant, we stress that it can be still described by conformal field theory techniques: the kinematical $W_{1+\infty}$ structure of the Hilbert space of edge excitations can in fact be used for diagonalizing the Hamiltonian.

Next, we discuss the contribution to the spectrum given by a generic short-range interaction, with Gaussian fall-off for $|\mathbf{x}| \gg \ell$. This contribution has the same form of the classical capillary frequency—a further indication of the semiclassical nature of the incompressible fluid [3]. The capillary frequency depends on one phenomenological parameter, which represents the *edge tension* of the one-dimensional boundary. This parameter is found to be a particular moment of the generic short-range interaction. The capillary energy, of order $O(1/R^3)$, is subdominant with respect to the leading contribution of the confining potential and Coulomb interaction. This shows that the spectrum of edge excitations is *universal*, i.e., independent of short-range effects. Universality implies experimental predictions that are *parameter-free* [2].

We conclude Section 4 by extending our results to the Laughlin incompressible fluids with fractional filling $\nu = 1/3, 1/5, \ldots$. These are described by other representations of $W_{1+\infty}$ [2, 4], and a Hamiltonian which is *form-invariant*.

Finally, in the conclusions we discuss the experiments that confirm the spectrum of edge excitations in the presence of Coulomb interactions [26, 27].

2. Dynamical Symmetry and Kinematics of Incompressible Fluids

In this section, we shall review the dynamical symmetry of *chiral*, two-dimensional, incompressible fluids and indicate how this leads uniquely to the construction of the Hilbert spaces of edge excitations. Next, we shall use their kinematical data to classify quantum Hall universality classes, without specifying the form of Hamiltonian.

2.1. *Classical Fluids*

A classical incompressible fluid is defined by its distribution function

$$\rho(z, \bar{z}, t) = \rho_0 \chi_{S_A(t)}, \qquad \rho_0 \equiv \frac{N}{A}, \tag{2.1}$$

where $\chi_{S_A(t)}$ is the characteristic function for a surface $S_A(t)$ of area A, and $z = x + iy$, $\bar{z} = x - iy$ are complex coordinates on the plane. Since the particle number N and the average density ρ_0 are constant, the area A is also *constant*. The only possible change in response to external forces is in the shape of the surface. The shape changes at constant area can be generated by *area-preserving diffeomorphisms* of the two-dimensional plane. Thus, the configuration space of a classical incompressible fluid can be generated by applying these transformations to a reference droplet.

Next we recall the Liouville theorem, which states that canonical transformations preserve the phase-space volume. Area-preserving diffeomorphisms are, therefore, canonical transformations of a two-dimensional phase space. In order to use the formalism of canonical transformations, we treat the original coordinate plane as a *phase space*, by postulating non-vanishing Poisson brackets between z and $\bar{z}$. We do this by defining the dimensionless Poisson brackets

$$\{f, g\} \equiv \frac{i}{\rho_0} (\partial f \, \bar{\partial} g - \bar{\partial} f \, \partial g), \tag{2.2}$$

where $\partial \equiv \partial/\partial z$ and $\bar{\partial} \equiv \partial/\partial \bar{z}$, so that

$$\{z, \bar{z}\} = \frac{i}{\rho_0}. \tag{2.3}$$

Note that the Poisson brackets select a preferred *chirality*, because they are not invariant under the two-dimensional parity transformation $z \to \bar{z}$, $\bar{z} \to z$; in the quantum Hall effect, the parity breaking is due to the external magnetic field.

Area-preserving diffeomorphisms, i.e., canonical transformations, are usually defined in terms of a generating function $\mathscr{L}(z, \bar{z})$ of both "coordinate" and "momentum," as follows:

$$\delta z = \{\mathscr{L}, z\}, \qquad \delta \bar{z} = \{\mathscr{L}, \bar{z}\}. \tag{2.4}$$

A basis of (dimensionless) generators is given by

$$\mathscr{L}^{(cl)}_{n,m} \equiv \rho_0^{(n+m)/2} \, z^n \, \bar{z}^m. \tag{2.5}$$

These satisfy the classical w_∞ algebra [15]

$$\{\mathscr{L}^{(cl)}_{n,m}, \mathscr{L}^{(cl)}_{k,l}\} = -i(mk - nl) \, \mathscr{L}^{(cl)}_{n+k-1,m+l-1}. \tag{2.6}$$

Let us now discuss how w_∞ transformations can be used to generate the configuration space of classical excitations above the ground state. These configurations have a classical energy due to the inter-particle interaction and the external confining potential, whose specific form is not needed here. Let us assume a generic convex and rotation-invariant energy function, such that the minimal energy configuration ρ_{GS} has the shape of a disk of radius R:

$$\rho_{GS}(z, \bar{z}) = \rho_0 \Theta(R^2 - z\bar{z}), \tag{2.7}$$

where Θ is the Heaviside step function. The classical "small excitations" around this ground state configuration are given by the infinitesimal deformations of ρ_{GS} under area-preserving diffeomorphisms,

$$\delta\rho_{n,m} \equiv \{\mathscr{L}^{(cl)}_{n,m}, \rho_{GS}\}. \tag{2.8}$$

Using the Poisson brackets (2.2), we obtain

$$\delta\rho_{n,m} = i(\rho_0 R^2)^{(n+m)/2}\,(m-n)\,e^{i(n-m)\theta}\delta(R^2 - z\bar{z}). \tag{2.9}$$

These correspond to density fluctuations localized on the sharp boundary (parametrized by the angle θ) of the classical droplet. Due to the dynamics provided by the energy function, they will propagate on the boundary with a frequency ω_k dependent on the angular momentum $k \equiv (n-m)$, thereby turning into *edge waves*. These are the eigenoscillations of the classical incompressible fluid.

Another type of excitations are classical vortices in the bulk of the droplet, which correspond to localized holes or dips in the density. The absence of density waves, due to incompressibility, implies that any localized density excess or defect is transmitted completely to the boundary, where it is seen as a further edge deformation. For each given vorticity in the bulk, we can then construct the corresponding basis of edge waves in a fashion analogous to (2.8). Thus, the configuration space of the excitations of a classical incompressible fluid (of a given vorticity) is spanned by infinitesimal w_∞ transformations. This is the property of *dynamical symmetry* described in the Introduction.

2.2. *Quantum Theory at the Edge*

The quantum[2] version of the chiral, incompressible fluids is given by the Laughlin theory of the plateaus of the quantum Hall effect [8]. The simplest example of such a macroscopic quantum state is the fully filled Landau level (filling fraction $\nu = 1$). Generically, it possesses three types of excitations. First, there are *gapless edge excitations*, which are the quantum descendants of the classical edge waves described before. These are particle-hole excitations across the Fermi surface represented by the edge of the droplet; therefore, they are called *neutral*. They are also *gapless* because their energy, of $O(1/R)$, vanishes for $R \to \infty$ [16]. Second, there are localized quasi-particle and quasi-hole excitations, which have a finite gap. These are the quantum analogs of the classical vortices and correspond to the anyon excitations with fractional charge, spin and statistics [8]. As in the classical case, they manifest themselves as charged excitations at the edge, owing to incompressibility. The third type of excitations are two-dimensional density waves in the bulk, the magnetoplasmons and (for $\nu < 1$) the magnetophonons [6, 28]. These have higher gaps and are not included in our effective field theory approach.

In the previous section, we have explained the connection between the classical edge waves and the generators of the algebra w_∞ of area-preserving diffeomorphisms. In the quantum theory, there is a corresponding relation between edge excitations and the generators of the quantum version of w_∞, called $W_{1+\infty}$ [15]. This algebra is obtained by replacing the Poisson brackets (2.2) with quantum commutators: $i\{\ ,\ \} \to [\ ,\ ]$, and by taking the thermodynamic limit [2].

[2] Throughout this paper we shall use units such that $c = 1$, $\hbar = 1$.

In this limit, the radius of the droplet grows as $R \propto \ell \sqrt{N}$, where $\ell = \sqrt{2/(eB)}$ is the magnetic length and B the magnetic field. Quantum edge excitations, instead, are confined to a boundary annulus of finite size $O(\ell)$. In order to construct an effective field theory for these low-lying excitations, the quantization should be followed by the limit of second-quantized operators into finite expressions localized on the boundary.[3] Let us consider, for example, the case of the fully-filled first Landau level. In Ref. [2], it was shown that the boundary limit amounts to a straightforward expansion in powers of $1/R \propto 1/\ell \sqrt{N}$. The *quantum* incompressible fluid, which corresponds to a Fermi sea of electrons, becomes effectively a Dirac sea. Correspondingly, the $(2+1)$-dimensional electrons on the boundary become Weyl fermions, i.e., relativistic, charged, chiral fermions in $(1+1)$-dimensions [10]. The field operator of electrons becomes[4] [2],

$$F_R(\theta) = \frac{1}{\sqrt{R}} \sum_{k=-\infty}^{\infty} e^{i(k-1/2)\theta} b_k, \qquad (|z| = R,\, t = 0), \tag{2.10}$$

where θ parameterizes the circular boundary, b_k and $b_k^\dagger$ are fermionic Fock space operators satisfying $\{b_l, b_k^\dagger\} = \delta_{l,k}$, and k is the angular momentum measured with respect to the ground state value.

The generators of the quantum algebra $W_{1+\infty}$ are represented in this Fock space by the bilinears

$$\begin{aligned} V_n^j &= \int_0^{2\pi} \frac{d\theta}{2\pi} : F^\dagger(\theta)\, e^{-in\theta}\, g_n^j(i\partial_\theta)\, F(\theta) : \\ &= \sum_{k=-\infty}^{\infty} p(k,n,j) : b_{k-n}^\dagger b_k :, \qquad j \geqslant 0. \end{aligned} \tag{2.11}$$

In this expression, $F(\theta) = \sqrt{R}\, F_R(\theta)\, e^{i\theta/2}$ is the canonical form of the Weyl field operator of conformal field theory. The factor $g_n^j(i\partial_\theta)$ is a j-th order polynomial in $i\partial_\theta$, whose form specifies the basis of operators and guarantees the hermiticity $(V_n^j)^\dagger = V_{-n}^j$. The coefficients $p(k,n,j)$ are also j-th order polynomials in k to be specified later (see Appendix A). The $W_{1+\infty}$ algebra reads

$$[V_n^i, V_m^j] = (jn - im)\, V_{n+m}^{i+j-1} + q(i,j,m,n)\, V_{n+m}^{i+j-3} + \cdots + c^i(n)\, \delta^{i,j} \delta_{n+m,0}. \tag{2.12}$$

Here, $i+1 = h \geqslant 1$ represents the "conformal spin" of the generator V_n^i, while $-\infty < n < +\infty$ is the angular momentum (the Fourier mode on the circle). The first term on the right-hand-side of (2.12) reproduces the classical w_∞ algebra (2.6) by the correspondence $\mathscr{L}_{i-n,i}^{(cl)} \to V_n^i$ and identifies $W_{1+\infty}$ as the algebra of "quantum area-preserving diffeomorphisms." The additional terms are quantum operator

[3] We refer to our previous papers for a complete discussion of quantization and the limit of the second-quantized operators to the boundary [2, 4].

[4] Hereafter, we choose units such that $\ell = 1$.

corrections with polynomial coefficients $q(i, j, n, m)$, due to the algebra of higher derivatives [15]. Moreover, the c-number term $c^i(n)$ is the quantum *anomaly*, a relativistic effect due to the renormalization of operators acting on the infinite Dirac sea. It is diagonal in the spin indices for our choice of basis for the g^i_k in Eq. (2.11) (see Appendix A). Finally, the normal ordering (: :) of the Fock operators takes care of the renormalization [10].

Let us analyse the generators V^0_n and V^1_n of lowest conformal spin. From (2.11) we see that the V^0_n are Fourier modes of the fermion density evaluated at the edge $|z| = R$; thus, V^0_0 measures the edge charge. Instead, the V^1_n are vector fields which generate angular momentum transformations on the edge, such that V^1_0 measures the angular momentum of edge excitations. Their algebra is given by

$$[V^0_n, V^0_m] = c\, n\, \delta_{n+m,0}, \tag{2.13}$$

and

$$\begin{aligned} [V^1_n, V^0_m] &= -m V^0_{n+m}, \\ [V^1_n, V^1_m] &= (n-m)\, V^1_{n+m} + \frac{c}{12}(n^3 - n)\, \delta_{n+m,0}, \end{aligned} \tag{2.14}$$

with $c = 1$. These equations show that the V^0_n and V^1_n operators satisfy an Abelian Kac–Moody algebra and a Virasoro algebra, respectively [10].

In conformal field theory [10], the Weyl fermion can be defined *algebraically* as the Hilbert space made by a set of irreducible, highest-weight representations of the $W_{1+\infty}$ algebra, closed under the *fusion rules* for making composite excitations. Any representation contains an infinite number of states, corresponding to all the neutral excitations above a bottom state, the so-called *highest weight* state. This can be, for example, the ground-state $|\Omega\rangle$ corresponding to the incompressible quantum fluid. The excitations can be written as

$$|k, \{n_1, n_2, ..., n_s\}\rangle = V^0_{-n_1} V^0_{-n_2} \cdots V^0_{-n_s} |\Omega\rangle, \qquad n_1 \geqslant n_2 \geqslant \cdots \geqslant n_s > 0, \tag{2.15}$$

while the positive modes $(n_i < 0)$ annihilate $|\Omega\rangle$. Here $k = \sum_j n_j$ is the total angular momentum of the edge excitation.

Furthermore, any charged edge excitation, together with its tower of neutral excitations, also forms an irreducible, highest-weight representation of $W_{1+\infty}$. The states in this representation have the same form of (2.15), but the bottom state $|Q\rangle$ now represents a quasi-particle inside the droplet. The charge and spin of the quasi-particle are given by the eigenvalues of the operators[5] $(-V^0_0)$ and V^1_0 respectively:

$$V^0_0\, |Q\rangle = Q\, |Q\rangle, \qquad V^1_0\, |Q\rangle = J\, |Q\rangle. \tag{2.16}$$

[5] The minus sign is due to the fact that V^0_0 measures the charge on the edge. Due to overall charge conservation, the charge of a quasi-particle in the bulk has the opposite sign of its edge counterpart.

Actually, all the operators V_0^i are simultaneously diagonal and assign other quantum numbers to the quasi-particle, $V_0^i \,|Q\rangle = m_i(Q)\,|Q\rangle$, $i \geqslant 2$, which are known polynomials in the charge Q (see Appendix A). These quantum numbers measure the radial moments of the charge distribution of a quasi-particle (see Eq. (3.18) below); their fixed functional form indicates the rigidity of density modulations of the quantum incompressible fluid.

2.3. *Classification of QHE Universality Classes*

Besides this explicit construction for $\nu = 1$, leading to a theory with $c = 1$, it has been shown in general that the algebra (2.12) is the unique quantization of the w_∞ algebra in the $(1+1)$-dimensional field theory on the circle [29]. This implies that the effective quantum theories of the incompressible fluids are completely specified by the $W_{1+\infty}$ symmetry for any value of the density, i.e., of the filling fraction. Although these general theories cannot be derived explicitly in terms of electrons, they can be completely constructed by using the algebraic methods of conformal field theory [10] and the $W_{1+\infty}$ representation theory. The latter was recently developed completely in Refs. [17, 19]: for unitary theories, the Virasoro central charge c can be any *positive integer*, which can be understood as the "number of components" in a many-fluid generalization of the previous $\nu = 1$ example.

In Refs. [4, 5], we *postulated* that *all* quantum Hall incompressible fluids are in one-to-one correspondence with $W_{1+\infty}$ theories, and we used the representation theory to obtain an algebraic classification of their *universality classes*. These classes are specified by the kinematical data of the incompressible fluid, which can be obtained from the quantum numbers (weights) of the $W_{1+\infty}$ representations, as follows:

(i) the charge Q and fractional statistics θ/π of the quasi-particle excitations are given by the eigenvalues of V_0^0 and V_0^1, respectively;

(ii) the Hall conductivity $\sigma_H = (e^2/h)\,\nu$, proportional to the filling fraction ν of the ground state, is obtained from the chiral anomaly;

(iii) the number of particle-hole excitations of given angular momentum, i.e., the *degeneracies* of states above the ground state (2.15) are obtained from the characters of the $W_{1+\infty}$ representations [5].

Let us also mention another kinematical data, the *topological order*, i.e., the degeneracy of the ground state on a toroidal geometry, which was used by X.-G. Wen and other authors to characterize the edge theories [16, 30]. Actually, this quantity can also be computed from the fusion rules [10] of the $W_{1+\infty}$ representations.

Here we wish to stress that the $W_{1+\infty}$ dynamical symmetry characterizes completely these kinematical properties of incompressible quantum fluids, without reference to a specific Hamiltonian. Note that fractional statistics is also independent of the dynamics, being a *static* property of correlators on the circle, i.e., in the plane. Moreover, these informations are sufficient to determine σ_H, and thus to describe the Hall conduction experiments. Actually, the Hall current is computable from the chiral anomaly of the $(1+1)$-dimensional edge theory [2], which is encoded in the charge commutator Eq. (2.13) (see also [4]).

The outcome of this classification program is quite interesting [4, 5]. The special class of *minimal* $W_{1+\infty}$ theories (minimal models) [5] was shown to yield a new *hierarchy* of the fractional Hall fluids. These models reproduce the Jain hierarchy [13] of stable, experimentally observed plateaus, and predict new, *non-Abelian*, properties for the edge excitations. These properties are rather different from those of the standard, Abelian theory of edge excitations [14], and will be hopefully tested by real and numerical experiments.

3. $W_{1+\infty}$ Dynamics of Edge Excitations

In the previous section, we have restricted ourselves to *kinematical* considerations. In the following, we are going to discuss the *dynamics* of the edge excitations. We start from the classical problem, the spectrum of eigenfrequencies ω_k of eigenoscillations.

3.1. *Classical Capillary Waves*

The dynamics of the classical incompressible fluid is fixed by a single parameter τ, with dimension of action, whose role is to translate between the purely kinematical, dimensionless Poisson brackets (2.2) and the actual Poisson brackets (with dimension (1/action))

$$\partial_t \, \delta\rho_{n,m} = \frac{1}{\tau} \{\delta\rho_{n,m}, H\} = i\omega_k \, \delta\rho_{n,m}, \qquad k = n - m. \tag{3.1}$$

The parameter τ is not generic, but depends on the specific model under consideration. In our case, it will be given by the classical limit of the Laughlin incompressible fluids.

First we study a Hamiltonian which contains only a rotational invariant confining potential $V(|\mathbf{x}|)$. From Eq. (3.1), we obtain the eigenfrequencies,

$$\omega_k = \frac{V'(R)}{2R\tau\rho_0} k, \tag{3.2}$$

where the prime denotes differentiation with respect to the radial variable $r = |\mathbf{x}|$. Note the sign correlation between k and ω_k, which means that the edge waves propagate only in one direction along the one-dimensional edge, i.e., they are *chiral*.

For applications to the quantum Hall effect, one should consider a Hamiltonian also containing the two-body repulsive interaction. A known method for handling the interaction at the classical level is the *hydrodynamic approximation* [20, 31]. The two-body interaction is replaced by the one-body effect of a *boundary pressure* arising from the imperfect cancelation of the repulsive forces near the edge of the sample. We expect this approximation to be good for inter-particle interactions which are not too long ranged, as in a liquid. This expectation will be confirmed by the quantum dynamics discussed below.

We thus have the following decomposition of the boundary force:

$$V'(R) = V'_{cp}(R) + V'_p(R). \tag{3.3}$$

The pressure contribution to the frequencies ω_k is known as *capillary frequency* [20], and can be computed as follows. The radial force due to the interactions is given by $F_p(r) = -p'(r)/\rho_0$, where $p(r)$ is the pressure. Therefore, we have $V'_p(R) = p'(R)/\rho_0$. Let us consider an harmonic edge oscillation parameterized by $r = R + \varepsilon(\theta)$, with $\varepsilon(\theta) = \varepsilon_0 \sin(k\theta)$, and expand

$$p(r) = p(R) + p'(R)\,\varepsilon. \tag{3.4}$$

We now use the following representation of the boundary pressure:

$$p = -\frac{\alpha}{R_c} = -\alpha\,\frac{|\partial_\theta \mathbf{x} \wedge \partial_\theta^2 \mathbf{x}|}{|\partial_\theta \mathbf{x}|^3} = -\alpha\,\frac{2(\partial_\theta \varepsilon)^2 + r(r - \partial_\theta^2 \varepsilon)}{(r^2 + (\partial_\theta \varepsilon)^2)^{3/2}}. \tag{3.5}$$

where α is the *edge tension*, which is positive for repulsive interactions, and R_c represents the local radius of curvature. Expanding this expression up to first order in ε we find,

$$-p(R+\varepsilon) = \frac{\alpha}{R} - \frac{\alpha}{R^2}\,(\varepsilon + \partial_\theta^2 \varepsilon) + O\left(\frac{\varepsilon^2}{R^3}\right). \tag{3.6}$$

Using $\partial_\theta^2 \varepsilon = -k^2 \varepsilon$, we obtain

$$V'_p(R) = -\frac{\alpha}{R^2 \rho_0}\,(k^2 - 1), \tag{3.7}$$

and the frequencies

$$\omega_k = \frac{V'_{cp}(R)}{2R\tau\rho_0}\,k - \frac{\alpha}{2R^3\tau\rho_0^2}\,k(k^2-1). \tag{3.8}$$

As a last step, it remains to determine the value of τ relevant to the classical limit of the Laughlin incompressible quantum fluids. Consider the Lagrangian for planar electrons of mass m in an external magnetic field B,

$$L = \sum_{i=1}^{N} \left(\frac{m}{2}\,\dot{\mathbf{x}}_i^2 + e\dot{\mathbf{x}}_i \cdot \mathbf{A}(\mathbf{x}_i) \right), \tag{3.9}$$

in the symmetric gauge $\mathbf{A}(\mathbf{x}) = B/2(-y, x)$, with $B > 0$. The Laughlin fluids involve electrons in the first Landau level only. Thus, we can project out the higher Landau levels in the limit of large B. As was shown in [32], this projection can be implemented by taking the limit $m \to 0$ of (3.9). The residual dynamics is then governed by the following Lagrangian:

$$L_R = \lim_{m \to 0} L = \frac{eB}{2} \sum_{i=1}^{N} \mathbf{x}_i \wedge \dot{\mathbf{x}}_i. \tag{3.10}$$

This Lagrangian is of first order in time derivatives, which implies that the projection on a subspace of constant (kinetic) energy amounts to a phase space reduction from a $4N$-dimensional phase-space to a $2N$-dimensional phase-space. The new symplectic structure is given by [32]

$$\{z_i, \bar{z}_j\}_{PB} = i \frac{2}{eB} \delta_{ij}, \tag{3.11}$$

and shows that the original coordinate plane of the electrons behaves indeed as a phase space for motions at constant kinetic energy. By comparing (3.11) with (2.2) we obtain the explicit expression for the parameter τ:

$$\tau = \frac{eB}{2\rho_0}. \tag{3.12}$$

Inserting this in (3.8), we obtain the final result for the dispersion relation of the classical edge waves:

$$\omega_k = \frac{v}{R} k - \frac{\alpha}{eB\rho_0 R^3} k(k^2 - 1), \qquad v = \frac{V'_{CP}(R)}{eB}. \tag{3.13}$$

This formula can be also obtained by solving the hydrodynamic equations in the presence of a magnetic field [23, 25]. Here, we showed that it can be derived in a much simpler way after realizing that the kinematics of the incompressible classical fluid is specified by w_∞ transformations.[6]

Equation (3.13) has some properties which are worth stressing and must be kept in mind for the quantum dynamics. First, there are zero modes, which correspond to symmetries: the $k=0$ radial breathing mode of the droplet is clearly absent in an incompressible fluid, as is evident from (2.9). In addition, a translation invariant two-body interaction gives rise to a vanishing capillary frequency for the modes $k = \pm 1$, which have $\varepsilon(\theta) \propto \sin\theta$ and thus correspond to an overall translation of the droplet. Another general property of the classical fluid is the degeneracy of its eigenoscillations, as it is also apparent from Eqs. (2.9) and (3.1): although a generic w_∞ deformation of the droplet is parameterized by two integers (n, m), the frequency depends on their difference k only. This is the consequence of the sharp

[6] Note, however, that the hydrodynamic equations also give a higher, gapful branch of frequencies for compression waves corresponding to the magneto-plasmon.

boundary of the classical droplet, which forbids infinitesimal radial fluctuations. On the contrary, the quantum incompressible fluid has a smooth boundary, and radial fluctuations produce a hierarchy of dynamical effects, which are described by the action of the quantum algebra $W_{1+\infty}$.

Finally, we note that ω_k contains two *phenomenological parameters* v and α, in addition to the generic parameters ρ_0, $A=\pi R^2$ and B. For large droplets, the confining potential term dominates over the capillary frequency and the dispersion relation becomes of relativistic nature, with *effective light-velocity* v.

3.2. *Quantum Hamiltonian in the First Landau Level*

In the following, we shall discuss the quantum Hamiltonian for the incompressible fluid at filling $\nu=1$. We consider the microscopic model of planar electrons confined to the first Landau level. This projection quenches the kinetic energy [33], so that the Hamiltonian contains only the confining potential $V_{CP}(|\mathbf{x}|)$ and the repulsive two-body interaction $V_I(|\mathbf{x}-\mathbf{y}|)$,

$$H=H_{CP}+H_I=\int d^2\mathbf{x}\; V_{CP}(\mathbf{x})\,\rho(\mathbf{x})+\tfrac{1}{2}\int d^2\mathbf{x}\int d^2\mathbf{y}\;\rho(\mathbf{x})\,V_I(\mathbf{x}-\mathbf{y})\,\rho(\mathbf{y}), \tag{3.14}$$

where $\rho(\mathbf{x})\equiv\Psi^\dagger(\mathbf{x})\,\Psi(\mathbf{x})$ is the density operator, constructed from the field operator

$$\Psi(\mathbf{x})=\sum_{j=0}^{\infty} a_j\psi_j(\mathbf{x}). \tag{3.15}$$

Here, a_j and $a_j^\dagger$ are fermionic Fock space annihilators and creators, satisfying the usual anticommutation rules $\{a_l, a_k^\dagger\}=\delta_{l,k}$, with all other anticommutators vanishing, and $\psi_j(\mathbf{x})$ are the first Landau level wave functions

$$\psi_j(\mathbf{x})=\frac{1}{\ell\sqrt{\pi j!}}\left(\frac{z}{\ell}\right)^j e^{-|z|^2/2\ell^2}. \tag{3.16}$$

In this equation, we have restored the magnetic length ℓ, for the sake of completeness.

In the following, we shall write the Hamiltonian in terms of quantum $W_{1+\infty}$ operators, which will be used later to find the spectrum of quantum edge excitations. The operators producing the quantum analogs of the classical deformations $\delta_{n,m}\rho$ have been defined in Ref. [1], by quantizing the classical generators (2.5) in the first Landau level:

$$\begin{gathered}\mathscr{L}_{n,m}=\int d^2\mathbf{x}\;\Psi^\dagger(\mathbf{x})(b^\dagger)^n\,(b)^m\,\Psi(\mathbf{x}),\\ b\equiv\frac{\bar{z}}{2}+\partial,\qquad b^\dagger\equiv\frac{z}{2}-\bar{\partial}.\end{gathered} \tag{3.17}$$

These operators satisfy a "non-relativistic" version of the quantum algebra $W_{1+\infty}$ (2.12) [1, 34]. The operators $\mathscr{L}_{nn}$ are mutually commuting, as in the classical case, and can be simultaneously diagonalized. The physical meaning of their eigenvalues can be inferred from the expectations values

$$\langle \mathscr{L}_{nn} \rangle = (-1)^n \, 2\pi n! \int_0^\infty dr \, r L_n(r^2) \langle \rho(r^2) \rangle, \tag{3.18}$$

where the $L_n(x)$ are the Laguerre polynomials [35]. Therefore, they measure certain *radial moments* of the density distribution in any given quantum state. In order to describe quantum numbers of excitations, they have to be normal ordered by subtracting the ground-state expectation values, $\mathscr{L}_{nn} \to :\mathscr{L}_{nn}: = \mathscr{L}_{nn} - \langle \Omega | \, \mathscr{L}_{nn} \, | \Omega \rangle$, where $|\Omega\rangle$ is the filled Fermi sea

$$|\Omega\rangle = a_0^\dagger \cdots a_N^\dagger \, |0\rangle, \tag{3.19}$$

of $(N+1)$ electrons. Given that $\langle \Omega | \, \mathscr{L}_{nn} \, | \Omega \rangle = O(N^{n+1})$, the $\mathscr{L}_{nm}$ operators, as well as their "non-relativistic" algebra, undergo a severe renormalization in the large N limit. It can be shown that specific linear combinations $\sum_{n=0}^{i} \lambda_n \mathscr{L}_{n-k,n}$ have a finite limit as the normal-ordered boundary operators V_k^i introduced in (2.11), satisfying the $W_{1+\infty}$ algebra (2.12). Therefore, we shall expand the Hamiltonian in the basis of the operators V_k^i.

Let us first consider the one-body term H_{CP}. As discussed in Ref. [2], for $\nu = 1$ the geometrical (semi)-classical picture of edge waves has a simple correspondence in the Fock space of fermions. Edge waves are particle-hole excitations above the filled Fermi sea (3.19) with angular momentum $k = \Delta J \sim O(1)$, $|k| \ll \sqrt{N}$ [2]. The boundary field theory is constructed by operators which measure these transitions and remain finite for large N. Let us consider the Fock expression of H_{CP} and expand the confining potential in a power series:

$$V_{CP}(|\mathbf{x}|) = \gamma_0 + \gamma_1 \, |\mathbf{x}|^2 + \gamma_2 \, |\mathbf{x}|^4 + \cdots, \tag{3.20}$$

$$H_{CP} = \sum_{k=0}^{\infty} \left[\gamma_0 + \gamma_1 (k+1) + \gamma_2 (k^2 + 3k + 2) + \cdots \right] a_k^\dagger a_k. \tag{3.21}$$

Following Ref. [2], it is convenient to redefine the Fock operators by shifting the angular momentum index,

$$b_r = a_{N+r}, \qquad b_s^\dagger = a_{N+s}^\dagger, \tag{3.22}$$

such that the first empty state has index $r = 1$. For large N, the excitations of electrons from the deep inside of the droplet have high energy and can be neglected. Thus, we can extend the summation in (3.21) from $k \in [-N, \infty)$ to $k \in (-\infty, +\infty)$, i.e., consider a relativistic Dirac sea. We can now compare H_{CP} in (3.21) to the explicit Fock space expression of the $W_{1+\infty}$ operators (2.11) (see Ref. [2] and Appendix A),

$$\begin{aligned} V_n^0 &= \sum_{r=-\infty}^{\infty} :b_{r-n}^\dagger b_r:, \\ V_n^1 &= \sum_{r=-\infty}^{\infty} \left(r - \frac{n+1}{2}\right) :b_{r-n}^\dagger b_r:, \\ V_n^2 &= \sum_{r=-\infty}^{\infty} \left(r^2 - (n+1)\, r + \frac{(n+1)(n+2)}{6}\right) :b_{r-n}^\dagger b_r:, \end{aligned} \tag{3.23}$$

and obtain the desired expansion,

$$\begin{aligned} H_{CP} - \langle \Omega |\, H_{CP} \,| \Omega \rangle = {} & \gamma_0 V_0^0 + \gamma_1 (V_0^1 + (N + \tfrac{3}{2})\, V_0^0) \\ & + \gamma_2 (V_0^2 + 2(N+2)\, V_0^1 + ((N+2)^2 - \tfrac{1}{3})\, V_0^0) + \cdots. \end{aligned} \tag{3.24}$$

In general, confining potentials growing as $V_{CP} = O(|\mathbf{x}|^{2i})$ involve all operators V_0^j with $j \leqslant i$.

We consider the confining potential V_{CP} to be generated dynamically at the edge of the sample. Therefore, we expect a self-tuning of the parameters γ_i for large $N \equiv R^2$, such that the normal-ordered energy is finite when written in terms of the quantum numbers $\langle V_0^i \rangle$ of edge excitations. We thus consider the normal-ordered Hamiltonian

$$H_{CP} = \alpha_0 (V_R)_0^0 + \alpha_1 (V_R)_0^1 + \alpha_2 (V_R)_0^2 + \cdots, \qquad N \to \infty, \tag{3.25}$$

where the $(V_R)_0^i$ are the generators of $W_{1+\infty}$ properly normal-ordered on the *cylinder* $(R\theta, t)$. They are related to the V_0^i of Eq. (2.11), conventionally defined on the *conformal plane*, by well-known transformation rules [36] (see Appendix A). The resulting expression is

$$H_{CP} = \alpha_0 V_0^0 + \frac{\alpha_1}{R} \left(V_0^1 - \frac{c}{24} \right) + \frac{\alpha_2}{R^2} \left(V_0^2 - \frac{1}{12} V_0^0 \right) + O\left(\frac{1}{R^3}\right), \qquad R \equiv \sqrt{N}. \tag{3.26}$$

Note that the $(V_R)_0^i$ are dimensionful, because their eigenvalues are of order $O((k/R)^i) = O(q^i)$ for states of angular momentum k and edge momentum q, as explained in Section 4. A sensible thermodynamic limit of the edge theory is achieved if the *phenomenological* coefficients α_i are asymptotically *independent* of R (modulo logarithmic corrections). In this case, the higher-spin operators V_0^i $(i \geqslant 2)$ measure the deviation from the massless, critical spectrum $\mathscr{E}_k = vk/R = vq$.

3.3. *Two-Body Interactions*

The interaction term H_I of the Hamiltonian (3.14) can be treated similarly. In Fock space, it reads

$$H_I = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=-m}^{n} \tfrac{1}{2}(M(n,m;k) - M(n,m;n-m-k))\, a^\dagger_{m+k} a^\dagger_{n-k} a_n a_m,$$

$$M(n,m;k) \equiv \tfrac{1}{2} \int d^2 z_1 \int d^2 z_2\, \psi^*_{n-k}(z_1,\bar{z}_1)\, \psi_n(z_1,\bar{z}_1)\, V_I(|z_1 - z_2|) \qquad (3.27)$$

$$\cdot\, \psi^*_{m+k}(z_2,\bar{z}_2)\, \psi_m(z_2,\bar{z}_2),$$

where we have explicitly anti-symmetrized the matrix element. We study again H_I in the large N limit for transitions corresponding to edge excitations, which involve indices $n = N + r$ and $m = N + s$ with $|r|$, $|s|$, $|k| \sim O(1)$. In these cases, the "transition density" describing each virtual electron in (3.27) can be approximated as follows:

$$\begin{aligned} \rho_{s,k}(z,\bar{z}) &\equiv \psi^*_{N+s+k}(z,\bar{z})\, \psi_{N+s}(z,\bar{z}) \\ &\simeq \frac{1}{\pi\sqrt{2\pi N}}\, e^{ik\theta} \exp\left(-\frac{(|z|^2 - (N+s+k/2))^2}{2N}\right), \end{aligned} \qquad (3.28)$$

where $z = |z|\, e^{i\theta}$. The phase of this expression is the chiral edge wave of momentum k. The modulus has the shape of a Gaussian, with center located at $|z| \simeq R + (2s+k)/4R$, and spread $\Delta\, |z| = O(1)$. This shows that all virtual electrons participating in edge transitions are localized on a boundary annulus of the size of the ultra-violet cut-off. We can develop some physical intuition on the form of the matrix element by thinking of the interaction between the densities of the two electrons through their phases (index k) and modulus (radial) fluctuations (indices r, s).

Consider in the first place the (three-dimensional) Coulomb potential[7]

$$V(|z_1 - z_2|) = \frac{e^2}{|z_1 - z_2|}. \qquad (3.29)$$

The two integrals in (3.27) are effectively restricted to the annular region. One can distinguish two contributions: when $|z_1 - z_2| > 1$, i.e., the relative angle is $|\theta| = |\theta_1 - \theta_2| > 1/R$, and its complement. In the first case, the two densities (3.28) are far apart, so that their radial fluctuations of $O(1/R)$ can be neglected. This allows the further approximation,

$$\rho_{s,k}(z,\bar{z}) \simeq \frac{1}{2\pi R}\, e^{ik\theta} \Theta((|z| - R)(R + 1 - |z|)). \qquad (3.30)$$

[7] We shall take the dielectric constant equal to one.

Therefore, the matrix element has only one non-trivial integration over the relative angle:

$$M(k) \equiv M(N+r, N+s; k) \simeq \frac{e^2}{2R} \int_{1/R}^{\pi} \frac{d\theta}{2\pi} \frac{\cos(k\theta)}{|\sin\theta/2|}. \tag{3.31}$$

In the other region, $|z_1 - z_2| < 1$, the integrals are fully two-dimensional. Nevertheless, the ultra-violet behaviour of (3.29) is integrable and the result is subleading with respect to (3.31). The integral in (3.31) can be computed by a recursion relation in k, and reads[8]

$$M(k) = M(|k|) \simeq M(0) - \frac{e^2}{\pi R} \sum_{l=1}^{k} \frac{1}{2l-1}. \tag{3.32}$$

The Coulomb matrix element $M(k)$ exhibits a logarithmic behaviour for $k \gg 1$,

$$M(k) = M(0) - \frac{e^2}{2\pi R} \log k, \qquad k \gg 1, \tag{3.33}$$

which is accessible within the validity of the edge wave approximation $k \ll R$ [2]. This behaviour is due to the infrared singularity of the electrostatic energy of charges in the annular region. Note that this singularity cannot be screened by the ion background, which ensures charge neutrality over distances larger than R.

For interactions that are not too long-ranged, both phase and radial fluctuations of $\rho_{s,k}(z, \bar{z})$ could be relevant in (3.27), and give an involved dependence of the matrix element on (r, s, k). We can establish a general result in the opposite regime of (very) *short-range* interactions, whose range is of the order of the ultra-violet cut-off. A convenient way to expand these potentials is given by the Gaussian basis of Ref. [33]

$$V_I(|z_1 - z_2|) = \int_0^{\infty} dt\, f(t)\, e^{-t\,|z_1 - z_2|^2}. \tag{3.34}$$

For short-range interactions, $f(t)$ has a compact support over values of $t \simeq O(1)$. The matrix element of the Gaussian interaction,

$$\begin{aligned} M_t(N+r, N+s; k) \equiv \tfrac{1}{2} \int d^2 z_1 \int d^2 z_2\, & \psi^*_{N+r-k}(z_1, \bar{z}_1)\, \psi_{N+r}(z_1, \bar{z}_1)\, e^{-t\,|z_1 - z_2|^2} \\ & \cdot \psi^*_{N+s+k}(z_2, \bar{z}_2)\, \psi_{N+s}(z_2, \bar{z}_2), \end{aligned} \tag{3.35}$$

[8] We have also obtained this result by more accurate two-dimensional integrations.

can be evaluated for $N \to \infty$ using the saddle-point technique, which is equivalent to the approximation in (3.28) (for details see Appendix B). The resulting expression is

$$M_t(N+r, N+s; k) \simeq \frac{1}{\sqrt{N}} \frac{1}{4\sqrt{\pi}\sqrt{t(1+t)}} \left\{ 1 + \frac{1}{4N} \left[-\frac{1}{t}\left(k^2 - \frac{1}{4}\right) \right.\right.$$

$$+ 2\left(k(r-s-k) - r^2 - s^2 - \frac{r+s}{2} - \frac{5}{24}\right)$$

$$\left.\left. + \frac{1}{1+t}\left(k^2 - 2k(r-s) + 2(r^2+s^2) + t(r+s)^2\right)\right]\right\}. \quad (3.36)$$

One can verify that this expression is symmetric under the interchange of the two particles, $(r, s, k) \leftrightarrow (s, r, -k)$. Moreover, the anti-symmetric matrix element in (3.27) becomes

$$\frac{1}{2}\left(M_t(k) - M_t(r-s-k)\right)$$

$$= -\frac{1}{R^3 32\sqrt{\pi}} \frac{(1+2t)}{(t(1+t))^{3/2}} \left[k^2 - (r-s-k)^2\right]. \quad (3.37)$$

We can now insert this result back into the Hamiltonian (3.27) and exploit again the antisymmetric in the indices k and $(r-s-k)$, to obtain a matrix element which depends on k only, as in the case of the Coulomb interaction. Actually, Eq. (3.37) has the unique index structure, at most quadratic in (r, s, k), which has the above symmetries.

The next step is to rewrite the Hamiltonian in terms of $W_{1+\infty}$ generators, as done for H_{CP} (see Eqs. (3.23)–(3.25)). For the Coulomb and short-range interactions, Eq. (3.27) takes the form

$$H_I = \sum_{r=-N}^{\infty} \sum_{s=-N}^{\infty} \sum_{k=-N-s}^{N+r} \frac{\lambda_k}{2} b^\dagger_{s+k} b^\dagger_{r-k} b_r b_s, \quad (3.38)$$

where the b's are the Fock operators with shifted index (3.22), and $\lambda_k/2$ is the matrix element in the limit of edge excitations, (3.32) and (3.37). Due to the explicit antisymmetrization of (3.27), however, these matrix elements are determined up to a k-independent additive term, which vanishes in any expectation value. This freedom will be useful later.

We manipulate Eq. (3.38) by letting $N \to \infty$ in the summation extrema and by replacing the four Fermi operators with a pair of V^0_k operators (3.23). Moreover, we introduce the normal ordering $::$ with respect to the incompressible ground state (3.19). This subtracts a one-body operator (assigned to H_{CP}) and a constant from (3.38). The first subtraction gives,

$$H_I = \sum_{k=-\infty}^{\infty} \frac{\lambda_k}{2} :V^0_{-k} V^0_k:. \tag{3.39}$$

The second subtraction requires some knowledge of $W_{1+\infty}$ representations. Let us recall from Section 2.2, Eq. (2.15), that the ground state is the highest weight state of a $W_{1+\infty}$ representation, which satisfies the conditions $V^i_k |\Omega\rangle = 0$, for $k \geqslant 0$ and $i \geqslant 0$. Therefore, the normal ordered expression is found to be[9]

$$H_I = \frac{\lambda_0}{2} (V^0_0)^2 + \sum_{k=1}^{\infty} \lambda_k V^0_{-k} V^0_k. \tag{3.40}$$

An important property of the microscopic Hamiltonian (3.27) is the translation invariance of two-body interactions. This is enforced on the edge approximation (3.40) as a further normal ordering condition, which fixes the up to now free additive constant in λ_k. As pointed out in Section 3.2, the edge excitations with $k = \pm 1$ correspond to infinitesimal overall translations of the droplet. At the quantum level, they are generated by $V^0_{\mp 1}$. Given that the translation of any state should cost no energy, we require

$$[H_I, V^0_{-1}] = 0 \rightarrow \lambda_1 = 0, \qquad \text{(translation invariance)}. \tag{3.41}$$

In conclusion, the Hamiltonian for edge excitations with the Coulomb and short-range interactions is given by (3.40), with

$$\begin{aligned} \lambda_k &= -\frac{2e^2}{\pi R} \sum_{l=2}^{k} \frac{1}{2l-1} \simeq -\frac{e^2}{\pi R} \log k, \quad \left(\lambda_0 = \frac{2e^2}{\pi R}\right), \qquad \text{(Coulomb interaction)}, \\ \lambda_k &= -\frac{k^2-1}{8\sqrt{\pi}\, R^3} \int_0^\infty dt\, f(t)\, \frac{(1+2t)}{(t(1+t))^{3/2}}, \qquad \text{(short-range interactions)}. \end{aligned} \tag{3.42}$$

Let us make a few remarks on the result (3.40) for the edge Hamiltonian. This form for H_I has been assumed in phenomenological and semiclassical descriptions of the edge excitations [21, 16, 24]: a one-dimensional *chiral boson* $\varphi(R\theta - vt)$, which expresses the fluctuations of the density at the edge by $\rho(\theta) = -(1/2\pi R)\, \partial_\theta \varphi(\theta)$ [2], is introduced. Moreover, its Hamiltonian is taken to be

$$H_I^{(\mathrm{boson})} = \tfrac{1}{2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2\, \rho(\theta_1)\, V_I^{(\mathrm{eff})}(|\theta_1 - \theta_2|)\, \rho(\theta_2). \tag{3.43}$$

[9] Note that finite-size effects are absent in this formula, in particular $(V_R)^0_k = V^0_k$ (see Appendix A).

Given that the chiral boson carries a representation of the Kac–Moody algebra (2.13), we can express $\rho(\theta)=\sum_k \exp(ik\theta)\, V_k^0$ and recover the operator form in (3.40). In this description, as well as in the semiclassical limit, the radial degree of freedom is completely neglected, thus the matrix element can only depend on the Fourier mode k on the edge. This *bosonization* of the edge waves in the quantum Hall effect was predicted by Stone [21] and later verified numerically [22].

Our derivation can be regarded as the first *proof* of the bosonization, for both the Coulomb and generic short-range interactions. Moreover, the $W_{1+\infty}$ dynamical symmetry encodes the physical picture and the subleading corrections. The bosonization formula (3.40) and the operators V_k^0 describe the leading semiclassical effect of phase fluctuations at the edge. Corrections can have the following structure: Eq. (3.18) shows that the V_0^i, $i>0$, measure the radial fluctuations of the density at the edge (3.28), which appear as subleading corrections in the $1/R$ expansion of the matrix elements. These corrections can be expressed in terms of the V_k^i, $i>0$, via their Fock expressions (3.23). For example, the first subleading correction to (3.40) has the form

$$\begin{aligned} H_I^{(\mathrm{sub})} &= \sum_{k>0} \eta_k (V_{-k}^0 V_k^1 + V_{-k}^1 V_k^0) \\ &= \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2\, \rho(\theta_1)\, V_I^{(\mathrm{sub})}(|\theta_1-\theta_2|)\, {:}\rho(\theta_2)^2{:}. \end{aligned} \tag{3.44}$$

The bosonization of edge waves in the presence of the Coulomb interaction is rather easy to understand, because radial fluctuations in (3.28), of size $O(1/R)$, are subleading for long-range $O(R)$ interactions. The bosonization of short-range interactions is, instead, due to the semiclassical nature of the incompressible fluid. As we show in the next section, their spectrum actually reproduces the classical capillary frequencies ω_k of Section 3.1. Although *a priori* radial excitations could be important at short distances, they are quenched by incompressibility, which is enforced by Fermi statistics (see Eq. (3.37)).

In conclusion, the bosonization of edge excitations is a semiclassical effect. Here we provided both a physical picture and a technique to derive it from the microscopic quantum theory. Moreover, we indicated the generic form and the origin of subleading corrections to the bosonization formula.

The $W_{1+\infty}$ description of bosonization of low-energy excitations is a rather general and can apply to other fermionic system. In particular, the Calogero–Sutherland model [37] of $(1+1)$-dimensional fermions with long-range interactions has been analysed in Ref. [38], and the Hamiltonian has been similarly expanded in terms of $W_{1+\infty}$ operators in the large N limit.

4. The Spectrum of Edge Excitations

Up to now we have derived the generic form of the Hamiltonian $H=H_{CP}+H_I$ governing the dynamics of quantum edge excitations for $\nu=1$, and we have shown that it can be written entirely in terms of $W_{1+\infty}$ generators. The piece H_{CP}, describing the effect of the confining potential, is a *linear* combination of the generators V_0^i of the Cartan subalgebra, see Eq. (3.26). The piece H_I, describing the two-body interactions, is quadratic, see Eq. (3.40).

The diagonalization of H follows in principle from the algebraic properties of the operators V_k^i only, owing to the dynamical $W_{1+\infty}$ symmetry. In practice, however, the diagonalization is rendered difficult by the fact that

$$[V_0^i, H_I] \neq 0, \qquad i \geqslant 2. \tag{4.1}$$

This implies that there is no general basis of quantum edge excitations, which diagonalizes simultaneously all terms of the confining potential and the interaction. Actually, each contribution to the Hamiltonian is diagonal in a known basis: the operators (V_0^0, V_0^1, H_I) are diagonal in the standard *bosonic* basis (2.15); the Cartan subalgebra $(V_0^i, i \geqslant 0)$, is diagonal in the *fermionic* basis to be discussed later. Therefore, we shall analyse each interaction separately, choosing the most appropriate basis according to the dominant operators in the $(1/R)$ expansion.

4.1. *Coulomb Interaction*

The Coulomb interaction $V(\mathbf{x}) = e^2/|\mathbf{x}|$ gives rise to an interacting Hamiltonian of order $O(1/R)$, Eqs. (3.40), (3.42). In this case, the generators V_0^i, $i \geqslant 2$, in the expansion of H_{CP} lead to subdominant contributions to the spectrum for large R and can be dropped. The only term which must be retained is the leading $1/R$ contribution, which is proportional to V_0^1. We thus consider the Hamiltonian:

$$H^{(\mathrm{Coulomb})} = \alpha_0 V_0^0 + \frac{\alpha_1}{R}\left(V_0^1 - \frac{c}{24}\right) + \frac{\lambda_0}{2}(V_0^0)^2 + \sum_{k=1}^{\infty} \lambda_k V_{-k}^0 V_k^0 + O\left(\frac{1}{R^2}\right), \tag{4.2}$$

where the conformal central charge for $\nu = 1$ is $c = 1$, and $\lambda_k \simeq -(e^2/\pi R)\log k$, $k \gg 1$. A contribution to the confining potential is given by the Coulomb interaction of the electrons with the neutralizing ion background, which has also a droplet shape. This determines [25] the phenomenological quantity $\alpha_1 = (e^2/\pi)\log \gamma R$, where γ is a non-universal parameter which depends on the details of the droplet edge.[10] Other contributions to the confining potential can be included in γ.

Let us now analyse the spectrum of (4.2) for edge excitations above the ground state $|\Omega\rangle$. Since $V_0^0 = 0$ for all these excitations, it drops out of Eq. (4.2). The Hamiltonian is diagonal in the bosonic basis (2.15), because the Kac-Moody

[10] Note that the term proportional to V_0^1 in (4.2) does have the correct $1/R$ dependence anticipated in (3.26), but also a logarithmic correction, due to the Coulomb infrared singularity discussed in the previous section.

generators satisfy the ladder relations $[H, V^0_{-k}] \propto V^0_{-k}$ (see Eqs. (2.13), (2.14)). The c-number term in (4.2),

$$E_0 = -\frac{e^2}{24\pi R} \log \gamma R, \tag{4.3}$$

represents the finite-size Casimir energy of the ground state [2].

The fundamental excitations in the bosonic basis are the one-boson states $V^0_{-k} |\Omega\rangle$, $k > 0$, whose energy is given by

$$\mathscr{E}_k \equiv E_k - E_0 = \frac{e^2}{\pi} \frac{k}{R} \log\left(\frac{\gamma R}{k}\right), \qquad 1 \ll k \ll R. \tag{4.4}$$

This spectrum has been previously found by using various approximations: phenomenological classical hydrodynamics [23, 25], edge bosonization [24] and a Feynman-like ansatz [25]. Here, we gave a complete proof of this result, free of unnecessary assumptions, in the framework of the $W_{1+\infty}$ dynamical symmetry. In the next section, we discuss how this spectrum has been confirmed experimentally.

A virtue of our approach is the complete knowledge of the Hilbert space of quantum edge excitations. Given that we have found the form of the Hamiltonian to leading order $O(1/R)$, we can find the complete spectrum of all many-body states (2.15). Due to the ladder property of the operators V^0_k with respect to the Hamiltonian (4.2), the energy of the many-boson states is simply additive in the one-boson components,

$$\begin{aligned} H^{(\text{Coulomb})} |k, \{n_1, n_2, ..., n_s\}\rangle &= \mathscr{E}_{\{n_1, ..., n_s\}} |k, \{n_1, n_2, ..., n_s\}\rangle, \\ \mathscr{E}_{\{n_1, ..., n_s\}} &= \sum_{i=1}^{s} \mathscr{E}_{n_i}. \end{aligned} \tag{4.5}$$

This property of the many-body spectrum was first shown by the numerical studies of Ref. [22]. Actually, it is the characteristic property of the one-dimensional bosonization of quantum edge excitations, which we have proved in this paper, and can be considered a success of the conformal field theory approach to the quantum Hall effect. Although the logarithmic deviation from the linear spectrum breaks scale and conformal invariance, the conformal field theory techniques remain still valid. In fact, the Hilbert space carrying the representations of the Virasoro algebra (angular momentum) also diagonalizes the non-conformal Hamiltonian (4.2). Actually, the $W_{1+\infty}$ algebra is richer than the Virasoro algebra, because it includes more operators. These can be used to construct non-conformal but still diagonalizable Hamiltonians. This is one of the major points we would like to emphasize here.

4.2. *Short-Range Interactions*

It is also interesting to study the dynamics of edge excitations in the presence of short-range interactions, which we have parametrized with Gaussians

$$V(\mathbf{x}) = \int_0^\infty dt\, f(t)\, e^{-t\,|\mathbf{x}|^2}, \tag{4.6}$$

and weight functions $f(t)$ of compact support centered around $t=1$.

We can add this interaction to the Hamiltonian (4.2), to describe short-distance deformations of the Coulomb interaction due to microscopic effects, like the thickness of the layer. Using the matrix elements of the short-range interaction in Eq. (3.42), we obtain the following contribution to the energy of the one-boson states (4.4),

$$\Delta\mathscr{E}_k = -\frac{\pi\alpha}{2R^3}\, k(k^2-1), \qquad \alpha = \frac{1}{4\pi^{3/2}} \int_0^\infty dt\, f(t)\, \frac{1+2t}{(t(1+t))^{3/2}}. \tag{4.7}$$

This spectrum coincides with the classical capillary frequency (3.13), for $\rho_0 = eB\nu/(2\pi) = \nu/(\ell^2\pi)$, and $\nu=1$, in units $\ell=1$. It identifies the quantum expression for the edge tension α as a "moment" of the interaction. This result shows that the quantum incompressible fluid behaves semiclassically in the presence of short-range interactions, as anticipated in the previous section.

Note that the result (4.7) includes the Haldane short-range interaction [39], $H_I(|\mathbf{x}_1-\mathbf{x}_2|) = a\,\Delta\delta^{(2)}(\mathbf{x}_1-\mathbf{x}_2)$, which has often been used in numerical and approximate analyses to stabilize the $\nu=1/3$ Laughlin incompressible fluid. In the present case, instead, it can be considered as a residual interaction for edge excitations of the stable $\nu=1$ ground state. The derivative of the delta function can be regularized by a Gaussian with weight function

$$f(t) = \lim_{t_0\to\infty} a(8t^2/\pi)(2+t\partial_t)\,\delta(t-t_0),$$

leading to the spectrum

$$\mathscr{E}_k = -\frac{a}{\pi\sqrt{\pi}\,R^3}\, k(k^2-1), \tag{4.8}$$

in agreement with previous results [22, 25].

Another important property of Eq. (4.7) is that the short-range contribution to the spectrum (4.4) is of $O(1/R^3)$ and thus negligible for large R. This proofs the *universality* of the dynamics of edge excitations: in the thermodynamic limit, short-range interactions become irrelevant for the dynamics of edge excitations, which is only governed by the long-distance physics. The spectrum (4.4) yields further universal quantities, which add to the kinematical data of Section 2; these are combinations of finite-size energies which are parameter-free (see also Ref. [2]).

Finally, we would like to discuss the excitation spectrum in the case of short-range interactions only. Since the experiments [26] confirm the Coulomb spectrum

(4.4), this is a rather academic issue. However, it allows us to clarify some aspects of the $W_{1+\infty}$ dynamics associated with the higher spin operator V_0^2, which measures the leading radial fluctuations of edge excitations. In this case, we can consider the following Hamiltonian

$$H^{(sr)} = \alpha_0 V_0^0 + \frac{\alpha_1}{R}\left(V_0^1 - \frac{1}{24}\right) + \frac{\alpha_2}{R^2}\left(V_0^2 - \frac{1}{12} V_0^0\right) + O\left(\frac{1}{R^3}\right). \tag{4.9}$$

The term proportional to V_0^2 produces subleading effects which break the leading scale-invariant spectrum of conformal field theory studied in [2]. The appropriate basis of edge excitations which diagonalizes V_0^2 (as well as all higher spin $W_{1+\infty}$ operators) is obtained from the fermionic Fock operators,

$$b_{n_1}^\dagger b_{m_1} \cdots b_{n_s}^\dagger b_{m_s} |\Omega\rangle, \quad n_1 > \cdots > n_s > 0 \geqslant m_1 > \cdots > m_s, \quad \sum_i (n_i - m_i) = k. \tag{4.10}$$

These are multiple particle-hole excitations with total angular momentum k. The operators V_0^i are diagonal in this basis because the pair $(b_n^\dagger b_m)$ is a ladder operator for them (see Eqs. (2.11), (3.23)). The spectrum of one particle-hole excitation $b_n^\dagger b_m |\Omega\rangle$ follows from (3.23):

$$\mathscr{E}_{nm} = E_{nm} - E_0 = \frac{\alpha_1}{R}(n-m) + \frac{\alpha_2}{R^2}(n-m)(n+m-1). \tag{4.11}$$

Since the indices (n, m) can be traced back to the ones of w_∞ classical deformations $\delta_{nm}\rho$ in (2.9), we find that quantum radial fluctuations indeed break the classical degeneracy at fixed $k = (n-m)$. Note that this degeneracy is still present in the leading conformal spectrum. For a given value of k, the non-degenerate eigenvalues are distributed in the interval

$$\frac{\alpha_1}{R} k - \frac{\alpha_2}{R^2} k(k-1) \leqslant \mathscr{E}_{nm} \leqslant \frac{\alpha_1}{R} k + \frac{\alpha_2}{R^2} k(k-1). \tag{4.12}$$

Note, however, that the breaking of conformal invariance generated by V_0^2 is different from the one induced by the Coulomb interaction, because the fermionic and bosonic bases are different.

4.3. *Charged Excitations*

We can easily extend the previous discussion to compute the spectrum of charged edge excitations. Let us consider states with q electrons ($q \in \mathbb{Z}$) added to the boundary and, correspondingly, q vortices in the interior of the droplet. These are the highest-weight states $|Q\rangle$ for other $W_{1+\infty}$ representations, with weights given by (2.16). In particular, the weights of V_0^0 and V_0^1 are given by $Q = q$ and $J = q^2/2$, respectively, and determine the normal-ordered energy of these states by Eq. (4.2),

$$H|Q=q\rangle = E_{0,q}|Q\rangle, \qquad E_{0,q} = \alpha_0 q + \frac{e^2}{\pi R}\left(\frac{q^2}{2} - \frac{1}{24}\right)\log \gamma R + \frac{e^2}{\pi R} q^2. \tag{4.13}$$

Clearly, this is only a contribution to the energy gap of these states. The main contribution comes from the physics at the core of the vortices in the deep interior of the droplet, which cannot be computed in the boundary theory [8]. However, Eq. (4.13) gives the universal part of this gap [2].

The edge excitations on top of both the charged and neutral states have the same structure. The energy levels for $Q \neq 0$ are only shifted by a Q-dependent constant. In particular, the Coulomb interaction yields the spectrum (4.4) for the energy differences $(E_{k,q} - E_{0,q})$.

4.4. *Filling* $v = 1/(2p+1)$

We can use the $W_{1+\infty}$ dynamical symmetry to generalize the previous results for the spectrum of edge excitations to the Laughlin fluids at fractional filling $v = 1/3$, $1/5, 1/7, \ldots$. As discussed in Section 2.2, these excitations are given by other representations of the $W_{1+\infty}$ algebra with central charge $c = 1$ [2, 4, 40]. In Ref. [2], we have shown that these representations have one free parameter, which is fixed by the value of the Hall conductivity. The operators V_k^0, normalized to measure the physical charge, satisfy the Kac-Moody algebra,

$$[V_n^0, V_m^0] = \frac{n}{2p+1} \delta_{n+m,0}, \qquad \left(v = \frac{1}{2p+1}, p = 1, 2, \ldots \right), \tag{4.14}$$

and the spectrum of anyon highest weights is given by $Q = q/(2p+1)$, $J = q^2/(4p+2)$, with $q \in \mathbb{Z}$ [2, 4].

Lacking a microscopical derivation of the Coulomb Hamiltonian for fractional filling, we shall use an argument based on the $W_{1+\infty}$ dynamical symmetry. Namely, we shall argue that the Hamiltonian for edge excitations must be *form invariant*, i.e., must take the same form[11] (4.2) in terms of the $W_{1+\infty}$ operators for any value of $v = 1/(2p+1)$. Indeed, the structure of edge excitations, i.e., of the $W_{1+\infty}$ highest-weight representations, is identical for all these values of v. Therefore, we can repeat the analysis after (4.2), and use the commutators (4.14) and (2.14) to compute the corresponding spectrum of one-boson neutral edge excitations,

$$\mathscr{E}_k = E_k - E_0 = \frac{e^2 v}{\pi} \frac{k}{R} \log\left(\frac{\eta R}{k}\right), \qquad 1 \ll k \ll R, \quad \left(v = \frac{1}{2p+1} \right), \tag{4.15}$$

where the parameter η may depend on v. This result is again semiclassical [24, 25].

Finally, we would like to mention the properties of the edge excitations for more general, hierarchical fractional fillings $v = m/(2mp \pm 1)$, where m, $p = 1, 2, \ldots$ [13], which were shown to be described by the $W_{1+\infty}$ minimal models [5]. The fractional charge and statistics of the excitations and the other kinematical data are

[11] Note that this corresponds to a complicated microscopic expression, which incorporates the idea of "attaching flux tubes to electrons" [13]; see also [40].

obtained from the *degenerate* representations of the $W_{1+\infty}$ algebra with central charge $c=m$ [19], whose weights belong to the weight lattice of the $U(m)$ Lie algebra [9]. Therefore, there still exists one elementary fractionally charged excitation and, in addition, there are $(m-1)$ *neutral* excitations which carry a non-Abelian quantum number. This implies that the charge does not characterize completely the $c>1$ edge excitations.

The integer p in ν parametrizes the unit of charge as in the $m=1$ case, while the $(\pm)$ sign corresponds to the same (respectively, opposite) chirality for the charged and neutral excitations. In Ref. [5], we described how to modify the $W_{1+\infty}$ algebra to incorporate excitations of opposite chirality (basically, the sign redefinition $V^i_n \to (-1)^i\, V^i_{-n}$). Opposite chiralities correspond to opposite signs of the slope of the confining potential $V'_{CP}(R^2)$ in (3.14), which is felt by the two components of the edge excitations; the modification of the $W_{1+\infty}$ algebra ensures that the V^1_0 term in the Hamiltonian (3.26) remains positive definite. Unfortunately, we do not yet know the form of the interaction Hamiltonian H_I (3.38) for general hierarchical fillings, and we cannot discuss here the spectrum of their edge excitations.

5. Conclusions

In this paper, we further developed the $W_{1+\infty}$ effective field theory for the plateaus of the quantum Hall effect.

We first reviewed how the $W_{1+\infty}$ representations describe the kinematical data of the Laughlin incompressible quantum fluids, like the fractional charges and statistics of anyon excitations. Secondly, we expressed the general Hamiltonian for edge excitations in terms of the $W_{1+\infty}$ generators for both cases of Coulomb and short-range interactions, and obtained its spectrum. We proved the one-dimensional bosonization of edge excitations by showing that the leading term in the $1/R$ expansion of the Hamiltonian contains the Kac–Moody and Virasoro operators only. Moreover, for large R, the spectrum is independent of short-range effects and thus possesses universal features.

Furthermore, we showed that the infinite tower of higher spin operators V^i_k, $i\geqslant 2$, parametrizes the subleading corrections to the bosonization, which arise from radial fluctuations of the edge excitations. This is the physical picture for the "$(2+1)$-dimensional nature" of the $W_{1+\infty}$ algebra recently remarked in Ref. [18].

Finally, we briefly discuss the experiment [26] of radio-frequency resonance, which confirmed the finite size spectra (4.4) and (4.15), for the case of the Coulomb interaction. Low dissipation excitations (called edge magnetoplasmons there) were observed as sharp resonance peaks in the transmitted signal, at the plateaus for $\nu=1$, 2 and 2/3. The observed resonance frequencies $\omega_i=\mathscr{E}_i/\hbar$ match the formulae (4.4) and (4.15), and verify both their logarithmic functional form and their ν dependence (see Table I and Eq. (7) of Ref. [26]). Let us also quote the recent experiment with very precise time resolution [27], which clearly establishes the one-dimensional and chiral nature of the edge excitations.

APPENDIX A: $W_{1+\infty}$ ALGEBRA AND WEYL FERMION REPRESENTATION

In this Appendix, we derive explicit expressions for the $W_{1+\infty}$ algebra, the form of the generators $(V_R)^i_n$ defined on the cylinder geometry, and the realization of the $W_{1+\infty}$ algebra in terms of a Weyl fermion.

$W_{1+\infty}$ *Algebra*

The $W_{1+\infty}$ algebra can be written in a compact first-quantized form [17],

$$[V(z^r f(D)), V(z^s g(D))] = V(z^{r+s} f(D+s)\, g(D)) - V(z^{r+s} f(D)\, g(D+r)) + c\Psi(z^r f(D), z^s g(D)), \tag{A.1}$$

where the central extension is

$$\Psi(z^r f(D), z^s g(D)) \equiv \delta_{r+s,0} \sum_{j=1}^{r} f(-j)\, g(r-j) \equiv -\Psi(z^s g(D), z^r f(D)), \qquad r > 0. \tag{A.2}$$

In these expressions, the $W_{1+\infty}$ operators $V(z^r f(D))$ are specified by the corresponding first-quantized differential expression $z^r f(D)$, where $f(D)$ is an entire function of $D \equiv z(\partial/\partial z)$. In the text, we introduced the operators V^i_k, characterized by mode index $k \in \mathbb{Z}$ and conformal spin $h = i + 1 \geqslant 1$: the two definitions are related by

$$V^i_k \equiv V(z^k f^i_k(D)), \tag{A.3}$$

where $f^i_k(D)$ are specific i-th order polynomials. The basis for these polynomials is chosen by requiring that the central extension (A.2) is diagonal in the spin indices. This can be done uniquely, because Ψ has the property of a symplectic metric. The generators of lower spin are thus found to be:

$$\begin{aligned}
V^0_n &= V(-z^n),\\
V^1_n &= V\left(-z^n\left(D + \frac{1}{2}(n+1)\right)\right),\\
V^2_n &= V\left(-z^n\left(D^2 + (n+1)\, D + \frac{1}{6}(n+1)(n+2)\right)\right),\\
V^3_n &= V\left(-z^n\left(D^3 + \frac{3}{2}(n+1)\, D^2 + \frac{6n^2+15n+11}{10} D + \frac{(n+3)(n+2)(n+1)}{20}\right)\right),
\end{aligned} \tag{A.4}$$

where V_k^0 and V_k^1 are, by definition, the Kac-Moody and Virasoro generators, respectively [10]. Some of the algebraic relations are:

$$
\begin{aligned}
&[V_n^0, V_m^0] = nc\delta_{n+m,0},\\
&[V_n^1, V_m^0] = -mV^0_{n+m},\\
&[V_n^1, V_m^1] = (n-m)\, V^1_{n+m} + \frac{c}{12}(n^3-n)\,\delta_{n+m,0},\\
&[V_n^2, V_m^0] = -2mV^1_{n+m},\\
&[V_n^2, V_m^1] = (n-2m)\, V^2_{n+m} - \frac{1}{6}(m^3-m)\, V^0_{n+m},\\
&[V_n^2, V_m^2] = (2n-2m)\, V^3_{n+m} + \frac{n-m}{15}(2n^2+2m^2-nm-8)\, V^1_{n+m}\\
&\qquad + c\,\frac{n(n^2-1)(n^2-4)}{180}\,\delta_{n+m,0}.
\end{aligned}
\tag{A.5}
$$

Any highest weight representation is formed by all the states obtained by applying $W_{1+\infty}$ generators with negative moding to the highest weight state $|Q\rangle$, which satisfies,

$$V_n^i\,|Q\rangle = 0, \qquad \forall n>0, \quad i\geqslant 0, \tag{A.6}$$

and is the eigenstate of V_0^i,

$$V_0^i\,|Q\rangle = m_i(Q)\,|Q\rangle. \tag{A.7}$$

We shall only discuss the unitary, irreducible representations for $c=1$, which describe the Laughlin incompressible fluids with $\nu = 1/(2p+1)$, $p=0, 1, 2, ...$ [2]. For these representations, the $m_i(Q)$ are $(i+1)$-th polynomials in Q, which are given by the generating function,

$$V(-e^{xD})\,|Q\rangle = \frac{e^{xQ}-1}{e^x-1}\,|Q\rangle. \tag{A.8}$$

For example, by expanding in powers of x, and using (A.4), one finds $m_0(Q)=Q$ and $m_1(Q)=J=Q^2/2$. The eigenvalues of $V_0^0/(2p+1)$ and V_0^1 measure the physical charge and the angular momentum of the states, respectively [2, 4]. For $c=1$, all the states in a unitary irreducible representation can obtained by using the Kac–Moody generators only [17]. These states have the form

$$|k, \{n_1, n_2, ..., n_s\}\rangle = V^0_{-n_1} V^0_{-n_2} \cdots V^0_{-n_s}\,|Q\rangle, \qquad n_1 \geqslant n_2 \geqslant \cdots \geqslant n_s > 0, \tag{A.9}$$

where $k = Q^2/2 + \sum_{i=1}^{s} n_i$ is their angular momentum.

$W_{1+\infty}$ Generators on the Cylinder

In the quantum Hall effect, fields and operators are naturally defined on the boundary circle ($0 \leqslant \theta < 2\pi$), i.e., on a compact space. In the mathematical literature, however, they are conventionally considered in an unbounded space. There is a conformal mapping between these two spaces, which are the *cylinder* ($u = \tau - iR\theta$) and the *conformal plane* (z), after inclusion of the euclidean time τ,

$$z = \exp\left(\frac{u}{R}\right) = \exp\left(\frac{\tau}{R} - i\theta\right). \tag{A.10}$$

For example, the operators (A.4) define the $W_{1+\infty}$ currents $V^i(z)$ on the conformal plane as follows,

$$V^i(z) \equiv \sum_n V^i_n z^{-n-i-1}. \tag{A.11}$$

On the other hand, the Hamiltonian of edge excitations should be expressed in terms of the $W_{1+\infty}$ generators $(V_R)^i_n$ on the cylinder. The latter can be obtained by transforming the $V^i(z)$ under the conformal mapping (A.10), as follows. The generator G of infinitesimal conformal transformations $z = u + \varepsilon(u)$, with $\varepsilon(u) = \sum_n \varepsilon_n u^{n+1}$, is parametrized by the Virasoro operators,

$$G = \sum_n \varepsilon_n V^1_n. \tag{A.12}$$

Therefore, the infinitesimal transformation of the $W_{1+\infty}$ currents, $V^i(u) = V^i(z) + \delta_\varepsilon V^i(z)$, are computed from the algebra (A.5),

$$\begin{aligned}
\delta_\varepsilon V^0 &\equiv [G, V^0] = \varepsilon\, \partial V^0 + (\partial\varepsilon)\, V^0, \\
\delta_\varepsilon V^1 &\equiv [G, V^1] = \varepsilon\, \partial V^1 + (\partial\varepsilon)\, V^1 + \frac{c}{12}\, \partial^3 \varepsilon, \\
\delta_\varepsilon V^2 &\equiv [G, V^2] = \varepsilon\, \partial V^2 + (\partial\varepsilon)\, V^2 + \frac{1}{6}\, (\partial^3\varepsilon)\, V^0.
\end{aligned} \tag{A.13}$$

Note that these transformations contain the differential operator $[\varepsilon\partial + h(\partial\varepsilon)]$ applied to the current, plus additional terms. If the latter are absent, as in the case of $V^0(z)$, the infinitesimal transformation can be integrated to yield the finite form

$$\Phi(u) = \left(\frac{dz}{du}\right)^h \Phi(z(u)). \tag{A.14}$$

Those fields $\Phi(z)$ which transform homogeneously are called *primary* [10]. The current V^i, $i>0$, have additional terms which, nevertheless, vanish for $\varepsilon=\varepsilon_{-1}+\varepsilon_0 z+\varepsilon_1 z^2$, i.e., for global conformal transformations generated by (V^1_{-1}, V^1_0, V^1_1). There is only one function $S(z(u), u)$ of the conformal mapping, which vanishes under the finite global transformations: the Schwartzian derivative [10]

$$S(f(z), z) \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2, \tag{A.15}$$

where the prime denotes differentiation. Owing to this property, all infinitesimal transformations in (A.13) can be integrated, and yield

$$\begin{aligned} V^0(u) &= \frac{dz}{du}\, V^0(z), \\ V^1(u) &= \left(\frac{dz}{du}\right)^2 V^1(z) + \frac{c}{12}\, S(z, u), \\ V^2(u) &= \left(\frac{dz}{du}\right)^3 V^2(z) + \frac{1}{6}\frac{dz}{du}\, S(z, u)\, V^0(z). \end{aligned} \tag{A.16}$$

The $W_{1+\infty}$ currents on the cylinder are thus found by using the mapping (A.10),

$$\begin{aligned} V^0_R(u) &= \frac{z}{R}\, V^0(z), \\ V^1_R(u) &= \frac{1}{R^2}\left(z^2 V^1(z) - \frac{1}{24}\right), \\ V^2_R(u) &= \frac{1}{R^3}\left(z^3 V^2(z) - \frac{z}{12}\, V^0(z)\right). \end{aligned} \tag{A.17}$$

Using the definition

$$(V_R)^j_0 \equiv \int_0^{2\pi i R} \frac{du}{-2\pi i}\, V^j_R(u), \tag{A.18}$$

we obtain the relation between the zero modes in the two geometries

$$(V_R)^0_0 = V^0_0, \quad (V_R)^1_0 = \frac{1}{R}\left(V^1_0 - \frac{c}{24}\right), \quad (V_R)^2_0 = \frac{1}{R^2}\left(V^2_0 - \frac{1}{12}\, V^0_0\right). \tag{A.19}$$

Weyl Fermion Representation of $W_{1+\infty}$

In the thermodynamic limit [2], the field operator of electrons in the lowest Landau level can be evaluated at the boundary of the droplet, and mapped into the $(1+1)$-dimensional field of the chiral, relativistic Weyl fermion [10],

$$F_R(\theta) = \frac{1}{\sqrt{R}} \sum_{k=-\infty}^{\infty} e^{i(k-1/2)\theta} b_k, \tag{A.20}$$

with Neveu-Schwarz boundary conditions on the circle (the b_k are the fermionic Fock operators). The vacuum of this theory corresponds to the ground state of the $\nu = 1$ incompressible fluid $|\Omega\rangle$, which satisfies,

$$b_l\,|\Omega\rangle = 0, \quad l > 0, \qquad b_l^\dagger\,|\Omega\rangle = 0, \quad l \leqslant 0. \tag{A.21}$$

The Hilbert space of a Weyl fermion consists of an infinity of $c = 1$, $W_{1+\infty}$ representations (A.6)–(A.9), all those corresponding to $Q \in \mathbb{Z}$. For example, $|\Omega\rangle$ is the highest weight state (A.6) with $Q = 0$.

The Weyl fermion is a primary conformal field (A.14) with $h = 1/2$, and on the plane z takes the form $(\tau = 0)$,

$$\begin{aligned} F(z) &= \left(\frac{du}{dz}\right)^{1/2} F_R(\theta) = \sum_l e^{il\theta} b_l, \\ \bar{F}(z) &= \left(\frac{du}{dz}\right)^{1/2} F_R^\dagger(\theta) = \sum_l e^{-i(l-1)\theta} b_l^\dagger. \end{aligned} \tag{A.22}$$

The representations of $W_{1+\infty}$ operators on this Hilbert space is obtained by sandwitching the first-quantized expressions (A.3) between the field operators,

$$V_n^i \equiv \oint \frac{dz}{2\pi i} : F(\theta)\, z^n f_n^i(D)\, \bar{F}(\theta) : = \oint \frac{dz}{2\pi i} : \bar{F}(\theta)\, z^n g_n^i(D)\, F(\theta) :, \tag{A.23}$$

where $g_n^i = (-1)^{i+1} f_n^i$, and the integration is carried clockwise over the unit circle. Since the anticommutator of fields is a delta function in Fock space, the V_k^i defined above clearly represent the algebra (A.1). One can verify that they also have the eigenvalues (A.8) when acting on Q fermion states. In Eq. (A.23), they are also written in the canonical form $(\bar{F}F)$ of quantum field theory, which identifies the polynomials g_n^i defined in Section 2.2. Moreover, the normal ordering prescription (: :) w.r.t. the ground state (A.21) is obtained by writing the annihilation operators b_l $(l > 0)$ and $b_l^\dagger$ $(l \leqslant 0)$ to the right-hand side of the creation operators b_l $(l \leqslant 0)$ and $b_l^\dagger$ $(l > 0)$.

From the expressions (A.4), we obtain the Fock space expression of the generators,

$$V_k^0 = \sum_l : b_{l-k}^\dagger b_l :,$$
$$V_k^1 = \sum_l \left(l - \frac{k+1}{2} \right) : b_{l-k}^\dagger b_l :, \tag{A.24}$$
$$V_k^2 = \sum_l \left(l^2 - (k+1)\, l + \frac{(k+1)(k+2)}{6} \right) : b_{l-k}^\dagger b_l :,$$

and the form of the currents (A.11) on the plane,

$$V^0(z) = : \bar{F} F :,$$
$$V^1(z) = \tfrac{1}{2} : \partial_z \bar{F} F : - \tfrac{1}{2} : \bar{F} \partial_z F :, \tag{A.25}$$
$$V^2(z) = - : \partial_z \bar{F} \partial_z F : + \tfrac{1}{6} : \partial_z^2 (\bar{F} F) :.$$

Let us finally remark that the different form (A.17) of the $W_{1+\infty}$ currents on the plane and the cylinder is due to a normal ordering effect [36]. Actually, we can recover (A.17) in the fermionic case ($c = 1$) as follows. We apply the conformal mapping (A.10) to each fermion field in (A.25), paying attention to the different normal ordering in the plane and the cylinder. For example,

$$V_R^0(u) = :\bar{F}_R(u)\, F_R(u): \equiv \lim_{u_1, u_2 \to u} \left(\bar{F}_R(u_1)\, F_R(u_2) - \frac{1}{u_1 - u_2} \right)$$
$$= \frac{dz}{du}\, V^0(z) + \lim_{u_1, u_2 \to u} \left[\left(\frac{dz_1}{du_1} \frac{dz_2}{du_2} \right)^{1/2} \frac{1}{z_1 - z_2} - \frac{1}{u_1 - u_2} \right] = \frac{z}{R}\, V^0(z). \tag{A.26}$$

Proceeding similarly for the other currents, we indeed recover (A.17).

Appendix B: The Gaussian Interaction Two-Body Matrix Element for $N \to \infty$

In this appendix, we evaluate the two-body matrix element for the Gaussian interaction (3.35) in the limit $N \to \infty$. It is given by the expression

$$M(N+r, N+s; k) = \frac{1}{2} \int \frac{d^2 z_1\, d^2 z_2}{\pi^2}\, e^{-|z_1|^2 - |z_2|^2} \frac{\bar{z}_1^{N+s} z_1^{N+s+k}}{\sqrt{(N+s)!\,(N+s+k)!}}\, e^{-t\,|z_1 - z_2|^2}$$
$$\cdot \frac{\bar{z}_2^{N+r} z_2^{N+r-k}}{\sqrt{(N+r)!\,(N+r-k)!}}. \tag{B.1}$$

The idea is to evaluate exactly the angular integrations in (B.1) and use the saddle-point technique to calculate the integrals that involve the modulus of the coordinates, when $N \to \infty$. We first rewrite (B.1) upon replacing $z_1 = xe^{i\theta}$, $z_2 = y$, with $x, y \geqslant 0$:

$$\begin{aligned} M(N+r, N+s; k) &= \frac{2}{C(N, s, r, k)} \int_0^\infty dx\, x^{2N+2s+k+1} \int_0^\infty dy\, y^{2N+2r-k+1} \\ &\quad \cdot e^{-(1+t)(x^2+y^2)} I_k(2txy), \\ C(N, s, r, k) &\equiv \sqrt{(N+s)!\,(N+s+k)!\,(N+r)!\,(N+r-k)!}, \\ I_k(z) &= \frac{1}{2\pi} \int_0^{2\pi} d\theta\, e^{ik\theta + z\cos\theta}. \end{aligned} \tag{B.2}$$

For $N \to \infty$, the most important contribution to the radial integrations in (B.2) arise from values of $x, y \sim O(\sqrt{N})$. Therefore, for values of t that are not too small, one can assume that the argument of the imaginary Bessel function $I_k(z)$ in (B.2) satisfies $z = 2txy \gg 1$. In this regime, one may use the asymptotic form [35]:

$$I_k(z) \simeq \frac{e^z}{\sqrt{2\pi z}} \left(1 - \frac{1}{2z}(k^2 - 1/4) + O\left(\frac{1}{z^2}\right)\right), \qquad z = 2txy \gg 1. \tag{B.3}$$

Replacing (B.3) in (B.2), one obtains

$$\begin{aligned} M(N+r, N+s; k) &= \frac{1}{C(N, s, r, k)} \frac{1}{\sqrt{\pi t}} \int_0^\infty \int_0^\infty dx\, dy\, e^{\Phi(x, y)}, \\ \Phi(x, y) &\equiv -(1+t)(x^2+y^2) + 2txy + (2N+2s+k+1/2)\ln x \\ &\quad + (2N+2r-k+1/2)\ln y. \end{aligned} \tag{B.4}$$

One then finds the two-dimensional saddle-point of (B.4) (x_0, y_0):

$$\begin{aligned} x_0^2 &= N + 1/4 + \frac{2s+k+t(r+s)}{2(1+t)} + O(1/N), \\ y_0^2 &= N + 1/4 + \frac{2r-k+t(r+s)}{2(1+t)} + O(1/N). \end{aligned} \tag{B.5}$$

Therefore, the saddle-point value of (B.4) is given by:

$$\begin{aligned} &M(N+r, N+s; k) \\ &\quad = \frac{1}{C(N, s, r, k)} \frac{1}{\sqrt{\pi t}} \\ &\qquad \cdot \left\{ e^{\Phi(x_0, y_0)} 2\pi \left(\det \left(\frac{\partial^2 \Phi}{\partial x\, \partial y} \right)_0 \right)^{-1/2} - \frac{(k^2 - 1/4)}{4t} \begin{pmatrix} r \to r - 1/2 \\ s \to s - 1/2 \end{pmatrix} \right\}. \end{aligned} \tag{6.6}$$

The value of the Jacobian is given by:

$$\det\left(\frac{\partial^2\Phi}{\partial x\,\partial y}\right)_0 = 16(1+t) + O(1/N^2). \tag{B.7}$$

Replacing now (B.5) and (B.7) into (B.6), and retaining terms up to $O(1/N^{3/2})$, one finds

$$\begin{aligned} M(N+r, N+s; k) = &\frac{1}{\sqrt{N}}\frac{1}{4\sqrt{\pi}\sqrt{t(1+t)}}\left(1-\frac{k^2-1/4}{4tN}\right)\\ &\cdot\left\{1+\frac{1}{2N}\left(k(r-s-k)-r^2-s^2-\frac{r+s}{2}-\frac{5}{24}\right)\right.\\ &\left.+\frac{k^2-2k(r-s)+2(r^2+s^2)+t(r+s)^2}{4N(1+t)}\right\}. \end{aligned} \tag{B.8}$$

which is formula (3.36).

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