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Boundary states of $c = 1$ and $3/2$ rational conformal field theories

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Abstract: We study the boundary states for the rational points in the moduli spaces of $c = 1$ conformal and $c = 3/2$ superconformal field theories, including the isolated Ginsparg points. We use the orbifold and simple-current techniques to relate the boundary states of different theories and to obtain symmetry-breaking, non-Cardy boundary states. We show some interesting examples of fractional and twisted branes on orbifold spaces.

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Contents

1. Introduction

One of the topics of present string theory investigations is the determination of D-branes on general backgrounds: their algebraic construction as boundary states of conformal field theory, their classification, the relevant consistency conditions and their geometrical interpretation [1]–[5].

In the rational conformal theories (RCFT), the boundary states should obey a closed set of modular covariance conditions [6, 7, 8], that involve the bulk-theory data specified by the torus partition function. A general solution was originally found by Cardy for the symmetry-preserving boundary states associated to the charge-conjugation partition function [6]; several authors have recently discussed other cases and examples with symmetry-breaking boundaries and general partition functions [9]–[12]. Among other solutions, a rather interesting pattern has emerged for the D-branes on group manifolds [2] and coset manifolds [4, 13]. Moreover, a theory for the symmetry-breaking boundary conditions has been introduced [11] and a large class of non-Cardy boundary states has been found [14] by extending the method of simple currents [15].

In this paper, we would like to contribute to these searches by presenting the detailed analysis of the rational conformal and superconformal theories at $c = 1$ and $3/2$, respectively [16, 17]; our study includes the isolated points of the non-abelian orbifolds $\mathrm{SU}(2)/G$, where G is the symmetry group of the tetrahedron ($\mathbf{T}$), octahedron ($\mathbf{O}$) and icosahedron ($\mathbf{I}$). These theories are interesting for their non-trivial, yet manageable, chiral algebras, involving several twisted sectors [18]. Their boundary states provide nice examples of "fractional" and "twisted" branes [19]. In our analysis of boundary states, we extensively use the method of simple currents [14], and we exemplify some properties of symmetry-breaking boundaries first discussed in ref. [11].

It is interesting to see these methods at work in elaborate examples and to discuss the resulting features. Whenever orbifold constructions, often implemented by simple currents, map pairs of conformal theories, we can find corresponding relations between the respective boundary states. Cardy-type boundaries are mapped to new, non-Cardy boundaries pertaining to the same or a different theory; these relations provide interesting hints and checks for the geometrical interpretation of the D-branes.

The paper is organized as follows. In section 2, we introduce the orbifold construction of boundary states in the rather well-known case of the $\widehat{\mathrm{SU}(2)}_k$ affine conformal theories [9, 7, 8], whose bulk field content is given by the ADE modular invariants [20]; this example also motivates the general formulae for the orbifold constructions based on simple currents [14][21]. In section 3, we discuss the boundary states for the rational $c = 1$ theories of the compactified boson and the $S^1/\mathbb{Z}_2$ orbifold [16]; most of the results are well known,[1] but they set the stage for the analysis of the **T**-**O**-**I** models.

In section 4, after (re)-deriving the **T**-**O**-**I** chiral algebras and S matrices [18], we discuss an interesting example of non-Cardy boundary states that pertains to the tetrahedron orbifold with diagonal partition function. Such boundaries are derived from the Cardy states of the octahedron by simple-current extension, namely by the inverse of the orbifold map: $\widehat{\mathrm{SU}(2)}_1/\mathbf{O} = (\widehat{\mathrm{SU}(2)}_1/\mathbf{T})/\mathbb{Z}_2$. The corresponding annulus amplitudes provide a 5-dimensional representation of the 21-dimensional fusion rules of the **T** model.

In section 5, we review the moduli space of $c = 3/2$ superconformal theories [17], we write their chiral algebras and characters, and find their boundary states. Finally, section 6 is devoted to the superconformal **T** and **O** orbifolds;[2] using the chiral data spelled out in appendix D, we find the boundary states for some of their non-charge-conjugate partition functions and the relations among them.

[1] See e.g. the refs. [1].

[2] The superconformal **I** orbifold is not discussed here.

The appendices contain some details of our work: appendix A reports the character tables of the **T-O-I** groups; appendix B discusses the amplitudes for non-orientable surfaces, the Klein bottle and the Möbius strip, [22] that complete the analysis of $c = 1$ theories. Appendix C and D contain the chiral algebras of the conformal and superconformal **T-O-I** models, respectively.

2. Orbifold constructions for boundary states

Rational conformal theories are characterized by the modular invariant partition function on the torus, that is a sesquilinear form in the characters χ_i of the representations of the chiral algebra $\mathcal{A}$ of the theory [23]:

$$Z = \sum_{i,j=1}^{N} Z_{ij}\ \chi_i(q)\overline{\chi_j(q)}\,, \qquad q = \exp(2i\pi\tau)\,. \tag{2.1}$$

In this expression, the trace over the states propagating in the bulk decomposes into the representations labeled by the indices i and j that occur with integer multiplicities Z_{ij}.

The determination of the conformal boundary conditions that are consistent with a given, generic bulk theory (generic Z) is a non-trivial problem that has been tackled by the recent literature [7]–[10]. Let us briefly recall the setting: the partition function on the annulus with boundary conditions of type (a) and (b) is a linear combination of characters,

$$A_{ab} = \sum_{i=1}^{N} A^i_{ab}\ \chi_i(q)\,, \tag{2.2}$$

where the non-negative integer multiplicities A^i_{ab} are in general unknown, and should be determined by the consistency conditions of modular covariance and by some physical requirements [6]. Upon performing the S modular transformation, $\tau \to -1/\tau$, the annulus amplitude (2.2) describes the propagation of bulk states between the two boundary states $|a\rangle$ and $|b\rangle$. The latter can be expended in the basis of the Ishibashi states $|m\rangle\rangle$ as follows [24]:

$$|a\rangle = \sum_{m=1}^{M} B_{am}\ |m\rangle\rangle\,. \tag{2.3}$$

There exists an Ishibashi state for any bulk representation that reflects at the boundary, namely for any m such that $Z_{mm^*} \neq 0$ in the partition function (2.1), with corresponding multiplicity (m^* is the representation conjugate to m).

The general form of the boundary states for the bulk theory of the charge-conjugation modular invariant, $Z_{ij} = \delta_{i,j^*}$, has been given by Cardy [6]: there are as many boundary states as representations of the chiral algebra, $a, m = 1, \ldots, N$, and the boundary coefficients are expressed in terms of the modular S matrix,

$$B_{am} = \frac{S_{am}}{\sqrt{S_{0m}}}\,. \tag{2.4}$$

As a consequence of the Verlinde formula [25], the corresponding annulus coefficients are equal to the fusion rules: $A^i_{ab} = \mathcal{N}^i_{ab}$.

For general torus partition functions, we can make the natural assumption of completeness of the boundary conditions [7], such that the boundary coefficients B_{am} define an invertible map in (2.3). Orthogonality is also required for the set of "pure" boundaries whose correlators obey the cluster decomposition. These conditions imply that the matrix $R_{am} \equiv \sqrt{S_{0m}}\ B_{am}$ is unitary. An equivalent condition is that the matrices of annulus coefficients[3] $(A_i)_a{}^b$ give rise to an integer-valued representation of the fusion algebra [7, 8]:

$$\sum_{b=1}^{M} A_{ia}{}^{b}\ A_{jb}{}^{c} = \sum_{k=1}^{N} \mathcal{N}_{ij}{}^{k}\ A_{ka}{}^{c}\,. \tag{2.5}$$

In this paper, we study boundary states at $c = 1$ and $3/2$: we discuss interesting theories not completely analyzed so far, and describe cases with non-charge-conjugate modular invariants. We extend the orbifold constructions of bulk theories to the determination of the boundary states, relying on the results of the refs. [11, 14, 4]. The method can be illustrated in the case of the $\widehat{\mathrm{SU}(2)}_k$ affine conformal theories with ADE modular invariant partition functions [20]. The D-theories with non-diagonal modular invariant can be obtained as orbifolds of the diagonal A-theories, by modding out the $\mathbb{Z}_2$ symmetry $\chi_i \mapsto (-1)^{i+1}\chi_i$, $i = 1, \ldots, k+1$. The invariant states are the integer-spin $\widehat{\mathrm{SU}(2)}_k$ representations, and the twisted states are added in a way that is consistent with modular invariance. The result is $(\ell = 1, 2, \ldots)$:

$$\begin{aligned} k &= 4\ell + 2\,, \qquad & Z_{D_{2\ell+3}} &= \sum_{i=1\ \mathrm{odd}}^{k+1} |\chi_i|^2 + |\chi_{(k+2)/2}|^2 + \sum_{i=2\ \mathrm{even}}^{(k-2)/2} \left(\chi_i \overline{\chi}_{k+2-i} + \mathrm{c.c.}\right); \\ k &= 4\ell\,, & Z_{D_{2\ell+2}} &= \sum_{i=1\ \mathrm{odd}}^{(k-2)/2} |\chi_i + \chi_{k+2-i}|^2 + 2|\chi_{(k+2)/2}|^2\,. \end{aligned} \tag{2.6}$$

The D-even partition functions are diagonal modular invariants for an extended chiral algebra, while the D-odd partition functions have left and right sectors paired by the permutation $i \to k+2-i$, that is an automorphism of the fusion rules. For $k = 6$ in particular, the orbifold construction relates the A_7 and D_5 partition functions:

$$Z_{A_7} = \sum_{i=1}^{7} |\chi_i|^2 \qquad \longrightarrow \qquad Z_{D_5} = \sum_{i=1\ \mathrm{odd}}^{7} |\chi_i|^2 + |\chi_4|^2 + (\chi_2\overline{\chi}_6 + \mathrm{c.c.})\ . \tag{2.7}$$

The even-index characters in Z_{D_5} correspond to the twisted sectors.

The orbifold operation can be applied to the Ishibashi states of the diagonal invariant since they are in one-to-one relation with the representations of the chiral algebra. First we should form combinations of boundary states that are invariant under the orbifold symmetry; in the present case, they are given by:

$$|i\rangle_D = \frac{1}{\sqrt{2}}\left(|i\rangle_A + |k+2-i\rangle_A\right) = \sum_{m=1}^{k+1} \frac{S_{im}}{\sqrt{S_{1m}}} \frac{1+(-1)^{m+1}}{\sqrt{2}} |m\rangle\!\rangle\,, \qquad i = 1, \ldots, \frac{k}{2}\,, \tag{2.8}$$

where $S_{ij} = \sqrt{2/(k+2)}\ \sin(\pi i j/(k+2))$. The states $|i\rangle_D$ can be called "invariant" boundaries because they only allow the propagation of odd-m bulk states in the closed channel.

[3] Note that the bulk (i) and boundary (a, b) indices of $A^i_{ab} = A^i_{ba}$ can be raised with the help of the bulk and boundary conjugation matrices, $C_{ij} = (S^2)_{ij} = \delta_{i,j^*}$ and $A^0_{ab} = \delta_{a,b^*}$, respectively.

JHEP02(2002)039

In the D-odd theories, we need two further boundaries to form a complete basis. These arise from the splitting of the boundary state of the diagonal theory $|f\rangle_A$, $f = (k+2)/2$, that is the fixed point of the orbifold action $i \to k+2-i$:

$$|f, \pm\rangle_D = \frac{1}{\sqrt{2}}\, |f\rangle_A \pm \frac{R}{\sqrt{S_{1f}}}\, |f\rangle\!\rangle\,. \tag{2.9}$$

These two boundary states are distinguished from the invariant combinations (2.8) by having non-vanishing coefficients for the Ishibashi $|f\rangle\!\rangle$ corresponding to the "twisted" sector $i = (k+2)/2$ in the D-odd modular invariants (2.6).

The coefficient $R = 1/\sqrt{2}$ is determined by the completeness condition for these boundaries (2.8), (2.9), i.e. by the orthogonality of the matrix R_{am}. The two boundary states $|f, \pm\rangle_D$ can be called *fractional branes*, using a string terminology [19]: owing to the splitting, their boundary coefficients are smaller than those of the invariant state $\sqrt{2}\, |f\rangle_A$ they originate from. Actually, there is a geometric interpretation of the $\widehat{\mathrm{SU}(2)}_k$ A-type boundary states as Dirichlet two-branes (two-spheres) on the SU(2) manifold, the S_3 sphere, [2]; in this picture, the previous orbifold action (2.8) is the antipodal map, and the $|f\rangle_A$ brane, localized at the equator, is left invariant and gets split. Simpler examples of this phenomenon will be found later among the $c = 1$ theories.

In the D-even theory, there are two degenerate Ishibashi states $|f, \pm\rangle\!\rangle$ corresponding to the fixed point $f = (k+2)/2$ of the orbifold action. One can similarly construct the invariant boundary states and the two fractional states; however, their boundary coefficients are not completely determined by the completeness condition. There remains a free rotation in the space of the states $|f, \pm\rangle\!\rangle$, that are degenerate in Virasoro dimension; a proper basis for these states is the one preserving the extended symmetry of the D-even theory. Such basis can be found by applying the Cardy formula (2.4) to the boundary states of the extended theory, involving the S-matrix for the extended characters [26]. The boundary states for all the ADE modular invariants have already been obtained by several methods [9, 10, 8], and the present discussion was just meant to be pedagogical; note that the orbifold construction can also be extended to the boundary operator-product expansion [27].

The orbifold construction can be generalized using the language of simple currents [15]. A simple current J is a primary field with one-term fusion rules with all the fields:

$$J \cdot \phi_i = \phi_{J(i)}\,, \qquad i = 1, \ldots, N\,. \tag{2.10}$$

The presence of the simple current implies an abelian discrete symmetry in the theory, that is generated by $\exp(2i\pi Q_J)$, with:

$$Q_J\,(\phi_i) = h_J + h_i - h_{J(i)} \qquad \mathrm{mod}\ 1\,. \tag{2.11}$$

This charge is the exponent for the monodromy of the current around the field ϕ_i and is conserved in the fusion rules. The fields ϕ_i can be organized in orbits, each orbit containing the fields generated by the repeated fusion with the simple current. The simple current and its powers generate an abelian group by fusion that is called the *center* $\mathcal{G}$ of the conformal field theory.

Starting from the charge-conjugation partition function, one can obtain a new modular invariant by modding out the abelian symmetry associated to the simple current. The result depends on the order of the center and on the (conformal) spin of the current. In particular, an integer-spin current J generates an extension-type modular invariant of the form [26]:

$$Z = \sum_{\text{orbits } a|\ Q_J(a)=0} |\mathcal{S}_a| \left| \sum_{J\in\mathcal{G}/\mathcal{S}_a} \chi_{J(i_a)} \right|^2 ; \tag{2.12}$$

in this equation, a labels the orbits, i_a is a representative point on each orbit, and $|\mathcal{S}_a|$ is the order of the stabilizer $\mathcal{S}_a$ of the orbit a, i.e. the subgroup of $\mathcal{G}$ acting trivially on any element i in a. Extension-type modular invariants can be considered as diagonal invariants with respect to the basis of the extended chiral algebra, and therefore the Cardy solution (2.4) can be used to obtain boundary states that preserve the extended symmetry. However, in many cases it is interesting to know also the boundaries that only respect the original chiral algebra [11].

Another example of simple-current modular invariant is of the automorphism type [15]:

$$Z = \sum_{i\ |Q_J(i)=0} |\chi_i|^2 + \sum_{i\ |Q_J(i)=1/2} \chi_i \overline{\chi}_{J(i)} , \tag{2.13}$$

and it is generated by an order-two current of half-integer spin. Both types of invariants, (2.12) and (2.13) are realized in the $\widehat{\mathrm{SU}(2)}_k$ D-series seen before, the simple current being the primary field ϕ_{k+1}.

The $\mathbb{Z}_2$ automorphism modular invariant (2.13) will appear frequently in our analysis of the boundary states of $c = 1$ and $c = 3/2$ theories. The corresponding boundary coefficients R_{am} have been found in general [10], and were shown to represent the so-called *classifying algebra* for boundary conditions that replaces the fusion algebra for the Cardy case. The general pattern of the boundary states is already apparent in the $\widehat{\mathrm{SU}(2)}_k$ example: there are $\mathbb{Z}_2$-invariant boundaries that are in one-to-one correspondence with length-two orbits of the simple current,

$$|a\rangle = \sum_i \frac{S_{a,i} + S_{J(a),i}}{\sqrt{2S_{0,i}}} |i\rangle\rangle . \tag{2.14}$$

These states are characterized by having vanishing coefficients for all the Ishibashi states $|i\rangle\rangle$ with $Q_J(i) = 1/2$, owing to the relation $S_{J(a),k} = S_{a,k} \exp(2\pi i Q_J(k))$ [15]. In addition there are fractional boundary states, two for each fixed point of the simple current, $J(f) = f, J(g) = g, \ldots$, of the form:

$$|f,\pm\rangle = \sum_i \frac{R_{f_\pm,i}}{\sqrt{S_{0i}}} |i\rangle\rangle . \tag{2.15}$$

These states are characterized by non-vanishing coefficients on the Ishibashi corresponding to fields with $Q_J = 1/2$ and fixed by J, that can be expressed in terms of a suitable fixed-point S matrix denoted by $\widetilde{S}$ [14]:

$$R_{f_\pm,g} = \pm\frac{\widetilde{S}_{fg}}{\sqrt{2}} . \tag{2.16}$$

The coefficients in (2.15) with respect to the Ishibashi states $|i\rangle\rangle$ with $J(i) \neq i$ are simply given by the S matrix of the model, $R_{f_\pm i} = S_{fi}/\sqrt{2}$. This ansatz for the boundary states was shown to satisfy the previous constraints of integrality and positivity for the annulus amplitude [10]. We finally mention that a general formula has been presented in ref. [14] for the boundary states of arbitrary simple-current modular invariants.

3. Boundary states at $c = 1$: circle and orbifold lines

We now turn to the analysis of the boundary states for the conformal field theories at $c = 1$: we should first recall some basic facts about these theories [16]. The first line of $c = 1$ models is realized by the free boson field X compactified on a circle of radius R. These models possess the affine U(1) symmetry and their field content can be organized in representations of this algebra, as summarized by the partition function:

$$Z_c(R) = \sum_{n,m\in\mathbb{Z}} \Gamma_{n,m}\,, \qquad \Gamma_{n,m} = \frac{1}{|\eta(q)|^2}\, q^{\frac{\alpha'}{4}\left(\frac{n}{R}+\frac{mR}{\alpha'}\right)^2}\, \overline{q}^{\frac{\alpha'}{4}\left(\frac{n}{R}-\frac{mR}{\alpha'}\right)^2}\,, \tag{3.1}$$

where η is the Dedekind function. By modding out the circle by the reflection $\mathcal{P}: X \mapsto -X$ one obtains the second line of the $S^1/\mathbb{Z}_2$ orbifold theories. On each line, the points R and α'/R are equivalent by T-duality and the two lines intersect at one point, corresponding to $R_c^2 = 4\alpha'$ and $R_o^2 = \alpha'$. The circle theory at the self-dual radius $R_c^2 = \alpha'$ possesses an $\widehat{\mathrm{SU}(2)}_1$ affine symmetry that can be modded by the discrete subgroups of SU(2). While the orbifolds by the cyclic and dihedral groups reproduce theories on the circle and orbifold lines, respectively, those by the three non-abelian groups **T**, **O**, **I**, (the symmetry groups of the tetrahedron, octahedron (cube) and icosahedron (dodecahedron)), give three new $c = 1$ CFT, that do not belong to the previous lines and have no marginal deformations. In the sequel, we will focus on the rational points on the $c = 1$ moduli space[4] [28].

3.1 Circle line

There are two natural boundary conditions $\overline{J} = \Omega(J)$ for the $\widehat{\mathrm{U}(1)}$ currents J and $\overline{J}$, namely $\Omega(J) = \pm J$, corresponding to Neumann and Dirichlet conditions, respectively. The corresponding Ishibashi states are:

$$\begin{aligned} |n,0\rangle\rangle_D &= \exp\left(\sum_{j=1}^{\infty} \frac{\alpha_{-j}\overline{\alpha}_{-j}}{j}\right) |n,0\rangle\,, \\ |0,m\rangle\rangle_N &= \exp\left(-\sum_{j=1}^{\infty} \frac{\alpha_{-j}\overline{\alpha}_{-j}}{j}\right) |0,m\rangle\,, \end{aligned} \tag{3.2}$$

where $|n,m\rangle$ is the highest weight state with n units of momentum and m units of winding and the α_n, $\bar{\alpha}_n$ are the bosonic modes. The boundary states [1][5],

$$|x\rangle = \left(\sqrt{\frac{\alpha'}{2}\frac{1}{R}}\right)^{1/2} \sum_{n\in\mathbb{Z}} e^{\frac{inx}{R}}\, |n,0\rangle\rangle_D\,,$$

[4]For recent results on the boundary states preserving the Virasoro algebra only, see ref. [12].

$$|\theta\rangle = \left(\frac{R}{\sqrt{2\alpha'}}\right)^{1/2} \sum_{m\in\mathbb{Z}} e^{im\theta}\ |0,m\rangle\rangle_N\ , \tag{3.3}$$

can be interpreted as D0 and D1 branes, respectively; they belong to two continuous families parametrized by the position on the circle x and by the value of a Wilson line θ.

Let us now focus on the RCFT at the radii $R = \sqrt{\alpha' k}$, $k \in \mathbb{N}$, where the $\widehat{\mathrm{U}(1)}$ chiral algebra is extended by the fields $e^{\pm i2\sqrt{k/\alpha'}X}$ and is usually referred to as the $\widehat{\mathrm{U}(1)}_k$ algebra [23]. There are $2k$ primary fields, whose characters and conformal dimensions are given by:

$$\chi_r = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} q^{k(n-\frac{r}{2k})^2}\ , \qquad r = -k+1,\ldots,k\,, \qquad h_r = \frac{r^2}{4k}\,. \tag{3.4}$$

The theory with charge conjugation modular invariant $Z = \sum_r \chi_r \overline{\chi}_{-r}$ possesses $2k$ boundary states that are specialization of the Dirichlet boundaries (3.3) for $x = 2\pi R(r/2k)$. In order to account for Neumann states in the rational theories, we should consider symmetry breaking boundary conditions: we postpone this discussion to the next section. The bulk theory described by the diagonal partition function $Z = \sum_r |\chi_r|^2$ can only have two boundary states, that are written as follows,

$$|\pm\rangle = \left(\frac{k}{2}\right)^{1/4} \left(|0\rangle\rangle \pm |k\rangle\rangle\right), \tag{3.5}$$

in terms of the Ishibashi states corresponding to the two self-conjugate fields, $r = 0, k$, of the rational theory. They can be realized as superpositions of D0 branes of the type (3.3), sitting at even (resp. odd) multiples of $2\pi R/(2k)$ [4]. This is the first occurrence of the orbifold relations of the previous section: actually, one can obtain the diagonal modular invariant from the charge conjugation one by the $S^1/\mathbb{Z}_k$ orbifold of the symmetry $\chi_l \mapsto \exp(2i\pi l/k)\chi_l$; thus, the new boundary states are given by invariant combinations of the old ones, in agreement with eq. (2.8).

3.2 Orbifold line

Let us first recall some general aspects of orbifold constructions [18] that will be useful in the following discussion. An orbifold theory is the quotient $\mathcal{C}/G$ of the CFT $\mathcal{C}$ by a discrete group G of symmetries of the theory, i.e. by an endomorphism group of the operator algebra, that commutes with Virasoro and respects the left-right decomposition of the Hilbert space:

$$\mathcal{H} = \sum_{j,\bar{j}} [\phi_j] \otimes [\overline{\phi}_{\bar{j}}]\,, \tag{3.6}$$

where the $[\phi_j]$ are the irreducible representations of the chiral algebra. When a σ-model description is available, the target space of the orbifold theory is the quotient of the original manifold by a subgroup G of its isometry group.

The partition function of the $\mathcal{C}/G$ orbifold is:

$$Z = \frac{1}{|G|} \sum_{g,h\in G\ |\ [g,h]=0} Z\left[\begin{smallmatrix} g\\ h\end{smallmatrix}\right]\ \epsilon(g,h)\,, \tag{3.7}$$

where $|G|$ is the order of the group and $Z\left[{g \atop h}\right]$ is the contribution to the partition function coming from the trace on the g-twisted sector with an insertion of the operator h. The restriction to commuting elements $ghg^{-1}h^{-1} = 1$ comes from the fact that the cycle $aba^{-1}b^{-1}$ is contractible on the torus. The phase $\epsilon(g,h)$ accounts for the so-called discrete torsion [29]. There is a clear hamiltonian interpretation: for each $g \in G$ there is an Hilbert space $\mathcal{H}_g$ of g-twisted states, that are projected by $S_g = \{h \in G | ghg^{-1}h^{-1} = 1\}$, the stabilizer of g. Recalling the relation $|G| = |S_g||C_g|$ between the order of the stabilizer and the dimension of the conjugacy class of g, we can restrict the sum in (3.7) to the conjugacy classes a and choose a representative element g_a for each class, leading to:

$$Z = \sum_a \frac{1}{|S_a|} \sum_{h \in S_a} Z[{g_a \atop h}] \epsilon(g_a, h) = \sum_a \operatorname*{Tr}_{\mathcal{H}_a} \Pi^a q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \,, \tag{3.8}$$

where,

$$\Pi^a = \frac{1}{|S_a|} \sum_{h \in S_a} h \; \epsilon(g_a, h) \,, \tag{3.9}$$

is a projection operator onto S_a-invariant states.

The expression (3.7) of the partition function does not show explicitly the chiral operator content of the orbifold theory, that is necessary for the construction of the boundary states. As stressed in ref. [18], several properties of the orbifold chiral sectors can be associated to the representations of the finite group G. With respect to its action, the original chiral algebra $\mathcal{A}$ decomposes into $\mathcal{A} = \bigoplus_\alpha \mathcal{A}_\alpha$, where $\mathcal{A}_\alpha$ contains the states that transform in the irreducible representation r_α of G. The chiral algebra of the orbifold is $\mathcal{A}_0 = \mathcal{A}/G$. Each $\mathcal{A}_\alpha$ is a representation of $\mathcal{A}_0$, in general reducible because G acts in $\mathcal{A}_\alpha$ and commutes with $\mathcal{A}_0$. We can then decompose each $\mathcal{A}_\alpha$ according to:

$$\mathcal{A}_\alpha = [\phi_\alpha] \otimes r_\alpha \,, \tag{3.10}$$

where $[\phi_\alpha]$ is an irreducible representation of $\mathcal{A}_0$.

Representations of the chiral algebra that are mapped by G into different representations are identified in the orbifold model, while representations that are fixed point of G are split. Morever, modular invariance requires the inclusion of twisted sectors $\mathcal{A}^g$, that are in one-to-one correspondence with the conjugacy classes of G. On $\mathcal{A}^g$, it is defined the action of the stabilizer S_g, and thus there is a decomposition analogous to (3.10):

$$\mathcal{A}^g = \bigoplus_\alpha \mathcal{A}^g_\alpha \,, \qquad \mathcal{A}^g_\alpha = [\phi^g_\alpha] \otimes r^g_\alpha \,, \tag{3.11}$$

where now α labels the irreducible representations r^g_α of S_g.

The characters of the orbifold theory associated to the decomposition $\mathcal{H}_g = \bigoplus_\alpha [\phi^g_\alpha] \otimes r^g_\alpha$ can be written as combinations of the traces,

$$z\left[{g \atop h}\right] = \operatorname*{Tr}_{\mathcal{H}_g} h \; q^{L_0 - \frac{c}{24}} \,, \qquad \left| z\left[{g \atop h}\right] \right|^2 = Z\left[{g \atop h}\right] \,, \tag{3.12}$$

as follows:

$$\chi^g_\alpha(q) = \frac{1}{|S_g|} \sum_{h \in S_g} \rho^g_\alpha(h^{-1}) \; z\left[{g \atop h}\right] \,, \tag{3.13}$$

where $\rho^g_\alpha(h^{-1})$ are the characters of the representation r^g_α. The characters $\chi^g_\alpha(q)$ have q-expansion with positive integers coefficients. The inverse relation is:

$$z\left[{g \atop h}\right] = \sum_\alpha \rho^g_\alpha(h)\chi^g_\alpha(q)\,. \tag{3.14}$$

The chiral algebra of the $S^1/\mathbb{Z}_2$ orbifold at rational radius can be obtained using the previous formulas [18]; it contains $(k+7)$ primary fields whose characters read:

$$\begin{aligned}
u_\pm &= \frac{1}{2}\left(\chi_0 \pm \sqrt{\frac{2\eta}{\theta_2}}\right), & & h = 0,1\,,\\
\phi_\pm &= \frac{1}{2}\chi_k\,, & & h = \frac{k}{4}\,,\\
\chi_r\,, & & r = 1,\ldots,k-1\,, \quad & h = \frac{r^2}{4k}\,,\\
\sigma_i &= \frac{1}{2}\left(\sqrt{\frac{\eta}{\theta_4}} + \sqrt{\frac{\eta}{\theta_3}}\right), & i = 0,1\,, \quad & h = \frac{1}{16}\,,\\
\tau_i &= \frac{1}{2}\left(\sqrt{\frac{\eta}{\theta_4}} - \sqrt{\frac{\eta}{\theta_3}}\right), & i = 0,1, \quad & h = \frac{9}{16}\,,
\end{aligned} \tag{3.15}$$

where the characters χ_r are defined in (3.4) and θ_α, $\alpha = 2,3,4$, are the Jacobi theta functions. The fields[5] χ_r, $r = 1,\ldots,k-1$, arise from the identification between the primaries χ_r and χ_{-r}. The fields $u_\pm$, $\phi_\pm$ arise from the splitting of the representations at the two fixed points $r = 0,k$:

$$\begin{aligned}
u_\pm &= \operatorname*{Tr}_{\chi_0}\left[\frac{1\pm\mathcal{P}}{2}q^{L_0-\frac{c}{24}}\right],\\
\phi_\pm &= \operatorname*{Tr}_{\chi_k}\left[\frac{1\pm\mathcal{P}}{2}q^{L_0-\frac{c}{24}}\right],
\end{aligned} \tag{3.16}$$

where the signs $\pm$ label the two irreducible representations of $\mathbb{Z}_2$. Finally there are four twisted fields σ_i, τ_i, $i = 0,1$, for the two fixed points, which are obtained by considering for each point the combinations:

$$\frac{1}{2}\left(z\left[{\mathcal{P} \atop 1}\right] \pm z\left[{\mathcal{P} \atop \mathcal{P}}\right]\right), \tag{3.17}$$

in agreement with (3.13). The S matrix in the basis (3.15) was found in ref. [18]: for k even, it reads (up to the factor $1/\sqrt{8k}$),

$$\begin{array}{c|ccccc}
 & u_\pm & \phi_\pm & \chi_s & \sigma_j & \tau_j \\
\hline
u_\pm & 1 & 1 & 2 & \pm\sqrt{k} & \pm\sqrt{k} \\
\phi_\pm & 1 & 1 & 2(-1)^s & \pm(-1)^j\sqrt{k} & \pm(-1)^j\sqrt{k} \\
\chi_r & 2 & 2(-1)^r & 4\cos(\pi rs/k) & 0 & 0 \\
\sigma_i & \pm\sqrt{k} & \pm(-1)^i\sqrt{k} & 0 & \delta_{ij}\sqrt{2k} & -\delta_{ij}\sqrt{2k} \\
\tau_i & \pm\sqrt{k} & \pm(-1)^i\sqrt{k} & 0 & -\delta_{ij}\sqrt{2k} & \delta_{ij}\sqrt{2k}
\end{array} \tag{3.18}$$

[5]Hereafter, the fields and the corresponding characters are labeled by the same symbols.

where $i, j = 0, 1$. The expression for k odd is [30]:

$$
\begin{array}{c|ccccc}
 & u_{\pm} & \phi_{\pm} & \chi_s & \sigma_j & \tau_j \\
\hline
u_{\pm} & 1 & 1 & 2 & \pm\sqrt{k} & \pm\sqrt{k} \\
\phi_{\pm} & 1 & -1 & 2(-1)^s & \pm i(-1)^j\sqrt{k} & \pm i(-1)^j\sqrt{k} \\
\chi_r & 2 & 2(-1)^r & 4\cos(\pi rs/k) & 0 & 0 \\
\sigma_i & \pm\sqrt{k} & \pm i(-1)^i\sqrt{k} & 0 & e^{i\sigma_{ij}\pi/8}\sqrt{k} & -e^{i\sigma_{ij}\pi/8}\sqrt{k} \\
\tau_i & \pm\sqrt{k} & \pm i(-1)^i\sqrt{k} & 0 & -e^{i\sigma_{ij}\pi/8}\sqrt{k} & e^{i\sigma_{ij}\pi/8}\sqrt{k}
\end{array}
\tag{3.19}
$$

where $\sigma_{ij} = (-1)^{i+j}(-1)^{(k+1)/2}$, $i, j = 0, 1$.

The Cardy boundary states of the $S^1/\mathbb{Z}_2$ orbifold can be read from these S matrices, eqs. (2.3) and (2.4); actually, they can be interpreted as the result of the orbifold action on the boundaries of the circle theory [1, 11], in agreement with the discussion of section 2. The states $|\chi_r\rangle$, $r = 1, \ldots, k-1$ come from the identification of D0 branes sitting at opposite points along the circle: in their spectrum there is an exactly marginal operator that allows displacements of the branes along the circle. The states $|u_+\rangle$, $|u_-\rangle$, $|\phi_+\rangle$ and $|\phi_-\rangle$ describe fractional D0-branes sitting at the fixed points of the interval. Actually, they have smaller coefficients than those of the states $|\chi_r\rangle$ and non-vanishing coefficients for the twisted Ishibashi's; these branes are forced to live on the fixed points because they have no marginal deformations, unless they combine with the other fractional brane. Finally the Cardy states corresponding to the twisted fields $|\sigma_i\rangle$, $|\tau_i\rangle$, $i = 0, 1$, can be interpreted as fractional D1-branes with suitable Wilson lines. Actually, the occurrence of both D0 and D1 branes is not unexpected, because both gluing conditions Ω and $\Omega\mathcal{P}$ should be present in the orbifold theory.

A general feature of abelian orbifolds is that they always contain a set of integer-spin simple currents, stemming from the decomposition of the chiral algebra of the original theory, that form a group isomorphic to the orbifold group G [15]. These currents allow for reconstructing the original theory as a simple current extension of the orbifold (see eq. (2.12)). Among the chiral fields of the $S^1/\mathbb{Z}_2$ orbifold (3.15), there is indeed the integer-spin current u_- that gives back the circle theory.

According to the discussion of section 2, we can use the simple-current map to transform the orbifold boundary states back to the circle theory; according to eq. (2.14), $|u_+\rangle_o$ and $|u_-\rangle_o$ combine into $|\chi_0\rangle_c$, while $|\phi_+\rangle_o$ and $|\phi_-\rangle_o$ give $|\chi_k\rangle_c$; the fixed points $|\chi_r\rangle_o$ split giving the two boundary states $|\chi_r\rangle_c$ and $|\chi_{-r}\rangle_c$, using eq. (2.15). These are the symmetry-preserving boundaries of the circle theory seen before.

Furthermore, two other boundary states are obtained from the fractional D1 branes $|\sigma_i\rangle_o, |\tau_i\rangle_o$:

$$
|+\rangle = \frac{1}{\sqrt{2}}(|\sigma_0\rangle_o + |\tau_0\rangle_o) = \left(\frac{k}{2}\right)^{1/4} (|u_+\rangle\rangle - |u_-\rangle\rangle + |\phi_+\rangle\rangle - |\phi_-\rangle\rangle)_o \,, \tag{3.20}
$$

and similarly for $|-\rangle = (|\sigma_1\rangle_o + |\tau_1\rangle_o)/\sqrt{2}$. The states $|\pm\rangle$ preserve only a $\mathrm{U}(1)_k/\mathbb{Z}_2$ orbifold subalgebra of the full $\mathrm{U}(1)_k$ symmetry of the circle [11] and can be interpreted as D1 branes with a particular Wilson line, namely they correspond to Neumann boundary conditions for the circle theory. This interpretation is confirmed by the expression of the annulus amplitude between these new states and the rational circle D0 states $|\chi_r\rangle$:

$$A_{+,r} = \sigma_0 + \tau_0\,, \tag{3.21}$$

which is the usual partition function for a free boson on a strip with Neumann-Dirichlet boundary conditions [5]. Actually the Ishibashi states $|u_+\rangle\!\rangle - |u_-\rangle\!\rangle$ and $|\phi_+\rangle\!\rangle - |\phi_-\rangle\!\rangle$ are precisely those associated to the Neumann gluing condition $\Omega\mathcal{P}$. According to the general analysis of symmetry breaking boundary conditions [11], we can move the automorphism $r \mapsto -r$ from the gluing condition to the modular invariant and consider the boundary states (3.20) as the complete set pertaining to the diagonal modular invariant of the circle theory. Actually, they coincide with the boundaries (3.5).

Two further simple currents exists in the orbifold theory, $\phi_\pm$, that, together with u_-, generate the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group for k even and the $\mathbb{Z}_4$ group for k odd. Let us discuss the other theories that they may yield. We note that there exists another orbifold modular invariant given by the automorphism $(\phi_+, \sigma_0, \tau_0) \leftrightarrow (\phi_-, \sigma_1, \tau_1)$, which coincides with charge conjugation for k odd. The boundary states for this modular invariant at $R^2 = \alpha' k$ can be constructed as before, by interpreting them as symmetry-breaking boundaries for the original theory, i.e. by shifting the automorphism from the modular invariant to the gluing condition. Consider the extension of the orbifold theory at $R^2 = 4\alpha' k$ given by the simple current ϕ_+ [11]: the boundaries with $Q_{\phi_+} = 0$ at $R^2 = 4\alpha' k$ give back the $(k+7)$ symmetry-preserving Cardy boundaries of the theory at $R^2 = \alpha' k$: these are obtained from the $k+1$ length-two orbits corresponding to (u_+, ϕ_+), (u_-, ϕ_-) and (χ_r, χ_{4k-r}), with r even, and from the three fixed points χ_{2k}, σ_0 and τ_0. Furthermore, the boundaries with $Q_{\phi_+} = 1/2$ give the $k+1$ symmetry breaking boundaries at $R^2 = \alpha' k$ that we were after: these are the k length-two orbits (χ_r, χ_{4k-r}), with r odd, and (σ_1, τ_1). They can be interpreted as k D0-branes at $x_r = \pi r R/2k$, r odd, and as another D1-brane (the symmetry preserving D0-branes are instead at $x_r = \pi r R/2k$, r even). The explicit expressions of the symmetry-breaking boundaries can be read from the matrices (3.18) and (3.19); the boundaries associated to the orbit (χ_r, χ_{4k-r}) with r odd are, for instance ($r = 1, 3, \ldots, 2k-1$):

$$|\chi_r\rangle = \frac{2}{(8k)^{1/4}} \left((|u_+\rangle\!\rangle - |\phi_+\rangle\!\rangle) + (|u_-\rangle\!\rangle - |\phi_-\rangle\!\rangle) + \sqrt{2} \sum_{s=1}^{k-1} \cos\left(\frac{\pi r s}{2k}\right) (|\chi_{2s}\rangle\!\rangle - |\chi_{4k-2s}\rangle\!\rangle) \right), \tag{3.22}$$

where the Ishibashi states are those of the orbifold theory at $R^2 = 4\alpha' k$. As said, these are also the boundaries of the automorphism modular invariant. Note that for k odd, these boundaries can also be obtained using directly the simple current ϕ_+ of the theory at $R^2 = \alpha' k$. Finally, for $k = 4l + 2$ the current ϕ_+ generates another automorphism modular invariant given by the exchange $\chi_r \leftrightarrow \chi_{k-r}$ for r odd.

In conclusion, we have seen that the orbifold map and its simple-current inverse relating the bulk theories have a clear extension to the boundary states. These maps allow to determine complete sets of boundaries for non-charge-conjugate modular invariants and some symmetry breaking boundaries; furthermore, they can indicate a geometric interpretation of the boundary states in the orbifold theories.

JHEP02(2002)039

The description of the circle and $S^1/\mathbb{Z}_2$ orbifold theories can be completed by determining the appropriate Klein and Möbius amplitudes, that project the bulk and boundary spectrum under the action of the worldsheet parity operation [22]. These are briefly discussed in appendix B.

4. Boundary states of $c = 1$ isolated points

4.1 Chiral algebras of T-O-I orbifolds

Let us now consider the $\widehat{\mathrm{SU}(2)}_1/G$ orbifolds of the circle theory at the self-dual point, where G is a discrete subgroup of SU(2) [16]. The series of the cyclic groups $G = \mathbf{C}_n$ have elements $g_{l/n}$, $l = 0, \ldots, n-1$, that rotate of the angle $2\pi l/n$ around the J^3 axis, where J^i, $i = 1, 2, 3$, are the three SU(2) generators. On the boson field X, this action simply amounts to the shift $X \mapsto X + 2\pi\sqrt{\alpha'} l/n$. The orbifold partition function is:

$$Z_n \equiv Z(\mathbf{C}_n) = \frac{1}{n} \sum_{k,l=0}^{n-1} Z \begin{bmatrix} g_{k/n} \\ g_{l/n} \end{bmatrix}, \tag{4.1}$$

where

$$Z \begin{bmatrix} g_{k/n} \\ g_{l/n} \end{bmatrix} = \frac{1}{|\eta(q)|^2} \sum_{p\in\mathbb{Z},\ m\in\mathbb{Z}+\frac{k}{n}} e^{\frac{2\pi i l p}{n}} q^{\frac{(p+m)^2}{4}} \bar{q}^{\frac{(p-m)^2}{4}}. \tag{4.2}$$

Actually these orbifolds coincide (up to T duality) with the compactified boson theories at the points $R^2 = \alpha' n^2$. The second series of orbifolds by the dihedral groups $\mathbf{D}_n$ similarly give points along the orbifold line at radius $R^2 = \alpha' n^2$; actually, the $\mathbf{D}_n$ groups are generated by adding the element $\exp(i\pi J_1)$ to $\mathbf{C}_n$ whose action on the bosonic field is precisely the reflection $X \mapsto -X$.

Finally there are the orbifolds by the symmetry groups of the regular solids $\mathbf{T}, \mathbf{O}, \mathbf{I}$, respectively A_4, S_4 and A_5, or more precisely their lifts to SU(2): $\mathrm{SL}_2(\mathbb{Z}_3)$, $\mathrm{GL}_2(\mathbb{Z}_3)$ and $\mathrm{SL}_2(\mathbb{Z}_5)$ (In appendix A we report their character tables). Following ref. [16], it is convenient to express the partition functions as sums over the abelian orbifolds of the mutually commuting subgroups of the non-abelian groups, with all overlappings removed. The mutually commuting elements of the non-abelian groups A_4, S_4 and A_5, can be easily visualized in terms of their action on the tetrahedron, cube and dodecahedron, respectively. For the tetrahedral group we have 4 $\mathbf{C}_3$ subgroups that acts by rotations around axes through the centers of the faces and a $\mathbf{D}_2$ generated by rotations of π around axes passing through the center of opposite edges. The partition function is then:

$$Z(\mathbf{T}) = \frac{1}{12} \left(4\,(3Z_3 - Z_1) + 4Z_o(2\sqrt{\alpha'}) \right). \tag{4.3}$$

A similar analysis for the octahedron and the icosahedron yields the partition functions [16]:

$$\begin{aligned} Z(\mathbf{O}) &= \frac{1}{24} \left(3(4Z_4 - 2Z_2) + 4(3Z_3 - Z_1) + 4Z_o(2\sqrt{\alpha'}) + 3(4Z_o(2\sqrt{\alpha'}) - 2Z_2) \right), \\ Z(\mathbf{I}) &= \frac{1}{60} \left(6(5Z_5 - Z_1) + 10(3Z_3 - Z_1) + 5(4Z_o(2\sqrt{\alpha'}) - Z_1) + Z_1 \right). \end{aligned} \tag{4.4}$$

There is an interesting relation between the orbifold line at radius $R^2 = 4\alpha'$ (4-state Potts model or $\widehat{\mathrm{SU}(2)}_1/\mathbf{D_2}$ model), the tetrahedron and the octahedron, that is due to the following chain of normal subgroups:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathbf{T} \subset \mathbf{O}\,, \tag{4.5}$$

with $\mathbf{O}/\mathbf{T} = \mathbb{Z}_2$ and $\mathbf{T}/\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_3$. Accordingly, these models are related among themselves by successive abelian orbifold operations and backward by simple current extensions. Moreover the $\mathbf{O}$ model can be considered as a non-abelian S_3 orbifold of the 4-state Potts model.

The characters of the chiral algebras of the three Ginsparg models can be found as follows [18]: we express the traces $z\left[{g \atop h}\right]$ in the various orbifold sectors in terms of θ functions, using the formulae,

$$Z_n = \frac{1}{n}\sum_{r=0}^{2n-1}\sum_{s=0}^{n-1}\left|\theta\left[{r/2n \atop s/n}\right](q)\right|^2\,, \tag{4.6}$$

where the θ functions are defined as follows,

$$\theta[{a \atop b}](q) \equiv \frac{\Theta[{a \atop b}](q^2)}{\eta(q)}\,, \qquad \Theta[{a \atop b}]\,(q) \equiv \sum_{m\in\mathbb{Z}} q^{\frac{1}{2}(m-a)^2} e^{-2\pi i m b}\,. \tag{4.7}$$

Then, we can express the orbifold characters using eq. (3.13) and the character tables for $\mathrm{SL}_2(\mathbb{Z}_3)$, $\mathrm{GL}_2(\mathbb{Z}_3)$ and $\mathrm{SL}_2(\mathbb{Z}_5)$ (appendix A). The result for the tetrahedron characters is shown hereafter [18], while the other cases are listed in appendix C.

T. The field content of the tetrahedron model consists of 21 chiral fields: 7 in the untwisted sector, 2 in the $\mathbb{Z}_2$-twisted sector and 6 for each of the two $\mathbb{Z}_3$-twisted sectors. In the untwisted sector, we find ($i = 0, 1, 2$):

$$\begin{aligned}
u_i &= \frac{1}{12}\theta[{0 \atop 0}] + \frac{\omega^i}{3}\theta\left[{0 \atop 1/3}\right] + \frac{\overline{\omega}^i}{3}\theta\left[{0 \atop 2/3}\right] + \frac{1}{4}\theta\left[{0 \atop 1/2}\right], && h = 0, 4, 4\,,\\
j &= \frac{1}{4}\theta[{0 \atop 0}] - \frac{1}{4}\theta\left[{0 \atop 1/2}\right], && h = 1\,,\\
\phi_i &= \frac{1}{6}\theta\left[{1/2 \atop 0}\right] - \frac{\omega^{i+2}}{3}\theta\left[{1/2 \atop 1/3}\right] - \frac{\overline{\omega}^{i+2}}{3}\theta\left[{1/2 \atop 2/3}\right], && h = \frac{1}{4}, \frac{9}{4}, \frac{9}{4}\,,
\end{aligned} \tag{4.8}$$

where $\omega = \exp(2i\pi/3)$. We can see that the identity representation of $\widehat{\mathrm{SU}(2)}_1$ decomposes according to the representation of $A_4 \subset \mathrm{SL}_2(\mathbb{Z}_3)$ while the spin one-half representation according to those $\mathrm{SL}_2(\mathbb{Z}_3)$ representations that are projective A_4 representations. The two characters in the $\mathbb{Z}_2$-twisted sector,

$$\begin{aligned}
\sigma &= \frac{1}{2}\theta\left[{1/4 \atop 0}\right] + \frac{1}{2}\theta\left[{1/4 \atop 1/2}\right], \;\; h = \frac{1}{16}\,,\\
\tau &= \frac{1}{2}\theta\left[{1/4 \atop 0}\right] - \frac{1}{2}\theta\left[{1/4 \atop 1/2}\right], \;\; h = \frac{9}{16}\,,
\end{aligned} \tag{4.9}$$

clearly reflect the structure of their $\mathbb{Z}_2$ stabilizer as in the orbifold line. Finally the $\mathbb{Z}_3$-twisted characters ($i = 0, 1, 2$),

$$
\begin{aligned}
\omega_i^+ &= \frac{1}{3}\theta\begin{bmatrix}1/3\\0\end{bmatrix} + \frac{\omega^i}{3}\theta\begin{bmatrix}1/3\\1/3\end{bmatrix} + \frac{\overline{\omega}^i}{3}\theta\begin{bmatrix}1/3\\2/3\end{bmatrix}, & h &= \frac{1}{9}, \frac{4}{9}, \frac{16}{9}, \\
\omega_i^- &= \frac{1}{3}\theta\begin{bmatrix}2/3\\0\end{bmatrix} + \frac{\overline{\omega}^{i-1}}{3}\theta\begin{bmatrix}2/3\\1/3\end{bmatrix} + \frac{\omega^{i-1}}{3}\theta\begin{bmatrix}2/3\\2/3\end{bmatrix}, & h &= \frac{1}{9}, \frac{4}{9}, \frac{16}{9}, \\
\theta_i^+ &= \frac{1}{3}\theta\begin{bmatrix}1/6\\0\end{bmatrix} + \frac{\omega^i}{3}\theta\begin{bmatrix}1/6\\1/3\end{bmatrix} + \frac{\overline{\omega}^i}{3}\theta\begin{bmatrix}1/6\\2/3\end{bmatrix}, & h &= \frac{1}{36}, \frac{25}{36}, \frac{49}{36}, \\
\theta_i^- &= \tfrac{1}{3}\theta\begin{bmatrix}5/6\\0\end{bmatrix} + \tfrac{\overline{\omega}^{i-1}}{3}\theta\begin{bmatrix}5/6\\1/3\end{bmatrix} + \tfrac{\omega^{i-1}}{3}\theta\begin{bmatrix}5/6\\2/3\end{bmatrix}, & h &= \frac{1}{36}, \frac{25}{36}, \frac{49}{36},
\end{aligned}
\tag{4.10}
$$

organize according to the representations of $\mathbb{Z}_3$. From the explicit form of the characters we can calculate the modular S matrix. The result (multiplied by $12\sqrt{2}$) is:

	u_j	j	ϕ_j	σ	τ	ω_j^+	ω_j^-	θ_j^+	θ_j^-
u_i	1	3	2	6	6	$4\omega^i$	$4\overline{\omega}^i$	$4\omega^i$	$4\overline{\omega}^i$
j	3	9	6	-6	-6	0	0	0	0
ϕ_i	2	6	-4	0	0	$-4\omega^i$	$-4\overline{\omega}^i$	$4\omega^i$	$4\overline{\omega}^i$
σ	6	-6	0	$6\sqrt{2}$	$-6\sqrt{2}$	0	0	0	0
τ	6	-6	0	$-6\sqrt{2}$	$6\sqrt{2}$	0	0	0	0
ω_i^+	$4\omega^j$	0	$-4\omega^j$	0	0	$4\overline{\alpha}^2\overline{\omega}^{i+j}$	$4\alpha^2\omega^{i+j}$	$4\alpha\omega^{2i+j}$	$4\overline{\alpha}\overline{\omega}^{2i+j}$
ω_i^-	$4\overline{\omega}^j$	0	$-4\overline{\omega}^j$	0	0	$4\alpha^2\omega^{i+j}$	$4\overline{\alpha}^2\overline{\omega}^{i+j}$	$4\overline{\alpha}\overline{\omega}^{2i+j}$	$4\alpha\omega^{2i+j}$
θ_i^+	$4\omega^j$	0	$4\omega^j$	0	0	$4\alpha\omega^{i+2j}$	$4\overline{\alpha}\overline{\omega}^{i+2j}$	$4\overline{\beta}\omega^{i+j}$	$4\beta\overline{\omega}^{i+j}$
θ_i^-	$4\overline{\omega}^j$	0	$4\overline{\omega}^j$	0	0	$4\overline{\alpha}\overline{\omega}^{i+2j}$	$4\alpha\omega^{i+2j}$	$4\beta\overline{\omega}^{i+j}$	$4\overline{\beta}\omega^{i+j}$

(4.11)

where $i, j = 0, 1, 2$, $\alpha = \exp(2i\pi/9)$ and $\beta = \exp(i\pi/9)$.

The complex S matrix implies a non-trivial conjugation for this model. Actually there are five self-conjugate fields corresponding to $\{u_0, j, \phi_0, \sigma, \tau\}$ while the fields $\{u_1, \phi_1, \omega_i^+, \theta_i^+\}$ are mapped to $\{u_2, \phi_2, \omega_i^-, \theta_i^-\}$. As a consequence there exist both the charge-conjugation and the diagonal modular invariant. The fields u_i form a $\mathbb{Z}_3$ group of simple currents that allow the extension of the tetrahedron to the 4-state Potts model. Furthermore the untwisted field fusion rules coincide with the representation algebra of the group $\mathbf{T}$ as expected from the general discussion in ref. [18].

O. The octahedron model contains 28 chiral fields: eight from the untwisted sector $\{u_\pm, u_f, j_\pm, \phi_\pm, \phi_f\}$, two from the $\mathbb{Z}_2$-twisted sector $\{\mu_r\}$, $r = 0, 1$, six from the $\mathbb{Z}_3$-twisted sector $\{\omega_i, \theta_i\}$, $i = 0, 1, 2$, and twelve from the $\mathbb{Z}_4$-twisted sector $\{\alpha_k, \beta_k, \sigma_\pm, \tau_\pm\}$, $k = 0, 1, 2, 3$. This notation reflects the $\mathbb{Z}_2$-orbifold relation with the $\mathbf{T}$ model. Actually the u_f and the ϕ_f characters originate from the identification of the u_1, u_2 and of the ϕ_1, ϕ_2 characters of the tetrahedron whereas $u_\pm$, $j_\pm$, $\phi_\pm$, $\sigma_\pm$ and $\tau_\pm$ arise from the splitting of the corresponding characters in the $\mathbf{T}$ model. To these five fixed points correspond 10 new twisted fields and finally the fields in the two $\mathbb{Z}_3$-twisted sectors are pairwise identified, resulting in only one $\mathbb{Z}_3$-twisted sector. The simple current that extends the octahedron model back to the tetrahedron is the chiral field u_-. The octahedron S matrix (multiplied by $24\sqrt{2}$) is:

	$u_\pm$	u_f	$j_\pm$	$\phi_\pm$	ϕ_f	μ_s	$\sigma_\pm$	$\tau_\pm$	ω_j	θ_j	α_l	β_l
$u_\pm$	1	2	3	2	4	± 12	6	6	8	8	± 6	± 6
u_f	2	4	6	4	8	0	12	12	-8	-8	0	0
$j_\pm$	3	6	9	6	12	± 12	-6	-6	0	0	∓ 6	∓ 6
$\phi_\pm$	2	4	6	-4	-8	0	0	0	-8	8	$\pm 6\sqrt{2}$	$\mp 6\sqrt{2}$
ϕ_f	4	8	12	-8	-16	0	0	0	8	-8	0	0
μ_r	± 12	0	± 12	0	0	$12\sqrt{2}\epsilon_{rs}$	0	0	0	0	0	0
$\sigma_\pm$	6	12	-6	0	0	0	$6\sqrt{2}$	$-6\sqrt{2}$	0	0	$\pm c_l$	$\pm s_l$
$\tau_\pm$	6	12	-6	0	0	0	$-6\sqrt{2}$	$6\sqrt{2}$	0	0	$\pm s_l$	$\mp c_l$
ω_i	8	-8	0	-8	8	0	0	0	a_{ij}	b_{ij}	0	0
θ_i	8	-8	0	8	-8	0	0	0	b_{ji}	d_{ij}	0	0
α_k	± 6	0	∓ 6	$\pm 6\sqrt{2}$	0	0	$\pm c_k$	$\pm s_k$	0	0	q_{kl}	r_{kl}
β_k	± 6	0	∓ 6	$\mp 6\sqrt{2}$	0	0	$\pm s_k$	$\mp c_k$	0	0	r_{lk}	s_{kl}

where we have introduced the matrices,

$$
\begin{aligned}
\epsilon_{rs} &= (-1)^{r+s}\,, & c_k &= (-1)^k 12\cos(\pi/8)\,, & s_k &= (-1)^k 12\sin(\pi/8)\,,\\
a_{ij} &= 16\mathrm{Re}(\overline{\alpha}^2\overline{\omega}^{i+j})\,, & b_{ij} &= 16\mathrm{Re}(\alpha\omega^{2i+j})\,, & d_{ij} &= 16\mathrm{Re}(\overline{\beta}\omega^{i+j})\,,\\
q_{kl} &= 12\mathrm{Re}\left(e^{-\frac{i\pi}{16}}i^{l+k}\right), & r_{kl} &= 12\mathrm{Re}\left(e^{-\frac{3i\pi}{16}}i^{l-k}\right),\\
s_{kl} &= 12\mathrm{Re}\left(e^{-\frac{9i\pi}{16}}(-i)^{l+k}\right),
\end{aligned}
\tag{4.12}
$$

with indices $i, j = 0, 1, 2$ and $k, l = 0, 1, 2, 3$. In this case the S matrix is real and there is only the diagonal modular invariant.

I. Finally the icosahedron field content amounts to 37 chiral fields whose characters are listed in appendix C together with the S matrix. There are nine fields in the untwisted sector $\{u_i, \phi_j\}$, with $i = 0, \ldots, 4$ and $j = 1, \ldots, 4$, two fields in the $\mathbb{Z}_2$-twisted sector $\{\sigma, \tau\}$, six in the $\mathbb{Z}_3$-twisted sector $\{\omega_i, \theta_i\}$, $i = 0, 1, 2$ and twenty in the $\mathbb{Z}_5$-twisted sector $\{\pi_k, \rho_k, \lambda_k, \xi_k\}$, $k = 0, \ldots, 4$. In this theory the S matrix is also real and there is the diagonal modular invariant only. Furthermore, there are no simple currents among the chiral fields of the icosahedron model, in agreement with the fact that it can not be obtained through a sequence of abelian orbifold operations as the **T** and the **O** models.

In studying these orbifolds one should also allow for the presence of discrete torsion [29]; the different possibilities are classified by $H^2(G, \mathrm{U}(1))$ that, in our case, is non-trivial and equal to $\mathbb{Z}_2$ for $G = \mathbf{D}_{2n}, \mathbf{T}, \mathbf{O}, \mathbf{I}$. Nevertheless, the $\mathbf{D}_{2n}$ orbifolds with or without discrete torsion are equivalent [17], and the same is expected for the $\mathbf{T}, \mathbf{O}, \mathbf{I}$ models.

4.2 Boundary states

In the previous section we have discussed the chiral sectors of the **T**, **O** and **I** orbifold models. The boundary coefficients for the charge-conjugation modular invariant can be read from the respective S matrices according to the Cardy formula (2.4). The result shows some interesting features: let us consider the tetrahedron, for example, eq. (4.11). Firstly,

JHEP02(2002)039

we can observe that boundary states corresponding to a given twisted field are uncharged with respect to fields in different twisted sectors: actually, the S matrix vanishes on the corresponding block entries.

Secondly, the untwisted boundary states can again be interpreted as fractional branes. The first four states of the tetrahedron, $|u_i\rangle$, $|j\rangle$, $i = 0, 1, 2$, come from the splitting of the D0 brane at the north pole of $\widehat{\mathrm{SU}(2)}_1$ while the next three, $|\phi_i\rangle$, come from the D0 brane at the south pole. The fractional nature of these branes is confirmed by the number of possible marginal deformations, respectively 0, 2 and 1, thus showing that the number of directions of motion is lower than the 3 displacements of the original D0 brane in the SU(2) three-sphere. The boundary operator content is given by the annulus amplitudes ($i = 0, 1, 2$):

$$\begin{aligned} A_{u_i,u_i^*} &= u_0 \,, \\ A_{j,j} &= u_0 + u_1 + u_2 + 2\, j \,, \\ A_{\phi_i,\phi_i^*} &= u_0 + j \,, \\ A_{\sigma,\sigma} = A_{\tau,\tau} &= \sum_{i=0}^{2} u_i + j + \sum_{i=0}^{2} \phi_i + 2\sigma + 2\tau \,, \\ A_{\omega_i^\pm,\omega_i^\mp} = A_{\theta_i^\pm,\theta_i^\mp} &= u_0 + j + \sigma + \tau \,. \end{aligned} \tag{4.13}$$

These amplitudes and the boundary states associated to the twisted fields $|\sigma\rangle$, $|\tau\rangle$, $|\omega_i^\pm\rangle$ and $|\theta_i^\pm\rangle$, $i = 0, 1, 2$, could be further understood by studying the geometry of the orbifold space. The Cardy states of the octahedron and icosahedron models present a similar pattern of fractional and twisted branes.

Let us now discuss the boundary states for the diagonal modular invariant of the tetrahedron: there are five Ishibashi states corresponding to the self-conjugate fields $\{u_0, j, \phi_0, \sigma, \tau\}$, and therefore we expect five boundary states. We use the same method applied to the diagonal modular invariant of the compactified boson, namely we obtain them from the octahedron by a simple current extension, the relevant integer-spin current being $J = u_-$. Actually, the $\mathbb{Z}_2$ symmetry $\mathcal{R}$ in $\mathbf{O} = \mathbf{T}/\mathcal{R}$ induces as automorphism of the tetrahedron fusion rules precisely the charge conjugation. The boundaries for the tetrahedron with conjugation modular invariant that preserve only the $\mathbf{T}/\mathcal{R}$ orbifold subalgebra then coincide with the symmetry preserving boundaries for the theory with diagonal modular invariant.

Under the action of J, the ten octahedron boundary states that correspond to the fields in the $\mathcal{R}$-twisted sector combine in pairs. Explicitly J maps the boundaries $|\mu_0\rangle$, $|\alpha_0\rangle$, $|\alpha_1\rangle$, $|\beta_0\rangle$ and $|\beta_1\rangle$ respectively to $|\mu_1\rangle$, $|\alpha_2\rangle$, $|\alpha_3\rangle$, $|\beta_2\rangle$ and $|\beta_3\rangle$. The boundary coefficients are:

$$B_{ai} = \frac{S_{a,i} + S_{J(a),i}}{\sqrt{2 S_{0,i}}} \,, \tag{4.14}$$

where S_{ij} is the S matrix of the octahedron. The resulting boundary coefficients are non zero only for the following five combinations of octahedron Ishibashi states:

$$\begin{array}{lll} |u_+\rangle\rangle - |u_-\rangle\rangle \,, & |j_+\rangle\rangle - |j_-\rangle\rangle \,, & |\phi_+\rangle\rangle - |\phi_-\rangle\rangle \,, \\ |\sigma_+\rangle\rangle - |\sigma_-\rangle\rangle \,, & |\tau_+\rangle\rangle - |\tau_-\rangle\rangle \,, & \end{array} \tag{4.15}$$

that are precisely the **T** Ishibashi states resulting from the gluing $\Omega\mathcal{R}$. In this basis the reflection coefficients are:

$$R = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & -1 & \sqrt{2} & \sqrt{2+\sqrt{2}} & \sqrt{2-\sqrt{2}} \\ 1 & -1 & \sqrt{2} & -\sqrt{2+\sqrt{2}} & -\sqrt{2-\sqrt{2}} \\ 1 & -1 & -\sqrt{2} & \sqrt{2-\sqrt{2}} & -\sqrt{2+\sqrt{2}} \\ 1 & -1 & -\sqrt{2} & -\sqrt{2-\sqrt{2}} & \sqrt{2+\sqrt{2}} \\ 2 & 2 & 0 & 0 & 0 \end{pmatrix} . \tag{4.16}$$

As already said, we can interpret these states as the five symmetry-preserving boundaries for the tetrahedron diagonal invariant. We have explicitly verified that the annulus amplitudes for the five boundaries in (4.16) are consistent and that the coefficient A^i_{ab} give a five dimensional representation of the fusion algebra (reported in appendix C). We have also verified that these boundaries, interpreted as symmetry breaking boundaries for the tetrahedron with conjugation modular invariant, have consistent overlaps with the usual Cardy states and that the corresponding annulus amplitudes contain, as expected, the twisted characters of the octahedron. For instance two of the $A_{i,a}$, $a = 1, \ldots, 5$ are:

$$A_{u_0,1} = \alpha_0 + \alpha_2 \, , \qquad A_{j,5} = 3\mu_0 + 3\mu_1 \, . \tag{4.17}$$

Finally, the Klein and Möbius amplitudes for the **T-O-I** models, in particular those for the diagonal **T** model, are discussed in appendix B.

5. $c = 3/2$ superconformal field theories

5.1 Moduli space

The partition functions of $N = 1$ superconformal theories can be written in general as follows [31]:

$$Z = \frac{1}{2}\left(Z_{NS} + Z_{\widetilde{NS}} + Z_R \pm Z_{\widetilde{R}}\right) , \tag{5.1}$$

where the four terms correspond to antiperiodic (a) and periodic (p) boundary conditions for the supercurrent G along the two non-trivial cycles of the torus: respectively, (a, a), (p, a), (a, p) and (p, p). The last term is the Witten index $Z_{\widetilde{R}} = \mathrm{Tr}_R(-1)^F$; the two choices of the sign are related by the $\mathbb{Z}_2$ symmetry $(-1)^{F_s}$, that takes the value $+1$ on states in the NS-NS sector and the value -1 on states in the R-R sector. Actually, one theory is the orbifold of the other by $(-1)^{F_s}$.

The simplest realization of superconformal symmetry at $c = 3/2$ is given by the theory of a free $N = 1$ superfield, made by the boson field X compactified on a circle of radius R and by the Majorana fermion ψ. The partition function for this system is the product of the familiar lattice sum (3.1) for the boson, and of the fermion partition function summed over the spin structures:

$$Z_c(R) = \sum_{n,m\in\mathbb{Z}} \Gamma_{n,m}\left(|o|^2 + |v|^2 + |s|^2\right) . \tag{5.2}$$

The fermion contribution is expressed in terms of the characters of the Ising model [23]:

$$\begin{aligned}
o &= \frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}} + \sqrt{\frac{\theta_4}{\eta}}\right), && h = 0\,, \\
v &= \frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}} - \sqrt{\frac{\theta_4}{\eta}}\right), && h = \frac{1}{2}\,, \\
s &= \sqrt{\frac{\theta_2}{2\eta}}\,, && h = \frac{1}{16}\,.
\end{aligned} \tag{5.3}$$

whose S matrix is

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}. \tag{5.4}$$

The free superfield compactified on a circle describes the first family of superconformal field theories, parametrized by the radius R. These theories possess one R-R ground state and the Witten index vanishes. Again, the radii R and α'/R are related by T-duality. At the self-dual radius $R = \sqrt{\alpha'}$ there is an $N = 3$ superconformal algebra, resulting from the combination of the affine $\widehat{\mathrm{SU}(2)}_1$ symmetry with the $N = 1$ superconformal symmetry.

The moduli space of $c = 3/2$ superconformal theories has been investigated in ref. [17] by identifying the discrete symmetries of a given family of models and by building new models by various orbifold constructions. Let us summarize these results. The discrete symmetries present at generic values of the circle radius are: the reflection,

$$\mathcal{P}: \qquad X \mapsto -X\,, \qquad \psi \mapsto -\psi\,, \tag{5.5}$$

and the previously mentioned $(-1)^{F_s}$. Furthermore, one can identify X modulo translations by integer fractions of the compactification radius,

$$\delta_n: \qquad X \sim X + \frac{2\pi R}{n}\,, \qquad \psi \mapsto \psi\,. \tag{5.6}$$

The resulting model is again the compactified superfield theory at radius R/n. New models were obtained by combining δ_2 and the other two involutions. As shown in [17], the relevant cases are $\mathcal{P}$, $(-1)^{F_s}\mathcal{P}$ and $(-1)^{F_s}\delta_2$.

The $\mathbb{Z}_2$ orbifold by the symmetry $\mathcal{P}$ is described by the superfield compactified on the interval of length πR, with partition function:

$$Z_o(R) = \frac{1}{2}\left(\sum_{n,m\in\mathbb{Z}} \Gamma_{n,m} + \left|\sqrt{\frac{2\eta}{\theta_2}}\right|^2 + \left|\sqrt{\frac{2\eta}{\theta_4}}\right|^2 + \left|\sqrt{\frac{2\eta}{\theta_3}}\right|^2\right)\left(|o|^2 + |v|^2 + |s|^2\right). \tag{5.7}$$

The T-duality is $Z_o(R) = Z_o\left(\alpha'/R\right)$; the circle line meets the orbifold line at the self-dual radius, $Z_o(\sqrt{\alpha'}) = Z_c(\sqrt{4\alpha'})$. Theories on the orbifold line possess three R-R ground states, $u_+ s\overline{u_+ s}$, $\sigma_0 o\overline{\sigma_0 o}$ and $\sigma_1 o\overline{\sigma_1 o}$ and the Witten index is equal to three.

Using $(-1)^{F_s}\mathcal{P}$ instead of $\mathcal{P}$, one obtains a very similar model that is nothing else than the orbifold by $(-1)^{F_s}$ of the previous theory, called the orbifold-prime theory [17].

Actually, the simple current $u_- v$ of the orbifold theory implements the $(-1)^{F_s}$ symmetry; the partition function reads:

$$Z_{o'}(R) = \frac{1}{2}\left(\sum_{n,m\in\mathbb{Z}}\Gamma_{n,m} + \left|\sqrt{\frac{2\eta}{\theta_2}}\right|^2\right)\left(|o|^2+|v|^2\right) + \frac{1}{2}\left(\sum_{n,m\in\mathbb{Z}}\Gamma_{n,m} - \left|\sqrt{\frac{2\eta}{\theta_2}}\right|^2\right)|s|^2 + \\ +\frac{1}{2}\left(\left|\sqrt{\frac{2\eta}{\theta_4}}\right|^2 - \left|\sqrt{\frac{2\eta}{\theta_3}}\right|^2\right)(o\overline{v}+v\overline{o}) + \frac{1}{2}\left(\left|\sqrt{\frac{2\eta}{\theta_4}}\right|^2 + \left|\sqrt{\frac{2\eta}{\theta_3}}\right|^2\right)|s|^2 . \tag{5.8}$$

At the rational points $R=\sqrt{\alpha' k}$, this partition function can be rewritten in terms of the characters of the rational $c=1$ $S^1/\mathbb{Z}_2$ orbifold (3.15):

$$Z_{o'}(\sqrt{\alpha' k}) = \left(|u_+|^2+|u_-|^2+|\phi_+|^2+|\phi_-|^2\right)\left(|o|^2+|v|^2\right) + \\ + \left(u_+\overline{u}_- + u_-\overline{u}_+ + \phi_+\overline{\phi}_- + \phi_-\overline{\phi}_+\right)|s|^2 + \sum_{r=1}^{k-1}|\chi_r|^2\left(|o|^2+|v|^2+|s|^2\right) + \\ + \sum_{i=0}^{1}\left[(\sigma_i\overline{\tau}_i + \tau_i\overline{\sigma}_i)(o\overline{v}+v\overline{o}) + \left(|\sigma_i|^2+|\tau_i|^2\right)|s|^2\right] . \tag{5.9}$$

From this expression, it is clear that there are no R-R ground states since the characters $u_+ s$, $\sigma_0 o$ and $\sigma_1 o$ now appear in off-diagonal combinations.

The orbifold of the circle theory by $(-1)^{F_s}\delta_2$ yields the so-called super-affine line, whose partition function is:

$$Z_{sa}(R) = \sum_{n,m\in\mathbb{Z}}\Gamma_{2n,m}\left(|o|^2+|v|^2\right) + \sum_{n,m\in\mathbb{Z}}\Gamma_{2n+1,m}|s|^2 + \\ + \sum_{n,m\in\mathbb{Z}}\Gamma_{2n+1,m+\frac{1}{2}}(o\overline{v}+v\overline{o}) + \sum_{n,m\in\mathbb{Z}}\Gamma_{2n,m+\frac{1}{2}}|s|^2 . \tag{5.10}$$

The rational theories at radii $R=\sqrt{2\alpha' m}$ display $12m$ sectors and their partition functions can be written, for m even ($r=-2m+1,\dots,2m$):

$$Z_{sa}\left(\sqrt{2\alpha' m}\right) = \sum_{r \text{ even}}|\chi_r|^2\left(|o|^2+|v|^2\right) + \sum_{r \text{ odd}}|\chi_r|^2|s|^2 + \\ + \sum_{r \text{ even}}\chi_r\overline{\chi}_{r+2m}|s|^2 + \sum_{r \text{ odd}}\chi_r\overline{\chi}_{r+2m}(o\overline{v}+v\overline{o}) , \tag{5.11}$$

and for m odd,

$$Z_{sa}\left(\sqrt{2\alpha' m}\right) = \sum_{r \text{ even}}|\chi_r o + \chi_{r+2m} v|^2 + \sum_{r=1,\text{ odd}}^{2m-1}|(\chi_r+\chi_{r+2m})s|^2 . \tag{5.12}$$

These modular invariants are easily understood noticing that they can be obtained as simple current constructions of the circle theory. The simple current is the field $\chi_{2m}v$ of the circle theory whose conformal dimension is $h=(m+1)/2$ and hence we have an automorphism or extension modular invariant for m even or odd, respectively. Note that the rational partition functions (5.11), (5.12) are written for the diagonal pairing of charges; one should also bear in mind the analogous expressions with charge-conjugation pairing, e.g. $|\chi_r|^2 \to \chi_r\overline{\chi}_{-r}$.

The T-duality of the super-affine line is $Z_{sa}(R) = Z_{sa}(2\alpha'/R)$. At the self-dual point $R = \sqrt{2\alpha'}$ there is a super-affine $\widehat{\mathrm{SO}(3)}_1$ symmetry and the partition function can be rewritten in terms of the affine characters:

$$
\begin{aligned}
O_3 &= \frac{1}{2}\left[\left(\frac{\theta_3}{\eta}\right)^{3/2} + \left(\frac{\theta_4}{\eta}\right)^{3/2}\right], \\
V_3 &= \frac{1}{2}\left[\left(\frac{\theta_3}{\eta}\right)^{3/2} - \left(\frac{\theta_4}{\eta}\right)^{3/2}\right], \\
S_3 &= \frac{1}{\sqrt{2}}\left(\frac{\theta_2}{\eta}\right)^{3/2}.
\end{aligned}
\tag{5.13}
$$

There are no R-R ground states on the super-affine line. The super-affine and the orbifold-prime lines intersect at the point: $Z_{sa}(\sqrt{4\alpha'}) = Z_{o'}(\sqrt{\alpha'})$.

The fifth and final family of theories is called the super-orbifold line and is obtained by orbifolding the super-affine line by $\mathcal{P}$ [17]. Let us present the partition function directly at the rational points $R = \sqrt{2\alpha' m}$: they are simple current modular invariants of the orbifold, the simple current being $\phi_+ v$ with conformal weight $h = (m+1)/2$. As for the super-affine line we have to distinguish between even and odd values of m: the first case gives the automorphism modular invariants,

$$
\begin{aligned}
Z_{so}\left(\sqrt{2\alpha' m}\right) = {} & \left(|u_+|^2 + |u_-|^2 + |\phi_+|^2 + |\phi_-|^2\right)\left(|o|^2 + |v|^2\right) + \\
& + \left(u_+\overline{\phi}_+ + \phi_+\overline{u}_+ + u_-\overline{\phi}_- + \phi_-\overline{u}_-\right)|s|^2 + \sum_{r=1}^{m-1}|\chi_{2r}|^2\left(|o|^2 + |v|^2\right) + \\
& + \sum_{r=1}^{m}|\chi_{2r-1}|^2|s|^2 + \sum_{r=1}^{m-1}\chi_{2r}\overline{\chi}_{2m-2r}|s|^2 + \sum_{r=1}^{m}\chi_{2r-1}\overline{\chi}_{2m-2r+1}\left(o\overline{v} + v\overline{o}\right) + \\
& + \left(|\sigma_0|^2 + |\tau_0|^2\right)\left(|o|^2 + |v|^2 + |s|^2\right) + \\
& + \left(\sigma_1\overline{\tau}_1 + \tau_1\overline{\sigma}_1\right)\left(o\overline{v} + v\overline{o}\right) + \left(|\sigma_1|^2 + |\tau_1|^2\right)|s|^2 .
\end{aligned}
\tag{5.14}
$$

The odd m case gives extension modular invariants:

$$
\begin{aligned}
Z_{so}\left(\sqrt{2\alpha' m}\right) = {} & |u_+ o + \phi_+ v|^2 + |\phi_+ o + u_+ v|^2 + |u_- o + \phi_- v|^2 + |\phi_- o + u_- v|^2 + \\
& + \sum_{r=1}^{m-1}|\chi_{2r} o + \chi_{2m-2r} v|^2 + \sum_{r=1}^{\frac{m-1}{2}}|\chi_{2r-1} + \chi_{2m-2r+1}|^2|s|^2 + 2|\chi_m s|^2 + \\
& + |\sigma_1 o + \tau_1 v|^2 + |\tau_1 o + \sigma_1 v|^2 + 2(|\sigma_0|^2 + |\tau_0|^2)|s|^2 .
\end{aligned}
\tag{5.15}
$$

Theories along this line possess one R-R ground state. The super-orbifold line crosses both the orbifold-prime and the circle line: $Z_{so}(\sqrt{4\alpha'}) = Z_{o'}(\sqrt{4\alpha'})$ and $Z_{so}(\sqrt{2\alpha'}) = Z_c(\sqrt{2\alpha'})$.

The five lines of theories are schematically drawn in figure 1 [17]. The pattern is more easily understood in terms of the chiral algebras underlying the various families of

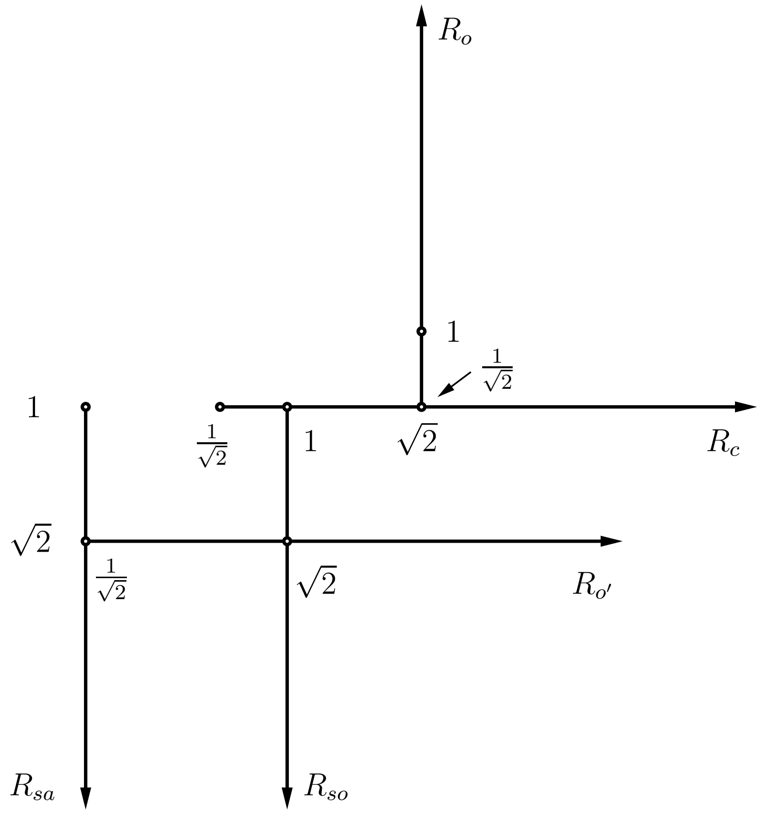

Figure 1: The continuous lines of $c = 3/2$ superconformal theories: the values of the compactification radii $R_c, R_o, R_{o'}, R_{so}$ and R_{sa}, are shown for $\alpha' = 1/2$: they parametrize the circle, orbifold, orbifold-prime, super-orbifold and super-affine theories, respectively [17].

rational theories, at the particular rational points considered before. There are only two independent chiral algebras: the direct product of the Ising-model algebra with either the $\widehat{\mathrm{U}(1)}_k$ or the $\widehat{\mathrm{U}(1)}_k/\mathbb{Z}_2$ algebra. The first one describes the circle and super-affine lines, the second applies to the orbifold, the orbifold-prime and the super-orbifold lines. Different lines for the same chiral algebra correspond to modular invariants generated by $\mathbb{Z}_2$ simple currents: the current $\chi_{2m}v$ of the circle theories yields the super-affine line, and the currents u_-v and ϕ_+v of the orbifold algebra produce the orbifold-prime and the super-orbifold lines, respectively.

We now consider orbifolds by discrete symmetries that exist at particular points on the moduli space [17]. One natural possibility is represented by the self-dual point along the circle line: modding out $Z_c(\sqrt{\alpha'})$ by the SU(2) discrete subgroups, one finds the product of the corresponding $c = 1$ theories times the Majorana fermion. More interesting $N = 1$ superconformal models can be constructed as orbifolds of the super-affine theory at the self-dual point $R = \sqrt{2\alpha'}$, that displays the $\widehat{\mathrm{SO}(3)}_1$ symmetry. Actually, this theory can be realized by three Majorana fermions ψ^i, $i = 1, 2, 3$, and the supercurrent $G = -\frac{1}{12}\epsilon_{ijk}\psi^i\psi^j\psi^k$ is SO(3) invariant; thus, the quotient by the discrete subgroups of SO(3) does not spoil the superconformal symmetry. The $\mathbf{C}_n$-orbifolds yield points along either the super-affine or the circle line (we set $2\alpha' = 1$ here):

$$Z(\mathbf{C}_{2n+1}) = Z_{sa}\,(2n+1)\,, \qquad Z(\mathbf{C}_{2n}) = Z_c\,(n/2)\,. \tag{5.16}$$

One similarly finds that:

$$Z(\mathbf{D}_{2n+1}) = Z_{so}\,(2n+1)\,, \quad Z(\mathbf{D}_{2n}) = Z_o\,(n/2)\,, \quad Z(\mathbf{D}'_{2n}) = Z_{o'}\,(n/2)\,. \tag{5.17}$$

The second theory for the $\mathbf{D}_{2n}$ orbifolds, namely $Z(\mathbf{D}'_{2n})$, is found by introducing a $\mathbb{Z}_2$ discrete torsion (in the following, the models with discrete torsion will be labeled by a prime). Finally, the orbifolds of the self-dual super-affine theory by the non-abelian groups A_4, S_4 and A_5 produce six isolated points: $\mathbf{T}, \mathbf{O}, \mathbf{I}$ and $\mathbf{T}', \mathbf{O}', \mathbf{I}'$ that will be discussed in section 6.

5.2 Boundary states for the superconformal lines

The boundary states for the circle line are easily obtained as combinations of the boundary states for the compactified boson and the Majorana fermion (see eqs. (3.3) and (5.4)). Restricting ourselves to the rational case $R^2 = \alpha' k$, we have:

$$\begin{aligned}
|r,o\rangle_c &= \frac{1}{(8k)^{1/4}} \sum_{\ell=-k+1}^{k} e^{-i\pi r l/k} \left(|\ell\rangle\!\rangle|o\rangle\!\rangle + |\ell\rangle\!\rangle|v\rangle\!\rangle + \sqrt{2}|\ell\rangle\!\rangle|s\rangle\!\rangle \right), \\
|r,v\rangle_c &= \frac{1}{(8k)^{1/4}} \sum_{\ell=-k+1}^{k} e^{-i\pi r l/k} \left(|\ell\rangle\!\rangle|o\rangle\!\rangle + |\ell\rangle\!\rangle|v\rangle\!\rangle - \sqrt{2}|\ell\rangle\!\rangle|s\rangle\!\rangle \right), \\
|r,s\rangle_c &= \frac{1}{(2k)^{1/4}} \sum_{\ell=-k+1}^{k} e^{-i\pi r l/k} \left(|\ell\rangle\!\rangle|o\rangle\!\rangle - |\ell\rangle\!\rangle|v\rangle\!\rangle \right),
\end{aligned} \tag{5.18}$$

where $|\ell\rangle\!\rangle|o\rangle\!\rangle$, $|\ell\rangle\!\rangle|v\rangle\!\rangle$ and $|\ell\rangle\!\rangle|s\rangle\!\rangle$, are the products of Ishibashi states for the rational boson and the Ising model (there are $6k$ boundary states in total). According to the values of the boundary coefficients for the R-R fields, the three types of boundary states in (5.18) can be considered as positively charged, negatively charged and uncharged boundary states, respectively.

On the super-affine line at $R^2 = \alpha' k$ with $k = 2m$, one expects $3m$ boundary states: they can be obtained acting on the boundaries of the circle with the operation $(-1)^{F_s}\delta_2$ or equivalently with the simple current $J = \chi_{2m} v$, that is freely acting. Therefore, there are neither fixed points nor fractional branes and the invariant boundary states are made of pairs, as follows:

$$\begin{aligned}
|r,o\rangle_{sa} &= \frac{1}{\sqrt{2}} \left(|r,o\rangle_c + |r+k,v\rangle_c \right), \qquad r = 0,\ldots,2k-1\,, \\
|r,s\rangle_{sa} &= \frac{1}{\sqrt{2}} \left(|r,s\rangle_c + |r+k,s\rangle_c \right), \qquad r = 0,\ldots,k-1\,.
\end{aligned} \tag{5.19}$$

For m odd the bulk modular invariant is of extension type and the boundaries $|r,o\rangle_{sa}$ with r odd are breaking the extended symmetry.

The Cardy boundary states of the orbifold line are again products of boundary states for the bosonic orbifold with states for the Ising model, for a total of $3(k+7)$ boundary states. They are labeled by the corresponding fields: in the untwisted NS sector, there are $u_\pm I, \phi_\pm I, \chi_r I$, with $I = o, v$; in the untwisted R sector, $u_\pm s, \phi_\pm s, \chi_r s$; in the twisted NS sector, $\sigma_i s, \tau_i s$, with $i = 0,1$; finally, in the twisted R sector, $\sigma_i I, \tau_i I$, $I = o, v$. We can distinguish them according to their R-R charges, taking into account that in this case both untwisted and twisted charges appear. For instance, the states $|u_\pm I\rangle$, have both untwisted and twisted charges, while the states $|u_\pm s\rangle$ carry only twisted charges.

Starting from this set of boundary states and acting with the $\mathbb{Z}_2$ simple current $J = \phi_+ v$ we can obtain the boundary states for the super-orbifold line. From the partition function (5.14), (5.15), we expect $(3m+15)$ Ishibashi states at radius $R^2 = \alpha' k = 2\alpha' m$. Under the action of the simple current, $6(m+3)$ boundaries are combined in pairs, while 3 of them are fixed, leading to the required $3m+15$ boundary states. In order to construct these boundary states, we must know the orbits of the simple current: the three fixed points are $\chi_m s$, $\sigma_0 s$ and $\tau_0 s$; the representations,

$$u_\pm o\,, \qquad \phi_\pm o\,, \qquad \chi_r o\,, \qquad \sigma_0 o\,, \qquad \tau_0 o\,, \qquad \sigma_1 o\,, \qquad \tau_1 o\,, \qquad\qquad r = 1, \ldots, 2m-1\,, \tag{5.20}$$

are respectively paired with the representations,

$$\phi_\pm v\,, \qquad u_\pm v\,, \qquad \chi_{2m-r} v\,, \qquad \sigma_0 v\,, \qquad \tau_0 v\,, \qquad \tau_1 v\,, \qquad \sigma_1 v\,, \tag{5.21}$$

while the representations,

$$u_\pm s\,, \qquad \chi_r s\,, \qquad \sigma_1 s\,, \qquad\qquad r = 1, \ldots, m-1\,, \tag{5.22}$$

are paired with

$$\phi_\pm s\,, \qquad \chi_{2m-r} s, \qquad \tau_1 s\,. \tag{5.23}$$

Again the boundary states of the super-orbifold follow, for m even, the general pattern described in [10] for automorphism modular invariants generated by a $\mathbb{Z}_2$ current of half-integer spin. In particular we have a set of $3m+9$ invariant boundaries in one-to-one correspondence with the length-two orbits of the simple current, as in equation (2.14). In addition there are six fractional boundary states, two for each of the fixed points of the simple current. These boundary states have the form displayed in (2.15) with the fixed-point $\widetilde{S}$-matrix equal to that of the Ising model (5.4) in the basis $\{|\sigma_0\rangle\rangle|s\rangle\rangle, |\tau_0\rangle\rangle|s\rangle\rangle, |\chi_m\rangle\rangle|s\rangle\rangle\}$.

Finally the boundary states for the orbifold-prime line can be obtained acting with the simple current $J = u_- v$ on the orbifold states. All the states labeled by NS fields are paired and give $(k+5)$ boundary states, while among those labeled by Ramond fields, 12 are paired and $(k-1)$ are fixed. The total number of boundaries is $(3k+9)$, in agreement with the number of Ishibashi states (see eq. (5.9)).

The $(k-1)$ fixed boundary states,

$$\begin{aligned} |r,s\rangle_o = &\left(\frac{2}{k}\right)^{1/4} \left[|u_+\rangle\rangle + |u_-\rangle\rangle + (-1)^r |\phi_+\rangle\rangle + (-1)^r |\phi_-\rangle\rangle + \sum_{l=1}^{k-1} \sqrt{2} \cos\left(\frac{\pi r l}{k}\right) |l\rangle\rangle \right] \times \\ &\times \left(|o\rangle\rangle - |v\rangle\rangle\right)\,, \qquad r = 1, \ldots, k-1\,, \end{aligned} \tag{5.24}$$

give rise to new boundaries that differ by their charges with respect to the $(k-1)$ R-R fields $\chi_r s$, with $r = 1, \ldots, k-1$:

$$|r, s, \pm\rangle_{o'} = \frac{1}{(2k)^{1/4}} \left[|u_+\rangle\!\rangle + |u_-\rangle\!\rangle + (-1)^r |\phi_+\rangle\!\rangle + (-1)^r |\phi_-\rangle\!\rangle + \sum_{l=1}^{k-1} \sqrt{2} \cos\left(\frac{\pi r l}{k}\right) |l\rangle\!\rangle \right] \times$$
$$\times \left(|o\rangle\!\rangle - |v\rangle\!\rangle\right) \pm \frac{1}{(2k)^{1/4}} \sum_{l=1}^{k-1} 2 \sin\left(\frac{\pi r l}{k}\right) |l\rangle\!\rangle |s\rangle\!\rangle \,, \qquad r = 1, \ldots, k-1 \,. \quad (5.25)$$

One can check that these states give consistent annulus amplitudes.

6. Superconformal T-O-I models

In this section we find the chiral fields for the superconformal **T**-**O** models and then describe their boundary states. The discussion parallels that of the three $c = 1$ models: we start from the self-dual point on the super-affine line and mod it by the symmetry groups A_4, S_4 and A_5. Taking into account the discrete torsion, one obtains three further models, $\mathbf{T}'$, $\mathbf{O}'$, $\mathbf{I}'$, that can also be realized as $(-1)^{F_s}$ orbifolds of the torsionless **T**, **O**, **I** models (recall that $(-1)^{F_s}$ also relates the orbifold and the orbifold-prime line). Note that the two triples differ in the number of R-R ground states.

The **T** model can also be realized as a $\mathbb{Z}_3$ orbifold of the theory made by the product of three Ising models, that is found at $R = \sqrt{2\alpha'}$ on the orbifold line. From the chain of inclusions (4.5), it is then clear that the **O** model can be obtained as a permutation orbifold of the triple Ising theory, *i.e.* as a non-abelian $\mathbf{S}_3$ orbifold: $(\text{Ising})^{\otimes 3}/\mathbb{Z}_3 = \mathbf{T}$ and $(\text{Ising})^{\otimes 3}/S_3 = \mathbf{O}$. General expressions for permutation orbifolds have been given in ref. [32], and agree with our findings.

The characters and S matrices for the superconformal **T** and **O** models are reported in appendix D; they are obtained as in the bosonic case (4.3) and (4.4), by first expanding the partition functions in terms pertaining to the mutually commuting subgroups of $\mathbf{T} - \mathbf{O}$ (5.16) and (5.17), and then by expressing the latter in terms of Θ functions (eq. (4.7)), using the identities (for $R^2 = 2\alpha' n^2$):

$$\sum_{p,w \in \mathbb{Z}} \frac{1 \pm (-1)^p}{2} \Gamma_{p,w} = \frac{1}{2n|\eta(q)|^2} \sum_{r=0}^{2n-1} \sum_{s=0}^{2n-1} \frac{1 \pm (-1)^r}{2} \left| \Theta\begin{bmatrix} r/2n \\ s/2n \end{bmatrix}(q) \right|^2 ,$$
$$\sum_{p,w \in \mathbb{Z}} \frac{1 \pm (-1)^p}{2} \Gamma_{p,w+1/2} = \frac{1}{2n|\eta(q)|^2} \sum_{r=0}^{2n-1} \sum_{s=0}^{2n-1} \frac{1 \pm (-1)^{r+n}}{2} (-1)^s \left| \Theta\begin{bmatrix} r/2n \\ s/2n \end{bmatrix}(q) \right|^2 . \quad (6.1)$$

Let us now describe the modular invariant partition functions and the corresponding boundaries for these models.

T. The field content of the superconformal **T** model consists of 35 fields: 22 of them belong to the NS sector and 13 to the R sector (see appendix D for notations and explicit expressions). There are several modular invariants: first of all the S matrix is complex and therefore the conjugation and diagonal invariants are distinct. Moreover, starting from the charge-conjugation modular invariant $Z(\mathbf{T}_c)$, we can obtain three further modular invariants using two $\mathbb{Z}_2$ operations. The first is the orbifold by $(-1)^{F_s}$: this is obtained through the action of the simple current ξ_0, that is the primary field containing the supercurrent, as usual. The second is the exchange $\sigma_v \leftrightarrow \tau_o$, that is an automorphism of the fusion rules. The resulting partition functions are:

$$\begin{aligned}
Z(\mathbf{T}_c) &= \left(|\sigma_o|^2 + |\sigma_v|^2 + |\tau_o|^2 + |\tau_v|^2\right) + \sum_i \chi_i \overline{\chi}_{i^*} \,, \\
Z(\mathbf{T}_{ca}) &= \left(|\sigma_o|^2 + \sigma_v\overline{\tau_o} + \tau_o\overline{\sigma_v} + |\tau_v|^2\right) + \sum_i \chi_i \overline{\chi}_{i^*} \,, \\
Z(\mathbf{T}'_c) &= \left(\sigma_o\overline{\tau_v} + \tau_v\overline{\sigma_o} + \sigma_v\overline{\tau_o} + \tau_o\overline{\sigma_v}\right) + \sum_i \chi_i \overline{\chi}_{i^*} \,, \\
Z(\mathbf{T}'_{ca}) &= \left(\sigma_o\overline{\tau_v} + \tau_v\overline{\sigma_o} + |\sigma_v|^2 + |\tau_o|^2\right) + \sum_i \chi_i \overline{\chi}_{i^*} \,. \qquad (6.2)
\end{aligned}$$

In these expressions, the index i runs over all the fields not explicitly present in the first parenthesis, the prime indicates the presence of discrete torsion and the subscript a stands for the previous automorphism. The diagonal modular invariant $Z(\mathbf{T}_d)$ similarly generates three other partition functions, that are denoted by $Z(\mathbf{T}_{da})$, $Z(\mathbf{T}'_d)$ and $Z(\mathbf{T}'_{da})$; they differ from the expressions (6.2) by the substitution $\sum_i \chi_i\overline{\chi}_{i^*} \to \sum_i \chi_i\overline{\chi}_i$.

The boundary states for the model $\mathbf{T}'_c$ with discrete torsion are given by the action of the simple current ξ_0 on the Cardy boundaries of the $\mathbf{T}_c$ model. The expected number of 31 boundary states is reproduced, because the simple current ξ_0 acting on the tetrahedron chiral fields forms 13 length-two orbits and 9 fixed points.

The boundary states for the $\mathbf{T}_{da}$ model can be obtained by using the simple-current extension from the $\mathbf{O}$ model, as already found in the bosonic case (section 4); this will be discussed further below. The $\mathbf{T}'_{da}$ model can also be analyzed by combining the previous two approaches. Unfortunately, the boundaries for the other four cases $\mathbf{T}_{ca}$, $\mathbf{T}'_{ca}$, $\mathbf{T}_d$ and $\mathbf{T}'_d$ do not seem to follow from simple-current constructions.

O. We now turn to the discussion of the supersymmetric octahedron model. This possesses 49 primary fields, 30 belonging to the NS sector and 19 to the R sector, that are all self-conjugate (see appendix D for the character list and the S matrix). There are three simple currents, u_-, v_+, v_-, that form the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and can be used to build several modular invariants. The current u_- has integer spin and give the extension of the octahedron to the tetrahedron; the current v_+ has half-integer spin and contains the supercurrent: the corresponding modular invariant coincide with the orbifold by $(-1)^{F_s}$, namely $\mathbf{O}'$. Finally, the current v_- has half-integer spin and yields a new automorphism modular invariant, that is named $Z(\widetilde{\mathbf{O}})$. The expressions of these modular invariants are, besides the diagonal one:

$$\begin{aligned}
Z(\mathbf{O}') &= \left(\mu_{0v}\overline{\mu_{1o}} + \mu_{0o}\overline{\mu_{1v}} + \sigma_{o+}\overline{\tau}_{v+} + \sigma_{o-}\overline{\tau}_{v-} + c.c.\right) + \sum_i |\chi_i|^2 \,, \\
Z(\widetilde{\mathbf{O}}) &= \left(\alpha_0\overline{\beta}_2 + \alpha_1\overline{\beta}_3 + \alpha_2\overline{\beta}_0 + \alpha_3\overline{\beta}_1 + \rho_+\overline{\rho}_- + \sigma_{o+}\overline{\tau}_{v-} + \sigma_{o-}\overline{\tau}_{v+} + c.c.\right) + \sum_i |\chi_i|^2 \,, \\
Z(\widetilde{\mathbf{O}}') &= (\alpha_0\overline{\beta}_2 + \alpha_1\overline{\beta}_3 + \alpha_2\overline{\beta}_0 + \alpha_3\overline{\beta}_1 + \rho_+\overline{\rho}_- + \mu_{0v}\overline{\mu}_{1o} + \mu_{0o}\overline{\mu}_{1v} + \sigma_{o+}\overline{\sigma}_{o-} + \\
&\quad + \tau_{v+}\overline{\tau}_{v-} + c.c.) + \sum_i |\chi_i|^2 \,, \qquad (6.3)
\end{aligned}$$

where again the sums over $|\chi_i|^2$ contain all the fields not explicitly written in the expressions. Let us discuss these partition functions in turn.

$\mathbf{O}'$. In this model there are 41 Ishibashi states and the simple current v_+ has precisely 19 length-two orbits and 11 fixed points, corresponding to the chiral fields $\rho_\pm$, ρ, osv, Λ_i, and γ_i.

$\widetilde{\mathbf{O}}$. There are 35 Ishibashi states: under the action of v_-, the octahedron fields form 21 orbits of length two and seven fixed points, that split developing charges with respect to the seven twisted characters ρ, osv, Λ_i, $i = 0, 1, 2$, and $\mu_{\alpha s}$, $\alpha = 0, 1$. The boundary coefficients R_{ai} (multiplied by $48\sqrt{2}$) of the resulting 14 boundaries w.r.t. to the Ishibashi states of the twisted fields (see eqs. (2.15) and (2.16)), are the following:

	$\cdots$	ρ	osv	Λ_0	Λ_1	Λ_2	$\mu_{\beta s}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\|\rho,\pm\rangle$	$\cdots$	0	0	$\pm16\sqrt{3}$	$\pm16\sqrt{3}$	$\pm16\sqrt{3}$	0
$\|osv,\pm\rangle$	$\cdots$	0	0	0	0	0	$\pm24\sqrt{2}(-1)^\beta$
$\|\Lambda_0,\pm\rangle$	$\cdots$	$\pm16\sqrt{3}$	0	$\mp32s_2$	$\mp32s_4$	$\pm32s_1$	0
$\|\Lambda_1,\pm\rangle$	$\cdots$	$\pm16\sqrt{3}$	0	$\mp32s_4$	$\pm32s_1$	$\mp32s_2$	0
$\|\Lambda_2,\pm\rangle$	$\cdots$	$\pm16\sqrt{3}$	0	$\pm32s_1$	$\mp32s_2$	$\mp32s_4$	0
$\|\mu_{\alpha s},\pm\rangle$	$\cdots$	0	$\pm24\sqrt{2}(-1)^\alpha$	0	0	0	±24

where $s_1 = \sin(\pi/9)$, $s_2 = \sin(2\pi/9)$ and $s_4 = \sin(4\pi/9)$.

$\widetilde{\mathbf{O}}'$. There are 31 Ishibashi states: the corresponding boundaries can be obtained by acting on the boundaries of the $\mathbf{O}$ models with the full simple-current group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The boundary states for the $\mathbf{T}_{da}$ model are obtained from the $\mathbf{O}$ model, as follows. The $\mathbb{Z}_2$ symmetry $\mathcal{R}$ relating the tetrahedron and octahedron models, $\mathbf{T}/\mathcal{R} = \mathbf{O}$, induces the automorphism of the fusion rules made by the charge-conjugation composed with the exchange $\sigma_v \leftrightarrow \tau_o$. The 9 boundary states for the $Z(\mathbf{T}_{da})$ modular invariant are obtained from the 18 boundary states of the octahedron corresponding to the $\mathcal{R}$-twisted sector, namely $\{\alpha_k, \beta_k, \gamma_k, \mu_{\alpha I}\}$, with $k = 0, \ldots, 3$, $\alpha = 0, 1$ and $I = o, v, s$. They are coupled by the simple current u_- according to:

$$(\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1, \mu_{0s}, \mu_{0o}, \mu_{0v}) \mapsto (\alpha_2, \alpha_3, \beta_2, \beta_3, \gamma_2, \gamma_3, \mu_{1s}, \mu_{1o}, \mu_{1v})\,. \tag{6.4}$$

Extending the $\mathcal{R}$-twisted sectors of the octahedron, we then obtain the boundary states for the $\mathbf{T}_{da}$ model. In a similar way one can obtain the boundary states for the $\mathbf{T}'_{da}$ model.

7. Conclusions

In this paper, we have shown a number of interesting features of boundary conformal field theories on orbifold spaces. The orbifold and simple-current relations between different theories can be extended to mappings for boundary states; these yield complete sets of boundaries for non-charge-conjugation modular invariants, furnish examples of symmetry-breaking boundary conditions and can be visualized geometrically.

The non-abelian **T**-**O**-**I** orbifold models at $c = 1$ and $3/2$ present some interesting features. The origin and properties of fractional Dirichlet-like branes are well understood, while the geometrical interpretation of the branes associated to the twisted sectors is not complete, lacking a clear picture of the **T-O-I** orbifold spaces. The analysis of supersymmetric models could also be developed; in particular, the study of the modular covariance conditions in the R-R sector, that relate the Witten index to the Ramond charges [33].

In conclusion, we hope that the models analyzed in this paper will provide a useful playground for future studies of D-branes.

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A. Character tables of the T-O-I groups

$\vert C_a\vert$	1	1	4	4	4	4	6
i_a	I	$-I$	T	T^{-1}	$-T^{-1}$	$-T$	S
u_0	1	1	1	1	1	1	1
u_1	1	1	ω	$\overline{\omega}$	ω	$\overline{\omega}$	1
u_2	1	1	$\overline{\omega}$	ω	$\overline{\omega}$	ω	1
j	3	3	0	0	0	0	-1
ϕ_0	2	-2	-1	-1	1	1	0
ϕ_1	2	-2	$-\overline{\omega}$	$-\omega$	$\overline{\omega}$	ω	0
ϕ_2	2	-2	$-\omega$	$-\overline{\omega}$	ω	$\overline{\omega}$	0

$\vert C_a\vert$	1	1	8	8	12	6	6	6
i_a	I	$-I$	T	$-T^{-1}$	E	S	F	$-F^T$
u_+	1	1	1	1	1	1	1	1
u_-	1	1	1	1	-1	1	-1	-1
u_f	2	2	-1	-1	0	2	0	0
j_+	3	3	0	0	1	-1	-1	-1
j_-	3	3	0	0	-1	-1	1	1
ϕ_+	2	-2	-1	1	0	0	$\sqrt{2}$	$-\sqrt{2}$
ϕ_-	2	-2	-1	1	0	0	$-\sqrt{2}$	$\sqrt{2}$
ϕ_f	4	-4	1	-1	0	0	0	0

$\lvert C_a \rvert$	1	1	12	12	12	12	30	20	20
i_a	I	$-I$	T	T^2	$-T^{-1}$	T^{-2}	$2E$	$2H$	$2H^{-1}$
u_0	1	1	1	1	1	1	1	1	1
u_1	3	3	A	$\widetilde{A}$	A	$\widetilde{A}$	-1	0	0
u_2	3	3	$\widetilde{A}$	A	$\widetilde{A}$	A	-1	0	0
u_3	4	4	-1	-1	-1	-1	0	1	1
u_4	5	5	0	0	0	0	1	-1	-1
ϕ_1	2	-2	$-A$	$-\widetilde{A}$	A	$\widetilde{A}$	0	-1	1
ϕ_2	2	-2	$-\widetilde{A}$	$-A$	$\widetilde{A}$	A	0	-1	1
ϕ_3	4	-4	-1	-1	1	1	0	1	-1
ϕ_4	6	-6	1	1	-1	-1	0	0	0

These are the character tables of the groups: $SL_2(\mathbb{Z}_3)$ (top), $GL_2(\mathbb{Z}_3)$ (center) and $SL_2(\mathbb{Z}_5)$ (bottom). We used the notations: $\omega = \exp(2i\pi/3)$, $A = (1+\sqrt{5})/2$ and $\widetilde{A} = (1-\sqrt{5})/2$. The representative elements i_a of each conjugacy class are:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

B. Amplitudes of non-orientable surfaces

We complete the discussion of the boundary states for the rational CFTs at $c = 1$ (sections 3 and 4) by considering the amplitudes on the Klein bottle and the Möbius strip [22]. These amplitudes project the closed and open spectra onto states invariant under the world-sheet parity operation Ω. The Klein bottle can be written as:

$$K = \frac{1}{2} \sum_i K^i \, \chi_i \,, \tag{B.1}$$

where the coefficients K^i are constrained by the requirement of integrality and positivity of the partition function $Z/2 + K$. Upon S modular transformation, this amplitude describes the propagation of states in the closed sector between two crosscap states:

$$\widetilde{K} = \sum_i \Gamma_i^2 \, \widetilde{\chi}_i \,, \tag{B.2}$$

where the Γ_i are the so-called crosscap coefficients.

The open spectrum is described by the annulus and Möbius amplitudes; the first one is:

$$A = \frac{1}{2} \sum_{i,a,b} n^a \, n^b \, A^i_{ab} \, \chi_i \,, \tag{B.3}$$

where we now sum over all boundaries with multiplicities n^a, n^b. The Möbius amplitude reads:

$$M = \pm \frac{1}{2} \sum_{i,a} n^a \, {M_a}^i \, \widehat{\chi}_i \,, \tag{B.4}$$

where the hatted characters [22] are defined as $\widehat{\chi} = T^{-1/2}\chi$ and the Möbius coefficients ${M_a}^i$ are again constrained by the requirement of integrality and positivity of the partition function $A + M$. The transverse Möbius amplitude is obtained through the modular transformation $P = T^{1/2}ST^2ST^{1/2}$, and describes the propagation of closed string states between a boundary and a crosscap:

$$\widetilde{M} = \pm \sum_{i,a} n^a \; \Gamma_i \; B_{ai} \; \widetilde{\widetilde{\chi}}_i \, . \tag{B.5}$$

In general, given a bulk conformal field theory one has several choices for the Klein bottle amplitude [22, 9]. These different Klein bottle projections correspond to acting on the closed and open spectra with a combination of Ω and some other involution of the theory, that can often be described by simple-current techniques [21, 35].

The Cardy solution for the annulus corresponding to the charge-conjugation modular invariant has been extended to the Klein and Möbius amplitudes [9]; it can be presented in a nice way by introducing the tensor:

$$Y_{ij}{}^{k} = \sum_{l} \frac{S_{il} P_{jl} P^{\dagger}_{kl}}{S_{0l}} \, . \tag{B.6}$$

The ansatz for the crosscap coefficients is:

$$\Gamma_i = \frac{P_{0i}}{\sqrt{S_{0i}}} \, , \tag{B.7}$$

from which one can derive,

$$K_i = Y_{i00} \, , \qquad M_{ai} = Y_{ai0} \, . \tag{B.8}$$

As shown in [21], one can define a modified Klein bottle projection whenever the model contains a simple current J. The corresponding crosscap coefficients are:

$$\Gamma_i = \frac{P_{J,i}}{\sqrt{S_{J,i}}} \, , \tag{B.9}$$

from which one can derive,

$$K_i = Y_{i,J}{}^{J} \, , \qquad M_a{}^{i} = Y_{J^*(a),J}{}^{i} \, . \tag{B.10}$$

Moreover, this ansatz has been extend to simple-current modular invariants in ref. [14].

Let us now discuss the compactified boson[6] at $c = 1$. We have two natural projections: Ω and $\Omega\delta_2$, where δ_2 is the half-radius shift operator $X \mapsto X + \pi R$. With the charge-conjugation modular invariant, only the representations χ_0 and χ_k can appear in the Klein bottle which reads, in the two cases:

$$K_r = \frac{1}{2}(\chi_0 + \chi_k) \, , \qquad K_c = \frac{1}{2}(\chi_0 + (-1)^k \chi_k) \, . \tag{B.11}$$

The two projections are really distinct for k odd.

[6]See ref. [34] for earlier studies.

We can then construct the annulus and Möbius amplitudes. For example, their explicit expressions for $k=3$ are:

$$
\begin{aligned}
A_r &= \frac{1}{2}\Big[\chi_0\left(n_0^2+n_3^2+2n\overline{n}+2m\overline{m}\right)+\chi_1\left(2n_0n+2m\overline{n}+2n_3\overline{m}\right)+\\
&\quad +\chi_2\left(n^2+\overline{m}^2+2n_0m+2n_3\overline{n}\right)+\chi_3\left(2n_0n_3+2nm+2\overline{n}\overline{m}\right)+\\
&\quad +\chi_{-2}\left(m^2+\overline{n}^2+2nn_3+2n_0\overline{m}\right)+\chi_{-1}\left(2n_0\overline{n}+2n\overline{m}+2n_3m\right)\Big],\\
M_r &= \pm\frac{1}{2}\left[\hat{\chi}_0(n_0-n_3)+\hat{\chi}_2(n-\overline{m})+\hat{\chi}_{-2}(\overline{n}-m)\right],
\end{aligned}
\tag{B.12}
$$

and

$$
\begin{aligned}
A_c &= \frac{1}{2}\Big[\chi_0\left(2l\bar{l}+2n\overline{n}+2m\overline{m}\right)+\chi_1\left(2l\overline{m}+2n\bar{l}+m^2+\overline{n}^2\right)+\\
&\quad +\chi_2\left(2\bar{l}\overline{n}+2n\overline{m}+2ml\right)+\chi_3\left(l^2+\bar{l}^2+2mn+2\overline{m}\overline{n}\right)+\\
&\quad +\chi_{-2}\left(2ln+2m\overline{n}+2\bar{l}\overline{m}\right)+\chi_{-1}\left(2\bar{l}m+2l\overline{n}+n^2+\overline{m}^2\right)\Big],\\
M_c &= \pm\frac{1}{2}\left[\hat{\chi}_1(m+\overline{n})+\hat{\chi}_3(l+\bar{l})+\hat{\chi}_{-1}(n+\overline{m})\right].
\end{aligned}
\tag{B.13}
$$

In eq. (B.13), boundaries that are not self-conjugate ($A^0_{aa}=0$) carry a pair of complex charges, e.g. $l,\bar{l}$.

It is clear from eq. (B.13) that different Ω projections lead to different boundary conjugation properties. When a geometric interpretation is available [35], the boundary conjugation properties simply reflect the action of Ω on the submanifolds wrapped by the brane world-volume. In the simple case analyzed before, Ω maps $X\mapsto -X$ and therefore all the branes sitting at opposite position form conjugate pairs and carry complex charges, except for the branes sitting at $X=0$ and $X=\pi R$ which are fixed under Ω and carry real charges. In a similar way, $\Omega\delta_2$ maps $X\mapsto -X+\pi R$ and the fixed branes are those sitting at $X=\pm\pi R/2$. From the point of view of the rational CFT, the second projection gives rise, for k odd, to an annulus amplitude involving only complex charges: the two fixed branes are missing, because they do not correspond to symmetry-preserving boundary conditions.

Let us consider now the diagonal modular invariant. In this case all the characters can appear in the Klein bottle and the two projections read:

$$
K_1=\frac{1}{2}\sum_{i=0}^{2k-1}\chi_i\,,\qquad K_2=\frac{1}{2}\sum_{i=0}^{2k-1}(-1)^i\chi_i\,. \tag{B.14}
$$

Recall that the annulus only contains the two boundaries in (3.5), obtained by the orbifold construction. For example, for $k=3$, we find:

$$
A_1 = n\overline{n}(\chi_0+\chi_2+\chi_{-2})+\frac{1}{2}\left(n^2+\overline{n}^2\right)(\chi_1+\chi_3+\chi_{-1})\,,
$$

$$
\begin{aligned}
M_1 &= \pm\frac{1}{2}\left(n+\overline{n}\right)\left(\hat{\chi}_1+\hat{\chi}_3+\hat{\chi}_{-1}\right), \\
A_2 &= \frac{1}{2}\left(n_+^2+n_-^2\right)\left(\chi_0+\chi_2+\chi_{-2}\right)+n_+n_-\left(\chi_1+\chi_3+\chi_{-1}\right), \\
M_2 &= \pm\frac{1}{2}\left(n_+-n_-\right)\left(\hat{\chi}_0-\hat{\chi}_2-\hat{\chi}_{-2}\right).
\end{aligned}
\tag{B.15}
$$

It is easy to verify that these amplitudes are consistent in the transverse channel.

A similar analysis can be performed for the orbifold line, in particular one can see that all the charges are real except for those corresponding to the fractional branes at the two fixed points, which can be real or complex depending on the action of Ω on the twisted sectors.

We now describe in some detail the Klein bottle projection for the **T** model. For the charge-conjugation modular invariant the standard Klein bottle projection is:

$$
K = \frac{1}{2}(u_0+\phi_0+j+\sigma+\tau). \tag{B.16}
$$

For the diagonal modular invariant, it is simply given by:

$$
K_d = \frac{1}{2}\sum_{i=1}^{21}\chi_i . \tag{B.17}
$$

It is interesting to notice that the crosscap coefficients for the diagonal case can be obtained from the crosscap coefficients of the **O** model by acting with the simple current u_-, as done for the annulus coefficients in section 4. The crosscap coefficients for the diagonal **T** model are thus found to be:

$$
\Gamma_i = \frac{P_{0,i}-P_{u_-,i}}{\sqrt{2S_{0,i}}} = \frac{1}{2^{1/4}}\left(\sqrt{6},\sqrt{2},0,1,1\right), \tag{B.18}
$$

where the P and S matrices are relative to the **O** model, and the five entries in the vector refer to the tetrahedron primaries $(u_0, j, \phi_0, \sigma, \tau)$, respectively. In the direct channel, these crosscap coefficients give the amplitude (B.17).

The other models at $c=1$ and at $c=3/2$, with various choices for the Klein bottle, can be discussed along similar lines, using the eqs. (B.7) and (B.9).

C. Chiral data of T-O-I models

C.1 Tetrahedron

The fusion rules of the theory, in the field basis given in section 4, are the following ($i, j = 0, 1, 2 \bmod 3$):

$$
u_i u_j = u_{i+j}, \qquad \phi_i\phi_j = u_{i+j}+j, \qquad jj = \sum_{i=0}^{2} u_i + 2j,
$$

$$
\sigma\sigma = \tau\tau = \sum_{i=0}^{2} u_i + j + \sum_{i=0}^{2}\phi_i + 2\sigma + 2\tau, \qquad \sigma\tau = 2j + \sum_{i=0}^{2}\phi_i + 2\sigma + 2\tau,
$$

$$
\begin{aligned}
\omega_i^{\pm}\omega_j^{\pm} &= \omega_{1-i-j}^{\mp} + \sum_{k=0}^{2}\theta_k^{\mp}\,, \qquad & \omega_i^{+}\omega_j^{-} &= u_{j-i} + j + \sigma + \tau\,,\\
\omega_i^{\pm}\theta_j^{\pm} &= \sum_{k=0}^{2}\omega_k^{\mp} + \theta_{i-j}^{\mp}\,, \qquad & \omega_i^{+}\theta_j^{-} &= \sigma + \tau + \sum_{k\neq 2-i-j}\phi_k\,,\\
\theta_i^{\pm}\theta_j^{\pm} &= \sum_{k=0}^{2}\theta_k^{\mp} + \omega_{i+j}^{\mp}\,, \qquad & \theta_i^{+}\theta_j^{-} &= j + \sigma + \tau + u_{i-j}\,.
\end{aligned}
\tag{C.1}
$$

Hereafter, we report the annulus amplitudes for the theory with diagonal modular invariant. There are 5 boundary states, described in section 4.2 (eqs. (4.15) and (4.16)); the corresponding annulus coefficients A^n_{ab}, with $a, b = 1, \ldots, 5$, can be written as 5×5 matrices; the index n runs over the 21 chiral sectors of the theory, ordered as in the S-matrix (4.11), $\{[\phi_n] \mid n = 1, \ldots, 21\} \equiv \{u_i, j, \phi_i, \sigma, \tau, \omega_i^+, \omega_i^-, \theta_i^+, \theta_i^- \mid i = 0, 1, 2\}$. The matrices are: $A^n = \mathbf{1}_5$, $n = 1, 2, 3$, and

$$
\begin{aligned}
A^4 &= \begin{pmatrix} 1&2&0&0&0\\ 2&1&0&0&0\\ 0&0&1&2&0\\ 0&0&2&1&0\\ 0&0&0&0&3 \end{pmatrix}, \qquad & A^m &= \begin{pmatrix} 0&0&1&1&0\\ 0&0&1&1&0\\ 1&1&0&0&0\\ 1&1&0&0&0\\ 0&0&0&0&2 \end{pmatrix}, \quad m = 5, 6, 7\,,\\
A^8 &= \begin{pmatrix} 1&0&1&0&2\\ 0&1&0&1&2\\ 1&0&0&1&2\\ 0&1&1&0&2\\ 2&2&2&2&2 \end{pmatrix}, \qquad & A^9 &= \begin{pmatrix} 0&1&0&1&2\\ 1&0&1&0&2\\ 0&1&1&0&2\\ 1&0&0&1&2\\ 2&2&2&2&2 \end{pmatrix}\\
A^l &= \begin{pmatrix} 0&0&1&1&1\\ 0&0&1&1&1\\ 1&1&0&0&1\\ 1&1&0&0&1\\ 1&1&1&1&2 \end{pmatrix}, \quad l = 10, \ldots, 15\,, \qquad & A^k &= \begin{pmatrix} 1&1&0&0&1\\ 1&1&0&0&1\\ 0&0&1&1&1\\ 0&0&1&1&1\\ 1&1&1&1&2 \end{pmatrix}, \quad k = 16, \ldots, 21\,.
\end{aligned}
\tag{C.2}
$$

One can check that they give a representation of the fusion rules (C.1).

C.2 Octahedron

Untwisted sector:

$$
\begin{aligned}
u_+ &= \frac{1}{24}\theta\begin{bmatrix}0\\0\end{bmatrix} + \frac{1}{3}\theta\begin{bmatrix}0\\1/3\end{bmatrix} + \frac{3}{8}\theta\begin{bmatrix}0\\1/2\end{bmatrix} + \frac{1}{4}\theta\begin{bmatrix}0\\1/4\end{bmatrix}\,, & h &= 0\,,\\
u_- &= \frac{1}{24}\theta\begin{bmatrix}0\\0\end{bmatrix} + \frac{1}{3}\theta\begin{bmatrix}0\\1/3\end{bmatrix} - \frac{1}{8}\theta\begin{bmatrix}0\\1/2\end{bmatrix} - \frac{1}{4}\theta\begin{bmatrix}0\\1/4\end{bmatrix}\,, & h &= 9\,,\\
u_f &= \frac{1}{12}\theta\begin{bmatrix}0\\0\end{bmatrix} - \frac{1}{3}\theta\begin{bmatrix}0\\1/3\end{bmatrix} + \frac{1}{4}\theta\begin{bmatrix}0\\1/2\end{bmatrix}, & h &= 4\,,\\
j_+ &= \frac{1}{8}\theta\begin{bmatrix}0\\0\end{bmatrix} + \frac{1}{8}\theta\begin{bmatrix}0\\1/2\end{bmatrix} - \frac{1}{4}\theta\begin{bmatrix}0\\1/4\end{bmatrix}, & h &= 4\,,\\
j_- &= \frac{1}{8}\theta\begin{bmatrix}0\\0\end{bmatrix} - \frac{3}{8}\theta\begin{bmatrix}0\\1/2\end{bmatrix} + \frac{1}{4}\theta\begin{bmatrix}0\\1/4\end{bmatrix}, & h &= 1\,,\\
\phi_+ &= \frac{1}{12}\theta\begin{bmatrix}1/2\\0\end{bmatrix} - \frac{\varpi}{3}\theta\begin{bmatrix}1/2\\1/3\end{bmatrix} + \frac{1}{4}\theta\begin{bmatrix}1/2\\1/4\end{bmatrix} + \frac{1}{4}\theta\begin{bmatrix}1/2\\3/4\end{bmatrix}, & h &= \frac{1}{4}\,,\\
\phi_- &= \frac{1}{12}\theta\begin{bmatrix}1/2\\0\end{bmatrix} - \frac{\varpi}{3}\theta\begin{bmatrix}1/2\\1/3\end{bmatrix} - \frac{1}{4}\theta\begin{bmatrix}1/2\\1/4\end{bmatrix} - \frac{1}{4}\theta\begin{bmatrix}1/2\\3/4\end{bmatrix}, & h &= \frac{25}{4}\,,\\
\phi_f &= \frac{1}{6}\theta\begin{bmatrix}1/2\\0\end{bmatrix} + \frac{\varpi}{3}\theta\begin{bmatrix}1/2\\1/3\end{bmatrix}, & h &= \frac{9}{4}\,.
\end{aligned}
\tag{C.3}
$$

$\mathbb{Z}_2$-twisted sector ($i = 0, 1$):

$$\mu_i = \frac{1}{2}\theta\begin{bmatrix}1/4\\0\end{bmatrix} + \frac{(-1)^i}{2}\theta\begin{bmatrix}1/4\\1/2\end{bmatrix}, \qquad h = \frac{1}{16}\ , \frac{9}{16}\,. \tag{C.4}$$

$\mathbb{Z}_3$-twisted sector ($i = 0, 1, 2$):

$$\begin{aligned}
\omega_i &= \frac{1}{3}\theta\begin{bmatrix}1/3\\0\end{bmatrix} + \frac{\omega^i}{3}\theta\begin{bmatrix}1/3\\1/3\end{bmatrix} + \frac{\overline{\omega}^i}{3}\theta\begin{bmatrix}1/3\\2/3\end{bmatrix}, \qquad && h = \frac{1}{9}, \frac{4}{9}, \frac{16}{9}\,,\\
\theta_i &= \frac{1}{3}\theta\begin{bmatrix}1/6\\0\end{bmatrix} + \frac{\omega^i}{3}\theta\begin{bmatrix}1/6\\1/3\end{bmatrix} + \frac{\overline{\omega}^i}{3}\theta\begin{bmatrix}1/6\\2/3\end{bmatrix}, \qquad && h = \frac{1}{36}, \frac{25}{36}, \frac{49}{36}\,.
\end{aligned} \tag{C.5}$$

$\mathbb{Z}_4$-twisted sector ($k = 0, 1, 2, 3$):

$$\begin{aligned}
\alpha_k &= \frac{1}{4}\theta\begin{bmatrix}1/8\\0\end{bmatrix} + \frac{i^k}{4}\theta\begin{bmatrix}1/8\\1/4\end{bmatrix} + \frac{(-1)^k}{4}\theta\begin{bmatrix}1/8\\1/2\end{bmatrix} + \frac{(-i)^k}{4}\theta\begin{bmatrix}1/8\\3/4\end{bmatrix}, && h = \frac{1}{64}, \frac{49}{64}, \frac{225}{64}, \frac{81}{64}\,,\\
\beta_k &= \frac{1}{4}\theta\begin{bmatrix}3/8\\0\end{bmatrix} + \frac{i^k}{4}\theta\begin{bmatrix}3/8\\1/4\end{bmatrix} + \frac{(-1)^k}{4}\theta\begin{bmatrix}3/8\\1/2\end{bmatrix} + \frac{(-i)^k}{4}\theta\begin{bmatrix}3/8\\3/4\end{bmatrix}, && h = \frac{9}{64}, \frac{25}{64}, \frac{169}{64}, \frac{121}{64}\,,\\
\sigma_\pm &= \frac{1}{4}\theta\begin{bmatrix}1/4\\0\end{bmatrix} + \frac{1}{4}\theta\begin{bmatrix}1/4\\1/2\end{bmatrix} \pm \frac{1}{4}\theta\begin{bmatrix}1/4\\1/4\end{bmatrix} \pm \frac{1}{4}\theta\begin{bmatrix}1/4\\3/4\end{bmatrix}, && h = \frac{1}{16}\ , \frac{49}{16}\,,\\
\tau_\pm &= \frac{1}{4}\theta\begin{bmatrix}1/4\\0\end{bmatrix} - \frac{1}{4}\theta\begin{bmatrix}1/4\\1/2\end{bmatrix} \pm \frac{i}{4}\theta\begin{bmatrix}1/4\\1/4\end{bmatrix} \mp \frac{i}{4}\theta\begin{bmatrix}1/4\\3/4\end{bmatrix}, && h = \frac{9}{16}\ , \frac{25}{16}\,.
\end{aligned} \tag{C.6}$$

C.3 Icosahedron

Untwisted sector:

$$\begin{aligned}
u_0 &= \frac{1}{60}\theta\begin{bmatrix}0\\0\end{bmatrix} + \frac{1}{3}\theta\begin{bmatrix}0\\1/3\end{bmatrix} + \frac{1}{5}\theta\begin{bmatrix}0\\1/5\end{bmatrix} + \frac{1}{5}\theta\begin{bmatrix}0\\2/5\end{bmatrix} + \frac{1}{4}\theta\begin{bmatrix}0\\1/2\end{bmatrix}, && h = 0\,,\\
u_1 &= \frac{1}{20}\theta\begin{bmatrix}0\\0\end{bmatrix} + \frac{1+\sqrt{5}}{10}\theta\begin{bmatrix}0\\1/5\end{bmatrix} + \frac{1-\sqrt{5}}{10}\theta\begin{bmatrix}0\\2/5\end{bmatrix} - \frac{1}{4}\theta\begin{bmatrix}0\\1/2\end{bmatrix}, && h = 1\,,\\
u_2 &= \frac{1}{20}\theta\begin{bmatrix}0\\0\end{bmatrix} + \frac{1-\sqrt{5}}{10}\theta\begin{bmatrix}0\\1/5\end{bmatrix} + \frac{1+\sqrt{5}}{10}\theta\begin{bmatrix}0\\2/5\end{bmatrix} - \frac{1}{4}\theta\begin{bmatrix}0\\1/2\end{bmatrix}, && h = 9\,,\\
u_3 &= \frac{1}{15}\theta\begin{bmatrix}0\\0\end{bmatrix} + \frac{1}{3}\theta\begin{bmatrix}0\\1/3\end{bmatrix} - \frac{1}{5}\theta\begin{bmatrix}0\\1/5\end{bmatrix} - \frac{1}{5}\theta\begin{bmatrix}0\\2/5\end{bmatrix}, && h = 9\,,\\
u_4 &= \frac{1}{12}\theta\begin{bmatrix}0\\0\end{bmatrix} - \frac{1}{3}\theta\begin{bmatrix}0\\1/3\end{bmatrix} + \frac{1}{4}\theta\begin{bmatrix}0\\1/2\end{bmatrix}, && h = 4\,,\\
\phi_1 &= \frac{1}{30}\theta\begin{bmatrix}1/2\\0\end{bmatrix} - \frac{1+\sqrt{5}}{10}\overline{\zeta}^2\theta\begin{bmatrix}1/2\\1/5\end{bmatrix} - \frac{1-\sqrt{5}}{10}\zeta\theta\begin{bmatrix}1/2\\2/5\end{bmatrix} - \frac{\overline{\omega}}{3}\theta\begin{bmatrix}1/2\\1/3\end{bmatrix}, && h = \frac{1}{4}\,,\\
\phi_2 &= \frac{1}{30}\theta\begin{bmatrix}1/2\\0\end{bmatrix} - \frac{1-\sqrt{5}}{10}\overline{\zeta}^2\theta\begin{bmatrix}1/2\\1/5\end{bmatrix} - \frac{1+\sqrt{5}}{10}\zeta\theta\begin{bmatrix}1/2\\2/5\end{bmatrix} - \frac{\overline{\omega}}{3}\theta\begin{bmatrix}1/2\\1/3\end{bmatrix}, && h = \frac{49}{4}\,,\\
\phi_3 &= \frac{1}{15}\theta\begin{bmatrix}1/2\\0\end{bmatrix} - \frac{1}{5}\overline{\zeta}^2\theta\begin{bmatrix}1/2\\1/5\end{bmatrix} - \frac{1}{5}\zeta\theta\begin{bmatrix}1/2\\2/5\end{bmatrix} + \frac{\overline{\omega}}{3}\theta\begin{bmatrix}1/2\\1/3\end{bmatrix}, && h = \frac{9}{4}\,,\\
\phi_4 &= \frac{1}{10}\theta\begin{bmatrix}1/2\\0\end{bmatrix} + \frac{1}{5}\overline{\zeta}^2\theta\begin{bmatrix}1/2\\1/5\end{bmatrix} + \frac{1}{5}\zeta\theta\begin{bmatrix}1/2\\2/5\end{bmatrix}, && h = \frac{25}{4}\,.
\end{aligned} \tag{C.7}$$

$\mathbb{Z}_2$-twisted sector:

$$\begin{aligned}
\sigma &= \frac{1}{2}\theta\begin{bmatrix}1/4\\0\end{bmatrix} + \frac{1}{2}\theta\begin{bmatrix}1/4\\1/2\end{bmatrix}, \quad h = \frac{1}{16}\,,\\
\tau &= \frac{1}{2}\theta\begin{bmatrix}1/4\\0\end{bmatrix} - \frac{1}{2}\theta\begin{bmatrix}1/4\\1/2\end{bmatrix}, \quad h = \frac{9}{16}\,.
\end{aligned} \tag{C.8}$$

$\mathbb{Z}_3$-twisted sector ($i = 0, 1, 2$):

$$\begin{aligned} \omega_i &= \frac{1}{3}\theta\begin{bmatrix}1/3\\0\end{bmatrix} + \frac{\omega^i}{3}\theta\begin{bmatrix}1/3\\1/3\end{bmatrix} + \frac{\overline{\omega}^i}{3}\theta\begin{bmatrix}1/3\\2/3\end{bmatrix}, \quad h = \frac{1}{9}, \frac{4}{9}, \frac{16}{9}, \\ \theta_i &= \frac{1}{3}\theta\begin{bmatrix}1/6\\0\end{bmatrix} + \frac{\omega^i}{3}\theta\begin{bmatrix}1/6\\1/3\end{bmatrix} + \frac{\overline{\omega}^i}{3}\theta\begin{bmatrix}1/6\\2/3\end{bmatrix}, \quad h = \frac{1}{36}, \frac{25}{36}, \frac{49}{36}. \end{aligned} \tag{C.9}$$

$\mathbb{Z}_5$-twisted sector ($k = 0, 1, 2, 3, 4$):

$$\begin{aligned} \pi_k &= \frac{1}{5}\theta\begin{bmatrix}1/5\\0\end{bmatrix} + \frac{\zeta^k}{5}\theta\begin{bmatrix}1/5\\1/5\end{bmatrix} + \frac{\zeta^{2k}}{5}\theta\begin{bmatrix}1/5\\2/5\end{bmatrix} + \\ &\quad + \frac{\overline{\zeta}^{2k}}{5}\theta\begin{bmatrix}1/5\\3/5\end{bmatrix} + \frac{\overline{\zeta}^k}{5}\theta\begin{bmatrix}1/5\\4/5\end{bmatrix}, \qquad h = \frac{1}{25}, \frac{16}{25}, \frac{81}{25}, \frac{121}{25}, \frac{36}{25}, \\ \rho_k &= \frac{1}{5}\theta\begin{bmatrix}1/10\\0\end{bmatrix} + \frac{\zeta^k}{5}\theta\begin{bmatrix}1/10\\1/5\end{bmatrix} + \frac{\zeta^{2k}}{5}\theta\begin{bmatrix}1/10\\2/5\end{bmatrix} + \\ &\quad + \frac{\overline{\zeta}^{2k}}{5}\theta\begin{bmatrix}1/10\\3/5\end{bmatrix} + \frac{\overline{\zeta}^k}{5}\theta\begin{bmatrix}1/10\\4/5\end{bmatrix}, \qquad h = \frac{1}{100}, \frac{81}{100}, \frac{361}{100}, \frac{441}{100}, \frac{121}{100}, \\ \lambda_k &= \frac{1}{5}\theta\begin{bmatrix}2/5\\0\end{bmatrix} + \frac{\zeta^k}{5}\theta\begin{bmatrix}2/5\\1/5\end{bmatrix} + \frac{\zeta^{2k}}{5}\theta\begin{bmatrix}2/5\\2/5\end{bmatrix} + \\ &\quad + \frac{\overline{\zeta}^{2k}}{5}\theta\begin{bmatrix}2/5\\3/5\end{bmatrix} + \frac{\overline{\zeta}^k}{5}\theta\begin{bmatrix}2/5\\4/5\end{bmatrix}, \qquad h = \frac{4}{25}, \frac{9}{25}, \frac{64}{25}, \frac{144}{25}, \frac{49}{25}, \\ \xi_k &= \frac{1}{5}\theta\begin{bmatrix}3/10\\0\end{bmatrix} + \frac{\zeta^k}{5}\theta\begin{bmatrix}3/10\\1/5\end{bmatrix} + \frac{\zeta^{2k}}{5}\theta\begin{bmatrix}3/10\\2/5\end{bmatrix} + \\ &\quad + \frac{\overline{\zeta}^{2k}}{5}\theta\begin{bmatrix}3/10\\3/5\end{bmatrix} + \frac{\overline{\zeta}^k}{5}\theta\begin{bmatrix}3/10\\4/5\end{bmatrix}, \qquad h = \frac{9}{100}, \frac{49}{100}, \frac{289}{100}, \frac{529}{100}, \frac{169}{100}, \end{aligned} \tag{C.10}$$

where $\zeta = \exp(2i\pi/5)$.

The S matrix ($\times 60\sqrt{2}$) in this basis is:

	u_0	u_1	u_2	u_3	u_4	ϕ_1	ϕ_2	ϕ_3	ϕ_4	σ	τ	ω_j	θ_j	π_l	ρ_l	λ_l	ξ_l
u_0	1	3	3	4	5	2	2	4	6	30	30	20	20	12	12	12	12
u_1	3	9	9	12	15	6	6	12	18	−30	−30	0	0	$\widetilde{g}$	g	g	$\widetilde{g}$
u_2	3	9	9	12	15	6	6	12	18	−30	−30	0	0	g	$\widetilde{g}$	$\widetilde{g}$	g
u_3	4	12	12	16	20	8	8	16	24	0	0	20	20	−12	−12	−12	−12
u_4	5	15	15	20	25	10	10	20	30	30	30	−20	−20	0	0	0	0
ϕ_1	2	6	6	8	10	−4	−4	−8	−12	0	0	−20	20	$-g$	$\widetilde{g}$	$-\widetilde{g}$	g
ϕ_2	2	6	6	8	10	−4	−4	−8	−12	0	0	−20	20	$-\widetilde{g}$	g	$-g$	$\widetilde{g}$
ϕ_3	4	12	12	16	20	−8	−8	−16	−24	0	0	20	−20	−12	12	−12	12
ϕ_4	6	18	18	24	30	−12	−12	−24	−36	0	0	0	0	12	−12	12	−12
σ	30	−30	−30	0	30	0	0	0	0	$30\sqrt{2}$	$-30\sqrt{2}$	0	0	0	0	0	0
τ	30	−30	−30	0	30	0	0	0	0	$-30\sqrt{2}$	$30\sqrt{2}$	0	0	0	0	0	0
ω_i	20	0	0	20	−20	−20	−20	20	0	0	0	a_{ij}	b_{ij}	0	0	0	0
θ_i	20	0	0	20	−20	−20	−20	20	0	0	0	b_{ji}	d_{ij}	0	0	0	0
π_k	12	$\widetilde{g}$	g	−12	0	$-g$	$-\widetilde{g}$	−12	12	0	0	0	0	P^1_{kl}	P^2_{kl}	P^3_{kl}	P^4_{kl}
ρ_k	12	g	$\widetilde{g}$	−12	0	$\widetilde{g}$	g	12	−12	0	0	0	0	P^2_{lk}	R^1_{kl}	R^2_{kl}	R^3_{kl}
λ_k	12	g	$\widetilde{g}$	−12	0	$-\widetilde{g}$	$-g$	−12	12	0	0	0	0	P^3_{lk}	R^2_{lk}	L^1_{kl}	L^2_{kl}
ξ_k	12	$\widetilde{g}$	g	−12	0	g	$\widetilde{g}$	12	−12	0	0	0	0	P^4_{lk}	R^3_{lk}	L^2_{lk}	X^1_{kl}

The submatrices are defined as follows:

$$
\begin{aligned}
&P^1_{kl} = \mathrm{Re}(e^{-4i\pi/25}\zeta^{2(k+l)})\,, &&P^2_{kl} = \mathrm{Re}(e^{-2i\pi/25}\zeta^{k+2l})\,, &&P^3_{kl} = \mathrm{Re}(e^{-8i\pi/25}\zeta^{2l-k})\,,\\
&P^4_{kl} = \mathrm{Re}(e^{-6i\pi/25}\zeta^{2(l-k)})\,, &&R^1_{kl} = \mathrm{Re}(e^{-i\pi/25}\zeta^{k+l})\,, &&R^2_{kl} = \mathrm{Re}(e^{-4i\pi/25}\zeta^{l-k})\,,\\
&R^3_{kl} = \mathrm{Re}(e^{-3i\pi/25}\zeta^{l-2k})\,, &&L^1_{kl} = \mathrm{Re}(e^{16i\pi/25}\zeta^{k+l})\,, &&L^2_{kl} = \mathrm{Re}(e^{12i\pi/25}\zeta^{2k+l})\,,\\
&X^1_{kl} = \mathrm{Re}(e^{9i\pi/25}\zeta^{2(k+l)})\,,
\end{aligned}
$$

with indices $i,j = 0,1,2$, $k,l = 0,1,2,3$, and $g = 6(1+\sqrt{5})$, $\tilde{g} = 6(1-\sqrt{5})$.

D. Superconformal T and O models

D.1 Super-Tetrahedron

In this appendix we display the characters and the S matrices for the superconformal **T** and **O** models. The characters are expressed in terms of the Θ functions (4.7) and of the Ising characters, o, v and s in eq. (5.3).

Untwisted NS sector ($i = 0,1,2$):

$$
\begin{aligned}
\chi_i \; &= \frac{1}{3}ooo + \frac{1}{6\eta}\left(\omega^i\Theta\begin{bmatrix}0\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}0\\2/3\end{bmatrix}\right)(o+v) + \\
&\quad + \frac{1}{6\eta}\left(\omega^i\Theta\begin{bmatrix}0\\5/6\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}0\\1/6\end{bmatrix}\right)(o-v)\;, && h = 0,2,2\,,\\
\chi_- &= ovv\,, && h = 1\,,\\
\xi_i \; &= \frac{1}{3}vvv + \frac{1}{6\eta}\left(\omega^i\Theta\begin{bmatrix}0\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}0\\2/3\end{bmatrix}\right)(o+v) - \\
&\quad - \frac{1}{6\eta}\left(\omega^i\Theta\begin{bmatrix}0\\5/6\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}0\\1/6\end{bmatrix}\right)(o-v)\;, && h = \frac{3}{2},\frac{5}{2},\frac{5}{2}\,,\\
\xi_- &= voo\,, && h = \frac{1}{2}\,.
\end{aligned}
\tag{D.1}
$$

Untwisted R sector ($i = 0,1,2$):

$$
\rho_i = \frac{1}{3}sss - \frac{1}{3\eta}\left(\omega^i\Theta\begin{bmatrix}1/2\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}1/2\\2/3\end{bmatrix}\right)s\,, \qquad h = \frac{19}{16},\frac{19}{16},\frac{3}{16}\,. \tag{D.2}
$$

$\mathbb{Z}_2$-twisted NS sector: $\{\lambda_s|\lambda = \sigma,\tau\}$,

$$
\begin{aligned}
\sigma_s &= oss\,, && h = \frac{1}{8}\,,\\
\tau_s &= vss\,, && h = \frac{5}{8}\,.
\end{aligned}
\tag{D.3}
$$

$\mathbb{Z}_2$-twisted R sector: $\{\sigma_I,\tau_I|I = o,v\}$,

$$
\begin{aligned}
\sigma_o &= soo\,, && h = \frac{1}{16}\,,\\
\sigma_v &= osv\,, && h = \frac{9}{16}\,,\\
\tau_o &= vso\,, && h = \frac{9}{16}\,,\\
\tau_v &= svv\,, && h = \frac{17}{16}\,.
\end{aligned}
\tag{D.4}
$$

$\mathbb{Z}_3$-twisted NS sector $(i = 0, 1, 2)$:

$$
\begin{aligned}
\omega_i^{\pm} &= \frac{1}{6\eta}\left(\Theta\begin{bmatrix}1/3\\0\end{bmatrix} + \omega^i\Theta\begin{bmatrix}1/3\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}1/3\\2/3\end{bmatrix}\right)(o+v)\pm \\
&\quad \pm\frac{1}{6\eta}\left(\Theta\begin{bmatrix}1/3\\1/2\end{bmatrix} + \omega^i\Theta\begin{bmatrix}1/3\\5/6\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}1/3\\1/6\end{bmatrix}\right)(o-v)\,, \qquad h = \frac{1}{18}, \frac{13}{18}, \frac{25}{18}, \frac{5}{9}, \frac{2}{9}, \frac{8}{9}\,, \\
\pi_i^{\pm} &= \frac{1}{6\eta}\left(\Theta\begin{bmatrix}2/3\\0\end{bmatrix} + \omega^i\Theta\begin{bmatrix}2/3\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}2/3\\2/3\end{bmatrix}\right)(o+v)\pm \\
&\quad \pm\frac{1}{6\eta}\left(\Theta\begin{bmatrix}2/3\\1/2\end{bmatrix} + \omega^i\Theta\begin{bmatrix}2/3\\5/6\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}2/3\\1/6\end{bmatrix}\right)(o-v)\,, \qquad h = \frac{2}{9}, \frac{8}{9}, \frac{5}{9}, \frac{13}{18}, \frac{25}{18}, \frac{1}{18}\,.
\end{aligned}
\tag{D.5}
$$

$\mathbb{Z}_3$-twisted R sector $(i = 0, 1, 2)$:

$$
\begin{aligned}
\lambda_i &= \frac{1}{3\eta}\left(\Theta\begin{bmatrix}5/6\\0\end{bmatrix} + \omega^i\Theta\begin{bmatrix}5/6\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}5/6\\2/3\end{bmatrix}\right)s\,, \qquad h = \tfrac{59}{144}, \tfrac{11}{144}, \tfrac{107}{144}\,, \\
\psi_i &= \frac{1}{3\eta}\left(\Theta\begin{bmatrix}1/6\\0\end{bmatrix} + \omega^i\Theta\begin{bmatrix}1/6\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}1/6\\2/3\end{bmatrix}\right)s\,, \qquad h = \tfrac{11}{144}, \tfrac{107}{144}, \tfrac{59}{144}\,.
\end{aligned}
\tag{D.6}
$$

The S matrix in this basis is reported in table 1.

	χ_j	χ_-	ξ_j	ξ_-	ρ_j	μ_s	σ_J	τ_J	$\omega_j^\pm$	$\pi_j^\pm$	λ_j	ψ_j
χ_i	1	3	1	3	$2\sqrt{2}$	6	$3\sqrt{2}$	$3\sqrt{2}$	$4\omega^i$	$4\overline{\omega}^i$	$4\sqrt{2}\omega^i$	$4\sqrt{2}\overline{\omega}^i$
χ_-	3	9	3	9	$6\sqrt{2}$	-6	$-3\sqrt{2}$	$-3\sqrt{2}$	0	0	0	0
ξ_i	1	3	1	3	$-2\sqrt{2}$	6	$-3\sqrt{2}$	$-3\sqrt{2}$	$4\omega^i$	$4\overline{\omega}^i$	$-4\sqrt{2}\omega^i$	$-4\sqrt{2}\overline{\omega}^i$
ξ_-	3	9	3	9	$-6\sqrt{2}$	-6	$3\sqrt{2}$	$3\sqrt{2}$	0	0	0	0
ρ_i	$2\sqrt{2}$	$6\sqrt{2}$	$-2\sqrt{2}$	$-6\sqrt{2}$	0	0	0	0	$\pm 4\sqrt{2}\omega^{i+1}$	$\mp 4\sqrt{2}\overline{\omega}^{i+i}$	0	0
λ_s	6	-6	6	-6	0	0	$-6\sqrt{2}\epsilon_{\lambda J}$	$6\sqrt{2}\epsilon_{\lambda J}$	0	0	0	0
σ_I	$3\sqrt{2}$	$-3\sqrt{2}$	$-3\sqrt{2}$	$3\sqrt{2}$	0	$-6\sqrt{2}\epsilon_{I\mu}$	$12\delta_{IJ}$	$12(1-\delta_{IJ})$	0	0	0	0
τ_I	$3\sqrt{2}$	$-3\sqrt{2}$	$-3\sqrt{2}$	$3\sqrt{2}$	0	$6\sqrt{2}\epsilon_{I\mu}$	$12(1-\delta_{IJ})$	$12\delta_{IJ}$	0	0	0	0
$\omega_i^\pm$	$4\omega^j$	0	$4\omega^j$	0	$\pm 4\sqrt{2}\omega^{j+1}$	0	0	0	$4\overline{\alpha}\omega^{i+j}$	$4\overline{\alpha}^2\overline{\omega}^{i+j}$	$\pm 4\sqrt{2}\beta\omega^{i+j-1}$	$\pm 4\sqrt{2}\overline{\beta}\overline{\omega}^{i+j}$
$\pi_i^\pm$	$4\overline{\omega}^j$	0	$4\omega^j$	0	$\mp 4\sqrt{2}\overline{\omega}^{j+1}$	0	0	0	$4\overline{\alpha}^2\overline{\omega}^{i+j}$	$4\overline{\alpha}^4\omega^{i+j}$	$\pm 4\sqrt{2}\beta^8\overline{\omega}^{i+j}$	$\pm 4\sqrt{2}\overline{\beta}^2\omega^{i+j}$
λ_i	$4\sqrt{2}\omega^j$	0	$-4\sqrt{2}\omega^j$	0	0	0	0	0	$\pm 4\sqrt{2}\beta\omega^{i+j-1}$	$\pm 4\sqrt{2}\beta^8\overline{\omega}^{i+j}$	0	0
ψ_i	$4\sqrt{2}\omega^j$	0	$-4\sqrt{2}\overline{\omega}^j$	0	0	0	0	0	$\pm 4\sqrt{2}\overline{\beta}\overline{\omega}^{i+j}$	$\pm 4\sqrt{2}\overline{\beta}^2\omega^{i+j}$	0	0

Table 1: Super-tetrahedron S matrix ($\times 24$): $\epsilon_{ij} = (-1)^{i+j}$, $\omega = \exp(2i\pi/3)$, $\alpha = \exp(2i\pi/9)$ and $\beta = \exp(i\pi/9)$, with indices $i, j = 0, 1, 2$; $\lambda, \mu = \sigma, \tau$ (resp. $0, 1$) and $I, J = o, v$ (resp. $0, 1$).

D.2 Super-Octahedron

Untwisted NS sector:

$$\begin{aligned}
u_\pm &= \frac{1}{6}ooo \pm \frac{1}{2}o(2\tau)o + \frac{1}{6\eta}\Theta\begin{bmatrix}0\\1/3\end{bmatrix}(o+v) + \frac{1}{6\eta}\Theta\begin{bmatrix}0\\5/6\end{bmatrix}(o-v)\,, && h=0,5\,,\\
u &= \frac{1}{3}ooo - \frac{1}{6\eta}\Theta\begin{bmatrix}0\\1/3\end{bmatrix}(o+v) - \frac{1}{6\eta}\Theta\begin{bmatrix}0\\5/6\end{bmatrix}(o-v)\,, && h=2\,,\\
j_\pm &= \frac{1}{2}vvo \pm \frac{1}{2}v(2\tau)o\,, && h=1,2\,,\\
v_\pm &= \frac{1}{6}vvv \pm \frac{1}{2}v(2\tau)v + \frac{1}{6\eta}\Theta\begin{bmatrix}0\\1/3\end{bmatrix}(o+v) - \frac{1}{6\eta}\Theta\begin{bmatrix}0\\5/6\end{bmatrix}(o-v)\,, && h=\frac{3}{2},\frac{9}{2}\,,\\
v &= \frac{1}{3}vvv - \frac{1}{6\eta}\Theta\begin{bmatrix}0\\1/3\end{bmatrix}(o+v) + \frac{1}{6\eta}\Theta\begin{bmatrix}0\\5/6\end{bmatrix}(o-v)\;, && h=\frac{5}{2}\,,\\
h_\pm &= \frac{1}{2}oov \pm \frac{1}{2}o(2\tau)v\,, && h=\frac{1}{2},\frac{5}{2}\,.
\end{aligned} \tag{D.7}$$

Untwisted R sector:

$$\begin{aligned}
\rho_\pm &= \frac{1}{6}sss \pm \frac{1}{2}s(2\tau)s - \frac{\bar\omega}{3\eta}\Theta\begin{bmatrix}1/2\\1/3\end{bmatrix}s\,, && h=\frac{3}{16},\frac{51}{16}\,,\\
\rho &= \frac{1}{3}sss + \frac{\bar\omega}{3\eta}\Theta\begin{bmatrix}1/2\\1/3\end{bmatrix}s\,, && h=\frac{19}{16}\,.
\end{aligned} \tag{D.8}$$

$\mathbb{Z}_3$-twisted NS sector ($i=0,1,2$):

$$\begin{aligned}
\omega_i^\pm = &\frac{1}{6\eta}\left(\Theta\begin{bmatrix}1/3\\0\end{bmatrix} + \omega^i\Theta\begin{bmatrix}1/3\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}1/3\\2/3\end{bmatrix}\right)(o+v)\pm\\
&\pm\frac{1}{6\eta}\left(\Theta\begin{bmatrix}1/3\\1/2\end{bmatrix} + \omega^i\Theta\begin{bmatrix}1/3\\5/6\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}1/3\\1/6\end{bmatrix}\right)(o-v)\;,\quad h=\frac{1}{18},\frac{13}{18},\frac{25}{18},\frac{5}{9},\frac{2}{9},\frac{8}{9}\,.
\end{aligned} \tag{D.9}$$

$\mathbb{Z}_3$-twisted R sector ($i=0,1,2$):

$$\lambda_i = \frac{1}{3\eta}\left(\Theta\begin{bmatrix}5/6\\0\end{bmatrix} + \omega^i\Theta\begin{bmatrix}5/6\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}5/6\\2/3\end{bmatrix}\right)s\,,\qquad h=\frac{59}{144},\frac{11}{144},\frac{107}{144}\,. \tag{D.10}$$

$\mathbb{Z}_4$-twisted NS sector ($k=0,1,2,3$):

$$\begin{aligned}
\sigma_{s\pm} &= \frac{1}{2}(sso \pm s(2\tau)o)\,, && h=\frac{1}{8},\frac{9}{8}\,,\\
\tau_{s\pm} &= \frac{1}{2}(ssv \pm s(2\tau)v)\,, && h=\frac{5}{8},\frac{13}{8}\,,\\
\alpha_k &= \frac{1}{4\eta}\left(\Theta\begin{bmatrix}1/4\\0\end{bmatrix} + i^k\Theta\begin{bmatrix}1/4\\1/4\end{bmatrix} + (-1)^k\Theta\begin{bmatrix}1/4\\1/2\end{bmatrix} + (-i)^k\Theta\begin{bmatrix}1/4\\3/4\end{bmatrix}\right)o\,, && h=\frac{1}{32},\frac{9}{32},\frac{49}{32},\frac{25}{32}\,,\\
\beta_k &= \frac{1}{4\eta}\left(\Theta\begin{bmatrix}1/4\\0\end{bmatrix} + i^k\Theta\begin{bmatrix}1/4\\1/4\end{bmatrix} + (-1)^k\Theta\begin{bmatrix}1/4\\1/2\end{bmatrix} + (-i)^k\Theta\begin{bmatrix}1/4\\3/4\end{bmatrix}\right)v\,, && h=\frac{17}{32},\frac{25}{32},\frac{65}{32},\frac{41}{32}\,.
\end{aligned} \tag{D.11}$$

$\mathbb{Z}_4$-twisted R sector ($I = o, v;\ k = 0, 1, 2, 3$):

$$
\begin{aligned}
\sigma_{o\pm} &= \frac{1}{2}(soo \pm o(2\tau)s)\,, && h = \frac{1}{16}, \frac{33}{16}\,,\\
\tau_{v\pm} &= \frac{1}{2}(svv \pm v(2\tau)s)\,, && h = \frac{17}{16}, \frac{33}{16}\,,\\
osv &= osv\,, && h = \frac{9}{16}\,,\\
\gamma_k &= \frac{1}{4\eta}\left(\Theta\begin{bmatrix}1/4\\0\end{bmatrix} + i^k\Theta\begin{bmatrix}1/4\\1/4\end{bmatrix} + (-1)^k\Theta\begin{bmatrix}1/4\\1/2\end{bmatrix} + (-i)^k\Theta\begin{bmatrix}1/4\\3/4\end{bmatrix}\right) s\,, && h = \frac{3}{32}, \frac{11}{32}, \frac{51}{32}, \frac{27}{32}\,.
\end{aligned}
\tag{D.12}
$$

$\mathbb{Z}_2$-twisted NS sector ($n = 0, 1$):

$$
\begin{aligned}
\mu_{0s} &= oss\,, && h = \frac{1}{8}\,,\\
\mu_{1s} &= vss\,, && h = \frac{5}{8}\,.
\end{aligned}
\tag{D.13}
$$

$\mathbb{Z}_2$-twisted R sector ($I = o, v,\ n = 0, 1$):

$$
\begin{aligned}
\mu_{0o} &= soo\,, && h = \frac{1}{16}\,,\\
\mu_{0v} &= osv\,, && h = \frac{9}{16}\,,\\
\mu_{1o} &= vso\,, && h = \frac{9}{16}\,,\\
\mu_{1v} &= svv\,, && h = \frac{17}{16}\,.
\end{aligned}
\tag{D.14}
$$

The S matrix in this basis is reported in table 2.

	$u_\pm$	u	$j_\pm$	$v_\pm$	v	$h_\pm$	$\rho_\pm$	ρ	$\sigma_{s\pm}$	$\tau_{s\pm}$	$\sigma_{I\pm}$	osv	ω_j^β	λ_j	α_l	β_l	γ_l	μ_{ms}	μ_{mJ}
$u_\pm$	1	2	3	1	2	3	$2\sqrt{2}$	$4\sqrt{2}$	6	6	$3\sqrt{2}$	$6\sqrt{2}$	8	$8\sqrt{2}$	± 6	± 6	$\pm 6\sqrt{2}$	± 12	$\pm 6\sqrt{2}$
u	2	4	6	2	4	6	$4\sqrt{2}$	$8\sqrt{2}$	12	12	$6\sqrt{2}$	$12\sqrt{2}$	-8	$-8\sqrt{2}$	0	0	0	0	0
$j_\pm$	3	6	9	3	6	9	$6\sqrt{2}$	$12\sqrt{2}$	-6	-6	$-3\sqrt{2}$	$-6\sqrt{2}$	0	0	± 6	± 6	$\pm 6\sqrt{2}$	∓ 12	$\mp 6\sqrt{2}$
$v_\pm$	1	2	3	1	2	3	$-2\sqrt{2}$	$-4\sqrt{2}$	6	6	$-3\sqrt{2}$	$-6\sqrt{2}$	8	$-8\sqrt{2}$	± 6	± 6	$\mp 6\sqrt{2}$	± 12	$\mp 6\sqrt{2}$
v	2	4	6	2	4	6	$-4\sqrt{2}$	$-8\sqrt{2}$	12	12	$-6\sqrt{2}$	$-12\sqrt{2}$	-8	$8\sqrt{2}$	0	0	0	0	0
$h_\pm$	3	6	9	3	6	9	$-6\sqrt{2}$	$-12\sqrt{2}$	-6	-6	$3\sqrt{2}$	$6\sqrt{2}$	0	0	± 6	± 6	$\mp 6\sqrt{2}$	∓ 12	$\pm 6\sqrt{2}$
$\rho_\pm$	$2\sqrt{2}$	$4\sqrt{2}$	$6\sqrt{2}$	$-2\sqrt{2}$	$-4\sqrt{2}$	$-6\sqrt{2}$	0	0	0	0	0	0	$\epsilon_\beta 8\sqrt{2}$	0	$\pm\epsilon_l 12$	$\mp\epsilon_l 12$	0	0	0
ρ	$4\sqrt{2}$	$8\sqrt{2}$	$12\sqrt{2}$	$-4\sqrt{2}$	$-8\sqrt{2}$	$-12\sqrt{2}$	0	0	0	0	0	0	$-\epsilon_\beta 8\sqrt{2}$	0	0	0	0	0	0
$\sigma_{s\pm}$	6	12	-6	6	12	-6	0	0	0	0	$6\sqrt{2}$	$-12\sqrt{2}$	0	0	$\pm\epsilon_l 6\sqrt{2}$	$\pm\epsilon_l 6\sqrt{2}$	$\pm\epsilon_l 12$	0	0
$\tau_{s\pm}$	6	12	-6	6	12	-6	0	0	0	0	$-6\sqrt{2}$	$12\sqrt{2}$	0	0	$\pm\epsilon_l 6\sqrt{2}$	$\pm\epsilon_l 6\sqrt{2}$	$\mp\epsilon_l 12$	0	0
$\sigma_{I\pm}$	$3\sqrt{2}$	$6\sqrt{2}$	$-3\sqrt{2}$	$-3\sqrt{2}$	$-6\sqrt{2}$	$3\sqrt{2}$	0	0	$6\sqrt{2}$	$-6\sqrt{2}$	$\epsilon_{IJ}12$	0	0	0	$\pm 6\sqrt{2}$	$\mp 6\sqrt{2}$	0	0	$\pm\epsilon_{IJ}$
osv	$6\sqrt{2}$	$12\sqrt{2}$	$-6\sqrt{2}$	$-6\sqrt{2}$	$-12\sqrt{2}$	$6\sqrt{2}$	0	0	$-12\sqrt{2}$	$12\sqrt{2}$	0	0	0	0	0	0	0	0	0
ω_i^α	8	-8	0	8	-8	0	$\epsilon_\alpha 8\sqrt{2}$	$-\epsilon_\alpha 8\sqrt{2}$	0	0	0	0	w_{ij}	$\epsilon_\alpha l_{ij}$	0	0	0	0	0
λ_i	$8\sqrt{2}$	$-8\sqrt{2}$	0	$-8\sqrt{2}$	$8\sqrt{2}$	0	0	0	0	0	0	0	$\epsilon_\beta l_{ij}$	0	0	0	0	0	0
α_k	± 6	0	± 6	± 6	0	± 6	$\pm\epsilon_k 12$	0	$\pm\epsilon_k 6\sqrt{2}$	$\pm\epsilon_k 6\sqrt{2}$	$\pm 6\sqrt{2}$	0	0	0	a_{kl}	a_{kl}	$\sqrt{2}a_{kl}$	0	0
β_k	± 6	0	± 6	± 6	0	± 6	$\mp\epsilon_k 12$	0	$\pm\epsilon_k 6\sqrt{2}$	$\pm\epsilon_k 6\sqrt{2}$	$\mp 6\sqrt{2}$	0	0	0	a_{kl}	a_{kl}	$-\sqrt{2}a_{kl}$	0	0
γ_k	$\pm 6\sqrt{2}$	0	$\pm 6\sqrt{2}$	$\mp 6\sqrt{2}$	0	$\mp 6\sqrt{2}$	0	0	$\pm\epsilon_k 12$	$\mp\epsilon_k 12$	0	0	0	0	$\sqrt{2}a_{kl}$	$-\sqrt{2}a_{kl}$	0	0	0
μ_{ns}	± 12	0	∓ 12	∓ 12	0	∓ 12	0	0	0	0	0	0	0	0	0	0	0	0	$\epsilon_{nmJ}12\sqrt{2}$
μ_{nI}	$\pm 6\sqrt{2}$	0	$\mp 6\sqrt{2}$	$\mp 6\sqrt{2}$	0	$\pm 6\sqrt{2}$	0	0	0	0	$\pm\epsilon_{IJ}$	0	0	0	0	0	0	$\epsilon_{nmI}12\sqrt{2}$	$\epsilon_{nm}12$

Table 2: Super-octahedron S matrix ($\times 48$). The sub-matrices are defined as follows: $w_{ij} = 16\mathrm{Re}\left(e^{-2i\pi/9}\omega^{i+j}\right)$, $l_{ij} = 16\sqrt{2}\mathrm{Re}\left(e^{-5i\pi/9}\omega^{i+j}\right)$, and $a_{kl} = 12\mathrm{Re}\left(e^{-i\pi/8}i^{k+l}\right)$, with indices $i, j = 0, 1, 2$; $k, l = 0, 1, 2, 3$, and $n, m = 0, 1$; the $\epsilon_{ijk\dots}$ are signs defined according to $\epsilon_{ijk\dots} = (-1)^{i+j+k+\cdots}$. The indices $\alpha, \beta = +, -$, in $\omega_i^\alpha, w_j^\beta$ and $I, J = o, v$, in $\sigma_{I\pm}$ and μ_{nI} should be considered as taking the values $0, 1$, respectively.

References

[1] A. Recknagel and V. Schomerus, *D-branes in Gepner models*, *Nucl. Phys.* **B 531** (1998) 185 [hep-th/9712186]; *Boundary deformation theory and moduli spaces of D-branes*, *Nucl. Phys.* **B 545** (1999) 233 [hep-th/9811237].

[2] A.Y. Alekseev and V. Schomerus, *D-branes in the WZW model*, *Phys. Rev.* **D 60** (1999) 061901 [hep-th/9812193];
G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, *The geometry of WZW branes*, *J. Geom. Phys.* **34** (2000) 162–190 [hep-th/9909030].

[3] I. Brunner, M.R. Douglas, A.E. Lawrence and C. Romelsberger, *D-branes on the quintic*, *J. High Energy Phys.* **08** (2000) 015 [hep-th/9906200].

[4] J. Maldacena, G.W. Moore and N. Seiberg, *Geometrical interpretation of D-branes in gauged WZW models*, *J. High Energy Phys.* **07** (2001) 046 [hep-th/0105038].

[5] For reviews, see: P. Di Vecchia and A. Liccardo, *D-branes in string theory. i*, hep-th/9912161;
M.R. Gaberdiel, *Lectures on non-BPS Dirichlet branes*, *Class. and Quant. Grav.* **17** (2000) 3483 [hep-th/0005029]; *D-branes from conformal field theory*, hep-th/0201113.

[6] J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, *Nucl. Phys.* **B 324** (1989) 581.

[7] G. Pradisi, A. Sagnotti and Y.S. Stanev, *Completeness conditions for boundary operators in 2D conformal field theory*, *Phys. Lett.* **B 381** (1996) 97 [hep-th/9603097].

[8] R.E. Behrend, P.A. Pearce, V.B. Petkova and J.-B. Zuber, *Boundary conditions in rational conformal field theories*, *Nucl. Phys.* **B 570** (2000) 525 [hep-th/9908036];
V.B. Petkova and J.B. Zuber, *The many faces of Ocneanu cells*, *Nucl. Phys.* **B 603** (2001) 449 [hep-th/0101151]; for a review, see: *Conformal boundary conditions and what they teach us*, hep-th/0103007.

[9] G. Pradisi, A. Sagnotti and Y.S. Stanev, *Planar duality in* SU(2) *WZW models*, *Phys. Lett.* **B 354** (1995) 279 [hep-th/9503207]; *The open descendants of nondiagonal* SU(2) *WZW models*, *Phys. Lett.* **B 356** (1995) 230 [hep-th/9506014].

[10] J. Fuchs and C. Schweigert, *A classifying algebra for boundary conditions*, *Phys. Lett.* **B 414** (1997) 251 [hep-th/9708141]; *Branes: from free fields to general backgrounds*, *Nucl. Phys.* **B 530** (1998) 99 [hep-th/9712257].

[11] J. Fuchs and C. Schweigert, *Symmetry breaking boundaries I: general theory*, *Nucl. Phys.* **B 558** (1999) 419 [hep-th/9902132]; *Symmetry breaking boundaries II: more structures, examples*, *Nucl. Phys.* **B 568** (2000) 543 [hep-th/9908025];
L. Birke, J. Fuchs and C. Schweigert, *Symmetry breaking boundary conditions and WZW orbifolds*, *Adv. Theor. Math. Phys.* **3** (1999) 671 [hep-th/9905038].

[12] M.R. Gaberdiel, A. Recknagel and G.M.T. Watts, *The conformal boundary states for* SU(2) *at level 1*, hep-th/0108102; *Conformal boundary states for free bosons and fermions*, *J. High Energy Phys.* **11** (2001) 016 [hep-th/0108238];
R.A. Janik, *Exceptional boundary states at* $c = 1$, *Nucl. Phys.* **B 618** (2001) 675 [hep-th/0109021].

[13] S. Elitzur and G. Sarkissian, *D-branes on a gauged WZW model*, *Nucl. Phys.* **B 625** (2002) 166 [hep-th/0108142].

[14] J. Fuchs, L.R. Huiszoon, A.N. Schellekens, C. Schweigert and J. Walcher, *Boundaries, crosscaps and simple currents*, *Phys. Lett.* **B 495** (2000) 427 [hep-th/0007174];
for a review, see: C. Schweigert, J. Fuchs and J. Walcher, *Conformal field theory, boundary conditions and applications to string theory*, hep-th/0011109.

[15] A.N. Schellekens and S. Yankielowicz, *Simple currents, modular invariants and fixed points*, *Int. J. Mod. Phys.* **A 5** (1990) 2903;
K.A. Intriligator, *Bonus symmetry in conformal field theory*, *Nucl. Phys.* **B 332** (1990) 541;
M. Kreuzer and A.N. Schellekens, *Simple currents versus orbifolds with discrete torsion: a complete classification*, *Nucl. Phys.* **B 411** (1994) 97 [hep-th/9306145].

[16] P. Ginsparg, *Curiosities at* $c = 1$, *Nucl. Phys.* **B 295** (1988) 153.

[17] L. J. Dixon, P. Ginsparg and J. A. Harvey, $c = 1$ *Superconformal field theory*, *Nucl. Phys.* **B 306** (1988) 470.

[18] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, *The operator algebra of orbifold models*, *Commun. Math. Phys.* **123** (1989) 485;
see also: V.G. Kac and I.T. Todorov, *Affine orbifolds and rational conformal field theory extensions of* $W_{1+infinity}$, *Commun. Math. Phys.* **190** (1997) 57 [hep-th/9612078].

[19] D. E. Diaconescu, M. R. Douglas and J. Gomis, *Fractional branes and wrapped branes*, *J. High Energy Phys.* **02** (1998) 013;
M. Billó, B. Craps and F. Roose, *Orbifold boundary states from Cardy's condition*, *J. High Energy Phys.* **01** (2001) 038 [hep-th/0011060].

[20] A. Cappelli, C. Itzykson and J.B. Zuber, *The ADE classification of minimal and* $A_1^{(1)}$ *conformal invariant theories*, *Commun. Math. Phys.* **113** (1987) 1.

[21] L.R. Huiszoon, A.N. Schellekens and N. Sousa, *Klein bottles and simple currents*, *Phys. Lett.* **B 470** (1999) 95 [hep-th/9909114]; *Open descendants of non-diagonal invariants*, *Nucl. Phys.* **B 575** (2000) 401 [hep-th/9911229].

[22] A. Sagnotti, *Non-perturbative quantum field theory*, in Cargèse '87, G. Mack et al. eds., Pergamon Press, Oxford 1988;
M. Bianchi and A. Sagnotti, *On the systematics of open string theories*, *Phys. Lett.* **B 247** (1990) 517;
For a review, see: A. Sagnotti and Y. S. Stanev, *Open descendants in conformal field theory*, *Fortschr. Phys.* **44** (1996) 585;
Y.S. Stanev, *Two dimensional conformal field theory on open and unoriented surfaces*, hep-th/0112222.

[23] For a textbook see: P. Di Francesco, P. Mathieu and D. Senechal, *Conformal field theories*, Springer-Verlag 1996.

[24] N. Ishibashi, *The boundary and crosscap states in conformal field theories*, *Mod. Phys. Lett.* **A 4** (1989) 251;
T. Onogi and N. Ishibashi, *Conformal field theories on surfaces with boundaries and crosscaps*, *Mod. Phys. Lett.* **A 4** (1989) 161 [Erratum-ibid. A **4** (1989) 885].

[25] E. Verlinde, *Fusion rules and modular transformations in 2-D conformal field theory*, *Nucl. Phys.* **B 300** (1988) 360.

[26] J. Fuchs, A.N. Schellekens and C. Schweigert, *A matrix S for all simple current extensions*, *Nucl. Phys.* **B 473** (1996) 323 [hep-th/9601078].

JHEP02(2002)039

[27] K. Matsubara, V. Schomerus and M. Smedback, *Open strings in simple current orbifolds*, hep-th/0108126.

[28] E.B. Kiritsis, *Proof of the completeness of the classification of rational conformal theories with* $c = 1$, *Phys. Lett.* **B 217** (1989) 427.

[29] C. Vafa, *Modular invariance and discrete torsion on orbifolds*, *Nucl. Phys.* **B 273** (1986) 592.

[30] See also: A.N. Schellekens and N. Sousa, *Open descendants of* $\mathrm{U}(2N)$ *orbifolds at rational radii*, *Int. J. Mod. Phys.* **A 16** (2001) 3659 [hep-th/0009100].

[31] D. Friedan, Z.-a. Qiu and S.H. Shenker, *Superconformal invariance in two-dimensions and the tricritical ising model*, *Phys. Lett.* **B 151** (1985) 37.

[32] P. Bantay, *Characters and modular properties of permutation orbifolds*, *Phys. Lett.* **B 419** (1998) 175 [hep-th/9708120].

[33] R. I. Nepomechie, *Consistent superconformal boundary states*, *J. Phys.* **A 34** (2001) 6509.

[34] M. Bianchi, G. Pradisi and A. Sagnotti, *Toroidal compactification and symmetry breaking in open string theories*, *Nucl. Phys.* **B 376** (1992) 365.

[35] I. Brunner, *On orientifolds of WZW models and their relation to geometry*, *J. High Energy Phys.* **01** (2002) 007 [hep-th/0110219];
L.R. Huiszoon, K. Schalm and A.N. Schellekens, *Geometry of WZW orientifolds*, *Nucl. Phys.* **B 624** (2002) 219 [hep-th/0110267];
C. Bachas, N. Couchoud and P. Windey, *Orientifolds of the 3-sphere*, *J. High Energy Phys.* **12** (2001) 003 [hep-th/0111002];
N. Couchoud, *D-branes and orientifolds of* SO(3), hep-th/0201089.