

# Classical scattering in 2 + 1 gravity with $N$ spinning sources

Andrea Cappelli<sup>1</sup>

*Theory Division, CERN, CH-1211 Geneva 23, Switzerland*

Marcello Ciafaloni and Paolo Valtancoli

*Dipartimento di Fisica, Università di Firenze and INFN, Sezione di Firenze, I-50125 Florence, Italy*

Received 8 October 1991

The classical dynamics of  $N$  spinning point sources in 2 + 1 Einstein–Cartan gravity is considered. It corresponds to the ISO(2, 1) Chern–Simons theory, in which the torsion source is restricted to its intrinsic spin part. A class of explicit solutions is found for the dreibein and the spin connection, which are torsionless in the spinless limit. By using the residual local Poincaré invariance of the solutions, we fix the gauge so that the metric is smooth outside the particles and satisfies proper asymptotic conditions at space and time infinity. We recover previous results for test bodies and find new ones for the scattering of two dynamical particles in the massless limit.

## 1. Introduction

In a recent paper<sup>#1</sup> [1], we have investigated the two-body classical scattering problem in 2 + 1 gravity [2–10], and we have provided an all-orders solution for the scattering angle of two massless and spinless particles in a properly defined center-of-mass frame. The basic observation was that the Einstein theory corresponds to a gauge fixing of the Chern–Simons theory of the Poincaré group [6], in which the metric is regular outside the particle trajectories, and satisfies proper asymptotic conditions at space and time infinity.

The solution of the Einstein theory just mentioned was based on a class of explicit  $N$ -particle singular solutions to the Chern–Simons theory. Similar solutions have been found independently in the literature [11–13], sometimes including spin. Some of them [11,13] are not immediately useful for the Einstein theory, because they contain torsion in the spinless limit.

In this note, we wish to generalize both the  $N$ -par-

ticle solution and the classical scattering gauge fixing to the case of non-vanishing intrinsic spin. Spinning particles can be thought of as localized sources of torsion in the gravitational field equations. Their introduction is natural in a first-order formalism and in a Chern–Simons framework, in which the spin connection is coupled to the angular momentum  $J^a$ .

However, the consequences of this treatment for the Einstein theory are not yet fully understood, and may lead to conflict with causality because of the appearance of time-like closed loops near to the torsion source [14]. Here we find that the metric theory reconstructed from our solution can also be interpreted as a torsionless Einstein theory, but with a very singular energy–momentum tensor [15,16]. Our final result will be that the classical scattering occurs at the same scattering angle, but with some time delay, with respect to its spinless counterpart.

## 2. Equations of motion and Einstein–Cartan theory

To begin with, we shall write the Chern–Simons action with sources,  $S = S_{CS} + S_m$ , as

<sup>1</sup> On leave of absence from INFN, Sezione di Firenze, I-50125 Florence, Italy.

<sup>#1</sup> Hereafter referred as I.

$$S = -\frac{1}{2} \int d^3x \epsilon^{\mu\nu\rho} \epsilon_{abc} e^a{}_\rho (\partial_{[\mu} \omega_{\nu]} + \omega_{[\mu} \omega_{\nu]})^{bc} - 2 \sum_{(r)} \int d\tau [\dot{\xi}^\mu (P_a e^a{}_\mu - \frac{1}{2} J_a \epsilon^{abc} \omega_{\mu,bc})]_{(r)}, \quad (1)$$

where  $P_{(r)}^a$ ,  $J_{(r)}^a$  are the internal momentum and angular momentum of the  $r$ th particle ( $r=1, \dots, N$ ), and the dreibein  $e^a{}_\mu$  and the spin connection  $\omega_{\mu b}^a$  are the components of the ISO(2, 1) gauge connection [6].

The field equations corresponding to the action (1) are given by

$$(\partial_{[\mu} \omega_{\nu]} + \omega_{[\mu} \omega_{\nu]})^a{}_b = -\epsilon_{\mu\nu\lambda} \sum_{(r)} v_{(r)}^\lambda \epsilon^a{}_{bc} P_{(r)}^c \delta^2(x - \xi_{(r)}(t)), \quad (2)$$

$$(\partial_{[\mu} e_{\nu]} + \omega_{[\mu} e_{\nu]})^a = \epsilon_{\mu\nu\lambda} \sum_{(r)} v_{(r)}^\lambda J_{(r)}^a \delta^2(x - \xi_{(r)}(t)), \quad (3)$$

where the momenta of the various particles are  $p_{(r)}^\mu = m_{(r)} d\xi_{(r)}^\mu/d\tau \equiv m_{(r)} \dot{\xi}_{(r)}^\mu$ ,  $v_{(r)}^\mu \equiv \dot{\xi}_{(r)}^\mu/\dot{\xi}_{(r)}^0$ . They show both an energy-momentum source and a torsion source.

The Bianchi identities for eqs. (2), (3) yield the covariant conservation constraints for the particle variables, which are

$$\dot{P}_{(r)}^a + \dot{\xi}_{(r)}^\mu \omega_{\mu b}^a P_{(r)}^b = 0, \quad (4)$$

$$\frac{dJ_{(r)}^a}{d\tau} + \dot{\xi}_{(r)}^\mu (\omega_{\mu b}^a J_{(r)}^b - \epsilon^a{}_{bc} P_{(r)}^c e^b{}_\mu) = 0. \quad (5)$$

As explained in I, we construct a metric theory of Einstein type from the solution to eqs. (2)–(5) by the “soldering conditions”

$$g_{\mu\nu} = e^a{}_\mu \eta_{ab} e^b{}_\nu, \quad (6)$$

$$m_{(r)} e^a{}_\mu (\dot{\xi}_{(r)}) \dot{\xi}_{(r)}^\mu = P_{(r)}^a, \quad (7)$$

which define the metric tensor  $g_{\mu\nu}$  and the relation between Lorentz and space-time momenta.

At this point, one should distinguish between the usual Levi-Civita connection deduced from eq. (6)

$$\Gamma_{\lambda,\mu\nu} = \frac{1}{2} (\partial_{(\mu} g_{\nu)\lambda} - \partial_\lambda g_{\mu\nu}), \quad (8)$$

which is symmetric in  $\mu$  and  $\nu$ , and the one reconstructed from the spin connection  $\omega_\mu$  in eqs. (2), (3), i.e.,

$$\tilde{\Gamma}_{\lambda,\mu\nu} = e_{a\lambda} (\eta^{ab} \partial_\mu + \omega_\mu^{ab}) e_{b\nu}, \quad (9)$$

which is not symmetric, due to the existence of the torsion in eq. (3). Notice that the parallel transport based on  $\tilde{\Gamma}$  also preserves the metric and thus the corresponding theory is of Einstein–Cartan type [15]. In general,  $\Gamma$  and  $\tilde{\Gamma}$  are not equal, but related by the identity

$$\Gamma_{\lambda,\mu\nu} - \tilde{\Gamma}_{\lambda,\mu\nu} = \frac{1}{2} (\tilde{\Gamma}_{\mu,[\nu\lambda]} + \tilde{\Gamma}_{\nu,[\mu\lambda]} + \tilde{\Gamma}_{\lambda,[\nu\mu]}), \quad (10)$$

where the right-hand side is given by eq. (3) in terms of the torsion tensor

$$T_{\lambda,\mu\nu} \equiv \tilde{\Gamma}_{\lambda,[\mu\nu]} = e_{a\lambda} \epsilon_{\mu\nu\rho} \sum_{(r)} v_{(r)}^\rho J_{(r)}^a \delta^2(x - \xi_{(r)}), \quad (11)$$

which is non-vanishing at the localized spin sources.

A few consequences follow from eqs. (10) and (11). First, in the spinless case we must have  $\Gamma = \tilde{\Gamma}$ , and thus no torsion. It follows that we cannot allow orbital angular momentum terms in  $J_{(r)}^a$ . Thus, we perform a Poincaré gauge choice by identifying  $J_{(r)}$  with its intrinsic component  $S_{(r)}$  as follows:

$$J_{(r)}^a = S_{(r)}^a \equiv \frac{\sigma_{(r)}}{m_{(r)}} P_{(r)}^a, \quad (12)$$

instead of using the more general form

$$J_{(r)}^a = \frac{\sigma_{(r)}}{m_{(r)}} P_{(r)}^a + \epsilon^a{}_{bc} b_{(r)}^b P_{(r)}^c. \quad (13)$$

Note that in the three-dimensional Poincaré group,  $S^a$  is proportional to the momentum in terms of the spin scalar  $\sigma_{(r)}$ . Conversely, if we start from eq. (13) in the Chern–Simons theory, it is always possible to perform a Poincaré gauge change of the form

$$\omega_\mu \rightarrow \omega_\mu, \quad e_\mu \rightarrow e_\mu + (\partial_\mu + \omega_\mu) b(x), \quad b(\xi_{(r)}) = b_{(r)}, \quad (14)$$

so as to eliminate the orbital part, leaving the Casimir  $(J_{(r)} P_{(r)}) = m_{(r)} \sigma_{(r)}$  invariant. However, only the choice (12) defines, by eqs. (6) and (7), a sensible Einstein theory having  $\Gamma = \tilde{\Gamma}$  for  $\sigma_{(r)} \rightarrow 0$ , as explained previously<sup>#2</sup>.

<sup>#2</sup> Thus we differ from refs. [11,13], which do not distinguish between orbital and spin parts. Another reason to restrict the torsion source is to obtain equivalence between Poincaré gauge transformations and diffeomorphisms, as discussed by Witten [6].

The second observation is that, even if  $\sigma_{(r)} \neq 0$ ,  $\Gamma$  and  $\tilde{\Gamma}$  coincide outside the particle trajectories, because the torsion is localized (eq. (11)). Thus, all consequences for parallel transporting a vector around the sources with the  $\Gamma$ -connection can be obtained with  $\tilde{\Gamma}$  as well. Furthermore, it is easy to check that  $(\Gamma_{\mu\nu}^{\lambda} - \tilde{\Gamma}_{\mu\nu}^{\lambda})\dot{\xi}_{(r)}^{\mu}\dot{\xi}_{(r)}^{\nu}$  vanishes on the particle sites, so that  $\Gamma$  and  $\tilde{\Gamma}$  are also equivalent for the particle motion.

Moreover, the Einstein equations for  $\Gamma$  are, of course, torsionless, but contain additional terms in the energy-momentum tensor with respect to eq. (2), because of the right hand side of eq. (10). Such terms from the spin sources have  $\delta'$ -singularities, which can be interpreted as localized sources of orbital angular momentum [16].

### 3. *N*-particle solution

Let us now construct the solution to eqs. (2)–(5) along the lines of I. To begin with, the particle motion is parametrized in terms of three arbitrary functions  $X^a(x^\mu)$  by the trajectory equations

$$\begin{aligned} X^2(\xi_{(r)}^\mu) &= B_{(r)} = \text{const.}, \\ X^1(\xi_{(r)}^\mu) &= V_{(r)} X^0(\xi_{(r)}^\mu) \end{aligned} \quad (15)$$

with the requirement

$$j_{(r)} = \left| \frac{\partial(X^1 - V_{(r)} X^0, X^2)}{\partial(x^1, x^2)} \right|_{(r)} > 0. \quad (16)$$

Here we notice that, in our spinning solution, some combinations of the  $X^a$ 's (actually  $X^0 - V_{(r)} X^1$ ) will acquire a singularity, at the particle sites. However, the combinations in eq. (15) turn out to be well defined, so that no problem arises.

The solutions to eqs. (2)–(5) for  $\omega$  and  $e$  in the gauge (12) are characterized by constant  $P_{(r)}^a$  and are functions of  $X^a(x)$ . The spin connection has the same form as in the case of spinless sources (I), i.e.,

$$\begin{aligned} \omega_\mu &= \sum_{(r)} \omega_\mu^{(r)}, \\ (\omega_\mu^{(r)})^a{}_b &= \epsilon^a{}_{bc} P_{(r)}^c (\partial_\mu X^2) \delta(X^2 - B_{(r)}) \\ &\quad \times \Theta(V_{(r)} X^0 - X^1) \\ &\equiv \epsilon^a{}_{bc} P_{(r)}^c f_{\mu(r)}, \end{aligned} \quad (17)$$

for tails running on the left.

The dreibein contains instead an additional term related to the torsion source,

$$\begin{aligned} e^a{}_\mu &= \partial_\mu X^a \\ &\quad + \sum_{(r)} (\omega_\mu^{(r)})^a{}_b (X^b - B_{(r)}^b) \\ &\quad - \sum_{(r)} \frac{\sigma_{(r)}}{m_{(r)}} P_{(r)}^a f_{\mu(r)}, \end{aligned} \quad (18)$$

where we have explicitly used the intrinsic spin identification in eq. (12).

We first need to check eq. (3), which is actually linear in  $e^a{}_\mu$ . Therefore, the additional spin term in eq. (18) directly gives rise to the torsion source, as in eq. (2) for the spin connection, because the quadratic terms vanish again. For  $\sigma \rightarrow 0$ , eq. (18) reduces, of course, to the (non-trivial) solution of the torsionless case, discussed in I, which already provides the relevant orbital angular momentum terms due to the translational parameters  $B_{(r)}$ .

Next, we consider the covariant conservation constraints. Eq. (4) is trivially verified by a constant  $P^a$ , because  $(1 + \omega_\mu)$  leaves it invariant (I). Eq. (5) is also satisfied for constant  $J^a$ , parallel to  $P^a$ , because the last term drops out, owing to eq. (7). The consistency equation (7) is less trivial, due to the extra singularities introduced by torsion. In fact, by using the explicit form of  $e^a{}_\mu$  in eq. (18) we obtain, for the  $r$ th particle,

$$\begin{aligned} P_{(r)}^a &= \left( m_{(r)} \frac{d}{d\tau} X^a(x) \right. \\ &\quad \left. - \sigma_{(r)} P_{(r)}^a \Theta(V_{(r)} X^0(x) - X^1(x)) \right. \\ &\quad \left. \times \frac{d}{d\tau} \Theta(X^2(x) - B_{(r)}) \right)_{x=\xi_{(r)}(\tau)}. \end{aligned} \quad (19)$$

Since  $P^a$  is constant by hypothesis, eq. (19) shows that  $X^a$  is ill-defined at the particle site by a quantity proportional to  $\sigma P^a/m$ , if the limit  $x \rightarrow \xi$  is performed on the tail side. Fortunately, the ill-defined term cancels for the quantities  $(X^1 - V_{(r)} X^0)$  and  $(X^2 - B_{(r)})$  (which are still defined and vanish on the trajectory), provided we set

$$V_r = \frac{P_{(r)}^1}{P_{(r)}^0}. \quad (20)$$

On the other hand, the combination  $(X^0 - V_{(r)}X^1)$  possesses a discontinuity on the tail given by

$$\Delta(\gamma(V_{(r)})(X^0 - V_{(r)}X^1)) = -\sigma_{(r)}, \quad (21)$$

which corresponds to the well-known time jump [2]. Such pathology of the  $X$ -variable will disappear in the regular  $x$ -variable to be discussed below.

The time jump feature is confirmed from the analysis of the matching conditions for geodesics crossing the tail. The Einstein spacetime is defined by eqs. (6), (8), but eq. (10) implies that the geodesics  $x^\mu(\tau)$  going across the tail, but not hitting the particle, are still determined by the Chern–Simons connection  $\tilde{F}$ . Thus they satisfy the first-order equation

$$\frac{d}{d\tau}(e^a{}_\nu \dot{x}^\nu) + \dot{x}^\mu \omega_{\mu}^a{}_b(e^b{}_\nu \dot{x}^\nu) = 0. \quad (22)$$

By integrating twice as in I, we obtain for a point just above ( $X_+$ ) and just below ( $X_-$ ) the tail (running on the left of the particle),

$$(X_- - X(\xi_{(r)}))^a = L_{(r)}^a{}_b \left( X_+ - \frac{\sigma_{(r)}}{m_{(r)}} P_{(r)} - X(\xi_{(r)}) \right)^b, \quad (23)$$

$$L_{(r)} = \exp(-P_{(r)}^a \cdot \mathcal{J}_a) \quad (r\text{th tail}) \quad (24)$$

(using the representation  $(\mathcal{J}_a)^b{}_c = \epsilon^b{}_{ac}$ ). This relation shows the well-known matching condition of Deser, Jackiw and 't Hooft [2], including its spacetime jump.

Let us finally note that our gauge choice  $J_{(r)} = ((\sigma/m)P)_{(r)}$  is consistent in the case of non-parallel velocities, i.e. when the particle (1) crosses the tail of particle (2). A rigorous discussion of this point needs the analysis of the Poincaré holonomies (see below). However, the final result is a simple consequence of the conservation constraints (4), (5), because, by eq. (7), the last term of eq. (5) vanishes identically. Thus, at the tail crossing, the  $\omega_2$ -term rotates both  $P_{(1)}$  and  $J_{(1)}$  in the same way, so that they are still proportional after crossing.

#### 4. Holonomies and particle exchanges

From the Chern–Simons point of view, only topological quantities are observable, and come from the invariants of Poincaré holonomies around the parti-

cles. The single-particle holonomies, already computed in I, are given by

$$U_{(r)}(x_{(r)}, x_{(r)}) = P \exp \left( - \oint_{\xi_{(r)} + \epsilon} \omega \cdot \mathcal{J} + e \cdot \mathcal{P} \right) \\ = \begin{pmatrix} \exp(-P_{(r)} \cdot \mathcal{J}) & -(\sigma_{(r)}/m_{(r)}) P_{(r)} \\ 0 & 1 \end{pmatrix}, \quad (25)$$

where we used the  $4 \times 4$  representation for the generators  $\mathcal{J}_a, \mathcal{P}_a$  of the Poincaré group and the gauge choice (12). By a gauge transformation at the basepoint  $g = g(x_{(r)})$ ,  $U_{(r)} \rightarrow g U_{(r)} g^{-1}$ , and its invariants are given by

$$m_{(r)} \leftrightarrow \text{Tr}[\exp(-P_{(r)} \cdot \mathcal{J})] = 1 + 2 \cos m_{(r)}, \quad (26)$$

$$\sigma_{(r)} \leftrightarrow J_{(r)} \cdot P_{(r)} = \sigma_{(r)} m_{(r)}. \quad (27)$$

Thus the  $U_{(r)}$  invariants correspond to the constants of motion of the  $r$ th particle, i.e. its mass and spin.

The two-particle system is characterized by the O-loop, i.e. by the holonomy encircling once (counterclockwise) both particles. Taking into account the translation between the two particles of impact parameters  $B_{(1)}$  and  $B_{(2)}$  and using the result (25) for each particle, we obtain

$$U_O = P \exp \left( - \oint_{1+2} \omega \cdot \mathcal{J} + e \cdot \mathcal{P} \right) \\ = \begin{pmatrix} \exp(\omega \cdot \mathcal{J}) & q \\ 0 & 1 \end{pmatrix}, \quad (28)$$

with invariants

$$\cos \frac{1}{2} \mathcal{M} = \cos \frac{1}{2} \sqrt{\omega^2} = c_{(1)} c_{(2)} - s_{(1)} s_{(2)} \frac{P_{(1)} \cdot P_{(2)}}{m_{(1)} m_{(2)}}, \\ \mathcal{S} = \frac{\omega \cdot q}{\mathcal{M}} = (\sin \frac{1}{2} \mathcal{M})^{-1} \\ \times \left[ 2 s_{(1)} s_{(2)} \epsilon_{abc} (B_{(1)} - B_{(2)})^a \frac{P_{(1)}^b P_{(2)}^c}{m_{(1)} m_{(2)}} \right. \\ \left. + \sigma_{(1)} \left( c_{(2)} s_{(1)} - c_{(1)} s_{(2)} \right) \frac{P_{(1)} \cdot P_{(2)}}{m_{(1)} m_{(2)}} \right. \\ \left. + \sigma_{(2)} \left( c_{(1)} s_{(2)} - c_{(2)} s_{(1)} \right) \frac{P_{(1)} \cdot P_{(2)}}{m_{(1)} m_{(2)}} \right], \quad (29)$$

where  $c_{(i)} = \cos \frac{1}{2} m_{(i)}$ ,  $s_{(i)} = \sin \frac{1}{2} m_{(i)}$ . In the special

relativity limit ( $m_{(i)} \equiv 8\pi G m_{(i)} \ll 1$ ), eq. (29) reduces to

$$\begin{aligned} \mathcal{M}^2 &= (P_{(1)} + P_{(2)})^a \eta_{ab} (P_{(1)} + P_{(2)})^b, \\ \mathcal{S} &= \frac{(P_{(1)} + P_{(2)})^a}{\mathcal{M}} \eta_{ab} [S_{(1)}^a + S_{(2)}^a \\ &\quad + \epsilon^a{}_{bc} (B_{(1)}^b P_{(1)}^c + B_{(2)}^b P_{(2)}^c)], \end{aligned} \quad (30)$$

consistent with the usual interpretation of the total mass and total angular momentum.

As discussed in I, the time evolution can lead to the collisions of two particles, which is represented by an “exchange” operation  $\sigma_{12}$ , corresponding to particle (2) crossing the tail of particle (1) (and vice versa for  $\sigma_{21}$ ). This exchange operation can be studied as in I by means of the non-abelian Stokes theorem for the Poincaré holonomies, and has the presentation

$$\sigma_{12}: U_{(1)} \rightarrow U_{(1)}, \quad U_{(2)} \rightarrow U_{(1)} U_{(2)} U_{(1)}^{-1}, \quad (31)$$

where the  $U_{(r)}$ 's are the single-particle holonomies (25) referred to a common basepoint, with a prescribed tail orientation, e.g. to the left of all particles<sup>#3</sup>.

By taking into account the spin part in the  $U_{(r)}$ 's, one obtains from (31) the following transformation properties:

$$\begin{aligned} \sigma_{12}: B_{(1)} &\rightarrow B_{(1)}, \\ B_{(2)} &\rightarrow B_{(1)} + L_{(1)}(B_{(2)} - B_{(1)}) - S_{(1)}, \\ P_{(1)} &\rightarrow P_{(1)}, \quad P_{(2)} \rightarrow L_{(1)} P_{(2)}, \\ S_{(1)} &\rightarrow S_{(1)}, \quad S_{(2)} \rightarrow L_{(1)} S_{(2)}. \end{aligned} \quad (32)$$

The first two equations are indeed equal to the geodesic limit for particle (2) discussed before (eq. (23)), and the third one implies that  $S_{(2)}$  stays parallel to  $P_{(2)}$ , as anticipated. Therefore, our gauge choice eq. (12) is consistent with the most general motion of the particles.

## 5. Smooth spacetimes and scattering properties

Let us finally discuss the spacetime corresponding to our solution in the Einstein theory, as defined by

<sup>#3</sup> The monodromy operation, given by  $\sigma_{12}\sigma_{21}$ , is instead gauge invariant.

eqs. (6) and (7). On physical grounds, we require the metric (6) to be (i) regular outside the particle trajectories, as explained in I. Furthermore, we impose (ii) asymptotic conditions defining single-particle and two-particle states at large negative times, and (iii) forbid rotating frames at spatial infinity.

The above conditions can be enforced by using the residual gauge freedom of our solutions, given by Poincaré transformations leaving  $\{P^a, S^a = (\sigma/m)P^a\}$  invariant, i.e. by (pseudo-)rotations around  $P^a$  and translations along  $P^a$ . We can thus look for gauge fixing on the coordinates  $X^A = \bar{X}^A$  ( $A=0, 1, 2, 3$ ), defined in the  $4 \times 4$  representation by

$$\bar{X}^A = (\mathcal{T}(x))^A{}_B x^B, \quad (33)$$

where the local Poincaré transformation  $\mathcal{T}$  is chosen so as to build a solution of the matching conditions around each particle site. Thus, for each ( $r$ ), we require

$$\mathcal{T} \simeq \mathcal{T}_{(r)}(x) S_{(r)}(x), \quad \text{for } x^\mu \simeq \xi_{(r)}^\mu(\tau), \quad (34)$$

where  $S_{(r)}$  is regular for  $x^\mu \rightarrow \xi_{(r)}^\mu$ . For a particle passing through the origin,  $\mathcal{T}$  performs the change of variables

$$\begin{aligned} \bar{X}^a &= A^a{}_b x^b + \frac{\sigma_{(r)}}{m_{(r)}} P_{(r)}^a \frac{\varphi_{(r)}^0(x)}{2\pi}, \\ A &\equiv \exp\left(\frac{\varphi_{(r)}^0(x)}{2\pi} P_{(r)} \cdot \mathcal{J}\right), \end{aligned} \quad (35)$$

$$\begin{aligned} \tan \varphi_{(r)}^0 &= \frac{1}{\gamma_{(r)}} \frac{y}{x - V_{(r)} t}, \\ P_{(r)}^\mu &= m(\gamma, V\gamma, 0)|_{(r)}, \end{aligned} \quad (36)$$

where  $\varphi_{(r)}^0$  is the azimuthal angle in the  $r$ th particle rest frame. The transformation (35) differs from the one given in I only for the spin-dependent translation, which reproduces the time jump in  $X$ -variables as  $\varphi_{(r)}^0$  varies from  $-\pi$  to  $+\pi$ , eq. (23), such that the  $x$ -variables are continuous everywhere, and fulfills the first requirement above.

If we have only one particle, it is easy to provide the explicit form of the dreibein resulting from the gauge fixing (35) of eq. (18), i.e.,

$$e^a{}_\mu = (\partial_\mu + \omega_\mu)^a{}_b \bar{X}^b - \frac{\sigma}{m} P^a f_\mu$$

$$= A^a{}_b \left[ \delta_\mu^b + \left( (P \cdot \mathcal{J})^b{}_c x^c + \frac{\sigma}{m} P^b \right) \frac{1}{2\pi} \partial_\mu \phi^0 \right]. \quad (37)$$

Notice that the explicit singular parts of  $\omega_\mu$  occurring in  $e^a{}_\mu$  have been cancelled by the singular derivative terms of the  $\bar{X}^a$ -variable in (35). More explicitly we obtain

$$e^a{}_\mu = A^a{}_b \left( \delta_\mu^b + \frac{m}{2\pi} \frac{N^b N_\mu}{|N^2|} + \frac{\sigma}{2\pi} \frac{P^b}{m} \frac{N_\mu}{|N^2|} \right),$$

$$N^a(P) = \left( \frac{P}{m} \cdot \mathcal{J} \right)^a{}_b x^b = \gamma(Vy, y, -x + Vt), \quad (38)$$

and, therefore, from eq. (6)

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{N_\mu N_\nu}{|N^2|} (1 - \alpha^2) + \frac{N_\mu N_\nu}{N^4} \left( \frac{\sigma}{2\pi} \right)^2$$

$$+ \frac{\sigma}{2\pi |N^2|} \left( N_\mu \frac{p_\nu}{m} + N_\nu \frac{p_\mu}{m} \right), \quad (39)$$

where  $\alpha = 1 - m/2\pi$ . This metric reduces to the well-known one of the spinning cone [2,5] in the static case ( $\phi^0 \equiv \phi$ ), i.e.,

$$ds^2 = \left( dt + \frac{\sigma}{2\pi} d\phi \right)^2 - dr^2 - r^2 \alpha^2 d\phi^2. \quad (40)$$

We can also study the massless limit in eq. (39), by setting

$$S^a = \lambda P^a \quad (m=0), \quad (41)$$

i.e.  $\sigma = m\lambda$ , where the (Poincaré) scalar  $\lambda$  is kept fixed, while the three-dimensional “helicity”  $\mathbf{S} \cdot \mathbf{P} / |\mathbf{P}| = \lambda E$  transforms as the zeroth component of a vector. In this limit, we obtain the “spinning” Aichelburg–Sexl metric [17]

$$ds^2 = 2 du dv - (dy)^2$$

$$+ \sqrt{2} E \delta(u) [ |y| + \lambda \text{sign}(y) ] (du)^2, \quad (42)$$

where

$$u = \frac{1}{\sqrt{2}} (t - x), \quad v = \frac{1}{\sqrt{2}} (t + x)$$

are light-cone variables.

Next, we consider the test particle scattering in the

one-particle metric given above. From eqs. (39) and (42), one sees that the spin modifications do not affect the scattering angle, because they are negligible at large distances; they do, however, give a time delay with respect to the spinless case of I. This can be seen more explicitly from the geodesic trajectory in the smooth metric produced by a particle of momentum  $P^a$  and spin  $\sigma$ , namely

$$x^a(\tau) = - \frac{\sigma}{m} P^a \frac{\phi^0(\tau)}{2\pi\alpha}$$

$$+ \exp \left( - \frac{\phi^0(\tau)}{2\pi\alpha} P \cdot \mathcal{J} \right)^a{}_b (U^b \tau + B^b), \quad (43)$$

where  $\phi^0(\tau) \equiv \alpha\phi^0$  is the azimuthal variable of the geodesic in minkowskian  $X$ -variables  $X(\tau) = U\tau + B$  and in the particle rest frame. During the evolution,  $\phi^0$  changes from  $\phi^0 = 0$  to  $\phi^0 = \pi$ , so that the classical “scattering matrix” has a Lorentz transformation and a translation given by

$$L(P)^{\pm 1/2\alpha}, \quad \Delta x^\mu = \mp \frac{\sigma}{2\alpha} \frac{p^\mu}{m}, \quad (44)$$

for geodesics running above (below) the spinning particle. In the static limit, the translation reduces to the geodesic “time delay” already found in ref. [5].

Finally, the scattering problem for two particles is set up along the same lines as in I, the only difference being that one has to keep track of the spacetime jump associated with the  $\mathcal{A}$ -transformation. For the massless case, by imposing the absence of impulsive rotations and translations at infinity, a condition follows for the collision at  $t=0$  of two Aichelburg–Sexl metrics. This yields the scattering parameters [18]

$$p_f = R(\theta) p_i, \quad \tan \frac{1}{2}\theta = \frac{1}{4}\sqrt{s},$$

$$\Delta x^\mu = - \left[ \frac{1}{2} (\lambda_1 p_1 + \lambda_2 p_2) \epsilon(b) + \frac{1}{4} |b| (p_1 + p_2) \right]$$

$$+ O(s), \quad (45)$$

in the center-of-mass defined in I. Eq. (45) shows the spacetime jump for the scattering of two dynamical particles, whose final (initial) momenta are denoted by  $p_f$  ( $p_i$ ),  $p_{(1)} = (E, E, 0)$ ,  $p_{(2)} = (E, -E, 0)$ ,  $s = (p_{(1)} + p_{(2)})^2$  and  $\epsilon = +1$  ( $-1$ ) corresponds to impact parameters of positive (negative) angular momenta.

## References

- [1] A. Cappelletti, M. Ciafaloni and P. Valtancoli, Classical scattering in 2+1 gravity with  $N$  point sources, preprint CERN-TH-6093/91 (May 1991).
- [2] S. Deser, R. Jackiw and G. 't Hooft, *Ann. Phys. (NY)* 152 (1984) 220.
- [3] G. 't Hooft, *Commun. Math. Phys.* 117 (1988) 685.
- [4] S. Deser and R. Jackiw, *Commun. Math. Phys.* 118 (1988) 495.
- [5] P. Sousa Gerbert and R. Jackiw, *Commun. Math. Phys.* 124 (1989) 229.
- [6] E. Witten, *Nucl. Phys. B* 311 (1988–89) 46; *B* 323 (1989) 113.
- [7] S. Carlip, *Nucl. Phys. B* 324 (1989) 106.
- [8] P. Sousa Gerbert, *Nucl. Phys. B* 346 (1990) 440.
- [9] K. Koehler, F. Mansouri, C. Vaz and L. Witten, *Nucl. Phys. B* 348 (1991) 373.
- [10] A. Cappelletti, talk at Workshop on Random surfaces and 2D quantum gravity (Barcelona, June 1991), preprint CERN-TH-6249/91, *Nucl. Phys. B (Proc. Suppl.)*, to appear.
- [11] K. Gupta and A. Stern, Spin and statistics in 2+1 gravity, Syracuse preprint SU-4228-463 (1991).
- [12] G. Grignani and G. Nardelli, Gravity in 2+1 dimensions coupled to point-like sources: A flat Chern–Simons gauge theory experimental to Einstein, preprint MIT CPT-1953 (1991).
- [13] C. Vaz and L. Witten, On the multiparticle phase space in Chern–Simons gravity, preprint UCTP-103/91.
- [14] G. Clement, *Intern. J. Theor. Phys.* 24 (1985) 267; S. Deser, R. Jackiw and G. 't Hooft, Physical cosmic strings do not generate closed time-like curves, MIT preprint CPT-2011 (1991).
- [15] F.W. Hehl, P. Heyde and G.D. Kerlick, *Rev. Mod. Phys.* 48 (1976) 393.
- [16] W. Arkuszewski, W. Kopczynski and V.N. Ponomarev, *Ann. Inst. H. Poincaré A* 21 (1974) 89.
- [17] P.C. Aichelburg and R.U. Sexl, *Gen. Rel. Grav.* 2 (1971) 303; T. Dray and G. 't Hooft, *Nucl. Phys. B* 253 (1985) 173.
- [18] A. Cappelletti, M. Ciafaloni and P. Valtancoli, to appear.