Matrix model description of Laughlin Hall states

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Abstract. We analyse Susskind's proposal of applying the non-commutative Chern–Simons theory to the quantum Hall effect. We study the corresponding regularized matrix Chern–Simons theory introduced by Polychronakos. We use holomorphic quantization and perform a change of matrix variables that solves the Gauss law constraint. The remaining physical degrees of freedom are the complex eigenvalues that can be interpreted as the coordinates of electrons in the lowest Landau level with Laughlin's wavefunction. At the same time, a statistical interaction is generated among the electrons that is necessary to stabilize the ground state. The stability conditions can be expressed as the highest-weight conditions for the representations of the W_∞ algebra in the matrix theory. This symmetry provides a coordinate-independent characterization of the incompressible quantum Hall states.

Keywords: integrable quantum field theory, fractional QHE (theory)

J. Stat. Mech. (2005) P05001

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Contents

1. Introduction

In 2001 Susskind wrote an interesting paper [1] where he suggested that the non-commutative Chern–Simons theory could describe the Laughlin incompressible fluids [2] in the fractional quantum Hall effect [3]. His work was inspired by the analogies between the physics of electrons in a strong magnetic field and the properties of D-branes in string theory [4]. In his argument, Susskind derived the semiclassical theory of the incompressible fluids in a magnetic field and showed that it corresponds to the non-commutative theory in the limit of small θ (high density); then, he proposed the fully quantum, non-commutative theory for the Laughlin Hall states made of discrete electrons.

Several authors [5]–[11] have then addressed the question of whether the non-commutative Chern–Simons theory really realizes the Laughlin wavefunctions and the physics of anyons [12]. If the answer is positive, this theory would provide a promising approach for understanding the open issues of the quantum Hall effect, in the form of an 'effective non-relativistic theory'. Moreover, it would offer an interesting physical realization of the new mathematical structures of non-commutative field theory [13] and non-commutative geometry [14].

Effective field theory descriptions of the quantum Hall effect has been extensively developed in the past few years using ordinary Chern–Simons theories [15], that correspond to conformal field theories of the low-energy excitations at the edge of the sample [16]. These approaches have been rather successful and have been experimentally

confirmed [17]. However, they present some limitations, such as the need for several conformal theories to describe the whole set of observed Hall plateaus and the presence of slightly different proposals [16, 18] for describing the less understood Jain plateaus [19]. Moreover, these effective descriptions do not incorporate the microscopic physics needed to understand the 'universality' of the Laughlin wavefunction [20], the Jain 'composite fermion' transformation[2] [19] and the phase transitions between plateaus [22]. The non-commutative Chern–Simons theory, if appropriate, could tackle these problems being actually non-relativistic, while the methods of non-commutative geometry could provide new theoretical tools.

Susskind's proposal was analysed by Polychronakos [5, 6], who introduced a finite-dimensional regularization of the non-commutative Chern–Simons theory, the so-called Chern–Simons matrix model or, more precisely, matrix quantum mechanics. This theory is suitable for describing finite systems, that expose the relevant boundary excitations; on the other hand, the original field theory of infinite fluids is a topological theory that is not fully defined without specifying the ultraviolet (and infrared) regularizations.

Furthermore, Polychronakos showed that the matrix model possesses a $U(N)$ gauge symmetry, where N is the size of the matrices, and that it can be reduced to $2N$ physical degrees of freedom living on a two-dimensional phase space. The resulting reduced theory is the Calogero model of one-dimensional non-relativistic fermions with repulsive interaction. This model has many features in common with the Laughlin theory in the lowest Landau level, but is not equivalent. More precisely, the two quantum problems have isomorphic sets of states but different measures of integration, that are real one-dimensional and complex two-dimensional, respectively. On the other hand, the classical solutions of the non-commutative theory present the expected features of the incompressible Hall fluid and its vortex excitations with fractional charge. The expected Hall conductivity was also derived in [10].

Karabali and Sakita [8] analysed the reduction of the matrix theory to the complex eigenvalues using the coherent states of the electrons in the lowest Landau (Bargmann–Fock space). They could not disentangle the electron coordinates (the complex eigenvalues) from the auxiliary variables of the boundary fields, but could perform some explicit calculations for low N. They found that the inner products of states contain the Laughlin wavefunction together with a non-trivial measure of integration that modifies the properties of the incompressible fluid at short distances. These authors concluded that either the matrix model does not describe the Laughlin physics at all, or it does it in another, unknown set of variables that is hard to find in general.

In this paper, we shall analyse the relation of the Chern–Simons matrix model with the Laughlin Hall states along similar lines. Using a canonical change of matrix variables, we solve the Gauss law constraint and reduce the theory to the physical degrees of freedom that are the complex eigenvalues and their canonical conjugate momenta. Thus, the path integral of the matrix theory becomes the holomorphic path integral of the lowest Landau level and the eigenvalues can be interpreted as electron coordinates. The states expressed in terms of the complex eigenvalues display the Laughlin wavefunction, while the overlaps of states are again different from the expected form. The latter fact can be explained as follows.

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[2] See [21] for the theories of composite fermions.

In the reduced variables, we find that the derivatives are replaced by covariant derivatives, similar to those caused by an ordinary Chern–Simons interaction that is solved instantaneously in terms of the sources (the so-called statistical interaction [12]). Therefore, a kind of statistical interaction is present among the electrons (yet keeping their fermion statistics). In the Bargmann space, the derivatives correspond to the conjugate variables; thus the statistical interaction modifies the rule of conjugation and leads to non-standard expressions for the overlap integrals.

The statistical interaction is actually necessary for the stability of the Laughlin ground state: upon acting with the covariant derivatives, it is not possible to create an excitation with energy and angular momentum lower than those of the Laughlin state. Since a reduction of angular momentum amounts to a compression of the fluid, energy stability also corresponds to incompressibility of the quantum Hall fluid.

The incompressibility of the ground state can be described by the highest-weight conditions for a representation of the (non-relativistic) W_∞ algebra of quantum area-preserving diffeomorphisms [23]: actually, this symmetry characterizes the quantum Hall fluids and their excitations [24]. We thus analyse the realization of the W_∞ algebra in the Chern–Simons matrix model, both in the original and the reduced variables, and prove the highest-weight conditions satisfied by the ground state using the covariant derivatives. However, we cannot derive the complete representation of the W_∞ algebra beyond the semiclassical leading order, owing to normal-ordering and finite-size problems. We remark that the state overlaps can be expressed as commutators of the W_∞ algebra, such that the W_∞ symmetry, once fully understood, can provide a complete algebraic characterization of the Laughlin incompressible fluids that is independent of the choice of coordinates.

In conclusion, the Chern–Simons matrix model exactly describes the Laughlin Hall states if it realizes the W_∞ symmetry. Although we cannot at present prove this, we believe that our results are rather positive and worth discussing. We should also remark that our work is not meant to prove experimental or numerical facts about the quantum Hall effect, but rather to set a new framework of effective field theory which can encompass the basic features of the many-body Hall states.

The plan of the paper is the following. In section 2, we recall the Susskind approach, the Polychronakos matrix model and the relation with the Calogero model. In section 3, we discuss the holomorphic quantization of the matrix model, solve the Gauss law and obtain the Laughlin wavefunction of the complex eigenvalues. In section 4, we discuss the realization of the W_∞ symmetry in the matrix model, its relation with the ground state stability and the physical interpretation of the incompressible Hall fluids. In section 5, we perform the reduction to the eigenvalues of the path integral and the state overlaps, and obtain the dynamics of electrons in the lowest Landau level. The same analysis in the case of real quantization yields the path integral of the Calogero model. In section 6, we discuss our results and suggest some developments.

2. Non-commutative Chern–Simons theory, Chern–Simons matrix model and Calogero model

The quantization of non-commutative field theories and especially of the topological Chern–Simons theory presents several subtle technical aspects, that sometimes have led to inconsistencies in the literature. Thus, we want to describe our approach as clearly as

possible, and start with a short but self-contained introduction to Susskind's derivation [1] and the developments by Polychronakos and other authors [5]–[9].

Let us begin with N first-quantized electrons with two-dimensional coordinates $X_\alpha^a(t)$, $a = 1, 2$, $\alpha = 1, \ldots, N$, subjected to a strong magnetic field B such that their action can be projected to the lowest Landau level [25],

$$S = \frac{eB}{2} \int \mathrm{d}t \sum_{\alpha=1}^{N} \epsilon_{ab} X_\alpha^a \dot{X}_\alpha^b. \tag{2.1}$$

Susskind considered the limit of the continuous fluid [1]:

$$\vec{X}_\alpha(t) \to \vec{X}(\vec{x}, t), \qquad \vec{X}(\vec{x}, t = 0) = \vec{x}, \tag{2.2}$$

where $\vec{x}$ are the coordinates of an initial, reference configuration of the fluid. The resulting fluid mechanics is in the Lagrangian formulation, because the field $\vec{X}$ follows the motion of the fluid [9]. For incompressible fluids, the constraint of constant density, $\rho(\vec{x}) = \rho_o$, can be written in terms of Poisson brackets $\{\cdot, \cdot\}$ of the $\vec{x}$ coordinate as follows:

$$\rho_o = \rho(\vec{x}) = \rho_o \left| \frac{\partial \vec{X}}{\partial \vec{x}} \right| = \frac{\rho_o}{2} \epsilon_{ab} \{X^a, X^b\}. \tag{2.3}$$

This constraint can be added to the action by using the Lagrange multiplier A_0,

$$S = \frac{eB\rho_o}{2} \int \mathrm{d}t \, \mathrm{d}^2x \, [\epsilon_{ab} X^a (\dot{X}^b - \theta\{X^b, A_0\}) + 2\theta A_0]; \tag{2.4}$$

in this equation, we introduced the constant θ,

$$\theta = \frac{1}{2\pi\rho_o}, \tag{2.5}$$

that will later parametrize the non-commutativity.

The action (2.4) is left invariant by reparametrizations of the $\vec{x}$ variable with unit Jacobian, the area-preserving diffeomorphism, also called w_∞ transformations [23, 24]: they correspond to changes of the original labels of the fluid at $t = 0$ (cf equation (2.2)) [1, 9]. The w_∞ symmetry can be put into the form of a gauge invariance by introducing the gauge potential $\vec{A}$, as follows:

$$X^a = x^a + \theta\epsilon_{ab} A_b(x). \tag{2.6}$$

The action (2.4) can be rewritten in the Chern–Simons form in terms of the three-dimensional gauge field $A_\mu = (A_0, A_a)$:

$$S = -\frac{k}{4\pi} \int \mathrm{d}t \, \mathrm{d}^2x \, \epsilon_{\mu\nu\rho} \left(\partial_\mu A_\nu A_\rho + \frac{\theta}{3} \{A_\mu, A_\nu\} A_\rho \right). \tag{2.7}$$

The coupling constant k parametrizes the filling fraction of this (semi)classical fluid:

$$\nu^{(cl)} = \frac{2\pi\rho_o}{eB} = \frac{1}{eB\theta} = \frac{1}{k}. \tag{2.8}$$

After the analysis of Lagrangian incompressible fluids, Susskind made a proposal for the complete theory of the fractional quantum Hall effect, that could hold beyond the continuous fluid approximation by accounting for the granularity of the electrons.

He suggested replacing the theory (2.7) with the non-commutative (Abelian) Chern–Simons theory [13],

$$S_{\text{NCCS}} = -\frac{k}{4\pi} \int \mathrm{d}t\, \mathrm{d}^2x\, \epsilon_{\mu\nu\rho} \left(\partial_\mu A_\nu \star A_\rho - \frac{2\mathrm{i}}{3} A_\mu \star A_\nu \star A_\rho \right), \tag{2.9}$$

where the Moyal star product is

$$(g \star f)(x) = \exp\left(\mathrm{i}\frac{\theta}{2}\epsilon_{ab}\frac{\partial}{\partial x_1^a}\frac{\partial}{\partial x_2^b} \right) f(x_1)g(x_2)\bigg|_{x_1=x_2=x}. \tag{2.10}$$

Actually, the two actions (2.9) and (2.7) agree to leading order in θ, i.e. for dense fluids. In the new action (2.9), the gauge fields with Moyal product have become Wigner functions of the non-commuting operators, $\hat{x}_1, \hat{x}_2$, the former spatial coordinates [14]:

$$[\hat{x}_1, \hat{x}_2] = x_1 \star x_2 - x_2 \star x_1 = \mathrm{i}\theta. \tag{2.11}$$

The corresponding quantization of the area can be thought of as a discretization of the fluid (at the classical level), with the minimal area θ allocated to a single electron [1]. Other motivations for this proposal were found in the study of D-brane dynamics [4].

Any non-commutative field theory corresponds to a theory of infinite-dimensional matrices, that represent the commutator $[\hat{x}_1, \hat{x}_2] = \mathrm{i}\theta$. The general map can be found in [13], while the specific case of the Chern–Simons theory has been discussed e.g. in [4]. The matrix theory equivalent to (2.9) is the Chern–Simons matrix quantum mechanics (Chern–Simons matrix model) with action

$$S_{\text{CSMM}} = \frac{eB}{2} \int \mathrm{d}t\, \text{Tr}[\epsilon_{ab}\hat{X}^a(\dot{\hat{X}}^b + \mathrm{i}[\hat{X}^b, \hat{A}_0]) + 2\theta\hat{A}_0]. \tag{2.12}$$

The variation of this action with respect to $\hat{A}_0$ yields the Gauss law constraint,

$$[\hat{X}^1, \hat{X}^2] = \mathrm{i}\theta, \tag{2.13}$$

that can actually be solved in terms of infinite-dimensional Hermitian matrices $\hat{X}^a$ (and thus $\hat{A}_0$). The previous relation (2.6) between the gauge field A_a and the coordinates X^a still holds for the hatted matrix variables: upon substituting it in the matrix action (2.12), one recovers the non-commutative theory (2.9) [4]. Let us finally note the correspondence between the matrix (2.12) and original (2.9) theories.

Susskind's proposal has been analysed by many authors who found several evidences of quantum Hall physics, both at the classical and at the quantum level. However, there remain some open problems; two of them will be particularly important for our discussion:

- In the extension from the fluid (2.4) to the non-commutative theory (2.12), the electron coordinates $\vec{X}$ become matrices and lose their physical interpretation. There is the question of defining the physical (gauge invariant) observables in the matrix theory that correspond to the electron coordinates, the density $\rho(\vec{x})$ and other quantities. This issue has been addressed in the [8, 10].

- The Gauss law (2.13) admits one solution modulo reparametrizations; therefore the matrix theory (2.12) possesses just one state, i.e. the ground state of an infinitely extended incompressible fluid with infinite electrons. This is a topological theory with rather peculiar quantum properties: for example, the number and type of physical degrees of freedom may depend on the regularization and the boundary conditions; they should be suitably chosen for the complete definition of the theory.

An answer to the second question was given by Polychronakos [5] who regularized the matrix model by introducing a 'boundary' vector field ψ^i, $i = 1, \ldots, N$, such that the modified Gauss law admits finite-dimensional matrix solutions. The modified action is

$$S_{\text{CSMM}} = \frac{B}{2} \int \mathrm{d}t \, \text{Tr}[\epsilon_{ab} X^a (\dot{X}^b - \mathrm{i}[A_0, X^b]) + 2\theta A_0 - \omega(X^a)^2] - \int \mathrm{d}t \, \psi^\dagger (\mathrm{i}\dot{\psi} + A_0 \psi). \tag{2.14}$$

In this equation, we suppressed the hats over the matrices, set $e = 1$ and also introduced a quadratic potential with coupling ω. The Gauss law now reads

$$G = 0, \qquad G = -\mathrm{i}B[X_1, X_2] - B\theta + \psi\psi^\dagger. \tag{2.15}$$

The condition $\text{Tr}\,G = 0$ can be satisfied by $N \times N$-dimensional matrices X^a, provided that $\|\psi\|^2 = NB\theta$. This reduction to a finite-dimensional quantum mechanical problem introduces both an ultraviolet and an infrared cut-off in the theory. As shown in [5], one can choose a gauge in which ψ^i has only one non-vanishing component, say the Nth one, resulting in a boundary effect that disappears for large N upon defining a suitable weak limit $N \to \infty$. The $U(N)$ symmetry of the matrix Chern–Simons theory (2.14) is given by $X^a \to U X^a U^\dagger$ and $\psi \to U\psi$, where U is a unitary matrix. Since G is the generator of infinitesimal transformations at the quantum level, the Gauss law condition requires that all physical states should be $U(N)$ singlets [5, 1].

Classical matrix solutions of this theory were found in [5] for the ground state and the quasi-hole excitation that possess the expected features of Hall incompressible fluids. The distribution of the matrix eigenvalues for the ground state in the isotropic potential $\omega\,\text{Tr}\,\vec{X}^2$ is a circular droplet with uniform density $\rho_o \sim 1/2\pi\theta$ for large N, as in the infinite theory. In the quasi-hole solution, a hole (vortex) is present in the density with the correct size. These solutions suggest the identification of the matrix eigenvalues with the electron coordinates: however, only one matrix can be diagonalized, say $(X^1)_{nm} = x_n^1 \delta_{nm}$, and their distribution $\rho(x^1)$ is actually meant to be integrated over the other coordinate x^2.

2.1. Quantization on the real line

The quantization of Polychronakos's theory (2.14) can be done before or after having solved the Gauss constraint: both approaches have been considered in [5] and we discuss the latter first. Since the Hermitian matrix X^1 can be diagonalized by a unitary transformation, we can fix the gauge:

$$(X^1)_{nm} = x_n \delta_{nm}. \tag{2.16}$$

The form of the other variables is obtained by solving the Gauss constraint (2.15) in this gauge. The result is

$$\begin{aligned} \psi_n &= \sqrt{B\theta}, \qquad \forall n, \\ (X^2)_{nm} &= y_n \delta_{nm} - \mathrm{i}\theta \frac{1 - \delta_{nm}}{x_n - x_m}; \end{aligned} \tag{2.17}$$

these two equations follow from the diagonal and off-diagonal components of $G_{nm} = 0$, respectively; in the diagonal components, we used the residual $U(1)^N$ gauge symmetry to

fix the phases of ψ_n. The variables y_n in (2.17) parametrize the components of X_2 that are left undetermined. The substitution of all variables back into the action yields

$$S = -\int \mathrm{d}t \, B \sum_{n=1}^{N} \dot{x}_n y_n + H,$$

$$H = \frac{B\omega}{2} \operatorname{Tr} \vec{X}^2 = \sum_{n=1}^{N} \left(\frac{\omega}{B} \frac{p_n^2}{2} + \frac{B\omega}{2} x_n^2 \right) + \sum_{n \neq m}^{N} \frac{\omega B \theta^2}{2} \frac{1}{(x_n - x_m)^2}. \tag{2.18}$$

That is, the interpretation of the N real variables x_n as particle coordinates has led to the identification of the conjugate momenta $p_n = -By_n$. Moreover, the Hamiltonian is found to be that of the Calogero model with coupling constant $B\theta = k$ taking integer values[3]. Therefore, the Chern–Simons matrix model has been reduced to the quantum mechanics of N particles on the line with two-body repulsion. Note the reduction of the degrees of freedom: starting from the $2N^2 + 2N$ real phase-space variables $(X_1, X_2, \psi, \psi^\dagger)$, the gauge fixings eliminate N^2 variables and the Hermitian constraint $G \equiv 0$ a further N^2 ones, leaving the conjugate variables of N particles.

The one-dimensional Calogero model is closely related to the theory of two-dimensional electrons quantized in the first Laudau level: it is integrable and the space of states is known [27] and isomorphic to that of the excitations over the Laughlin state at filling fraction $\nu = 1/(k+1)$ [2]; the Calogero particles satisfy selection rules of an enhanced exclusion principle [28] that allow one to define a one-dimensional analogue of the fractional statistics of anyons [12]. On the other hand, the Hilbert spaces of the two problems are different, because the one-dimensional norm of the Calogero model is different from that of the first Landau level [27, 29].

Therefore, Polychronakos's analysis found strong analogies between the Chern–Simons matrix model and the Laughlin Hall states but not a complete equivalence. In the next section, we shall discuss another quantization scheme and will perform a change of matrix variables that let the electron coordinates and the Laughlin wavefunction emerge rather naturally.

3. Holomorphic quantization of the Chern–Simons matrix model

We now discuss the 'covariant quantization' of the Chern–Simons matrix model (2.14): we solve the Gauss law constraint at the quantum level, elaborating on the results of [5, 7]. It is convenient to introduce the complex matrices:

$$X = X_1 + \mathrm{i}X_2, \qquad X^\dagger = X_1 - \mathrm{i}X_2. \tag{3.1}$$

The matrix action (2.14) in the $A_0 = 0$ gauge is

$$S_{\mathrm{CSMM}}|_{A_0=0} = \int \mathrm{d}t \, \frac{B}{2} \operatorname{Tr}(X_1 \dot{X}_2 - \dot{X}_1 X_2) - \mathrm{i}\psi^\dagger \dot{\psi} - \mathcal{H}(X_a)$$

$$= \int \mathrm{d}t \, \frac{B}{2\mathrm{i}} \sum_{nm} \dot{X}_{nm} \overline{X}_{nm} - \mathrm{i} \sum_n \dot{\psi}_n \overline{\psi}_n - \mathcal{H}(X^\dagger X), \tag{3.2}$$

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[3] The quantization of the Chern–Simons coupling k follows from the requirement of invariance of the action (2.9) under a large gauge transformation [26].

where $\mathcal{H} = B\omega \operatorname{Tr} X_a^2/2$. This action implies the following oscillator commutation relations between the components of the matrices and vectors:

$$[[\overline{X}_{nm}, X_{kl}]] = \frac{2}{B}\delta_{nk}\delta_{ml},$$
$$[[\overline{\psi}_n, \psi_m]] = \delta_{nm}. \tag{3.3}$$

In these equations, we represented the quantum commutator with double brackets to distinguish it from the classical matrix commutator.

The form of the action (3.2) is that of $N^2 + N$ 'particles' in the lowest Landau level with complex 'coordinates' X_{nm} and ψ_n, that can be quantized in the Bargmann–Fock space of holomorphic wavefunctions $\Psi(X, \psi)$, with integration measure [30]

$$\langle \Psi_1 | \Psi_2 \rangle = \int \mathcal{D}X \, \mathcal{D}\overline{X} \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \, \exp\left(-\frac{B}{2}\operatorname{Tr} X^\dagger X - \psi^\dagger \psi\right)\overline{\Psi_1(X, \psi)}\Psi_2(X, \psi). \tag{3.4}$$

The conjugate variables act as derivative operators on the wavefunctions,

$$\overline{X}_{nm} \to \frac{2}{B}\frac{\partial}{\partial X_{nm}}, \qquad \overline{\psi}_n \to \frac{\partial}{\partial \psi_n}, \tag{3.5}$$

and the (properly normal-ordered) Gauss law (2.15) becomes a differential equation for the wavefunctions of physical states ($B\theta = k$):

$$G_{ij}\Psi_{\text{phys}}(X, \psi) = 0,$$
$$G_{ij} = \sum_\ell \left(X_{i\ell}\frac{\partial}{\partial X_{j\ell}} - X_{\ell j}\frac{\partial}{\partial X_{\ell i}}\right) - k\delta_{ij} + \psi_i\frac{\partial}{\partial \psi_j}. \tag{3.6}$$

We now come to a crucial point of our analysis: we are going to perform a change of matrix variables that leaves invariant the commutation relations (3.3). This Bäcklund (or Bogoliubov) transformation is defined as follows:

$$X = V^{-1}\Lambda V, \qquad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N),$$
$$\psi = V^{-1}\phi. \tag{3.7}$$

Here we used the fact that a complex matrix can be diagonalized by a $GL(N, \mathbb{C})$ transformation (up to the zero-measure set of matrices with degenerate eigenvalues). For the transformation of the derivative operators (3.5), we should consider the linear transformation in the $(N^2 + N)$-dimensional tangent space:

$$\{dX_{nm}, d\psi_n\} \to \{d\lambda_n, dv_{ij} \, (i \neq j), d\phi_n\}, \qquad dv = dV \, V^{-1}. \tag{3.8}$$

From the transformation of the covariant vector $\{dX_{nm}, d\psi_n\}$,

$$dX = V^{-1}(d\Lambda + [\Lambda, dv])V,$$
$$d\psi = V^{-1}(d\phi - dv \, \phi), \tag{3.9}$$

we can compute the inverse transformation of the contravariant vector $\{\partial/\partial X_{nm}, \partial/\partial \psi_n\}$. The result is the following:

$$\frac{\partial}{\partial X_{ij}} = V_{ni}V_{jm}^{-1}\frac{\partial}{\partial \Lambda_{nm}}, \qquad \frac{\partial}{\partial \Lambda_{nm}} = \frac{\partial}{\partial \lambda_n}\delta_{nm} + \frac{1 - \delta_{nm}}{\lambda_n - \lambda_m}\left(\frac{\partial}{\partial v_{nm}} + \phi_m\frac{\partial}{\partial \phi_n}\right),$$
$$\frac{\partial}{\partial \psi_j} = V_{nj}\frac{\partial}{\partial \phi_n}. \tag{3.10}$$

In this equation, $\mathrm{d}/\mathrm{d}v_{ij}$ is the vector field implementing the left translations of V, namely $\langle \mathrm{d}/\mathrm{d}v_{ij}, \mathrm{d}v_{kl} \rangle = \delta_{ik}\delta_{jl}$, and more explicitly $\mathrm{d}/\mathrm{d}v_{ij} = V_{jn}\mathrm{d}/\mathrm{d}V_{in}$. The transformation (3.9) is invertible for $\det V \neq 0$ and distinct eigenvalues $\lambda_n \neq \lambda_m$. One can explicitly check that the commutation relations (3.3) are left invariant by the transformation, namely that each new variable (λ, V, ϕ) satisfies canonical commutators with the corresponding derivative[4].

The oscillator vacuum state is left invariant by the transformation (3.7): indeed, the vacuum wavefunctions for the original and new oscillators, $\Psi_o^{(\mathrm{old})} = 1$ and $\Psi_o^{(\mathrm{new})}$, respectively, should satisfy

$$\frac{\partial}{\partial X_{ij}}\Psi_o^{(\mathrm{old})} = 0, \qquad \frac{\partial}{\partial \psi_j}\Psi_o^{(\mathrm{old})} = 0,$$

$$\frac{\partial}{\partial v_{ij}}\Psi_o^{(\mathrm{new})} = 0, \qquad i \neq j, \qquad \frac{\partial}{\partial \lambda_j}\Psi_o^{(\mathrm{new})} = \frac{\partial}{\partial \phi_j}\Psi_o^{(\mathrm{new})} = 0. \tag{3.11}$$

The comparison of these expressions with the transformation (3.10) shows that it is consistent to keep the same vacuum: $\Psi_o^{(\mathrm{new})} = 1$. This result is at variance with the usual Bogoliubov transformations, where the new vacuum contains an infinite number of old particles. Therefore, the transformation (3.10) preserves the number operators associated with both X_{ij} and ψ_i oscillators.

The substitution of the new matrix variables (3.7) and derivatives (3.10) in the Gauss law (3.6) yields the following result:

$$G_{ij} = V_{im}^{-1}V_{nj}\tilde{G}_{nm}, \qquad \tilde{G}_{nm}\Psi(\Lambda, V, \phi) = 0,$$

$$\tilde{G}_{nm} = \begin{cases} -\dfrac{\partial}{\partial v_{nm}} & n \neq m, \\[2ex] \phi_n\dfrac{\partial}{\partial \phi_n} - k & n = m. \end{cases} \tag{3.12}$$

Rather remarkably, the change of variables diagonalizes the constraint and allows for the elimination of the unphysical degrees of freedom:

- The $N^2 - N$ off-diagonal components of V are killed, that is $\Psi^{(\mathrm{phys})}(\lambda, V, \phi)$ can only depend on V through quantities like $\det V$.
- The N degrees of freedom of ψ are also frozen, because all physical wavefunctions should contain the same homogeneous polynomial of degree k in each component of the vector, which is $\prod_{n=1}^N (\phi_n)^k$.

The remaining dynamical variables are the N complex eigenvalues λ_n, that can be interpreted as coordinates of electrons in lowest Landau level.

3.1. Wavefunctions

The general solution of the Gauss law for the wavefunctions of physical states in the (X, ψ) coordinates, equation (3.6), has been found in [5, 7]: we should form $U(N)$ singlet polynomials made of the N-component epsilon tensor and an arbitrary number of X matrices; moreover, the condition $\mathrm{Tr}\, G = 0$ in (3.6) implies that the vector ψ should occur to the power Nk. For $k = 1$, these wavefunctions take the form [7]

$$\Psi_{\{n_1,\ldots,n_N\}}(X, \psi) = \varepsilon^{i_1 \ldots i_N}(X^{n_1}\psi)_{i_1} \cdots (X^{n_N}\psi)_{i_N}, \qquad 0 \leq n_1 < n_2 < \cdots < n_N \tag{3.13}$$

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[4] The transformation of the matrix derivative has been suitably normal ordered in equation (3.10).

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for any ordered set of positive integers $\{n_i\}$. The ground state in the confining potential $\mathrm{Tr}(XX^\dagger)$ is given by the closest packing $\{0, 1, \ldots, N-1\}$ that has the lowest degree in X. For $k \neq 1$, one can multiply k terms of this sort, leading to $\Psi_{\{n_1^1,\ldots,n_N^1\}\cdots\{n_1^k,\ldots,n_N^k\}}$. As shown in [7], there is an equivalent basis for these states that involves the 'bosonic' powers of X:

$$\Psi(X, \psi) = \sum_{\{m_k\}} \mathrm{Tr}(X^{m_1})\cdots\mathrm{Tr}(X^{m_k})\Psi_{k-gs},$$

$$\Psi_{k-gs} = [\varepsilon^{i_1\ldots i_N}\psi_{i_1}(X\psi)_{i_2}\cdots(X^{N-1}\psi)_{i_N}]^k, \tag{3.14}$$

where the positive integers $\{m_1, \ldots, m_k\}$ are now unrestricted. This second basis (3.14) also makes sense in the $k = 0$ case, where $\Psi_{0-gs} = 1$.

Let us now perform the change of matrix variables in the wavefunctions (3.14): the excitations made by the invariant powers $\mathrm{Tr}(X^r)$ became the power sums of the eigenvalues, $\sum_n \lambda_n^r$; in the ground state wavefunction, the dependence on V and ϕ factorizes and the powers of the eigenvalues make up the Vandermonde determinant $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$:

$$\begin{aligned}\Psi_{k-gs}(\Lambda, V, \psi) &= [\varepsilon^{i_1\ldots i_N}(V^{-1}\phi)_{i_1}(V^{-1}\Lambda\phi)_{i_2}\cdots(V^{-1}\Lambda^{N-1}\phi)_{i_N}]^k \\ &= [(\det V)^{-1}\det(\lambda_j^{i-1}\phi_j)]^k \\ &= (\det V)^{-k}\prod_{1\le n<m\le N}(\lambda_n - \lambda_m)^k\left(\prod_i \phi_i\right)^k.\end{aligned} \tag{3.15}$$

We thus obtain the Laughlin wavefunction for the ground state of the Hall effect with the electron coordinates corresponding to the complex eigenvalues of X [8]. The value of the filling fraction is

$$\nu = \frac{1}{k+1}, \tag{3.16}$$

and is renormalized from the classical value (2.8) because the wavefunction should acquire one extra factor of $\Delta(\lambda)$ from the integration measure, as shown later. The factorized dependence on V and ψ in (3.15) is the same for all the states (3.14), since it is the unique solution to the Gauss law (3.12); that is, these degrees of freedom are frozen. The bosonic power sums $\sum_n \lambda_n^r$ are the natural basis of symmetric polynomials forming the excitations over the Laughlin state[5].

Therefore, we have shown that the change of matrix variables (3.7) allows one to explicitly eliminate the gauge degrees of freedom and reduce the Chern–Simons matrix model to the quantum mechanics of N variables with the ground state given by the Laughlin wavefunction. In the next section, we discuss the meaning of the covariant derivatives (3.10) that have emerged in the reduction.

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[5] The counting of these states is given by the number of partitions and it shows the correspondence between the edge excitations of incompressible Hall fluids and the states of a one-dimensional bosonic field [16].

4. Stability, incompressibility and W_∞ symmetry

4.1. Introduction

In a series of papers [24, 31, 32, 18], the incompressible Hall fluids have been characterized by the symmetry under W_∞ transformations, that are the quantization of the ω_∞ area-preserving diffeomorphisms of the plane [23]. Actually, the deformations of a classical droplet of fluid of constant density all have the same area and can be mapped one into another by ω_∞ reparametrizations. In the quantum theory of the first Landau level, the $\nu = 1$ ground state is a circular droplet of quantum incompressible fluid that admits the further interpretation of a filled Fermi sea [24]: the electrons occupy all the available one-particle states of angular momentum $J = 0, 1, \ldots, N - 1$, leading to a droplet of radius $R \sim \sqrt{N}$.

The deformations of the droplet are obtained from the W_∞ operators: they are the moments of the generating function of classical ω_∞ transformations that are quantized in the Bargmann space [24]:

$$\mathcal{L}_{nm} = \sum_{\alpha=1}^{N} \lambda_\alpha^n \left(\frac{\partial}{\partial \lambda_\alpha} \right)^m, \qquad \alpha = 1, \ldots, N, \tag{4.1}$$

where $\alpha = 1, \ldots, N$ is the particle index, $\bar{\lambda} \to \partial/\partial \lambda$ when acting on holomorphic wavefunctions and n, m are non-negative integers.

The W_∞ operators generate small fluctuations of the $\nu = 1$ ground state[6], $\Phi_{gs} = \Delta(\lambda)$, that possess angular momentum $\Delta J = n - m$ with respect to the ground state. Excitations with $\Delta J < 0$ are forbidden in the filled Fermi sea, because they would correspond to violations of the exclusion principle. Therefore, the ground state should satisfy the conditions

$$\mathcal{L}_{nm} \Phi_{gs} = 0, \qquad 0 \le n < m \le N - 1. \tag{4.2}$$

Furthermore, the generators with $m \ge N$ also vanish because they would correspond to particle–hole transitions outside the Fermi sea [24].

The $N(N-1)/2$ conditions (4.2) express the incompressibility of the $\nu = 1$ quantum Hall ground state and represent the highest-weight conditions for the representation of the W_∞ algebra. In the quadratic confining potential, $\mathcal{H} = \sum_\alpha \bar{\lambda}_\alpha \lambda_\alpha \propto J$, the incompressibility of the ground state is equivalent to its energy stability.

The other states in the infinite-dimensional representations are the excitations obtained by acting with $\mathcal{L}_{nm}$, $n > m$, on the ground state. One can show that these operators generate all the bosonic power sums $\sum_\alpha \lambda_\alpha^k$ described before, such that the $\mathcal{L}_{nm}$ carry over the bosonization of the incompressible fluid in the non-relativistic theory[7].

The W_∞ algebra is [24]

$$[\mathcal{L}_{nm}, \mathcal{L}_{kl}] = \sum_{s=1}^{\min(m,k)} \frac{m!k!}{(m-s)!(k-s)!s!} \mathcal{L}_{n+k-s, m+l-s} - (m \leftrightarrow l, n \leftrightarrow k). \tag{4.3}$$

---

[6] We use the notation Φ for the wavefunction of the physical electron variables, to distinguish it from that of the matrix model Ψ (3.15).

[7] Actually, these operators become the bosonic current and its normal-ordered powers when evaluated in the relativistic effective theory of edge excitations [31].

The first term in the rhs, $[\mathcal{L}_{nm}, \mathcal{L}_{kl}] = \hbar(mk - nl)\mathcal{L}_{n+k-1,m+l-1}$, corresponds to the quantization of the classical algebra w_∞ of area-preserving diffeomorphism, while the other terms are quantum corrections $\mathrm{O}(\hbar^p)$, $p \geq 2$. Finally, the operators with equal indices, $\mathcal{L}_{nn}$, are the Casimirs of the representation, e.g. $\mathcal{L}_{00} = N$, $\mathcal{L}_{11} = J$. The representations of the W_∞ symmetry for N electrons can be related to the representation of the $U(N)$ algebra [23].

The W_∞ algebra is also useful for expressing the inner product of states. Consider two excitations, e.g. $\mathcal{L}_{mn}\Phi_{gs}$ and $\mathcal{L}_{kl}\Phi_{gs}$, for $m > n$ and $k > l$: thanks to the conjugation rule,

$$\mathcal{L}_{nm}^\dagger = \mathcal{L}_{mn}, \tag{4.4}$$

and the incompressibility conditions (4.2), the overlap of these two states can be rewritten as a commutator:

$$\langle \mathcal{L}_{mn}\Phi_{gs}|\mathcal{L}_{kl}\Phi_{gs}\rangle = \langle\Phi_{gs}|[\mathcal{L}_{nm}, \mathcal{L}_{kl}]|\Phi_{gs}\rangle, \qquad m > n, \quad k > l, \tag{4.5}$$

that can be reduced to the Casimirs of the algebra, if non-vanishing. In particular, one finds that the $\nu = 1$ representation is unitary, thanks to the positivity of the Casimirs [24].

In conclusion, the use of the W_∞ symmetry allows a complete algebraic description of the $\nu = 1$ incompressible quantum Hall fluid and its excitations, that does not rely on the coordinate representation of wavefunctions and overlaps.

Let us now review previous analyses of the W_∞ symmetry of the fractional Laughlin states:

$$\Phi_{k-gs} = \Delta(\lambda)^{k+1}, \qquad \nu = \frac{1}{k+1}. \tag{4.6}$$

As proposed in [32], one can perform a similarity transformation on the $\nu = 1$ generators, as follows:

$$\mathcal{L}_{nm}^{(k)} = \Delta(\lambda)^k \mathcal{L}_{nm} \Delta(\lambda)^{-k} = \sum_{\alpha=1}^{N} \lambda_\alpha^n \left(\frac{\partial}{\partial\lambda_\alpha} - \sum_{\beta,\beta\neq\alpha} \frac{k}{\lambda_\alpha - \lambda_\beta} \right)^m . \tag{4.7}$$

These operators are non-singular when acting of the Laughlin wavefunction and its excitation, obey the same algebra (4.3) and realize exactly the same $\nu = 1$ representation (same values of the Casimirs); in particular, the incompressibility conditions read again

$$\mathcal{L}_{nm}^{(k)}\Phi_{k-gs} = 0, \qquad 0 \leq n < m \leq N-1. \tag{4.8}$$

One problem of these operators is the Hermiticity relation (4.4), that is not manifestly satisfied and thus the unitarity of the representation is not guaranteed (unless an exotic measure of integration is introduced [32]). Nevertheless, a couple of remarks are suggested by this analysis:

- The similarity between Laughlin states of different ν values and the bosonization of the corresponding relativistic theory on the edge [31] indicate that all Laughlin states should realize representations of the W_∞ algebra (4.3).

- In the fractional case, the suggested W_∞ generators (4.7) contain covariant derivatives:

$$D_z = \partial_z + A_z, \qquad A_z = -\sum_\beta \frac{k}{z - \lambda_\beta}, \tag{4.9}$$

that assign a magnetic charge of k fluxes to each electron, the excess magnetic field being $\mathcal{B} = [D_z, D_{\bar{z}}] = k\pi \sum_\beta \delta^2(z - \lambda_\beta)$ ($A_{\bar{z}} = 0$). Therefore, the covariant derivatives introduce a 'statistical interaction' among the electrons [12], given by the Aharonov–Bohm phases between electric and magnetic charges; note, however, that this interaction does not change the statistics of electrons for even integer values of k [20].

4.2. W_∞ symmetry of the Chern–Simons matrix model

In the Chern–Simons matrix model, we can introduce two kinds of polynomial generators that generalize (4.1) and are gauge invariant (we set $B = 2$ in this section):

$$\begin{aligned}
\mathcal{L}_{nm} &= \mathrm{Tr}(X^n X^{\dagger m}), \\
\mathcal{P}_{nm} &= \psi^\dagger X^n X^{\dagger m} \psi.
\end{aligned} \tag{4.10}$$

The second operators can be considered as finite-N corrections to the first ones, because they involve the boundary vectors.

Both families of operators satisfy the incompressibility conditions on the matrix ground states (3.14) for all k:

$$\begin{aligned}
\mathcal{L}_{nm} \Psi_{k-gs} &= 0, \qquad 0 \le n < m, \\
\mathcal{P}_{nm} \Psi_{k-gs} &= 0, \qquad 0 \le n < m.
\end{aligned} \tag{4.11}$$

The proof is easily obtained in graphical form. Represent the matrices X_{ij} as oriented links, the vectors ψ_i by dots, the epsilon tensor as the N-branching root of a tree and attach the extrema according to the summations of matrix indices; then, the $k = 1$ wavefunction Ψ_{1-gs} is represented by a tree with N branches of different lengths ranging from zero to $N-1$. Note that their total length $N(N-1)/2$ is the minimal one for having a non-vanishing expression, owing to presence of the epsilon tensor. The $\mathcal{L}_{nm}$ operators, e.g.,

$$\mathcal{L}_{12} = X_{ij} \frac{\partial}{\partial X_{ik}} \frac{\partial}{\partial X_{kj}}, \tag{4.12}$$

act on Ψ_{1-gs} as follows: the derivative $\partial/\partial X_{ij}$ removes one link in a branch of the tree and identifies the indices (i, j) at the free extrema. Then, the matrices X_{ij}^n rejoin the segments and form new branches of different lengths or nucleate closed rings; after this cut and paste, the tree is reformed. Under the action of $\mathcal{L}_{nm}$ with $n < m$, the total length of the branches is lower than that of the ground state; thus the expression vanishes. This proves the first of equation (4.10) for $k = 1$. For general k, the wavefunction Ψ_{k-gs} contains k independent trees. The action of $\mathcal{L}_{nm}$ can cut and paste branches of different trees, but trees cannot be joined because the orientation of the lines would be violated. Thus, independent trees are reformed and the previous length counting applies again.

The action of the $\mathcal{P}_{nm}$ operators, e.g.,

$$\mathcal{P}_{01} = \psi_i \frac{\partial}{\partial X_{ij}} \frac{\partial}{\partial \psi_j}, \tag{4.13}$$

is analogous, with the addition that branches can be cut and joined at their end points, and terminated at some point. The resulting tree is similarly shortened for $n < m$.

Therefore, the incompressibility conditions (4.11) are verified. It is interesting to note the existence of a generalized exclusion principle in the ground state, that actually follows from the $SU(N)$ singlet condition. The different branches of the tree can be associated with 'states' and there cannot be more than k 'particles' of the same type in Ψ_{k-gs}. The W_∞ generators map branches into branches, i.e. make particle–hole transitions as in the $\nu = 1$ filled Fermi sea, some of which are forbidden by the close packing conditions[8].

We now discuss the algebra of two $\mathcal{L}_{nm}$ operators. Some care should be taken in dealing with objects that are both operator and matrix ordered: in fact, one should abandon the convention of implicitly summing over contiguous matrix indices and leave them explicit. The $\mathcal{L}_{nm}$ commutator reads (the graphical representation is still useful)

$$[[\mathcal{L}_{nm}, \mathcal{L}_{kl}]] = X_{ij}^n \left[\left[\left(\frac{\partial}{\partial X}\right)_{ij}^m, X_{pq}^k\right]\right]\left(\frac{\partial}{\partial X}\right)_{pq}^l - \begin{pmatrix} n \leftrightarrow k, m \leftrightarrow l \\ i \leftrightarrow p, j \leftrightarrow q \end{pmatrix}. \tag{4.14}$$

Again one derivative kills one matrix and identifies its pair of indices: the result is an operator containing $(n + k - s)$ times X and $(m + l - s)$ times $\partial/\partial X$, with $s = 1, 2, \ldots$. Operator orderings are dealt with as in the case of the $\nu = 1$ W_∞ algebra (4.3) and they create further terms with $s \geq 1$. However, the matrix summations in the resulting operators may not be properly ordered for identifying them as $\mathcal{L}_{n+k-s,m+l-s}$; an example is $X_{ij}^p X_{kl}^q X_{jk}^\dagger X_{li}^\dagger$. Here we can use the Gauss law to perform the matrix reorderings because it is an identity in gauge invariant expressions: from equation (3.6), we read that the reordering of the pair $X_{lj} X_{il}^\dagger \to X_{il} X_{lj}^\dagger$ creates the extra terms $k\delta_{ij}$ and $\psi_i \psi_j^\dagger$, leading to the descendants $\mathcal{L}_{n+k-s,m+l-s}$ and the operators $\mathcal{P}_{n+k-s,m+l-s}$, with $s > 1$.

Therefore, the rhs of the $\mathcal{L}_{nm}$ algebra (4.14) contains the characteristic leading $\mathrm{O}(\hbar)$ term for the semiclassical interpretation, but also involves the finite-N descendants $\mathcal{P}_{n-s,m-s}$, i.e. it does not close. Presumably, for $N \to \infty$ the latter terms can be disregarded and the algebra closes as in the $\nu = 1$ case (4.3), up to possible redefinitions of the higher, non-classical structure constants. For finite N, there remains the open problem of selecting the right basis,

$$\tilde{\mathcal{L}}_{nm} = \mathcal{L}_{nm} + \gamma_1 \mathcal{P}_{n-1,m-1} + \gamma_2 \mathcal{P}_{n-2,m-2} + \cdots, \tag{4.15}$$

that forms a closed algebra. The incompressibility conditions (4.11) are satisfied anyway. This solution of the matrix ordering is also important for identifying physical quantities like the one-particle density, that is the generating function of $\tilde{\mathcal{L}}_{nm}$ [10].

In conclusion, we have shown that the Chern–Simons matrix model realizes highest-weight representations of the type occurring in the quantum Hall effect, but we cannot at present account for the complete form of the W_∞ algebra.

4.3. W_∞ symmetry in physical coordinates

The realization of the symmetry in the electron coordinates is more interesting because it has a direct physical interpretation in the quantum Hall effect. The form of the W_∞

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[8] The use of $SU(N)$ singlets for building states with exclusion statistics was also proposed in [33].

generators is obtained by replacing the canonical transformations (3.7), (3.10) into the matrix expressions (4.10):

$$\mathcal{L}_{nm}^{(k)} = \sum_{j=1}^{N} \lambda_j^n D_{jj}^m, \qquad D_{pq} = \delta_{pq} \frac{\partial}{\partial \lambda_p} - \frac{1 - \delta_{pq}}{\lambda_p - \lambda_q} \phi_p \frac{\partial}{\partial \phi_q},$$

$$\mathcal{P}_{nm}^{(k)} = \sum_{i,j=1}^{N} \phi_i \lambda_i^n D_{ij}^m \frac{\partial}{\partial \phi_j}. \tag{4.16}$$

The matrix covariant derivative D_{ij} enforces a kind of statistical interaction among the electrons similar to the one discussed in section 4.1. The form of D_{ij} is obtained from (3.10), by suppressing the V dependence absent in physical states, but leaving the ϕ_i derivatives to allow for possible normal orderings: one should eventually replace $\phi_i \partial/\partial \phi_i \to k$, $\forall i$. In writing the expressions (4.16), we neglected further operator ordering problems, that are not well defined anyway for non-linear transformations. Therefore, the present operators are not guaranteed to satisfy the W_∞ highest-weight conditions of the previous section, which should be checked again. Here, we cannot provide a general argument, but shall present some sample calculations that have been done for $N = 3, 4, 5$, $k = 1, \ldots, 8$ and low values of the (n, m) indices with the help of computer algebra.

The check of the incompressibility conditions (4.8) on the shifted Laughlin wavefunction (4.6) gives the following results:

$$\mathcal{L}_{nm}^{(k)} \Phi_{k-gs} = 0, \qquad \text{for } 0 \le n < m = 1, 2, \tag{4.17}$$

with

$$\mathcal{L}_{n1}^{(k)} = \sum_i \lambda_i^n \frac{\partial}{\partial \lambda_i}, \qquad \mathcal{L}_{n2}^{(k)} = \sum_i \lambda_i^n \left(\frac{\partial^2}{\partial \lambda_i^2} - \sum_{n, n \ne i} \frac{k^2 + k}{(\lambda_n - \lambda_i)^2} \right). \tag{4.18}$$

We see that the covariant derivatives are already effective for stabilizing the incompressible fluid at second order. Note the similarities, but also the differences, of the covariant derivatives (4.18) and (4.7), in the present and earlier proposals [32] of W_∞ generators at fractional filling. In the eigenvalue representation, both the operators and the wavefunctions depend explicitly on k: this allows us to check the alleged shift $k \to k + 1$ in $\Phi_{k-gs} = \Delta(\lambda)^{k+1}$.

Next, the operators $\mathcal{L}_{03}^{(k)}$ and $\mathcal{P}_{01}^{(k)} \sim \mathcal{L}_{01}^{(k)}$ also annihilate the ground state. For higher values of the indices, the incompressibility conditions are only satisfied by specific superpositions of the two kinds of operators (4.16). For example, we empirically find

$$\tilde{\mathcal{L}}_{n3}^{(k)} \Phi_{k-gs} = \left(\mathcal{L}_{n3}^{(k)} + \frac{n}{2} \frac{2k^3 + k^2 - k}{(k+1)^3} \mathcal{P}_{02}^{(k)} \right) \Phi_{k-gs} = 0, \qquad n = 1, 2. \tag{4.19}$$

Note that the coefficient of proportionality is independent of N, as it should be. The general pattern that emerges from these examples is that the $\mathcal{L}_{nm}^{(k)}$ operators combine with their finite-N descendants $\mathcal{P}_{n-s,m-s}^{(k)}$ (that are missing or are too simple in the cases (4.17)). We guess that the same combinations also obey a closed W_∞ algebra.

In conclusion, we have seen that the W_∞ generators express the stability (incompressibility) of the Laughlin ground state in the Chern–Simons matrix model, in both the gauge invariant and gauge-fixed forms. In the latter case, the covariant derivatives are instrumental for this result and express a form of statistical interaction.

5. Path integral and integration measure

5.1. Real quantization

In this section we perform the reduction to the physical degrees of freedom both in the path integral and the state overlaps. We start by discussing the real case leading to the Calogero model, basically repeating the analysis in [34].

The path integral of the Chern–Simons matrix model is

$$
\langle f|i\rangle = \int \mathcal{D}X_1(t)\,\mathcal{D}X_2(t)\,\mathcal{D}\psi(t)\,\mathcal{D}\overline{\psi}(t)
$$
$$
\times\,\exp\int \mathrm{d}t\,(-\mathrm{i}B\,\mathrm{Tr}(X_2\dot{X}_1) + \psi^\dagger\dot{\psi} - \mathrm{i}\mathcal{H})\prod_t \delta(G(t))\,\mathrm{FP}, \tag{5.1}
$$

where G is the Gauss law condition (2.15) and FP is the Faddeev–Popov term for the gauge fixing (2.16) reducing X_1 to its eigenvalues. Such gauge fixing can be written in the path integral as follows:

$$
\delta(\chi) = \prod_{i\neq j}\delta(X^1_{ij}) = \int \mathcal{D}\Lambda \prod_{ij}\delta(X^1_{ij} - \Lambda_{ij}), \tag{5.2}
$$

where $\Lambda = \mathrm{diag}(x_1,\dots,x_N)$ is a real diagonal matrix. The corresponding Faddeev–Popov term is

$$
1 = \mathrm{FP} = \int \mathcal{D}U\,\delta(\chi^U)\det\left(\frac{\partial\chi^U}{\partial\omega}\right) = \int \mathcal{D}U\,\mathcal{D}\Lambda\,\delta(U^\dagger X_1 U - \Lambda)\Delta(x)^2. \tag{5.3}
$$

In this expression, $\mathcal{D}U$ is the $U(N)$ Haar measure, $\mathrm{d}\omega = U^\dagger\,\mathrm{d}U$ and the Faddeev–Popov determinant is easily computed to be the square of the Vandermonde. Upon inserting (5.3) in the path integral and performing the gauge transformation, $X_1 = U\Lambda U^\dagger$, $X_2 = U\tilde{X}_2 U^\dagger$, $\psi = U\phi$, the Gauss constraint can be rewritten:

$$
\delta(G) = \delta[U(B[\Lambda,\tilde{X}_2] - \mathrm{i}kI + \mathrm{i}\psi\otimes\psi^\dagger)U^\dagger]
$$
$$
= \frac{1}{\Delta(x)^2}\prod_{i\neq j}\delta\left(\tilde{X}_{2ij} + \frac{\mathrm{i}}{B}\frac{\phi_i\phi_j^\dagger}{x_i - x_j}\right)\prod_i \delta\left(\phi_i\phi_i^\dagger - k\right). \tag{5.4}
$$

Therefore, we find that the Faddeev–Popov determinant is cancelled by the Jacobian coming from the solution of the Gauss law. The conditions (5.2), (5.4) do not involve time derivatives and can be substituted in the path integral (5.1) at every time step: they eliminate the $X_1(t), X_2(t)$ integrations in favour of their diagonal elements, $\{x_i(t), y_i(t)\}$. The $\psi, \psi^\dagger$ integrations can be performed by substituting $\phi_i = \rho_i\exp(\mathrm{i}\varphi_i)$, using the Gauss law for ρ_i and fixing the residual $U(1)^N$ symmetry with the linear conditions $\prod_i \delta(\varphi_i)$ causing no FP determinant.

The kinetic terms in the action can be rewritten using the constraint (5.4) as follows:

$$
-\mathrm{i}B\,\mathrm{Tr}(X_2\dot{X}_1) + \psi^\dagger\dot{\psi} = -\mathrm{i}B\,\mathrm{Tr}(\tilde{X}_2\dot{\Lambda} + [\Lambda,\tilde{X}_2]U^\dagger\dot{U}) + \phi^\dagger\dot{\phi} + \phi^\dagger U^\dagger\dot{U}\phi
$$
$$
= \sum_n(-\mathrm{i}By_n\dot{x}_n + \overline{\phi}_n\dot{\phi}_n) + k\,\mathrm{Tr}(U^\dagger\dot{U}). \tag{5.5}
$$

The term proportional to k is a total derivative that expresses the variation of phase of the determinant over the time interval, $\mathrm{i}k \operatorname{Arg}(\det U)|_{t_f}^{t_i}$, and does not contribute to the path integral for integer k [26]. Therefore, the $U(N)$ integrals factors out and one is left with the phase-space path integral of the $2N$ conjugate variables of Calogero model, $\{x_i, p_i = -By_i\}$, with Hamiltonian (2.18):

$$\langle f|i\rangle = \int \prod_i \mathcal{D}p_i(t)\,\mathcal{D}x_i(t) \exp \int \mathrm{d}t \left(\mathrm{i} \sum_{i=1}^{N} p_i \dot{x}_i - H(p_i, x_i) \right). \qquad (5.6)$$

In the real quantization, one is interested in the wavefunctions which depend on the coordinate X_1 and then, after reduction, on its eigenvalues. These wavefunctions can be obtained by replacing $X = X_1 + X_2$ in the matrix expressions (3.14) with X_{1ij} coordinates and X_{2ij} derivatives w.r.t. them. In particular, for the ground state, the derivatives vanish [8] and one recovers the same determinant expression (3.14) with $X \to X_1$, i.e. $\Psi_{k-gs}(X_1, \psi)$.

The ground state overlap is defined by (after freezing the vector to $\phi_i = \sqrt{k}$)

$$\langle \Psi_{k-gs}|\Psi_{k-gs}\rangle = \int \mathcal{D}X_1 \,\mathrm{e}^{-B\operatorname{Tr}X_1^2} \|\Psi_{k-gs}(X_1)\|^2. \qquad (5.7)$$

This expression is actually a Hermitian matrix model, whose reduction to the eigenvalues is well known [35]: nothing depends on the unitary group that factorizes, leaving the Jacobian $\Delta(x)^2$ for the volume of the $U(N)$ gauge group. The wavefunction becomes $\Psi_{k-gs}(U\Lambda U^\dagger) = \exp(\mathrm{i}\sigma)\Delta(x)^k k^{Nk/2}$, leading to [8]

$$\langle \Psi_{k-gs}|\Psi_{k-gs}\rangle = \mathcal{N} \int \mathcal{D}\Lambda\, \Delta(x)^2 \mathrm{e}^{-B\sum_i x_i^2} \Delta(x)^{2k}. \qquad (5.8)$$

After reduction, we find that the wavefunction of the Calogero model should be defined with an additional factor of the Vandermonde, $\Phi_{k-gs} = \Delta(x)\Psi_{k-gs}$, corresponding to the shift $k \to k+1$ found by several authors [5, 4].

Another way to understand this shift is through the comparison of the ground state energies computed in the original matrix theory and the Calogero model (the case $k = 0$ is already significant). In the Chern–Simons matrix model, the Hamiltonian $\mathcal{H} = \omega \operatorname{Tr}(X_1^2 + X_2^2)$ (setting $B = 2$) is a collection of N^2 harmonic oscillators and the $k = 0$ ground state is the Fock vacuum. The quantization of these bosonic oscillators gives the ground state energy $E_0 = \omega N^2/2$. On the other hand, the $k = 0$ Calogero model contains N oscillators that would give $E_0 = \omega N/2$ if quantized as bosons and $E_0 = \omega \sum_{n=0}^{N-1}(n + 1/2) = \omega N^2/2$ if they are fermions. Therefore, the second choice should be made, leading to the ground state wavefunction $\Phi_{0-gs} = \exp(-B\sum x_i^2)\Delta(x)$ as the result of the Slater determinant of the first N harmonic oscillator states.

5.2. Holomorphic quantization

The path integral of the Chern–Simons matrix model in holomorphic form (3.2) is

$$\langle f|i\rangle = \int \mathcal{D}X(t)\,\mathcal{D}\overline{X}(t)\,\mathcal{D}\psi(t)\,\mathcal{D}\overline{\psi}(t)$$
$$\times \exp \int \mathrm{d}t \left(\frac{B}{2}\operatorname{Tr}(X^\dagger \dot{X}) + \psi^\dagger \dot{\psi} - \mathrm{i}\mathcal{H} \right) \prod_t \delta(G(t))\,\mathrm{FP}. \qquad (5.9)$$

The analysis goes in parallel with that of the previous section, with some differences regarding the reality conditions. The classical change of variables that corresponds to the canonical transformation to the complex eigenvalues[9] (3.7) and (3.10) is

$$
\begin{aligned}
X &= V^{-1}\Lambda V, & \psi &= V^{-1}\phi, \\
X^{\dagger} &= V^{-1}\tilde{\Lambda}V, & \psi^{\dagger} &= \tilde{\phi}V,
\end{aligned}
\tag{5.10}
$$

where Λ is diagonal and V belongs to the quotient of linear complex matrices modulo the real diagonal ones, $V \in GL(N,\mathbb{C})/\mathbb{R}^N$. In the $k = 0$ case, the normal matrices are diagonalized by a unitary transformation, $V^{-1} = V^{\dagger}$, and thus $\tilde{\Lambda} = \Lambda^{\dagger}$ is also diagonal; for general k, the matrix $\tilde{\Lambda}$ will be different from the conjugate of Λ and non-diagonal. Therefore, we should perform a transformation of the integration measure in (5.9) that does not respect the complex conjugation of matrices at the classical level, i.e. a analytic continuation of the matrix integral. The same remark applies to the integration of ψ. Nonetheless, the final result will be real and well defined.

In the matrix models arising from $N = 2$ topological string theory [36], one encounters an analogous situation of real (so-called A-model) and holomorphic (B-model) quantizations. The holomorphic matrix model has been analysed in depth by Lazaroiu in [37]. Following his approach, we will consider X and $\overline{X}$ as independent complex matrices and $\int \mathcal{D}X$ as the holomorphic integral on a curve $\gamma \in \mathbb{C}^{N^2}$ that is left invariant by the holomorphic gauge transformations V. The reduction to the eigenvalues is in this case [37]

$$
\int_{\gamma} \mathcal{D}X = \int_{\gamma} \mathcal{D}\Lambda \, \mathcal{D}v \, \Delta(\lambda)^2, \qquad \mathrm{d}v = V^{-1}\,\mathrm{d}V.
\tag{5.11}
$$

Note that all three terms in the rhs of this equation are holomorphic, thus respecting the counting of degrees of freedom.

Therefore, the holomorphic version of the Faddeev–Popov term (5.3) is

$$
1 = \int \mathcal{D}v \, \mathcal{D}\Lambda \, \delta(VXV^{-1} - \Lambda)\Delta(\lambda)^2,
\tag{5.12}
$$

where the real delta function has been extended to complex holomorphic arguments. After the transformation (5.10) in the path integral, the Gauss constraint can be solved for $\tilde{\Lambda}$ (cf (5.4)), leading to

$$
\begin{aligned}
\delta(G) &= \prod_{i,j=1}^{N} \delta\left(\frac{B}{2}\left(\lambda_i - \lambda_j\right)\tilde{\Lambda}_{ij} - k\delta_{ij} + \phi_i\tilde{\phi}_j\right) \\
&= \Delta(\lambda)^{-2}\prod_{i\neq j}\delta\left(\tilde{\Lambda}_{ij} + \frac{2}{B}\frac{\phi_i\tilde{\phi}_j}{\lambda_i - \lambda_j}\right)\prod_{i}\delta\left(\phi_i\tilde{\phi}_i - k\right).
\end{aligned}
\tag{5.13}
$$

The Faddeev–Popov determinant and the Jacobian of the Gauss constraint cancels out in the path integral as in the previous case of real quantization. In the solution $\tilde{\Lambda}$, we recognize the off-diagonal terms of the covariant derivative (3.10): the complete

---

[9] Note that the canonical commutators (3.3) are left invariant by these $GL(N,\mathbb{C})$ transformations that are more general than the $U(N)$ gauge invariance of the complete theory.

parametrization is

$$\tilde{\Lambda}_{ij} = \tilde{\lambda}\delta_{ij} - \frac{2}{B}\frac{1-\delta_{ij}}{\lambda_i - \lambda_j}\phi_i\tilde{\phi}_j. \tag{5.14}$$

The diagonal elements are unconstrained and become the canonical conjugate variables of the eigenvalues. Actually, the substitution of the transformation (5.10) and of the constraints (5.12), (5.13) in the path integral (5.9) yield the action

$$\frac{B}{2}\operatorname{Tr}(X^\dagger \dot{X}) + \psi^\dagger\dot{\psi} = \frac{B}{2}\operatorname{Tr}\left(\tilde{\Lambda}\dot{\Lambda} - [\Lambda,\tilde{\Lambda}]\dot{V}V^{-1}\right) + \tilde{\phi}\dot{\phi} - \tilde{\phi}\dot{V}V^{-1}\phi$$
$$= \sum_n\left(\frac{B}{2}\tilde{\lambda}_n\dot{\lambda}_n + \tilde{\phi}_n\dot{\phi}_n\right) - k\operatorname{Tr}(V^{-1}\dot{V}), \tag{5.15}$$

and finally (setting $B = 2$ hereafter)

$$\langle f|i\rangle = \int \prod_n \mathcal{D}\tilde{\lambda}_n(t)\,\mathcal{D}\lambda_n(t)\,\mathcal{D}\tilde{\phi}_n(t)\,\mathcal{D}\phi_n(t)\prod_{i,t}\delta(\tilde{\phi}_i(t)\phi_i(t) - k)$$
$$\times \exp\int \mathrm{d}t\left[\sum_n\tilde{\lambda}_n\dot{\lambda}_n + \tilde{\phi}_n\dot{\phi}_n - \mathcal{H}\left(\Lambda,\tilde{\Lambda}\right)\right]\Bigg|_{\tilde{\Lambda}_{ij}=\tilde{\lambda}_i\delta_{ij}-(1-\delta_{ij})\phi_i\tilde{\phi}_j/(\lambda_i-\lambda_j)}. \tag{5.16}$$

The variables $\{\phi_n, \tilde{\phi}_n\}$ are actually frozen to $\sqrt{k}$, $\forall t$, but are left indicated in (5.16) to keep track of possible normal orderings in the Hamiltonian. Once they are eliminated, we recognize that this path integral describes electrons in the lowest Landau level, with coordinates $\{\lambda_n, \tilde{\lambda}_n\}$. The replacement of $X^\dagger$ by $\tilde{\Lambda}$ amounts to a 'twisted' rule of complex conjugation that will better analysed in the next section.

The result (5.16) can also be obtained in another way that uses the Ginibre decomposition of complex matrices respecting reality conditions [35]. After diagonalization of X, the measure of integration can be expressed in terms of the eigenvalues and their complex conjugates, as follows [38]:

$$\mathcal{D}X\,\mathcal{D}\overline{X}\,\mathcal{D}\psi\,\mathcal{D}\overline{\psi} = |\Delta(\lambda)|^4\prod_{i=1}^N\mathrm{d}\overline{\lambda}_i\,\mathrm{d}\lambda_i\prod_{i\neq j=1}^N\mathrm{d}v_{ij}\,\mathrm{d}\overline{v}_{ij}\prod_{i=1}^N\mathrm{d}\phi_i\,\mathrm{d}\overline{\phi}_i. \tag{5.17}$$

The $\mathrm{d}v_{ij}$ integral can be further elaborated, thanks to the decomposition [35]

$$V = DYU, \tag{5.18}$$

where D is a diagonal, positive real matrix ($D \to I$ for the above-mentioned quotient), Y is upper triangular with diagonal elements equal to one and U is unitary. The measure $\prod_{i\neq j}\mathrm{d}v_{ij}\,\mathrm{d}\overline{v}_{ij}$ factorizes into the unitary measure $\prod_{i>j}\mathrm{d}\omega_{ij}\,\mathrm{d}\overline{\omega}_{ij}$ and the measure $\prod_{i<j}\mathrm{d}\alpha_{ij}\,\mathrm{d}\overline{\alpha}_{ij}$ for $\mathrm{d}\alpha = Y^{-1}\mathrm{d}Y$. The earlier variables (5.10) are expressed in the coordinates (5.18) as follows:

$$\tilde{\Lambda} = H\overline{\Lambda}H^{-1}, \qquad \tilde{\phi} = \phi^\dagger H^{-1}, \qquad H = YY^\dagger, \tag{5.19}$$

and the Gauss constraint reads

$$\prod_{i \neq j} \delta((\lambda_i - \lambda_j)(H\overline{\Lambda}H^{-1})_{ij} + \phi_i(\phi^\dagger H^{-1})_j) \prod_i \delta(\phi_i(\phi^\dagger H^{-1})_i - k). \qquad (5.20)$$

We see that $(N^2 - N)$ gauge degrees of freedom have disappeared, while the $(N^2 - N)$ variables of Y, i.e. of the similarity transformation H, should be fixed by the Gauss constraint in terms of $\{\lambda_n, \overline{\lambda}_n, \phi_m, \overline{\phi}_m\}$.

Although in complex conjugate pairs, these coordinates do not have a simple dynamics and the reduced path integral cannot be interpreted in the quantum Hall effect. This set of variables was also considered by Karabali and Sakita [8], with the difference that they allowed $\{H_{nm}, \phi_n\}$ to fluctuate because no gauge fixing was included in the path integral. Although fully legitimate for finite N, this approach does not allow one to disentangle the eigenvalues from the remaining variables and makes the physical interpretation harder. Therefore, we are led to reintroduce the earlier variables $\{\tilde{\lambda}_n, \tilde{\phi}_n\}$ in the path integral via the identities

$$1 = \int \prod_n \mathrm{d}\tilde{\lambda}_n \, \mathrm{d}\tilde{\phi}_n \, \delta(\tilde{\lambda}_n - (H\overline{\Lambda}H^{-1})_{nn})\delta(\tilde{\phi}_n - (\phi^\dagger H^{-1})_{nn}), \qquad (5.21)$$

and solve for $\{\overline{\lambda}_n, \overline{\phi}_n\}$ and $\{Y_{nm}\}$ (or $\{H_{nm}\}$ without the extra Jacobian, $Y^{-1}\mathcal{D}Y = H^{-1}\mathcal{D}H$ [35]). The result is the following: the first two deltas in (5.20), (5.21) combine into $\delta(\overline{\Lambda} - H^{-1}\tilde{\Lambda}H)$ times the Jacobian $|\Delta(\lambda)|^{-4}$ that cancels the contribution from the integration measure (5.17). This result is derived in analogy with the other deltas occurring before and explicitly checked for $N = 2$.

5.3. Complex overlaps

The integration measure for the Chern–Simons matrix model in holomorphic quantization is obtained from the coherent states of the matrix components (3.4), with the inclusion of the Gauss law constraint and Faddeev–Popov term ($B = 2$):

$$\langle \Psi_1 | \Psi_2 \rangle = \int \mathcal{D}X \, \mathcal{D}\overline{X} \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \, \mathrm{e}^{-\operatorname{Tr} X^\dagger X - \psi^\dagger \psi} \delta(G) \, \mathrm{FP} \, \overline{\Psi_1(X, \psi)} \Psi_2(X, \psi). \qquad (5.22)$$

Using the previous solutions of the constraints (5.12) and (5.13), we find the reduced measure:

$$\langle \Psi_1 | \Psi_2 \rangle = \int \prod_n \mathrm{d}\tilde{\lambda}_n \, \mathrm{d}\lambda_n \, \mathrm{d}\tilde{\phi}_n \, \mathrm{d}\phi_n \, \mathrm{e}^{-\sum_n (\tilde{\lambda}_n \lambda_n + \tilde{\phi}_n \phi_n)}$$
$$\times \overline{\Psi_1(\tilde{\Lambda}, \tilde{\phi})} \Psi_2(\Lambda, \phi)\big|_{\tilde{\phi}_i \phi_i = k, \tilde{\Lambda}_{ij} = \tilde{\lambda}_i \delta_{ij} - (1 - \delta_{ij})(\phi_i \tilde{\phi}_j / (\lambda_i - \lambda_j))}. \qquad (5.23)$$

In this expression, the wavefunction $\Psi_1(X, \psi)$ is first complex conjugated and then the matrix $X^\dagger$ is replaced by $\tilde{\Lambda}$. The frozen $\{\tilde{\phi}_n, \phi_n\}$ are again maintained for allowing normal orderings. For example, the $N = 2$ ground state overlap reads (up to constants)

$$\langle \Psi_{k-gs} | \Psi_{k-gs} \rangle_{N=2} = \int \mathrm{d}\tilde{\lambda}_1 \, \mathrm{d}\lambda_1 \, \mathrm{d}\tilde{\lambda}_2 \, \mathrm{d}\lambda_2 \, \mathrm{e}^{-\tilde{\lambda}_1 \lambda_1 - \tilde{\lambda}_2 \lambda_2} \left(\tilde{\lambda}_1 - \tilde{\lambda}_2 + \frac{2k}{\lambda_1 - \lambda_2} \right)^k (\lambda_1 - \lambda_2)^k.$$
$$(5.24)$$

Owning to the twisted rule of conjugation, this overlap does not have the standard expression for the Laughlin wavefunction in the quantum Hall effect:

$$\langle \Psi_{k-gs} | \Psi_{k-gs} \rangle_{N=2} = \int d\overline{z}_1 \, dz_1 \, d\overline{z}_2 \, dz_2 \, e^{-\overline{z}_1 z_1 - \overline{z}_2 z_2} (\overline{z}_1 - \overline{z}_2)^k (z_1 - z_2)^k. \tag{5.25}$$

Here, we remark that the physically relevant quantities are the values of the overlap integrals, normalized to that of the ground state: thus, the discrepancy between (5.24) and (5.25) does not immediately imply that the matrix theory does not describe the Laughlin states.

The rule of conjugation in the overlap integral (5.23) is consistent with the result of the change of variables of section 3: actually, we see that the covariant derivative (3.10) acting on the holomorphic wavefunction Ψ_2 can be adjoined as follows:

$$\langle \Psi_1 | \operatorname{Tr}(X^{\dagger n}) \Psi_2 \rangle = \langle \operatorname{Tr}(X^n) \Psi_1 | \Psi_2 \rangle. \tag{5.26}$$

Therefore, the W_∞ operators satisfy the correct Hermiticity rule, $\mathcal{L}_{nm}^\dagger = \mathcal{L}_{mn}$, both in the matrix (4.10) and reduced (4.16) coordinates. In conclusion, the twisted conjugation rule guarantees the Hermiticity of the covariant derivative that shows up in the reduced coordinates.

This result is rather important for the physical interpretation of the Chern–Simons matrix model. The main physical aspect of this theory is the realization of the Laughlin wavefunction in the ground state thanks to the conditions set by the Gauss law, following from the classical non-commutativity of fields (section 3.1). Actually, $U(N)$ group theory arguments [7] prove that all wavefunctions should contain the Laughlin ground state as a factor (cf equation (3.14)); thus the ground state is stable (incompressible). Unwanted states representing smaller droplets of fluid are possible in the matrix Fock space, but they do not respect the Gauss law and are not physical states. In conclusion, stability comes from gauge invariance and the Gauss law condition.

In the reduced eigenvalue basis, the gauge symmetry has been projected out, but the stability of the ground state is still proven by the W_∞ incompressibility conditions, that are satisfied thanks to the statistical interaction, i.e. to the covariant derivatives. As in the Chern–Simons matter–field theory approach of Fradkin and Lopez [20], the stability of the Laughlin ground state is realized by adding a further gauge interaction, rather than a two-body repulsion between the electrons [39]. In the reduced theory (5.16), (5.23), ordinary derivatives $\partial/\partial\lambda_n$ could create states of lower energy and higher density, i.e. generate instabilities; however, these operators are forbidden because they are not Hermitian under the twisted conjugation rule (5.23), and thus create non-unitary states; the W_∞ generators $\mathcal{L}_{nm}$ are the only Hermitian operators available for creating excitations. In conclusion, the Hermitian conjugation of (5.23) is consistent with the presence of the statistical interaction that stabilizes the Laughlin ground state.

It remains to be proven that the unconventional overlaps (5.23) take the same value (for physical states only) of the ordinary quantum Hall expressions. Actually, as discussed in section 4.1, equation (4.5), a coordinate-free characterization of the overlaps in the quantum Hall effect is obtained by expressing them as commutators of the W_∞ algebra. Thus, the proof of the correspondence between the matrix and Hall overlaps is equivalent to the derivation of the unitary representations of the W_∞ algebra in the Chern–Simons matrix model.

Another aspect of the measure of integration (5.23) is the lack of the shift $k \to k+1$ that was found in the real quantization and was crucial for the analysis of the W_∞ symmetry in section 4.3. In particular, for $k = 0$, we do not recover the known result[10] $|\Delta(\lambda)|^2$ for the measure of the ensemble of normal complex matrices [35] that should correspond to the $\nu = 1$ quantum Hall effect [38]. The reason for this discrepancy is the different definitions of the measure of integration: in our case, we solved for the delta function of the Gauss constraint coming from the integration of the A_0 field in the Lagrangian; in [38], they computed the induced metric on the manifold of the classical constraint $[X, X^\dagger] = 0$ in the space $\mathbb{C}^{N^2}$ of complex matrices. The two analyses are not in contradiction: indeed, the Gauss constraint is degenerate for $k = 0$, where it admits both types of solution, depending on how the quantity $\delta(0)$ is regularized. In contrast, for $k > 0$, the constraint is made non-degenerate by the presence of the extra field ψ and our results are not ambiguous and hold in the limit $k \to 0^+$.

The study of the ground state energy provides another way to analyse this issue, as discussed at the end of section 5.1. In the present case of a quadratic Hamiltonian, the ground state energy is also related to the filling fraction, i.e. to the size of the droplet of fluid [4]: for a circular droplet of uniform fluid, one finds $\langle R^2 \rangle = (N/2\pi)\mathrm{Area}$, and therefore

$$\frac{1}{\nu} = \frac{B\mathrm{Area}}{2\pi N} = \frac{B}{N^2}\langle \mathrm{Tr}(X_1^2 + X_2^2) \rangle = \frac{2\omega}{N^2}E_0. \tag{5.27}$$

In the real case, we got $\langle \mathrm{Tr}(X_1^2 + X_2^2) \rangle = \langle \mathrm{Tr}\, XX^\dagger + N^2/2 \rangle$, namely a shift in the ground state energy leading to $1/\nu = k + 1$. In the holomorphic matrix quantization of section 3, the normal ordering is $\langle \mathrm{Tr}\, X_a^2 \rangle = \langle \sum_{ij} X_{ij}\partial/\partial X_{ij} \rangle$, yielding no shift (and vanishing E_0 for $k = 0$). Thus, it seems that our result (5.23) is consistent with the operator ordering of matrices in holomorphic quantization. On the other hand, the ordering adopted for the W_∞ operators in the reduced coordinates (4.16) seems to be different, because the incompressibility conditions are satisfied by the shifted wavefunction Φ_{k-gs} (cf equation (4.17)). The shift in the wavefunctions is also important for changing the statistics of the reduced particle degrees of freedom from bosonic to fermionic.

Although we do not at present understand these facts completely, they seem to indicate that the derivation of the reduced holomorphic path integral and overlap should be modified by allowing for ground state fluctuations. One possibility is to include an additional measure by hand in the starting expression for the complex matrix overlap (5.22). This should be a gauge invariant, self-adjoint (real positive) quantity, carrying no charge for the ψ field, i.e. made of $X, X^\dagger$ only, and should reduce to $|\Delta(\lambda)|^2$ for $k = 0$. The following measure satisfies these rather stringent requirements and is inspired by the work of [40]:

$$\mathcal{M} = \det(\mathrm{Tr}(X^{\dagger i-1}X^{j-1})) = \det((\tilde{\Lambda}^{i-1})_{jj})\Delta(\lambda). \tag{5.28}$$

This quantity inserted in the overlap (5.22) realizes the shift $k \to k+1$ in the eigenvalue wavefunction Φ of section 4.3. Note also that a similar determinant expression involving the ψ field, $\det(\psi^\dagger X^{\dagger i-1}X^{j-1}\psi)$, is actually the squared modulus of the ground state wavefunction. The measure (5.28) should also be included in the holomorphic path integral (5.9) for consistency, where it defines an instantaneous repulsive interaction that enforces the fermionic character of the electrons.

---

[10] Note that $\tilde{\lambda}_n \to \bar{\lambda}_n$ for $k \to 0$.

6. Conclusions

In this paper, we described the holomorphic quantization of the Chern–Simons matrix model; we performed a change of matrix variables that allowed us to solve the Gauss law constraint and to map the N^2-dimensional quantum mechanics of $D0$-branes with Chern–Simons dynamics into the problem of N particles in the lowest Landau level. We found that the matrix theory describes the Laughlin ground state wavefunction and the corresponding excitations, together with a statistical interaction that is crucial for stabilizing the ground state and verifying the W_∞ incompressibility conditions.

Some gaps are still present in our analysis: we did not obtain the full representation of the W_∞ algebra in the theory and could not completely determine the measure in the overlap integrals. Nonetheless, we showed that the eventual proof of the W_∞ symmetry in the matrix theory would definitely establish the correspondence with the Laughlin Hall states, independently of coordinate choices. We can at present confirm that the long distance properties of the Laughlin states are faithfully reproduced, because the matrix W_∞ algebra (4.14) closes to the leading semiclassical order; this result also follows from the original semiclassical argument given by Susskind combined with the reduction of the matrix path integral to the lowest Landau level (section 5.2).

The non-commutative Chern–Simons theory has shown rather interesting features: besides incorporating the discrete nature of the electrons [1], it enhances the quantum repulsion of the electrons in the first Landau level,

$$[[\overline{\lambda}_n, \lambda_m]] = \frac{2\hbar}{B}\delta_{nm}, \tag{6.1}$$

by means of the non-commutativity of classical matrices,

$$[X, X^\dagger] = 2\theta, \tag{6.2}$$

thus reducing the filling fraction from $\nu = 1$ to $\nu = 1/(1+B\theta)$. Moreover, the electrons are placed in the non-trivial quantum state of Laughlin, that follows by gauge invariance and the Gauss law condition (2.15): it corresponds to a $SU(N)$ gauge singlet with $k = B\theta\,U(1)$ sources of the ψ field (section 3.1). This is another consequence of the non-commutativity relation (6.2).

The (integer) statistical interaction among the electrons found in holomorphic quantization can also be deduced from non-commutativity. Equation (6.2) can be interpreted as a gauge field strength by representing non-commutative fields, i.e. matrices, as covariant derivatives. Setting $X = Z + R$, where Z is a normal matrix (unitarily equivalent to Λ), we can view Z as the ordinary derivative ($[Z, Z^\dagger] = 0$) and R as the gauge potential. The solution of (6.2) found in section 3 is thus

$$X = Z, \qquad (R = 0), \qquad X^\dagger = Z^\dagger + R^\dagger, \qquad [Z, R^\dagger] = 2\theta. \tag{6.3}$$

Note that such decomposition of the covariant derivative is preserved under $GL(N, \mathbb{C})$ transformations $X \to V^{-1}XV$ provided that the gauge potential transforms as $R \to V^{-1}RV + V^{-1}[Z, V]$.

A known property of the Laughlin Hall states is the presence of non-trivial quantum mechanical long range correlations that cause the fractional statistics of excitations [12] and the topological order of the ground state on compact surfaces [16]. An interesting open

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question is how to recover these features in the matrix theory. The $1/(k+1)$ fractional statistics of two quasi-hole excitations, with matrix wavefunction

$$\Psi_{2q-hole}(z_1, z_2; X, \psi) = \det(z_1 - X) \det(z_2 - X)\Psi_{k-gs}(X, \psi), \qquad (6.4)$$

should come out as a (re)normalization effect in the limit $N \to \infty$. Note that for N finite this state belongs to the same (unique) W_∞ representation including the ground state (cf equation (3.14)).

The Chern–Simons matrix model could provide a concrete setting for studying the 'exactness' and 'universality' of the Laughlin wavefunction, upon using the sophisticated tools of matrix models [38, 41], as well as the new features of non-commutative field theories [13, 14]. Among the possible developments, we mention:

- The description of the Jain states at filling fraction $\nu = (k + 1/m)^{-1}$ by means of a suitable extension of the boundary terms (cf [6]).
- The study of Morita's equivalences in the non-commutative theories [13, 14, 11], that change the non-commutativity parameter θ via a $SL(2, \mathbb{Z})$ transformation, and could actually relate the matrix theories of different plateaus.

Acknowledgments

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