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2011 J. Phys. A: Math. Theor. 44 075401

(http://iopscience.iop.org/1751-8121/44/7/075401)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **44** (2011) 075401 (36pp) doi:10.1088/1751-8113/44/7/075401

Partition functions of non-Abelian quantum Hall states

Andrea Cappelli[1] **and Giovanni Viola**[1,2]

[1] INFN, Sezione di Firenze, Via G. Sansone 1, 50019 Sesto Fiorentino (FI), Italy
[2] Dipartimento di Fisica e Astronomia, Università di Firenze, Via G. Sansone 1, 50019 Sesto Fiorentino (FI), Italy

E-mail: andrea.cappelli@fi.infn.it and giovanni.viola@fi.infn.it

Received 30 July 2010, in final form 22 December 2010
Published 20 January 2011
Online at stacks.iop.org/JPhysA/44/075401

Abstract
Partition functions of edge excitations are obtained for non-Abelian Hall states in the second Landau level, such as the anti-Read–Rezayi state, the Bonderson–Slingerland hierarchy and the Wen non-Abelian fluid, as well as for the non-Abelian spin-singlet state. The derivation is straightforward and unique starting from the non-Abelian conformal field theory data and solving the modular invariance conditions. The partition functions provide a complete account of the excitation spectrum and are used to describe experiments of Coulomb blockade and thermopower.

PACS numbers: 73.43.Cd, 11.25.Hf, 73.23.Hk, 73.43.Jn

(Some figures in this article are in color only in the electronic version)

1. Introduction

Quantum Hall states [1] in the second Landau level are intensively investigated both theoretically [2, 3] and experimentally [4, 5], because they might possess excitations with non-Abelian fractional statistics [6]. The latter could provide concrete realizations of quantum gates that are topologically protected from decoherence and thus implement the topological quantum computation scheme of [7].

Besides the original Moore–Read Pfaffian state at the filling fraction $\nu = 5/2$ [6] and its parafermion generalization by Read and Rezayi (RR) [8], other non-Abelian states have been proposed that could also explain the observed plateaux for filling fractions $2 < \nu < 3$. These are the non-Abelian spin-singlet states (NASS), introduced in [9] and further developed in [10], the charge-conjugates of Read–Rezayi's states ($\overline{\mathrm{RR}}$) [11], the Bonderson–Slingerland (BS) hierarchy built over the Pfaffian state [12] and Wen's $SU(n)$ non-Abelian fluids (NAF) [13], the $SU(2)$ case in particular.

In this paper, we continue the study of partition functions for edge excitations of quantum Hall states. Issued from conformal field theory (CFT) data [14], the partition functions are found in the geometry of the annulus where they enjoy the symmetry under modular transformations [15]: they provide a complete identification of the Hilbert space of excitations and can be used to describe experiments searching non-Abelian statistics. Modular invariant partition functions were obtained in [16, 17] for the Abelian hierarchical and non-Abelian Read–Rezayi states; here, we provide the expressions for the other non-Abelian states in the second Landau level.

In general, the non-Abelian states are described by rational conformal field theories (RCFT) of the type $U(1) \times G/H$, where the non-Abelian part is characterized by the affine symmetry group G or the coset G/H [14], and the Abelian part $U(1)$ accounts for the charge of excitations. The parameters specifying the second part (compactification radius, charges and filling fractions) can be determined by the standard requirements on the charge and statistics of the electron and its relative statistics with respect to the other excitations [18]; in some cases, further conditions are suggested by the physics of the specific Hall state [19].

In our analysis, the construction of partition functions of non-Abelian Hall states is completely straightforward. The inputs are (i) the conformal field theory G/H of the neutral non-Abelian part of excitations and (ii) the choice of Abelian field in this theory representing the neutral part of the electron. From these data, the charge and statistics of all excitations can be self-consistently found without any further physical hypothesis. Actually, modular invariance requires a non-trivial pairing between the sectors of the neutral and charged RCFTs and reproduces the standard physical conditions on the spectrum.

Modular invariance is one of the defining properties of RCFT; when combined with the exchange (duality) symmetry of correlators, it implies the Moore–Seiberg identities among n-point functions on Riemann surfaces [20]. In particular, the matrix elements S_{ab} of the S modular transformation of (extended) conformal characters θ_a, corresponding to the partition functions on the disk [16], determine the fusion rules of quasiparticles *via* the Verlinde formula [14], as well as entropy numbers describing both the degeneracy of non-Abelian quasiparticle many-body states (quantum dimensions $d_a = S_{a0}/S_{00}$) [14] and the bipartite entanglement of topological fluids, $\mathcal{S} = \alpha L - \gamma$, where L is the length of the boundary and γ is the universal term, $\gamma = \frac{1}{2}\log\left(\sum_a d_a^2\right)$ [21].

In recent literature, the partition functions of Abelian hierarchical and non-Abelian Read–Rezayi states were successfully employed for describing the Coulomb Blockade (CB) current peaks [22], both at zero [16, 17] and non-zero [23, 24] temperatures. Here we extend these analyses to the other non-Abelian fluids; we also point out that the thermal-activated off-equilibrium CB current can measure the degeneracies of neutral states and distinguish between different Hall states with equal peak patterns at zero temperature [25].

Another proposed signature of non-Abelian statistics that could be experimentally accessible [5] is the thermopower, namely the ratio of the thermal and electric gradients at equilibrium [26, 27]: this could measure the quantum dimension d_1 of the basic quasiparticle in the Hall fluid. We show that the thermopower can be easily described by the edge partition function, with the S modular matrix playing an important role again.

Let us stress that in this paper we describe Hall states by means of rational CFTs and exploit the modular invariance of their partition functions; we do not discuss other approaches involving non-rational theories that could be relevant for the Jain hierarchical states in particular [19, 28].

The paper is organized as follows. In section 2, we recall the main features of partition functions in the quantum Hall effect (QHE) [17]: we rederive their expression for the Read–Rezayi states starting from the non-Abelian CFT data only. In section 3, we obtain the partition

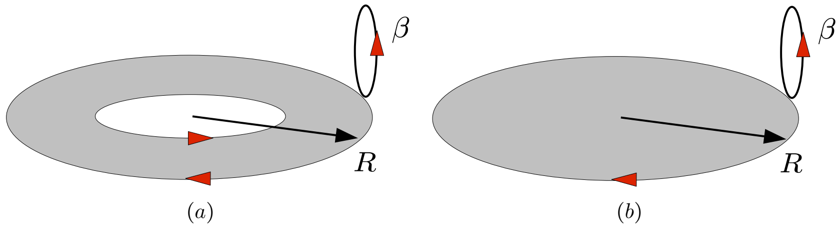

Figure 1. The (*a*) annulus and (*b*) disk geometries.

functions corresponding to the other non-Abelian states. In section 4, we use the partition functions to compute the Coulomb blockade current peaks at non-vanishing temperatures, the thermopower and the associated non-Abelian entropies. In section 5, we discuss further model building based on the study of partition functions. The appendix contains more technical data of non-Abelian RCFT and modular transformations.

2. Building partition functions

2.1. Modular invariance

Partition functions are best defined for the Hall geometry of an annulus (see figure 1(*a*)) to which we add a compact Euclidean time coordinate for the inverse temperature β. This space geometry allows for the measurement of the Hall current and is equivalent to the bar geometry (corresponding to $R \to \infty$), while enjoying some special symmetries. The disk geometry (figure 1(*b*)), describing isolated Hall droplets, can be obtained from the annulus by shrinking the inner radius to zero.

As a spacetime manifold, the annulus at finite temperature has the topology: $\mathcal{M} = S^1 \times S^1 \times I$, where I is the finite interval of the radial coordinate. The edge excitations live on the boundary $\partial\mathcal{M}$, corresponding to two copies of a spacetime torus $S^1 \times S^1$: they are chiral and anti-chiral waves on the outer (R) and inner (L) edges, respectively. This geometry is particularly convenient owing to the symmetry under modular transformations acting on the two periodic coordinates and basically exchanging the space and time periods. The partition functions should be invariant under this geometrical symmetry, which actually implies several physical conditions on the spectrum of the theory [17]. For simplicity, we consider the case of no bulk excitations inside the annulus: later we shall see how to include them.

The (grand canonical) partition function on the annulus is defined by [14]

$$Z_{\text{annulus}}(\tau,\zeta) = \mathcal{K}\,\mathrm{Tr}\left[\mathrm{e}^{\mathrm{i}2\pi(\tau(L_0^L-c/24)-\bar{\tau}(L_0^R-c/24)+\zeta Q^L+\bar{\zeta}Q^R)}\right], \tag{2.1}$$

where the trace is over the states of the Hilbert space, $\mathcal{K}$ is a normalization and (τ,ζ) are complex numbers with $\mathrm{Im}\,\tau > 0$. The total Hamiltonian and spin are given by

$$\begin{aligned} H &= \frac{v_R}{R_R}\left(L_0^R - \frac{c}{24}\right) + \frac{v_L}{R_L}\left(L_0^L - \frac{c}{24}\right) + V_o\left(Q^L - Q^R\right) + \text{ const.},\\ J &= L_0^L - L_0^R. \end{aligned} \tag{2.2}$$

The energy on both L and R edges, $E = (v/R)(L_0 - c/24)$, is proportional to the dilatation operator in the plane L_0, with eigenvalue being the conformal dimension h; it also includes the Casimir energy proportional to the Virasoro central charge c [14]. The real and imaginary parts of τ are respectively given by the 'torsion' η and the inverse temperature β times the

Fermi velocity v; the parameter ζ is proportional to the chemical potential μ and electric potential difference between the edges V_o:

$$\mathrm{i}2\pi\tau = -\beta\,\frac{v + \mathrm{i}\eta}{R}, \qquad \mathrm{i}2\pi\zeta = -\beta\,(V_o + \mathrm{i}\mu)\,. \tag{2.3}$$

Note that the charges Q^L and Q^R are defined in (2.1) in such a way as to obtain the correct coupling to the electric potential in (2.2). In the following, we shall momentarily choose a symmetric Hamiltonian for the two edges by adjusting the velocities of propagation of excitations, $v_L/R_L = v_R/R_R$.

Most theories of edge excitations in the QHE are RCFT [14]: their Hilbert space is divided into a finite number of 'sectors', each one describing a basic quasiparticle, with rational charge and statistics, together with the addition of electrons; for example, in the Laughlin states with filling $\nu = 1/p$, there are p sectors, $\lambda = 1, \ldots, p$, and the associated charge is $Q = \lambda/p + \mathbb{Z}$. In mathematical terms, each sector provides a representation of the maximally extended chiral symmetry algebra, which contains the Virasoro conformal algebra as a subalgebra.

The trace over the Hilbert space in (2.1) can be divided into sub-sums relative to pairs of sectors, left and right for the inner and outer edges; the sum of states in each sector gives rise to a character $\theta_\lambda(\tau, \zeta)$ of the extended algebra [14]. As a result, the partition function reduces to the finite-dimensional sum:

$$Z_{\text{annulus}} = \sum_{\lambda,\mu=1}^{N} \mathcal{N}_{\lambda,\mu}\, \theta_\lambda(\tau, \zeta)\, \overline{\theta^c_\mu(\tau, \zeta)}. \tag{2.4}$$

In this equation, the bar denotes complex conjugation and the suffix (c) is the charge conjugation C, acting by $Q \to -Q$, $\theta \to \theta^c$. The inner (resp. outer) excitations are described by θ_λ (resp. θ^c_μ), according to definition (2.1). The coefficients $\mathcal{N}_{\lambda,\mu}$ are positive integer numbers giving the multiplicities of sectors of excitations to be determined by imposing modular invariance and some physical requirements.

In the case of the Laughlin fluids, the RCFT is an extension of the affine $\widehat{U(1)}$ algebra of the chiral Luttinger liquid ($c = 1$). The characters are given by theta functions with rational characteristics [15]:

$$\begin{aligned}\theta_\lambda(\tau, \zeta) &= \mathrm{e}^{-\frac{\pi}{p}\frac{(\mathrm{Im}\zeta)^2}{\mathrm{Im}\tau}}\; K_\lambda(\tau, \zeta; p)\\ &= \mathrm{e}^{-\frac{\pi}{p}\frac{(\mathrm{Im}\zeta)^2}{\mathrm{Im}\tau}}\; \frac{1}{\eta(\tau)} \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi\left(\tau\frac{(pk+\lambda)^2}{2p} + \zeta\left(\frac{\lambda}{p} + k\right)\right)}.\end{aligned} \tag{2.5}$$

Note that each term in the summation is a character of a $\widehat{U(1)}$ representation with charge $Q = \lambda/p + k$, i.e. it describes the addition of k electrons to the basic anyon. The Dedekind function $\eta(\tau)$ in the denominator describes particle–hole excitations and the non-analytic exponential as a measure factor; both terms are explained in [15].

Owing to scale invariance, the doubly periodic geometry of the torus is specified by the modular parameter $\tau = \omega_2/\omega_1$, the ratio of the two periods ($\mathrm{Im}\,\tau > 0$). The same toroidal geometry can be described by different sets of coordinates respecting periodicity that are related among themselves by integer linear transformations with a unit determinant. The transformation of the modular parameter is

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad a, b, c, d \in \mathbb{Z}, \qquad ad - bc = 1. \tag{2.6}$$

Modular transformations belong to the infinite discrete group $\Gamma \equiv PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$ (the quotient is over the global sign of transformations) [14]. There are two generators, $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$, satisfying the relations $S^2 = (ST)^3 = C$, where C is the charge conjugation matrix, $C^2 = 1$ [14].

The invariance of the partition function under modular transformations is therefore given by

$$Z_{\text{annulus}}\left(\frac{-1}{\tau}, \frac{-\zeta}{\tau}\right) = Z_{\text{annulus}}(\tau+1,\zeta) = Z_{\text{annulus}}(\tau,\zeta). \tag{2.7}$$

Let us briefly recall from [15] the physical conditions corresponding to these symmetries. The S invariance amounts to a completeness condition for the spectrum of the RCFT: upon exchanging time and space, it roughly imposes that the set of states at a given time is the same as that ensuring time propagation [14]. The S transformation acts by a unitary linear transformation in the finite basis of the characters (2.4):

$$\theta_a\left(-\frac{1}{\tau}, -\frac{\zeta}{\tau}\right) = \mathrm{e}^{\mathrm{i}\varphi} \sum_{b=1}^{N} S_{ab}\, \theta_b(\tau,\zeta) \tag{2.8}$$

(φ is an overall phase and $N = p$ for Laughlin states). The matrix S_{ab} determines the fusion rules of the RCFT through the Verlinde formula [14]; moreover, the dimension N of the matrix is equal to the Wen topological order of the Hall fluid [1].

The action of the T^2 transformation is

$$T^2: \qquad Z(\tau+2,\zeta) \equiv \mathrm{Tr}\left[\cdots \mathrm{e}^{\mathrm{i}2\pi\ 2(L_0^L - L_0^R)}\right] = Z(\tau,\zeta); \tag{2.9}$$

namely, it allows states with integer or half-integer spin on the whole system. These are electron-like excitations that carry the electric current in and out of the system: thus, anyon excitations present on both edges should combine between them to form global fermionic states. Note that the presence of fermions in the QHE implies the weaker invariance under T^2 rather than T: actually, S and T^2 generate a subgroup of the modular group, $\Gamma_\theta \subset \Gamma$ [14].

The partition function should also be invariant under the transformations of the ζ variable, $\zeta \to \zeta + 1$ and $\zeta \to \zeta + \tau$, that can be geometrically interpreted as a coordinate on the torus. In physical terms, these correspond to conditions on the charge spectrum. The first one,

$$U: \qquad Z(\tau,\zeta+1) \equiv \mathrm{Tr}\left[\cdots \mathrm{e}^{\mathrm{i}2\pi(Q^R+Q^L)}\right] = Z(\tau,\zeta), \tag{2.10}$$

requires that excitations possess total integer charge, $Q^L + Q^R \in \mathbb{Z}$ (no quasiparticles are present in the bulk). Thus, fractionally charged excitations at one edge must pair with complementary ones on the other boundary. Consider, for example, a system at $\nu = 1/3$ and add one electron to it: it can split into one pair of excitations with $(Q^L, Q^R) = (1/3, 2/3), (0, 1), (2/3, 1/3), (1, 0)$. The different splittings are related one to another by tuning the electric potential V_o in (2.2).

The potential at the edges can be varied by adding the localized magnetic flux inside the annulus; the addition of one flux quantum leads to a symmetry of the spectrum, as first observed by Laughlin [1]. This is actually realized by the transformation $\zeta \to \zeta + \tau$, corresponding to $V_o \to V_o + 1/R$ (in our notations $e = c = \hbar = 1$) [15]. The corresponding invariance of the partition function is

$$V: \qquad Z(\tau,\zeta+\tau) = Z(\tau,\zeta). \tag{2.11}$$

This transformation is called the ‘spectral flow’, because it amounts to a drift of each state of the theory into another one.

In the case of the Laughlin theory, the T^2, S, U, V transformations of the characters θ_λ (2.5) are given by [15]

$$
\begin{aligned}
T^2 :\ \theta_\lambda(\tau+2,\zeta) &= \mathrm{e}^{\mathrm{i}2\pi\left(\frac{\lambda^2}{p}-\frac{1}{12}\right)}\theta_\lambda(\tau,\zeta),\\
S :\ \theta_\lambda\left(-\frac{1}{\tau},-\frac{\zeta}{\tau}\right) &= \mathrm{e}^{\mathrm{i}\frac{\pi}{p}\mathrm{Re}\frac{\zeta^2}{\tau}}\frac{1}{\sqrt{p}}\sum_{\lambda'=0}^{p-1}\mathrm{e}^{\mathrm{i}2\pi\frac{\lambda\lambda'}{p}}\,\theta_{\lambda'}(\tau,\zeta),\\
U :\ \theta_\lambda(\tau,\zeta+1) &= \mathrm{e}^{\mathrm{i}2\pi\lambda/p}\,\theta_\lambda(\tau,\zeta),\\
V :\ \theta_\lambda(\tau,\zeta+\tau) &= \mathrm{e}^{-\mathrm{i}\frac{2\pi}{p}\left(\mathrm{Re}\zeta+\mathrm{Re}\frac{\tau}{2}\right)}\theta_{\lambda+1}(\tau,\zeta).
\end{aligned}
\tag{2.12}
$$

These formulas show that the generalized characters θ_λ carry a unitary representation of the modular group, which is projective for $\zeta \neq 0$ (the composition law is verified up to a phase).

Note that the charge transported between the two edges by adding one flux quantum in the center (V transformation) is equal to the Hall conductivity: indeed, $\theta_\lambda(\zeta+\tau) \propto \theta_{\lambda+1}(\zeta)$ corresponds to $\nu = 1/p$. This provides a method to determine the value of ν from the partition function.

The corresponding sums of right $\widehat{U(1)}$ representations are given by $\overline{\theta}^c_\mu$ carrying charge $Q^R = \mu/p + \mathbb{Z}$. Finally, the U condition (2.10), applied to the Z_{annulus} (2.4), requires that left and right charges obey $\lambda + \mu = 0 \bmod p$. This form of the partition function also satisfies the other conditions, T^2, S, V, by unitarity.

Finally, the modular invariant partition function of Laughlin's states is

$$
Z_{\text{annulus}} = \sum_{\lambda=1}^{p} \theta_\lambda\,\overline{\theta}_\lambda. \tag{2.13}
$$

The partition function for the disk geometry is obtained from that of the annulus (2.13) by letting the inner radius to vanish, $R_L \to 0$ (see figure 1(b)). To this effect, the variable $\overline{\tau}$ in $\overline{\theta}_\lambda$ should be taken independent of τ: $\mathrm{Im}\tau \neq -\mathrm{Im}\overline{\tau}$, $v_R/R_R \neq v_L/R_L$. The annulus partition function is no longer a real positive quantity but remains modular invariant, up to a global phase. In the limit $R_L \to 0$, the $\overline{\theta}_\lambda$ are dominated by their $|q| \to 0$ behavior: therefore, only the ground-state sector remains in (2.13), leading to $\overline{\theta}_\lambda \to \delta_{\lambda,0}$, up to zero-point energy contributions. One finds $Z^{(0)}_{\text{disk}} = \theta_0$. In the presence of quasiparticles in the bulk of the disk, with charge $Q_{\text{Bulk}} = -a/q$, the condition of the total integer charge selects another sector, leading to

$$
Z^{(a)}_{\text{disk}} = \theta_a. \tag{2.14}
$$

This is the desired result for the partition function on the disk geometry; there are N such functions, $a = 1, \ldots, N$, that transforms unitarily among themselves under S as an N-dimensional vector. Note here a manifestation of the general correspondence between bulk and edge excitations that can be proven by using Chern–Simons theory but is actually valid for general RCFTs [29]: bulk excitations are equivalent, as much as the low-energy theory is concerned, to an edge with the radius shrinking to zero; in this limit, the tower of excitations in that sector decouples.

2.2. *Physical conditions for the spectrum*

The construction of RCFTs for quantum Hall states, both Abelian and non-Abelian, has been relying on a set of conditions for the spectrum of charge and statistics that implement the properties of electron excitations [18]; they should have

(A) integer charge;
(B) Abelian fusion rules with all excitations;
(C) fermionic statistics among themselves (half-integer spin);
(D) integer statistics with all other excitations (integer exponent of mutual exchange).

The (B) and (D) conditions characterize the operator–product expansion of the conformal field $\Phi_e(z)$, representing the electron, with the field $\Phi_i(w)$ of a quasiparticle: for $z \to w$, this is

$$\Phi_e(z)\ \Phi_i(w) \sim (z-w)^{h_{ei}}\ \Phi_{e\times i}(w). \tag{2.15}$$

The mutual statistics exponents are given by the conformal dimensions

$$h_{ei} = -h_e - h_i + h_{e\times i}, \tag{2.16}$$

respectively, of the electron field, the ith quasiparticle and their fusion product, $\Phi_{e\times i} = \Phi_e \times \Phi_i$. In general, excitations are made of a neutral part, described by a non-trivial RCFT, typically a coset theory G/H, and by a charged part (Luttinger liquid, i.e. a $\widehat{U(1)}$ RCFT) [14]: the fields in the above formulas are made of neutral and charged parts, and their dimensions h have contributions from both of them. The requirement of integer statistics of the electron with all excitations, $h_{ei} \in \mathbb{Z}$, is motivated by the properties of many-body wavefunctions (describing, e.g., states with a quasiparticle of ith type) that are described by correlators of the same RCFT for edge excitations. This bulk-edge correspondence follows again from the description of Hall fluids in terms of the Chern–Simons theory and is believed to be true for all rational CFTs [29].

The condition of Abelian fusion rules restricts to one the number of terms on the rhs of the operator product expansion (2.15): indeed, if there were more terms, all the corresponding h_{ei} exponents would need to be integers, a condition generically impossible to achieve. Even if it were satisfied, this would lead to a degeneracy of n-electron wavefunctions and an unacceptable degenerate ground state. Non-Abelian fusion rules and associated degeneracies (the quantum dimensions d_a) are possible for quasiparticles but not for electrons (within the RCFT description, at least [30]).

In the following, we show that the electron conditions (A)–(D) can be rederived from the requirement of modular invariance of the partition function. We find that upon choosing the RCFT for the neutral part of excitations and identifying the field representing the electron, the modular conditions are sufficient to self-consistently determine the charge and statistics spectrum, as well as the filling fraction[3], without the need of additional physical hypotheses.

Let us compare (A)–(D) with the modular conditions introduced in the previous section for Laughlin states. Condition (A) is clearly the same as the U modular invariance (2.10), whose solution is the extended character θ_λ (2.4) that resums electron excitations added to the λ quasiparticle.

Condition (B) of Abelian fusion rules of the electron has a natural correspondent in RCFT, where a field with such property is called a 'simple current' J ($\equiv \Phi_e$). The notion of a simple current was introduced for orbifold theories and their modular invariant partition functions, as we now briefly recall [14, 31].

The action of the simple current is indicated by $J \times \Phi_i = \Phi_{J(i)} (\equiv \Phi_{e\times i})$; it implies an Abelian discrete symmetry in the theory that is generated by $\exp(2\mathrm{i}\pi Q_J)$, with

$$Q_J(\Phi_i) = h_J + h_i - h_{J(i)} \qquad \mathrm{mod}\ 1. \tag{2.17}$$

This charge is the exponent for the monodromy discussed in (2.16) and is conserved in the fusion rules. The fields Φ_i can be organized in orbits, each orbit containing the fields generated

[3] And the spin parts for non-polarized Hall fluids.

by the repeated fusion with the simple current. The simple current and its powers generate an Abelian group by fusion that is called the 'center' $\mathcal{G}$ of the CFT.

The modular invariant partition function can be obtained by the orbifold construction corresponding to modding out the symmetry associated with the simple current. The result has the general form [31]

$$Z = \sum_{\text{orbits } a|\ Q_J(a)=0} |\mathcal{S}_a| \left| \sum_{J\in\mathcal{G}/\mathcal{S}_a} \chi_{J(i_a)} \right|^2 = \sum_a |\theta_a|^2 ; \tag{2.18}$$

in this equation, a labels the orbits, i_a is a representative point on each orbit and $|\mathcal{S}_a|$ is the order of the stabilizer $\mathcal{S}_a$ of the orbit a, i.e. the subgroup of $\mathcal{G}$ acting trivially on any element i in a. The proof of the general expression (2.18) is not trivial and it involves the symmetry of the S matrix under the J action: $S_{i,J(k)} = S_{i,k} \exp(2\mathrm{i}\pi Q_J(i))$. The modular invariants (2.18) can be considered as diagonal invariants with respect to the basis of the extended chiral algebra, whose characters are θ_a. In the QHE case, the stabilizer is trivial and the J action has no fixed points, owing to the additivity of the physical charge carried by the electron [32]. We recognize in (2.17), (2.18), the (D) condition (2.16) derived from QHE wavefunctions. Note that in the Abelian case this condition follows immediately from U invariance [17].

Let us further analyze solution (2.18); the T^2 invariance of the extended characters for the ground state and ath quasiparticles, respectively θ_0 and $\overline{\theta}_a$, implies that

$$2\, h_e = M, \qquad 2\, h_{e\times i} - 2\, h_i = N, \qquad M, N \text{ integers.} \tag{2.19}$$

Condition (C) of the half-integer electron spin requires M to be odd, i.e the even case is excluded for physical reasons (although sometimes allowed for bosonic fluids). Therefore, we should consider algebra extensions by half-integer spin currents, as in the case of the Neveu–Schwarz sector of supersymmetric theories [14]. (Note that the integer N in (2.19) is also odd by the (D) condition.)

In conclusion, we have re-derived the standard physical conditions (A)–(D) on Hall excitations from modular invariance of RCFT partition functions. They have been found to be diagonal invariants for extended symmetry algebras that are obtained in the orbifold construction through simple currents (in the present case without fixed points) [31].

In the following analysis of non-Abelian Hall fluids, we shall find that the (A)–(D) conditions (i.e. modular invariance) straightforwardly determine the charge and statistics spectrum of excitations for a given choice of neutral RCFT and electron field (simple current). These results are relevant for model building: in the literature, the derivation of the theory pertaining to a given plateau often involves a combination of technical arguments and physical arguing that might suggest a certain degree of arbitrariness in the construction of the theory, which is however not present.

We remark that the neutral RCFT may possess more than one simple current that could be used as the electron field, although a preferred choice may exist, e.g. the lowest-dimensional field. This cannot be considered as an ambiguity of the construction, because the choice of electron field is part of the definition of the theory: different electrons correspond to different Hall states, with different charge spectra, filling fraction, etc, all quantities being determined self-consistently.

Another possibility for RCFTs with two simple currents is that of using both of them simultaneously for building a modular invariant with further extended symmetry. This issue will be discussed in section 5.

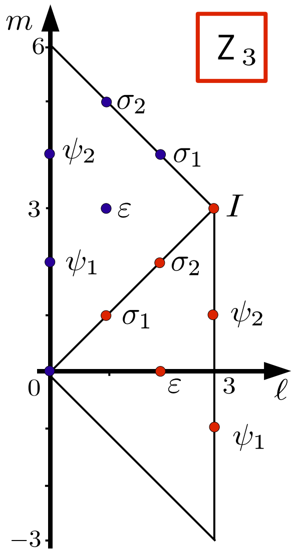

Figure 2. Diagram of $\mathbb{Z}_3$ parafermion fields with field symbols.

2.3. Example: Read–Rezayi states

In the following, conditions (A)–(D) will be illustrated by rederiving the spectrum and partition functions [17, 19] of Read–Rezayi states [8] with filling fractions:

$$\nu = 2 + \frac{k}{kM+2}, \qquad k = 2, 3, \ldots, \qquad M = 1, 3, \ldots. \tag{2.20}$$

The Read–Rezayi theory is based on the neutral $\mathbb{Z}_k$ parafermion theory (PF_k) with the central charge $c = 2(k-1)/(k+2)$ that can be described by the standard coset construction $\mathrm{PF}_k = \widehat{SU(2)}_k/\widehat{U(1)}_{2k}$ [33]. From the coset, we find that neutral sectors are characterized by a pair of quantum numbers for the representations of the algebras in the numerator and denominator: these are (ℓ, m), equal to twice the $SU(2)$ spin and spin component, respectively ($m = \ell \bmod 2$).

The dimensions of parafermionic fields ϕ^ℓ_m are given by

$$\begin{aligned} h^\ell_m &= \frac{\ell(\ell+2)}{4(k+2)} - \frac{m^2}{4k}, \\ \ell &= 0, 1, \ldots, k, \qquad -\ell < m \leqslant \ell, \quad \ell = m \bmod 2. \end{aligned} \tag{2.21}$$

The $\mathbb{Z}_3$ parafermion fields are shown in figure 2: the coset construction implies that the m charge is defined modulo $2k$ [33]; indeed, the fields are repeated once outside the fundamental (ℓ, m) domain (2.21) by the reflection-translation, $(\ell, m) \to (k-\ell, m+k)$,

$$\phi^\ell_m = \phi^{k-\ell}_{m-k}, \qquad \ell = 0, 1, \ldots, k, \qquad l < m \leqslant 2k - l, \tag{2.22}$$

also called the 'field identification' [34].

The fusion rules are given by the addition of the $\widehat{SU(2)}_k$ spin and $\widehat{U(1)}_{2k}$ charge:

$$\phi^\ell_m \cdot \phi^{\ell'}_{m'} = \sum_{\ell''=|\ell-\ell'|}^{\min(\ell+\ell', 2k-\ell-\ell')} \phi^{\ell''}_{m+m' \bmod 2k}. \tag{2.23}$$

The fields in the theory are called parafermions, $\psi_j = \phi^0_{2j}$, $j = 1, \ldots, k-1$; spin fields, $\sigma_i = \phi^i_i$; and other fields. The parafermions have Abelian fusion rules with all the fields; among themselves, these are $\psi_i \times \psi_j = \psi_n$, with $n = i+j \bmod k$. The basic parafermion ψ_1 represents the neutral component of the electron in the Read–Rezayi states: the fusion $(\psi_1)^k = I$ describes the characteristic clustering of k electrons in the ground-state wavefunction [8].

The excitations of the full theory are found by attaching $\widehat{U(1)}$ vertex operators to the parafermion fields: for the electron and a generic quasiparticle, we write

$$\Phi_e = \mathrm{e}^{\mathrm{i}\alpha_0\varphi}\,\psi_1, \qquad \Phi_i = \mathrm{e}^{\mathrm{i}\alpha\varphi}\,\phi^\ell_m, \tag{2.24}$$

with triplets of quantum numbers $(\alpha_0, 0, 2)$ and (α, ℓ, m), respectively. The mutual statistics exponent (2.16) is

$$h_{ei} = \frac{(\alpha+\alpha_0)^2 - \alpha^2 - \alpha_0^2}{2} + h^\ell_{m+2} - h^\ell_m - h^0_2. \tag{2.25}$$

Upon substituting (2.21), condition (D) reads

$$(D)\ :\ \alpha\alpha_0 - \frac{m}{k} = N, \qquad \text{integer}. \tag{2.26}$$

In particular, for the electron with itself, we find $\alpha_0^2 - 2/k = M$ integer; in combination with condition (C) of the half-integer electron spin

$$(\mathrm{C})\ :\ 2h_e = 2h^0_2 + \alpha_0^2 = 2 + M, \qquad \text{odd integer}, \tag{2.27}$$

it determines that

$$\alpha_0^2 = \frac{2+kM}{k}, \qquad M \text{ odd integer}. \tag{2.28}$$

The electric charge Q of excitations is proportional to the $\widehat{U(1)}$ charge, $Q = \mu\alpha$: the constant μ is fixed by assigning $Q = 1$ to the electron, i.e. $\mu = 1/\alpha_0$ ((A) condition).

In conclusion, (A)–(D) conditions determine the charge and spin (i.e. half-statistics) of excitations in the Read–Rezayi theory as follows:

$$Q = \frac{\alpha\alpha_0}{\alpha_0^2} = \frac{Nk+m}{2+Mk} = \frac{q}{2+Mk}, \tag{2.29}$$

$$J = h^\ell_m + \frac{1}{2}Q^2\alpha_0^2 = h^\ell_m + \frac{q^2}{2k(2+kM)}. \tag{2.30}$$

The excitations are characterized by triplets of integer labels (q, ℓ, m), where q is the charge index. From the Abelian part of conformal dimensions (2.30), we find that the Luttinger field is compactified into p sectors (cf (2.5)), $p = k(2+kM)$, and is indicated by $\widehat{U(1)}_p$; the charge index q is thus defined modulo p. We write $p = k\hat{p}$, where $\hat{p} = 2+kM$ is the denominator of the fractional charge; moreover, equation (2.29) shows that q is coupled to the $SU(2)$ spin component m by the selection rule

$$q = m \mod k, \qquad (q \mod \hat{p} = (kM+2), \qquad m \mod 2k). \tag{2.31}$$

We thus recover the $\mathbb{Z}_k$ ‘parity rule’ of [19] that constraints charge and neutral quantum numbers of Read–Rezayi quasiparticle excitations. Earlier derivations of this rule were based on some physical conditions, such as a relation with a ‘parent Abelian state’, that were useful as motivations but actually not necessary (see, however, section 4.1.1). The present derivation shows that the relevant information is the type of RCFT for neutral excitations and the identification in this theory of the field ψ_1 representing the electron.

As outlined in the previous section, the derivation of annulus partition functions requires the solution of the modular invariance conditions: in particular, the U symmetry (2.10) requires us to put each basic anyon together with its electron excitations, leading to sectors of charge $Q = \lambda/p + \mathbb{Z}$ described by the RCFT extended character θ_λ as in (2.5). The addition of an electron changes the integer labels of excitations as follows:

$$(q, m, \ell) \;\rightarrow\; (q + \hat{p}, m + 2, \ell)\,. \tag{2.32}$$

Therefore, the extended characters θ_λ of the theory $PF_k \otimes \widehat{U(1)}_p$ are made of products of characters of the charged and neutral parts, whose indices obey the parity rule (2.31) and are summed over according to (2.32). The charged characters are given by the functions $K_q(\tau, k\zeta; k\hat{p})$ introduced earlier in (2.5), with parameters chosen to fit the fractional charge and the Abelian conformal dimension in (2.29). The $\mathbb{Z}_k$ parafermionic characters are denoted by $\chi^\ell_m(\tau; 2k)$ and have rather involved expressions; actually it is enough to know their periodicities,

$$\begin{aligned} &\chi^\ell_m = \chi^\ell_{m+2k} = \chi^{k-\ell}_{m+k}, && m = \ell \bmod 2,\\ &\chi^\ell_m = 0, && m = \ell + 1 \bmod 2, \end{aligned} \tag{2.33}$$

and modular transformation

$$\begin{aligned} &\chi^\ell_m(-1/\tau; 2k) = \frac{1}{\sqrt{2k}} \sum_{\ell'=0}^{k} \sum_{m=1}^{2k} \mathrm{e}^{-\mathrm{i}2\pi\frac{mm'}{2k}}\, s_{\ell,\ell'}\, \chi^{\ell'}_{m'}(\tau; 2k),\\ &s_{\ell,\ell'} = \sqrt{\frac{2}{k+2}}\, \sin\left(\frac{\pi(\ell+1)(\ell'+1)}{k+2}\right). \end{aligned} \tag{2.34}$$

As recalled in the appendix, this transformation can be obtained from the coset construction $PF_k = \widehat{SU(2)}_k / \widehat{U(1)}_{2k}$ [33].

The extended characters are thus given by products $K_q\chi^\ell_m$ with $q = m \bmod k$ and $\ell = m \bmod 2$: adding one electron to the earlier product gives the term $K_{q+\hat{p}}\chi^\ell_{m+2}$; by continuing to add electrons until a periodicity is found, one obtains the expressions[4]

$$\begin{aligned} &\theta^\ell_a = \sum_{b=1}^{k} K_{a+b\hat{p}}(\tau, k\zeta; k\hat{p})\; \chi^\ell_{a+2b}(\tau; 2k),\\ &a = 0, 1, \ldots, \hat{p} - 1,\;\; \hat{p} = kM + 2,\\ &\ell = 0, 1, \ldots, k,\\ &a = \ell \bmod 2, \end{aligned} \tag{2.35}$$

that reproduce the charge and statistics spectrum (2.29) and (2.30).

Expression (2.35) corresponds to the following solution of the parity rule (2.31):

$$\begin{aligned} &q = a + b\hat{p}, \qquad a = 1, \ldots, \hat{p}, \qquad b = 1, \ldots, k,\\ &m = a + 2b \end{aligned} \tag{2.36}$$

(note $\hat{p} = 2 \bmod k$). The other solution $m = k + a + 2b \bmod 2k$ would lead to the same expressions with the shifted index, $\theta^{k-\ell}_a$, owing to the field identification of parafermion fields $(\ell, m) \sim (k - \ell, m \pm k)$.

The θ^ℓ_a characters have the periodicity, $\theta^\ell_{a+\hat{p}} = \theta^{k-\ell}_a$, that explains the range of indices indicated in (2.35); the dimension of the basis of θ^ℓ_a characters is therefore given by $\hat{p}(k+1)/2$, in agreement with the value of the topological order of Read–Rezayi states [8, 19]. Moreover,

[4] Hereafter, we disregard the non-analytic prefactor of K for ease of presentation.

the V transformation of these characters reads $\theta_a^\ell(\zeta+\tau)\sim\theta(\zeta)_{a+k}^{k-\ell}$, showing that the charge $\Delta Q = k/\hat{p}$ is created by adding one flux quantum: we thus recover the value of the filling fraction $\nu = k/(Mk+2)$ with M odd.

The final step is to find the modular transformations of θ_a^ℓ that is given by [17] (see the appendix):

$$\theta_a^\ell(-1/\tau) = \delta_{a,\ell}^{(2)}\,\frac{1}{\sqrt{\hat{p}}}\sum_{a'=1}^{\hat{p}}\sum_{\ell'=0}^{k}\,\mathrm{e}^{-\mathrm{i}2\pi\frac{aa'M}{2\hat{p}}}\;s_{\ell,\ell'}\;\theta_{a'}^{\ell'}(\tau), \tag{2.37}$$

where the delta modulo 2 denotes that $\theta_a^\ell(-1/\tau)$ vanishes for $a = \ell+1 \bmod 2$ (we also disregard the global phase $\propto \mathrm{Re}(\zeta^2/\tau)$ acquired by the characters). In the appendix, the check of unitarity of the S matrix confirms that the extended characters θ_a^ℓ form an independent basis.

We remark that the coupling of neutral and charged parts by the $\mathbb{Z}_k$ parity rule and the sum over electron excitations amounts to a projection in the full $K_\lambda\chi_m^\ell$ tensor space to a subspace of dimension $1/k^2$ smaller: this reduction by a square factor is a standard property of S transformations (i.e. discrete Fourier transforms). We also note that the S matrix is factorized into charged and neutral parts, where the latter has the naive expression for the $\widehat{SU(2)}_k$ theory, although the sectors are not factorized at all, as shown by the extended characters θ_a^ℓ.

Finally, the annulus partition function of Read–Rezayi states is given by the diagonal sesquilinear form

$$Z^{\mathrm{RR}}_{\mathrm{annulus}} = \sum_{\ell=0}^{k}\;\sum_{\substack{a=0\\ a=\ell \bmod 2}}^{\hat{p}-1}\;\left|\,\theta_a^\ell\,\right|^2 \tag{2.38}$$

that solves the (S, T^2, U, V) conditions of section 2.2.

For example, the expression of the $k=2$ Pfaffian state is as follows. The $\mathbb{Z}_2$ parafermions are the three fields of the Ising model: $\phi_0^0=\phi_2^2=I$, $\phi_1^1=\phi_3^1=\sigma$ and $\phi_2^0=\phi_0^2=\psi$ of dimensions $h = 0, 1/16, 1/2$, respectively. For $\nu=5/2$, i.e. $M=1$ in (2.20), the Pfaffian theory possesses six sectors. The partition function is

$$\begin{aligned} Z^{\mathrm{Pfaffian}}_{\mathrm{annulus}} = {} & |K_0 I + K_4\psi|^2\;+|K_0\psi+K_4 I|^2\;+|(K_1+K_{-3})\sigma|^2\\ & +|K_2 I+K_{-2}\psi|^2\;+|K_2\psi+K_{-2}I|^2\;+|(K_3+K_{-1})\sigma|^2, \end{aligned} \tag{2.39}$$

where the neutral characters are written with the same symbol of the field and the charged ones are $K_\lambda = K_\lambda(\tau, 2\zeta; 8)$, $K_\lambda = K_{\lambda+8}$, with charge $Q = \lambda/4 + 2\mathbb{Z}$. The first square term in Z describes the ground state and its electron excitations, such as those in $K_4\psi$ with $Q = 1+2\mathbb{Z}$; in the third and sixth terms, the characters $K_{\pm1}\sigma$ contain the basic quasiparticles with charge, $Q=\pm1/4$, and non-Abelian fusion rules $\sigma\cdot\sigma = I+\psi$. The other three sectors are less familiar: the second one contains a neutral Ising-fermion excitation (in $K_0\psi$) and the fourth and fifth sectors describe $Q=\pm1/2$ Abelian quasiparticles.

As in section 2.1, the partition function on the disk is given by the extended character θ_a^ℓ, with indices selected by the quasiparticle type in the bulk; if there are many of them, their indices are combined by using the fusion rules to find the edge sector (a,ℓ).

In conclusion, the annulus and disk partition functions completely determine the Hilbert space of edge excitations and the fusion rules through the Verlinde formula. Physical applications to current experiments will be described in section 4.

3. Partition functions of non-Abelian Hall states

In this section we obtain the partition functions of other non-Abelian states that have been proposed to describe plateaus with $2<\nu<3$: the Wen non-Abelian fluids (NAF) [13], the anti-Read–Rezayi states ($\overline{\mathrm{RR}}$) [11], the Bonderson–Slingerland hierarchy (BS) [12] and finally the non-Abelian spin-singlet state (NASS) [9, 10]. All these states have been considered as phenomenologically interesting in the recent literature searching for signatures of non-Abelian statistics in the QHE.

From the technical point of view, non-Abelian states can be built out of the $\widehat{U(1)}_p$ charged part and a neutral part given by any RCFT that possess at least one 'simple current' [14], a field with Abelian fusion rules with all the others that can be associated with the electron excitation. However, only a limited number of such constructions have received support by (2+1)-dimensional microscopic physics that is based on wavefunctions and analytic/numerical study of spectra searching for corresponding incompressible states.

3.1. $SU(2)$ non-Abelian fluids

Some time ago, Block and Wen [13] considered the natural choice of RCFT with affine symmetry $\widehat{SU(m)}_k$ and the associated $SU(m)$ non-Abelian Chern–Simons theory, starting from the physical idea of breaking the electron excitation into k fermions called 'partons'. In such theories, there always are one or more simple currents. We shall limit ourselves to the simplest $SU(2)$ case that has been recently considered in relation with the physics of the second Landau level [3, 25] and also serves as a starting point for other non-Abelian fluids.

We shall obtain the annulus partition function for the RCFT $\widehat{SU(2)}_k \otimes \widehat{U(1)}_p$. The $\widehat{SU(2)}_k$ theory is characterized by the primary fields ϕ^ℓ for $\ell = 0, 1, \ldots, k$, with dimensions $h_\ell = \ell(\ell+2)/(4(k+2))$. The simple current is ϕ^k with $h_k = k/4$ and fusion rules given by (cf (2.23))

$$\phi^k \times \phi^\ell = \phi^{k-\ell}, \qquad 0 \leqslant \ell \leqslant k, \tag{3.1}$$

which realizes a $\mathbb{Z}_2$ parity among the neutral sectors. Following the steps outlined in section 2.2, we introduce the fields $\Phi_e = \phi^k \exp(\mathrm{i}\alpha_0\varphi)$, $\Phi_i = \phi^\ell \exp(\mathrm{i}\alpha\varphi)$ corresponding to the electron and to a quasiparticle excitation, respectively.

Condition (D) on the mutual statistics exponent (2.16) of these excitations and of the electron with itself imply, respectively,

$$\alpha\alpha_0 = \frac{2N+\ell}{2}, \qquad \alpha_0^2 = \frac{2M+k}{2}, \qquad N, M \in \mathbb{Z}. \tag{3.2}$$

The charge and spin of excitations are therefore

$$\begin{aligned} Q &= \frac{\alpha}{\alpha_0} = \frac{2N+\ell}{2M+k} = \frac{q}{2M+k}, \qquad M+k \ \text{ odd integer}, \\ J &= h^\ell + \frac{1}{2}Q^2\alpha_0^2 = \frac{\ell(\ell+2)}{4(k+2)} + \frac{q^2}{4(2M+k)}. \end{aligned} \tag{3.3}$$

In particular, condition (C) of the half-integer electron spin requires that $(M+k)$ is odd.

The $\widehat{U(1)}_p$ contribution to the J spectrum identifies the compactification parameter $p = 2\hat{p}$, with $\hat{p} = (2M+k)$ the fractional-charge denominator. A quasiparticle is characterized by the integer pair $(\ell, 2N+\ell)$ of the $SU(2)$ spin and charge that are constrained by the parity rule: $q = \ell \bmod 2$. The additions of one (two) electron to a quasiparticle cause the following shifts (i.e. changes of sector):

$$(\ell, 2N+\ell) \;\to\; (k-\ell, 2N+\ell+\hat{p}) \;\to\; (\ell, 2N+\ell+2\hat{p}), \tag{3.4}$$

which involves two non-Abelian sectors only.

Therefore, we are led to consider extended characters θ_a^ℓ of the full theory, labeled by spin ℓ and charge a indices, that are made of products of charged characters, $K_q = K_q(\tau, 2\zeta; 2\hat{p})$ in the notation of (2.5), and $\widehat{SU(2)}_k$ characters $\chi^\ell(\tau)$, as follows:

$$\theta_a^\ell = K_a\, \chi^\ell + K_{a+\hat{p}}\, \chi^{k-\ell}, \qquad \begin{array}{l} a = 0, 1, \ldots, \hat{p}-1, \;\; \hat{p} = 2M+k, \\ \ell = 0, 1, \ldots, k, \\ a = \ell \bmod 2. \end{array} \tag{3.5}$$

Note that only two terms are needed in the sums, owing to the mentioned $\mathbb{Z}_2$ symmetry. As explained in section 2.2, the topological order of the $SU(2)$ NAF is given by the number of independent θ_a^ℓ characters: the symmetry $\theta_{a+\hat{p}}^{k-\ell} = \theta_a^\ell$ confirms the index ranges indicated in (3.5). Furthermore, the filling fraction can be obtained from the V transformation of K_a characters: in summary, the $SU(2)$ NAF fluids are characterized by the values

$$T.O. = (k+1)\frac{2M+k}{2}, \qquad \nu = \frac{2}{2M+k}, \qquad M+k \text{ odd}. \tag{3.6}$$

The modular transformations of extended characters can be obtained by those of the components K_a and χ^ℓ introduced in earlier sections: the result is (see the appendix)

$$\theta_a^\ell(-1/\tau) = \frac{1}{\sqrt{\hat{p}}} \sum_{a'=0}^{\hat{p}-1} \sum_{\ell'=0}^{k} \delta_{a,\ell}^{(2)}\, \mathrm{e}^{\mathrm{i}2\pi \frac{aa'}{2\hat{p}}}\, s_{\ell,\ell'}\, \delta_{a',\ell'}^{(2)}\, \theta_{a'}^{\ell'}(\tau), \tag{3.7}$$

where $s_{\ell,\ell'}$ is the $\widehat{SU(2)}_k$ S-matrix (2.34). The S-matrix of the full theory is again factorized in Abelian and non-Abelian parts (up to details) and is unitary.

In conclusion, the annulus partition function is given by the diagonal combination of extended characters

$$Z_{\text{annulus}}^{\text{NAF}} = \sum_{\ell=0}^{k} \sum_{\substack{a=0 \\ a=\ell \bmod 2}}^{\hat{p}-1} \left| \theta_a^\ell \right|^2, \tag{3.8}$$

with ranges of parameters given by (3.5). Each individual θ_a^ℓ is the partition function on the disk geometry in the presence of a specific bulk excitation.

3.2. *Anti-Read–Rezayi fluids*

It has been recently proposed [11] that a particle–hole conjugate of the $M = 1$ Read–Rezayi fluids could be realizable in the second Landau level, with filling fractions $\nu - 2 = 1 - k/(k+2) = 2/(k+2)$. In particular, for $\nu = 5/2$ the Pfaffian state may compete with its conjugate state. The fluid of RR holes inside the $\nu = 3$ droplet possesses an additional edge, leading to the CFT $\overline{SU(2)_k/U(1)_{2k} \otimes U(1)_p} \otimes \widehat{U(1)}$. The Luttinger liquids on the edges of opposite chirality interact through impurities and re-equilibrate: the result of this process was shown to lead to the $\overline{\widehat{SU(2)}_k} \otimes \widehat{U(1)}_p$ edge theory [11] for the so-called anti-Read–Rezayi state ($\overline{\text{RR}}$).

The partition function of this theory can be obtained as in the NAF case of the previous section, with little modifications due to the different chirality of the neutral sector. The electron and quasiparticle fields are given by $\Phi_e = \overline{\phi^k} \exp(\mathrm{i}\alpha_0\varphi)$ and $\Phi_i = \overline{\phi^\ell} \exp(\mathrm{i}\alpha\varphi)$, respectively (note that ϕ^k is the unique simple current of the $\widehat{SU(2)}_k$ theory).

In the mutual statistics exponent, the chiral and antichiral conformal dimensions should be subtracted leading to the conditions

$$\alpha\alpha_0 = \frac{2N-\ell}{2}, \qquad \alpha_0^2 = \frac{2M+k}{2}, \qquad N, M \in \mathbb{Z}. \tag{3.9}$$

The charge and spin of excitations are therefore

$$\begin{aligned} Q &= \frac{\alpha}{\alpha_0} = \frac{2N-\ell}{2M+k} = \frac{q}{2M+k}, \qquad M \ \text{odd} \\ J &= -h^\ell + \frac{1}{2}Q^2\alpha_0^2 = -\frac{\ell(\ell+2)}{4(k+2)} + \frac{q^2}{4(2M+k)}. \end{aligned} \tag{3.10}$$

In particular, condition (C) of half-integer electron spin requires that M is odd.

The $\widehat{U(1)}_p$ compactification parameter is $p = 2\hat{p}$, where $\hat{p} = (2M+k)$ is the denominator of the fractional charge. A quasiparticle is characterized by the integer pair $(\ell, 2N-\ell)$ of the $SU(2)$ spin and charge that are again constrained by $q = \ell \bmod 2$.

Following the same steps of the NAF case (cf (3.4) and (3.5)), we obtain the extended characters:

$$\theta_a^\ell = K_a\,\overline{\chi^\ell} + K_{a+\hat{p}}\,\overline{\chi^{k-\ell}}, \qquad \begin{aligned} &a = 0, 1, \ldots, \hat{p}-1, \ \ \hat{p} = 2M+k, \\ &\ell = 0, 1, \ldots, k, \\ &a = \ell \bmod 2, \end{aligned} \tag{3.11}$$

which are products of charged characters $K_q = K_q(\tau, 2\zeta; 2\hat{p})$ of the same period $p = 2(2M+k)$ as in the NAF case, and conjugate $\widehat{SU(2)}_k$ characters.

The values of topological order and filling fractions of $\overline{\text{RR}}$ fluids are given by the NAF expressions (3.6):

$$T.O. = (k+1)\frac{2M+k}{2}, \qquad \nu = \frac{2}{2M+k}, \quad M \ \text{odd}; \tag{3.12}$$

only the parity of M is different. The $M = 1$ case mentioned at the beginning is recovered.

The modular transformations of $\overline{\text{RR}}$ (3.11) are the same as in the NAF case (3.7), because the $\widehat{SU(2)}_k$ S-matrix is real and thus not affected by conjugation. The annulus partition function is finally given by

$$Z^{\overline{\text{RR}}}_{\text{annulus}} = \sum_{\ell=0}^{k} \sum_{\substack{a=0 \\ a=\ell \bmod 2}}^{\hat{p}-1} \left| \theta_a^\ell \right|^2. \tag{3.13}$$

3.3. Bonderson–Slingerland hierarchy

In [12], the authors considered the realization of hierarchical Hall states in the second Landau level that are built over the Pfaffian $\nu = 5/2$ state. In the Jain construction [35], the wavefunctions are given by $\Psi = \Delta^{2p}\chi_n$, where χ_n is relative to n filled Landau levels and Δ^{2p} is an even power of the Vandermonde factor, leading to the filling fraction $\nu = n/(2pn+1)$. The (projected) Jain wavefunction, interpreted within the Haldane–Halperin hierarchical construction, can be transposed into the second Landau level, and give rise to the Bonderson–Slingerland wavefunctions of the form $\Psi = \text{Pf}(1/(z_i - z_j))\Delta^M\chi_n$, with filling fractions $\nu - 2 = n/(nM+1)$, with M odd.

From earlier studies of edge excitations of Jain states [15, 18], we know that the associated CFT has the central charge $c = n$ and is Abelian with the extended symmetry $\widehat{SU(n)}_1 \otimes \widehat{U(1)}$: this theory is described by a specific n-dimensional charge lattice that includes the $SU(n)$ root lattice. The Bonderson–Slingerland hierarchy is thus realized by the CFT $\widehat{U(1)}_p \otimes \widehat{SU(n)}_1 \otimes \text{Ising}$, where the last two factors are neutral. The respective conformal dimensions are [14, 17]

$$
\begin{aligned}
&\widehat{U(1)}_p : h_\alpha = \frac{\alpha^2}{2}, \\
&\widehat{SU(n)}_1 : h_\beta = \frac{\beta(n-\beta)}{2n}, \qquad \beta = 0, 1, \ldots, n-1, \\
&\text{Ising} : h_m = 0, \frac{1}{16}, \frac{1}{2}, \qquad m = 0, 1, 2.
\end{aligned} \tag{3.14}
$$

The electron and quasiparticle excitations are made by triplets of fields

$$
\Phi_e = \rho_1\, \phi^0_2 \exp(\mathrm{i}\alpha_0\varphi), \qquad \Phi_i = \rho_\beta\, \phi^\ell_m \exp(\mathrm{i}\alpha\varphi). \tag{3.15}
$$

In this equation, we indicated the $\widehat{SU(n)}_1$ fields by ρ_β, with β mod n, and the Ising fields by ϕ^ℓ_m in the notation of section 2.3. The index ℓ in ϕ^ℓ_m can be omitted because it is determined by m: indeed, the three Ising fields, the identity, the parafermion and the spin are, respectively, $\phi^0_0 = I$, $\phi^0_2 = \psi$ and $\phi^1_1 = \sigma$, with periodicities $\phi^0_{m+4} = \phi^0_m$ and $\phi^1_{m+2} = \phi^1_m$. The electron excitation is associated with the parafermion, obeying $\psi \times \psi = 1$, and to the $\widehat{SU(n)}_1$ field with the smallest charge $\beta = 1$ (with Abelian fusion rules over β modulo n) [17]. The other fusion rules with the parafermion field are given by $\psi \times \psi = I$ and $\sigma \times \psi = \sigma$.

The excitations are thus associated with triplets (α, β, m), where the index β is mod n and m is mod 4 (2) if even (odd). The condition of integer statistics with the electron $(\alpha_0, 1, 2)$ should be independently computed for the three Ising sectors $m = 0, 1, 2$, leading to conditions that can be summarized into

$$
\alpha\alpha_0 = \frac{2nN + 2\beta + n\delta_{m,1}}{2n}, \qquad \alpha_0^2 = \frac{nM+1}{n}, \qquad N, M \text{ integers}. \tag{3.16}
$$

The resulting spectrum is

$$
\begin{aligned}
Q &= \frac{2nN + 2\beta + n\delta_{m,1}}{2nM+2} = \frac{q}{2nM+2}, \\
J &= \frac{\beta(n-\beta)}{2n} + h_m + \frac{q^2}{4n(2nM+2)}.
\end{aligned} \tag{3.17}
$$

This spectrum identifies the number of $\widehat{U(1)}$ sectors and the value of the charge denominator $p = 2n\hat{p}$ and $\hat{p} = 2nM + 2$, respectively. The parity rules relating the neutral and charge sectors are

$$
\begin{aligned}
q &= 2\beta \qquad \text{mod } 2n,\ m \text{ even}, \\
q &= 2\beta + n \qquad \text{mod } 2n,\ m \text{ odd}.
\end{aligned} \tag{3.18}
$$

In particular, the condition on the half-integer electron spin requires that M is odd.

As in the previous cases, the extended characters should resum the spectra obtained by adding any number of electrons to each anyon: in the Bonderson–Slingerland states, the addition of one electron amounts to the following index shifts:

$$
(q, \beta, m) \;\to\; (q + \hat{p}, \beta + 1, m + 2). \tag{3.19}
$$

Therefore, we are led to consider the extended characters

$$
\begin{aligned}
\theta_{q,m} &= \sum_{b=1}^{2n} K_{q+b\hat{p}}\,(\tau, 2n\zeta; 2n\hat{p})\ \Theta_{q/2+b}\ \chi^0_{m+2b}, \qquad m = 0, 2, \\
\theta_{q,m} &= \sum_{b=1}^{2n} K_{q+b\hat{p}+n}\,(\tau, 2n\zeta; 2n\hat{p})\ \Theta_{q/2+b}\ \chi^1_{m}, \qquad m = 1,
\end{aligned} \tag{3.20}
$$

where q is even. In this equation the factors $K_a = K_{a+p}$ and $\chi^{\ell}_m = \chi^{\ell}_{m+4}$ denote the $\widehat{U(1)}$ and $\mathbb{Z}_2$ parafermion characters introduced earlier, respectively, and the $\Theta_\beta = \Theta_{\beta+n}$ are the $\widehat{SU(n)}_1$ characters described in [17]. From the periodicity property $\theta_{q+\hat{p},m} = \theta_{q,m+2}$, the identity of sectors $m = 1 \sim 3$ and the V transformation of K characters, we obtain the values for the topological order and filling fraction of the Bonderson–Slingerland states

$$T.O. = 3(nM+1), \qquad \nu = 2 + \frac{n}{nM+1}, \qquad M \text{ odd}, \tag{3.21}$$

which reduce to those of the Pfaffian state in the $n = 1$ case.

Owing to the fact that $(n, nM+1) = 1$, a representative set of the $(nM+1)$ values of the charge q in the extended characters can be chosen to be $q = 2an$, $a = 0, 1, \ldots, nM$, leading to the characters

$$\begin{aligned}
\theta_{a,0} &= \sum_{b=1}^{2n} K_{2an+b\hat{p}}(\tau, 2n\zeta; 2n\hat{p})\ \Theta_b\big(I\ \delta^{(2)}_{b,0} + \psi\ \delta^{(2)}_{b,1}\big), && m = 0,\\
\theta_{a,1} &= \sum_{b=1}^{2n} K_{(2a+1)n+b\hat{p}}(\tau, 2n\zeta; 2n\hat{p})\ \Theta_b\ \sigma, && m = 1,\\
\theta_{a,2} &= \sum_{b=1}^{2n} K_{2an+b\hat{p}}(\tau, 2n\zeta; 2n\hat{p})\ \Theta_b\big(\psi\ \delta^{(2)}_{b,0} + I\ \delta^{(2)}_{b,1}\big), && m = 2,\\
&\qquad a = 0, 1, \ldots, nM.
\end{aligned} \tag{3.22}$$

where we write the Ising characters with the same symbols of the fields.

The annulus partition function is therefore given by the diagonal combination of these characters:

$$Z^{\mathrm{BS}}_{\mathrm{annulus}} = \sum_{a=0}^{nM} |\ \theta_{a,0}\ |^2 + |\ \theta_{a,1}\ |^2 + |\ \theta_{a,2}\ |^2. \tag{3.23}$$

In particular, the expression earlier found for the Pfaffian state (2.39) is recovered for $n = M = 1$.

As before, the final step is to verify that this set of extended characters is closed under S modular transformation that is unitarily represented. As discussed in the appendix in more detail, the transformation of Ising characters is particularly simple in the following basis:

$$\begin{aligned}
&\widetilde{\chi}_m = \left\{\frac{I+\psi}{\sqrt{2}}, \sigma, \frac{I-\psi}{\sqrt{2}}\right\},\\
&\frac{1}{\sqrt{2}}(I(-\tau^{-1}) + \psi(-\tau^{-1})) = \frac{1}{\sqrt{2}}(I(\tau) + \psi(\tau)),\\
&\sigma(-\tau^{-1}) = \frac{1}{\sqrt{2}}(I(\tau) - \psi(\tau)).
\end{aligned} \tag{3.24}$$

The full S matrix in this basis turns out to be

$$S_{(a,m),(a',m')} = \frac{\mathrm{e}^{\mathrm{i}2\pi\frac{aa'n}{nM+1}}}{\sqrt{nM+1}} \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & \mathrm{e}^{\mathrm{i}2\pi a'n/2(nM+1)}\\ 0 & \mathrm{e}^{\mathrm{i}2\pi an/2(nM+1)} & 0 \end{pmatrix}, \tag{3.25}$$

which is unitary and again factorized in neutral and charged parts, up to details.

3.4. Non-Abelian spin-singlet states

In the states, the clustering property of the Read–Rezayi states is generalized to spinful electrons: namely, the ground-state wavefunction must not vanish when k electrons with spin-up and k with spin-down are brought to the same point [9]. In terms of the CFT operator product expansion, we need two species of parafermions, $\psi_\uparrow, \psi_\downarrow$, that obey $(\psi_\uparrow)^k = (\psi_\downarrow)^k = I$. This possibility is offered by the generalized parafermions that are obtained by the coset construction $\widehat{SU(3)}_k/\widehat{U(1)}^2$, first discussed in [34]. Let us recall their main features.

The $\widehat{SU(3)}_k/\widehat{U(1)}^2$ parafermion fields ϕ^Λ_λ are characterized by the pair of $SU(3)$ weights (Λ, λ) that belong to the two-dimensional lattice generated by the positive fundamental weights μ_1, μ_2, with scalar products $(\mu_1, \mu_1) = (\mu_2, \mu_2) = 2/3$ and $(\mu_1, \mu_2) = 1/3$. The dual lattice is generated by the positive roots α_1, α_2, with $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$ and $(\alpha_1, \alpha_2) = -1$, i.e. $\alpha_1 = 2\mu_1 - \mu_2$ and $\alpha_2 = -\mu_1 + 2\mu_2$. The weight Λ takes values inside the so-called truncated Weyl chamber, $\Lambda \in P^+_k$, while λ is a vector of the weight lattice P quotiented by the k-expanded root lattice Q , $\lambda \in P/kQ$. In more detail, we have the following values of the weights (Λ, λ) and integer labels $(n_1, n_2, \ell_1, \ell_2)$:

$$
\begin{aligned}
&\phi^\Lambda_\lambda \equiv \phi^{n_1,n_2}_{\ell_1,\ell_2},\\
&\Lambda = n_1\mu_1 + n_2\mu_2, \quad 0 \leqslant n_1, n_2, \ n_1 + n_2 \leqslant k,\\
&\lambda = \ell_1\mu_1 + \ell_2\mu_2, \quad (\ell_1, \ell_2) \bmod (2k, -k), (k, -2k), \quad n_1 - n_2 = \ell_1 - \ell_2 \mod 3.
\end{aligned}
\tag{3.26}
$$

The last condition in this equation states that the $SU(3)$ triality of two weights is the same, $(\Lambda - \lambda) \in Q$. Another trivalent condition is the coset field identification [34], saying that the following labelings are equivalent:

$$
\phi^{n_1,n_2}_{\ell_1,\ell_2} = \phi^{k-n_1-n_2,n_1}_{\ell_1+k,\ell_2} = \phi^{n_2,k-n_1-n_2}_{\ell_1,\ell_2+k}. \tag{3.27}
$$

The number $k^2(k+1)(k+2)/6$ of parafermion fields is easily determined from these data: the product of the independent Λ values is $(k+1)(k+2)/2$, and that of the λ's ones is $3k^2$ (from the areas of lattice fundamental cells $|k\alpha_1 \wedge k\alpha_2|/|\mu_1 \wedge \mu_2|$), while the two conditions account for a factor $1/9$.

Within this theory, the two fundamental parafermions are $\psi_\uparrow = \phi^0_{\alpha_1}$ and $\psi_\downarrow = \phi^0_{-\alpha_2}$, which have Abelian fusion rules with all fields: actually, the lower index λ of ϕ^Λ_λ belongs to an Abelian charge lattice and is thus additive modulo kQ, leading to the fusion rules $\phi^\Lambda_\lambda \times \phi^0_{\alpha_i} = \phi^\Lambda_{\lambda+\alpha_i \bmod kQ}$. (The choices of the root sign in the definition of the fundamental parafermions will be relevant in (3.32).)

The conformal dimension of parafermion fields are given by [33]

$$
h^\Lambda_\lambda = \frac{(\Lambda, \Lambda + 2\rho)}{2(k+3)} - \frac{|\lambda|^2}{2k}, \tag{3.28}
$$

where $\rho = \mu_1 + \mu_2$ is half the sum of positive roots and $|\lambda|^2 = (\lambda, \lambda)$; finally, the central charge of the theory is $c = 8k/(k+3)$.

The excitations of the NASS state are characterized by two Abelian quantum numbers, the charge Q and the intrinsic spin S (not to be confused with the orbital J, equal to half the statistics). The full description is based on the RCFT $\left(\widehat{SU(3)}_k/\widehat{U(1)}^2\right) \otimes \widehat{U(1)}_n \otimes \widehat{U(1)}_p$: in the following, we shall determine the spin and charge compactification parameters (n, p) and the selection rules relating the (S, Q) values to the non-Abelian weights (Λ, λ) by extending the conditions (A)–(D) for physical excitations of section 2.2. The NASS quasiparticles are described by parafermion fields and vertex operators with α and β charges for the electric charge and spin, respectively:

$$\Psi_i = \phi^{\Lambda}_{\lambda}\, e^{i\alpha\varphi}\, e^{i\beta\varphi'}; \tag{3.29}$$

in particular, the two electron fields are

$$\Psi^{\uparrow}_e = \phi^0_{\alpha_1}\, e^{i\alpha_0\varphi}\, e^{i\beta_0\varphi'}, \qquad \Psi^{\downarrow}_e = \phi^0_{-\alpha_2}\, e^{i\alpha_0\varphi}\, e^{-i\beta_0\varphi'}. \tag{3.30}$$

The conditions of integer statistics of each excitation with the two electrons read

$$\begin{aligned} &\alpha\alpha_0 + \beta\beta_0 + h^{\Lambda}_{\lambda+\alpha_1} - h^{\Lambda}_{\lambda} - h^0_{\alpha_1} \in \mathbb{Z}, \\ &\alpha\alpha_0 - \beta\beta_0 + h^{\Lambda}_{\lambda-\alpha_2} - h^{\Lambda}_{\lambda} - h^0_{-\alpha_2} \in Z; \end{aligned} \tag{3.31}$$

from the conformal dimensions (3.28), one finds that the weight Λ does not enter in these conditions. We now analyze the various cases of mutual statistics in turn: for the electrons with/among themselves, we obtain

$$\alpha_0^2 - \beta_0^2 = \frac{1}{k} + N, \qquad \alpha_0^2 + \beta_0^2 = \frac{2}{k} + N'; \tag{3.32}$$

for the electrons with a quasiparticle with labels $(\ell_1, \ell_2, \alpha, \beta)$, they are

$$2\alpha\alpha_0 = M + M' + \frac{\ell_1 - \ell_2}{k}, \qquad 2\beta\beta_0 = M - M' + \frac{\ell_1 + \ell_2}{k}, \tag{3.33}$$

with N, N', M, M' integers.

Applying these quantization conditions, we obtain the spectrum

$$\begin{aligned} Q &= \frac{\alpha}{\alpha_o} = \frac{(M+M')k + \ell_1 - \ell_2}{3+2kN} = \frac{q}{3+2kN}, \qquad N \text{ odd}, \\ S &= \frac{\beta}{2\beta_o} = \frac{(M-M')k + \ell_1 + \ell_2}{2} = \frac{s}{2}, \\ J &= h^{\Lambda}_{\lambda} + \frac{\alpha^2+\beta^2}{2} = h^{\Lambda}_{\lambda} + \frac{q^2}{4k(3+2kN)} + \frac{s^2}{4k}. \end{aligned} \tag{3.34}$$

In these equations, the condition of the electron intrinsic spin $S = \pm 1/2$ has fixed $N = N'$ (i.e. $\beta_0^2 = 1/2k$), and that of a half-integer orbital spin J has selected N odd.

From the spectrum, we identify the compactification parameters for s and q,

$$s \mod n = 2k, \qquad q \mod p = 2k\hat{p}, \;\; \hat{p} = 3 + 2kN, \tag{3.35}$$

and the parity rules

$$\ell_1 = \frac{s+q}{2} \mod k, \qquad \ell_2 = \frac{s-q}{2} \mod k \tag{3.36}$$

that also imply $s = q \bmod 2$. In summary, a quasiparticle is characterized by the integer labels $(\Lambda, \lambda, q, s) \equiv (n_1, n_2; \ell_1, \ell_2; q, s)$ obeying these parity rules and further constrained by triality, $n_1 - n_2 = \ell_1 - \ell_2 = q \bmod 3$.

The addition of electrons to a quasiparticle causes the following index shifts:

$$\begin{array}{c|cccc} & \Delta\ell_1 & \Delta\ell_1 & \Delta q & \Delta s \\ \hline +\Psi^{\uparrow}_e & 2 & -1 & \hat{p} & 1 \\ +\Psi^{\downarrow}_e & 1 & -2 & \hat{p} & -1. \end{array} \tag{3.37}$$

The extended characters are thus obtained by products of Abelian characters, $K^{(Q)} = K(\tau, 2k\zeta; 2k\hat{p})$, $K^{(S)} = K(\tau, 0; 2k)$, for charge and intrinsic spin, respectively, and of parafermion characters $\chi^{\Lambda}_{\lambda}(\tau)$, summed over electron excitations:

$$\theta^{\Lambda}_{q,s} = \sum_{a,b} K^{(Q)}_{q+(a+b)\hat{p}} \; K^{(S)}_{s+a-b} \; \chi^{n_1,n_2}_{\ell_1,\ell_2}, \qquad (3.38)$$

where the values of (ℓ_1, ℓ_2) are constrained by the parity rules (3.36).

Further specifications/conditions on expression (3.38) are as follows:

(1) The (a, b) ranges are fixed by checking the periodicity of the summand; upon inspection, a, b are surely $2k$ periodic, but also shifts by $(a, b) \to (a + k, b + k)$ and $(a, b) \to (a + k, b - k)$ maps the sums into themselves: thus, we can take $a = 1, \ldots, k$ and $b = 1, \ldots, 2k$.

(2) The parity rule (3.36) has three solutions for $\lambda \in P/kQ$, i.e. for (ℓ_1, ℓ_2) mod $(2k, -k)$, $(k, -2k)$; these are

$$\begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = \begin{pmatrix} \frac{s+q}{2} \\ \frac{s-q}{2} \end{pmatrix}, \quad \begin{pmatrix} \frac{s+q}{2} + k \\ \frac{s-q}{2} \end{pmatrix}, \quad \begin{pmatrix} \frac{s+q}{2} \\ \frac{s-q}{2} + k \end{pmatrix}. \qquad (3.39)$$

However, using the field identifications (3.27), these three solutions can be traded for Λ changes and should not be considered as independent. We thus obtain

$$\theta^{\Lambda}_{q,s} = \frac{1}{2} \sum_{a,b=1}^{2k} K^{(Q)}_{q+(a+b)\hat{p}} \; K^{(S)}_{s+a-b} \; \chi^{\Lambda}_{\frac{s+q}{2}+2a+b,\, \frac{s-q}{2}-a-2b}. \qquad (3.40)$$

(3) Independent values of (q, s) indices, $q = s \bmod 2$, are found by checking the periodicities of $\theta^{\Lambda}_{q,s}$; we find that

$$\theta^{\Lambda}_{q,s+2} = \theta^{\Lambda}_{q,s}, \qquad \theta^{n_1,n_2}_{q+\hat{p},s+1} = \theta^{k-n_1-n_2,n_1}_{q,s}. \qquad (3.41)$$

Therefore, the intrinsic spin index is not independent and can be taken to be $s = 0$ (1) for q even (odd), while the charge range is $q = 1, \ldots, \hat{p} = 3 + 2kN$.

The number of independent extended NASS characters is given by the values of q and Λ constrained by triality, $n_1 - n_2 = q \bmod 3$. We obtain

$$T.O. = (3 + 2kN)\frac{(k+1)(k+2)}{6}, \qquad \nu = \frac{2k}{2kN+3}, \quad N \text{ odd}, \qquad (3.42)$$

where the filling fraction follows from the V transformation of $K^{(Q)}$.

In conclusion, the NASS partition function on the annulus geometry is

$$Z^{\text{NASS}}_{\text{annulus}} = \sum_{q=1}^{2kN+3} \; \sum_{\substack{s=0,1 \\ s=q \bmod 2}} \; \sum_{\substack{0 \leqslant n_1+n_2 \leqslant k \\ n_1-n_2=q \bmod 3}} \left|\theta^{n_1,n_2}_{q,s}\right|^2. \qquad (3.43)$$

The modular transformation of the NASS characters is derived in the appendix and reads

$$\theta^{\Lambda}_{q,s} = \delta^{(2)}_{q,s}\, \delta^{(3)}_{n_1-n_2,q} \sum_{q'=1}^{\hat{p}} \; \sum_{\substack{s'=0,1 \\ s'=q' \bmod 2}} \; \sum_{\Lambda' \in P^+_k} \frac{1}{\sqrt{\hat{p}}}\, \mathrm{e}^{-\mathrm{i}2\pi \frac{qq'N}{2\hat{p}}}\, \delta^{(2)}_{q',s'}\, s_{\Lambda,\Lambda'}\, \theta^{\Lambda'}_{q',s'}, \qquad (3.44)$$

where $s_{\Lambda,\Lambda'}$ is the $\widehat{SU(3)}_k$ modular S-matrix.

3.4.1. The $k = 2$, $M = 1$ case. Let us discuss the simplest NASS state with $k = 2$ and $M = 1$, corresponding to $\nu = 4/7$. There are eight parafermionic fields

$$
\begin{aligned}
&I = \Phi^{0,0}_{0,0}, \quad \psi_1 = \phi^{0,0}_{2,-1}, \quad \psi_2 = \phi^{0,0}_{-1,2} = \phi^{0,0}_{1,-2}, \quad \psi_{12} = \Phi^{0,0}_{1,1} = \Phi^{0,0}_{3,-3},\\
&\sigma_\downarrow = \Phi^{0,1}_{0,1} = \Phi^{1,1}_{2,-1}, \quad \sigma_\uparrow = \Phi^{1,0}_{1,0} = \Phi^{1,1}_{-1,2}, \quad \sigma_3 = \Phi^{1,1}_{1,1}, \quad \rho = \Phi^{1,1}_{0,0}
\end{aligned}
\tag{3.45}
$$

that are also written in the notation of [10].

There are 14 sectors in the theory, and corresponding extended characters $\theta^{\Lambda}_{\lambda}$, that are made of the parafermion characters χ^{Λ}_{λ} combined with Abelian characters $K^{(Q)}_m K^{(S)}_s$, with m mod $p = 28$ and s mod 4; quasiparticles have charge $m/7 + 4\mathbb{Z}$ and intrinsic spin $s/2 + 2\mathbb{Z}$. The partition function (3.43) can be rewritten as

$$
\begin{aligned}
Z^{\mathrm{NASS}}_{\mathrm{annulus}}(k = 2) = &\sum_{m=0,4,8,12,16,20,24} \left|\chi_I Q_{m,0} + \chi_{\psi_1} Q_{m+7,1} + \chi_{\psi_2} Q_{m+7,3} + \chi_{\psi_{12}} Q_{m+14,0}\right|^2\\
&+ \sum_{m=0,4,8,16,20,24,26} \left|\chi_\rho Q_{m,0} + \chi_{\sigma_\downarrow} Q_{m+7,1} + \chi_{\sigma_\uparrow} Q_{m+7,3} + \chi_{\sigma_3} Q_{m+14,0}\right|^2,
\end{aligned}
\tag{3.46}
$$

where we denoted

$$
Q_{m,s} = Q_{m+14,s+2} = K^{(Q)}_m K^{(S)}_s + K^{(Q)}_{m+14} K^{(Q)}_{s+2}. \tag{3.47}
$$

The $m = 0$ term in the first sum contains the identity I and the two electron excitations made of ψ_1 and ψ_2 parafermions, obeying the fusions $(\psi_1)^2 = (\psi_2)^2 = I$ and also $\psi_1 \times \psi_2 = \psi_{12}$, leading to the forth term in that sector. The term $m = 8$ in the second sum contains the basic quasiparticle made of spin fields $\sigma_\uparrow, \sigma_\downarrow$, with the smallest charge $Q = 1/8$ and spin $S = \pm 1/2$. The other terms are further quasiparticle excitations. Note that the division into extended characters is in agreement with the fusion subalgebras made by multiple fusing of ψ_1, ψ_2 with all the fields; the complete table of fusion rules can be found in [10].

4. Physical applications of partition functions

Besides providing a complete definition of the Hilbert space of edge excitations, their quasiparticle sectors and fusion rules, the partition functions are useful for computing physical quantities that could be measured in current experiments on non-Abelian statistics: in particular Coulomb blockade [22] and entropy measurements through the thermopower effect [27]. We first recapitulate the main features of partition functions discussed in the previous sections, using a standard notation for all of them (inspired by the Read–Rezayi states). The partition function on the annulus is a sum of squared terms:

$$
Z_{\mathrm{annulus}} = \sum_{a,\ell} \left|\theta^{\ell}_a\right|^2, \tag{4.1}
$$

where a and ℓ are the Abelian and non-Abelian indices, respectively, possibly obeying some relative conditions. The total number of terms in the sum is equal to the topological order of the Hall state. Each extended character takes the form

$$
\theta^{\ell}_a = K_a \chi^{\ell}_m + K_{a+\hat{p}} \chi^{\ell}_{m+\hat{m}} + \cdots, \tag{4.2}
$$

where the Abelian indices (a, m) are related by a selection rule (parity rule) and the various terms in the sum describe the addition of electrons (quantum numbers $(\hat{p}, \hat{m})$) to the basic quasiparticle of that sector.

The extended character transforms linearly under the modular S transformation

$$
\theta^{\ell}_a(-1/\tau) = \sum_{a',\ell'} S_{aa'}\, s_{\ell\ell'}\, \theta^{\ell'}_{a'}(\tau), \tag{4.3}
$$

where the S-matrix is basically factorized into an Abelian phase, $S_{aa'} \sim \exp(\mathrm{i}2\pi aa'N/M)$, and a less trivial non-Abelian part $s_{\ell\ell'}$.

Finally, the expression of charged characters $K_a(\tau, n\zeta; n\hat{p})$ is well known and given by the theta functions (2.5) with parameters defined in (2.3), while that of the non-Abelian ones $\chi_m^\ell(\tau_n)$ is less explicit: nonetheless, the knowledge of their leading low-temperature behavior $\tau_n \to \mathrm{i}\infty$ is usually sufficient:

$$\chi_m^\ell(\tau_n) \sim d_m^\ell \, \mathrm{e}^{\mathrm{i}2\pi\tau_n(h_m^\ell - c/24)}, \qquad \mathrm{Im}\,\tau_n = \frac{\beta}{2\pi}\frac{v_n}{R}. \tag{4.4}$$

In this equation, h_m^ℓ is the conformal dimension of the corresponding non-Abelian field and c is the central charge of the CFT; d_m^ℓ is the multiplicity of the low-lying state in that sector, that is present in some theories due to their extended symmetry. We also changed the modular parameter $\tau \to \tau_n$ to account for a different Fermi velocity v_n of neutral excitations.

4.1. Coulomb blockade in quantum Hall droplets

The study of Coulomb blockade current peaks has been initiated in [22], where they were shown to provide interesting information on the spectrum of edge excitations. Indications of experimental feasibility have been reported in [4]. The physical system is an isolated droplet of the Hall fluid and the current peaks are due to electrons that tunnel in and out the system, one by one because the voltage bias V_o is counterbalanced by the electrostatic energy E_C of the droplet; the latter is relevant for small droplets of size $2 - 20\ \mu$m. We shall discuss the cases of $T = 0$ and $T > 0$, both at and out of the equilibrium.

4.1.1. $T = 0$. At very small temperatures and low bias $V_o \sim 0$, the conductance peaks can only occur for exact energy matching: the spectrum of droplet edge excitations should possess degenerate energy levels under the addition of one electron, $Q \to Q + 1$. The use of the partition function at $T = 0$ has already been discussed in earlier publications [16, 17] and will be briefly summarized here.

The discrete spectrum, such as (2.30) in the Read–Rezayi state, can be continuously deformed by varying the size of the droplet, which modifies the background charge and adds a capacitive energy to the $\widehat{U(1)}$ part of the spectrum, $E \propto (Q - Q_{\mathrm{bkg}})^2$. In the character $K_\lambda(\tau, \zeta; p)$, the deformed energy is

$$E_{\lambda,\sigma}(n) = \frac{v}{R}\frac{(\lambda + pn - \sigma)^2}{2p}, \qquad \sigma = \frac{B\Delta S}{\phi_o}, \tag{4.5}$$

where σ is a dimensionless measure of area deformation. The neutral part of the spectrum is unaffected. In the sector θ_a^ℓ (4.2), corresponding to the partition function on disk geometry with the (ℓ, a) quasiparticle in the bulk, one should compare the lowest energies on pairs of consecutive terms, i.e. with consecutive electron numbers, and obtain the values $\sigma = \sigma_i^\ell$ for energy matching at which the current can occur. One finds that the distance among them $\Delta\sigma_i^\ell$ is not constant owing to the contribution of neutral energies according to the following formula:

$$\Delta\sigma_i^\ell = \sigma_{i+1}^\ell - \sigma_i^\ell = \frac{1}{\nu} + \frac{v_n}{v}\left(h_{a+2\mathrm{i}+2}^\ell - 2h_{a+2\mathrm{i}}^\ell + h_{a+2\mathrm{i}-2}^\ell\right). \tag{4.6}$$

For example, in the $\mathbb{Z}_k$ Read–Rezayi states, the following peak patterns were obtained [17]:

$$\begin{aligned} &\ell = 0, k: \\ &\Delta\sigma = (\Delta + 2r, \Delta, \ldots, \Delta), \qquad (k)\ \text{groups}, \end{aligned}$$

$$
\begin{aligned}
&\ell = 1, \ldots, k-1 : \\
&\quad \Delta\sigma = (\Delta + r, \Delta, \ldots, \Delta + r, \Delta, \ldots, \Delta), \qquad (\ell)(k-\ell) \ \text{groups}, \\
&\quad \Delta = \frac{1}{\nu} - \frac{v_n}{v}\frac{2}{k}, \qquad r = \frac{v_n}{v};
\end{aligned}
\tag{4.7}
$$

they depend on the non-Abelian index ℓ (counting the number of basic quasiparticle σ_1 in the bulk). The modulation of peak distances is equal to the ratio of velocities of neutral to charged excitations $v_n/v = r \sim 1/10$. Coulomb blockade peaks are also obtained by tuning the magnetic field but their pattern is the same once expressed in terms of flux change [17]. Let us stress that this is an equilibrium phenomenon caused by adiabatic variation of the Hamiltonian. The analysis of peak patterns for the non-Abelian theories discussed in the previous sections has been carried out in [17, 22, 25] and will not be repeated here.

In [25], it was argued that the modulation of Coulomb peaks at $T = 0$ is not a signature of non-Abelian states, since similar or equal patterns can also be found in Abelian states that have non-trivial neutral excitations. Indeed, at $T = 0$, one is probing only the leading behavior (4.4) of the neutral characters χ_m^ℓ that can be the same for different theories, Abelian and non-Abelian. For example, the Read–Rezayi states and the (331) Haldane–Halperin Abelian hierarchical fluids have this property.

One physical explanation of this fact was given in [19], where it was shown that the (331) state can be considered as a 'parent' Abelian theory of the Read–Rezayi state in the following sense. The Haldane–Halperin theory possesses the same kth electron clustering property but it is realized through the addition of a k-fold quantum number, called e.g. 'color'. Since $n \leqslant k$ electrons can have different color and be distinguishable, the wavefunction does not vanish when k of them coincide; this behavior can be achieved in a standard Abelian multicomponent theory specified by a charge lattice. In this setting, the Read–Rezayi state is reobtained when electrons are made indistinguishable, i.e. when wavefunctions are antisymmetrized with respect to all electrons independently of the color.

In the CFT language, such antisymmetrization amounts to a projection in the Hilbert space that does not change the sectors and selection rules over which the partition function is built; the neutral CFT undergoes a coset reduction, from the Abelian lattice with $\widehat{SU(k)}_1 \otimes \widehat{SU(k)}_1$ symmetry to the non-Abelian theory $\widehat{SU(k)}_1 \otimes \widehat{SU(k)}_1/\widehat{SU(k)}_2$, the latter being another realization of $\mathbb{Z}_k$ parafermions [19]. Therefore, the Abelian and non-Abelian fluids have the same form of the partition function, only the neutral characters are different: however, their leading behavior (4.4) is unaffected by the projection, leading to the same patterns of Coulomb peaks at $T = 0$. In conclusion, the presence of equal Coulomb peaks' patterns in two different theories, Abelian and non-Abelian, is not accidental and it may suggest a physical mechanism behind it.

4.1.2. $T > 0$ at equilibrium. The analysis of the temperature-dependent Coulomb blockade in Hall droplets has been initiated in [23, 24]. From the disk partition function, one obtains the thermal average of the charge

$$
\langle Q \rangle = -\frac{1}{\beta}\frac{\partial}{\partial V_o} \log \theta_a^\ell,
\tag{4.8}
$$

whose qualitative expression at low temperature is

$$
\langle Q \rangle_\ell = \frac{\sum_i \ \mathrm{i}\, \delta_\beta\big(\sigma - \sigma_i^\ell\big)}{\sum_i \delta_\beta\big(\sigma - \sigma_i^\ell\big)},
\tag{4.9}
$$

where the $\delta_\beta(x)$ is a Gaussian representation of the delta function with spread proportional to $1/\beta$. Upon varying the droplet area σ by $\Delta\sigma_i^\ell$, the dominant term in the sum changes from the ith to the $(i+1)$st one and the value of $\langle Q\rangle$ jumps by one unit. Therefore, this quantity has a characteristic staircase shape for electrons entering the droplet; the effect of the temperature is that of rounding the corners that become bell-shaped peaks in the derivative of $\langle Q\rangle$ with respect to the control parameter.

We should distinguish two temperature scales:

$$T_{\rm n} = O\left(\frac{v_n}{R}\right) \sim 50\ \text{mK}, \qquad T_{\rm ch} = O\left(\frac{v}{R}\right) \sim 250\ \text{mK}, \tag{4.10}$$

which correspond to typical energies of neutral and charged excitations of a small droplet ($R \sim 10\ \mu$m), respectively. The discussion above applies to the range $T < T_{\rm n}$, where one sees smoothing of the $T = 0$ peaks.

A new feature [16, 17, 24] is associated with the multiplicity factor d_m^ℓ in the leading behavior of the neutral characters (4.4): if $d_m^\ell > 1$, the electron entering the droplet finds more than one available degenerate state. At equilibrium for $T \simeq 0$, the probability of one-electron tunneling under parametric variation of the Hamiltonian is either 1 (at degeneracy) or 0 (off degeneracy); thus, the presence of more than one available empty state is not relevant. Inclusion of a small finite-size energy splitting among the d_m^ℓ states does not substantially modify the peak shape. In particular, the earlier analysis of [16, 17], indicating the possibility of a comb-like sub-structure of peaks for $d_m^\ell > 1$ is not actually correct at equilibrium (there is an off-equilibrium effect to be discussed in the next section). In conclusion, the presence of level multiplicities does not influence the $T = 0$ peak pattern.

On the other hand, for $T > 0$, these multiplicities lead to a displacement of the peak centers, as follows [24]:

$$\sigma_i^\ell \to \sigma_i^\ell + \frac{T}{T_{ch}} \log\left(\frac{d^\ell_{a+2{\rm i}}}{d^\ell_{a+2{\rm i}+2}}\right), \tag{4.11}$$

as is clear by exponentiating the d_m^ℓ factor into the energy. The effect is observable for $T \leqslant T_n$ by increasing the experimental precision. In particular, the multiplicities and peak displacements are present in the (331) states (due to color multiplicity) and absent in the Read–Rezayi states, thus providing a difference in the Coulomb peaks of these two theories for $T > 0$ [24].

The level multiplicities of the other non-Abelian states are found by the leading expansion of the respective neutral characters:

$$\begin{aligned} &d^\ell = (\ell+1), && \chi^\ell \sim d^\ell\, q^{h^\ell - c/24}, && SU(2)\ \text{NAF and}\ \overline{\text{RR}},\\ &d_\beta = \binom{n}{\beta}, && \Theta_\beta \sim d_\beta\, q^{\beta(n-\beta)/2n - n/24}, && SU(n)\ \text{BS and Jain},\\ &d_m^\ell = d_\lambda^\Lambda = 1, && && \text{RR and NASS}. \end{aligned} \tag{4.12}$$

The peak displacements are particularly interesting when they can be used to distinguish between competing theories with the same filling fraction and Coulomb peak pattern at $T = 0$. In the case of the Jain hierarchical states, one can test the multiplicities of states predicted by the multicomponent $\widehat{SU(n)}_1 \otimes \widehat{U(1)}$ Abelian theories versus the $W_{1+\infty}$ minimal models possessing no degeneracy, as discussed in earlier works [30, 36]. The peak pattern including displacements of the $\widehat{SU(n)}_1 \otimes \widehat{U(1)}$ theory are, for $\nu = n/(2sn \pm 1)$,

$$\begin{aligned} \Delta\sigma_i &= \frac{1}{\nu} - \frac{v_n}{v}\frac{1}{n} + \frac{T}{T_{\rm ch}} \log\frac{(i+1)(n-i+1)}{i(n-i)}, \qquad i = 1,\ldots,n-1,\\ \Delta\sigma_n &= \frac{1}{\nu} + \frac{v_n}{v}\frac{n-1}{n} + \frac{T}{T_{\rm ch}} \log\frac{1}{n^2}. \end{aligned} \tag{4.13}$$

The $W_{1+\infty}$ minimal models possess the same pattern without the temperature-dependent term for $T < T_{\rm n}$. Formula (4.13) supersedes our earlier analyses of this problem [16, 17].

Another interesting case is the competition between the Pfaffian state at $\nu = 5/2$ and its particle hole conjugate state $\overline{\mathrm{RR}}$, possessing multiplicities due to its $SU(2)$ symmetry, as indicated in (4.12). Let us discuss this point in some detail. The Coulomb peak distances in the $\overline{\mathrm{RR}}$ states, parameterized by (ℓ, a), $a = \ell \bmod 2$ (cf (3.11)), have period 2 for any k and their expressions including the shift due to level multiplicity are, for $\nu = 2/(2M + k)$,

$$\Delta\sigma_a^\ell = \frac{1}{\nu} + (-1)^a \left[\frac{v_n}{v} \frac{2\ell - k}{2} + 2\frac{T}{T_{\mathrm{ch}}} \log\left(\frac{k - \ell + 1}{\ell + 1} \right) \right], \qquad \ell = 0, 1, \ldots, k. \tag{4.14}$$

In particular, for $\nu = 5/2$ ($k = 2$) there is the so-called even/odd effect [22] for both the RR, the $\overline{\mathrm{RR}}$ and (331) states (no modulation for $\ell = 1$, pairwise modulation for $\ell = 0, 2$). The peak shifts are present in the last two theories for $\ell = 0$ but are absent in the Pfaffian state. It follows that the observation of a temperature-dependent displacement in the position of paired peaks could support both the (331) and the anti-Pfaffian at $\nu = 5/2$ (the last case was not discussed before).

At higher temperatures, in the region $T_{\mathrm{n}} < T < T_{\mathrm{ch}}$, a new feature appears: the Boltzmann factors relative to higher neutral excitations can be of order 1 and can contribute beyond the leading $T \to 0$ term in (4.4). As observed in [23], it is convenient to perform the S modular transformation on the neutral characters $\chi(\tau_n)$, $\tau_n \to -1/\tau_n \sim \mathrm{i}T/T_{\mathrm{n}}$, and expand them for $\tau_n \to 0$.

Using again the Read–Rezayi case as an example, we find from (2.35) and (2.37) upon keeping the first three terms

$$\begin{aligned}
\chi_m^\ell(\tau_n) &\sim \frac{1}{\sqrt{2k}} \left(s_{\ell 0}\, \chi_0^0\left(\frac{-1}{\tau_n}\right) + s_{\ell 1}\, \mathrm{e}^{-\mathrm{i}\pi m/k}\, \chi_1^1\left(\frac{-1}{\tau_n}\right) + s_{\ell 1}\, \mathrm{e}^{\mathrm{i}\pi m/k}\, \chi_{-1}^1\left(\frac{-1}{\tau_n}\right) \right) \\
&\sim s_{\ell 0} \left(1 + \frac{s_{\ell 1}}{s_{\ell 0}} \mathrm{e}^{-\mathrm{i}(2\pi/\tau_n)h_1^1}\, 2\cos\left(\frac{\pi m}{k}\right) \right) \\
&\sim s_{\ell 0}\, \mathrm{e}^{D_m^\ell}, \qquad\qquad \tau_n \to 0.
\end{aligned} \tag{4.15}$$

In this limit, the extended characters (2.35) become

$$\theta_a^\ell \sim \sum_{b=1}^{k} K_{a+b\hat{p}}(\tau)\, s_{\ell 0}\, \mathrm{e}^{D_{a+2b}^\ell}, \tag{4.16}$$

where

$$D_{a+2b}^\ell = \mathrm{e}^{-4\pi^2 h_1^1 T/T_n}\, 2\cos\frac{\pi(\ell+1)}{k+2}\, 2\cos\frac{\pi(a+2b)}{k}. \tag{4.17}$$

The distance between Coulomb peaks in RR states for $T > T_{\mathrm{n}}$ is therefore given by (cf (4.6))

$$\Delta\sigma_i^\ell = \frac{1}{\nu} - \frac{T}{T_{\mathrm{ch}}} \mathrm{e}^{-4\pi^2 h_1^1 T/T_n}\, 8\cos\frac{\pi(\ell+1)}{k+2} \cos\frac{\pi(a+2\mathrm{i})}{k} \left(\cos\frac{2\pi}{k} - 1 \right). \tag{4.18}$$

Although exponentially small, this temperature effect is full-fledged non-Abelian, since it involves the ratio $s_{\ell 1}/s_{\ell 0}$ of components of the S-matrix for neutral states; such ratios also characterizes other non-Abelian probes, including the most popular Fabry–Perot interference phase [3].

The corresponding correction terms for the other Hall states are

$$SU(n)\ \text{Jain} : D_b = \mathrm{e}^{-4\pi^2 h_1 T/T_n} 2n\ \cos\frac{2\pi b}{n}, \tag{4.19}$$

$$SU(2)\ \text{NAF and}\ \overline{\mathrm{RR}} : D_a^\ell = \mathrm{e}^{-4\pi^2 h^1 T/T_n} (-1)^a\, 4\cos\frac{\pi(\ell+1)}{k+2}, \tag{4.20}$$

$$SU(n)\times \text{Ising BS}: D_{a,m} = \mathrm{e}^{-4\pi^2 h_1 T/T_{SU(n)}} 2n \;\cos\frac{2\pi a}{n} + \begin{cases} (-1)^{a+m/2}\sqrt{2}\,\mathrm{e}^{-\pi^2 T/(4T_{\text{Ising}})}, & m=0,2, \\ -\mathrm{e}^{-2\pi^2 T/T_{\text{Ising}}}, & m=1. \end{cases} \tag{4.21}$$

In all these expressions, we used the notations introduced earlier in section 3; $T_{SU(n)}$ and T_{Ising} are proportional to the neutral edge velocities of the corresponding terms in the BS theory. We remark that the difference between the Jain (Abelian) (4.19) and Read–Rezayi (non-Abelian) (4.17) correction is precisely due to the neutral S-matrix term.

The corresponding $T > T_n$ Coulomb peak separations are

$$\begin{aligned} &SU(n)\ \text{Jain}: \\ &\qquad \Delta\sigma_i = \frac{1}{\nu} - \frac{T}{T_{\text{ch}}}\mathrm{e}^{-4\pi^2 h_1 T/T_n}\ 4n\cos\frac{2\pi \mathrm{i}}{n}\left(\cos\frac{2\pi}{n}-1\right), \\ &\text{NAF and } \overline{\text{RR}}: \\ &\qquad \Delta\sigma_i^{\ell} = \frac{1}{\nu} - \frac{T}{T_{\text{ch}}}\mathrm{e}^{-4\pi^2 h^1 T/T_n}(-1)^i\ 16\cos\frac{\pi(\ell+1)}{k+2}, \\ &SU(n)\times\text{Ising BS}: \\ &\qquad \Delta\sigma_i^{m} = \frac{1}{\nu} - \frac{T}{T_{\text{ch}}}\mathrm{e}^{-4\pi^2 h_1 T/T_{SU(n)}} 4n\cos\frac{2\pi \mathrm{i}}{n}\left(\cos\frac{2\pi}{n}-1\right) \\ &\qquad\qquad + \begin{cases} (-1)^{\mathrm{i}+m/2} 4\sqrt{2}\,\mathrm{e}^{-\pi^2 T/(4T_{\text{Ising}})}, & m=0,2, \\ 0, & m=1. \end{cases} \end{aligned} \tag{4.22}$$

Finally, in the temperature range $T > T_{\text{ch}}$, the Coulomb peaks could be similarly predicted from the partition function by performing the S transformation on both τ_n and τ. However, the experimental values of T_{ch} for small droplets are comparable to the bulk gap Δ of Hall states in the second Landau level, such that the CFT description is doubtful for higher temperatures.

4.1.3. $T > 0$ off-equilibrium. In this section we discuss the $T > 0$ Coulomb blockade in the presence of a finite potential ΔV_o between the Hall droplet and left (L) and right (R) reservoirs, leading to a steady flow of electrons through the droplet. This setting resembles a scattering experiment and is sensible to the multiplicity of edge states d_a^{ℓ} discussed in the previous section.

We study the problem within the phenomenological approach of the master equation of [37] that was successfully applied to studying the fluctuations of the CB current in a quantum dot, in particular its suppression due to Fermi statistics when the transmission rates are not too small. Clearly, a more precise analysis of off-equilibrium physics would require knowledge of the finite-temperature real-time current–current correlation function which is beyond the scope of this work.

The starting point of [37] is the evolution equation of the semiclassical density matrix ρ, $\mathrm{d}\rho/\mathrm{d}t = M\rho$, where M is the matrix of transition rates. The latter can be written in the basis of states with a definite electron number n inside the dot: its components are the rates Γ_{ij} that can be computed by the Fermi golden rule and have the following factor for ideal-gas statistics:

$$\gamma(\varepsilon) = \frac{\varepsilon}{1-\mathrm{e}^{-\beta\varepsilon}}, \tag{4.23}$$

where $\varepsilon = E_i - E_f$ is the energy of one-particle transitions. This expression has two natural limits:

$$\gamma(\varepsilon) \sim \begin{cases} \varepsilon, & \beta\varepsilon \gg 1, \\ |\varepsilon| \mathrm{e}^{-\beta|\varepsilon|}, & \varepsilon < -T, \end{cases} \tag{4.24}$$

showing the $T \to 0$ phase–space enhancement for allowed electron transition and the distribution of thermal-activated forbidden transitions, respectively.

The transition rates for one electron entering ($n \to n+1$) or leaving ($n \to n-1$) the dot, coming from the (L) or (R) reservoirs are respectively given by [37]

$$\Gamma^{\mathrm{L(R)}}_{n\to n\pm 1} = \frac{1}{e^2 R_{\mathrm{L(R)}}} \gamma[\mp e(V - V_{\mathrm{L(R)}}) - E_C], \tag{4.25}$$

where $R_{\mathrm{L(R)}}$ and $V_{\mathrm{L(R)}}$ are the resistance and potential levels of the reservoirs, and V is the dot potential (here we assume $V_L - V = V - V_R = \Delta V_o$). In the case of Hall droplets, the Coulomb energy E_C should be replaced by the difference of edge energies $\Delta E_\sigma = E_\sigma(Q+1) - E_\sigma(Q)$ (4.5), including neutral parts, that are obtained from the CFT partition function as seen before (we also set $e = 1$ in the following).

Among the different regimes considered in [37], that of $T < \Delta V_o < \Delta E_\sigma$ is the most relevant for our purposes: this is the situation of thermal-activated CB conduction, where few electrons can enter the droplet from the left by thermal jumps and then quickly get out to the right. The rate for the combined process is

$$\Gamma = \Gamma^L_{n\to n+1}\Gamma^R_{n+1\to n} \sim ((\Delta E_\sigma)^2 - (\Delta V_o)^2)\, \mathrm{e}^{-\beta(\Delta E_\sigma - \Delta V_o)}, \tag{4.26}$$

and the time interval between the peaks is $\Delta t \sim 1/\Gamma$.

The main observation in this section is the following: if the Hall states possess multiplicities d^ℓ_a (4.12) for edge electron states, the corresponding transition rates are amplified accordingly:

$$\Gamma \;\to\; d^\ell_a\, \Gamma. \tag{4.27}$$

Therefore, a real-time experiment of the peak rate can provide a direct test of edge multiplicities and be useful to distinguish between Hall states with otherwise equal CB peak patters. Clearly, formula (4.26) depends on several unknown and state-dependent phenomenological parameters, such as reservoir–droplet couplings. Nevertheless, the qualitative signal should be detectable: upon parametric variation of σ one can find the points of level matching $\sigma = \sigma_i$, where $\Delta E_\sigma \to 0$ and the rate Γ saturates. From these points, one can tune σ at the midpoints in-between and test formula (4.26) of thermal-activated CB conduction with less uncertainty. The signal is more significant when the values of d^ℓ_a change considerably from one σ interval to the following (as e.g. for $SU(n) \times U(1)$ Jain states with $d_a = \binom{n}{a}$).

4.2. Thermopower and $T = 0$ entropies

Quasiparticles with non-Abelian statistics are characterized by degenerate energy levels that are due to the multiplicity of fusion channels [2]. Their wavefunctions, described by Euclidean RCFT correlators in the plane, form multiplets whose dimension can be obtained by repeated use of the fusion rules, $\phi_a \times \phi_b = \sum_c (N_a)^c_b\, \phi_c$, where (N_a) is the fusion matrix. The number of terms obtained by fusing n_{qp} quasiparticles of ath type is given by

$$\mathcal{D}_a(n_{qp}) = \sum_{b=1}^{N} [(N_a)^{n_{qp}-1}]^b_a \sim d_a^{n_{qp}-1}, \qquad n_{qp} \to \infty, \qquad d_a = \frac{S_{a0}}{S_{00}}, \tag{4.28}$$

where d_a is the so-called quantum dimension (not to be confused with other multiplicities discussed in the previous sections). This result was obtained by using the Verlinde formula [14] for diagonalizing the fusion matrix and by keeping the contribution of the largest eigenvalue for $n_{qp} \to \infty$.

From the thermodynamical point of view, the presence of $n_{qp} \gg 1$ quasiparticles in the system implies a quantum entropy at $T = 0$:

$$\mathcal{S}_a(T = 0) \sim n_{qp} \log(d_a), \tag{4.29}$$

that is 'unexpected' because the state is completely determined (gapped, fixed control parameters, fixed quasiparticle positions, etc).

This entropy can be obtained from the RCFT partition function described in this paper that encodes all the low-energy physics of static bulk quasiparticles and massless edge excitations. Let us consider an isolated droplet of the Hall fluid, i.e. the disk geometry, and compute the entropy first for one quasiparticle and then for many of them.

The disk partition function for a quasiparticle of type a in the bulk is given by the generalized character $\theta_a(\tau)$ (cf (2.14) and section 2.1), owing to the condition of the global integer charge $Q_{\text{bulk}} + Q_{\text{edge}} \in \mathbb{Z}$. The $T = 0$ entropy is a many-body effect that manifests itself in the thermodynamic limit; therefore, we should first send $R \to \infty$ and then $T \to 0$, i.e. expand the partition function for $\tau \to 0$ [14]. Upon performing the S modular transformation (of both neutral and charged parts), we obtain the leading behavior, $\theta_a(\tau) \sim S_{a0}\, \theta_0(-1/\tau) \sim S_{a0} \exp(\mathrm{i}2\pi c/(24\tau))$, and compute the entropy

$$\mathcal{S}_a(T \to 0) = \left(1 - \tau \frac{\mathrm{d}}{\mathrm{d}\tau}\right) \log \theta_a \sim \log \frac{S_{a0}}{S_{00}} - \log \frac{1}{S_{00}}. \tag{4.30}$$

The first contribution indeed reproduces the entropy for one non-Abelian quasiparticle, $\mathcal{S}_a = \log d_a$; note that the Abelian part of the S-matrix cancels out (e.g. $S_{0a}^{(\text{Abelian})} = 1/\sqrt{\hat{p}}$, $\forall a$, in the Read–Rezayi state) and one obtains the quantum dimension (4.28).

The last term in (4.30) is the (negative) boundary contribution to the entropy that is also present without quasiparticles; it involves the 'total quantum dimension', $\mathcal{D} = 1/S_{00} = \sqrt{\hat{p}}\sqrt{\sum_a d_a^2}$ [38], where the sum extends over all quasiparticles and receives a contribution from the Abelian part. Note that a single quasiparticle is not usually associated with a bulk entropy, as described at the beginning of this section, because multiple fusion channels only appear for $n_{pq} \geqslant 4$ quasiparticles. On the other hand, from the topological point of view, the edge divides the infinite system into two parts, interior and exterior, with the edge keeping track of the missing part [38].

The entropy for several quasiparticles in the bulk can be obtained from the corresponding disk partition function; for two quasiparticles, for example, this is obtained by fusing the two particles and summing over the resulting edge sectors, as follows:

$$Z_{aa} = \sum_b N_{aa}^b\, \theta_b(\tau). \tag{4.31}$$

In general, the repeated fusion of several particles reproduces the computation of the bulk entropy contribution (4.28), leading again to $\mathcal{S} \sim n_{qp} \log(d_a)$.

Several aspects of the non-Abelian entropy have been discussed in [21, 38]; here we deal with the proposal of observing it in thermopower measurements [27] that could be feasible [5]. Let us consider the annulus geometry and introduce both an electric potential difference ∇V_o and a temperature gradient ∇T between the two edges. The electric current takes the form

$$\mathbf{J} = -\sigma \cdot \nabla V_o - \alpha \cdot \nabla T, \tag{4.32}$$

where σ, α are the electric and Peltier conductivity tensors. The thermopower (or Seebeck coefficient) is defined as the tensor, $S_{\text{Seebeck}} = \sigma^{-1} \cdot \alpha$, i.e. the ratio of transport coefficients pertaining to the two gradients [26].

Here we consider the case of exact compensation between the gradients, such that the current vanishes, $\mathbf{J} = 0$. In this case, the thermopower component in the annulus geometry is given by

$$S_{\text{Seebeck}} = \frac{\alpha}{\sigma} = -\frac{\Delta V_o}{\Delta T} = \frac{\mathcal{S}}{eN_e}. \tag{4.33}$$

In the last part of this equation, we also wrote the desired relation that the thermopower is equal to the entropy per electron [26, 27].

In the following, this result is recovered by adapting the near-equilibrium description by Yang and Halperin [27] to our setting. We consider a small variation of the grand-canonical potential

$$\mathrm{d}\Omega = -\mathcal{S}\mathrm{d}T - N_e(\mathrm{d}\mu + e\mathrm{d}V_o), \tag{4.34}$$

involving both the chemical μ and electric V_o potentials coupled to the N_e electrons. Note that equation (4.34) takes the non-relativistic form because we are describing bulk effects related to adding the quasiparticles. For vanishing current, the gradients induce an excess of charge at the edges that is equivalent to the pressure effect considered in [27].

From definition (4.32) and the grand-canonical potential (4.34), we can express the two conductivities in terms of second derivatives (implying the Maxwell relations), as follows:

$$\sigma = -\frac{\partial Q}{\partial V_o} = e^2 \frac{\partial^2 \Omega}{\partial \mu^2} = -e^2 \frac{\partial N}{\partial \mu}, \qquad \alpha = -\frac{\partial Q}{\partial T} = e \frac{\partial^2 \Omega}{\partial \mu \partial T} = -e \frac{\partial \mathcal{S}}{\partial \mu}. \tag{4.35}$$

Upon taking their ratio, the result (4.33) is recovered for $\mathcal{S}$ linear in the number of electrons, as shown momentarily.

From the experimental point of view, quasiparticles of smaller charge ($a = 1$) are induced in the system by varying infinitesimally the magnetic field from the center of the plateau $B = B_o$. In the diluted limit, ν remains constant and the number of quasiparticles and of electrons can be related as follows:

$$n_{qp} = \frac{e(B - B_o)}{e^* B_o} N_e. \tag{4.36}$$

The entropy associated with the non-Abelian quasiparticles is given by the partition function as explained before: in the annulus geometry, its expression for $T \to 0$ is given by the one-edge expression, e.g. (4.31), multiplied by the ground-state contribution for the other edge $\overline{\theta(\tau)_0}$. The result is again given by (4.29) up to a constant.

Finally, the value of the thermopower is found by taking the ratio (4.33) of the entropy over the total electron charge [27]:

$$S_{\text{Seebeck}} = \left| \frac{(B - B_o)}{e^* B_o} \right| \log(d_1). \tag{4.37}$$

Upon measuring the two gradients $\nabla T, \nabla V_o$ at vanishing current, one can observe a characteristic V-shaped behavior of S_{Seebeck} near the center of the plateau, signaling the non-Abelian state [5]; other sources of entropy are under control in the gapped state. Clearly, Abelian quasipartides ($d_1 = 1$) do not contribute.

In conclusion, we have shown that the RCFT partition function of edge excitations is useful to obtain the $T = 0$ entropies associated with the non-Abelian quasiparticles.

5. Conclusions

In this paper, we have obtained the modular invariant partition functions of several non-Abelian states that could describe the Hall plateaux in the second Landau level. We have extended and simplified the earlier derivations [15, 17, 19] and showed that they straightforwardly follow from the choice of the RCFT for neutral states and of the electron field, corresponding to a so-called simple current.

The physical applications to Coulomb blockade experiments [4] have been discussed in section 4: for $T > 0$ there are two corrections to the periodic peak positions found at $T = 0$ [22], in the ranges $T < T_{\mathrm{n}}$ and $T_{\mathrm{n}} < T < T_{\mathrm{ch}}$, respectively, where T_{n} and T_{ch} are typical energies of neutral and charged excitations ($T_{\mathrm{ch}} \sim 10\, T_{\mathrm{n}}$). These corrections are sensible to the multiplicity of low-lying neutral states [24] and to their S-matrix of modular transformation [23]: therefore, they can give a richer and unambiguous signal of non-Abelian statistics [25].

Using a phenomenological approach, in section 4.1.3 we also argued that the multiplicity of neutral states could be better seen in experiments observing the Coulomb-peak time rate off-equilibrium, i.e. at finite bias $V_0 > 0$.

Finally, partition functions were used to compute the thermopower and the associated entropy of non-Abelian quasiparticles; here, earlier topological approaches were recovered and extended by using the physical partition function of the Hall system with an edge in the $T = 0$ limit.

A new perspective in our RCFT approach based on annulus partition functions is offered by the possibility of finding other solutions of the modular invariant conditions, where the neutral excitations of the left and right edges are paired differently. Actually, the U condition (2.10) requires the matching of fractional charges on the two edges, but leaves the possibility on non-trivial pairing of neutral states, still constrained by the other modular conditions.

For example, if the neutral CFT possesses a second simple current, the chiral symmetry can be further extended by orbifolding; the resulting partition function is diagonal with respect to the new extended-symmetry basis but is non-diagonal in the original basis. The new theory possesses less excitations than the original one and an additional neutral excitation (the second simple current). Such a solution is possible for example in the Read–Rezayi states with $k \geqslant 4$ even: its physical interpretation remains to be understood.

Non-diagonal modular invariants have been much analyzed in the RCFT literature [14]: in the application to critical phenomena in two dimensions, they describe new universality classes that are different from those of diagonal invariants [39]. It would be interesting to study this kind of model building in the quantum Hall setting because it may reveal new non-Abelian features.

Acknowledgments

We thank D Ferraro, N Magnoli, I D Rodriguez, M Sassetti, A Stern and G R Zemba for interesting discussions. AC would like to thank the hospitality of LPTHE, ENS, Paris. This work was partially supported by the ESF program INSTANS and by a PRIN grant of the Italian Ministry of Education and Research.

Appendix. Modular transformations

In this appendix, we collect some definitions, properties and modular transformations of the non-Abelian characters used in the text. A brief review on modular invariance and modular forms can be found in [17] and references therein; more extensive material is presented in [14].

A.1. Read–Rezayi

The properties and transformations of parafermion characters (2.33) and (2.34) are obtained from the coset construction [14, 33, 34]. The basic identity of the coset $\widehat{SU(2)}_k/\widehat{U(1)}$ (resp. $\widehat{SU(3)}_k/\widehat{U(1)}^2$ for the NASS state) is the following expansion of the affine $\widehat{SU(2)}_k$ (resp. $\widehat{SU(3)}_k$) characters χ^Λ:

$$\chi^\Lambda(\tau,\zeta)=\sum_{\lambda\in P/kQ}\chi_\lambda^\Lambda(\tau)\,\vartheta_\lambda(\tau,\zeta),\tag{A.1}$$

where χ_λ^Λ are the parafermionic characters and ϑ_λ are the classical theta functions at level k associated with the root lattice Q of the Lie algebra [14]. The indices Λ and λ belong to the weight lattice P (cf section 3.4).

For the $SU(2)$ lattice, $\vartheta_{m/\sqrt{2}}(\tau,\zeta)=K_m(\tau,k\zeta;2k)$, the Abelian theta function (2.5), and $\chi_\lambda^\Lambda=\chi_m^\ell$, the $\mathbb{Z}_k$ parafermion characters. From (A.1) we also obtain the embedding index for the cosets, $2k$ and $2k\times 6k$ for $\widehat{SU(2)}_k/\widehat{U(1)}$ and $\widehat{SU(3)}_k/\widehat{U(1)}^2$, respectively. The modular transformations of the characters χ_m^ℓ are determined in such a way that (A.1) reproduces the correct transformation of the $\widehat{SU(2)}_k$ characters:

$$\chi^\ell\left(\frac{-1}{\tau}\right)=\sum_{\ell'=0}^{k}s_{\ell,\ell'}\,\chi^{\ell'}(\tau),\qquad s_{\ell,\ell'}=\sqrt{\frac{2}{k+2}}\,\sin\frac{\pi(\ell+1)(\ell'+1)}{k+2}.\tag{A.2}$$

(For simplicity, we fix $\zeta=0$ in neutral characters.) The combination of (A.1) and (A.2) yields the transformation of the characters χ_m^ℓ in (2.34).

The field identifications (2.33) and (3.27), leading to symmetries among the parafermionic characters, follow from the properties of the modular transformations [34]. For example, in the $\widehat{SU(2)}_k$ case the matrix $s_{\ell,\ell'}$ obeys the following symmetry under $\ell\to\mathcal{A}(\ell)=k-\ell$:

$$\mathcal{A}(s_{\ell',\ell})\equiv s_{\mathcal{A}(\ell'),\ell}=s_{k-\ell',\ell}=(-1)^\ell\,s_{\ell',\ell}.\tag{A.3}$$

We now describe the modular transformation (2.37) of the Read–Rezayi extended characters θ_a^ℓ (2.35). After transformation of each term in their sum, the sum over the running index b yields

$$\theta_a^\ell(-1/\tau)=\frac{k}{\sqrt{2kp}}\sum_{q'=0}^{p-1}\sum_{m'=0}^{2k-1}\sum_{\ell=0}^{k}\delta_{m',q'}^{(k)}\,\mathrm{e}^{2\pi\mathrm{i}\frac{2aq'-\hat{p}am'}{p}}\,s_{\ell,\ell'}K_{q'}\chi_{m'}^{\ell'}.\tag{A.4}$$

The mod-k delta function is solved by $m'=q'+\sigma k$ with $\sigma=0,1$; then, q' is re-expressed as $q'=a'+b'\hat{p}$, with a' mod $\hat{p}$ and b' mod k. The sum on q' and m' can be decomposed into $\sum_{\sigma=0}^{1}\sum_{a'=0}^{\hat{p}-1}\sum_{b'=0}^{k-1}$, and the phase is rewritten as $(-1)^{\sigma a}\,\mathrm{e}^{-2\pi\mathrm{i}\frac{Maa'}{2\hat{p}}}$. Using the identifications (2.33) and (A.3), the sum on σ becomes $2\delta_{a,\ell}^{(2)}$ and finally the result (2.37) is obtained.

Calling the whole S-transformation of the θ_a^ℓ characters in (2.37) as $S_{a,a'}^{\ell\ell'}$, we now check its unitarity. We find that

$$\sum_{\ell',a'}S_{a,a'}^{\ell\ell'}(S^\dagger)_{a',a''}^{\ell'\ell''}=\frac{1}{\hat{p}}\sum_{\ell'=0}^{k}\delta_{a,\ell}^{(2)}\delta_{a'',\ell''}^{(2)}s_{\ell\ell'}s_{\ell'\ell''}\sum_{a'=0}^{\hat{p}-1}\mathrm{e}^{2\pi\mathrm{i}\frac{a'(a-a'')M}{2\hat{p}}}.\tag{A.5}$$

Since the non-Abelian part is unitary, we obtain $\delta_{\ell,\ell''}^{(2)}$ and thus $\delta_{(a-a'')M/2,0}^{(\hat{p})}$; the latter condition is equivalent to $\delta_{aa''}^{(2\hat{p})}$ because $\hat{p}$ and M are coprime, $(\hat{p},M)=1$.

A.2. NAF and $\overline{\mathrm{RR}}$

We now derive the modular transformation of the NAF characters θ_a^ℓ in (3.5). Transformation of the two terms in their expression leads to

$$\theta_a^\ell(-1/\tau) = \frac{1}{\sqrt{2\hat{p}}}\sum_{q'=0}^{2\hat{p}-1}\sum_{\tilde{\ell}=0}^{k} \mathrm{e}^{2\pi\mathrm{i}\frac{aq'}{2\hat{p}}}(s_{\ell,\tilde{\ell}} + s_{k-\ell,\tilde{\ell}}\mathrm{e}^{\pi\mathrm{i}q'})\chi^{\tilde{\ell}}K_{q'}. \tag{A.6}$$

The term in the parenthesis is $2s_{\ell\tilde{\ell}}\delta^{(2)}_{q',\tilde{\ell}}$, using (A.3); this condition can be solved by the following parameterization: $q' = a' + b'\hat{p}$ and $\tilde{\ell} = \ell'$ for $b' = 0$, and $\tilde{\ell} = k - \ell'$ for $b' = 1$, with $a' = 0 \bmod \hat{p}$ and $\ell' = a' \bmod 2$. The sums on $q'$and $\tilde{\ell}$ become sums on a', ℓ' and b'. The sum on b' is

$\mathrm{e}^{2\pi\mathrm{i}\frac{aa'}{2\hat{p}}}(s_{\ell,\ell'}K_{a'}\chi^{\ell'} + s_{\ell,k-\ell'}\mathrm{e}^{\mathrm{i}\pi a}\chi^{k-\ell'}K_{a'+\hat{p}}) = \mathrm{e}^{2\pi\mathrm{i}\frac{aa'}{2\hat{p}}}\delta^{(2)}_{a\ell}s_{\ell,\ell'}(K_{a'}\chi^{\ell'} + \chi^{k-\ell'}K_{a'+\hat{p}})$,

finally leading to (3.7).

The unitary of the modular transformation in (3.7) can be verified following the same steps of the previous section. For $\overline{\mathrm{RR}}$ fluids the computation is the same due to the reality of the $\widehat{SU(2)}_k$ *S*-matrix; a small difference is that *k*+*M* is odd for NAF and *M* is odd for $\overline{\mathrm{RR}}$.

A.3. Bonderson–Slingerland states

The BS case is easier by changing the basis of characters from (3.22) to the one in (3.24), because the new Ising characters possess simpler transformations. This new basis is ($a = 0, 1, \ldots, nM$)

$$\begin{aligned}
\widetilde{\theta}_{a,0} &= \frac{1}{\sqrt{2}}(\theta_{a,0} + \theta_{a,2}) = \sum_{b=1}^{2n} K_{2an+b\hat{p}}(\tau, 2n\zeta; 2n\hat{p})\,\Theta_b\widetilde{\chi}_0, && m = 0\\
\widetilde{\theta}_{a,1} &= \theta_{a,1} = \sum_{b=1}^{2n} K_{(2a+1)n+b\hat{p}}(\tau, 2n\zeta; 2n\hat{p})\,\Theta_b\,\widetilde{\chi}_1, && m = 1,\\
\widetilde{\theta}_{a,2} &= \frac{1}{\sqrt{2}}(\theta_{a,0} - \theta_{a,2}) = \sum_{b=1}^{2n} \mathrm{e}^{\mathrm{i}\pi b}K_{2an+b\hat{p}}(\tau, 2n\zeta; 2n\hat{p})\Theta_b\widetilde{\chi}_2, && m = 2.
\end{aligned} \tag{A.7}$$

The computation requires the modular transformation of $\widehat{SU(n)}_1$ characters [15]:

$$\Theta_b\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{n}}\sum_{b'=1}^{n} \mathrm{e}^{-\mathrm{i}2\pi\frac{bb'}{n}}\,\Theta_{b'}(\tau). \tag{A.8}$$

Combining all the transformations in the factors of $\widetilde{\theta}_{a,i}$, we write

$$\begin{aligned}
\widetilde{\theta}_{a,i} = \frac{1}{\sqrt{n^2 2\hat{p}}}\sum_{b=1}^{2n}\sum_{q'=1}^{p}\sum_{\beta'}^{n}\exp\left(2\pi\mathrm{i}\frac{(2an + n\delta_{i,1} + b\hat{p})q' - 2\hat{p}b\beta'}{2n\hat{p}} + \pi\mathrm{i}\delta_{i,2}b\right)\\
\times K_{q'}\Theta_{\beta'}\sum_{i'=0}^{2} S^{\mathrm{Ising}}_{i,i'}\,\widetilde{\chi}_{i'},
\end{aligned} \tag{A.9}$$

where the modular transformation,

$$S^{\mathrm{Ising}} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},$$

is defined according to (3.24). The sum on b gives the factor $\delta^{(2n)}_{q',2\beta'+n\delta_{i,2}}$, whose solution requires us to parameterize q' as $q' = r' + \widetilde{b}\hat{p}$, with r' mod $\hat{p}$ and $\widetilde{b}$ mod $2n$. The delta function imposes

$$r' - n\delta_{i,2} + 2\widetilde{b} - 2\beta' = 2ln, \qquad \text{i.e.} \qquad r' - n\delta_{i,2} = 2a', \qquad \widetilde{b} - \beta' = \sigma n,$$

since $\widetilde{b}$ and β' are defined mod $2n$ and n, respectively, and $\sigma = 0, 1$. It is convenient to write $r' = 2a' + n\delta_{i,2}$, with a' mod $\hat{p}/2$, which is coprime with n; therefore, we can write $r' = 2a'n + n\delta_{i,2}$. Note that β' is the index mod n of $\Theta_{\beta'}$; thus, we can replace it with $\widetilde{b}$. After these substitutions in (A.9), we obtain the modular transformation of the Bonderson–Slingerland characters (A.7), with S reported in (3.25). The unitarity of this matrix is

$$\begin{aligned}\sum_{a',m'} S_{(a,m),(a',m')} S^{\dagger}_{(a',m'),(a'',m'')} &= \sum_{a'} \mathrm{e}^{2\pi\mathrm{i}\frac{a'(a-a'')}{mM+1}} \\ &\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}2\pi a'n/2(nM+1)} \\ 0 & \mathrm{e}^{\mathrm{i}2\pi an/2(nM+1)} & 0 \end{pmatrix} \\ &\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}2\pi an/2(nM+1)} \\ 0 & \mathrm{e}^{-\mathrm{i}2\pi a'n/2(nM+1)} & 0 \end{pmatrix} = \delta^{(mM+1)}_{aa''}\delta_{mm''}. \end{aligned} \tag{A.10}$$

A.4 NASS states

The modular transformations of the $\widehat{SU(3)}_k/\widehat{U(1)}^2$ parafermion characters is found from (A.1): the S-matrix of the $\widehat{SU(3)}_k$ characters in the numerator of the coset is

$$\begin{aligned}\chi^{\Lambda}\left(\frac{-1}{\tau}\right) &= \sum_{\Lambda' \in P^+_k} s_{\Lambda\Lambda'}\, \chi^{\Lambda'}(\tau), \\ s_{\Lambda\Lambda'} &= \frac{\mathrm{i}}{\sqrt{3}(k+3)} \sum_{w\in W} (-1)^{|w|} \exp\left[2\pi\mathrm{i}\frac{(w(\Lambda+\rho), \Lambda'+\rho)}{k+3}\right], \end{aligned} \tag{A.11}$$

where w is an element of the Weyl group W of $SU(3)$, $|w|$ is its length and ρ is half of the sum of the positive roots [14]. The coset decomposition yields the following transformations of the parafermionic characters:

$$\chi^{\Lambda}_{\lambda}\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{3k}} \sum_{\lambda' \in P/kQ} \mathrm{e}^{-2\pi\mathrm{i}\frac{(\lambda,\lambda')}{k}} \sum_{\Lambda' \in P^k_+} s_{\Lambda\Lambda'}\, \chi^{\Lambda'}_{\lambda'}(\tau)\,, \tag{A.12}$$

where the ranges of the indices are explained in section 3.4. Useful properties of $s_{\Lambda\Lambda'}$ are its transformations under the automorphism $\mathcal{A}$ [34]:

$$\begin{aligned}\Lambda = (n_1, n_2) \mapsto \mathcal{A}(\Lambda) &= (k - n_1 - n_2, n_1), \\ \mathcal{A}(s_{\Lambda',\Lambda}) \equiv s_{\mathcal{A}\Lambda',\Lambda} &= s_{\Lambda',\Lambda}\mathrm{e}^{2\pi\mathrm{i}\frac{2n_1+n_2}{3}}; \end{aligned} \tag{A.13}$$

this is the $\widehat{SU(3)}_k$ version of (A.3): note that this map obeys $\mathcal{A}^3 = 1$. The field identifications (3.27) are deduced by studying the modular matrix of the parafermionic characters [34]. Equation (3.27) can be rewritten as $\chi^{\Lambda}_{\lambda} = \chi^{\mathcal{A}(\Lambda)}_{\mathcal{A}(0)+\lambda} = \chi^{\mathcal{A}^2(\Lambda)}_{2\mathcal{A}(0)+\lambda}$.

In order to derive the modular transformation of NASS characters (3.44), we rewrite (3.40) as follows:

$$\Theta^{n_1 n_2}_{q,s}(\tau,\zeta) = \sum_{a,b=0}^{k-1} \sum_{\iota=0,1} K^{(Q)}_{q+\hat{p}(a+b)+\iota k\hat{p}} K^{(S)}_{s+(a-b)+\iota k}\, \chi^{n_1,n_2}_{\frac{q+s}{2}+2a+b,\,\frac{-q+s}{2}-a-2b}. \tag{A.14}$$

After transformation of each term in this sum, we find

$$\Theta^{n_1 n_2}_{q,s}\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{2kp}} \sum_{\iota=0,1} \sum_{a,b=0}^{k-1} \sum_{s'=0}^{2k-1} \sum_{c'=0}^{p-1} \sum_{\mu\in\frac{P}{kQ}} \sum_{\Lambda'\in P^k_+} S_{\Lambda\Lambda'}$$
$$\times \exp\left[2\pi \mathrm{i}\left(\frac{(m+\hat{p}(a+b)+\iota k\hat{p})c'}{q} + \frac{(s+(a-b)+\iota k)s'}{2k} - \frac{(\lambda,\mu)}{k}\right)\right]$$
$$\times K^{(Q)}_{c'}\, K^{(S)}_{s'}\, \chi^{\Lambda'}_{\mu}(\tau) \qquad \text{(A.15)}$$

where λ is an abbreviation for the subscript of χ^{Λ}_{λ} in (A.14). The sum on ι gives $2\delta^{(2)}_{c',s'}$. The next step requires to write the product (λ,μ) explicitly; then, the sums on a and b give the conditions

$$\frac{c'+s'}{2} - \mu_1 = 0 \mod k, \qquad \frac{c'-s'}{2} + \mu_2 = 0 \mod k. \qquad \text{(A.16)}$$

These equations have three solutions in the fundamental domain $\mu \in P/kQ$ that should be taken into account. As before, we reparameterize the indices c' and s':

$$c' = \tilde{q} + \hat{p}(\tilde{a}+\tilde{b}) + \tilde{\iota}k\hat{p}, \qquad s' = \tilde{s} + (\tilde{a}-\tilde{b}) + \tilde{\iota}k. \qquad \text{(A.17)}$$

Note that since $c' = s' \mod 2$ also $\tilde{q} = \tilde{s} \mod 2$. In this parameterization, the three solutions of (A.16) are

$$\mu_1 = \frac{\tilde{q}+\tilde{s}}{2} + 2kN(\tilde{a}+\tilde{b}) + 2\tilde{a} + \tilde{b} + \tilde{n}k, \qquad \mu_2 = \frac{-\tilde{q}+\tilde{s}}{2} - 2\tilde{b} - \tilde{a} - \tilde{n}'k, \qquad \text{(A.18)}$$

where the possible values of $(\tilde{n},\tilde{n}')$ are $(0,0)$, $(1,0)$ and $(1,1)$. The weight μ in (A.18) can be rewritten as

$$\mu = \tilde{\mu} + tA(0), \qquad t = \tilde{n} + \tilde{n}' + 2N(\tilde{a}+\tilde{b}). \qquad \text{(A.19)}$$

Using the automorphism (A.13), we find

$$S_{\Lambda\Lambda'}\,\chi^{\Lambda'}_{\tilde{\mu}+tA(0)} = S_{\Lambda\Lambda'}\,\chi^{A^{-t}(\Lambda')}_{\tilde{\mu}} = S_{\Lambda A^t(\Lambda'')}\,\chi^{(\Lambda'')}_{\tilde{\mu}} = \mathrm{e}^{2\pi \mathrm{i}t(2n_1+n_2)/3} S_{\Lambda\Lambda''}\,\chi^{\Lambda''}_{\tilde{\mu}},$$

with $\Lambda'' = A^{-t}(\Lambda')$. Upon substituting in (A.15), the phase becomes

$$2\pi \mathrm{i}\left(-\frac{q\tilde{q}N}{3p} - \frac{q}{6}(\tilde{n}+\tilde{n}') - \frac{s}{2}(\tilde{n}-\tilde{n}) + \frac{qN(\tilde{a}+\tilde{b})}{3} + \tilde{\iota}\frac{q+s}{2} + \frac{t(2n_1+n_2)}{3}\right).$$

The sum on $\tilde{\iota}$ yields $\delta^{(2)}_{q,s}$. The sum on the three values of (n,n') leads to

$$\delta^{(3)}_{-\frac{q+3s}{2}+2n_1+n_2,0} = \delta^{(3)}_{n_1-n_2,q}, \qquad \text{(A.20)}$$

which is the triality condition. Finally, the result in (3.44) is recovered.

The unitary of the the S-matrix is

$$\sum_{q'\Lambda'} S^{\Lambda\Lambda'}_{qq',ss'}(S^{\dagger})^{\Lambda'\Lambda''}_{q'q'',s's''} = \delta^{(3)}_{n_1-n_2,q}\delta^{(3)}_{n''_1-n''_2,q''}\delta^{(2)}_{q,s}\delta^{(2)}_{q'',s''}\sum_{\Lambda'} s_{\Lambda\Lambda'}\, s^{\dagger}_{\Lambda'\Lambda''}\sum_{q'}\mathrm{e}^{-2\pi \mathrm{i}M\frac{q'(q-q'')}{3\hat{p}}}. \qquad \text{(A.21)}$$

This expression is zero for $n_1 - n_2 \neq n''_1 - n''_2 \mod 3$. Then, for $n_1 - n_2 = n''_1 - n''_2$, i.e. $\Lambda - \Lambda'' \in Q$, the two delta mod 3 impose $q - q'' = 3n$ for an integer n; the sum on the two indices q' and Λ' plus the unitary of $s_{\Lambda\Lambda'}$ show that the lhs of (A.21) is equal to $\delta^{(\hat{p})}_{(q-q'')N/3,0}\delta_{\Lambda,\Lambda''}$. For $N \neq 0 \mod 3$, $\delta^{(\hat{p})}_{(q-q'')N/3,0} = \delta^{(\hat{p})}_{q-q'',0}$, since $(\hat{p}, N) = 1$, thus proving unitarity. Note that the case $N = 1$ includes the physically relevant fractions, $\nu = 2 + \frac{4}{7}$ and $\nu = 2 + \frac{2}{3}$ for $k = 2, 3$.

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