Stable hierarchical quantum Hall fluids as $W_{1+\infty}$ minimal models

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Abstract

In this paper, we pursue our analysis of the $W_{1+\infty}$ symmetry of the low-energy edge excitations of incompressible quantum Hall fluids. These excitations are described by $(1 + 1)$-dimensional effective field theories, which are built by representations of the $W_{1+\infty}$ algebra. Generic $W_{1+\infty}$ theories predict many more fluids than the few, stable ones found in experiments. Here we identify a particular class of $W_{1+\infty}$ theories, the minimal models, which are made of degenerate representations only – a familiar construction in conformal field theory. The $W_{1+\infty}$ minimal models exist for specific values of the fractional Hall conductivity, which nicely fit the experimental data and match the results of the Jain hierarchy of quantum Hall fluids. We thus obtain a new hierarchical construction, which is based uniquely on the concept of quantum incompressible fluid and is independent of Jain's approach and hypotheses. Furthermore, a surprising non-abelian structure is found in the $W_{1+\infty}$ minimal models: they possess neutral quark-like excitations with $SU(m)$ quantum numbers and non-abelian fractional statistics. The physical electron is made of anyon and quark excitations. We discuss some properties of these neutral excitations which could be seen in experiments and numerical simulations.

1. Introduction

In the past few years, the fractional quantum Hall effect (for a review, see Ref. [1]) has become an exciting arena for new physics. Very precise measurements have stimulated the theoretical physicists beyond the borders of the solid-state community. Many
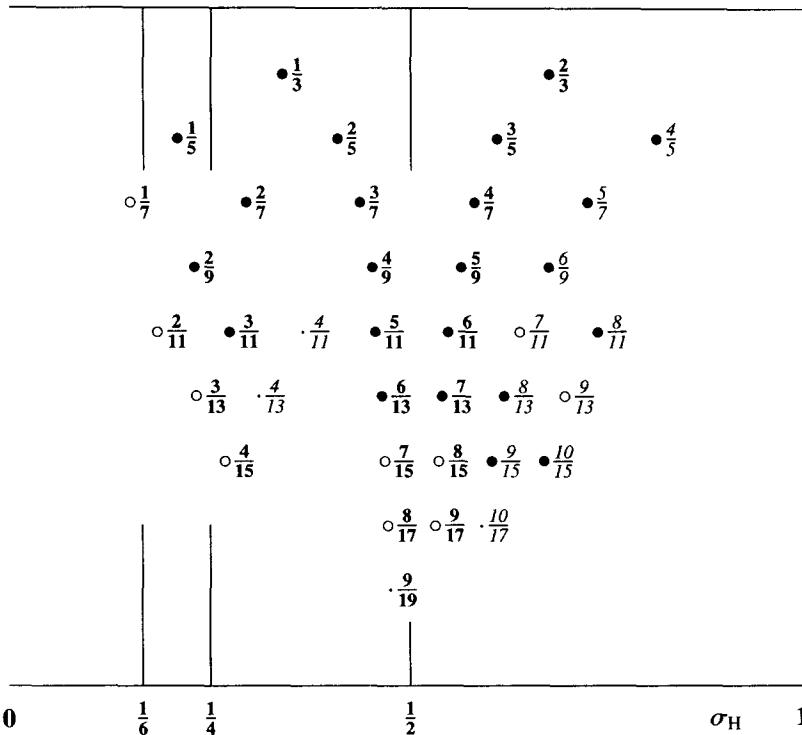
Fig. 1. Experimentally observed plateaus in the range $0 < \nu < 1$: their Hall conductivity $\sigma_H = (e^2/h)\nu$ is displayed in units of (e^2/h). The points denote stability: (•) very stable, (○) stable, and (·) less stable plateaus. Theoretically understood plateaus are in **bold**, unexplained ones are in *italic*. Observed cases of coexisting fluids are displayed as $\nu = \frac{2}{3}, \frac{6}{9}, \frac{10}{15}$, $\nu = \frac{3}{5}, \frac{9}{15}$ and $\nu = \frac{5}{7}, \frac{15}{21}$ (but $\frac{15}{21}$ is not displayed). (Adapted from Ref. [27]).

ideas, and sophisticated mathematical tools, have been proposed to describe the non-trivial quantum many-body ground states revealed by the experiments. More recently, new experiments have remarkably confirmed some of the new key ideas. For example, the new concepts of *edge excitation* and *composite fermion* have acquired the status of real quantum many-body states.

1.1. Hierarchical trial wave functions

There are many aspects of the quantum Hall effect, which reflect the dynamics at different scales of energy and ranges of the control parameters. The problem of our concern here is the description of the stable ground states of the electrons, corresponding to the plateaus of the Hall conductivity, and their spectrum of low-lying excitations. These are relevant for the precise conduction experiments.

Laughlin (for a review, see Ref. [2]) introduced the key concept of the *quantum incompressible fluid* of electrons, which is a ground state with uniform density and a gap for density fluctuations. This ground state, as well as its low-lying excitations, was originally described by trial wave functions for the Hall conductivities $\sigma_{xy} = (e^2/h)\nu$,

where $\nu = 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ are the filling fractions. Afterwards, a hierarchical generalization of these trial wave functions was introduced by Haldane and Halperin [3], in order to describe other observed filling fractions. Therefore, by the *hierarchy* problem we usually mean the classification of stable ground states (and their excitations) corresponding to all observed plateaus. Naturally, the stability is related to the order of iteration of the hierarchical construction, starting from the integer fillings, then the Laughlin fillings and so forth.

The Haldane–Halperin hierarchy is not completely satisfactory, because it produces ground states for too many filling fractions, already at low order of iteration. On the contrary, the experiments show only some stable ground states (see Fig. 1). Although numerical experiments show that the hierarchical wave functions are rather accurate [4], their construction lacks a good control of stability.

Another hierarchical construction of wave functions, which match most of the experimental plateaus to lowest order of the hierarchy, has been proposed by Jain (for a review, see Ref. [5]). Jain abstracted from Laughlin's work the concept of *composite fermion*, a local bound state of the electron and an even number of flux quanta. Due to yet unknown dynamical reasons, the composite fermions are stable quasi-particles, which interact weakly among themselves. Moreover, the strongly interacting electrons at fractional filling can be mapped into composite fermions at effective integer filling [5,6]. Therefore, the stability of the observed ground states with fractional filling can be related with the stability of completely filled Landau levels. The composite fermion picture was successfully applied to the independent dynamics of the compressible fluid at $\nu = \frac{1}{2}$. This strongly interacting, gapless ground state can be described as a Fermi liquid of composite fermions with vanishing effective magnetic field. Experiments [8] have confirmed this theory by observing the free motion of the composite fermions.

1.2. Theories of edge excitations

In open domains of typical size R, like an annulus, the Laughlin incompressible fluid possesses "gapless" excitations of energy $O(1/R)$ located at the boundary [9–11]. These are shape deformations of the fluid edge at constant density (bulk density modulations are suppressed by the gap). The dynamics of the quantum incompressible fluid for energies below the gap is completely described by a $(1 + 1)$-dimensional field theory of the edge excitations, which is defined on the two edge circles. The propagation of low-energy waves along the edge, with definite chirality (either clockwise or anti-clockwise), has been clearly established by an experiment with precise time resolution [12]. Moreover, their $O(1/R)$ energy spectrum has been measured by radio-frequency resonance [13]. Finally, the spectrum of edge excitations and their $(1 + 1)$-dimensional interaction has been confirmed by the experiment of resonant tunnelling through a point contact [14]. Two edges of the electron fluid are pinched by applying a localized voltage, such that edge excitations can interact; the resonant conductance is a computable universal scaling function [15], which fits the experimental data.

The theories of edge excitations are effective field theories of the quantum incompressible fluids, whose low-energy, long-distance dynamics is universal in the sense of the renormalization group [16,17]. The long-distance *scaling* limit defining the edge theories can be explicitly performed at integer fillings [10,18,19]. These $(1+1)$-dimensional field theories are conformal field theories, which can be solved exactly (for a review, see the second reference of Ref. [20]). Although for a limited energy range, they yield exact kinematical properties of the incompressible fluid, which are sufficient to describe the precise conduction experiments.

After the original works of Halperin [9] and Stone [10], a general theory of edge excitations, corresponding to the hierarchical constructions of wave functions, has been formulated [21,11,23]. This is the $(1+1)$-dimensional theory of the chiral boson [22]. An equivalent description is given by abelian Chern–Simons theories on $(2+1)$-dimensional open domains [23]. The edge excitations of the Laughlin fluid are described by a one-component chiral boson, while the hierarchical fluids require many components. Every boson describes an independent edge current, and thus the incompressible fluids have generically a composite edge structure. Each current gives rise to the abelian current algebra in $(1+1)$-dimensions, denoted by $\widehat{U}(1)$, which implies the Virasoro algebra with central charge $c = 1$ [20]. These are well-known examples of the infinite-dimensional symmetries in $(1+1)$ dimensions.

The m-edge theory has $\widehat{U}(1)^{\otimes m}$ symmetry, $c = m$, and is parametrized by an integer, symmetric $(m \times m)$ matrix, with odd diagonal elements, the K-matrix [23]. This determines the Hall conductivity and the fractional charge, spin and statistics [24] of the edge excitations, which correspond to the anyon quasi-particles of the incompressible fluid [2]. The Haldane–Halperin and the Jain hierarchies of wave functions lead to the same long-distance physics, which is described by the chiral boson theories for two different hierarchical constructions of the K-matrix [23].

The lowest-order Jain hierarchy corresponds to the matrices $K_{ij} = \pm\delta_{ij} + pC_{ij}$, where p is an even, positive integer and $C_{ij} = 1$, $\forall i,j = 1,\ldots,m$ [23]. These describe the experimentally observed filling fractions $\nu = m/(mp \pm 1)$. For these K-matrices, additional currents can be defined which enlarge the symmetry from $\widehat{U}(1)^{\otimes m}$ to $\widehat{U}(1) \otimes \widehat{SU}(m)_1$ [25,23], the last being the non-abelian current algebra of level one [20]. This specific many-component chiral-boson theory explains all the experiments discussed above, after the inclusion of the interaction with impurities [26].

Nevertheless, in this framework, there is no a priori reason why the most stable hierarchical fluids should have the $\widehat{SU}(m)_1$ symmetry. Actually, there are also K-matrices with $SO(k)_1$ and $(E_n)_1$ symmetry [27], but their filling fractions do not match well the experimental pattern. Another weak point is that the concept of composite fermion has not yet been translated in the language of edge excitations.

1.3. The $W_{1+\infty}$ symmetry of edge excitations

In a recent series of papers, we have developed the idea that a specific symmetry characterizes the Laughlin incompressible fluids and their edge excitations. At the classical

level, a droplet of incompressible fluid can take different shapes of the same area, i.e. the same density. These configurations are mapped into each other by reparametrizations of the coordinates of the plane which preserve the area. This is the *dynamical symmetry* of classical incompressible fluids under *area-preserving diffeomorphisms* [28,29], whose algebra is called w_∞ (for a review, see the last reference of Ref. [30]). In particular, the infinitesimal shape changes, which are the classical edge waves, can be produced by applying the w_∞ infinitesimal generators to the ground-state droplet configuration [31].

In our first two papers [28,31], we have shown that the quantum incompressibility of the Laughlin ground states can be expressed as *highest-weight conditions* of the infinite-dimensional $W_{1+\infty}$ algebra, the quantum analogue of w_∞ (see also Refs. [32]). Edge excitations are obtained by applying $W_{1+\infty}$ generators with negative mode index to the ground state. Moreover, in the thermodynamic limit of infinite particle number $(N \propto R^2 \to \infty)$, the Laughlin ground state becomes classical and possesses the w_∞ symmetry according to the above picture [31].

In a subsequent paper [18], we explicitly constructed the quantum theory of edge excitations of the incompressible fluid at integer fillings. This was achieved by taking the thermodynamic limit of the states near the edge. The resulting edge theory was shown to correspond to the chiral-boson theory [10,11]. Moreover, any microscopic interaction of the electrons can be expanded in the same *scaling* limit, leading to a universal form for the dispersion relation of the edge excitations [33], which agrees with experiments [13]. As expected, the one-component chiral-boson theory is characterized by the $W_{1+\infty}$ symmetry. In this case, the $W_{1+\infty}$ algebra is actually generated by polynomials of the edge current; thus it includes the current algebra $\widehat{U}(1)$ as a subalgebra.

Having identified the correct symmetry for the simplest examples of incompressible fluids, we proposed to characterize all quantum incompressible fluids as $(1 + 1)$-dimensional $W_{1+\infty}$ theories [34]. The general edge theory can be constructed by using the algebraic methods of conformal field theory: the complete Hilbert space of the theory is built by grouping representations of the symmetry algebra which are closed under the fusion rules, the composition rules for representations [20]. All unitary, irreducible $W_{1+\infty}$ representations were obtained in the fundamental work by Kac and collaborators [35,36]: they exist for integer central charge ($c = m$) and can be regular, i.e. *generic*, or *degenerate*. In Ref. [34], we used the generic representations to build the *generic* $W_{1+\infty}$ theories, which were shown to correspond to m-component chiral-boson theories parametrized by generic $(m \times m)$ K-matrices [34].

1.4. The content of this paper

Here, we pursue our study of $W_{1+\infty}$ edge theories by constructing the special theories made by *degenerate* $W_{1+\infty}$ representations. These theories were not treated in Ref. [34] because the complete mathematical theory of $W_{1+\infty}$ representations was only made available afterwards [36].

Degenerate representations are common in conformal field theory: if the central charge and the weight of a given representation satisfy certain algebraic relations, some of its

states decouple, and should be projected out to obtain an irreducible representation. Degenerate representations form closed sets under the fusion rules, which are called *minimal models*. There are specific minimal models for any symmetry algebra: the well-known ones are the $c < 1$ Virasoro minimal models [20]; larger symmetry algebras, like $W_{1+\infty}$, have $c > 1$ minimal models. The minimal models have less states than the generic theories with the same symmetry, due to the projection; for the same reason, they have a richer dynamics.

In this paper, we show that the $W_{1+\infty}$ minimal models correspond to the edge theories of the Jain hierarchy, which fit the experimental data. The mathematical rules for building the degenerate representations have a hierarchical structure similar to the Jain construction: here, we fully explain this correspondence to the lowest order of the hierarchies.

This result has far-reaching consequences, both theoretical and experimental. The physical mechanism which stabilizes the observed quantum Hall fluids has both short- and long-distance manifestations. At the microscopic level, it can be described by the Jain composite-electron picture and by the size of the gaps; in the scaling limit, by the minimality of the $W_{1+\infty}$ edge theory. Actually, we find it rather natural that the theories with a minimal set of excitations are also dynamically more stable. This long-distance stability principle leads to a logically self-contained edge theory of the fractional Hall effect: a thoroughful derivation of experimental results is obtained from the principle of $W_{1+\infty}$ symmetry, which is the basic property of the Laughlin incompressible fluid.

The $W_{1+\infty}$ minimal models are *not* realized by the multi-component chiral-boson theories with $\widehat{U}(1)^{\otimes m}$ symmetry, because the latter do not incorporate the projection for making irreducible the $W_{1+\infty}$ representations of degenerate type. Nevertheless, reducible and irreducible degenerate representations have the same quantum numbers of fractional charge, spin and statistics. The existing experiments at hierarchical filling fractions were sensible to these data only; therefore their successful interpretation within the chiral-boson theory is consistent with our theory. More refined experiments are needed to test the difference.

The $W_{1+\infty}$ minimal theories are realized by the $\widehat{U}(1) \otimes \mathcal{W}_m(p = \infty)$ conformal theories [36], where the $\mathcal{W}_m(p)$ are the Zamolodchikov–Fateev–Lykyanov models with $c = (m - 1)\left[1 - m(m + 1)/p(p + 1)\right]$ [37]. The main differences with respect to the chiral-boson theory are as follows:

(i) There is a *single* abelian current, instead of m independent ones, and therefore a single elementary (fractionally) charged excitation; there are neutral excitations, but they cannot be associated to $(m - 1)$ independent edges.

(ii) The dynamics of these neutral excitations is new: they have an SU(m) (not $\widehat{SU}(m)_1$) "isospin" quantum number, because their fusion rules are given by the branching rules of this group. Therefore, they are quark-like and their fractional statistics is non-abelian [38]. For example, the edge excitation corresponding to the electron, for the filling fractions $\nu = m/(mp \pm 1)$, is a composite made of $(m \pm 1p)$ anyons and one "quark", and carries both the additive electric charge and the SU(m) isospin.

(iii) The degeneracy of particle–hole excitations at fixed angular momentum is modified by the projection of the minimal models. This counting of states is provided by the characters of degenerate $W_{1+\infty}$ representations, which are known [36]. If the neutral SU(m) excitations have a bulk gap, the particle–hole degeneracy of the ground state (the Wen topological order on the disk [39]) is different from the corresponding one of $\widehat{U}(1)^{\otimes m}$ excitations. This can be tested in numerical diagonalizations of few electron systems; existing data are not accurate enough (for a review, see Ref. [11]).

The plan of the paper is the following. In Section 2, we review the generic $W_{1+\infty}$ theories [34], and their chiral-boson realizations. In Section 3, we construct the $W_{1+\infty}$ minimal models out of degenerate representations. In Section 4, we describe some physical properties of the minimal models and propose some tests for them. In the concluding remarks, we discuss the open problem of the higher-order hierarchical construction of $W_{1+\infty}$ minimal models.

2. Existing theories of edge excitations and experiments

In this section, we shall review the essential points of the *generic* $W_{1+\infty}$ theories of edge excitations [34]. We shall moreover show in detail how these $W_{1+\infty}$ field theories are realized as theories of multi-component chiral bosons, thus making contact with previous treatments [23,11] and providing a symmetry ground for them.

2.1. The generic $W_{1+\infty}$ theories

At the classical level, a two-dimensional droplet of incompressible fluid can take different shapes of the same area. The configuration space of a classical (chiral) incompressible fluid can thus be generated by *area-preserving diffeomorphisms* from a reference droplet, say a disk [28,29]. The infinitesimal generators of area-preserving diffeomorphisms satisfy an infinite-dimensional Lie algebra called w_∞ [30]. When they are applied on the classical ground-state distribution function, they produce small excitations corresponding to (chiral) edge waves localized on the one-dimensional boundary [31][2].

This picture is essentially maintained at the quantum level, in the thermodynamic limit $N \to \infty$. In this case, the relevant symmetry algebra is $W_{1+\infty}$, the quantum analogue of w_∞. The generators V_n^i of $W_{1+\infty}$ are characterized by a mode index $n \in \mathbb{Z}$ and an integer *conformal spin* $h = i + 1 \geqslant 1$. They satisfy the algebra [30]

$$[V_n^i, V_m^j] = (jn - im)V_{n+m}^{i+j-1} + q(i,j,n,m)V_{n+m}^{i+j-3} + \ldots + \delta^{ij}\delta_{n+m,0}cd(i,n), \quad (2.1)$$

where the structure constants q and d are polynomial of their arguments, c is the central charge, and the dots denote a finite number of similar terms involving the operators

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[2] A complete introduction can be found in Section 2 of Ref. [33].

$V_{n+m}^{i+j-1-2k}$ (the complete expression of (2.1) is a bit cumbersome and will be given later). The operators V_n^0 satisfy the abelian current algebra (Kac–Moody algebra) $\widehat{U}(1)$ and the operators V_n^1 the Virasoro algebra [20], respectively,

$$[V_n^0, V_m^0] = nc\delta_{n+m,0} \,, \tag{2.2}$$

$$[V_n^1, V_m^0] = -mV_{n+m}^0 \,,$$

$$[V_n^1, V_m^1] = (n-m)V_{n+m}^1 + \tfrac{1}{12}cn(n^2-1)\delta_{n+m,0} \,. \tag{2.3}$$

V_n^0 and V_n^1 are identified as the charge and angular-momentum modes of the edge excitations, respectively. In the classical limit, all terms but the first in the r.h.s. of (2.1) vanish; the resulting algebra is the classical algebra w_∞ of area-preserving diffeomorphisms. Conversely, the algebra (2.1) is the unique quantization (up to changes of basis) of w_∞ on the circle [40].

A $W_{1+\infty}$ theory is defined as a *Hilbert space* constructed as a set of irreducible, unitary, highest-weight representations of $W_{1+\infty}$, which are closed under the fusion rules for making composite states. This is the well-known algebraic construction of conformal field theories [20]. No reference to a hamiltonian governing the dynamics of excitations is needed. In this formalism, the incompressible quantum fluid ground state appears as a highest-weight state $|\Omega\rangle_W$ satisfying

$$V_n^i|\Omega\rangle_W = 0 \,, \qquad \forall n > 0 \,, \quad i \geqslant 0 \,, \tag{2.4}$$

and

$$V_0^i|\Omega\rangle_W = 0 \,, \qquad i \geqslant 0 \,. \tag{2.5}$$

Particle–hole edge excitations above the ground state are obtained by applying generators with negative mode index to $|\Omega\rangle_W$.

The *quasi-particle* excitations are the Laughlin anyons [2], which are localized deformations of the bulk density. Due to incompressibility, their charge excess (or defect) is entirely transmitted to the boundary, where it is seen as a charged edge excitation [18]. These excitations appear as further highest-weight states, which have non-vanishing eigenvalues for all the V_0^i. Specifically, V_0^0 and V_0^1 determine (minus) the charge and the spin of the quasi-particle, respectively [34]. The eigenvalues of V_0^i ($i \geqslant 2$) measure the radial moments of the charge distribution of quasi-particles [33].

For integer filling, one can show explicitly that the excitations above the incompressible ground state can be organized as a $W_{1+\infty}$ representation [28,18]. This fact, combined with the classical picture, has led us to characterize *any* quantum incompressible fluid as a $W_{1+\infty}$ theory. The hierarchy problem reduces therefore to a complete classification of $W_{1+\infty}$ theories.

This classification can be achieved thanks to the crucial work [35], in which all irreducible, unitary, *quasi-finite* highest-weight representations of $W_{1+\infty}$ have been constructed. Such representations exist only if the central charge is a positive integer, $c = m$, $m \in \mathbb{Z}_+$. They are characterized by a *m-dimensional weight vector r* with real elements, and are built on top of a highest-weight state $|r\rangle_W$, which satisfies

$$V_n^i |r\rangle_\text{W} = 0 , \qquad \forall n > 0 , \quad i \geqslant 0 , \tag{2.6}$$

and is an eigenstate of the V_0^i,

$$V_0^i |r\rangle_\text{W} = \sum_{n=1}^{m} m^i(r_n) |r\rangle_\text{W} , \tag{2.7}$$

where $m^i(r)$ are ith order polynomials of a weight component (see Section 3). In particular, the charge V_0^0 and Virasoro V_0^1 eigenvalues are given by

$$\sum_{n=1}^{m} m^0(r_n) = r_1 + \ldots + r_m ,$$

$$\sum_{n=1}^{m} m^1(r_n) = \tfrac{1}{2} \left[(r_1)^2 + \ldots + (r_m)^2 \right] . \tag{2.8}$$

For *generic* $W_{1+\infty}$ representations, the fusion rules require that all weight vectors r forming a $W_{1+\infty}$ theory span a *lattice* Γ,

$$\Gamma = \left\{ r \,\bigg|\, r = \sum_{i=1}^{m} n_i v_i , \quad n_i \in \mathbb{Z} \right\} , \tag{2.9}$$

whose points satisfy $(r_i - r_j) \notin \mathbb{Z}$, $\forall i \neq j$, as better explained below. The resulting $W_{1+\infty}$ theory describes an m-component incompressible fluid with m interacting edges. The v_i are identified as the basis vectors, representing a physical elementary excitation in the ith component. The *physical charge* of an excitation with labels $n_i \in \mathbb{Z}$ is given by the sum of the components in the physical basis [34][3],

$$Q = \sum_{i,j=1}^{m} K_{ij}^{-1} n_j , \qquad n_i \in \mathbb{Z} , \tag{2.10}$$

where $K_{ij}^{-1} = (v_i \cdot v_j)$ is the metric of the lattice. It is related to the highest-weight r by a linear basis transformation. The spin J and the statistics θ/π of this excitation are given by the Virasoro V_0^1 eigenvalue (2.8),

$$2J = \frac{\theta}{\pi} = \sum_{i,j=1}^{m} n_i K_{ij}^{-1} n_j , \tag{2.11}$$

while the Hall conductivity can be computed via the chiral anomaly equation and is found to be [34]:

$$\sigma_\text{H} = \frac{e^2}{h} \nu , \qquad \nu = \sum_{i,j=1}^{m} K_{ij}^{-1} . \tag{2.12}$$

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[3] As also explained afterwards in Eq. (2.33).

The presence of m electron excitations in the spectrum (2.10), (2.11), with unit charge, fermionic statistics and integer monodromy relative to any other excitation requires the matrix K to have *integer* entries, odd on the diagonal.

Note that there are many bases for the lattice Γ (2.9), which are related by integer linear transformations, the *modular transformations*,

$$u_i = \sum_{j=1}^{m} \Theta_{ij} v_j, \qquad \Theta_{ij} \in \mathrm{SL}(m, \mathbb{Z}). \tag{2.13}$$

Actually, different bases correspond to different definitions of the charge units of the fluids, and give different spectra for the total charge Q, while the fractional statistics spectrum is independent of the basis.

2.2. The chiral-boson theories

We now recall the main properties of the multi-component chiral-boson theories [11,23]. On an annulus geometry, with edge circles $|\mathbf{x}| = R_1$ and $|\mathbf{x}| = R_2$, one introduces m independent one-dimensional *chiral* currents

$$J^i (R_1\theta - v_i t) = -\frac{1}{2\pi R_1} \frac{\partial}{\partial \theta} \phi^i \qquad (|\mathbf{x}| = R_1), \tag{2.14}$$

and corresponding ones with opposite chirality $J^i (R_2\theta + v_i t)$ at the other edge $|\mathbf{x}| = R_2$. The dynamics of these currents on the edge circle $|\mathbf{x}| = R_1$ is governed by the action

$$S = -\frac{1}{4\pi} \int \mathrm{d}t\, \mathrm{d}x \sum_{i=1}^{m} \kappa_i (\partial_t \phi^i + v_i \partial_x \phi^i) \partial_x \phi^i, \qquad \text{for } x \equiv R_1\theta, \tag{2.15}$$

for the m $(1+1)$-dimensional *chiral boson* fields ϕ^i [22]. The corresponding action for the other circle $x \equiv R_2\theta$ is obtained by replacing $v_i \to (-v_i)$. The dynamics on the two edges are identical and independent, only constrained by the conservation of the total charge: thus we describe one of them only. We can change the normalization of the fields, and reduce each coupling constant to a sign, $\kappa_i \to \pm 1$. The equations of motion imply that the fields are chiral, $\phi^i = \phi^i (x - v_i t)$, and canonical quantization implies the following commutation relations for the currents:

$$[J^i(x_1), J^k(x_2)] = \frac{1}{2\pi\kappa_i} \delta^{ik} \delta'(x_1 - x_2) \qquad (t_1 = t_2), \tag{2.16}$$

which are those of the multi-component abelian current algebra $\widehat{U}(1)^{\otimes m}$ [20]. The hamiltonian is

$$H = \pi \int \mathrm{d}x \sum_{i=1}^{m} \kappa_i v_i : J^i J^i :, \tag{2.17}$$

where the double dots denote normal ordering. Its positive definiteness requires the signs of the velocities v_i and the couplings κ_i to be related:

$$v_i \kappa_i > 0, \qquad i = 1, \ldots, m. \tag{2.18}$$

Let us first discuss one particular chiral current, $v_i > 0$ (i.e. $\kappa_i = 1$). The quantization of the chiral boson is equivalent to the construction of the representations of the current algebra (2.16). Actually, all the states in the Hilbert space of the theory can be fitted into a set of representations [20]. To this end, we introduce the Fourier modes of the currents,

$$J^i(R\theta - v_it) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \alpha_n^i \exp(in(\theta - v_it)) \,, \tag{2.19}$$

which satisfy,

$$[\alpha_n^i, \alpha_m^j] = \delta^{ij} \frac{n}{\kappa_i} \delta_{n+m,0} \,. \tag{2.20}$$

The positivity of the ground-state expectation value

$$\langle \Omega | \alpha_n^i \alpha_{-n}^i | \Omega \rangle \equiv \| \alpha_{-n}^i | \Omega \rangle \|^2 \geq 0 \,, \tag{2.21}$$

and the commutation relations (2.20) with $\kappa_i = 1$ imply the conditions

$$\alpha_n^i | \Omega \rangle = 0 \,, \qquad n > 0 \quad (v_i > 0) \,. \tag{2.22}$$

An irreducible highest-weight representation of the $\widehat{U}(1)$ current algebra is made by the highest-weight state $|\Omega\rangle$ and by all states obtained by applying any number of α_n^i, $n < 0$, to it. The weight of the representation is given by the eigenvalue of α_0^i, which is the single-edge charge, in units to be specified later. For the ground state, we have

$$\alpha_0^i | \Omega \rangle = 0 \,. \tag{2.23}$$

Other unitary highest-weight representations can be similarly built on top of other highest-weight states $|r\rangle$, $r \in \mathbb{R}$, which satisfy

$$\alpha_n^i | r \rangle = 0 \,, \qquad n > 0 \,, \qquad \alpha_0^i | r \rangle = r | r \rangle \,, \tag{2.24}$$

and are built by applying the *vertex operators* to the ground state [20]. These representations correspond to the quasi-particle excitations of this edge theory. The Virasoro generators are defined by the Sugawara construction [20],

$$L_n^i = \tfrac{1}{2}\kappa_i \sum_{l=-\infty}^{\infty} : \alpha_{n-l}^i \alpha_l^i : \,, \tag{2.25}$$

and act on the highest-weight states as follows:

$$L_n^i | r \rangle = 0 \,, \quad n > 0 \,, \qquad L_0^i | r \rangle = \tfrac{1}{2}\kappa_i r^2 | r \rangle \,. \tag{2.26}$$

They give rise to the Virasoro algebra (2.3) with $c = 1$, for each current component i.

Let us now discuss an antichiral edge current, $v_i < 0$ (i.e. $\kappa_i = -1$). The modes of the antichiral current are now called $\bar{\alpha}_n^i$,

$$J^i(R\theta + v_it) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \bar{\alpha}_n^i \exp(in(\theta + v_it)) \,. \tag{2.27}$$

The $\bar{\alpha}_n^i$ satisfy the current algebra (2.20) with $\kappa_i = -1$. The positivity of (2.21) requires in this case

$$\bar{\alpha}_n^i |\Omega\rangle = 0, \qquad n < 0 \quad (v_i < 0). \tag{2.28}$$

The Virasoro generators $\overline{L}_n^i$ are again defined in terms of the $\bar{\alpha}_n^i$ by the Sugawara construction (2.25), and similarly satisfy $\overline{L}_n^i |\Omega\rangle = 0$, $n < 0$. These are non-standard ground-state conditions, which also imply that the operators $\overline{L}_n$ do not satisfy the Virasoro algebra (2.3). Nevertheless, it is possible to define operators,

$$\alpha_n^i \equiv \bar{\alpha}_{-n}^i, \qquad L_n^i \equiv -\overline{L}_{-n}^i \quad (v_i < 0). \tag{2.29}$$

which satisfy the standard algebras (2.25), (2.3) and the standard h.w.s. conditions (2.24), (2.26) with $|\kappa_i| = 1$. On the other hand, $\overline{L}_0^i$ retains its physical meaning of fractional spin and statistics and thus has negative spectrum for $v_i < 0$.

The hamiltonian (2.17) can be finally expressed in terms of the Virasoro generators as follows [20]:

$$H = \sum_{i=1}^{m} \frac{|v_i|}{R} \left(L_0^i - \frac{1}{24} \right); \tag{2.30}$$

its spectrum is positive for both chiralities, as originally required.

Let us now define the unit of charge carried by the ith edge by coupling the current J^i minimally to an electric field along the edge $E^i = -\partial A_0^i(\theta, t)/\partial\theta$. In general, an edge excitation can have components on all edges. Therefore, we introduce new currents ρ^i, carrying one unit of charge in the ith edge. These are general linear combinations of the J^i, which create instead energy eigenstates of definite chirality:

$$H_{\text{e.m.}} = \int dx \sum_{i=1}^{m} \rho^i A_0^i, \qquad \rho^i = \sum_{j=1}^{m} \Lambda_{ij} J^j, \qquad \Lambda \in \text{GL}(m, \mathbb{R}). \tag{2.31}$$

The total charge is

$$Q = \sum_{i,j=1}^{m} \Lambda_{ij} \alpha_0^j, \tag{2.32}$$

In a chiral theory, the minimal coupling (2.31) produces the chiral anomaly [20]. This means that the charge is not conserved, i.e. the ground state evolves into some charged state. By integrating this evolution over asymptotic times, we can generate all charged states in the theory, and obtain the spectrum of α_0^i:

$$\Delta\alpha_0^i \Big|_{t=-\infty}^{t=\infty} = i \int dt \, [\alpha_0^i, H_{\text{e.m.}}]$$

$$= \frac{1}{|\kappa_i| 2\pi} \sum_{j=1}^{m} \Lambda_{ji} \int dt \, dx \, E^j = \sum_{j=1}^{m} \Lambda_{ij}^{\text{T}} n_j, \qquad n_j \in \mathbb{Z}. \tag{2.33}$$

In this equation, the total integral of the $(1+1)$-dimensional electric field is a topological invariant quantity, which takes integer values only. Indeed, this electric field can be produced by adding a solenoid in the center of the annulus, and by switching on an integer number of flux quanta [2]. These results show that $\widehat{U}(1)^{\otimes m}$ representations are characterized by weight vectors $r = \{r_1, \ldots, r_m\}$, which span the same lattice Γ of the $W_{1+\infty}$ weights (2.9), with basis $(v_i)_j = \Lambda_{ij}$.

In conformal field theories, the *fusion rules* are the laws for making composite excitations [20]. For any pair of representations, i.e. of charged excitations, the representations obtained by fusing them should also be present in the theory – the representations must form a closed set under the action of the fusion rules. For the abelian current algebra, this rule is simply the addition of weight vectors [20]. Denoting by $M(g, c, r)$ the irreducible representations of the algebra g with central charge c and weight r, we have

$$M\left(\widehat{U}(1)^{\otimes m}, m, r\right) \bullet M\left(\widehat{U}(1)^{\otimes m}, m, s\right) = M\left(\widehat{U}(1)^{\otimes m}, m, r+s\right). \tag{2.34}$$

The lattice Γ (2.9) is the closed set for this rule, because for any pair $r, s \in \Gamma$, the vector $(r + s)$ also belongs to Γ.

The spectrum of the physical charge and fractional statistics of any edge excitation is obtained by replacing the spectrum of α_0^i (2.33) into Eqs. (2.32), (2.26):

$$Q = \sum_{i,j=1}^{m} K_{ij}^{-1} n_j,$$

$$\frac{\theta}{\pi} = \sum_{i,j=1}^{m} n_i K_{ij}^{-1} n_j, \qquad n_i \in \mathbb{Z}, \tag{2.35}$$

where

$$K_{ij}^{-1} = \sum_{l=1}^{m} \Lambda_{il} \frac{1}{\kappa_l} \Lambda_{lj}^{\mathrm{T}} = \left(v_i \cdot \eta \cdot v_j\right). \tag{2.36}$$

In general, the metric K^{-1} of the lattice Γ in the basis v_i is pseudo-euclidean with signature $\eta_{ij} = \delta_{ij}\kappa_i$, due to the possible presence of excitations with different chiralities.

The Hall conductivity in the annulus geometry can be measured by applying a uniform electric field along all the edges, $E^i = E$. The chiral anomaly of the edge theory actually corresponds to a radial flow of particles in the annulus, which move from the inner edge to the outer edge. From Eq. (2.33), the Hall conductivity can be thus identified as

$$\sigma_{\mathrm{H}} = \frac{e^2}{h} \nu, \qquad \nu = \sum_{i,j=1}^{m} K_{ij}^{-1}. \tag{2.37}$$

Eqs. (2.35), (2.36), (2.37) for the Hall conductivity and the spectrum of the charge and fractional statistics of the chiral-boson theories reproduce those of the generic $W_{1+\infty}$ theories (2.10), (2.11), (2.12). As before, the existence of m electron excitations with unit charge and integer statistics relative to all excitations, requires that K has integer entries with odd integers on the diagonal.

Note that the action (2.15) can be also presented in the basis of charge eigenstates ρ_i,

$$S = -\frac{1}{4\pi} \int dt\, dx \sum_{i,j=1}^{m} \left(K_{ij} \partial_t \tilde{\phi}^i + V_{ij} \partial_x \tilde{\phi}^i \right) \partial_x \tilde{\phi}^j , \qquad (2.38)$$

where we define $\rho^i = -\partial_x \tilde{\phi}^i / 2\pi$, and $V_{ij} = \sum_{l=1}^{m} A_{il}^{\mathrm{T}} |v_l| A_{lj}$. Note that K can be any symmetric matrix, while V has a specific form. More general forms of V would give a non-diagonal hamiltonian in the previous basis. These interactions among the edges have been discussed in Ref. [26]. Let us remark that the quantities Q, θ/π and ν derived before are *universal kinematical data* of the quantum incompressible fluid, which are independent of the dynamical data encoded in V, i.e. in the hamiltonian. Actually, only $H_{\mathrm{e.m.}}$ (2.31) and the current algebra (2.20) were used in their derivation. Other physical quantities can depend on V, like the time dependence of the correlators and the Hall conductivity in the bar geometry [11]. On the other hand, these edge interactions were shown to be irrelevant, due to the random effect of disorder, when K describes the physical Jain fluids [26], which we shall deal with in the following.

2.3. $W_{1+\infty}$ theories and chiral-boson theories

We have introduced two kinds of edge theories, the generic $W_{1+\infty}$ theories and the chiral-boson theories, which have the same spectra of filling fractions and fractional charge and statistics of the excitations. We now show that:
  (i) the generic $W_{1+\infty}$ theories are equivalent to chiral-boson theories;
  (ii) there are *more* $W_{1+\infty}$ symmetric theories, which are different.
  In the algebraic approach, the first point is proven by identifying the generic $W_{1+\infty}$ representations with $\widehat{U}(1)^{\otimes m}$ representations – we must qualify the word "generic". We already saw that both representations are labelled by the same weight vectors r. We must also map one-to-one the states built on top of the respective highest-weight states.
  The general theory of unitary, irreducible (quasi-finite) $W_{1+\infty}$ representations, developed in Refs. [35,36], leads to the following relations between irreducible representations of the two algebras,

$$M\left(W_{1+\infty}, 1, r\right) \sim M\left(\widehat{U}(1), 1, r\right) ;$$

$$M\left(W_{1+\infty}, m > 1, r\right) \sim M\left(\widehat{U}(1)^{\otimes m}, m, r\right) , \qquad \text{for } (r_i - r_j) \notin \mathbb{Z}, \forall i \neq j ;$$

$$M\left(W_{1+\infty}, m > 1, r\right) \subset M\left(\widehat{U}(1)^{\otimes m}, m, r\right) , \qquad \text{if } \exists\, (r_i - r_j) \in \mathbb{Z}. \qquad (2.39)$$

Generically, $W_{1+\infty}$ and $\widehat{U}(1)^{\otimes m}$ representations are one-to-one equivalent. The exceptions appear for $c > 1$, when the weight has some integer components $(r_i - r_j)$. In these cases, the relation is many-to-one, i.e. an irreducible $\widehat{U}(1)^{\otimes m}$ representation is *reducible* with respect to the $W_{1+\infty}$ algebra. We call *generic* the $W_{1+\infty}$ representations which are

one-to-one equivalent to $\widehat{U}(1)^{\otimes m}$ ones $((r_i - r_j) \notin \mathbb{Z}, \forall i \neq j)$, and *degenerate* the remaining $W_{1+\infty}$ representations $(\exists (r_i - r_j) \in \mathbb{Z})$.

The results (2.39) allow the construction of several types of $W_{1+\infty}$ symmetric theories. The generic $W_{1+\infty}$ theories [34] are defined by lattices Γ (2.9) which contain generic $W_{1+\infty}$ representations only: for these, the basis vectors satisfy $((v_\alpha)_i - (v_\alpha)_j) \notin \mathbb{Q}, \forall \alpha, i \neq j = 1, \ldots, m$. These theories are thus equivalent to chiral-boson theories[4]. Other $W_{1+\infty}$ theories, containing only degenerate representations, are instead different. These are the *minimal models*, which we shall explicitly construct in the next section. They are the basic new $W_{1+\infty}$ theories, and are actually very important, because they will be shown to correspond to the experimentally observed Jain fluids.

On the other hand, the chiral-boson theories of the Jain hierarchy have been widely used in the literature and partially confirmed by the experiments, as we discuss hereafter. These are also $W_{1+\infty}$ symmetric, but are not the simplest realizations of this symmetry, because their $\widehat{U}(1)^{\otimes m}$ representations are reducible. The $W_{1+\infty}$ minimal models will be shown to have basically the same spectrum of excitations, but to differ in very important properties.

2.4. The Jain hierarchy

The Jain fluids have been described by the subset of the chiral-boson theories characterized by the following K-matrices [23]:

$$K_{ij} = \pm \delta_{ij} + pC_{ij}, \qquad C_{ij} = 1 \,\forall i, j = 1, \ldots, m, \quad p > 0 \text{ even}, \tag{2.40}$$

and the following spectra of edge excitations (Eqs. (2.35), (2.37)):

$$\nu = \frac{m}{mp \pm 1}, \qquad p > 0 \text{ even} \quad (c = m),$$

$$Q = \frac{1}{pm \pm 1} \sum_{i=1}^{m} n_i,$$

$$\frac{\theta}{\pi} = \pm \left(\sum_{i=1}^{m} n_i^2 - \frac{p}{mp \pm 1} \left(\sum_{i=1}^{m} n_i \right)^2 \right). \tag{2.41}$$

Note that K has $(m-1)$ degenerate eigenvalues $\lambda_i = 1, i = 1, \ldots, m-1$ (resp. $\lambda_i = -1$), and a single value $\lambda_m = \pm 1 + mp$. If the sign $\pm$ is negative, one edge has opposite chirality to the others. There is one basic charged quasi-particle excitation with label $n_i = (1, \ldots, 1)$ and $m(m-1)/2$ *neutral* excitations for $n_i = (\delta_{ik} - \delta_{il})$, $1 \leqslant k < l \leqslant m$, with identical integer statistics.

The corresponding trial wave functions for the ground state have been constructed by Jain as [5]

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[4] The ground-state representation $(r = 0)$ must also be a $\widehat{U}(1)^{\otimes m}$ representation for the closure of the fusion rules.

$$\Psi_\nu = D^{p/2} L^m \, \mathbf{1} \, , \qquad p \text{ even} \, , \tag{2.42}$$

where $L^m \, \mathbf{1}$ represents schematically the wave function of m filled Landau levels and $D^{p/2}$ multiplies the wave function by the pth power of the Vandermonde determinant, which "attaches p flux tubes to each electron", and transforms them into "composite fermions". This construction has been implemented in the multi-component chiral-boson theory (2.38) in Refs. [21,23].

The Jain hierarchy covers most of the experimentally observed plateaus, as we discuss in the next section. However, within the chiral-boson approach, there is no clear motivation for selecting the special K-matrices (2.40). The size of the gap for bulk density waves is usually invoked for solving this puzzle: the observed fluids are supposed to have the largest gaps, while the general K fluids have small gaps and are destroyed by thermal fluctuations and other effects. It is also true that edge theories give kinematically possible incompressible fluids and their universal properties, but cannot describe the size of the gaps, which is determined by the microscopic bulk dynamics.

Nevertheless, in this paper, we show that there is a natural way to select the Jain hierarchy within the $W_{1+\infty}$ edge theory approach. Indeed, the Jain fluids correspond to the $W_{1+\infty}$ minimal models, which are characterized by possessing less states than their chiral-boson counterparts. We propose this reduction of available states as a natural stability principle.

2.5. Experiments

We first discuss the spectrum of fractional Hall conductivities in Eq. (2.41). According to Jain, the stability of the ground states (2.42) should be approximately independent of m, which counts the number of Landau levels filled by the composite fermion. This is in analogy with the integer Hall plateaus, which are all equally stable. On the other hand, the stability decreases by increasing $|p|$, as observed for the Laughlin fluids ($m = 1$). Therefore, the most stable family of plateaus is,

$$\nu = \frac{m}{2m \pm 1} \qquad (p = 2) \, , \tag{2.43}$$

which accumulate at $\nu = \frac{1}{2}$. The next less stable family is

$$\nu = \frac{m}{4m \pm 1} \qquad (p = 4) \, , \tag{2.44}$$

which accumulate at $\nu = \frac{1}{4}$. This behavior is clearly seen in the experimental data of Fig. 1. For these filling fractions, the Jain wave functions for the ground state and the simplest excited states have a good overlap with those obtained numerically by diagonalizing the microscopic hamiltonian with a small number of electrons. Another confirmation of the composite fermion picture comes from the independent dynamics of the *compressible* fluid at $\nu = \frac{1}{2}$: a theory of weakly interacting composite fermions has been proposed [7], which has been confirmed by the experiments [8].

A closer look into the experimental values of the filling fractions in Fig. 1 shows other points (in italic), like $\nu = \frac{4}{5}, \frac{5}{7}, \frac{8}{11}$, which fall outside the Jain main series (2.43), (2.44) (in bold). These points were originally interpreted as "charge conjugates" of these series [5],

$$\nu = 1 - \frac{m}{2m \pm 1}, \qquad \nu = 1 - \frac{m}{4m \pm 1}. \tag{2.45}$$

A charge-conjugated fluid is a fluid of holes in a ($\nu = 1$) electron fluid. The corresponding K-matrix is easily obtained as the $((m + 1) \times (m + 1))$-dimensional matrix [23],

$$\overline{K} = \begin{pmatrix} 1 & 0 \\ 0 & -K \end{pmatrix}. \tag{2.46}$$

These charge-conjugate models actually belong to the second iteration of the Jain hierarchy [5]. Unfortunately, the charge conjugate states do not fit well the data in Fig. 1. The $\nu = \frac{1}{2}$ family would be self-conjugate; thus there should be two fluids per filling fraction, which are not observed, apart from two cases. Actually, coexisting fluids can be detected by experiments where the magnetic field is tilted from the orthogonal direction to the plane [41]. Furthermore, the conjugate of the observed fractions in the $\nu = \frac{1}{4}$ family are not observed in half of the cases. Finally, there are fractions which do not belong to any previous group: $\nu = \frac{4}{11}, \frac{7}{11}, \frac{4}{13}, \frac{8}{13}, \frac{9}{13}, \frac{10}{17}$.

In conclusion, all the fractions outside the main Jain families (2.43), (2.44) are not well understood at present (and will not be explained in this paper). Any known extension of the previous theory which explains these few extra fractions, also introduces many more unobserved fractions, with an unclear pattern of stability. Besides the second iteration of the Jain hierarchy [5], we also quote the approach proposed by Fröhlich and collaborators [27]. They analyzed all lattices Γ (2.9), with positive-definite, integer (inverse) metric K, for small values of $\det(K)$, whose classification is known in the mathematical literature. These lattices can be related to the $SU(m)$, $SO(k)$ and exceptional Lie algebras. The stability of the corresponding fluids does not follow a clear pattern related to these algebras, besides the case of the chiral Jain fluids (2.40), whose $SU(m)$ symmetry will be explained in the next section. Moreover, in this approach, the K-matrices for the Jain filling fractions $\nu = m/(mp - 1)$, $p > 0$, are different from the Jain proposal (2.40) which is not positive definite.

In Fig. 2, we study the limitations of phenomenological descriptions of the stability of the fluids. Besides all the observed (bold) fractions of Fig. 1, we report the unobserved (italic) ones $\nu = p/q$, which satisfy the conservative cuts of the "phase space"

$$\frac{2}{11} < \nu = \frac{p}{q} < \frac{4}{5}, \quad \text{and} \quad p \leqslant 10, \quad q \leqslant 17. \tag{2.47}$$

Namely, we display all fractions which would be observed if the gap were a smooth function of the parameters (ν, p, q) interpolating the data, a typical phenomenological hypothesis. Fig. 2 shows that, besides the families (2.43), (2.44), about half of the fractions are unexplained observed fillings and half are unobserved but phenomenologi-
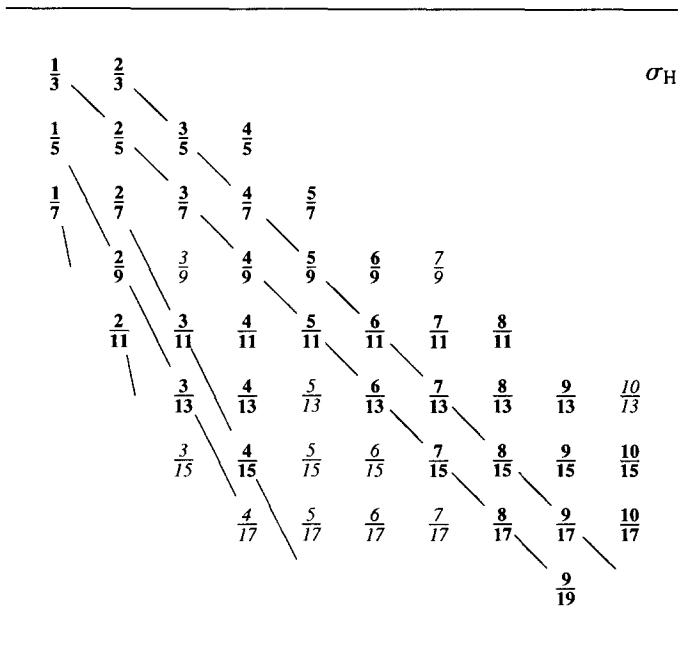
$\frac{1}{3}$ $\frac{2}{3}$ σ_{H}

$\frac{1}{5}$ $\frac{2}{5}$ $\frac{3}{5}$ $\frac{4}{5}$

$\frac{1}{7}$ $\frac{2}{7}$ $\frac{3}{7}$ $\frac{4}{7}$ $\frac{5}{7}$

$\frac{2}{9}$ $\frac{3}{9}$ $\frac{4}{9}$ $\frac{5}{9}$ $\frac{6}{9}$ $\frac{7}{9}$

$\frac{2}{11}$ $\frac{3}{11}$ $\frac{4}{11}$ $\frac{5}{11}$ $\frac{6}{11}$ $\frac{7}{11}$ $\frac{8}{11}$

$\frac{3}{13}$ $\frac{4}{13}$ $\frac{5}{13}$ $\frac{6}{13}$ $\frac{7}{13}$ $\frac{8}{13}$ $\frac{9}{13}$ $\frac{10}{13}$

$\frac{3}{15}$ $\frac{4}{15}$ $\frac{5}{15}$ $\frac{6}{15}$ $\frac{7}{15}$ $\frac{8}{15}$ $\frac{9}{15}$ $\frac{10}{15}$

$\frac{4}{17}$ $\frac{5}{17}$ $\frac{6}{17}$ $\frac{7}{17}$ $\frac{8}{17}$ $\frac{9}{17}$ $\frac{10}{17}$

$\frac{9}{19}$

Fig. 2. List of all fractions $\nu = p/q$, with $\frac{2}{11} < \nu < \frac{4}{5}$, $1 < p \leqslant 10$ and $3 \leqslant q \leqslant 17$, q odd. The fractions corresponding to experimental values of the Hall conductivity $\sigma_{\mathrm{H}} = (e^2/h)\nu$ are in **bold**; the unobserved fractions are in *italic*. Observed fractions joined by lines are explained by the Jain hierarchy (2.43), (2.44).

cally possible. This implies that the gap is not a smooth function of simple parameters like (ν, p, q) – deeper theories are needed to explain stability.

Actually, a major virtue of the Jain hierarchy is that of representing one-parameter families of Hall states, within which the gap *is* a smooth function of the above parameters. We think of these families as the set of *kinematically allowed* quantum incompressible fluids (at first level of the hierarchy).

More specific confirmations of the edge theory (2.40) come from the experimental tests of the spectrum of excitations (2.41). An experiment with high time resolution [12] has measured the propagation of a single chiral charge excitation for $\nu = \frac{1}{3}$ ($m = 1$, $p = 2$), and $\nu = \frac{2}{3}$ ($m = 2$, $p = 2$); this is in agreement with the Jain theory, although the neutral excitations have not been seen yet. The resonant tunnelling experiment [14] has verified the conformal dimensions (2.41) for the simplest Laughlin fluid $\nu = \frac{1}{3}$ [15]. Extensions of this experiment to ($m > 1$) fluids have been suggested, as well as tests of the neutral edge spectrum [26]. We shall discuss more these experiments in Section 4.

Let us finally remark that experiments support the $W_{1+\infty}$ symmetry of the edge excitations, because there is no evidence for more general conformal field theories without this symmetry, like orbifolds and coset models [20]. Actually, these could easily be obtained by adding degrees of freedom to the edge excitations. Such generalizations

are not relevant to the fractional Hall effect for spin-polarized, single-layer electrons.

3. $W_{1+\infty}$ minimal models

In this section, we review the work of Ref. [36] on degenerate $W_{1+\infty}$ representations and use these to build the $W_{1+\infty}$ minimal models, which are then shown to correspond to the Jain fluids [5].

3.1. Degenerate representations

The complete $W_{1+\infty}$ algebra can be given in compact form by using a parametric sum of the V_n^i current modes, denoted by $V(-z^n \exp(\lambda D))$, where $D \equiv z\partial/\partial z$ [35]. This satisfies the algebra

$$[V\left(-z^r e^{\lambda D}\right), V\left(-z^s e^{\mu D}\right)] = \left(e^{\mu r} - e^{\lambda s}\right) V\left(-z^{r+s} e^{(\lambda+\mu)D}\right)$$
$$+ \delta_{r+s,0} C \frac{e^{-\lambda r} - e^{-\mu s}}{1 - e^{\lambda+\mu}}. \tag{3.1}$$

The currents V_n^i are identified by an expansion of this parametric operator in λ,

$$V_n^i \equiv V\left(-z^n f_n^i(D)\right), \tag{3.2}$$

where $f_n^i(D)$ are specific ith order polynomials which diagonalize the central term of (3.1) in the i, j indices [35,33]. For example,

$$V_n^0 \equiv V(-z^n), \qquad V_n^1 \equiv V\left(-z^n \left(D + \frac{n+1}{2}\right)\right). \tag{3.3}$$

The unitary irreducible *quasi-finite* highest-weight representations [36], denoted by $M(W_{1+\infty}, c, r)$, exist for $c = m$ and are characterized by the highest-weight state $|r\rangle_W$, which satisfies

$$V\left(-z^n e^{\lambda D}\right)|r\rangle_W = 0, \qquad n > 0, \tag{3.4}$$

and

$$V\left(-e^{\lambda D}\right)|r\rangle_W = \Delta(\lambda)|r\rangle_W \equiv \sum_{i=1}^m \frac{e^{\lambda r_i} - 1}{e^\lambda - 1}|r\rangle_W, \tag{3.5}$$

where $r = \{r_1, \ldots, r_m\} \in \mathbb{R}$. In particular, the charge V_0^0 and Virasoro V_0^1 eigenvalues given before in (2.8) are recovered by expanding $\Delta(\lambda)$ and comparing to (3.3).

The infinite tower of states in each representation is generated by expansion in the $\{\lambda_i\}$ of

$$V\left(-z^{-n_1} e^{\lambda_1 D}\right) \ldots V\left(-z^{-n_k} e^{\lambda_k D}\right)|r\rangle_W, \qquad n_1 \geqslant n_2 \ldots \geqslant n_k > 0, \tag{3.6}$$

where $n = \sum_{i=1}^k n_i$ is the *level* of the states. The quasi-finite representations have only a finite number of independent states at each level, thus there are an infinity of polynomial

relations among the generators V_n^k, whose explicit form depends on the values of c and r. The number of independent states $d(n)$ at level n is encoded in the (specialized) character of the representation ($|q| < 1$) [20],

$$\chi_{M(W_{1+\infty},m,r)}(q) \equiv \mathrm{tr}_{M(W_{1+\infty},m,r)}\left(q^{V_0^1-m/24}\right) = q^{\sum_{i=1}^m (r_i^2/2-1/24)} \sum_{n=0}^{\infty} d(n)q^n . \quad (3.7)$$

We call the representation *generic* if the weight r has components $(r_i - r_j) \notin \mathbb{Z}, \forall i \neq j$, and *degenerate* if it has $(r_i - r_j) \in \mathbb{Z}$ for some $i \neq j$. The weight components $\{r_i\}$ of the degenerate representations can be grouped and ordered in *congruence classes* modulo $\mathbb{Z}$ [36],

$$\{r_i, \ldots, r_m\} = \left\{s_1 + n_1^{(1)}, \ldots, s_1 + n_{m_1}^{(1)}\right\} \cup \ldots \cup \left\{s_k + n_1^{(k)}, \ldots, s_k + n_{m_k}^{(k)}\right\},$$

$$n_j^{(i)} \in \mathbb{Z}, \quad n_1^{(i)} \geqslant n_2^{(i)} \geqslant \ldots \geqslant n_{m_i}^{(i)}, \quad m = \sum_{i=1}^k m_i, \quad s_i \in \mathbb{R}. \quad (3.8)$$

A representation with two classes is the tensor product of two one-class representations. Therefore, the one-class degenerate representations are the basic building blocks, which we shall use for the $W_{1+\infty}$ minimal models.

The character for the generic representations is

$$\chi_{M(W_{1+\infty},m,r)}(q) = \prod_{i=1}^m \frac{q^{r_i^2/2}}{\eta(q)} = \prod_{i=1}^m \chi_{M(\widehat{U}(1),1,r_i)}(q), \quad (3.9)$$

where $\eta(q)$ is the Dedekind function [20],

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) . \quad (3.10)$$

Eq. (3.9) also shows the expression of the $W_{1+\infty}$ character in terms of m characters of the $\widehat{U}(1)$ algebra, which confirms the one-to-one equivalence of generic $W_{1+\infty}$ and $\widehat{U}(1)^{\otimes m}$ representations in Eq. (2.39).

The character for the one-class degenerate representations is [36]

$$\chi_{M(W_{1+\infty},m,r)}(q) = \eta(q)^{-m} q^{\sum_{i=1}^m r_i^2/2} \prod_{1 \leqslant i < j \leqslant m} \left(1 - q^{n_i - n_j + j - i}\right),$$

$$r = \{r_1, \ldots, r_m\} = \{s + n_1, \ldots, s + n_m\}, \quad n_1 \geqslant \ldots \geqslant n_m . \quad (3.11)$$

Note that the number of independent states $d(n)$ at level n is lower for degenerate representations (3.11) than for generic ones (3.7), because the former have additional relations among the states, leading to *null vectors* [20]. This is the origin of reducibility of the $\widehat{U}(1)^{\otimes m}$ representations with respect to the $W_{1+\infty}$ algebra. On the other hand, the one-class degenerate $W_{1+\infty}$ representations are one-to-one equivalent to those of the $\widehat{U}(1) \otimes \mathcal{W}_m(p = \infty)$ minimal models, where $\mathcal{W}_m$ is the Fateev–Lykyanov–

Zamolodchikov algebra [37]. The $\mathcal{W}_m(p)$ minimal models exist for the values of the central charge

$$c = (m - 1) \left(1 - \frac{m(m+1)}{p(p+1)} \right), \qquad p > m \geqslant 2, \tag{3.12}$$

therefore we are interested in their limit $p = \infty$.

3.2. The c = 2 case

The nature of the $W_{1+\infty}$ degenerate representations and their equivalence to the $\widehat{U}(1) \otimes \mathcal{W}_m$ ones [5] can be understood in simple terms for $m = 2$, where the $\mathcal{W}_2$ algebra is the $c = 1$ Virasoro algebra. By explicit construction, we shall derive the relations

$$M\left(W_{1+\infty}, 2, \{r+n, r\}\right) \sim M\left(\widehat{U}(1) \otimes \text{Vir}, 2, \{(2r+n)/\sqrt{2}\}, \{n^2/4\}\right)$$
$$\subset M\left(\widehat{U}(1)^{\otimes 2}, 2, \{(2r+n)/\sqrt{2}, n/\sqrt{2}\}\right), \tag{3.13}$$

where we characterized the Virasoro representations by the L_0 eigenvalue.

For $c = 1$, the $W_{1+\infty}$ algebra is the enveloping algebra of the $\widehat{U}(1)$ algebra, i.e. all V_n^i can be written as polynomials of one current mode α_n (2.20) [18,33]. Thus, the states of a $W_{1+\infty}$ representation (3.6) can be built by applying any number of α_n, with $n < 0$. Their degeneracy $d(n)$ is thus equal to the number of partitions of n, whose generating function is the U(1) character in (3.9). Given the linearity in c of the $W_{1+\infty}$ algebra, a pair of current modes α_n^1 and α_n^2 similarly build $W_{1+\infty}$ representations for $c = 2$. The V_n^0 and V_n^1 generators can be written,

$$V_n^0 = \alpha_n^1 + \alpha_n^2, \qquad V_n^1 = \frac{1}{2} \sum_{l=-\infty}^{\infty} : \alpha_{n-l}^1 \alpha_l^1 : + \frac{1}{2} \sum_{l=-\infty}^{\infty} : \alpha_{n-l}^2 \alpha_l^2 : . \tag{3.14}$$

The degeneracy of states in these $W_{1+\infty}$ representations is given by the product of two $\widehat{U}(1)$ characters. Therefore, this construction is useful to show the equivalence of generic $W_{1+\infty}$ representations with $\widehat{U}(1)^{\otimes 2}$ ones, by Eq. (3.9).

The degenerate $W_{1+\infty}$ representations are obtained by another construction,

$$V_n^0 = \sqrt{2}\alpha_n, \qquad V_n^1 = \frac{1}{2} \sum_{l=-\infty}^{\infty} : \alpha_{n-l}\alpha_l : + L_n, \tag{3.15}$$

where $\alpha_n \in \widehat{U}(1)$ and $L_n \in \text{Vir}$. On the degenerate highest-weight state $|\{r+n, r\}\rangle_W$, $n \in \mathbb{Z}_+$, their eigenvalues are, respectively,

$$\{\alpha_0\} \to (2r+n)/\sqrt{2}, \qquad \left\{ \frac{1}{2}\alpha_0^2 + \sum_{l=1}^{\infty} \alpha_{-l}\alpha_l \right\} \to \frac{1}{4}(2r+n)^2,$$

$$\{L_0\} \to \frac{1}{4}n^2. \tag{3.16}$$

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[5] From now on, we simply denote by $\mathcal{W}_m$ the $p \to \infty$ limit of $\mathcal{W}_m(p)$.

The $\widehat{U}(1)$ mode can be identified with the diagonal $\widehat{U}(1) \subset \widehat{U}(1)^{\otimes 2}$, generated by $(J^1 + J^2)/\sqrt{2}$. The orthogonal current $(J^1 - J^2)/\sqrt{2}$ does not exist in $W_{1+\infty}$, but gives the Virasoro L_n in (3.15) by the Sugawara construction (2.25). The higher $W_{1+\infty}$ generators V_n^i for $i \geqslant c = 2$ are functions of the operators α_n and L_n, and satisfy the rest of the $W_{1+\infty}$ algebra without further conditions [36]. The states of these $W_{1+\infty}$ representations are built by polynomials of α_n and L_n, with $n < 0$; thus, their character is the product of a $\widehat{U}(1)$ character and a $c = 1$ Virasoro one. Actually, the Virasoro representations with dimension $h = n^2/4$ are degenerate, and correspond to those of the well-known Virasoro $c \leqslant 1$ minimal models [20]. They have a single null vector at level $(n + 1)$, and their character is,

$$\chi_{M(\mathrm{Vir},1,\{n^2/4\})} = \eta^{-1} q^{n^2/4} \left(1 - q^{n+1}\right) . \tag{3.17}$$

The product of this character and the $\widehat{U}(1)$ character in Eq. (3.9) matches the general formula of the $W_{1+\infty}$ character (3.11) for $m = 2$ and $r = \{r + n, r\}$. This confirms the one-to-one relation in (3.13), and shows that the $W_{1+\infty}$ null vectors at $c = 2$ are those of the Virasoro algebra at $c = 1$.

Moreover, the decomposition of $\widehat{U}(1)^{\otimes 2}$ representations into $W_{1+\infty}$ degenerate representations can be inferred from the previous characters: the Virasoro character (3.17) can be written as,

$$\chi_{M(\mathrm{Vir},1,\{n^2/4\})} = \chi_{M\left(\widehat{U}(1),1,\{n/\sqrt{2}\}\right)} - \chi_{M\left(\widehat{U}(1),1,\{(n+2)/\sqrt{2}\}\right)} , \tag{3.18}$$

and solved for the abelian character. This gives the decomposition

$$M\left(\widehat{U}(1)^{\otimes 2}, 2, \{r + n, r\}\right) = \sum_{\ell=0}^{\infty} M\left(W_{1+\infty}, 2, \{r + n + \ell, r - \ell\}\right) . \tag{3.19}$$

The fusion rules of the degenerate $W_{1+\infty}$ representations are different from the abelian addition of weights (2.34), due to the non-trivial structure of excited states. For $c = 2$, we can deduce them from the fusion rules of the Virasoro minimal models for $c \to 1$ [20]. By denoting $M(\mathrm{Vir}, 1, \{n^2/4\}) \equiv [n]$, the latter are

$$[n] \bullet [k] = [n + k] + [n + k - 2] + \ldots + [|n - k|] , \tag{3.20}$$

and actually correspond to the addition of two SU(2) isospin quantum numbers, identified as $s_1 = n/2$ and $s_2 = k/2$, respectively. From the previous construction of $W_{1+\infty} \sim \widehat{U}(1) \otimes \mathrm{Vir}$, we deduce the fusion rules

$$M\left(W_{1+\infty}, 2, r\right) \bullet M\left(W_{1+\infty}, 2, s\right) = \sum_{t \in \Lambda_{r,s}} M\left(W_{1+\infty}, 2, t\right) ,$$

$$\Lambda_{r,s} = \left\{ t \mid t = r + s - \ell\alpha , 0 \leqslant \ell \leqslant \frac{n + m - |n - m|}{2} \right\} ,$$

$$r = \begin{pmatrix} r + n \\ r \end{pmatrix} , \qquad s = \begin{pmatrix} s + m \\ s \end{pmatrix} , \qquad \alpha = \begin{pmatrix} +1 \\ -1 \end{pmatrix} , \qquad n, m, \ell \in \mathbb{Z}_+ . \tag{3.21}$$

A general fact in conformal field theory is that the fusion of degenerate representations only gives degenerate representations of the same type; thus it is possible to find sets of degenerate representations which are closed under the fusion rules [20].

3.3. Construction of the minimal models

The $W_{1+\infty}$ minimal models are constructed by assembling the minimal set of one-class degenerate representations (3.11) which is closed under fusion. This algebraic method of construction, widely applied in conformal field theory, was already introduced for the abelian model of Section 2. In that case, the lattice Γ (2.9) was shown to be the minimal set of $\widehat{U}(1)^{\otimes m}$ weights which is closed under the addition as fusion rule (2.34). The fusion rule of $c = 2$ degenerate $W_{1+\infty}$ weights (3.21) is again the addition, but modulo the special weight $\boldsymbol{\alpha} = \{1, -1\}$ (with vanishing $\widehat{U}(1)$ charge). As a consequence, these rules close again on a lattice P^+ which satisfies some conditions: (i) the points in P^+, in particular the basis vectors, should all be degenerate weights; (ii) the weight $\boldsymbol{\alpha}$ should be in P^+, such that the result of the fusion (3.21) is a sum of weights in P^+, which belongs to P^+; (iii) the ordering of weight components in (3.11), $r_1 \geqslant r_2$, being preserved by fusion, should be imposed to avoid double counting. The following basis:

$$\boldsymbol{v}_1 = \begin{pmatrix} s+k \\ s \end{pmatrix}, \qquad \boldsymbol{v}_2 = \begin{pmatrix} +1 \\ -1 \end{pmatrix}, \qquad s \in \mathbb{R}, \quad k \in \mathbb{Z}\,\text{odd}, \tag{3.22}$$

conveniently expresses these properties of P^+,

$$P^+ = \{\boldsymbol{r} \,|\, \boldsymbol{r} = n_1\boldsymbol{v}_1 + n_2\boldsymbol{v}_2, n_1, n_2 \in \mathbb{Z}, kn_1 + n_2 \geqslant 0\} \quad (c = 2). \tag{3.23}$$

Note that k is taken odd in Eq. (3.22) in order to span both integer and half-integer isospin representations.

Next, we extend these results to $m > 2$ by using some results of the $\mathcal{W}_m$ minimal models at $c = (m - 1)$ [37] :

(i) The $\mathcal{W}_m$ degenerate representations are in one-to-one relation with the representations of the SU(m) Lie algebra, which are characterized by a $(m-1)$-dimensional highest-weight vector Λ ,

$$\Lambda = \sum_{a=1}^{m-1} \Lambda^{(a)} \ell_a, \qquad \ell_a \in \mathbb{Z}, \tag{3.24}$$

where $\Lambda^{(a)}$ are the fundamental weights of SU(m) (see, for example, Ref. [42]). In particular, the $\mathcal{W}_m$ fusion rules are isomorphic to the decomposition of SU(m) tensor representations.

(ii) The eigenvalue h of the Virasoro generator L_0 for these representations is given by

$$2h = \left\| \sum_{a=1}^{m-1} \Lambda^{(a)} \ell_a \right\|^2 , \tag{3.25}$$

i.e. by the norm of the highest weight. Due to the equivalence $W_{1+\infty} \sim \widehat{U}(1) \otimes W_m$, the m-dimensional $W_{1+\infty}$ weight r can be written in terms of the W_m weight. To this extent, we introduce the new basis $q = \{q, \Lambda\}$ for the $W_{1+\infty}$ weights by the orthogonal transformation $q_i = \sum_{j=1}^{m} U_{ij} r_j$, defined by

$$q = \frac{1}{\sqrt{m}} (r_1 + r_2 + \ldots + r_m) ,$$

$$\Lambda_a = \sum_{i=1}^{m} u_a^{(i)} r_i , \qquad a = 1, \ldots, m - 1 , \tag{3.26}$$

where the m, $(m-1)$-dimensional vectors $\mathbf{u}^{(i)}$ are

$$\mathbf{u}^{(1)} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \ldots, \frac{1}{\sqrt{k(k+1)}}, \ldots, \frac{1}{\sqrt{m(m-1)}} \right) ,$$

$$\mathbf{u}^{(2)} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \ldots \ldots, \ldots \ldots \ldots, \frac{1}{\sqrt{m(m-1)}} \right) ,$$

$$\mathbf{u}^{(k)} = \left(\overbrace{0, \ldots \ldots \ldots, 0}^{k-2}, -\frac{k-1}{\sqrt{k(k-1)}}, \ldots, \frac{1}{\sqrt{m(m-1)}} \right) ,$$

$$\mathbf{u}^{(m)} = \left(0, \ldots \ldots \ldots, \ldots \ldots \ldots, 0, -\frac{m-1}{\sqrt{m(m-1)}} \right) . \tag{3.27}$$

These vectors satisfy the rules

$$\sum_{i=1}^{m} \mathbf{u}^{(i)} = 0 , \qquad \sum_{i=1}^{m} u_a^{(i)} u_b^{(i)} = \delta_{ab} , \qquad \mathbf{u}^{(i)} \cdot \mathbf{u}^{(j)} = \delta_{ij} - \frac{1}{m} , \tag{3.28}$$

and can be used to build the lattice of roots and weights of $SU(m)$ [42]. The simple roots $\alpha^{(a)}$ and the fundamental weights $\Lambda^{(a)}$ are

$$\alpha^{(a)} = \mathbf{u}^{(a)} - \mathbf{u}^{(a+1)} ,$$

$$\Lambda^{(a)} = \sum_{i=1}^{a} \mathbf{u}^{(i)} , \qquad a = 1, \ldots, m - 1 ,$$

$$\Lambda^{(a)} \cdot \alpha^{(b)} = \delta_{ab} , \qquad \Lambda^{(a)} \cdot \Lambda^{(b)} = a - \frac{ab}{m} , \qquad a \leqslant b . \tag{3.29}$$

After these definitions, it easy to see that the Eqs. (3.24) and (3.26) are made equal by identifying the integer components of the $W_{1+\infty}$ degenerate weight $(r_a - r_{a+1})$ with those of the $SU(m)$ weight ℓ_a:

$$\ell_a = r_a - r_{a+1} , \qquad a = 1, \ldots, m - 1 . \tag{3.30}$$

It is useful to check the $m = 2$ case in this $SU(m)$ notation:

$$u^{(1)} = \frac{1}{\sqrt{2}}, \qquad u^{(2)} = -\frac{1}{\sqrt{2}}, \qquad \alpha^{(1)} = \sqrt{2}, \qquad \Lambda^{(1)} = \frac{1}{\sqrt{2}},$$

$$q = \frac{1}{\sqrt{2}} (r_1 + r_2),$$

$$\Lambda = \frac{1}{\sqrt{2}} (r_1 - r_2) = \ell_1 \Lambda^{(1)}, \qquad \ell_1 \in \mathbb{Z}_+. \tag{3.31}$$

In the $\{q, \Lambda\}$ basis, the $m = 2$ fusion rules (3.21) are

$$q \cdot p = p + q \quad \mathrm{mod} \ \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha^{(1)} \end{pmatrix} \quad (m = 2). \tag{3.32}$$

Based on these data, we can identify the components of the $W_{1+\infty}$ weight in the basis $q = \{q, \Lambda\}$ (3.26) as the $\hat{U}(1)$ charge and the $SU(m)$ weight, respectively, which label the tensor representations $\hat{U}(1) \otimes \mathcal{W}_m$. We can thus deduce the $W_{1+\infty}$ fusion rules from the $SU(m)$ fusion rules, and find their general structure:

$$q \cdot p = p + q \quad \mathrm{mod} \ \left\{ \begin{pmatrix} 0 \\ \alpha^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \alpha^{(m-1)} \end{pmatrix} \right\}, \tag{3.33}$$

namely the $SU(m)$ weights add up modulo an integer combination of the simple roots [42].

Similarly to the $(m = 2)$ case, we can find lattices P^+ which are closed under these fusion rules because they contain the $(m - 1)$ simple roots. In the original r basis (3.11), these lattices can be generated as follows (see Eqs. (3.29), (3.11)):

$$r = \sum_{i=1}^{m} n_i v_i = n_1 \begin{pmatrix} s + k_1 \\ \vdots \\ \vdots \\ \vdots \\ s + k_{m-1} \\ s \end{pmatrix} + n_2 \begin{pmatrix} +1 \\ -1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + n_3 \begin{pmatrix} 0 \\ +1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + n_m \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ +1 \\ -1 \end{pmatrix}, \tag{3.34}$$

with $n_i, k_j \in \mathbb{Z}$. The condition

$$\xi \equiv -\frac{(-1)^m + k_1 + \dots + k_{m-1}}{m} \in \mathbb{Z}, \tag{3.35}$$

ensures that $SU(m)$ representations of minimal n-ality, e.g. the fundamental one, are contained in P^+. The ordering of components $r_a \geqslant r_{a+1}$ will be discussed later on.

In summary, the $W_{1+\infty}$ minimal models are made of one-class degenerate representations (3.11) with weights belonging to the lattices (3.34), which are parametrized by $s \in \mathbb{R}$ and $\{k_j\} \in \mathbb{Z}$, $j = 1, \dots, m - 1$, with the constraints (3.35) and $r_a \geqslant r_{a+1}$, $a = 1, \dots, m - 1$.

Further conditions on the parameters of these lattices, i.e. on the $W_{1+\infty}$ minimal models, come from a physical requirement already discussed in the $\widehat{U}(1)^{\otimes m}$ theory. We must identify the electron excitation in the lattice (3.34) and define the corresponding unit of charge. It is convenient to introduce another simpler basis u_i for the lattice (3.35), by the following modular transformation (2.13) $v_i = \sum_{j=1}^{m} \varXi_{ij} u_j$:

$$\varXi = \begin{pmatrix} \xi + k_1 & \xi + k_2 & \ldots & \ldots & \xi + k_{m-1} & \xi \\ 1 & -1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & -1 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & 1 & -1 \end{pmatrix} \in \mathrm{SL}(m, \mathbb{Z}) \,. \tag{3.36}$$

This leads to the vectors

$$(u_i)_j = \delta_{ij} + rC_{ij}\,, \qquad r = (-1)^m (\xi - s) \in \mathbb{R}\,. \tag{3.37}$$

Given that the $W_{1+\infty}$ and the $\widehat{U}(1)^{\otimes m}$ representations are labelled by the same type of highest-weight vector r, the spectrum of the corresponding quantum numbers $(\nu, Q, \theta/\pi)$ of the $W_{1+\infty}$ minimal models can be again described by the K-matrix (2.35), (2.37), which is the metric of the lattice P^+ in the basis (3.37):

$$K_{ij}^{-1} = u_i \cdot u_j = \delta_{ij} + \lambda C_{ij}\,, \qquad \lambda = mr^2 + 2r \in \mathbb{R}\,, \tag{3.38}$$

These quantum numbers can be written (see Eq. (2.35)),

$$\begin{aligned} \nu &= t^{\mathrm{T}} \cdot K^{-1} \cdot t\,, \\ Q &= t^{\mathrm{T}} \cdot K^{-1} \cdot n\,, \qquad n_i \in \mathbb{Z}\,, \qquad n_1 \geqslant n_2 \geqslant \ldots \geqslant n_m\,, \\ \frac{\theta}{\pi} &= n^{\mathrm{T}} \cdot K^{-1} \cdot n\,. \end{aligned} \tag{3.39}$$

In this equation, the arbitrary vector $t^{\mathrm{T}} = (1, \ldots, 1) \cdot \Theta$, $\Theta \in \mathrm{Sl}(m, \mathbb{Z})$, parametrizes any possible choice of basis for the lattice P^+ (3.34). This vector defines the physical charge Q as a linear functional of the $W_{1+\infty}$ weights, and is determined as follows. The $W_{1+\infty}$ degenerate representations, being equivalent to the $\widehat{U}(1) \otimes \mathcal{W}_m$ ones, are characterized by a unique additive quantum number q in (3.26), which can be interpreted as the electric charge. Actually, the $W_{1+\infty}$ minimal models do not possess the notion of m individual edges as in the abelian hierarchical theory (2.15), corresponding to its m currents. Therefore, the physical total charge Q in (3.39) should be taken proportional to q. By working through the previous coordinate changes, this condition can be translated into the condition $t \propto \{1, \ldots, 1\}$, whose unique solution is

$$t = \{1, \ldots, 1\}\,, \tag{3.40}$$

because all Θ have unit determinant. We have thus obtained K-matrices of the $W_{1+\infty}$ minimal models (3.38) which are almost equal to those of the Jain hierarchy (2.40); only the quantization of the parameter λ is to be found and the ordering $n_a \geqslant n_{a+1}$ to be explained.

To this end, we require that the spectrum (3.39), with $t = \{1,\ldots,1\}$, contains *one* excitation $n_i = e_i \in \mathbb{Z}$, $i = 1,\ldots,m$, with the quantum numbers and monodromy properties of the electron. The monodromy properties of a pair of excitations are dictated by the fusion rules of the corresponding $W_{1+\infty}$ degenerate representations [20]. In the monodromy process of the pair (n_i, m_i), more than one intermediate state k_i is obtained by fusion: each one acquires a phase proportional to the Virasoro eigenvalues, which is $\theta[n_i] + \theta[m_i] - \theta[k_i]$, where $\theta[n_i]$ is defined in (3.39). This is the *non-abelian* fractional statistics which will be better analyzed in the next section. It can be shown that the Virasoro eigenvalues of the intermediate states only differ by integers, thus the locality conditions for the electron excitations reduce to those for the abelian fusion rules ($k_i = n_i + m_i$). These conditions are:

(i) $Q\left[n_i = e_i\right] = 1$,

(ii) $\dfrac{\theta[e_i]}{\pi} \in \mathbb{Z}\,\text{odd}$,

(iii) $\dfrac{1}{2\pi}\left(\theta[e_i] + \theta[n_i] - \theta[e_i + n_i]\right) = e^{\mathrm{T}} \cdot K^{-1} \cdot n \in \mathbb{Z}\,\forall\,n_i$.   (3.41)

The condition (iii) requires that $\sum_{j=1}^{m} K_{ij}^{-1} e_j = \eta_i \in \mathbb{Z}\,\forall\,i$, and the condition (i) that $\sum_i \eta_i = 1$. Note that the K-matrices (2.40) are invariant under permutations of the basis components,

$$K_{ij} = K_{\sigma(i)\sigma(j)}\,, \qquad \sigma \in \mathcal{S}_m\,,\tag{3.42}$$

where $\mathcal{S}_m$ is the symmetric group. Due to this symmetry, we can choose the solution of (i) in the form $\eta_i = (1,0,0,\ldots,0)$, and obtain the electron excitation as

$$e_i = (1 + p, p,\ldots,p)\,,\tag{3.43}$$

where $p = -\lambda/(1 + m\lambda) \in \mathbb{Z}$. Moreover, the condition (ii) requires $e \cdot \eta = 1 + p$ odd, i.e. p even.

We have thus obtained the Jain K-matrices $K_{ij} = \delta_{ij} + pC_{ij}$, in the purely chiral case ($p > 0$ even). These are positive definite by construction (Eq. (3.38)). Mixed chiral and antichiral propagation of the edge excitations can be introduced by extending the arguments used in the discussion of the chiral-boson hierarchy of Section 2 (see Eqs. (2.21)–(2.29). The data of the $W_{1+\infty}$ representations, summarized in the lattice Γ (3.34), are independent of the choice of chirality – only the identification of the physical quantities changes. For chiral (antichiral) excitations, we must identify the fractional statistics spectrum θ/π with plus (minus) the Virasoro eigenvalue, while the charge eigenvalue does not change. In the $W_{1+\infty}$ minimal models, made by $\widehat{U}(1) \times \mathcal{W}_m$ representations, two types of excitations can be identified: the charged excitations associated to the $\widehat{U}(1)$ factor and the neutral, non-standard, excitations described by the $\mathcal{W}_m$ factor. Therefore, there are only two possible choices, corresponding to whether the chiralities of charged and neutral excitations are equal or opposite. The $(m - 1)$ neutral excitations described by $\mathcal{W}_m$ are *interacting* and cannot be assigned mixed chiralities.

In order to change the sign of the corresponding Virasoro eigenvalues, we diagonalize the quadratic form of θ/π in (3.39) by the orthogonal transformation $K = U^T D U$, where U was defined in (3.26). We can thus summarize the relations among the integer labels $\{n\}$ in Eq. (3.39), the $W_{1+\infty}$ weights $\{r\}$ in Eq. (3.11) and the $\hat{U}(1) \otimes W_m$ weights $\{q = (q, \Lambda)\}$ in Eq. (3.26), as follows:

$$r = U^T D^{-1/2} U \cdot n \,,$$

$$q = D^{-1/2} U \cdot n = U \cdot r \,,$$

$$K = U^T D U \,, \qquad D^{1/2} \equiv \mathrm{diag}\left(\sqrt{1+mp}, 1, \ldots, 1\right) \,, \qquad p > 0 \,. \tag{3.44}$$

In components,

$$q = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} r_i = \frac{1}{\sqrt{m}} \frac{1}{\sqrt{1+mp}} \sum_{i=1}^{m} n_i \,,$$

$$\Lambda = \sum_{i=1}^{m} \mathbf{u}^{(i)} r_i = \sum_{i=1}^{m} \mathbf{u}^{(i)} n_i \,,$$

i.e. $r_a - r_{a+1} = n_a - n_{a+1} \,, \qquad a = 1, \ldots, m-1 \,.$ \tag{3.45}

Therefore, in the chiral case, the diagonal form of the Virasoro eigenvalue is

$$\frac{\theta}{\pi} = 2h = n^T \cdot K^{-1} \cdot n = \sum_{i=1}^{m} r_i^2 = q^2 + \|\Lambda\|^2 \,. \tag{3.46}$$

We can change the relative chirality of the neutral and charged excitations by defining

$$\frac{\tilde{\theta}}{\pi} = q^2 - \|\Lambda\|^2 = n^T \cdot \tilde{K}^{-1} \cdot n \,. \tag{3.47}$$

The solution of the discretization electron conditions now gives

$$\tilde{D} = \mathrm{diag}\left((mp-1), -1, \ldots, -1\right) \,,$$

i.e. $\tilde{K} = pC_{ij} - \delta_{ij}$, with p even positive integer, which we recognize as the Jain matrices for mixed chiralities (2.40). Finally, the ordering of r components in (3.11) translates into the announced conditions $n_a \geqslant n_{a+1}$, which identify the minimal lattice P^+ (3.34).

4. Physical properties of $W_{1+\infty}$ minimal models

In the previous section, we have constructed the $W_{1+\infty}$ minimal models out of one-congruence-class degenerate representations (3.11) [36]. We have found their filling fractions and the spectrum of fractional charge and statistics of their edge excitations (3.39), (3.44),

$$\nu = \frac{m}{mp \pm 1} \,, \qquad p > 0 \text{ even} \,, \qquad c = m \,,$$

$$Q = \frac{1}{pm \pm 1} \sum_{i=1}^{m} n_i, \qquad n_1 \geqslant n_2 \geqslant \ldots \geqslant n_m,$$

$$\frac{\theta}{\pi} = \pm \left(\sum_{i=1}^{m} n_i^2 - \frac{p}{mp \pm 1} \left(\sum_{i=1}^{m} n_i \right)^2 \right). \tag{4.1}$$

These spectra agree with the experimental data and match the results of the lowest-order Jain hierarchy (2.41)[6] discussed in Section 2. Our derivation was independent and self-consistent within the theory of edge excitations; we only used the principle of $W_{1+\infty}$ symmetry of the incompressible fluids and the hypothesis of stability of the minimal theories, and we did not make any reference to microscopic physics and wave functions. This independent hierarchical construction is the main result of this paper.

Furthermore, the detailed predictions of the $W_{1+\infty}$ minimal theories are different from those of the chiral-boson theories. The neutral excitations are associated to $\mathcal{W}_m$ representations and carry an $SU(m)$ quantum number, leading to new physical effects. We shall discuss two of them:

(i) The non-abelian fusion rules and non-abelian fractional statistics;
(ii) The degeneracy of excitations at fixed angular momentum above the ground state (the Wen topological order on the disk geometry [39]).

4.1. Non-abelian fusion rules and non-abelian statistics

In the previous section, we have identified the physical electron as the minimal set of $W_{1+\infty}$ representations with unit charge and integer statistics relative to all excitations. These conditions are fulfilled by a composite edge excitation $n_i = (1 + p, p, \ldots, p)$, which is made of $(mp + 1)$ elementary charged *anyons* and the *quark* elementary neutral excitation, i.e. the fundamental $SU(m)$ isospin representation, due to $(n_i - n_{i+1}) = \delta_{i,1}$.

A conduction experiment which could show the composite nature of the electron has been proposed [26]. It is a modification of the "time-domain" experiment [12], in which a very fast electric pulse was injected at the boundary of a disk sample and a chiral wave was detected at another boundary point. The proposed experiment will also detect the neutral excitation in the electron, which propagates at a different speed.

The compositeness of the electron also plays a role in the resonant tunnelling experiment [14], in which two edges of the sample are pinched at one point, such that the corresponding edge excitations, having opposite chiralities, can interact. At $\nu = \frac{1}{3}$, the point interaction of two elementary anyons is relevant and determines the *scaling law* $T^{2/3}$ for the conductance [15]. This scaling of the tunnelling resonance peaks is verified experimentally. On the other hand, off-resonance and at low temperature, the conductance is given by the tunnelling of the whole electron, with a different scaling law in temperature [11].

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[6] Note, however, the reduced multiplicities of Eq. (4.1).

These experiments involve processes with one or two electrons: their quark compositeness can be seen in four-electron processes, like scattering. Indeed, the expansion of the four-point function of the electrons in intermediate channels is determined by the fusion ((3.20) for $m = 2$) of an electron pair. This is, schematically,

$$\langle \Omega | \Psi^\dagger(1) \Psi^\dagger(2) \Psi(3) \Psi(4) | \Omega \rangle = \sum_{s=0,1} \langle \Omega | \Psi^\dagger(1) \Psi^\dagger(2) | \{s\} \rangle$$
$$\times \langle \{s\} | \Psi(3) \Psi(4) | \Omega \rangle , \tag{4.2}$$

where the two channels follow from the addition of two one-half isospin values. More than one intermediate channel are also created in the adiabatic transport of two electrons around each other, in presence of two other excitations, because the amplitude for this process is again a four-point function. For generic excitations, the monodromy phases form a matrix, which gives a non-abelian representation of the braid group [24]. This is precisely the notion of non-abelian statistics[7]. These monodromy properties also determine the degeneracy of the ground state on a torus geometry, the so-called topological order [11]. This depends on the type of the representations carried by the edge excitations [20], and should be computed for the $\widehat{U}(1) \otimes \mathcal{W}_m$ ones. We hope to develop these issues in a separate work.

4.2. The degeneracy of excitations above the ground state

In order to discuss this point, we must rewrite the spectrum (4.1). Let us consider the $m = 2$ chiral theories, relevant for $\nu = \frac{2}{5}, \ldots$; the extension of the analysis to any m and mixed chiralities is straightforward. Recall that any excitation (n_1, n_2) is associated to a $\widehat{U}(1) \otimes$ Vir representation, labelled by the $\widehat{U}(1)$ charge $Q \propto (n_1 + n_2)$ and the SU(2) isospin $s = |n_1 - n_2|/2$ (3.20). Divide the square lattice (n_1, n_2) into charged excitations and their neutral daughter excitations by introducing the change of integer variables $(n_1, n_2) \rightarrow (l, n)$:

$$\text{I}: \begin{cases} 2l = n_1 + n_2 \\ 2n = n_1 - n_2 > 0 \\ \quad (n_1 + n_2 \text{ even}), \end{cases} \qquad \text{II}: \begin{cases} 2l + 1 = n_1 + n_2 \\ 2n + 1 = n_1 - n_2 > 0 \\ \quad (n_1 + n_2 \text{ odd}). \end{cases} \tag{4.3}$$

The spectrum (4.1) can be rewritten, for $\nu = 2/(2p + 1)$,

$$\text{I}: \begin{cases} Q = \dfrac{2l}{2p+1} , \\ \dfrac{1}{2}\dfrac{\theta}{\pi} = \dfrac{1}{2p+1} l^2 + n^2 \end{cases} \qquad \text{II}: \begin{cases} Q = \dfrac{2}{2p+1} \left(l + \tfrac{1}{2}\right) \\ \dfrac{1}{2}\dfrac{\theta}{\pi} = \dfrac{1}{2p+1} \left(l + \tfrac{1}{2}\right)^2 + \tfrac{1}{4}(2n + 1)^2 . \end{cases}$$
$$\tag{4.4}$$

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[7] For a general discussion of non-abelian statistics in the quantum Hall effect, see Ref. [38].

The $\widehat{U}(1)$ charged excitations have the same spectrum $Q = \nu k$, $\theta/\pi = \nu k^2$, of the simpler Laughlin fluids ($m = 1$). Moreover, the infinite tower of neutral daughters ($n > 0$) are characterized by the conformal dimensions $h = (2n)^2/4$ (resp. $h = (2n + 1)^2/4$).

The number of excitations above the ground state depends on whether the neutral excitations have a bulk gap or not. This affects also the thermodynamic quantities like the specific heat.

As said before, the charged edge excitations correspond to Laughlin quasi-particles vortices in the bulk of the incompressible fluid, which spill their density excess or defect to the boundary. They have a (non-universal) gap proportional to the electrostatic energy of the vortex core, which is not accounted for by the edge theory [2]. On the other hand, the bulk excitations corresponding to *neutral* edge excitations are not well understood yet. If they have a gap, they could exhibit the internal structure of the quasi-particle vortex, or be bound states of a quasi-particle and a quasi-hole; these would be localized two-dimensional excitations. Neutral and charged gapful excitations can be thought of as analogs of the *breathers* and *solitons* of one-dimensional integrable models, respectively, see, for example, Ref. [43]. On the other hand, gapless neutral excitations would be pure effects of the structured edge.

In the gapful case, the excitations above the ground state are particle–hole excitations described by the $W_{1+\infty}$ representation of the ground state ($n = l = 0$) in (4.4). In the gapless case, there are also contributions from the neutral daughter Virasoro representations ($n > 0$, $l = 0$), because they have integer spin (Virasoro dimension) and are indistinguishable. Actually, the infinite tower of Virasoro representations ($n > 0$, l fixed) of each charged parent state ($l, n = 0$) can be summed (with their multiplicity one) into a single $\widehat{U}(1)$ representation by using backward the decomposition (3.19):

$$\mathrm{I}: \quad \sum_{n=0}^{\infty} M\left(\mathrm{Vir}, 1, \{n^2\}\right) = M\left(\widehat{U}(1), 1, \{0\}\right),$$

$$\mathrm{II}: \quad \sum_{n=0}^{\infty} M\left(\mathrm{Vir}, 1, \{(2n + 1)^2/4\}\right) = M\left(\widehat{U}(1), 1, \{1/\sqrt{2}\}\right). \qquad (4.5)$$

In this case, the $m = 2$ $W_{1+\infty}$ square-lattice spectrum (4.4) reduces to a one-dimensional array of $\widehat{U}(1)^{\otimes 2}$ representations, with spectrum

$$\mathrm{I}: \quad \frac{1}{2}\frac{\theta}{\pi} = \frac{1}{2p + 1}l^2,$$

$$\mathrm{II}: \quad \frac{1}{2}\frac{\theta}{\pi} = \frac{1}{2p + 1}\left(l + \tfrac{1}{2}\right)^2 + \tfrac{1}{4}, \qquad (4.6)$$

where the second $\widehat{U}(1)$ eigenvalue is not observable.

Let us repeat this analysis for the corresponding chiral-boson theory of the Jain fluid. The spectrum of charge and fractional statistics is again given by (4.4), with multiplicities given by $n \in \mathbb{Z}$: each (l, n) value corresponds to a $\widehat{U}(1) \otimes \widehat{U}(1)$ representation now. If they are gapless, the neutral daughter $\widehat{U}(1) \otimes \widehat{U}(1)$ representations ((l, n), $n \neq 0 \in \mathbb{Z}$) of each charged representation ($l, 0$) can be similarly summed up into one

Table 1

ΔJ	0	1	2	3	4	5
$\widehat{U}(1) \otimes \text{Vir}$	1	1	3	5	10	16
$\widehat{U}(1) \otimes \widehat{U}(1)$	1	2	5	10	20	36
$\widehat{U}(1) \otimes \widehat{SU}(2)_1$	1	4	9	20	42	80

representation of the larger algebra $\widehat{U}(1) \otimes \widehat{SU}(2)_1$, the non-abelian current algebra of level one [20]. Using the following expansion of the $\widehat{SU}(2)_1$ characters of spin $s = 0$ and $s = \frac{1}{2}$ [20,44]:

$$\text{I}: \quad \chi_{M(\widehat{SU}(2)_1,1,s=0)} = \chi_{M(\widehat{U}(1),1,r=0)} + 2 \sum_{n=1}^{\infty} \chi_{M(\widehat{U}(1),1,r=\sqrt{2}n)},$$

$$\text{II}: \quad \chi_{M(\widehat{SU}(2)_1,1,s=1/2)} = 2 \sum_{n=0}^{\infty} \chi_{M(\widehat{U}(1),1,r=(2n+1)/\sqrt{2})}, \quad (4.7)$$

we sum all neutral states ($n \in \mathbb{Z}$) with the correct multiplicity. The spectrum of $\widehat{U}(1) \otimes \widehat{SU}(2)_1$ representations is again given by (4.6).

We can now compare the predictions of the $W_{1+\infty}$ minimal models and the chiral-boson theories for the degeneracy of the excitations above the ground state. This degeneracy can be measured in numerical simulations of a few electron system in the disk geometry, by charting the eigenstates of the hamiltonian below the bulk gap [11,39]. Consider, for example, the $\nu = \frac{2}{5}$ ($m = p = 2$) ground state (($l = 0, n = 0$) in (4.4)). In Table 1, we report the degeneracies encoded in the $\widehat{U}(1) \otimes \text{Vir}$ character (3.11) and the $\widehat{U}(1)^{\otimes 2}$ character (3.9), for $r = 0$, as well as those of the $\widehat{U}(1) \otimes \widehat{SU}(2)_1$ one (4.7) for $r = s = 0$.

If neutral daughter excitations have a gap, they should not be counted, and the degeneracy is only given by the particle–hole excitations encoded in the ground-state character of the theory. On the other hand, gapless neutral excitations contribute and the total degeneracy is given by the resummed characters (4.5), (4.7). We conclude that:

   (i) The observation of $\widehat{U}(1) \otimes \text{Vir}$ degeneracies confirms the $W_{1+\infty}$ minimal theory with gapful neutral excitations;

  (ii) The $\widehat{U}(1) \otimes \widehat{U}(1)$ degeneracies are found both in the $W_{1+\infty}$ minimal theory with gapless neutral excitations and in the chiral-boson theory with gapful ones;

 (iii) The $\widehat{U}(1) \otimes \widehat{SU}(2)_1$ degeneracies support the chiral-boson theory with gapless neutral excitations.

Numerical results known to us [11] are not accurate enough to see the differences in Table 1. Note the characteristic reduction of states of $W_{1+\infty}$ minimal models.

These remarks on the gap for neutral excitations do not affect the previous discussion of the conduction experiments, where excitations move along one edge or are transferred between two edges at the same Fermi energy, such that bulk excitations are never produced. Although the resummation of the neutral daughter W_m excitations gives abelian

excitations, these are not $W_{1+\infty}$ irreducible, and thus unlikely to be produced experimentally. We think that only irreducible $W_{1+\infty}$ excitations, i.e., the elementary ones, can be naturally produced in a real system by an external probe, for example by injecting an electron at the edge.

4.3. Remarks on the $SU(m)$ and $\widehat{SU}(m)_1$ symmetries

We would like to explain the type of non-abelian symmetry of the $W_{1+\infty}$ minimal models and clarify the differences with the chiral-boson theories of the Jain hierarchy, which have been also assigned the $SU(m)$ and $\widehat{SU}(m)_1$ symmetries [25,23,27,26].

Due to the $\widehat{U}(1) \otimes \mathcal{W}_m$ construction of the $W_{1+\infty}$ minimal models, their excitations carry a quantum number which adds up as a $SU(m)$ isospin. This *does not* imply that these models have the full $SU(m)$ symmetry, in the usual sense of, say, the quark model of strong interactions, because the states in each $\mathcal{W}_m$ representation do not form $SU(m)$ multiplets. As shown by the $m = 2$ case, the quantum number $s = n/2$ of Virasoro representations is like the total isospin $S^2 = s(s + 1)$, but the S_z component is missing. In some sense, the effects of the $\mathcal{W}_m$ non-abelian fusion rules can be thought of as a *hidden* $SU(m)$ symmetry.

On the other hand, it has been claimed that the chiral-boson theories of the Jain hierarchy have a $SU(m)$ symmetry. The correct statement is, however, that they possess $\widehat{U}(1) \otimes \widehat{SU}(m)_1$ symmetry. This means that their $\widehat{U}(1)^{\otimes m}$ representations can be rearranged into representations of the $\widehat{U}(1) \otimes \widehat{SU}(m)_1$ current algebra, as shown before in (4.7). In the $\widehat{SU}(m)_k$ current algebra, the weights cannot be arbitrary, but are cut-off by the *level* k (e.g. for $m = 2$, the spin s can be $0 \leqslant s \leqslant k/2$) [20]. The level-one non-abelian current algebra has very elementary representations and their fusion rules are made abelian by this cut-off.

Therefore, the $\widehat{SU}(m)_1$ symmetry has no non-abelian physical effect, it is only a convenient reorganization of the abelian current algebra. The non-abelian character of the excitations is a characteristic feature of the $W_{1+\infty}$ minimal models.

5. Concluding remarks

In this paper, we constructed the *simplest* $W_{1+\infty}$ minimal models, which are made of one-congruence-class degenerate representations, with weight $r = \{s + n_1, \ldots, s + n_m\}$, $n_1 \geqslant \ldots \geqslant n_m \in \mathbb{Z}$ (3.11). It would be interesting to generalize this construction, in view of describing the experimentally observed filling fractions $\frac{4}{5}$, $\frac{5}{7}$ $\frac{4}{11}$, $\frac{7}{11}$, $\frac{8}{11}$, $\frac{4}{13}$, $\frac{8}{13}$, $\frac{9}{13}$, $\frac{10}{17}$, ..., not explained here. The $W_{1+\infty}$ minimal models can be generalized by considering two (or more) congruence classes, $r = \{s+n_1, \ldots, s+n_m; t+k_1, \ldots, t+k_l\}$, with $k_1 \geqslant \ldots \geqslant k_l \in \mathbb{Z}$ and $c = m + l$. There are analogies between this mathematical construction and the Jain hierarchical construction of wave functions, which read

$$\Psi_\nu = D^{q/2} L^l D^{p/2} L^m \mathbf{1}, \qquad p, q \text{ even}, \tag{5.1}$$

to second order of iteration [5]. The number of fluids in any $W_{1+\infty}$ congruence class corresponds to the number of Landau levels in (5.1); in both constructions, there are two independent elementary anyons, each one accompanied by neutral excitations. However, we have not yet proven a complete equivalence of the two second-order hierarchies: the Jain construction assigns a definite filling fraction to each wave function (5.1), while we have a large modular degeneracy (of the group $SL(2, \mathbb{Z})$) in the definition of the physical charge of the two independent anyons, leading to many values of the filling fraction for each minimal model. On the contrary, we would like to find *more* constraints than in the Jain construction, because most of its second-order filling fractions are not observed experimentally. We guess that our algebraic construction of $W_{1+\infty}$ minimal model Hilbert spaces should be supplemented by the construction of other physical quantities, like the partition function [44], which could impose further conditions on the physical theories.

Acknowledgements

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