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$W_{1+\infty}$ minimal models and the hierarchy of the quantum Hall effect

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We review our recent work on the algebraic characterization of quantum Hall fluids. Specifically, we explain how the incompressible quantum fluid ground states can be classified by effective edge field theories with the $W_{1+\infty}$ dynamical symmetry of "quantum area-preserving diffeomorphisms". Using the representation theory of $W_{1+\infty}$, we show how all fluids with filling factors $\nu = m/(pm+1)$ and $\nu = m/(pm-1)$ with m and p positive integers, p even, correspond exactly to the $W_{1+\infty}$ *minimal models*.

1. INTRODUCTION

We would like to briefly report on a research program on the quantum Hall effect which we have been developing over the last three years. The main points of our program are:

- The high degree of *universality* and *precision* in the measures of the fractional Hall conductivity [1] strongly motivates an effective field theory approach to describe the low-energy, long-distance physics of the quantum Hall effect.

- The Laughlin theory of the *incompressible quantum fluid* of electrons at the Hall plateaus [2] is assumed here. Classical incompressible fluids are characterized by the *dynamical* symmetry of *area-preserving diffeomorphisms*, obeying the w_∞ algebra [3]. Correspondingly, the effective field theory for the quantum incompressible fluid possesses the quantum analogue of this symmetry, obeying the $W_{1+\infty}$ algebra.

- The low-energy excitations of a *droplet* of incompressible fluid reside at its boundary. Thus, the effective field theory is $(1+1)$-dimensional [4] - actually, it is a conformal field theory [5].

- The conformal field theories with $W_{1+\infty}$ symmetry can be exactly solved by *algebraic* methods and classified completely [6]. This gives an algebraic characterization of the quantum Hall fluids. The necessary mathematical tools were provided by the works [7].

- The special class of *minimal* $W_{1+\infty}$ field theories (minimal models) was recently constructed [8] and shown to give a new hierarchy of the fractional Hall fluids. These models reproduce the Jain hierarchy [9], and predict new, *non-Abelian*, properties for the low-energy edge excitations.

- The detailed experimental predictions of the $W_{1+\infty}$ minimal models are being developed. They must be compared with the experimental data and with the predictions of the standard, Abelian theory of edge excitations [10].

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2. THE $W_{1+\infty}$ DYNAMICAL SYMMETRY OF THE LAUGHLIN INCOMPRESSIBLE FLUID

In a seminal work, Laughlin proposed [2] that the ground-state density for the electrons at the plateaus of the quantum Hall effect is uniform *(fluid character)*, and has an energy gap *(incompressibility)*. The latter property accounts for the lack of low-energy modes for the longitudinal conduction ($\sigma_{xx}=0$), while the Hall conductivity is realised as a rigid motion of the uniform droplet. Most of the characteristics of the incompressible fluid can be understood at the (semi)-classical level [2][3]. A *classical* incompressible fluid is defined by its distribution function,

$$\rho(z,\bar{z},t)=\rho_0\ \chi_{S_A(t)}\ ,\qquad \rho_0\equiv\frac{N}{A}\ , \tag{1}$$

where $\chi_{S_A(t)}$ is the characteristic function for a surface $S_A(t)$ of area A, and $z=x+iy$, $\bar{z}=x-iy$ are complex coordinates on the plane. Since the particle number N and the average density ρ_0 are constant, the area A is also *constant*. The only possible change in response to external forces is in the shape of the surface. The shape changes at constant area can be generated by *area-preserving diffeomorphisms* of the two-dimensional plane. Thus, the configuration space of a *droplet* of incompressible fluid can be generated by applying these transformations to a reference droplet. This can be the ground state in a rotation-invariant potential,

$$\rho_{GS}(z,\bar{z})=\rho_0\ \Theta\left(R^2-z\bar{z}\right)\ . \tag{2}$$

Next we recall the Liouville theorem, which states that canonical transformations preserve the phase-space volume. Area-preserving diffeomorphisms are, therefore, canonical transformations of a two-dimensional *phase space*. If we identify the $(z,\bar{z})$ coordinate plane with a phase space, we can describe the area-preserving diffeomorphisms in terms of familiar formulas of canonical transformations. We define the dimensionless Poisson brackets,

$$\{f,g\}\equiv\frac{i}{\rho_0}\ \left(\partial f\bar{\partial}g-\bar{\partial}f\partial g\right)\ , \tag{3}$$

where $\partial\equiv\partial/\partial z$ and $\bar{\partial}\equiv\partial/\partial\bar{z}$. Thus, area-preserving diffeomorphisms are defined by $\delta z=\{\mathcal{L},z\}$ and $\delta\bar{z}=\{\mathcal{L},\bar{z}\}$, in terms of generating functions $\mathcal{L}(z,\bar{z})$ of both "coordinate" and "momentum". The basis of generators $\mathcal{L}^{(cl)}_{n,m}\equiv\rho_0^{\,n+m/2}\ z^n\bar{z}^m$ satisfy the w_∞ algebra [3],

$$\left\{\mathcal{L}^{(cl)}_{n,m},\mathcal{L}^{(cl)}_{k,l}\right\}=-i(mk-nl)\mathcal{L}^{(cl)}_{n+k-1,m+l-1}\ . \tag{4}$$

The *small* excitations above the ground state are given by the infinitesimal w_∞ transformations of ρ_{GS} in (1), $\delta\rho_{n,m}\equiv\left\{\mathcal{L}^{(cl)}_{n,m},\rho_{GS}\right\}$. Using the Poisson brackets (3) , we obtain

$$\delta\rho_{n,m}\propto i(m-n)\ \mathrm{e}^{i(n-m)\theta}\ \delta\left(R^2-z\bar{z}\right)\ . \tag{5}$$

These correspond to density fluctuations localized on the sharp boundary (which is parametrized by the angle θ) of the classical droplet. Due to the dynamics provided by the energy function, they will propagate on the boundary with a frequency ω_k dependent on the angular momentum $k\equiv(n-m)$, thereby turning into *edge waves*.

Another type of excitations are classical *vortices* in the bulk of the droplet, which correspond to localized holes or dips in the density. The absence of density waves, due to incompressibility, implies that any localized density excess or defect is transmitted completely to the boundary, where it is seen as a further edge deformation. For each given vorticity in the bulk, we can then construct the corresponding basis of edge waves as in (5). Thus, the configuration space of the excitations of a classical incompressible fluid (of a given vorticity) is spanned by infinitesimal w_∞ transformations. This is the *dynamical symmetry* of classical incompressible fluids.

The classical edge waves and vortices have quantum analogues in the Laughlin theory [2]. These are easily understood in the simplest case of incompressible quantum fluid, corresponding to the fully filled Landau level (filling fraction $\nu=1$) [3]. The edge waves are particle-hole excitations across the Fermi surface represented by the edge of the droplet of radius R. The vortices correspond to localized quasi-particle and quasi-hole excitations in the bulk of the fluid, the so-called *anyons* with fractional charge, spin and statistics. As in the classical case, they manifest themselves as charged excitations at the edge, owing to incompressibility.

In the quantum theory, there is also a relation between edge excitations and the generators of the quantum version of w_∞, called $W_{1+\infty}$ [3]. This algebra is obtained by replacing the Poisson brackets with quantum commutators: $i\{\ ,\ \} \to [\ ,\ ]$, and by taking the thermodynamic limit of large droplets [4]. In this limit, the radius of the droplet grows as $R \propto \ell\sqrt{N} \to \infty$, where $\ell = \sqrt{2/(eB)}$ is the magnetic length and B the magnetic field. Quantum edge excitations, instead, are confined to a boundary annulus of finite size $O(\ell)$. Therefore, edge excitations become the particle hole excitations of the $(1+1)$-dimensional relativistic theory of the Weyl (chiral) fermion [4]. The quantum incompressible fluid becomes the Dirac sea for this long-distance, "scaling" theory. Moreover, the charged states in the relativistic theory represent the edge effect of quasi-particle excitations.

It is not difficult to understand the $W_{1+\infty}$ symmetry of the Weyl fermion. The generators of the $W_{1+\infty}$ algebra are built by bilinears of the Weyl fermion field $F(\theta)$:

$$V_n^j = \int_0^{2\pi} \frac{d\theta}{2\pi}\ :F^\dagger(\theta)\ \mathrm{e}^{-in\theta}\ g_n^j\,(i\partial_\theta)\ F(\theta):\ , \quad (6)$$

where $g_n^j\,(i\partial_\theta)$ are j-th order polynomials in $i\partial_\theta$, whose form specifies the basis of V_n^i generators. The $W_{1+\infty}$ algebra reads [4],

$$\begin{aligned} [\ V_n^i, V_m^j\ ] &= (jn - im)\ V_{n+m}^{i+j-1} \\ &+ q(i,j,m,n)\ V_{n+m}^{i+j-3} + \dots \\ &+ c\ d^i(n)\ \delta^{i,j}\delta_{n+m,0}\ . \end{aligned} \quad (7)$$

Here, $i+1 = h \geq 1$ represents the "conformal spin" of the generator V_n^i, while $-\infty < n < +\infty$ is the angular momentum (the Fourier mode on the circle). The first term on the right-hand-side of (7) reproduces the classical w_∞ algebra (4) by the correspondence $\mathcal{L}_{i-n,i}^{(cl)} \to V_n^i$ and identifies $W_{1+\infty}$ as the algebra of "quantum area-preserving diffeomorphisms". There are also a finite number of quantum operator corrections with polynomial coefficients $q(i,j,n,m)$. Finally, the c-number term $c\ d^i(n)$ is the quantum *anomaly*, a relativistic effect [5]. All terms in the commutation relations (7) are known and uniquely determined by the closure of the algebra; only the *central charge* c is a free parameter, which takes the value one in the Weyl fermion theory.

The generators V_n^0 are Fourier modes of the fermion density evaluated at the edge $|z| = R$ and V_0^0 measures the edge charge. Instead, the V_n^1 generate angular momentum transformations on the edge, such that V_0^1 measures the angular momentum of edge excitations. The generators V_n^0 and V_n^1 obey the Abelian current algebra $\widehat{U(1)}$ and the Virasoro algebra, respectively [5].

In conformal field theory [5], the Weyl fermion theory can be defined *algebraically* as the Hilbert space made by a set of irreducible, highest-weight representations of the $W_{1+\infty}$ algebra, which is closed under the *fusion rules* for making composite excitations. Any representation contains an infinite number of states, corresponding to all the particle-hole excitations above a bottom state, the so-called *highest weight* state. This can be, for example, the ground-state $|\Omega\rangle$ corresponding to the incompressible quantum fluid. The particle-hole excitations can be written as

$$\begin{aligned} &V_{-n_1}^{i_1}\ V_{-n_2}^{i_2} \cdots V_{-n_s}^{i_s} |\ \Omega\ \rangle\ , \\ &n_1 \geq n_2 \geq \cdots \geq n_s > 0\ ,\quad i_1, \dots, i_s \geq 0\ , \end{aligned} \quad (8)$$

while the positive modes ($n_i < 0$) annihilate $|\ \Omega\ \rangle$. Here $k = \sum_j n_j$ is the total angular momentum of the edge excitation [3].

Furthermore, any charged edge excitation, together with its tower of particle-hole excitations, also forms an irreducible, highest-weight representation of $W_{1+\infty}$. The states in this representation have the same form of (8), but the bottom state $|Q\rangle$ now represents a quasi-particle inside the droplet. The charge Q and the spin J of the quasi-particle are given by the eigenvalues of the operators V_0^0 and V_0^1, respectively:

$$V_0^0\ |Q\rangle\ =\ Q\ |Q\rangle\ ,\ V_0^1\ |Q\rangle\ =\ J\ |Q\rangle\ . \quad (9)$$

Moreover, the statistics θ/π of quasi-particles is equal to twice the spin J. For $W_{1+\infty}$ representations , all the operators V_0^i are simultaneously diagonal and assign other quantum numbers to the quasi-particle, $V_0^i|Q\rangle = m_i(Q)|Q\rangle$, $i \geq 2$, which are known polynomials in the charge Q [7]. These quantum numbers measure the radial moments of the charge distribution of the quasi-particles [4].

2.1. Classification of QHE universality classes

Besides the previous explicit construction for $\nu = 1$, we have found a general method for constructing the effective edge theories corresponding to fractional fillings [6][8]. A known mathematical result is that the $W_{1+\infty}$ algebra (7) is the unique quantization of the w_∞ algebra in the $(1+1)$-dimensional field theory on the circle - the droplet boundary [7]. This implies that the effective quantum theories of the incompressible fluids are completely specified by the $W_{1+\infty}$ symmetry, for any value of the density, i.e., of the filling fraction. Although these general theories cannot be derived explicitly in terms of electrons, they can be completely constructed by using the algebraic methods of conformal field theory [5] and the $W_{1+\infty}$ representation theory [7].

We shall therefore *postulate* that *all* quantum Hall incompressible fluids are in one-to-one correspondence with $W_{1+\infty}$ theories [6]. This characterization provides a powerful classification scheme for quantum Hall universality classes.

These classes are specified by the *kinematical data* of the incompressible fluid, which can be obtained from the quantum numbers (weights) of the $W_{1+\infty}$ representations, as follows:

- the charge Q and fractional statistics θ/π of the quasi-particle excitations are given by the eigenvalues of V_0^0 and V_0^1, respectively (see (9)) (the other $V_0^i, i > 1$ eigenvalues are of secondary interest).

- the Hall conductivity $\sigma_H = (e^2/h)\nu$, proportional to the filling fraction ν of the ground state, is obtained from the chiral anomaly of the $W_{1+\infty}$ $(1+1)$-dimensional theory [4] [6].

- the number of particle-hole excitations of given angular momentum, *i.e.*, the *degeneracies* of states above the ground state (8) are obtained from the characters of the $W_{1+\infty}$ representations [8].

3. THE STANDARD THEORY OF EDGE EXCITATIONS AND THE HIERARCHY

There is nowadays a "standard model" for the description of edge excitations, which is given by the $(1+1)$-dimensional theory of the chiral boson field $\phi(\theta)$ [10]. An equivalent description is given by Abelian Chern-Simons theories on $(2+1)$-dimensional open domains. While one field describes the edge excitations of the Laughlin fluid $\nu = 1, 1/3, 1/5, \ldots$, many components have been introduced for describing more general fractional fillings. Each boson gives rise to an independent current $J^i \propto \partial_\theta \phi^i$, which satisfy the algebra,

$$[J^i(\theta_1), J^k(\theta_2)] = \delta^{ik}\, \delta'(\theta_1 - \theta_2) , \qquad i,k = 1,\ldots,m . \tag{10}$$

This multi-component Abelian current algebra is denoted by $\widehat{U(1)}^{\otimes m}$ and implies the Virasoro algebra with integer central charge equal to the number of current components $c = m$ [5].

Besides the central charge, each Abelian edge theory is specified by a $(m \times m)$ symmetric, integer-valued matrix K_{ij} (with K_{ii} odd); K is pseudo-Euclidean if the theory has excitations with both types of chiralities. The observables quantities of edge excitations are obtained as follows [10]: the spectrum of charges and fractional statistics are given by,

$$Q = \sum_{i,j=1}^{m} K_{ij}^{-1}\, n_j , \qquad \frac{\theta}{\pi} = \sum_{i,j=1}^{m} n_i\, K_{ij}^{-1}\, n_j , \qquad n_i \in \mathbf{Z} , \tag{11}$$

respectively, where the integers n_i label the edge excitation. The Hall conductivity, given by the chiral anomaly [4], is similarly parametrized by:

$$\sigma_H = \frac{e^2}{h}\,\nu , \qquad \nu = \sum_{i,j=1}^{m} K_{ij}^{-1} . \tag{12}$$

The one-component theory has one free parameter ($K_{11} =$ odd), which spans the simplest Laughlin fluids, and its predictions have been confirmed by recent experiments[3]. Instead, the

[3] See the references quoted in [8].

multi-component theory has many free parameters (K_{ij}) and is thus *too generic*. Actually, the experimental values of the fractional Hall conductivity show a characteristic *hierarchical pattern* (see figure 1). This pattern is not captured by the standard edge theory and new physical inputs must be introduced for making this theory predictive [10].

On the other hand, this pattern has been understood in the completely different language and methods of trial wave functions and microscopic dynamics, which has culminated in the Jain hierarchical theory [9]. For filling fractions,

$$\nu = \frac{m}{mp \pm 1}, \qquad m = 1, 2, \ldots, \qquad p = 2, 4, \ldots, \tag{13}$$

Jain devised a generalization of the Laughlin wave functions based on the physical picture of the *composite fermion*, a local bound state of the electron and an even number (p) of flux quanta. The strongly-interacting electrons can be mapped into weakly-interacting composite fermions at effective integer filling (m). The stability of the fluids at the fractional fillings (13) can be related with the stability of m completely filled Landau levels. Therefore, at fixed p, equation (13) gives one-parameter families of approximately equally stable Hall fluids, which accumulate at $\nu \to (1/p)$. This is the hierarchical pattern shown by the experimental points in figure 1 (**bold** fractions); the other points (*italic* fractions) are not presently well understood [10].

4. $W_{1+\infty}$ MINIMAL MODELS

The $W_{1+\infty}$ classification of incompressible quantum Hall fluids outlined in section two can be carried through because all unitary, irreducible, highest-weight representations of the $W_{1+\infty}$ algebra (7) were recently obtained in the mathematical work [7]. These representations exist for positive integer central charge $c = m = 1, 2, \ldots$ and are labeled by a m-component weight vector $\vec{r} = \{r_1, \ldots, r_m\}$, which is the multi-component generalization of the charge Q in (9). There exist two kinds of representations, depending on the value of the weight vector:

- if $(r_i - r_j) \notin \mathbf{Z}$, $\forall\, i \neq j$, the representation is said *generic*, and is actually *equivalent* to a representation of the $\widehat{U(1)}^m$ algebra (10) having the same weight;

- if $(r_i - r_j) \in \mathbf{Z}$, for some i, j, the representation is called *degenerate* and is *contained* in the corresponding $\widehat{U(1)}^m$ representation

In ref. [6], the generic representations have been used to build the *generic* $W_{1+\infty}$ theories, which were shown to correspond to the previously discussed m-component Abelian edge theories.

In ref. [8], the fully degenerate representations, with $(r_i - r_j) \in \mathbf{Z}$, $\forall\, i \neq j$, were used to build the $W_{1+\infty}$ *minimal models*, which we shall now describe. The minimal models are maximally different from the standard Abelian edge theories, being *quantum reductions* of the latter theories. This means that they have less excited states because the $\widehat{U(1)}^m$ representations decompose into many $W_{1+\infty}$ representations.

The most relevant result is that *the minimal models exists only for the filling fractions (13) of the Jain fluids* [8]. The parameters m and p are, respectively, the central charge and the free charge unit in these theories. This result has far-reaching consequences, both theoretical and experimental. The physical mechanism which stabilizes the observed quantum Hall fluids has both *short* and *long distance* manifestations. At the microscopic level, it can be described by the Jain composite-electron picture and by the size of the gaps; in the scaling limit, by the minimality of the $W_{1+\infty}$ edge theory. It is rather natural that the theories with a minimal set of excitations are also dynamically more stable. This long-distance stability principle leads to a logically self-contained edge theory of the fractional Hall effect: a thorough derivation of experimental results is obtained from the principle of $W_{1+\infty}$ symmetry, which is the basic property of the Laughlin incompressible fluid. This independent hierarchical construction is the main result of our approach.

The spectrum of charge and fractional statistics of edge excitations of the $W_{1+\infty}$ minimal models

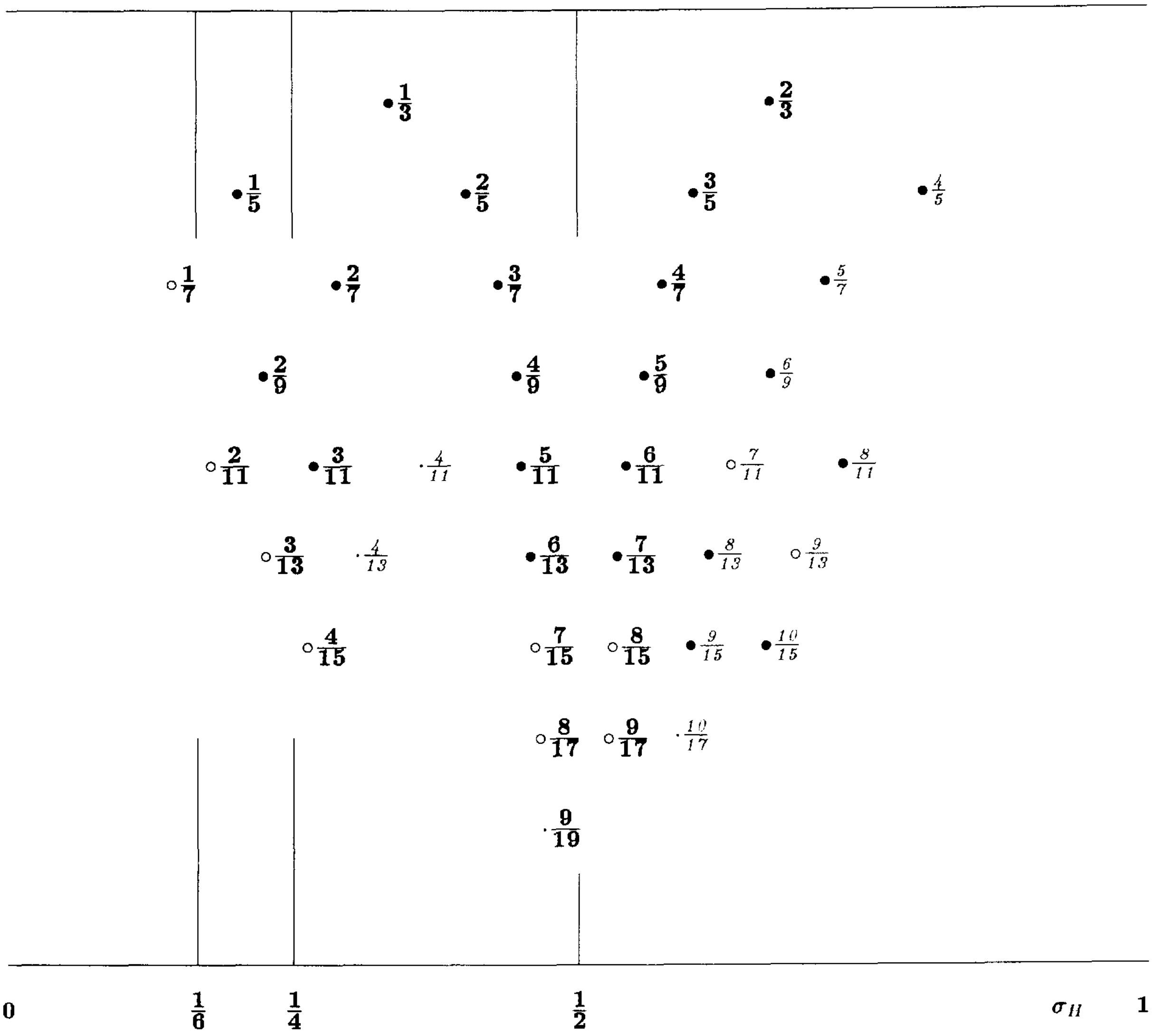

Figure 1. Experimentally observed plateaus in the range $0 < \nu < 1$: their Hall conductivity $\sigma_H = (e^2/h)\nu$ is displayed in units of (e^2/h). The points denote stability: ($\bullet$) very stable, ($\circ$) stable, and ($\cdot$) less stable plateaus. Theoretically understood plateaus are in **bold**, unexplained ones are in *italic*. Observed cases of coexisting fluids are displayed as $\nu = 2/3, 6/9, 10/15$, $\nu = 3/5, 9/15$ and $\nu = 5/7, 15/21$ (but $15/21$ is not displayed).

is given by [8],

$$
\begin{aligned}
Q &= \frac{1}{pm \pm 1} \sum_{i=1}^{m} n_i \,, \\
\frac{\theta}{\pi} &= \pm \left(\sum_{i=1}^{m} n_i^2 - \frac{p}{mp \pm 1} \left(\sum_{i=1}^{m} n_i \right)^2 \right) , \\
& \quad p = 2, 4, \ldots, \quad m = 1, 2, \ldots, \\
& \quad n_1 \geq n_2 \geq \cdots \geq n_m \,, \quad n_i \in \mathbf{Z} \,, \qquad (14)
\end{aligned}
$$

Let us remark that the spectra (14) of quantum numbers have also been obtained in the standard Abelian $\widehat{U(1)}^m$ theory [10], by choosing a specific form of the K_{ij} matrix in (11-12), inspired by the wave-function construction [9]. However, the properties of the excitations in the two theories are rather different, due to the degeneracy of $W_{1+\infty}$ representations .

Degenerate representations are rather common in conformal field theory: if the central charge and the weight of a given representation satisfy certain algebraic relations, some of its states decouple, and should be projected out from the Hilbert space. The minimal models are made of sets of degenerate representations , which are closed under the fusion rules [5]. There are specific minimal models for any symmetry algebra: the well-known ones are the $c < 1$ Virasoro minimal models; larger symmetry algebras, like $W_{1+\infty}$, have $c > 1$ minimal models.

The $W_{1+\infty}$ minimal models are *not* realised by the multi-component chiral boson theories with $\widehat{U(1)}^{\otimes m}$ symmetry, because the latter do not incorporate the projection that makes irreducible the $W_{1+\infty}$ degenerate representations. They are instead realised by the $\widehat{U(1)} \otimes \mathcal{W}_m(p = \infty)$ conformal theories [7], where the $\mathcal{W}_m(p)$ are the Zamolodchikov-Fateev-Lykyanov models with $c = (m-1)\left[1 - m(m+1)/p(p+1)\right]$. The weight $\vec{r}$ of these representations belongs to the weight lattice of the Lie algebra $U(1) \times SU(m)$. As a consequence, the $W_{1+\infty}$ minimal models give a natural explanation of the $SU(m)$ "symmetry" of the Jain fluids often observed before [10]. Moreover, this non-Abelian character implies the following properties of the edge excitations:

i) There is a *single* Abelian current, instead of m independent ones, and therefore a single elementary (fractionally) charged excitation; there are neutral excitations, but they cannot be associated to $(m-1)$ independent edges.

ii) The dynamics of these neutral excitations is new: they have associated an $SU(m)$ (not $\widehat{SU(m)}_1$ [10]) *"isospin"* quantum number, given by the highest weight [8],

$$
\Lambda = \sum_{a=1}^{m-1} \Lambda^{(a)} \left(n_a - n_{a+1} \right) \,, \qquad (15)
$$

where $\Lambda^{(a)}$ are the fundamental weights of $SU(m)$ and $\{n_i\}$ are the integer labels of eq. (14). Therefore, the neutral excitations are quark-like and their statistics is non-Abelian. For example, the edge excitation corresponding to the electron is a composite carrying both the additive electric charge and the fundamental "quark" $SU(m)$ representation . It is possible that a direct experimental test of the non-Abelian properties of the neutral excitations could be made in the future [8].

iii) The degeneracy of particle-hole excitations at fixed angular momentum is modified by the projection of the minimal models. This counting of states is provided by the characters of degenerate $W_{1+\infty}$ representations, which are known [7]. If the neutral $SU(m)$ excitations have a bulk gap, the particle-hole degeneracy of the ground state (the Wen topological order on the disk) is different from the corresponding one of $\widehat{U(1)}^{\otimes m}$ excitations [10]. This can be tested in numerical diagonalizations of few electron systems [8].

5. CONCLUSIONS

These results show that the hierarchical pattern of fractional Hall fluids emerges naturally from the $W_{1+\infty}$ symmetry. Moreover, they suggest that the $W_{1+\infty}$ minimal models are the correct edge theories corresponding to the Jain wave functions. This hypothesis can be confirmed by specific experimental and numerical tests of the properties of edge excitations in the minimal models [8]. Finally, there is a hierarchical structure of intermediate $W_{1+\infty}$ theories, in between the generic and minimal theories, which has yet to be explored, and shows some similarities with the higher orders in the Jain hierarchy.

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