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MEAN FIELD METHODS FOR STRINGS AND MONOPOLES IN ABELIAN LATTICE GAUGE THEORIES

A. CAPPELLI

Dipartimento di Fisica, Università di Firenze, and INFN, Sezione di Firenze, Largo E. Fermi 2, 50125 Firenze, Italy, and Service de Physique Théorique, CEN-Saclay, 91191 Gif-sur-Yvette Cedex, France*

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Selfconsistent approximations, which join the Bethe-Peierls method and duality transformation, are applied to disorder parameters related to strings (domain boundaries) and monopoles in $Z(N)$ and U(1) lattice gauge theories. The two-phase and the three-phase diagrams are reproduced in three and four dimensions. Nice results are obtained for the internal energy and the monopole charge density. A formulation for gauge theories of the selfconsistent Monte Carlo method is introduced in order to improve these approximations.

1. Introduction

The phase diagrams of both spin models with global $Z(N)$ symmetry and gauge theories with local $Z(N)$ symmetry can be described in terms of two types of disorder configurations, called topological excitations (TE) of strings and monopoles [1]. Strings (domain boundaries in spin models) form a condensate in the vacuum below a phase transition which occurs at the value $\beta \sim O((N/2\pi)^2)$ of the coupling constant. Monopoles (vortices in spin models) condense at another phase transition point occurring at $\beta \sim O(1)$.

These excitations are manifestly gauge invariant and therefore this description of the phase diagram is particularly suitable for gauge theories. Unfortunately, the methods applied by Kosterlitz and Thouless (KT) to the condensation of vortices in the two-dimensional (2d) XY spin model [2, 3] have not yet been extended to gauge theories. Other analytical methods for studying the condensation mechanism have not been introduced [4] and the effect of TE has been considered for locating approximately the transition points [1, 3] and for rigorous bounds [5].

* Present address.

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In this paper we shall derive analytical tools for the description of the condensation of strings and monopoles based on Mean Field-like (MF) selfconsistent hypotheses. These methods provide results which are manifestly gauge invariant and in good agreement with numerical Monte Carlo (MC) data [6]; they are generally better than the results of the standard MF for the link variable [7]. Moreover, this phenomenological description of TE may be useful for gaining intuition over the mechanism of the phase transition and may set the basis for a more sophisticated theoretical study of the regions very near the transition points.

In previous papers [8] (referred to as I from now on) we have already presented this selfconsistent MF approach for Z(2) symmetric models; here we shall extend it to other abelian models.

Our approach is based on the relation between spin theories and gauge theories which is provided by the duality transformation [1] in 3d. A Bethe-Peierls approximation [9] is considered for the spin model: it is defined on a finite cluster with a source term on the border and the consistency equation is imposed between internal and external magnetizations. Duality transformation on this cluster yields, in the corresponding gauge theory, a selfconsistent approximation for the disorder parameter, which is commonly called "frustration" [10]. This parameter measures the probability of configurations with an infinite string; therefore, it is different from zero when the strings condense.

In subsect. 2.1 we shall clarify the physical meaning of the frustration and we shall derive our selfconsistent approximation.

In subsects. 2.2 and 2.3 we shall present the results: in 3d gauge theory a second order phase transition is obtained at $\beta \sim O(N^2)$ and there is no signal of a transition at $\beta \sim O(1)$: this is in agreement with other numerical and theoretical arguments. In 4d gauge theory a first-order phase transition is obtained at $\beta \sim O(N^2)$ and there are indications for a phase transition at $\beta \sim O(1)$.

In subsect. 3.2, a second-order phase transition is indeed obtained at $\beta \sim O(1)$ by an improved selfconsistent method, which describes properly the monopole condensation. The strategy of sect. 2 is applied again and selfduality of the 4d theory is exploited. A consistency equation is considered for the monopole disorder parameter, which is dual to the order parameter of the $\beta \sim O(N^2)$ transition introduced in ref. [11].

As an introduction to the last part, subsect. 3.1 contains the study of the monopole density: our results are in good agreement with MC data.

Finally, the improvement of our approximations by increasing the size of the clusters is presented in sect. 4: expectation values are computed by a formulation for gauge theories of the selfconsistent Monte Carlo method applied by Binder et al. [12] to spin models.

The conclusions contain a discussion on MC simulations with open boundary conditions, including selfconsistent border terms, as an alternative to traditional simulations with periodic boundary conditions.

2. Selfconsistent approximation for strings in Z(N) models

2.1. INTRODUCTION AND DERIVATION OF THE APPROXIMATION

Let us recall the definitions of Z(N) strings and frustrations; let us consider, for example, the 2d Z(N) spin model, whose action is

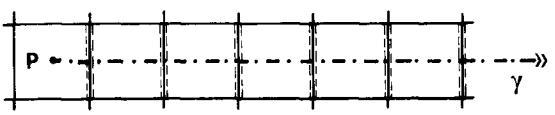
$$A = \beta \sum_{(x,\mu)} \cos\big(\Delta_\mu \theta(x)\big), \tag{2.1}$$

where $\beta = T^{-1}$, $\Delta_\mu \theta(x) = \theta(x+\mu) - \theta(x)$, $x = (x_0, x_1)$, and $\theta = 2\pi n/N$, $n = 0, \ldots, N-1$, is the spin angle on every site.

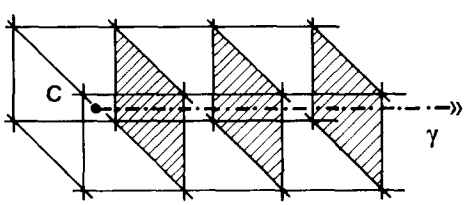
In this model the strings are the domain contours [1]: they are closed or extend from border to border through the lattice. The frustration is the following expectation value (see fig. 1a)

$$\langle \mathscr{F}(x) \rangle_\beta \underset{(2d)}{\equiv} \left\langle \exp\left\{ \beta \sum_{(y,\mu)\perp\gamma} \left[\cos\left(\Delta_\mu \theta - \frac{2\pi}{N} \right) - \cos\big(\Delta_\mu \theta\big) \right] \right\} \right\rangle_\beta . \tag{2.2}$$

The spin angles are twisted on links (y, μ) orthogonal to a (dual) line γ starting at x and going to the border of the lattice, where free conditions are needed. $\langle \mathscr{F} \rangle$ depends only on x since the line γ can be displaced by a change of variables. The



(a)



(b)

Fig. 1. (a) Spin model frustration at the plaquette p in the 2d lattice; (b) gauge theory frustration at the cube c in 3d. The dashed links and plaquettes have $2\pi/N$ twists in the action.

twist at the plaquette dual to x cannot be removed and it "frustrates" the alignment of spins.

The frustration measures the probability of configurations which contain an open string from x to the border. Let us estimate the quantity eq. (2.2) by a Peierls argument. In the large β ordered vacuum state, we have $\langle \mathscr{F} \rangle \sim \exp(-\Delta F)$, where ΔF is the excess of free energy of maximally ordered configurations with "twisted" action, i.e. open domain contours which continue γ or neutralize it. As in ref. [1], we estimate their energy E, by a spin-flip expansion, and entropy S, as follows:

$$\Delta F = \beta(\Delta E - \Delta S) \approx (\beta\eta - \log 3)L, \tag{2.3}$$

where $\eta = (1 - \cos(2\pi/N))$ and L is the length of such contours, which is of the order of the size of the lattice. In the limit $L \to \infty$, the frustration vanishes for $\beta > \beta_c = (\log 3)/\eta \approx O((N/2\pi)^2)$. On the contrary, it can be non-zero for $\beta < \beta_c$, becuase the entropy dominates the energy in eq. (2.3); strings of infinite length are allowed configurations and "condense" in the vacuum.

Therefore, the frustration is a disorder parameter of the phase transition occurring at $\beta \sim O(N^2)$.

Also 3d and 4d $Z(N)$ gauge theories show phase transitions at $\beta \sim O(N^2)$ due to the condensation of the corresponding string excitations. In 3d the strings are again contours; the frustration is defined on a cube by twisting plaquettes $\theta_p(x) = \Delta_\alpha \theta_\beta(x) - \Delta_\beta \theta_\alpha(x)$, $p \equiv (\alpha, \beta)$, $x = (x_0, x_1, x_2)$, in the gauge action

$$A = \beta \sum_p \cos(\theta_p) \qquad \left(\theta = \frac{2\pi}{N} n,\ n = 0, \ldots, N-1\right) \tag{2.4}$$

as follows (see fig. 1b):

$$\langle \mathscr{F}(x) \rangle_\beta \underset{(3d)}{\equiv} \left\langle \exp\left\{ \beta \sum_{p \perp \gamma} \left[\cos\left(\theta_p - \frac{2\pi}{N} \right) - \cos\theta_p \right] \right\} \right\rangle_\beta . \tag{2.5}$$

In eq. (2.5) the contour γ starts at x and extends to infinity: these twisted plaquettes induce open strings on the ordered vacuum and the Peierls argument gives $\beta_c \approx (\log 5)/\eta$.

The 4d frustration is a path operator $\langle \mathscr{F}_\Gamma \rangle$, since cubes along a dual infinite path Γ are frustrated. The twisted plaquettes belong to a "string sheet", i.e. they are orthogonal to a dual half plane Σ^* bordered by Γ. When Γ is a closed loop, the frustration is also called 't Hooft Loop [1].

Let us now introduce the selfconsistent approximation for $Z(N)$ frustrations in the 3d gauge theory. As in I, we consider the Bethe-Peierls approximation for the 3d $Z(N)$ spin model on a finite cluster Λ, as in fig. 2. The cluster action reads

$$A_\Lambda = \beta \sum_{(x,\mu) \in \Lambda} \cos(\Delta_\mu \theta(x)) + b \sum_{x \in \partial\Lambda} \cos\theta(x), \tag{2.6}$$

where, for example, $\theta_1 - \theta_0 = \Delta_\mu \theta(x)$ for $x = 0$, $\mu = \hat{1}$ in fig. 2.

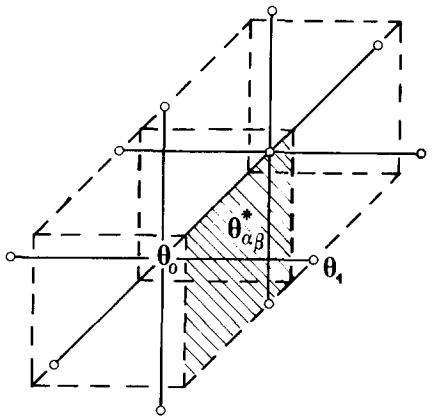


Fig. 2. 3d clusters: the spin model cluster Λ has full links and white sites, the gauge model cluster Λ^* has dashed links.

The second term of eq. (2.6) on border sites $x \in \partial\Lambda$ breaks the global symmetry $\theta(x) \to \theta(x) + 2\pi q/N$, $q = 0, \ldots, N-1$ of the action. In the Bethe-Peierls approach [9], the following consistency equation is imposed between the internal magnetization (site 0) and external one (site 1, for example):

$$\langle \cos\theta_0 \rangle_{\beta,b} = \langle \cos\theta_1 \rangle_{\beta,b}; \qquad \langle (\cdot) \rangle_{\beta,b} = \frac{\sum_{\{\theta\}} (\cdot) \exp(A_\Lambda)}{\sum_{\{\theta\}} \exp(A_\Lambda)}. \tag{2.7}$$

Eq. (2.7) has, as usual, two solutions: the symmetric solution $b \equiv 0$, which is always present, and, below the critical temperature ($\beta > \beta_c$), the broken phase solution $b \neq 0$.

Let us perform the duality transformation on the cluster partition function. This is realized by a Fourier expansion of each term in $\exp(A_\Lambda)$, with A_Λ given by eq. (2.6):

$$e^{\beta\cos(\Delta_\mu\theta)} = \frac{e^\beta}{\sqrt{2\pi\beta}} \sum_{\kappa_\mu=-\infty}^{+\infty} \exp\left(i\kappa_\mu \Delta_\mu \theta - \frac{\kappa_\mu^2}{2\beta} \right), \tag{2.8a}$$

$$e^{b\cos\theta} = \frac{e^b}{\sqrt{2\pi b}} \sum_{r=-\infty}^{+\infty} \exp\left(ir\theta - \frac{r^2}{2b} \right). \tag{2.8b}$$

Expressions (2.8) are valid within the Villain approximation [1] and for N large enough. Substituting eq. (2.8) in the partition function and summing by parts we

obtain

$$Z_\Lambda = \sum_{\{\theta\}} e^{A_\Lambda} = \sum_{\{\theta\}} \sum_{\{\kappa_\mu, r\}} \exp\left\{ -\frac{1}{2\beta} \sum \kappa_\mu^2 - \frac{1}{2b} \sum{}' r^2 \right\}$$

$$\times \exp\left\{ i \sum{}' (\kappa_\nu + r)\theta - i \sum \theta (\Delta_\mu \kappa_\mu) \right\}, \tag{2.9}$$

where Σ runs over the internal sites of Λ, Σ' run over the border sites $x \in \partial\Lambda$ and the direction of the free index $\hat{\nu}$ is normal to the surface of $\partial\Lambda$. Integration over θ variables leads to the following conditions:

$$\sum_\mu \Delta_\mu \kappa_\mu(x) = 0 \qquad \text{(internal sites } x\text{)},$$

$$\kappa_\nu(y) = -r(y) \qquad \text{(border sites } y \text{ and } \nu \perp \partial\Lambda\text{)}. \tag{2.10}$$

These conditions are solved in terms of gauge variables θ_ρ^* defined on the dual cluster Λ^* (see fig. 2):

$$\frac{2\pi}{N} \kappa_\mu = \sum_{\nu,\rho} \varepsilon_{\mu\nu\rho} \Delta_\mu \theta_\rho^*, \qquad \theta_\rho^* = \frac{2\pi}{N} n_\rho, \qquad n_\rho = 0, \cdots, N-1. \tag{2.11}$$

Therefore duality transformation leads to Z(N) 3d gauge theory defined on the dual cluster Λ^*. By eqs. (2.9)–(2.10), the border term in the spin action (eq. (2.6)) provides the following border term in the gauge action:

$$A_{\Lambda^*} = -\tfrac{1}{2}\beta^* \sum_{(\mu,\nu)\in\Lambda^*} (\Delta_\mu \theta_\nu^* - \Delta_\nu \theta_\mu^*)^2 - \tfrac{1}{2} b^* \sum_{(\mu,\nu)\in\partial\Lambda^*} (\Delta_\mu \theta_\nu^* - \Delta_\nu \theta_\mu^*)^2. \tag{2.12}$$

In eq. (2.12), $\beta^* = (N/2\pi)^2/\beta$ and $b^* = (N/2\pi)^2/b$ are the dual couplings. In fig. 2, $\partial\Lambda^*$ contains all the plaquettes but one, which is shared by the two cubes.

By a similar procedure we can derive the dual order parameters. For the cluster in fig. 2, we have the relations

$$\langle \cos\theta_0 \rangle_{\beta,b} = \left\langle \exp\left\{ -(\beta^* + b^*) \left[\left(\theta_{\alpha\beta}^* - \frac{2\pi}{N} \right)^2 - \theta_{\alpha\beta}^{*2} \right] \right\} \right\rangle_{\beta^*, b^*}, \tag{2.13a}$$

$$\langle \cos\theta_1 \rangle_{\beta,b} = \left\langle \exp\left\{ -b^* \left[\left(\theta_{\alpha\beta}^* - \frac{2\pi}{N} \right)^2 - \theta_{\alpha\beta}^{*2} \right] \right\} \right\rangle_{\beta^*, b^*}, \tag{2.13b}$$

where (α, β) is a border plaquette like the shaded one in fig. 2, and the expectation values $\langle (\cdot) \rangle_{\beta^*, b^*}$ are evaluated by the action eq. (2.12).

Let us now compare the expectation values on a finite cluster eq. (2.13) with eq. (2.5): apart from details of the action, they can be interpreted as finite cluster representations of the frustration, where the external infinite line of twisted plaquettes is replaced by the border term. The l.h.s. of eq. (2.13a) represents the frustration on a cube of Λ^*, the l.h.s. of eq. (2.13b) is the frustration on the cube external to Λ^*, which shares the plaquette (α, β) with the internal one.

A selfconsistent equation for these frustrations on the cluster is imposed by equating the l.h.s.'s of eqs. (2.13). The analysis of solutions confirms the previous interpretation: by comparison with eq. (2.7), the symmetric solution $b^* = \infty$ gives vanishing frustrations, due to the infinite energy of the string, while the broken solution $b^* < \infty$ for $\beta^* < \beta_c^*$ gives $\langle \mathscr{F} \rangle \neq 0$, i.e. a finite string energy.

Therefore, this selfconsistent approach allows a description of the condensation of strings in 3d gauge theories. Let us now remark that frustrations in eq. (2.13) are gauge invariant operators, then our method is manifestly gauge invariant. Moreover, it shows that a global $Z(N)$ symmetry breaks spontaneously at the transition; this symmetry is related to the conservation (modulo N) of the topological charge Q of frustrations. The charge Q is the magnetic flux out of unitary cubes and takes fractional values multiples of $2\pi/N$ on frustrated cubes (see I on this point).

2.2. RESULTS IN 3d GAUGE THEORY

Let us now discuss in details the results of the selfconsistent approach for frustrations. The lowest order approximation is provided by a cluster made of only one cube. It can be shown that the two-cube cluster of fig. 2 gives equivalent results, because it is an iteration of the same Cayley tree underlying the Bethe-Peierls approximation [9].

On the one-cube cluster we consider the Wilson action with border terms (see eq. (2.12)):

$$A \underset{(3d)}{\equiv} (\beta + b) \sum_{(\mu,\nu)} \cos \theta_{\mu\nu}, \tag{2.14}$$

where the sum runs over the six plaquettes (μ, ν) of the cube and we drop the stars in the couplings. The consistency equation is

$$\langle \mathscr{F}_0 \rangle_{\beta, b} = \langle \mathscr{F}_1 \rangle_{\beta, b} \tag{2.15}$$

between the internal and external frustrations, which are respectively (see eq. (2.13))

$$\langle \mathscr{F}_0 \rangle_{\beta, b} \underset{(3d)}{\equiv} \left\langle \exp\left\{ (\beta + b) \left[\cos\left(\theta_{\alpha\beta} - \frac{2\pi}{N} \right) - \cos \theta_{\alpha\beta} \right] \right\} \right\rangle_{\beta, b}, \tag{2.16a}$$

$$\langle \mathscr{F}_1 \rangle_{\beta, b} \underset{(3d)}{\equiv} \left\langle \exp\left\{ b \left[\cos\left(\theta_{\alpha\beta} - \frac{2\pi}{N} \right) - \cos \theta_{\alpha\beta} \right] \right\} \right\rangle_{\beta, b}, \tag{2.16b}$$

where the plaquette (α, β) is orthogonal to the $\hat{1}$ direction.

In order to evaluate these expectation values it is convenient to use Fourier series expansions. The Fourier coefficients of the Z(N) Wilson action are

$$a_p^{(N)}(x) = \frac{1}{N}\sum_{\theta} \exp(ip\theta + x\cos\theta), \qquad p = 0, \ldots, N-1, \tag{2.17}$$

where $\theta = 2\pi n/N$, $n = 0, \ldots, N-1$; $p = 0, \ldots, N-1$ because $a_p = a_{p+N}$ and also $a_p = a_p^* = a_{-p}$. For N large enough these coefficients have the following different behaviours (the index N is implicit):

$$a_p(x) = \sum_{\kappa=-\infty}^{+\infty} I_{p+\kappa N}(x) \sim \left(\frac{x}{2}\right)^p \frac{1}{p!}, \qquad x \ll 1, p > 0 \qquad \text{(SC)}, \tag{2.18a}$$

$$\frac{a_p(x)}{a_0(x)} \sim \exp\left(-\frac{p^2}{2x}\right), \qquad \mathrm{O}(1) < x < \mathrm{O}(N^2) \qquad \text{(SW)}, \tag{2.18b}$$

$$a_p(x) \sim \frac{e^x}{N}\left(1 + 2\cos\left(\frac{2\pi}{N}p\right)e^{-\eta x}\right), \qquad x > \mathrm{O}(N^2) \qquad \text{(WC)}, \tag{2.18c}$$

where $\eta = (1 - \cos(2\pi/N))$ and $I_p(x)$ are the modified Bessel functions. These behaviours are related to the three phases which may arise in Z(N) models: they are the strong coupling phase (SC), the weak coupling phase with continuous fluctuations, or spin waves (SW), and discrete fluctuations (WC).

Inserting the Fourier series into eqs. (2.16a, b) we obtain the simple expressions

$$\langle \mathscr{F}_0 \rangle_{\beta, b} = \frac{1}{Z}\sum_{p=0}^{N-1} a_p^6(\beta + b)\cos\left(\frac{2\pi}{N}p\right), \tag{2.19a}$$

$$\langle \mathscr{F}_1 \rangle_{\beta, b} = \frac{1}{Z}\sum_{p=0}^{N-1}\sum_{r=0}^{N-1} a_p^5(\beta + b)a_{p-r}(\beta)a_r(b)\cos\left(\frac{2\pi}{N}r\right), \tag{2.19b}$$

where $Z = \Sigma_p[a_p(\beta + b)]^6$ is the cluster partition function. For each value of N, the a_p's can be tabulated and the consistency equation (eq. (2.15)) can be studied numerically on a small computer: below β_c, it provides $b(\beta)$ needed in computing expectation values. Analytical calculations may be done within the three behaviours of eq. (2.18).

Since we expect a second-order phase transition, the critical value of β can be obtained by an expansion for small disorder parameter of eq. (2.15) ($b \to \infty$), around the solution $\langle \mathscr{F} \rangle \equiv 0$ ($b \equiv \infty$). Expanding it to $\mathrm{O}(e^{-\eta b})$ by using eq. (2.18c), we obtain

$$6e^{-\eta\beta_c} = 5e^{-2\eta\beta_c} + 1 \qquad (N > 3), \tag{2.20}$$

with solutions

$$\beta_c = \alpha/\eta = (\log 5)/\eta \tag{2.21}$$

and the trivial one $\beta_c = \infty$. The inclusion of the next spin flip term in eq. (2.18c) and the expansion to $O(e^{-3\eta b})$ of eq. (2.15) allow to recover the standard form of the Landau theory:

$$\begin{cases} \langle \mathscr{F} \rangle \big(O(\beta_c - \beta) - \langle \mathscr{F} \rangle^2 \big) = 0 ; & (\beta \lesssim \beta_c, N \geqslant 6) \\ \langle \mathscr{F} \rangle \sim e^{-\eta b}, & b \to \infty . \end{cases} \tag{2.22}$$

It is worth observing that the value of β_c in eq. (2.21) is identical to the estimate in subsect. 2.2. Furthermore, this is also in good agreement with MC simulations [6], both for the scaling behaviour with N and for the value of α: MC $\alpha \simeq 1.5$, our method $\alpha = 1.61$. Comparisons of numerical values are collected in table 1.

For $N = 2$ the transition is second order (see I). For $N = 3$, there are two additional solutions of eq. (2.15) in the region between $\beta_c = 1.07$ and the other singular point $\beta_s = 1.11$ ($\partial b/\partial \beta|_{\beta_s} = \infty$): therefore we have a first-order phase transition at $\beta \lesssim 1.1$ and the second-order point β_c occurs in the unphysical region. Numerical simulations also indicate a weak first-order transition [13].

The mean value of the plaquette

$$P = \langle \cos \theta_{\alpha\beta} \rangle_{\beta, b} = \frac{1}{Z} \sum_{p=0}^{N-1} a_p^5(\beta + b)\, a_{p+1}(b + \beta) \tag{2.23}$$

is reported in fig. 3 and compared with MC data for $N = 10$, as an example. The numerical data are very well fitted in both the SC and SW phases; the WC solution is instead trivial, i.e. $P \equiv 1$ (it is not reported in fig. 3) and the critical point singularity is a finite discontinuity in the first derivative. Further terms in the WC expansion of P can be recovered by applying our method to larger clusters. For example, in I for $N = 2$ it was shown that the first term $O(e^{-2\eta(d-1)\beta})$ is obtained by the cluster containing a plaquette whose links do not belong to the border.

Using eq. (2.18a), we obtain the correct SC expansion of the plaquette up to $O(\beta^7)$. High order terms corresponding to tree-like plaquette surfaces are resummed, as it happens in the Cayley-tree approach [14]. According to subsect. 2.1, the two approaches are exactly equivalent within the Villain approximation.

The SW expansion, for N large enough, is also correct up to first order: $P \sim 1 - 1/\kappa\beta$, $\kappa = 3$. This result holds also for the internal energy of the 2d spin model, but it does not hold in 4d in the lowest order approximation. Other quantities, like frustrations, have also SW behaviours with the wrong coefficient. We shall discuss this problem in the next section.

Fig. 4 shows a remarkable agreement of the U(1) MC data with our result for $N = 20$ in the crossover between the SC and SW regions. ($N = 20$ is a good

TABLE 1
Numerical constants in our approach and the MC simulations are compared with analytical results, estimates and SC expansions

2d spin model

	α	(Order) $N=2$	3	4	>4	κ	c_0	c_1	λ
our method	1.1	(II)	(I)	(II)	(II)	4	$\frac{1}{6}$	$1/\pi^2$	5.3
analytic	0.67	(II)	(II)	(II)	(KT)	4	$\frac{1}{6}$	$1/\pi^2$	10.2

3d gauge theory

	β_c (Order) $N=2$	3	4	5	6	α ($N>6$)	κ	c_0	c_1	λ
our method	0.80(II)	1.1(I)	1.61(II)	2.33(II)	3.22(II)	1.61(II)	4	$\frac{7}{30}$	0.109	3.7
Monte Carlo	0.76(II)	1.08(I)	1.52(II)	2.1(II)	2.8(II)	1.5(II)				
analytic						1.61	4	$\frac{7}{30}$	0.109	6.6

4d gauge theory

	β_c (Order) $N=2$	3	4	5	6	α ($N>6$)	κ	c_0	c_1	λ
our (cube)	0.45(I)	0.70(I)	0.91(I)	~ 1.03(I)	~ 1.07(I)	~ 0.48(I)	6	$\frac{7}{30}$	0.109	7.8
our (hypercube)	0.45(I)	0.70(I)	0.92(I)	~ 1.03(I)	~ 1.29(I)	~ 0.65(I)	$\frac{80}{17}$			
Monte Carlo	0.44(I)	0.67(I)	0.88(I)	1.2(II)	1.56(II)	0.78(II)				
analytic	0.44(I)		0.88(I)				4	$\frac{7}{30}$	0.109	9.9

Entries: (β_c) critical coupling for the $O(N^2)$ transitions; (α) scaling coefficient eq. (2.21); (κ) SW behaviour of the plaquette; (c_0, c_1) first two SC terms of the monopole density equation (3.10); and (λ) its SW coefficient eq. (3.7).

approximation for the large N behaviour within these values of β.) Moreover, our method is consistent with the absence of a phase transition in 3d U(1) theory [5]: both $\partial h/\partial\beta$ and $\partial P/\partial\beta$ show very broad maxima for $\beta \sim 1.5$–2. The Peierls argument suggests that the monopole excitations in 3d condense at the same point ($\beta \sim O(N^2)$) of frustrations [1].

In summary, this selfconsistent approximation provides rather good results in 3d, even in the lowest approximation (smaller cluster): the order of the phase transition at $\beta \sim O(N^2)$ is second, there are no indications for the $\beta \sim O(1)$ transition and the behaviour of the plaquette is almost matched.

Let us recall for a comparison that the MF methods for the link variable [7] fail to describe the 3d abelian gauge theories, since they predict a first-order transition at $\beta \sim O(1)$ and are unable to treat the $\beta \sim O(N^2)$ transition point.

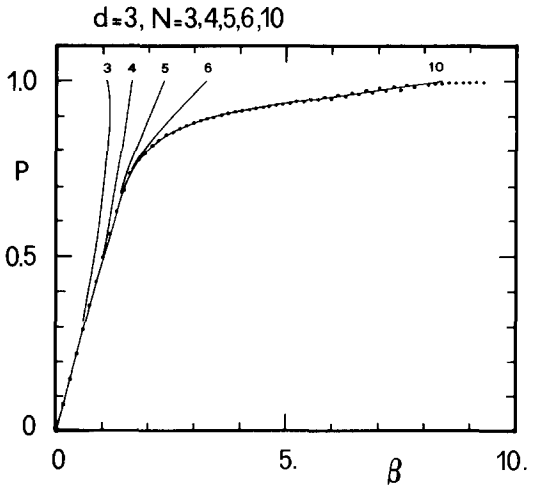


Fig. 3. Mean plaquette in 3d Z(N) gauge theories. Our MF results for $N = 3,4,5,6,10$ (lines) and MC hysteresis loop for $N = 10$ (points).

Before moving to the more interesting 4d case, let us show in summary the result of the method for frustrations in 2d spin models. We consider the following action on a one-plaquette cluster:

$$A \underset{(2d)}{\equiv} (\beta + b) \sum_{(x,\mu)\in p} \cos\bigl(\Delta_\mu \theta(x)\bigr) \tag{2.24}$$

(see also eq. (2.1)) and the consistency equation $\langle \mathscr{F}_0 \rangle_{\beta,b} = \langle \mathscr{F}_1 \rangle_{\beta,b}$ for the internal

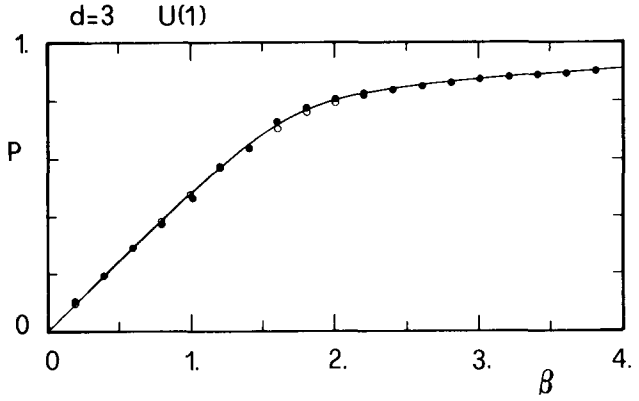


Fig. 4. Mean plaquette in 3d Z(N) gauge theories. $N = 20$ MF results (line) and U(1) MC points (black points = cooling, white = heating).

and external frustrations:

$$\langle \mathscr{F}_0 \rangle_{\beta,b} \underset{(2d)}{\equiv} \left\langle \exp\left\{ (\beta + b)\left[\cos\left(\Delta_\alpha \theta - \frac{2\pi}{N} \right) - \cos(\Delta_\alpha \theta) \right] \right\} \right\rangle_{\beta,b}, \tag{2.25a}$$

$$\langle \mathscr{F}_1 \rangle_{\beta,b} \underset{(2d)}{\equiv} \left\langle \exp\left\{ b\left[\cos\left(\Delta_\alpha \theta - \frac{2\pi}{N} \right) - \cos(\Delta_\alpha \theta) \right] \right\} \right\rangle_{\beta,b} \qquad (\hat{\alpha} \perp \hat{1}). \tag{2.25b}$$

A second-order phase transition is obtained at $\beta_c \sim O(N^2)$ for any N (apart from the $N = 3$ case which has a first-order transition [13]) and the critical temperature is given by eq. (2.21) with $\alpha = \log 3$. In fig. 5, the MC results for the internal energy in the XY spin model (ref. [15])

$$E = -2\langle \cos(\theta(x + \alpha) - \theta(x)) \rangle \tag{2.26}$$

are compared with our MF results: for $N = 20$ the two behaviours match well in the SC and the SW phases.

However, our method gives incorrect results near the phase transition points, because a Kosterlitz-Thouless singularity is expected at both $\beta \sim O(1)$ and $\beta \sim O(N^2)$ for N large enough [1, 2]. This kind of singularity is related to the infrared behaviour of fluctuations in 2d, and it cannot be reproduced in our semiclassical MF approach. In spite of this, the singular part of the KT free energy vanishes at the transitions as well as its derivatives [2]: the absence of singularities in E can explain the rather good agreement shown in fig. 5.

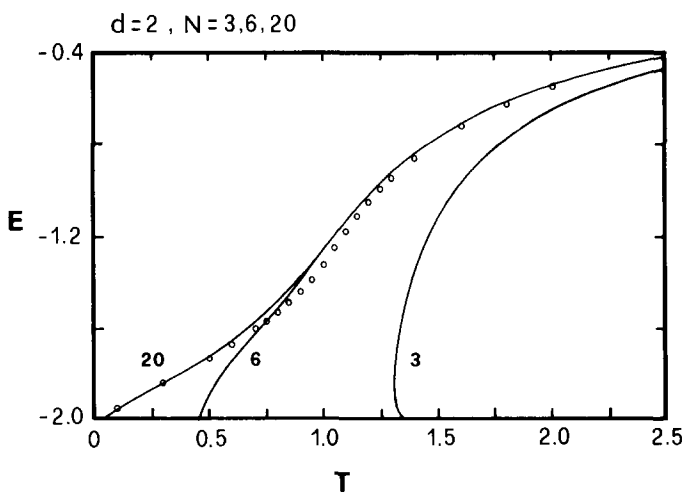


Fig. 5. Internal energy in 2d $Z(N)$ spin models. MF results for $N = 3, 6, 20$ (lines) and U(1) MC (points).

2.3. RESULTS IN 4d GAUGE THEORY

$Z(N)$ frustrations in 4d can be analyzed to the lowest order on a cube cluster, whose action is

$$A \underset{(4d)}{\equiv} (\beta + 3b) \sum_{(\mu,\nu)=1}^{6} \cos\theta_{\mu\nu}, \tag{2.27}$$

where the factor 3 counts the number of external cubes per plaquette. We shall follow I in extending our selfconsistent approach to 4d. There, it was suggested that the frustration path operator has a "reduced" form for a finite cluster on a path crossing the cluster from border to border. This reduced operator is effective in detecting the condensation transition on the cluster as well as the frustration in the infinite lattice, and, eventually, they can match in a large volume limit of the cluster.

On the one-cube cluster we are led to consider a consistency equation between one-cube frustrations, as follows:

$$\langle \mathscr{F}_\nu \rangle_{\beta,b} = \langle \mathscr{F}_\nu' \rangle_{\beta,b}, \tag{2.28}$$

where

$$\langle \mathscr{F}_\nu \rangle_{\beta,b} \underset{(4d)}{\equiv} \left\langle \exp\left\{ (\beta + 3b)\left[\cos\left(\theta_{\alpha\beta} - \frac{2\pi}{N} \right) - \cos\theta_{\alpha\beta} \right] \right\} \right\rangle_{\beta,b}, \tag{2.29a}$$

$$\langle \mathscr{F}_\nu' \rangle_{\beta,b} \underset{(4d)}{\equiv} \left\langle \exp\left\{ b\left[\cos\left(\theta_{\alpha\beta} - \frac{2\pi}{N} \right) - \cos\theta_{\alpha\beta} \right] \right\} \right\rangle_{\beta,b}, \tag{2.29b}$$

are the internal and external 4d frustrations, respectively, $\hat{\nu}$ is the direction dual to the cube and (α, β) one of its plaquettes.

The analysis of eq. (2.28) reveals a first-order transition at $\beta \sim O(N^2)$. The expansion to $O(e^{-\eta b})$, as in the previous section, provides a singular point at $\beta \to -\infty$, while the numerical analysis provides the other singular point β_s at a finite value: for large N it has a scaling law like eq. (2.21) with $\alpha \simeq 0.48$ (see table 1). The critical point $\beta_c \lesssim \beta_s$ is determined by equating the free energy densities f of the two phases: in our approach they are obtained by integration of the mean plaquette $P(\beta, b(\beta))$ and the integration constants are determined by comparison with the infinite lattice f in a computable limit. In SC we normalize f at $\beta = 0$, as follows:

$$f_{SC}(\beta) = \int_0^\beta P(\beta, b(\beta))\, d\beta. \tag{2.30}$$

In WC the plaquette $P \equiv 1$ for the cube cluster. We normalize f at $\beta \to \infty$, as follows:

$$f_{WC}(\beta) = \beta - \tfrac{1}{2} \ln N. \tag{2.31}$$

The value of β_c by equating $f_{SC} = f_{WC}$ are reported in table 1 up to $N = 4$. For $N > 4$ we find that $f_{SC} > f_{WC}$ for any β, then our approximation only provides the upper bound $\beta_c \lesssim \beta_s$. On the other hand, we have an indication that the 4d gauge theory has a richer structure for $N > 4$, since a description in terms of two phases is not adequate.

In fig. 6 our results are compared with MC data for the mean plaquette for $N = 4, 5, 6$ [6]. Numerical data show a first-order phase transition up to $N = 4$ and two phase transitions of higher order above (perhaps second, but still unclear [16]),

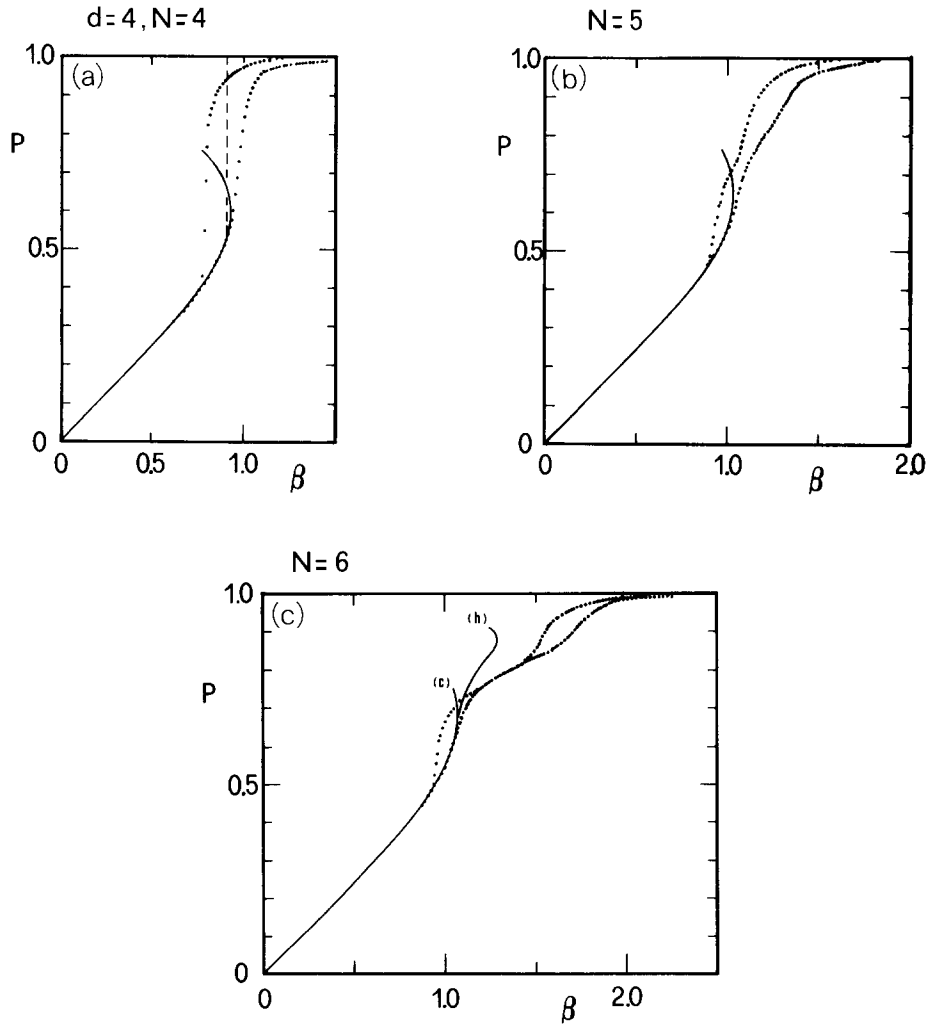


Fig. 6. Mean plaquette in 4d Z(N) gauge theories. MF results (line) and MC hysteresis loop (points): (a) $N = 4$; (b) $N = 5$; (c) $N = 6$; lines (c) and (h) are MF results for the cubic and the hypercubic cluster, respectively.

one at $\beta \simeq 1$ and the other at $\beta \sim O(N^2)$ with scaling coefficient $\alpha = 0.78$ in eq. (2.21). Our results agree with MC results up to $N = 4$. The agreement is worse at the onset of the three-phase structure ($N = 5, 6$), since the singular point β_s should follow the higher transition instead of lying between them. For $N = 6$ the curve (h) shows the improvement of our method on a hypercubic cluster, which will be discussed in sect. 4: it shows that higher order approximations can provide the desired accuracy in the location of β_s. In sect. 4, we also compute $\alpha \sim 0.64$ for the hypercube and verify that still $f_{SC} > f_{WC}$ for $N > 4$.

In fig. 7 we compare the MC data for U(1) (dashed line) with our results around the $\beta \sim O(1)$ transition for $N = 20$ on the cube (curve (c)) and $N = 10$ on the hypercube (h). (These values of N already approximate the large N behaviour for these values of β.) Near the expected transition, a maximum of variation is observed in the plaquette and $\partial b/\partial\beta$ shows a narrow peak rising about an order of magnitude: a comparison with the flat variation in 3d supports the absence of a transition in the latter case.

On the other hand, this transition in 4d will be obtained in subsect. 3.2 by the inclusion of a further consistency equation for the disorder parameter of monopoles.

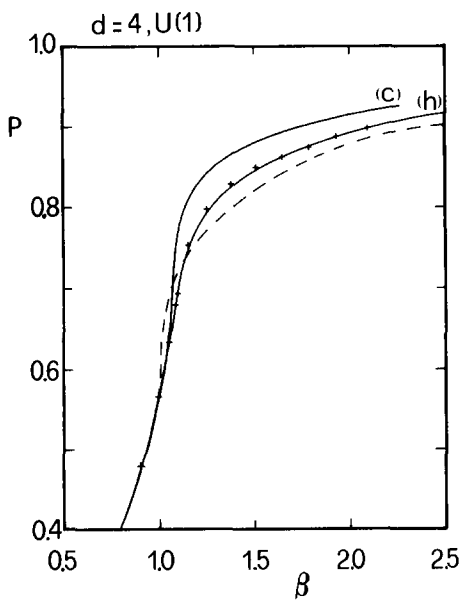


Fig. 7. Mean plaquette in 4d Z(N) gauge theory. $N = 20$ MF on the cube cluster (line (c)); MF on the hypercube (line (h) $N = 10$, with points of the selfconsistent MC for $N = 20$); U(1) MC data (dashed line).

The WC of the plaquette, $P \simeq 1 - 1/\kappa\beta$, is different from the correct value $\kappa = 4$: one obtains $\kappa = 6$ for the cube and $\kappa = \frac{80}{17}$ for the hypercubic clusters. We see that the first correction of our approach tends to match the correct behaviour but is not sufficient.

Let us compare with the results of the MF method for the link variable [7,11]: a first-order transition is obtained at $\beta \sim O(1)$ and a second order one at $\beta \sim O(N^2)$; in SC a high order expansion is required to match the plaquette behaviour and in WC this method yields $\kappa = 3$ without gauge fixing and $\kappa = 4$ in the axial gauge. This gauge dependence is related to the problem of counting the degrees of freedom, since $\frac{1}{2}\kappa$ is equal to the ratio of number of plaquettes to the number of independent link variables [7]: without choosing a gauge, the correct ratio cannot be obtained in the mean link MF. Our cluster method has clearly the same problem: even if it is gauge invariant, the correct counting is not obtained on small clusters.

3. Selfconsistent approximations for abelian monopoles

3.1. INTRODUCTION AND RESULTS FOR THE MONOPOLE DENSITY

In the following, we shall study the monopole excitations which are relevant for the abelian phase transition at $\beta \sim O(1)$.

Monopoles are disorder configurations allowed by the periodicity of the lattice action, which become singular in the continuum limit. In order to detect them, we must look at the period of the plaquette angle $\theta_{\mu\nu}$ as follows [17]. We fix the period of the link, say $-\pi \leqslant \theta_\mu < \pi$, and define a "physical" plaquette $\bar{\theta}_{\mu\nu}$:

$$\bar{\theta}_{\mu\nu} = \theta_{\mu\nu} - 2\pi n_{\mu\nu}, \qquad -\pi \leqslant \bar{\theta}_{\mu\nu} < \pi, \tag{3.1}$$

i.e. $\bar{\theta}_{\mu\nu}$ is small in the continuum limit. The number of monopole m is the magnetic flux out of every cube:

$$2\pi m(x) = \sum \bar{\theta}_{\mu\nu} = \sum_{\rho,\mu,\nu} \varepsilon_{\rho\mu\nu} \Delta_\rho \bar{\theta}_{\mu\nu} = -2\pi \sum_{\rho,\mu,\nu} \varepsilon_{\rho\mu\nu} \Delta_\rho n_{\mu\nu}. \tag{3.2}$$

In eq. (3.2), m is a scalar field in 3d and a conserved vector field in 4d, $2\pi m_\mu = \sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu \bar{\theta}_{\rho\sigma}$; $m, m_\mu = 0, \pm 1, \pm 2$ with these conventions.

This definition of m was used in MC calculations with Wilson action [15,17]: it yields the Coulomb gas representation in the Villain approximation [1,3] and it has a well defined SC behaviour.

The $m(x)$ field appears naturally in the formulation of the theory in terms of independent plaquette variables. In such a formulation, the partition function is

written in the following form*:

$$Z = \int_{\pi}^{\pi} \mathscr{D}\theta_{\mu\nu} \exp\Big(\beta \sum_{(\mu,\nu)} \cos\theta_{\mu\nu}\Big)\Big\{\prod_{c} \delta_{2\pi}\Big(\sum_{\mu,\nu,\rho} \varepsilon_{\mu\nu\rho}\Delta_{\mu}\theta_{\nu\rho}\Big)\Big\}. \tag{3.3}$$

In eq. (3.3) the periodic delta functions $\delta_{2\pi}$ imposes the lattice Bianchi identity $\theta_{\mu\nu} = 0, \pm 2\pi, \ldots$ on every cube c. The 2π solutions of $\delta_{2\pi}$ are clearly related to the values of m in eq. (3.2), therefore we can write

$$Z = \sum_{\{m_{\mu}\}=-\infty}^{+\infty} \int_{-\pi}^{\pi} \mathscr{D}\theta_{\mu\nu} \exp\Big(\beta \sum_{(\mu,\nu)} \cos\theta_{\mu\nu}\Big)\Big\{\prod_{c} \delta_{D}\Big(\sum \theta_{\mu\nu} - 2\pi m\Big)\Big\}, \tag{3.4}$$

where now δ_D is the Dirac delta function. The Coulomb gas representation for the action in the m variable is recovered from eq. (3.3) by a gaussian approximation of the action and a Fourier representation for δ_D in $\delta_{2\pi}(x) = \sum_m \delta_D(x - 2\pi m)$, $m \in \mathbb{Z}$.

The statistical properties of monopoles can be analyzed from the probabilities

$$P(\{m\}) = \frac{1}{Z}\int \mathscr{D}\theta_{\mu\nu} \exp\Big(\beta \sum_{(\mu,\nu)} \cos\theta_{\mu\nu}\Big)\Big\{\prod_{(\mu,\nu)} \delta_{D}\Big(\sum \theta_{\mu\nu} - 2\pi m\Big)\Big\}, \tag{3.5}$$

which are zero if $|m(x)| > 2$ for some x.

The monopole density is the following local quantity:

$$\rho = \tfrac{1}{2}\big\langle |m(0)| \big\rangle = \frac{1}{2Z} \sum_{\{m\}=-\infty}^{+\infty} |m(0)| P(\{m\}), \tag{3.6}$$

where the absolute value is taken because of the symmetry $m \to -m$.

The Coulomb gas representation provides the following SW behaviour

$$\rho \cong \exp(-\lambda\beta) \qquad (\beta \gg 1), \tag{3.7}$$

where $\lambda \cong 6.6$ in 3d [17].

Let us now derive in few steps the SC expansion for ρ. First we insert into eqs. (3.4) and (3.5) the Fourier integrals and series for δ_D and $\delta_{2\pi}$, respectively, and we have

$$\rho = \frac{1}{2} \frac{\displaystyle\sum_{m(0)=-\infty}^{+\infty} |m(0)| \int_{-\infty}^{+\infty} \mathrm{d}p(0) \exp(2\pi i p(0) m(0)) \sum_{\{p \neq p(0)\}=-\infty}^{+\infty} \prod_{(\mu,\nu)} a_{\varepsilon_{\mu\nu\rho}\Delta_{\rho}p}(\beta)}{\displaystyle\sum_{\{p\}=-\infty}^{+\infty} \prod_{(\mu,\nu)} a_{\varepsilon_{\mu\nu\rho}\Delta_{\rho}p}(\beta)} \tag{3.8}$$

* See e.g. [18] or use duality forward and backward.

The following formula for the Fourier coefficients is useful for non-integer argument ν:

$$a_p(x) = \sum_{n=-\infty}^{+\infty} I_n(x) \frac{\sin \pi(n+p)}{\pi(n+p)} . \tag{3.9}$$

The SC diagrammatic rules are obtained by associating to every plaquette the weights I_n, which have a source n of "momentum" p along the dual link and a "wave function" $(\sin \pi(p_x - p_{x+\mu} - n))/\pi(p_x - p_{x+\mu} - n)$. For integer values of p, say at the denominator of eq. (3.8), the wave function is a delta function and momentum is conserved on every site: the expansion of Z in closed surfaces of plaquettes is recovered [7]. For continuous p there are overlappings of wave functions with different p's: it follows that the numerator of eq. (3.8) may have open surfaces at the cube $x = 0$. We obtain to the first order

$$\rho = c_0 + \beta c_1 + O(\beta^2), \tag{3.10}$$

where $c_0 = \frac{7}{30}$ and $c_1 = (3 + 4\pi^2)/4\pi^4 \simeq 0.109$ in $d = 3$ (see table 1).

The 4d partition function with explicit monopole field $m_\mu(x)$ is, like eq. (3.4),

$$Z = \sum_{\{m_\mu\}=-\infty}^{+\infty} \int_{-\pi}^{\pi} \mathcal{D}\theta_{\mu\nu} \exp\Big(\beta \sum_{(\mu,\nu)} \cos\theta_{\mu\nu}\Big) \Big\{ \prod_c \delta_D\Big(\sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu \theta_{\rho\sigma} - 2\pi m_\mu \Big) \Big\} . \tag{3.11}$$

The monopole charge density is defined $\rho = \frac{1}{2}\langle |m_\mu| \rangle$ as in eq. (3.6) and a similar SC expansion holds: the result to the first order is again eq. (3.10).

Let us now discuss the cluster approximation for the monopole density ρ within the methods of the previous section. In 3d, the one-cube cluster partition function with monopole number m is (see eqs. (2.14), (3.4))

$$Z_\Lambda = \sum_{m=-2}^{2} \int_{-\pi}^{\pi} \Big\{ \prod_{(\mu,\nu)=1}^{6} \mathrm{d}\theta_{\mu\nu} \Big\} \exp\Big((\beta + b(\beta)) \sum_{(\mu,\nu)} \cos\theta_{\mu\nu} \Big) \delta_D\Big(\sum \theta_{\mu\nu} - 2\pi m \Big) \tag{3.12}$$

and the cluster monopole density is, by eq. (3.8),

$$\rho_\Lambda = \frac{1}{2Z_\Lambda} \sum_{m=-2}^{2} |m| \int_{-\infty}^{+\infty} \mathrm{d}p \, \mathrm{e}^{i2\pi m p} a_p^6(\beta + b(\beta)) . \tag{3.13}$$

The 4d case (one-cube cluster) is obtained by replacing $b \to 3b$ in eqs. (3.12), (3.13). By repeating the SC analysis, we see that the cluster monopole density is

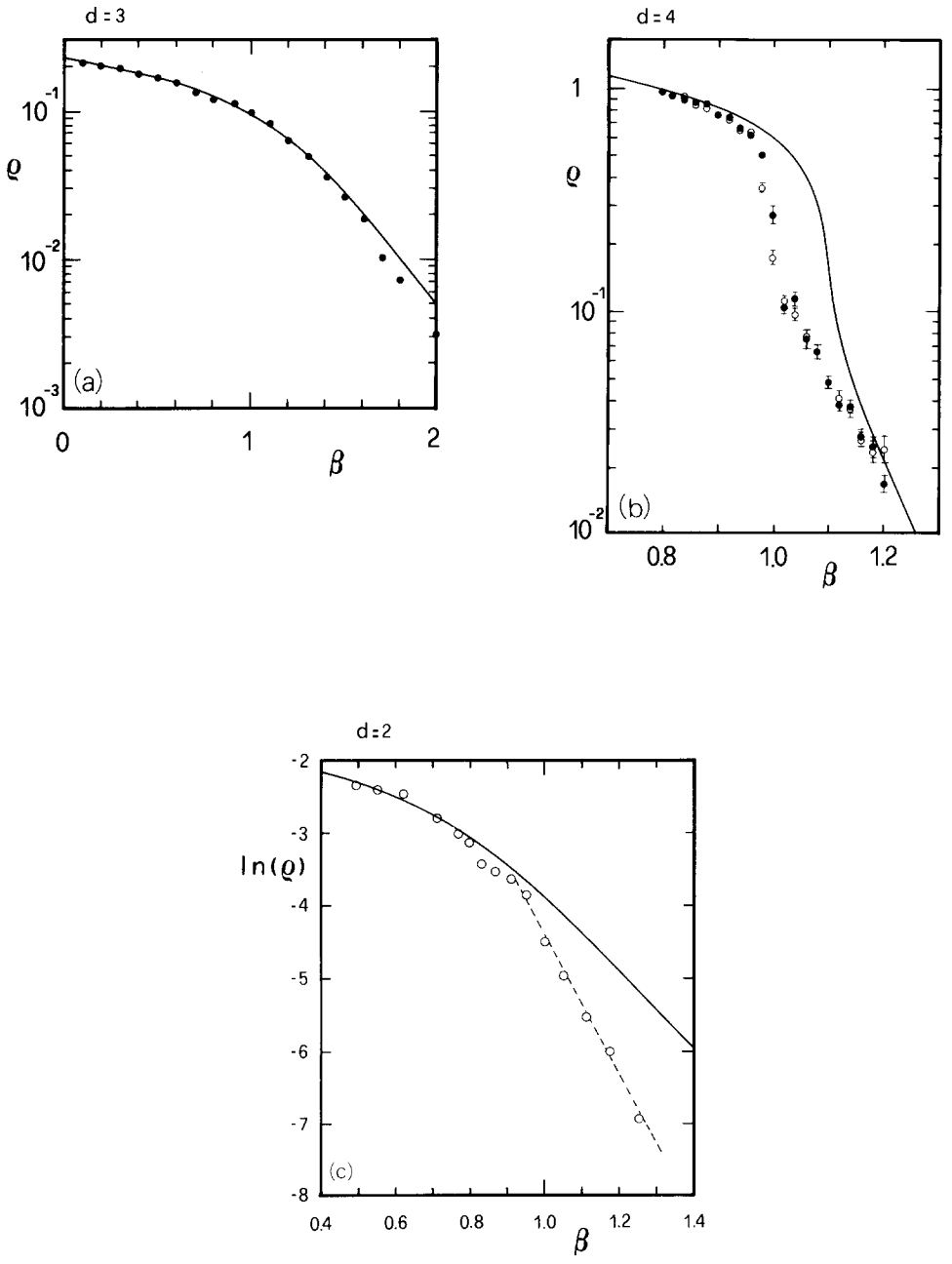


Fig. 8. Monopole density. MF results (lines) and MC results (points): (a) monopole density in 3d gauge theory; (b) monopole loops length density in 4d; (c) vortex density, with Coulomb gas approximation (dashed).

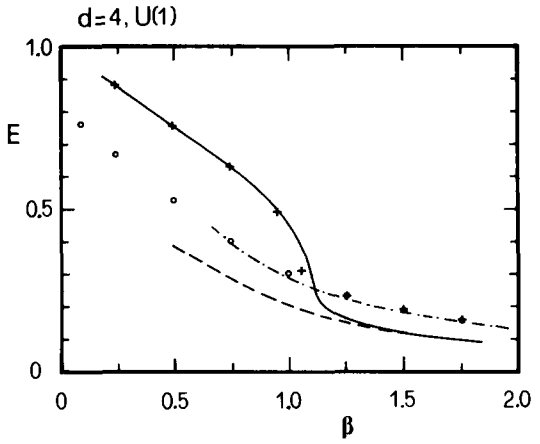


Fig. 9. Internal energy in 4d U(1) gauge theory. MF results with (line) and without (dashed line) monopoles; MC results with (crosses) and without (open points) monopoles; SW expansion (pointed-dashed line).

correct up to $O(\beta^7)$: higher order terms are partially resumed by our method, as well as for the plaquette expectation value.

The results for ρ_Λ are reported in fig. 8a and b for $d = 3, 4$ and compared with MC data* [17] (in 4d we report $8\rho_c$ for comparison with data referring to the total length of monopole loops). The data are well reproduced across the transition, in particular the higher suppression of monopoles in 4d. The WC exponential behaviour in eq. (3.7) is also obtained in our approach: the values of λ are lower than the Coulomb gas estimate (see table 1), because the $m - \bar{m}$ pairs are less suppressed on the small volume of the cluster. This effect is larger for the vortex density of the 2d spin model, fig. 8c, obtained on a one-plaquette cluster by similar methods.

A complete description would have provided a singularity in ρ_Λ at the phase transition occurring in the 4d gauge theory. However, it should be pointed out that the monopole density is not a disorder parameter of this phase transition: from eqs. (3.6) and (3.11) we see that it takes contributions not only from configurations of monopole loops with infinite length which condense at the transition [1], but also from loops of small length which are allowed in SW and provide the exponential behaviour in eq. (3.7). In the next section, a disorder parameter for the condensation of monopole loops of infinite length will be presented.

As a final exercise on the monopole density, we shall repeat the analysis of Barber et al. [19] on the fate of the phase transition when monopoles are excluded from the theory. By a MC simulation with the constraint $\{m_\mu(x) = 0, \forall x\}$ in eq. (3.11), they

* Our numerical evaluation of eq. (3.13) is done by approximating U(1) by Z(29) and the integral in p is evaluated exactly by a series of values $p = n/\kappa$, which amounts to replacing $\delta_D(x)$ by $\delta_{2\pi\kappa}(x)$ in eq. (3.12) for x values not exceeding the period $2\pi\kappa$.

observed that the internal energy follows the SW behaviour $E = 1 - P \simeq 1/4\beta$ well below the transition and goes smoothly to a finite value in the SC limit.

Fig. 9 reports E with and without constraints, for the MC and for our cluster method: the same qualitative behaviour is observed, even if our method is not quantitatively correct in WC ($E \sim 1/6\beta$, see subsect. 2.3). The constrained MF curve dashed stops at $\beta \sim 0.5$, where the consistency equation looses the solution. Let us observe that in the unconstrained theory a SC analysis of the consistency equation shows that a solution $b(\beta)$ always exists. However, in the constrained theory the action is no more periodic and then the SC expansion does not holds.

3.2. SECOND-ORDER PHASE TRANSITION BY MONOPOLE CONDENSATION IN 4d

The selfconsistent approach for monopole condensation will be derived by the same method as in subsect. 2.1; we shall follow two steps:

(i) a selfconsistent approach will be applied to the order parameter of the $\beta \sim O(N^2)$ transition, by repeating the analysis by Alessandrini [11];

(ii) the selfconsistent approach for the disorder parameter related to monopole condensation will be obtained by exploiting the selfduality of $Z(N)$ gauge theories in 4d.

The results of the MF for the link variable [7] can be obtained by a Bethe-Peierls approach on the cluster in fig. 10, with action (see I also)

$$A = \beta \sum_{\mu} \cos\left(\theta_\mu + \theta_\nu + \theta'_\mu + \theta'_\nu\right) + b \sum_{\mu} \left(\cos\theta_\mu + \cos\theta'_\mu + \cos\theta'_\nu\right), \qquad (3.14)$$

for a $Z(N)$ model (N is assumed to be large), $\theta = 2\pi n/N$, $n = 0, \ldots, N-1$, and the summation over $\mu = \pm 1; \pm 2, \pm 3$, $\mu \perp \nu$, $\nu = 4$, is employed from now on. The consistency equation between the internal and external mean links is

$$\langle \cos\theta_\nu \rangle_{\beta, b} = \langle \cos\theta'_\nu \rangle_{\beta, b} \qquad (3.15)$$

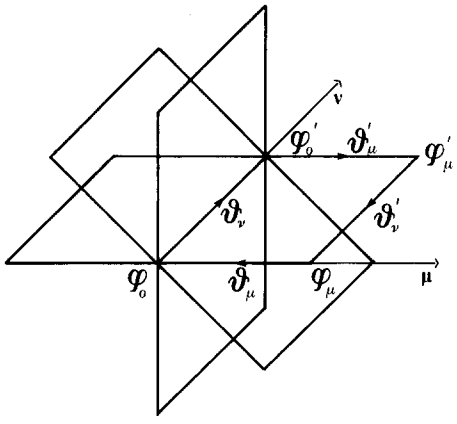


Fig. 10. 4d gauge theory cluster for studying fluctuations in the Coulomb phase (see the text).

and it yields a first-order phase transition at $\beta \sim O(1)$. It can be shown that it reduces to the consistency equation by functional MF methods [7] provided that $6 \to 2(d-1)$ and $d \to \infty$ with $\beta d \sim O(1)$.

Let us now recall the MF analysis of the $\beta \sim O(N^2)$ transition which was done in ref [11]. In the SW (Coulomb) phase of $Z(N)$ theories, transverse fluctuations are dynamically massless, i.e. the photon mass is zero; this happens also in the U(1) theory by continuous gauge invariance. Therefore, in ref. [11] U(1) gauge transformations were explicitly considered in $Z(N)$ theories through the introduction of a continuous field; the order parameter was identified in the expectation value of this field, because it vanishes for massless gauge fluctuations and it is non-zero for massive fluctuations above the phase transition.

In order to repeat this analysis by our cluster methods, let us consider a U(1) gauge transformation of the θ_μ field in the action eq. (3.14):

$$\theta_\mu \to \theta_\mu + (\phi_0 - \phi_\mu). \tag{3.16}$$

In eq. (3.16), the relevant values of ϕ are in the interval $(0, 2\pi/N)$, due to $Z(N)$ gauge invariance: it is convenient to rescale $\phi = \varphi/N$ and have $0 < \varphi < 2\pi$. According to ref. [11], massless transverse fluctuations will be present if the theory is globally U(1) invariant in the "Higgs" field φ.

A further MF analysis of this "Higgs" model can be done by including source terms for the field φ in the action eq. (3.14)

$$A(\theta_\mu) \to A\left(\theta_\mu + \frac{1}{N}\Delta_\mu \varphi\right) + y \sum_\mu (\cos \varphi_\mu + \cos \varphi'_\mu) \tag{3.17}$$

and studying the consistency equation

$$\langle \cos \varphi_0 \rangle_{\beta, b, y} = \langle \cos \varphi_\mu \rangle_{\beta, b, y}, \tag{3.18}$$

where $\mu = 1$, for example. Expectation values are now taken as follows

$$\langle (\cdot) \rangle_{\beta, b, y} = \frac{\int \mathscr{D}\varphi \sum_{\{\theta_\mu\}} e^{A(\theta, \varphi)} (\cdot)}{\int \mathscr{D}\varphi \sum_{\{\theta_\mu\}} e^{A(\theta, \varphi)}}. \tag{3.19}$$

In eqs. (3.17), (3.18) the form of the source term and of the order parameter is given by U(1) invariance. The value of β_c at the expected second-order phase transition is obtained by expanding eq. (3.18) to $O(y)$. For an estimate of the effective action for φ we may take $\{\theta_\mu\} = 0$:

$$e^A \sim \prod_\mu \left\{ \left[\left(I_0(b) + 2 I_N(b) \cos(\varphi_0 - \varphi_\mu) \right) e^{y \cos \varphi_\mu} \right] [\varphi \to \varphi'] \right\}. \tag{3.20}$$

To O(y) we get the equation $[I_N(b_c)/I_0(b_c)] \sim \exp(-N^2/2b_c) = \frac{1}{5}$; since $\beta \sim \frac{1}{2}b$ from the WC of eq. (3.15), we have $\beta_c \sim O(N^2)$. Clearly, in eq. (3.20) the φ field has $I_N(\beta)$ coupling because it is the Fourier amplitude for O($2\pi/N$) fluctuations of the θ_μ field.

Let us now derive the dual formulation. In performing the duality transformation on the θ_μ variables in eq. (3.17), we follow the same steps as in subsect. 2.1 but we do not discard the modulo N periodicity of the $Z(N)$ delta conditions (see eq. (2.10)).

The dual partition function can be written as follows:

$$
\begin{aligned}
Z_{\Lambda^*} = \int_0^1 \mathscr{D}\lambda \sum_{\{m_\mu, m'_\mu, m'_\nu, s_\mu, s'_\mu\} = -\infty}^{+\infty} \sum_{m_\nu = -\infty}^{+\infty} \sum_{\{\theta^*_{\alpha\delta}\}} \delta\Big(\sum_{\mu,\alpha,\delta} \varepsilon_{\nu\mu\alpha\delta} \Delta_\mu \theta^*_{\alpha\delta} - 2\pi m_\nu \Big)
\\
\times \exp\Big\{ \tfrac{1}{2} - \beta^* \sum_{(\alpha,\delta)} (\theta^*_{\alpha\delta})^2
\\
- \tfrac{1}{2} b^* \sum_{(\alpha,\delta)} \Big[(\theta^*_{\alpha\delta} + 2\pi m_\mu)^2 + (\theta^*_{\alpha\delta} + 2\pi m'_\mu)^2 + (\theta^*_{\alpha\delta} + 2\pi m'_\nu)^2 \Big] \Big\}
\\
\times \exp\Big\{ i\lambda_0 \Big(2\pi m_\nu + 2\pi \sum_\mu m_\mu \Big) - i\lambda'_0 \Big(2\pi m_\nu + 2\pi \sum_\mu m'_\mu \Big) \Big\}
\\
\times \exp\Big\{ \sum_\mu \Big[i2\pi\lambda_\mu (m_\mu - m'_\nu - s_\mu) + i2\pi\lambda'_\mu (m'_\nu - m'_\mu - s'_\mu) - \tfrac{1}{2} y^* \Big(\frac{2\pi}{N} \Big)^2 (s_\mu^2 + s_\mu'^2) \Big] \Big\}
\end{aligned}
\tag{3.21}
$$

On the dual one-cube cluster Λ^*, $\{\lambda\} = \{\varphi/2\pi\}$ are variable on hypercubes (dual sites), the $\{m_\mu, m'_\mu, m'_\nu\}$ and m_ν are integer variables on border cubes (dual border links) accounting for N periods, as in eq. (3.12), and the $\theta^*_{\alpha\delta}$ are $Z(N)$ plaquette variables, $\theta^* = 2\pi n/N$, $n = 0, \ldots, N-1$. The dual couplings are $x^* = (1/x)(N/2\pi)^2$, $x \equiv \beta, b, y$. According to the discussion of the previous section, the $\{m_\mu\}$ variables are interpreted as monopole numbers, like m_ν, on border cubes and the λ_0, λ'_0 integrations impose the conservation law $\Sigma_\mu m_\mu = 0$ on each of the two hypercubes containing the one-cube cluster.

The dual of the order parameters in eq. (3.18) add to the partition function (3.21) a unit of monopole number in one hypercube and in one of the neighbours,

respectively:

$$\mathrm{Re}\langle e^{i2\pi\lambda_0}\rangle_{\beta,b,y} = \frac{Z_{\Lambda^*}\{(\Sigma_\mu m_\mu)\to(1+\Sigma_\mu m_\mu)\}}{Z_{\Lambda^*}} = \langle v_0\rangle_{\beta^*,b^*,y^*}, \tag{3.22a}$$

$$\mathrm{Re}\langle e^{i2\pi\lambda_\mu}\rangle_{\beta,b,y} = \frac{Z_{\Lambda^*}\{(m_\mu - m'_\nu)\to(m_\mu - m'_\nu + 1)\}}{Z_{\Lambda^*}} = \langle v_\mu\rangle_{\beta^*,b^*,y^*}. \tag{3.22b}$$

As for frustrations in subsect. 2.1, the disorder parameter $\langle v_0\rangle$ in eq. (3.22a) is the probability of configurations which have a topological excitation added to the vacuum by modifying the action. Again, we can use a Peierls argument to have same indications. On a large cluster Λ^* the field λ will couple only at the border, by gauge invariance: we remove the y^*, b^* sources in eqs. (3.21), (3.22) and consider the border-to-border correlation function

$$\begin{aligned}\mathrm{Re}\langle\exp(i2\pi(\lambda(x)-\lambda(y)))\rangle_{\beta^*} = \frac{1}{Z}\sum_{\{\theta^*_{\alpha\delta}\}} e^{A(\theta^*_{\alpha\delta})}\Big\{\prod_c \delta_D\Big(\sum\theta^*_{\alpha\delta} - 2\pi m_\mu\Big)\Big\} \\ \times\Big\{\prod_{z\neq x,y}\delta_D\big(\Delta_\mu m_\mu(z)\big)\Big\}\delta_D\big(\Delta_\mu m_\mu(x)+1\big) \\ \times\delta_D\big(\Delta_\mu m_\mu(y)+1\big),\end{aligned} \tag{3.23}$$

where x, y and z are sites on the border of Λ^*. The delta functions in eq. (3.23) induce a monopole loop through the cluster, joining the two boundary points x and y. Above the transition, in the SW phase, $\mathrm{Re}\langle\exp\{i2\pi(\lambda(x)-\lambda(y)\}\rangle \sim \exp(-\Delta F)$, where ΔF is the free energy of an isolate monopole loop: it can be estimated as

$$\Delta F \cong (2\pi^2\beta^* g(0) - \log 7)L \qquad (L\to\infty), \tag{3.24}$$

by repeating the evaluation shown in ref. [3] (Villain approximation); L is of the order of the linear length of the cluster and $g(0) = 0.155$ is the lattice gaussian propagator at the origin.

In the limit $L\to\infty$, the estimate in eq. (3.24) and the assumed factorization of the correlation (3.23) lead to a non-zero disorder parameter $\mathrm{Re}\langle\exp(i2\pi\lambda(x))\rangle_{\beta^*} \approx \langle v_0\rangle_{\beta^*}$ below the transition $\beta^* < \beta_c^*$ ($\beta_c^* \simeq 0,6$), due to the condensation of infinite monopole loops.

Let us further observe that in the Coulomb phase the inclusion of continuous fluctuations in the Fourier conjugate variables simply means that the action is no more periodic in its $\theta_{\mu\nu}$ variables: their 2π jumps enter in the dynamics, as we have seen in discussing the monopole density in the previous subsection.

Let us go further and compute the prediction for the second-order transition coupling β_c; an analytical computation is possible in the Villain approximation.

From eqs. (3.15) and (3.18), we can solve the following system

$$\begin{cases} \langle \cos \theta_\nu \rangle_{\beta, b, y} - \langle \cos \theta'_\nu \rangle_{\beta, b, y} = 0, & y \equiv 0 \\ \langle \cos \varphi_0 \rangle_{\beta, b, y} - \langle \cos \varphi_\mu \rangle_{\beta, b, y} = 0, & \mathrm{O}(y), \end{cases} \tag{3.25}$$

and use the duality relation for the β_c solution. The first equation provides the SW behaviour $\beta \sim \frac{1}{2}b$. The second equation must be evaluated to order $\mathrm{O}(y)$: performing all gaussian integrations we obtain

$$-5y^3 - 5y^{25/8} + 6y^2 + 5y^{13/8} - 1 = 0, \tag{3.26}$$

where

$$y = \exp\left(-\frac{N^2}{9\beta} \right) \simeq \exp\left(-\beta^* \cdot \tfrac{4}{9}\pi^2 \right) \simeq 1 - \beta^* \cdot \tfrac{4}{9}\pi^2 .$$

It yields $\beta_c^* \simeq 0.16$, a value four times lower than the MC estimate $\beta_c^* \sim 0.66$ for the Villain action, at the same level of accuracy of functional methods [11].

In order to study the Wilson action, it is not sufficient to replace the cosines in the action of eq. (3.21), because of the b^* border terms. According to the discussion in the previous section, this problem is solved by replacing the border plaquettes by cubes, whose monopole number has the desired value. The replacement in eq. (3.21) is as follows:

$$\exp\left\{ -\tfrac{1}{2}b^*(\theta^*_{\sigma\tau} + 2\pi m_\alpha) \right\}$$

$$\to \int \left\{ \prod_{\substack{(\mu,\nu)\neq(\sigma,\tau) \\ (\mu,\nu)\in c}} \mathrm{d}\theta^*_{\mu\nu} \right\} \exp\left(\sum_{(\mu,\nu)\neq(\sigma,\tau)} (\beta^* + 3b^*)\cos\theta_{\mu\nu} \right) \delta_{\mathrm{D}}\left(\sum \theta_{\mu\nu} + \theta_{\sigma\tau} - 2\pi m_\alpha \right) \tag{3.27}$$

The system (3.25) in the dual case is analyzed numerically and the critical coupling is found to be $\beta_c^* \sim 0.53$ for the Wilson action, about a factor 2 smaller than the MC estimate $\beta_c^* \sim 1$. The SW behaviour is maintained up to the transition point.

In conclusion, a selfconsistent approximation has been found for the disorder parameter related to monopole loops condensation in 4d which predicts a second-order transition. The poor β_c^* value is possibly due to the mean field estimate we have done of fluctuations. The inclusion, in the same scheme, of a computational method more sensitive to fluctuations could improve this result.

4. Inclusion of corrections: the selfconsistent Monte Carlo

In this section we shall discuss briefly the inclusion of corrections in the approach of sect. 2, for 4d gauge theory. We shall present numerical MC results on the hypercubic cluster, as a trial for larger clusters. Similar numerical extensions of the approach presented in sect. 3 are possible.

The hypercubic cluster action is (see I also)

$$A = (\beta + 2b) \sum_{(\alpha,\delta)=1}^{24} \cos\theta_{\alpha\delta} . \tag{4.1}$$

According to the discussion of subsect. 2.3, the consistency equation involves two-cube frustrations, i.e. frustrations on paths of the maximal length allowed by the cluster. We have

$$\langle \mathscr{F}_\nu \rangle_{\beta,b} = \langle \mathscr{F}_\nu' \rangle_{\beta,b} . \tag{4.2}$$

The internal and external frustrations are respectively,

$$\langle \mathscr{F}_\nu \rangle_{\beta,b} = \left\langle \exp\left\{ (\beta + 2b)\left[T(\theta_{\mathrm{p}}) + T(\theta_{\bar{\mathrm{p}}}) \right] \right\} \right\rangle_{\beta,b}, \tag{4.3a}$$

$$\langle \mathscr{F}_\nu' \rangle_{\beta,b} = \left\langle \exp\left\{ b\left[T(\theta_{\mathrm{p}}) + T(\theta_{\bar{\mathrm{p}}}) \right] \right\} \right\rangle_{\beta,b}, \tag{4.3b}$$

where $T(\theta) = \cos(\theta - 2\pi/N) - \cos\theta$ and $\mathrm{p}, \bar{\mathrm{p}}$ are a plaquette for each cube along the dual direction ν. By Fourier expansion, the sum over configurations reduces to the eight links joining at the site dual to the hypercube, which are labelled q_μ, $\mu = \pm 1, \ldots, \pm 4$. For example, one has for the partition function

$$Z = \sum_{\{q_\mu\}=0}^{N-1} \left\{ \prod_{(\mu,\nu)=1}^{24} a_{\mathrm{p}_\mu - \mathrm{p}_\nu}(\beta + 2b) \right\}, \tag{4.4}$$

where the product runs over the (μ, ν) dual to the plaquette (α, δ) of the hypercube in eq. (4.1).

Analytical calculations and checks can be done in the Coulomb phase: the gaussian approximation for expectation values has the form

$$\begin{aligned} \langle \mathscr{O} \rangle &= \sum_{\{q_\mu\}} \exp\left\{ -\frac{1}{2(\beta + 2b)} \sum_{\mu,\nu} q_\mu A_{\mu\nu} q_\nu + i \sum_\mu J_\mu q_\mu - c \right\} \\ &= \exp\left\{ -c - \tfrac{1}{2}(\beta + 2b) \sum_{\mu,\nu} J_\mu (A^{-1})_{\mu\nu} J_\nu^* \right\}, \end{aligned} \tag{4.5}$$

where J_μ and c depends on $\mathcal{O}$. For example, eq. (4.5) allows computing the behaviour of P quoted in subsect. 2.3.

Numerical evaluation of eq. (4.4) requires N^7 cycles at least; therefore for large N it is useful to set up a MC integration routine.

This procedure is implemented by the Metropolis algorithm [6] applied to action in eq. (4.4) and it is an extension of the selfconsistent MC of ref. [12]. It computes expectation values as functions of (β, b) and looks for the solution of eq. (4.2): at fixed b it is convenient to solve eq. (4.2) in β by repeated hints β_i, according to

$$\beta_{i+1} = \beta_i \left(1 - \frac{\langle \mathscr{F}_\nu' \rangle - \langle \mathscr{F}_\nu \rangle}{|\langle \mathscr{F}_\nu \rangle|} \frac{(S-1)}{S} \right). \tag{4.6}$$

Eq. (4.6) follows by proportionality arguments and $\partial x / \partial b < 0$, where $x = \langle \mathscr{F}_\nu' \rangle - \langle \mathscr{F}_\nu \rangle$; S is a significance parameter for x, $S = |x|/\sigma_x$ and σ_x is its standard deviation. Expectation values in eq. (4.6) are averages of the MC simulation, which we stop when $S > 1$, i.e. when the sign of x is determined within one standard deviation; the upgrading of β completes a cycle of this procedure. It follows that after an initial drift the β_i values oscillates around the solution of $x(\beta(b), b) = 0$ and a mean value of β_i is taken for $\beta(b)$.

The result for the plaquette at $N = 30$ are the crosses in fig. 7 superimposed to the $N = 10$ exact curve, which should differ very little for these values of β. Since the summation variables are dual, the MC converges better in WC than in SC: near the crossover, a 3σ statistical error $< 1\%$ is obtained by 20 minutes of CPU time of a VAX 750, while in SW 1‰ error can be obtained.

In general, we can say that the computation of one value of the consistency function x requires 10^2 more time than an expectation value O(1) with the same accuracy.

Moreover, frustrations have modified actions which take large contributions from configurations of small probability. Therefore, the range for a new value in the Metropolis upgrading must not be limited around the old value, because such a constraint on tunnelling of configurations may lead a convergence to a wrong equilibrium state.

5. Conclusion

In this paper, we have obtained good phenomenological descriptions of the two kinds of topological excitations which exist in $Z(N)$ gauge theories: in sect. 2 our selfconsistent approach has been applied to string excitations and in sect. 3 it has been extended to monopoles.

Our local approximation has clarified that these excitations are related to local operators in 3d, while in 4d there exist good local "representatives" of the extended

operators. Therefore it is possible that more refined theoretical methods can apply successfully, and work in that direction is required.

The monopole charge density in subsect. 3.1 admits also a local approximation. This, in turn, indicates that correlation functions of m variables or disorder correlation functions as eq. (3.23) can be better suited for numerical simulation around the transition, since they can better show the relevant dynamics of long monopole loops.

A practical implementation of these ideas has been presented in sect. 4: the selfconsistent MC method on a larger cluster can provide a valid alternative to MC with periodic boundary conditions. The selfconsistent border gives a tree-like structure to the finite cluster which is rather efficient in reproducing thermodynamical properties of larger lattices. Furthermore, the open boundary conditions do not induce finite size effects related to topological excitations on a torus, like monopole loops and string-sheets winding around the lattice, which hide the characters of the deconfining transition (see e.g. ref. [16]). Attempts in this direction are under study [20].

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