Noncommutative Field Theory

and

Quantum Hall Effect

(A. Cappelli, M. Riccardi, hep-th/0410151)

Outline

- Intro: Laughlin wave function, CFT, etc.
- Incompressibility and W_∞ symmetry
- NC Chern-Simons theory (Susskind (01))

  & Chern-Simons Matrix Model (Polychronakos)
- Complex quantization of CSMM:

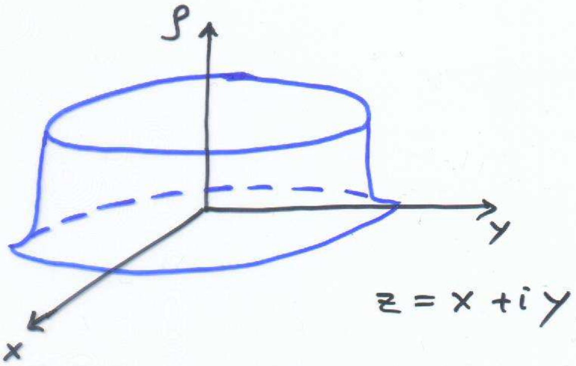
  → Laughlin w.f., QHE, etc.

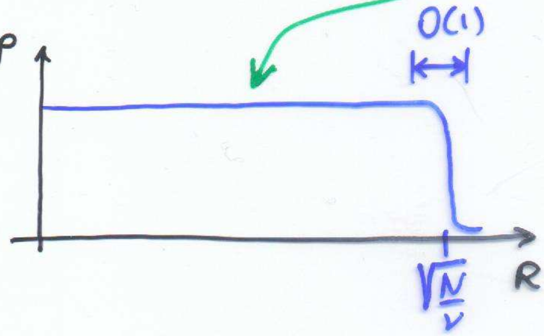
  → "non-relativistic effective field theory"
- Hopes

Laughlin's quantum incompressible fluid

Electrons form a droplet of liquid without sound waves

$$\begin{cases} \text{Incompressible} \equiv \text{density waves have a gap} \\ \text{Fluid} \equiv \rho(\vec{x}) = \rho_0 = \text{const.} \end{cases}$$





A = area of the droplet

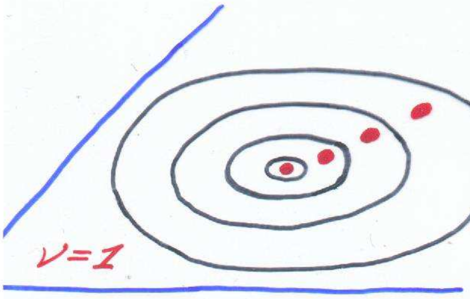
N = # of electrons

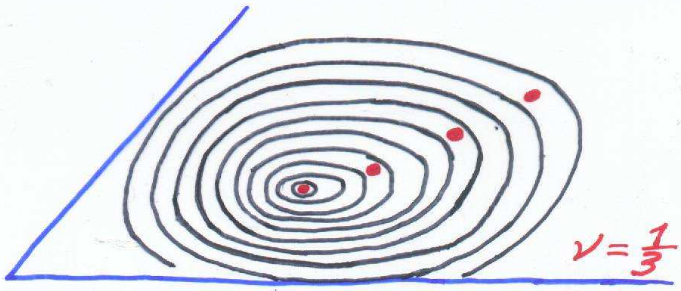
$\mathcal{D}_A = \dfrac{BA}{\frac{hc}{e}}$ = # of degenerate Landau orbitals

$\rho = \dfrac{N}{A}$ = electron density

$\nu = \dfrac{N}{\mathcal{D}_A} = \dfrac{N}{BA/\Phi_0}$ = filling fraction $= 1, \frac{1}{3}, \frac{1}{5}, \dots$

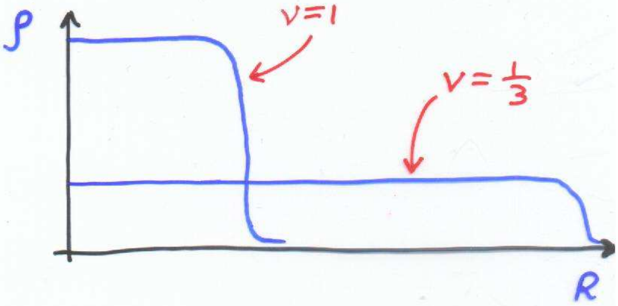
= density for quantum-mech. problem





- Laughlin's trial wave function $\nu = \frac{1}{2k+1} = 1, \frac{1}{3}, \frac{1}{5}, \dots$

$$\Psi_{g.s.}(z_1, \dots, z_N) = \prod_{1=i<j}^{N} (z_i - z_j)^{2k+1} \; e^{-\sum |z_i|^2/2\ell^2}$$

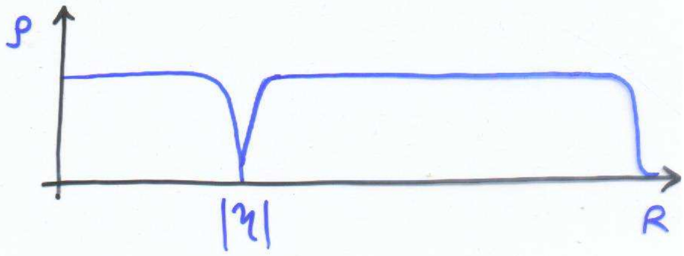


- $\nu = 1$
  obvious
  filled Landau level
  gap $= \omega_c = \frac{eB}{mc} \gg k_B T$

- $\nu = \frac{1}{3}$
  highly non-trivial
  due to repulsive electron-electron interaction
  gap $= O(\frac{e^2}{\ell})$ $\qquad \ell = \sqrt{\frac{2\hbar c}{eB}}$ "magnetic length"

- quasi-hole excitation $\approx$ vortex

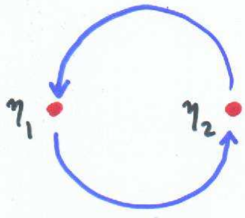
$$\Psi_{q\text{-}h}(\eta; z_1, \dots, z_N) = \prod_{i=1}^{N} (\eta - z_i) \prod_{i<j} (z_i - z_j)^{2k+1} \; e^{-\sum |z_i|^2/2\ell^2}$$



- $\nu = \frac{1}{2k+1}$ it has fractional charge $\qquad Q = \frac{e}{2k+1}$

  and fractional statistics $\qquad \frac{\theta}{\pi} = \frac{1}{2k+1}$

$$\Psi_{2q\text{-}h}(\eta_1, \eta_2; z_1, \dots z_N) = (\eta_1 - \eta_2)^{\frac{1}{2k+1}} \prod_i (\eta_1 - z_i) \prod_i (\eta_2 - z_i)\, \Psi_{g.s.}$$



$$\Psi\left(\eta_1 - \eta_2 \to e^{i\pi}(\eta_1 - \eta_2)\right) = e^{i\frac{\pi}{2k+1}}\, \Psi(\eta_1, \eta_2)$$

Fractional statistics $\frac{\theta}{\pi} = \frac{1}{2k+1} = \frac{1}{3}, \frac{1}{5}, \dots$

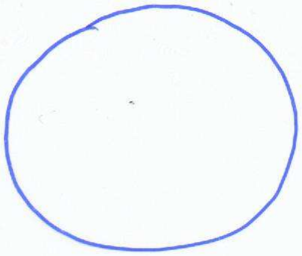
- quasi-hole is an <u>anyon</u> (Wilczek etal.)

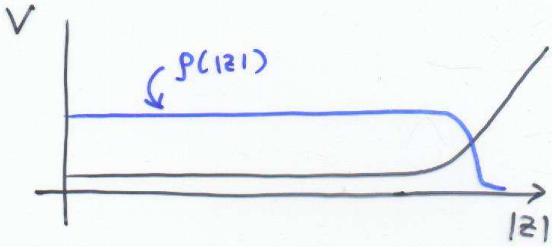
- fractional statistics is a long-range correlation of vortices, independent of distance, i.e. "topological"

- both fractional charge & statistics are nicely modelled by Chern-Simons gauge theory and conformal field theory

Edge excitations of the incompressible fluid

The sample has a boundary, with a confining potential; take e.g. a disk:

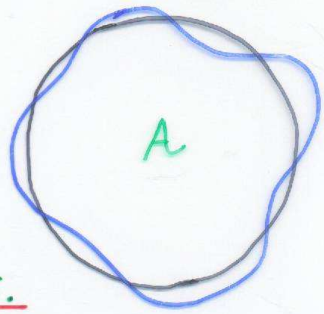




The incompressible fluid satisfies $\rho(|z|) = \rho_0$, i.e. (2+1)-dimensional waves have a high gap and can be neglected.
But:

- the boundary shape can fluctuate:

→ "neutral" edge excitations



- almost gapless
- excitations satisfy $A = \text{const.}$

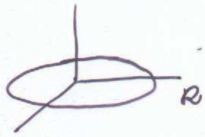
$$N = \int d^2x\, \rho(x) = \rho_0 \cdot A \quad , \quad N, \rho_0 = \text{const.}$$

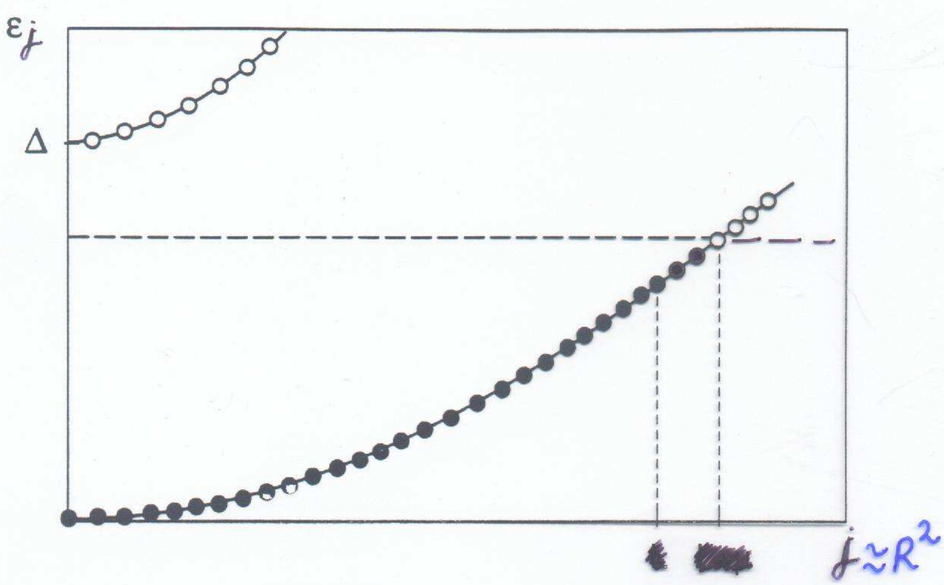
→ area-preserving diffeomorphisms of the plane
$\equiv W_\infty$ symmetry (my work with C. Trugenberger & G. Zemba (92))

- $\nu = 1$ obvious:
the filled Landau level is like a Fermi sea
edge excitations = particle-hole excitations at the Fermi surface

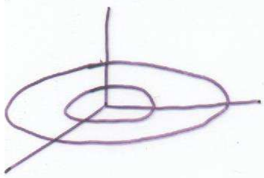
The $\nu=1$ quantum incompressible fluid is like a Fermi sea in coordinate space

Disk

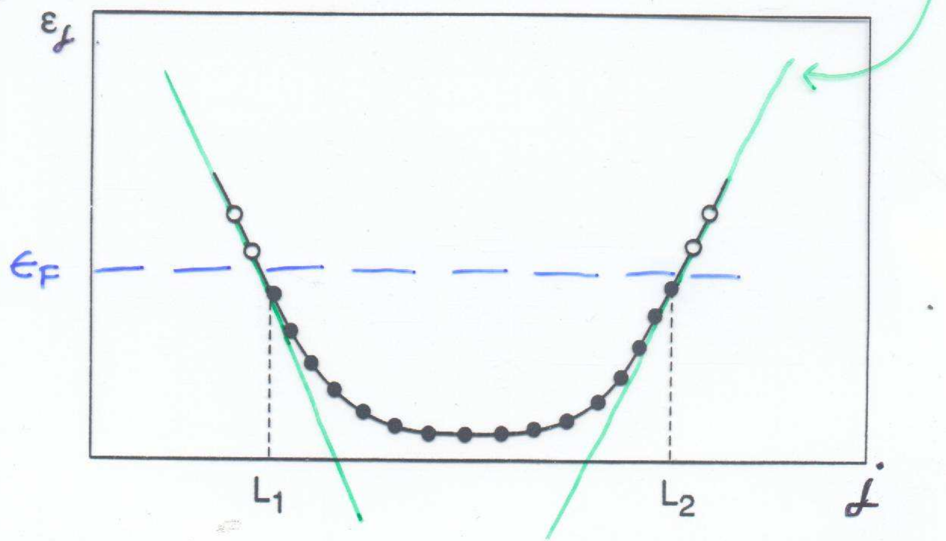




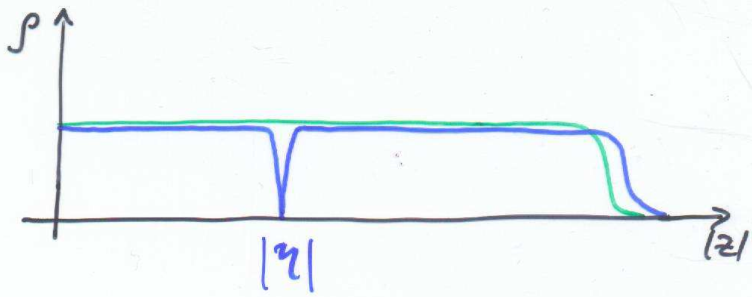
Annulus



linear approx. at the boundary
$\epsilon_j - \epsilon_F \sim v_F(j - j_F) \propto k_{boundary}$
relativistic spectrum



- charged quasi-hole excitation also observed at the edge



depleted density is spilled at the edge
$\longrightarrow$ charged excitation at the edge

Conclusion: both excitations of the incompressible fluid can be detected at the edge $z = Re^{i\theta}$

Idea: conformal field theory description of edge excitations $(\xi^1, \xi^2) = (t, R\theta)$

chiral Luttinger theory $(c=1, \bar{c}=0)$

(compactified chiral boson)

Main features

- effective description valid at low energy, it does explain the conduction experiments ('97, '98)

- spectrum at $\nu = 1/3$:

$$Q = \frac{n}{3} \qquad n = \pm 1, \pm 2, \ldots \qquad \text{fractional charge}$$

- vertex operators $:e^{in\varphi}:$ anyons at the edge

$$\langle :e^{i\varphi(z_1)}::e^{i\varphi(z_2)}: \rangle = (z_1 - z_2)^{1/3}, \qquad z_i = Re^{i\theta_i}$$

fractional statistics

Conclusion:

Effective field theory description of incompressible Hall fluid as

CFT (edge) or Chern-Simons theory (bulk)
$c=1$ $U(1)$

Generalizations:

Rich model building during the 90's,

e.g. Jain's plateaus $\nu = \frac{m}{mp \pm 1}$, $c=m$,

and paired Hall states (parafermions)

(our proposal: CFTs with W_∞ symmetry)

Drawbacks

- different effective theory for each plateaus
- more than one proposal for Jain's plateaus
- no handle of transitions between plateaus

Drawbacks of microscopic theories

- Laughlin wave function of which Hamiltonian?
- mystery of Jain's transformation: "composite fermion"

many experiments and no theory, only numerics

Non-Commutative Chern-Simons Theory

(Susskind (01))

Reconsider electrons in the lowest Landau level $\vec{X}_\alpha(t)$

$$S = \frac{eB}{2}\int dt \sum_{\alpha=1}^{N} \varepsilon_{ab}\, \dot{X}^a_\alpha X^b_\alpha \qquad \sim \int e\,\vec{v}\cdot\vec{A} + \cancel{\tfrac{1}{2} m \vec{v}^{\,2}}$$

$A_a = \frac{B}{2}\varepsilon_{ab}x^b$

- take the limit of the continuous fluid

$$\vec{X}_\alpha(t) \to \vec{X}(x,t)\,, \qquad \vec{X}(t=0) = \vec{x} \quad \text{initial configuration}$$

→ fluid mechanics in the Lagrangian approach

- impose incompressibility

$$\rho(x) = \rho_0 = \rho_0 \left|\frac{\partial \vec{X}}{\partial \vec{x}}\right| = \rho_0 \frac{1}{2}\varepsilon_{ab}\{X^a, X^b\}_{x\,PB}$$

via the Lagrange multiplier A_0

$$S = \frac{eB\rho_0}{2}\int dt\, d^2x \left[\varepsilon_{ab}\left(\dot{X}^a - \theta\{X^a, A_0\}\right)X^b - 2\theta A_0\right]$$

where $\boxed{\theta = \frac{1}{2\pi\rho_0}}$ $\qquad \nu = \frac{2\pi\rho_0}{eB} = \frac{1}{eB\theta}$

- introduce w_∞ gauge field $\vec{A}(x)$

$$X^a = x^a + \theta\,\varepsilon^{ab} A_b(x)$$

$$S = \frac{1}{4\pi\nu}\int \varepsilon_{\mu\nu\rho}\left(\partial_\mu A_\nu A_\rho + \frac{\theta}{3}\{A_\mu, A_\nu\}A_\rho\right)$$

Chern-Simons theory of w_∞

Susskind's guess:

The full theory beyond the continuous fluid approx. is

$$\boxed{S_{NCCS} = \frac{1}{4\pi\nu}\int \varepsilon_{\mu\nu\rho}\left(\partial_\mu A_\nu * A_\rho - \frac{2i}{3} A_\mu * A_\nu * A_\rho\right)}$$

$$(g*f)(x) = \exp\left(i\frac{\theta}{2}\varepsilon_{ij}\frac{\partial}{\partial x_1^i}\frac{\partial}{\partial x_2^j}\right) f(x_1)\, g(x_2)\Big|_{x_1=x_2=x}$$

that agrees to leading order in θ

- guess motivated by the analogies QHE $\Leftrightarrow$ D-branes

Remarks

- Very nice! But should we believe it?
- lost track of electron coordinates: observables?
- how to compute ?

Equivalent Matrix theory

Every NC theory is equivalent to a $N=\infty$ matrix theory

$$S_{CSMM} = \frac{eB}{2}\int dt\, \mathrm{Tr}\left[\varepsilon_{ab}\left(\dot{\hat{X}}^a + i[\hat{X}^a, \hat{A}_0]\right)\hat{X}^b - 2\theta \hat{A}_0\right]$$

$X^a \to \hat{X}^a$ $\quad N=\infty$ Hermitean matrices , $a=1,2$

$\{X, Y\} \to -i\,[\hat{X}, \hat{Y}]$

$\hat{X}^a = \hat{x}^a + \theta\,\varepsilon^{ab}\hat{A}_b(x) \longrightarrow$ Wigner Function $A_b(x)$

Chern-Simons Matrix Model (Polychronakos 01)

- Introduce a "boundary" vector field ψ^i to get a well-defined matrix theory of finite $N\times N$ matrices

$$\boxed{\begin{aligned} S_{CSMM} &= \frac{eB}{2}\int dt\, \mathrm{Tr}\{\epsilon_{ab}(\dot{X}_a + i[A_0, X_a])X_b + 2\theta A_0 - \omega X_a^2\} \\ &+ \int dt\, \psi^\dagger(i\dot{\psi} - A_0\psi) \end{aligned}}$$

confining potential

- $U(N)$ symmetry $\quad X_a \to U X_a U^\dagger \quad , \quad \psi \to U\psi$
- Gauss law constraint : states are $U(N)$ singlets

$$G\,|\text{phys}\rangle = 0 \,, \qquad G = -iB[X_1, X_2] - B\theta\mathbb{1} + \psi\psi^\dagger$$

$$[X_1, X_2] \approx i\theta\mathbb{1} - \frac{i}{B}\psi\psi^\dagger$$

- trace $\to \quad 0 = i\theta N - i\frac{|\psi|^2}{B}$ $\qquad$ finite N solution exists
- ψ can be neglected for $N\to\infty$ getting back NCCS

Quantization on the Real line

- choose a gauge in which $(X_1)_{nm} = x_n\delta_{nm}$
- solve the constraint $G=0$

$$(X_2)_{nm} = y_n\delta_{nm} - i\frac{\theta(1-\delta_{nm})}{x_n - x_m} \,, \qquad \psi_n = \sqrt{B\theta} \;\; \forall n$$

- counting d.o.f. $\quad X_1, X_2, \psi, \psi^\dagger, U(N), G \qquad x_n, y_n$

$$N^2 + N^2 + 2N - N^2 - N^2 = 2N$$

OK

- substitute back into the action

$$S = \int dt\, B \sum_{n=1}^{N} \dot{x}_n y_n - \mathcal{H}$$

$$\mathcal{H} = \frac{B\omega}{2} \operatorname{tr}(X_a^2) = \sum_n \frac{\omega}{B}\frac{p_n^2}{2} + \frac{B\omega}{2} x_n^2 + \sum_{n\neq m} \frac{\omega B \theta^2}{2} \frac{1}{(x_n - x_m)^2}$$

→ Calogero model in 1 dimension, $x_n \in \mathbb{R}$

Remarks

- space of states is known and isomorphic to that of Laughlin states $\nu = \frac{1}{B\theta+1} = \frac{1}{n+1}$
- but 1d norm is different from 2d Landau levels
- exact relation of CSMM to Laughlin not cleared by subsequent literature (Hellerman, Van Raamsdonk; Karabali, Sakita; Jackiw et al; Susskind et al.; ..)

SOLUTION

- choose holomorphic quantization

$$X = X_1 + i X_2 = V^{-1} \Lambda V \quad , \quad \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_N)$$

➡ find path-integral of electrons in lowest LL with coordinates λ_i

➡ find Laughlin's wave function

➡ matrix theory of QHE

Holomorphic quantization of Chern-Simons MM

(A. Cappelli, M. Riccardi)

- consider complex matrices $X = X_1 + iX_2$, $X^+ = X_1 - iX_2$

$$S\Big|_{A_0=0} = \frac{B}{2}\int dt\, \mathrm{tr}(\dot X_1 X_2 - X_1 \dot X_2) = \frac{B}{2i}\int dt\, \mathrm{tr}\sum_{nm} \dot X_{nm}\bar X_{nm}$$

$$[\![\bar X_{nm}, X_{kl}]\!] = \delta_{nk}\,\delta_{ml} \qquad (eB=2)$$

- consider Bargmann – Fock space for all complex components X_{nm}

$$\langle \Psi_1 | \Psi_2 \rangle = \int \mathcal{D}X\mathcal{D}\bar X\, e^{-\mathrm{tr}(XX^+)}\, \overline{\Psi_1(X)}\, \Psi_2(X)$$

then $\bar X_{nm} \to \frac{\partial}{\partial X_{nm}}$

Gauss law $G_{ij} = X_{ik}\frac{\partial}{\partial X_{jk}} - X_{kj}\frac{\partial}{\partial X_{ki}} - n\,\delta_{ij} + \psi_i \frac{\partial}{\partial \psi_j}$

$$= [X, X^+] - 2\theta\,\mathbb{1} + \psi\psi^+ \qquad B\theta = n$$

- consider complex linear change of variables $\{dX, d\psi\} \to \{d\lambda, dv, d\phi\}$

$$\begin{cases} X = V^{-1}\Lambda V \\ \psi = V^{-1}\phi \end{cases} \qquad \begin{matrix} \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N) \\ dv = dV\,V^{-1} \end{matrix}$$

dX is contravariant , $\frac{d}{dX}$ is covariant

$[\![\ ,\ ]\!]$ are invariant : Bogoliubov – Backlund transformation

- Gauss law is diagonalized

$$G_{ij} = \begin{cases} -\frac{\partial}{\partial v_{ij}} & i \neq j \\ \phi_i \frac{\partial}{\partial \phi_i} - n & i = j \end{cases} \qquad \text{on } \Psi(\Lambda, V, \phi)$$

$\rightarrow$ the $2(N^2-N)$ d.o.f. of V are frozen $\qquad dv = dV\,V^{-1}$

$\rightarrow$ the ground state w.f. of the matrix theory reduces to the Laughlin w.f.

$$\Psi_{g.s.} = \left[\varepsilon_{i_1 \dots i_N}\, \psi^{i_1} (X\psi)^{i_2} \dots (X^{N-1}\psi)^{i_N}\right]^n |0\rangle$$

(Hellermann, Von-Raamsdonk '01)

$$= \left[\det V \prod_{i<j}(\lambda_i - \lambda_j) \prod_i \phi_i\right]^n |0\rangle$$

$B\theta = n$

$\rightarrow$ after elimination of V

$$X_{ij} \rightarrow \lambda_i \delta_{ij}$$

$$X^+_{ij} = \left(\frac{\partial}{\partial X}\right)^T_{ij} \rightarrow \frac{\partial}{\partial \lambda_i}\delta_{ij} - \frac{1-\delta_{ij}}{\lambda_i - \lambda_j}\left(\cancel{\frac{\partial}{\partial v_{ji}}} + \underbrace{\phi_i \frac{\partial}{\partial \phi_j}}^{n}\right) \equiv \frac{\partial}{\partial \lambda_{ij}}$$

$\rightarrow$ the $2N$ d.o.f. of ϕ_i are frozen too: $\phi_i \rightarrow \sqrt{n}\ \forall i$

- for $n=0$ one recovers the MM of Normal Complex Matrices $[X, X^+] = 0$ and the vector ψ_i decouples

(Wiegmann, Zabrodin et al.)

- Laughlin ground state is a consequence of gauge symmetry and non commutativity

$$G\,\Psi_{gs} = 0, \qquad G = [X, X^+] - 2\theta + \psi\psi^+$$

SU(N) singlet $+\ n$ U(1) sources of ψ_i

Features of the NC quantum theory

- path integral reduces to N electrons in the lowest Landau level with coordinates λ_i

$$\begin{cases} X = V^{-1} \Lambda V & \Lambda_{ij} = \delta_{ij} \lambda_i \\ X^+ = V^{-1} \tilde{\Lambda} V & \tilde{\Lambda}_{ij} = \delta_{ij} \tilde{\lambda}_i - (1-\delta_{ij}) \dfrac{n}{\lambda_i - \lambda_j} \end{cases}$$

$$S = \int dt \, \mathrm{tr}(X^+ \dot{X}) \rightarrow \int dt \sum_{n=1}^{N} \tilde{\lambda}_i \dot{\lambda}_i$$

+ Fateev-Popov trick

- ground state is the Laughlin wave function $\nu = \frac{1}{n}$
- "statistical interaction" with integer statistics n

$$D_\lambda = \frac{\partial}{\partial \lambda} - \sum_i \frac{n}{\lambda - \lambda_i} \quad , \quad \Delta B = [D_\lambda, \bar{D}_\lambda] = n\pi \sum_i \delta^{(2)}(\lambda - \lambda_i)$$

technically important but physically harmless (?)

$$\Psi(\lambda_i - \lambda_j \rightarrow e^{i\pi}(\lambda_i - \lambda_j)) = e^{i\pi n} \Psi(\lambda_i, \lambda_j)$$

(cf. Fradkin-Lopez)

- modified inner product for statistical interaction

$$\langle \Psi_1 | \Psi_2 \rangle = \int \prod_i d\lambda_i d\tilde{\lambda}_i \, e^{-\sum \lambda_i \tilde{\lambda}_i} \, \bar{\Psi}_1(\tilde{\Lambda}) \Psi_2(\Lambda) \Big|_{\substack{\Lambda_{ij} = \delta_{ij} \lambda_j \\ \tilde{\Lambda}_{ij} = \delta_{ij} \tilde{\lambda}_i - \frac{(1-\delta_{ij}) n}{\lambda_i - \lambda_j}}}$$

that respects Hermiticity, e.g.

$$\langle \Psi_1 | \mathrm{tr}(X^+) \Psi_2 \rangle = \langle \mathrm{tr}(X) \Psi_1 | \Psi_2 \rangle$$

Non-Relativistic W_∞ symmetry & incompressibility

- classical transformation of w_∞

$$\delta z = \{z, L\} \quad , \quad \delta \bar{z} = \{\bar{z}, L\} \qquad L = \sum_{n,m=0}^{\infty} c_{nm} z^n \bar{z}^m$$

- quantum generators in the lowest Landau level

$$\mathcal{L}_{nm} = \sum_{\alpha=1}^{N} z_\alpha^n \left(\frac{\partial}{\partial z_\alpha}\right)^m \qquad N = \#\text{ of electrons}$$

- NR W_∞ algebra

$$[\mathcal{L}_{nm}, \mathcal{L}_{k\ell}] = \hbar (mk - n\ell)\, \mathcal{L}_{n+k-1,\, m+\ell-1} + \hbar^2 (\ ) + \ldots$$

equivalent to $U(N)$ for N electrons

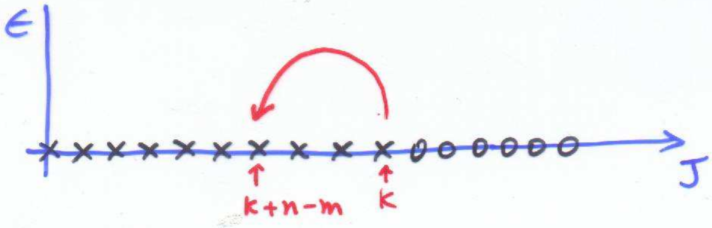
(Fairlie, Fletcher, Zackos)

- the ground state satisfies the h.w.s. conditions

$$\mathcal{L}_{nm} \Psi_{gs}(z_1, \ldots, z_N) = 0 \qquad 0 \leq n < m \leq N-1$$

- they can be interpreted as incompressibility conditions: $n<m$ lowers the angular momentum, compression $\rightarrow$ impossible

Fock-space picture $\qquad \mathcal{L}_{nm} \approx \sum_k a^+_{k+n-m}\, a_k$



- symmetry can be implemented in CFT and fully exploited

Incompressibility and W_∞ symmetry

- Incompressibility (stability) of ground state: cannot create excitations with lower angular momentum that would reduce the size of the droplet

- W_∞ highest weight conditions

$$\mathcal{L}_{nm} \Psi_{gs} = 0 , \qquad 0 \le n < m , \qquad \Delta J = n - m < 0$$

- they are satisfied in the matrix model, by

$$\mathcal{L}_{nm} = \mathrm{tr}\left(X^{n} X^{+m} \right)$$

→ these operators obey the W_∞ algebra $O(\hbar)$ but the $O(\hbar^2) \approx O(1/N)$ corrections $\quad \rho_{nm} = \psi^+ X^n X^{+m} \psi$ are not yet under control

→ in the reduced electron coordinates λ_i, they read

$$\mathcal{L}_{nm} = \sum_j \lambda_j^n \left(\frac{\partial}{\partial \lambda_p} \delta_{pq} - (1-\delta_{pq}) \frac{n}{\lambda_p - \lambda_q} \right)^m_{jj}$$

→ statistical interaction is crucial for verifying the incompressibility conditions

- the incompressibility conditions in the matrix form show a generalized exclusion principle due to the $SU(N)$ singlet condition $\varepsilon_{i_1 \cdots i_N}$

(cf. Haldane-Pasquier)

Conclusions

- Matrix model : electrons (N d.o.f.) $\rightarrow$ D0 branes (N^2)
  \+ huge gauge symmetry $U(N)$

- Noncommutative quantum theory:
  $[X, X^+] = 2\theta \mathbb{1}$, $B\theta = n$ $\rightarrow$ Gauss law condition

  LLL electrons already noncommutative,
  $$[[\bar{\lambda}_n, \lambda_m]] = \frac{2\hbar}{B}\delta_{nm}$$
  are forced to repel each other further , $\nu = \frac{1}{B\theta} < 1$

  and are placed in the entangled Laughlin state

- long-distance physics of Laughlin state is reproduced ; short-distance could be modified by the statistical interaction $R^2 = \frac{\ell^2 N}{\nu} \sim \hbar N$

Perspectives

- reproduce fractional charge & statistics
  $$\Psi_{qh}(\eta; z_1, \ldots z_N) = \det(\eta - X)\,\Psi_{gs}(X)$$

- describe Jain's plateaus $\frac{1}{\nu} = k + \frac{1}{m}$ by suitable extension of boundary terms (Moriaru, Polychronakos)

- Morita's equivalences in NC theories
  $\theta \leftrightarrow \theta'$ by $SL(2,\mathbb{Z})$ transformations
  maybe this is just Jain's transformation
  ($SL(2,\mathbb{Z})$ already proposed in QHE, cf. Lütken-Latorre)