

Renormalization group patterns and c -theorem in more than two dimensions

Andrea Cappelli * and José Ignacio Latorre **

Theory Division, CERN, Geneva, Switzerland

Xavier Vilasís-Cardona ***

*Departament d'Estructura i Constituents de la Matèria, University of Barcelona, Av. Diagonal 647,
08028 Barcelona, Spain*

Received 20 September 1991

Accepted for publication 10 January 1992

We elaborate on a previous attempt to prove the irreversibility of the renormalization group flow in more than two dimensions. This involves the construction of a monotonically decreasing c -function using a spectral representation. The missing step of the proof is a good definition of this function at the fixed points. We argue that for all kinds of perturbative flows the c -function is well defined and the c -theorem holds in any dimension. We provide examples in multicritical and multicomponent scalar theories for dimension $2 < d < 4$. We also discuss the non-perturbative flows in the yet unsettled case of the $O(N)$ sigma model for $2 \leq d \leq 4$ and large N .

1. Introduction

There is a common belief which says that the renormalization group (RG) flows are irreversible. Intuitively, short-distance degrees of freedom are integrated out in order to obtain a long-distance effective description of a physical system and are, therefore, irrecoverable. Materializing this intuition into a theorem has proven to be quite a hard task, so far unfinished for dimensions $d > 2$.

Zamolodchikov produced a theorem in two dimensions based on the explicit construction of a function which decreases monotonically along RG trajectories [1]. This is obtained from the correlator of two stress tensors and it is called “ c -function” because it coincides at fixed points with the central charge of the corresponding conformal field theory. In his original proof, Zamolodchikov assumed Poincaré invariance, locality, renormalizability and, notably, unitarity. Actually, the very

* On leave of absence from INFN, Sezione di Firenze, Italy.

Bitnet: cappelli@cernvm

** On leave of absence from Departament ECM, Barcelona, Spain.

Bitnet: latorre@ebubecm1

*** Bitnet: druida@ebubecm1

irreversibility of the RG flow, i.e. the monotonicity of the c -function, is due to unitarity, which implies a positive correlator.

The c -theorem is by now a valuable tool for non-perturbative field theory in two dimensions. A striking example is the proof of spontaneous breaking of supersymmetry in the flow from the tricritical to the critical Ising model [2]. A similar tool would be very interesting in higher dimensions, to investigate long-standing non-perturbative problems like confinement, chiral symmetry breaking and the Higgs mechanism. The explicit ingredients in Zamolodchikov's proof are not specific to two dimensions, but so far problems have been found to extend it to higher dimensions.

There are no known counterexamples against a c -theorem in higher dimensions. Nonmonotonic RG flows with complex behaviour, like limit cycles and chaos, have been discussed in ref. [8]. They all violate some hypotheses of the would-be theorem, like Poincaré invariance (spin glasses, hierarchical models) or unitarity (polymers, models with replica trick *).

The efforts to extend the theorem were roughly of two types. One group looked for analogs of the central charge in four-dimensional critical theories [3,4,7]. The c -numbers parametrizing the gravitational trace anomaly were naturally considered, in particular the Euler topological term [3]. Clearly, one needs a mechanism for proving the monotonicity of these quantities along the RG, such as unitarity. Unfortunately, the usual condition from positive two-point functions did not constrain these numbers [6,7], and one should hope for a more advanced analysis.

A second group developed a formalism which manifestly has all the constraints on the RG flow coming from unitarity [6]. It is based on the Lehmann spectral representation of the stress-tensor correlator and it was originally used by Friedan for an alternative proof of the two-dimensional c -theorem. Actually, this proof almost holds in any higher dimension, the unclear point being the definition of the corresponding c -function at the fixed points. Thus, we cannot claim that the theorem is conquered yet.

In this paper, we intend to present the state of the art of the spectral approach to the c -theorem in more than two dimensions. In sect. 2, we recall its main points. We stress that the c -function is actually well-defined for perturbative flows, where it changes infinitesimally between the ultraviolet (UV) and the infrared (IR) fixed points, $\Delta c = c_{UV} - c_{IR} \ll 1$.

Therefore the RG flow is irreversible in any perturbative expansion **. This fact is of great practical importance, due to the major role of perturbative expansions in higher dimensions, even if it is far from the non-perturbative goals we mentioned before.

In sect. 3, we substantiate this claim by studying the c -function in arbitrary

* They are not unitarity in two dimensions [9] and likely not in any dimension [10].

** See also ref. [5].

dimension using Wilson's epsilon expansion in scalar theories as a bench-matrix [11]. In this framework, we compute the RG patterns of the $\lambda\phi^4$ theory [12], of the multicritical $\lambda\phi^{2r}$ [13] and of the multicomponent $g_{ijkl}\phi_i\phi_j\phi_k\phi_l$ theories [14]. A typical variation of the c -function is $\Delta c \sim \epsilon^3 \ll 1$ in $d = 4 - \epsilon$. As expected, the c -theorem works perfectly. It supports the belief that the multicritical pattern of the minimal conformal models in two dimensions [15] extends smoothly to higher dimensions. Moreover, the c -functions correctly add up for chains of flows in the multicomponent theory.

In sect. 4, we try to understand more difficult flows, which are “non-perturbative” in the following sense. We investigate the $O(N)$ sigma-model in the large N limit for $2 \leq d \leq 4$ [14,16]. In this case, the expansion is perturbative in $1/N$ (the coupling of the theory given by the connected 4-point function), but it is actually non-perturbative for what concerns the c -function. Indeed, the flow in the massive phase gives $\Delta c \sim N \gg 1$ and is definitely different from that of free massive bosonic particles. Our results agree with the known RG pattern of the model for $d = 2$ and $d = 4$, but disagree with it for $2 < d < 4$. We sketch some possible explanations of this fact which deserve further investigation. In the conclusions, we comment on four-dimensional physical theories like QCD. Appendix A is devoted to setting the technique of conformal perturbation expansion, valid in any dimension.

2. Steps towards the proof of the c -theorem

Let us first recall the main features of the c -theorem in two dimensions [6]. We define the stress tensor $T_{\mu\nu}(x)$ as the response of the action to small fluctuations of the space-time metric. Then we consider the spectral representation of the euclidean correlator

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0) \rangle = \frac{\pi}{3} \int_0^\infty d\mu \, c(\mu) \int \frac{d^2p}{(2\pi)^2} e^{ipx} \frac{(g_{\mu\nu}p^2 - p_\mu p_\nu)(g_{\rho\sigma}p^2 - p_\rho p_\sigma)}{p^2 + \mu^2}. \quad (2.1)$$

In this expression, the spectral density $c(\mu)$ is a scalar function by Poincaré invariance and it is positive-definite by unitarity, $c(\mu) \geq 0$. In the short-distance limit, the theory is described by a UV conformal invariant theory, where eq. (2.1) reduces to the single component $\langle T_{zz}(x)T_{zz}(0) \rangle_{\text{CFT}} = c_{\text{UV}}/2z^4$, parametrized by the Virasoro central charge c_{UV} . The same happens in the long-distance (IR) limit. Working out these two limits in eq. (2.1), it follows that

$$c_{\text{UV}} = \int_0^\infty d\mu \, c(\mu) \geq c_{\text{IR}} = \lim_{\epsilon \rightarrow 0} \int_0^\epsilon d\mu \, c(\mu). \quad (2.2)$$

where the inequality stems from unitarity. Thus $c(\mu) d\mu$ is a dimensionless measure of degrees of freedom off-criticality.

The proof is completed by considering the RG flow of the spectral density. The previous equation implies that

$$c(\mu) = c_{\text{IR}} \delta(\mu) + c_1(\mu, \Lambda), \quad \Delta c \equiv c_{\text{UV}} - c_{\text{IR}} = \int_0^\infty d\mu c_1(\mu). \quad (2.3)$$

The evolution of $c(\mu) d\mu$ under the RG flow is governed by the flow of the physical mass scale Λ of the theory. As we leave the UV fixed point $\Lambda = 0$, the delta term in eq. (2.3) stays constant, because it measures the states of the Hilbert space which remains massless for $\Lambda \neq 0$. Instead, the states acquiring mass contribute to the smooth density $c_1(\mu, \Lambda)$, roughly bell-shaped and peaked at $\mu \sim \Lambda$. In the IR limit $\Lambda \rightarrow \infty$, c_1 is pushed away and contributes no more to observables. Thus its integral Δc gives a quantitative measure of the loss of degrees of freedom along the RG flow.

In a typical application of the theorem [17,18], c_{UV} and c_{IR} are first determined from $\langle TT \rangle_{\text{CFT}}$ in the corresponding conformal field theories. Next, the correlator $\langle \Theta \Theta \rangle$ of the interpolating off-critical theory is considered. By eq. (2.3), the variation Δc is independently measured by the sum rule [19]

$$\Delta c = \frac{3}{4\pi} \int_{|x| > \epsilon} d^2x x^2 \langle \Theta(x) \Theta(0) \rangle. \quad (2.4)$$

Thus the data of the critical and off-critical theories are compared and the RG pattern is verified.

Let us stress some virtues of this proof. The c -theorem expresses a geometrical property of the space of theories, parametrized by some coupling coordinates $\{g^i\}$. But notice that our proof was given in a coordinate-free language. We did not need to talk of bare lagrangians and renormalization conditions on fields and couplings, nor care about their associated reparametrization invariance. Actually, the Lehmann representation gives us the spectral density expressed in terms of matrix elements of the stress tensor, belonging to the Hilbert space of the theory, so that renormalization problems are by-passed.

Moreover, these matrix elements are necessarily non-vanishing, because any matter couples to the stress tensor. Thus, this is a good measure of degrees of freedom in the theory. Any other current would not couple to all of them (depending on their charges) and would not detect their flow. Finally, the density $c(\mu)$ summarizes all the unitarity conditions on the two-point function, because any positive quantity can be obtained by integrating it against positive smearing functions.

We believe that these are rather unique features, which should necessarily appear in any generalization of the theorem to higher dimensions.

For later reference, let us also express the theorem in terms of the coordinates $\{g^i\}$ and beta-functions $\Lambda(dg^i/d\Lambda) = \beta^i(g)$ [1]. One has to expand the trace of the stress tensor $\Theta(x)$ in the basis of renormalized fields Φ_i at the UV fixed point

$$\Theta(x) = 2\pi\beta^i\Phi_i. \quad (2.5)$$

Next, the c -function $c = c(g)$ and the Zamolodchikov metric $G_{ij}(g) \propto \langle \Phi_i(x)\Phi_j(0) \rangle|_{|x|=1}$ are introduced by smearing $c(\mu)$ against appropriate positive functions, which contain a fixed scale [6]. It follows

$$\frac{d}{dt}c \equiv -\beta^i \frac{\partial}{\partial g^i} c(g) = -\beta^i \beta^j G_{ij}(g) \leq 0, \quad (2.6)$$

Thus, the previous flow of $c(\mu)$ driven by a physical mass Λ is now traded for the change of $c(g)$ along the flow curve of affine parameter t .

The spectral form of the c -theorem can be generalized to higher dimensions, where it shows a new feature. There are two spectral densities, $c^{(0)}(\mu)$, related to spin-zero intermediate states, and $c^{(2)}(\mu)$ for spin-two ones. Both densities are, in principle, candidates for a c -theorem, since they display some of the properties of the unique density in two dimensions. $c^{(2)}(\mu)$ determines the correlator of two stress tensors at the conformal invariant points [20], and it could define an analog of the central charge. However, by inspection this does not correspond to a monotonically decreasing function along RG trajectories and it must be discarded [6].

On the other hand, $c^{(0)}(\mu)$ is related to changes of scale off-criticality, because

$$\langle \Theta(x)\Theta(0) \rangle = A \int_0^\infty d\mu \, c^{(0)}(\mu) \int \frac{d^d p}{(2\pi)^d} e^{ipx} \frac{p^4}{p^2 + \mu^2}, \quad (2.7)$$

$$A = \frac{V}{\Gamma(d)(d+1)2^{d-1}}, \quad V \equiv \text{Vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \quad (2.8)$$

Therefore, we can generalize the sum rule (2.4). Limiting ourselves to theories with vanishing Θ at fixed points, i.e. conformally invariant fixed points, a dimensional analysis led us to define

$$\Delta c = \int_\epsilon^\infty d\mu \frac{c^{(0)}(\mu, \Lambda)}{\mu^{d-2}} = \frac{d+1}{Vd} \int_{|x|>\epsilon} d^d x \, x^d \langle \Theta(x)\Theta(0) \rangle. \quad (2.9)$$

The normalization of $c^{(0)}(\mu)$ in eq. (2.7) assigns $c=0$ to the trivial theory and $c=1$ to the free bosonic theory in any dimension, the latter being computed by the sum rule (2.9) in the free massive phase. By smearing $c^{(0)}(\mu)$ we can also

generalize eq. (2.6), and get a new c -function $c(g)$ in d dimensions, which is monotonically decreasing along the RG flow and stationary at fixed points, in close analogy with the two-dimensional case [6].

However, a point is missing, the characterization of $c(g)$ at fixed points, the would-be central charge, or c -charge. A closer inspection shows that this is defined as a limit from off-criticality of the spin-zero density,

$$\lim_{A \rightarrow 0} \frac{c^{(0)}(\mu, A)}{\mu^{d-2}} d\mu = c\delta(\mu) d\mu. \quad (2.10)$$

In general, this limit may depend on the path approaching the fixed point, implying unacceptable non-universal effects on the c -charge. If c is monotonic but multivalued, we can still have closed cycles violating the theorem. On the other hand, the limit is universal if the c -charge is related to an observable of the fixed-point theory – we could not prove this fact so far. Note that $\langle \Theta \Theta \rangle_{\text{CFT}} = 0$ for $d > 2$, and is thus independent of c , owing to the factor μ^{d-2} in eq. (2.10).

As we pointed out above, in two dimensions c has indeed an independent characterization at the fixed point from $\langle TT \rangle$. The unique density is both responsible for controlling changes of scale (genuine spin-0 in $d \geq 2$) and for the coefficient of the short distance singularity of $T_{zz}(x)T_{zz}(0)$, (genuine spin 2 in $d \geq 2$). At the conformal point, the trace of the stress tensor still sees the central charge through contact terms. This is actually enforced by Lorentz invariance since T_{zz} and Θ are just different components of the same Lorentz structure [6]. Above two dimensions, the roles of controlling scale transformations and the short-distance singularity are related to two different spin structures. These two structures are independent of one another since Lorentz invariance acts separately on each one. This is the reason why we could not find an independent characterization of our c -charge.

The previous problem can be made quantitative by checking of Δc in a chain of RG flows. For three fixed points (fig. 1), this reads

$$(\Delta c)_{1 \rightarrow 2} + (\Delta c)_{2 \rightarrow 3} = (\Delta c)_{1 \rightarrow 3}. \quad (2.11)$$

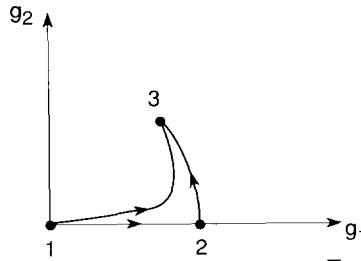


Fig. 1. RG pattern which exemplifies the additivity property of the c -charge.

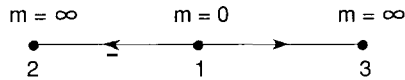


Fig. 2. Schematic picture of two RG flows leading to the purely massive phase which are not deformable into each other.

Additivity of the c -charge also amounts to integrability of the system of beta functions. Eq. (2.11) holds if c at the theory 2 has the same value independently of whether we approach this theory from 1 or from 3. This is not obvious, because $(\Delta c)_{i \rightarrow j}$ is computed in the i th theory, so that we are comparing calculations in two a priori different off-critical theories.

Nevertheless, suppose that the three fixed points lie in a region of the space of theories parametrized by smooth coordinates. Indeed, this is the case of perturbative calculations (by definition). Δc is a polynomial in the renormalized couplings, thus the limit from off-criticality exists, trajectories can be deformed at will to prove additivity. Therefore, we can say that the c -theorem is proven for all kinds of perturbation expansions, namely $\Delta c \ll 1$ and polynomial in the couplings. Two examples of this kind will be discussed in sect. 3.

The drawback to the above comments is that the space of theories has singularities and is probably not a manifold. For example, in fig. 2 we imagine comparing Δc for two non-perturbative flows into the massive phase ($\Delta c \sim 1$). The trivial theory ($c = 0$) can appear in several points of the coupling space, thus the two paths can be topologically inequivalent. The additivity property, $(\Delta c)_{1 \rightarrow 2} = (\Delta c)_{1 \rightarrow 3}$, is not all trivial in this case. This kind of situation will be discussed in sect. 4.

3. Perturbative flows in scalar theories for $2 < d < 4$

In this section, we provide examples of perturbative RG patterns. Two or more fixed points appear in a small region of coupling space, and the d -dimensional c -function varies infinitesimally while flowing among them. The validity of the c -theorem is verified by using various versions of the ϵ -expansion.

First we study the Landau–Ginsburg–Wilson RG pattern of the r th multicritical points in the $\lambda \varphi^{2r}$ theory. The comparison of the c -charges of the (r) and $(r - 1)$ theories suggests that the known multicritical RG flows in two dimensions extend smoothly to higher dimensions. Next, the multicomponent $\lambda \varphi^4$ theory is considered, and eq. (2.11) for additivity of the c -charge is verified.

3.1. MULTICRITICAL POINTS AND LANDAU–GINSBURG THEORY

The Landau–Ginsburg action

$$S_{\text{LG}} = \int d^d x \left(\frac{1}{2} (\partial_\mu \varphi)^2 - \sum_{k=1}^r \frac{\lambda_{k,0}}{2k!} \varphi^{2k} \right), \quad (3.1)$$

describes the qualitative features of generic r -multicritical points with parity symmetry (only), which appear for $2 \leq d < 4$ [11]. At the multicritical point $\{\lambda_{k,0}\} = \{0, 0, \dots, \lambda_{r,0}\}$, r minima of the potential merge, which correspond to r coexisting phases of the theory. The flow to lower r' -critical points, $r' < r$, is described by switching on the relevant perturbation $\lambda_{r',0} \varphi^{2r'}$. At the end of the flow, the higher powers φ^{2k} , $r' < k \leq r$ become irrelevant fields and can be neglected in the action.

In two dimensions, the minimal conformal theories [21] with central charge

$$c(r) = 1 - \frac{6}{r(r-1)} < 1,$$

are the exact renormalization of the above Landau–Ginsburg actions [15]. The renormalized fields φ^{2k} , $k < r$, are the primary conformal fields which appear in the first two diagonals of the Kac table. Off-criticality, this picture has been confirmed along the flow between the (r) and the $(r-1)$ models, driven by the least relevant field φ^{2r-2} , for $r \gg 1$ [1,9] *. The dimension of this field is $2 - \epsilon$, $\epsilon \sim 1/r \ll 1$, thus it is the typical perturbation situation encountered in the ϵ -expansion. The IR fixed point, $(r-1)$, appears infinitesimally close to the UV one, (r) , in coupling space. The invariant definition of distance is given by the Zamolodchikov metric, eq. (2.6), or, equivalently, by the change of the central charge, $\Delta c = c_r - c_{r-1} \sim O(1/r^3)$.

The c -theorem is a nice complement to the Landau–Ginsburg picture. Higher multicritical points have higher values of c , thus flowing downhill corresponds to going to lower multicritical points. Vafa has further developed this picture [22]. The c -function can be considered as the height function in the space of theories Q , so that Morse theory can be applied and the Poincaré polynomial gives some information on the holonomy of Q . In short, the qualitative Landau–Ginsburg description together with the c -theorem give some grasp of the geometry of this space of theories.

Following Wilson, the Landau–Ginsburg picture holds for all dimensions up to the upper critical dimension d_c

$$2 \leq d \leq d_c(r) \equiv 2 + \frac{2}{r-1}, \quad (3.2)$$

the dimension for which φ^{2r} becomes marginal, i.e. the r th multicritical point merges with the gaussian one. A natural question to ask is whether our candidate for a c -theorem extends above two dimensions as well. The higher multicritical points should continue to have larger values of c . In such a case, the space of multicritical points and their flows would have an analytical continuation in dimensions.

* As recalled in example 1 of appendix A.

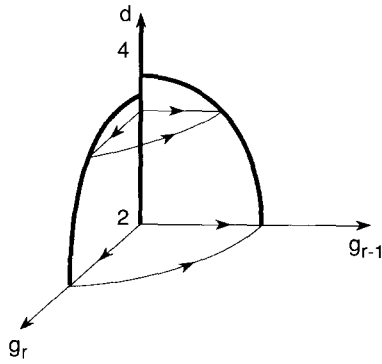


Fig. 3. RG flows for the r th and $(r-1)$ th Ginsburg-Landau models between two and four dimensions.

3.2. THE c -CHARGE OF THE $\lambda\varphi^{2r}$ THEORY

Above two dimensions, we compute our candidate c -charge c_r of the r th multicritical point by applying the sum rule eq. (2.9) to the flow from the gaussian theory, $c_r = 1 - \Delta c$ (fig. 3). We use the ϵ -expansion at dimension

$$d = d_c(r) - \frac{\epsilon_r}{r-1}, \quad 0 < \epsilon_r \ll 1, \quad (3.3)$$

where $\epsilon_r = \dim(\lambda_{r,0})$ is the small parameter.

Let us start with the familiar example of $\lambda\varphi^4$ theory, for $d = 4 - \epsilon$, and derive the first-order term of the ϵ -expansion. The action is

$$S = \int d^d x \left[\frac{1}{2} (\partial_\mu \varphi_0)^2 - \frac{\lambda_0}{4!} \varphi_0^4 \right], \quad (3.4)$$

where λ_0 and φ_0 are the bare coupling constant and field respectively. At $d = 4 - \epsilon$ the dimension of the coupling constant is $\dim(\lambda_0) = \epsilon$, i.e. the field φ_0^4 is slightly relevant and it produces a flow from the gaussian fixed point $\lambda_0 = g = 0$ ($c = 1$, by definition) to the Wilson fixed point at $g = g^* \sim \epsilon$, where g is the renormalized coupling constant.

The trace of the energy-momentum tensor is

$$\Theta = -\epsilon \frac{\lambda_0}{4!} V \varphi^4. \quad (3.5)$$

From the computation of the two leading Feynman diagrams, we can extract

$$\text{Im} \langle \Theta(p) \Theta(-p) \rangle |_{p^2 = -\mu^2} = \epsilon^2 \frac{(\lambda_0 S)^2 V^2 S \pi}{3 \cdot 128} \left(\frac{1}{6} \mu^{4-3\epsilon} + \frac{\lambda_0 S}{\epsilon} \mu^{4-4\epsilon} \right). \quad (3.6)$$

Upon insertion of this imaginary part in eq. (2.9) and integration, one obtains

$$c(\lambda_0 \kappa^{-\epsilon}) = 1 - a(\lambda_0 S \kappa^{-\epsilon})^2 (\epsilon + (\lambda_0 S \kappa^{-\epsilon}) b), \quad (3.7)$$

$$a = \frac{5}{48} VS, \quad b = 4, \quad S \equiv \frac{V}{(2\pi)^d} = \frac{2}{(4\pi)^{d/2} \Gamma(d/2)}. \quad (3.8)$$

This flowing c -function depends on the bare coupling and the IR cut-off κ , which appears in the intermediate steps of calculations of massless perturbations, as usual [9]. Since c has no anomalous dimension, its renormalization is simply achieved by replacing $\lambda_0 \kappa^{-\epsilon}$ with the renormalized coupling g . This is a change of coordinates in coupling space which removes the unphysical singularity in the Zamolodchikov metric $G(\lambda_0)$ at the IR fixed point [23]. Because of eq. (2.6), all the information needed to find such a transformation is contained in the c -function itself, and we need not carry out the renormalization of fields. This produces economical formulae for the flow. For one-coupling flows, eq. (2.6) gives

$$\frac{\partial}{\partial \lambda_0} c(\lambda_0) = G(\lambda_0) \beta(\lambda_0), \quad (3.9)$$

where the beta function in terms of the bare coupling constant is $\beta(\lambda_0) = -\epsilon \lambda_0$, by eq. (3.5). The relation between bare to renormalized couplings can then be cast into an elegant geometrical condition – the invariant distance in coupling space remains the same whatever coordinate system is chosen,

$$ds^2 = G(\lambda_0) d\lambda_0^2 = G(g) dg^2. \quad (3.10)$$

The renormalized coupling is defined by requiring that $G(g) = 2a$, where the specific value chosen for the constant is of later convenience. Then $g(\lambda_0)$ is obtained by integrating eq. (3.10). The final form for the c -function reads

$$c(g) = 1 - ag^2 \left(\epsilon + \frac{1}{4}bg \right) = 1 - \frac{5}{3 \cdot 64} g^2 (\epsilon + g). \quad (3.11)$$

Its derivative is proportional to the beta function,

$$\beta(g) = -\epsilon g - \frac{3}{8}bg^2 = -\epsilon g - \frac{3}{2}g^2, \quad (3.12)$$

which agrees with standard derivations (see e.g. ref. [16]). In agreement with the general discussion of eq. (2.6) [6], we have obtained a monotonic decreasing c -function, which is stationary at the fixed points $g = 0$ and $g = g^* = -\frac{2}{3}\epsilon$. These results are an explicit illustration of Zamolodchikov's ideas in more than two dimensions [1].

The value of the c -function at the Wilson (2)-critical point is

$$c_2 = c(g^*) = 1 - \frac{5}{16 \cdot 81} \epsilon^3, \quad d = 4 - \epsilon. \quad (3.13)$$

We can generalize this analysis to the flow between the gaussian and the (r) -critical point. The action is

$$S = \int d^d x \left[\frac{1}{2} (\partial_\mu \varphi)^2 - \frac{\lambda_{r,0}}{2r!} \varphi^{2r} \right], \quad (3.14)$$

where the dimension now is slightly below $d_c(r)$, eqs. (3.2), (3.3). The trace of the stress tensor reads

$$\Theta = -\epsilon_r \frac{\lambda_{r,0}}{2r!} V \varphi^{2r}. \quad (3.15)$$

Again the computation of the two-point correlator to leading perturbative order involves two Feynman diagrams which have the same singularity structure as the $r = 2$ case implying again stationarity of $c(g)$. The only numerical changes are

$$\begin{aligned} a_r &= 2^{d-3} (d+1) V \left(\frac{S[r][1]}{2} \right)^{2r-3} \frac{[1]^3 [r]^2}{[2r-1] 2r!}, \\ b_r &= \frac{4}{3} \left(\frac{S[r][1]}{2} \right)^{r-2} \frac{[1]^2 2r!}{(r!)^3}, \end{aligned} \quad (3.16)$$

where the notation $[\alpha] = \Gamma(\alpha/(r-1))$ has been introduced *. Again we find a non-trivial fixed point at $g_r = g_r^* \equiv -8\epsilon_r/3b_r$, with c -charge

$$c_r = c(g_r^*) = 1 - \frac{3r-1}{3r} \left(\frac{r!^2}{2r!} \epsilon_r \right)^3, \quad d = \frac{2r-\epsilon_r}{r-1}, \quad 0 < \epsilon_r \ll 1. \quad (3.17)$$

Notice that the ϵ_r -expansion to first order is not good enough for reproducing the known values of the charge in two dimensions, especially for large r . We got $c_r \sim 1 - O(2^{-6r})$ instead of $c_r \sim 1 - O(1/r^2)$. Actually, the first few terms of this asymptotic expansion give an accurate result for $0 \leq \epsilon_r < O(A^{-r+1})$, where A is a positive constant, so that we cannot use it for two dimensions ($\epsilon_r = 2$) **.

* The corresponding beta function agrees with ref. [24] by rescaling $g \rightarrow g/S$.

** This bound can be obtained from the fact that the k th term in the expansion grows like $(k!)^{r-1}$ [13].

3.3. THE HEIGHT OF THE (r) -THEORY VERSUS THE $(r-1)$ ONE

We have now the elements to compare the c -number or “height” of two neighbour multicritical points. From the Landau–Ginsburg picture and the two-dimensional c -theorem we expect a flow from the (r) -theory to the $(r-1)$ one (see fig. 3). If the c -theorem holds in any dimension we expect $c_r > c_{r-1}$. At dimension

$$d = d_c(r) - \frac{\epsilon_r}{r-1} = d_c(r-1) - \frac{1}{r-2} \left(\frac{2}{r-1} + \frac{r-2}{r-1} \epsilon_r \right), \quad 0 < \epsilon_r \ll 1, \quad (3.18)$$

both critical points are present (fig. 3) and we find

$$c_r - c_{r-1} = \left(\frac{r!^2}{2r!} \right)^3 \left[\frac{\left(1 - \frac{4}{3r}\right) \left(1 - \frac{1}{2r}\right)^3}{\left(1 - \frac{1}{r}\right)} \left(\frac{8}{r-1} + 4 \frac{r-2}{r-1} \epsilon_r \right)^3 - \left(1 - \frac{1}{3r}\right) \epsilon_r^3 \right] > 0. \quad (3.19)$$

This bound is satisfied for all ϵ_r within the range of validity of the ϵ -expansion at first order. As shown by fig. 3, both multicritical points exist for ϵ_r small at will, and $\epsilon_{r-1} \sim O(1/r^2)$ at least, the latter going outside the range of accuracy for large r . Therefore our result in eq. (3.19) is probably numerically good for small $r=2, 3$, but only heuristic for large r . Nevertheless, the comparison of the two charges at the same perturbative order is certainly better than the absolute value of each one. Eq. (3.19) shows that both c -charges have exponentially small corrections, but differ in an algebraic factor.

Moreover, $c_r > c_{r-1}$ necessarily holds for $\epsilon_r \rightarrow 0$. Indeed, the nucleation of multicritical points ordered in dimension (see fig. 3), the fact that $c=1$ for the free theory in any dimension and its monotonicity property imply $c_r - 1 < 1$ and $c_r \sim 1$ in this limit.

Let us finally quote Felder’s investigation of this RG pattern in more than two dimensions [25]. Working in a different perturbative approach, he was also able to build a monotonic function along the RG flows from his system of beta functions.

In conclusion, these are good indications that the geometry of the space of multicritical scalar theories extends smoothly above two dimensions.

3.4. THE MULTICOMPONENT φ^4 THEORY

Another well-known RG flow pattern is provided by the multicomponent φ^4 model in $4 - \epsilon$ dimensions [14],

$$S = \int d^d x \left(\frac{1}{2} (\partial_\mu \varphi_i)^2 - \frac{1}{4!} g_{ijkl} \varphi_i \varphi_j \varphi_k \varphi_l \right), \quad (3.20)$$

where the sum over N components φ_i , $i = 1, \dots, N$ is implicit. In this theory, Wallace and Zia [12] first showed by the RG trajectories are gradient flows to three-loop order in the ϵ -expansion, thus ensuring that the flow is driven to the IR fixed points – a non-trivial fact when we look at the multi-loop β -function!

These results can easily be framed into the c -theorem philosophy. A simple modification of our previous Feynman rules leads to

$$c(g_{ijkl}) = 1 - \frac{5}{64 \cdot 3} (\epsilon g_{ijkl} g_{ijkl} + g_{ijkl} g_{ijrs} g_{klrs}) \quad (3.21)$$

and

$$\beta_{ijkl}(g_{mnrs}) = \frac{1}{2a} \frac{\partial}{\partial g_{ijkl}} c = -\epsilon g_{ijkl} - \frac{3}{2} g_{ijrs} g_{klrs}, \quad (3.22)$$

which necessarily agree with the Wallace and Zia result to one loop. To higher orders, our c -function would correspond to a definite choice of the free parameters in their gradient function Φ , $c \sim 1 - \text{const.} \times \Phi$.

The RG flows given by eq. (3.22) correspond to trajectories leaving the gaussian point in different directions and reaching IR fixed points. One of them is stable and, therefore, displays a minimum of c . The other ones are unstable, the stability changing with N . Let us recall the results of ref. [14]. A two-dimensional subspace of the coupling space is given by the $O(N)$ -symmetric perturbation $(\sum_i \varphi_i^2)^2$ leading to the $O(N)$ -symmetric Wilson point, and the hypercubic symmetry one $\sum_i \varphi_i^4$, leading to N decoupled Ising models. This pattern describes the breaking of $O(N)$ symmetry to the hypercubic one in lattice ferromagnets. After computing the location of the $O(N)$ -symmetric and hypercubic fixed points, it is convenient to introduce rescaled variables x and y

$$g_{ijkl} \varphi^i \varphi^j \varphi^k \varphi^l = -x \frac{6\epsilon}{N+8} \left(\sum_i \varphi_i^2 \right)^2 - y \frac{2\epsilon}{3} \sum_i \varphi_i^4, \quad (3.23)$$

such that the $O(N)$ symmetric point is at $(x, y) = (1, 0)$ and the hypercubic symmetric is at $(x, y) = (0, 1)$ (fig. 4).

The c -function in this parametrization is

$$c(x, y) = 1 - \frac{5}{16 \cdot 3} \epsilon^3 \left[\frac{N(N+2)}{(N+8)^2} (3x^2 - 2x^3) + \frac{2N}{N+8} xy(1-x-y) + \frac{N}{27} (3y^2 - 2y^3) \right]. \quad (3.24)$$

Inspection of the extrema of c (which correspond to $\beta_x = 0$, $\beta_y = 0$) shows that there is a fourth fixed point at $(x^*, y^*) = ((N+8)/3N, (N-4)/N)$. The stability of these points is also deduced from c . For any value of N , there are two unstable

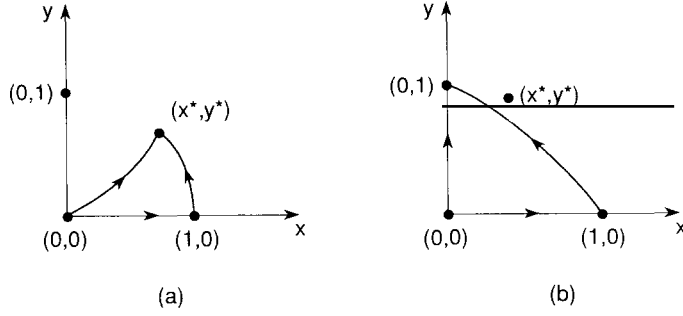


Fig. 4. Chains of RG flows for the multicomponent φ^4 theory (see text): (a) gaussian \rightarrow symmetric $\rightarrow (x^*, y^*)$, for $N > 4$; (b) gaussian \rightarrow symmetric \rightarrow decoupled, for $N > 10$.

points besides the gaussian one and a stable fixed point. For $N < 4$, the $O(N)$ symmetric point is stable; for $N > 4$, (x^*, y^*) is stable. An example of the shape of $c(x, y)$ for $N = 8$ is shown in the level plot fig. 5.

3.5. ADDITIVITY OF THE c -CHARGE

Let us now verify the additivity property of the d -dimensional charge for compositions of flows among three fixed points, eq. (2.11),

$$(\Delta c)_{1 \rightarrow 2} + (\Delta c)_{2 \rightarrow 3} = (\Delta c)_{1 \rightarrow 3}. \quad (3.25)$$

In the previous RG pattern, let us consider two possible chains:

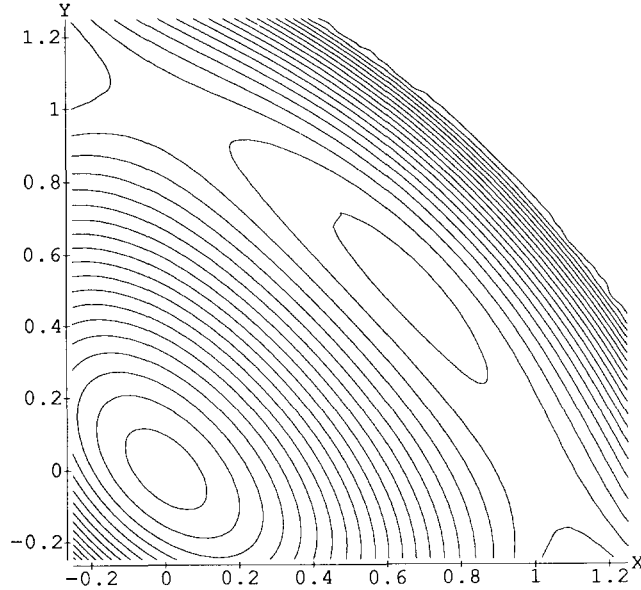


Fig. 5. Level map of the c -function for the multicomponent $\lambda\varphi^4$ theory with $N = 8$ and $d = 3.8$.

(i) For $N > 4$, we can reach the stable IR fixed point (x^*, y^*) in two different ways, see fig. 4a,

$$(0, 0) \rightarrow (1, 0) \rightarrow (x^*, y^*) \quad \text{or} \quad (0, 0) \rightarrow (x^*, y^*), \quad (3.26)$$

corresponding to the l.h.s. and r.h.s. of eq. (3.25).

(ii) For $N > 10$, we can also compare, see fig. 4b,

$$(0, 0) \rightarrow (1, 0) \rightarrow (0, 1) \quad \text{versus} \quad (0, 0) \rightarrow (0, 1). \quad (3.27)$$

Remember that each $(\Delta c)_{i \rightarrow j}$ in eq. (3.25) has to be computed using the sum rule eq. (2.9) applied to the corresponding off-critical theory, having the i th fixed point as UV limit. Within the coordinates (x, y) considered so far, the UV fixed point was the free gaussian theory $(0, 0)$. Thus, without extra work we can particularize eq. (3.24) for the flows

$$\begin{aligned} c(0, 0) - c(1, 0) &= \frac{5}{48} \frac{N(N+2)}{(N+8)^2} \epsilon^3, \\ c(0, 0) - c(0, 1) &= \frac{5}{48} \frac{N}{27} \epsilon^3, \\ c(0, 0) - c(x^*, y^*) &= \frac{5}{48} \frac{(N+2)(N-1)}{27N} \epsilon^3. \end{aligned} \quad (3.28)$$

The other flows appearing in eqs. (3.26), (3.27) should be described in another coordinate patch, having $(1, 0)$ as UV fixed point. A more advanced perturbative technique is needed, because the starting theory is not free. Nevertheless, we can make use of the fact that the $(1, 0)$ fixed point is conformal invariant (the trace of the stress tensor vanishes) and use the “conformal perturbation theory” (see appendix A and ref. [5]). This technique generalizes the one used for flowing off conformal theories in two dimensions [1,17]. Conformal invariance in higher dimensions fixes the form of 2- and 3-point functions, up to some coefficients, the conformal dimensions of fields and the structure constants. The first-order perturbative expansion only requires these ingredients, thus it can be given for any flow in any dimension.

Let us first build a suitable basis of operators around the $(0, 0)$ fixed point. We choose the orthonormal basis

$$\begin{aligned} \phi_s^{(0,0)} &\equiv A \left(\sum_i \varphi_i^2 \right)^2, & \phi_\perp^{(0,0)} &\equiv B \left(\sum_i \varphi_i^4 - \frac{3}{N+2} \left(\sum_i \varphi_i^2 \right)^2 \right), \\ A &= \sqrt{\frac{3}{N(N+2)}}, & B &= \sqrt{\frac{N+2}{N(N-1)}}, \end{aligned} \quad (3.29)$$

which we call “symmetric” and “orthogonal” respectively. It is easy to compute the three-point functions made out of these two operators, and obtain the following structure constants

$$\begin{aligned} C_{\text{sss}}^{(0,0)} &= \frac{A(N+8)}{9} C, & C_{\text{ss}\perp}^{(0,0)} &= 0, \\ C_{\text{s}\perp\perp}^{(0,0)} &= \frac{A(N-2)}{N+2} C, & C_{\perp\perp\perp}^{(0,0)} &= \frac{B(N-2)}{N+2} C, \end{aligned} \quad (3.30)$$

where C is the N -independent part of the structure constant. Note that, to the order in ϵ we are working, the structure constants remain the same at the $(1, 0)$ fixed point, i.e.

$$C_{ijk}^{(0,0)} = C_{ijk}^{(1,0)} \quad i, j, k = s, \perp, \quad (3.31)$$

and therefore we can omit these superindices from now on.

Using the orthonormal basis we have just presented, we use the ϕ_s field as a perturbation off the gaussian fixed point. This perturbation leads us to the $(1, 0)$ point. In general, conformal perturbations are subject to RG mixing of operators. Nevertheless, the flow between $(0, 0)$ and $(1, 0)$ is free of this intricacy because $\langle \phi_s \phi_\perp \rangle^{(0,0)} = 0$ and $C_{\text{s}\perp\text{s}} = 0$, i.e. ϕ_s and ϕ_\perp stay orthogonal all the way along the RG flow ^{*}.

It is easy to compute the scaling dimensions of the symmetric and the orthogonal operators which get renormalized,

$$\begin{aligned} \Delta_s^{(1,0)} &= \Delta_s^{(0,0)} + 2(d - \Delta_s^{(0,0)}) = d + \epsilon \equiv d - y_s, \\ \Delta_\perp^{(1,0)} &= \Delta_\perp^{(0,0)} + 2(d - \Delta_s^{(0,0)}) \frac{C_{\perp\perp\text{s}}}{C_{\text{sss}}} = d - \frac{N-4}{N+8} \epsilon \equiv d - y_\perp. \end{aligned} \quad (3.32)$$

The above equations tell us that the symmetric operator becomes irrelevant at the $(1, 0)$ fixed point whereas the orthogonal one is relevant for $N > 4$, in agreement with the discussion of the phase diagram.

Having established the new basis of fields at $(1, 0)$, we can now compute the flows off this point driven by $\phi_\perp^{(1,0)}$ for $N > 4$. In this case, though $\langle \phi_s \phi_\perp \rangle^{(1,0)} = 0$, we note that $C_{\text{p}\perp\perp} \neq 0$, which implies that $\phi_s^{(1,0)}$ and $\phi_\perp^{(1,0)}$ do mix along the new flow. Thus, we are forced to consider a general two-parameter deformation of the $(1,0)$ fixed point, namely

$$g_\perp \phi_\perp^{(1,0)} + g_s \phi_s^{(1,0)}.$$

^{*} Actually, in this case the general system of beta functions given in appendix A simplifies to a single equation.

The system of beta functions one obtains in conformal perturbation theory is (see appendix A)

$$\begin{aligned}\beta_{\perp} &= -y_{\perp} g_{\perp} - (C_{\perp\perp\perp} g_{\perp}^2 + 2C_{\perp\perp s} g_{\perp} g_s) \\ \beta_s &= -y_s g_s - (C_{s\perp\perp} g_{\perp}^2 + C_{sss} g_s^2).\end{aligned}\tag{3.33}$$

The above set of equations correspond to the new c -function for flows off the $(1, 0)$ fixed point, call it \tilde{c} ,

$$\tilde{c}(g_{\perp}, g_s) = c(1, 0) - y_{\perp} \frac{g_{\perp}^2}{2} - y_s \frac{g_s^2}{2} - \left(C_{\perp\perp\perp} \frac{g_{\perp}^3}{3} + C_{\perp\perp s} g_{\perp}^2 g_s + C_{sss} \frac{g_s^3}{3} \right).\tag{3.34}$$

It turns out that the system of beta functions has two solutions

$$(G_{\perp}^*, g_s^*) = \begin{cases} -\frac{\epsilon}{C} \left(\frac{N-4}{BN}, \frac{A(N-4)^2}{(N+8)} \right), \\ -\frac{\epsilon}{C} \left(\frac{1}{B}, \frac{2AN(N-1)}{(N+8)} \right), \end{cases}\tag{3.35}$$

Finally, we can substitute back the solutions of the $\beta_s, \beta_{\perp} = 0$ system in the \tilde{c} -function and get, for the two solutions,

$$\Delta\tilde{c} = \begin{cases} \frac{5}{48} \frac{(N+2)(N-4)^3}{27N(N+8)} \epsilon^3 = c(1, 0) - c(x^*, y^*) \\ \frac{5}{48} \frac{N(N-1)(N-10)}{(N+8)^2} \epsilon^3 = c(1, 0) - c(0, 1) \end{cases}\tag{3.36}$$

The comparison of this result to eq. (3.24) shows that the two solutions correspond to the two flows in eqs. (3.26), (3.27) and, therefore, the additivity property of our c -charge, eq. (3.25), does hold.

We are now confident of the limiting procedure eq. (2.10) which defines the c -charge at fixed points from the spectral function $c(\mu, \Lambda)$ away from criticality ($\Lambda \neq 0$). In a perturbative domain, it produces consistent results for inequivalent coupling coordinates, so that the c -charge is indeed a universal quantity attached to each fixed point.

These results, though expected, have taught us that, in general, we have to allow for irrelevant fields in the expansion of Θ in eq. (2.5), if they mix with the relevant one driving the flow.

4. The $O(N)$ sigma-model in the large- N expansion

In sect. 3 we considered RG flows between infinitesimally close fixed points, $\Delta c \ll 1$. The large- N expansion allows, instead, to describe RG flows which run over a large distance in coupling space, e.g. $\Delta c \sim N$, thus non-perturbative with respect to the coupling. Actually, the saddle-point method amounts to the resummation of an infinite set of diagrams of conventional perturbation theory, and it leads to beta and c functions which are non-analytic in the coupling (the mass for the sigma-model). Further corrections in the $1/N$ expansion are similarly non-analytic for what concerns the c -theorem and the sum rule. In the following, we study the flow in the symmetric phase of the $O(N)$ sigma-model, for large N and $2 \leq d \leq 4$.

The $O(N)$ symmetric non-linear sigma-model is defined by an action which contains N fields φ_i and a Lagrange multiplier α_0

$$Z = \int \mathcal{D}\alpha_0 \mathcal{D}\varphi^i e^{-S}, \quad (4.1)$$

$$S = \int d^d x \frac{1}{2} \left[\partial_\mu \varphi^i \partial_\mu \varphi^i + \alpha_0 \left(\varphi^i \varphi^i - \frac{N}{g_0^2} \right) \right],$$

where the fields and the couplings are conveniently rescaled for the large- N expansion. The integration over φ_i produces the effective action

$$S_{\text{eff}} = \frac{N}{2} \ln \det_A (-\partial^2 + \alpha_0(x)) - \frac{N}{2g_0^2} \int d^d x \alpha_0(x). \quad (4.2)$$

which, for large N , can be estimated using the saddle-point approximation. The saddle-point equation is

$$\frac{1}{g_0^2} = \int^\Lambda \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2}, \quad m^2 = \langle \alpha_0 \rangle_{\text{s.p.}}, \quad (4.3)$$

where Λ is the cut-off, and m^2 is the translational invariant value of the field α_0 at the saddle point. The $1/N$ expansion is obtained by setting $\alpha_0(x) = m^2 + (\alpha(x)/\sqrt{N})$ and expanding S_{eff} around the saddle point. The Feynman diagrams and the basic properties of the theory are discussed in ref. [26]. Investigations of the c -theorem for this theory were initiated in ref. [6].

For $2 \leq d \leq 4$, the saddle point defines the physical mass m for φ , in terms of the bare coupling g_0^2 , while the higher-order corrections to the saddle point are

weighted with the coupling $1/N$. From the heat kernel regularization of the determinant in eq. (4.2) we obtain

$$\frac{1}{g_0^2} = \frac{2}{(4\pi)^{d/2}} \left[\frac{\Lambda^{d-2}}{d-2} - \frac{m^{d-2}}{d-2} \Gamma\left(2 - \frac{d}{2}\right) - \frac{\Lambda^{d-4} m^2}{d-4} + \sum_{k=2}^{\infty} \frac{1}{k!} \frac{\Lambda^{d-2k-2} (-m^2)^k}{d-2k-2} \right]. \quad (4.4)$$

In particular, for $d = 2$,

$$\frac{1}{g_0^2} = \frac{1}{4\pi} \ln\left(\frac{\Lambda^2 + m^2}{m^2}\right). \quad (4.5)$$

The critical point $g_{0,\text{cr}}^2$ is obtained for $m = 0$

$$g_{0,\text{cr}}^2 = 0 \quad (d = 2), \quad \frac{1}{g_{0,\text{cr}}^2} = \frac{2}{(4\pi)^d} \frac{\Lambda^{d-2}}{d-2} \quad (d > 2), \quad (4.6)$$

and the massive phase corresponds to $g_0^2 > g_{0,\text{cr}}^2$. On the other hand, for $g_0^2 < g_{0,\text{cr}}^2$, there is no solution to the $O(N)$ -symmetric saddle-point equation. There are non-symmetric saddle points, obtained by integration only $N - 1$ fields in eq. (4.1), giving $\langle \varphi \rangle \neq 0$ [26]. In short, the phase diagram of the sigma-model for $2 \leq d \leq 4$ contains the $O(N)$ -symmetric phase $g_0^2 > g_{0,\text{cr}}^2$, with $\langle \alpha_0 \rangle = m^2$, $\langle \varphi \rangle = 0$, and (for $d > 2$), the spontaneously broken one $g_0^2 < g_{0,\text{cr}}^2$, with $\langle \alpha_0 \rangle = 0$ and $\langle \varphi \rangle \neq 0$.

4.1. THE STRESS TENSOR AND THE SPECTRAL DENSITY

Let us consider the RG flow in the symmetric phase which leaves the critical point and reaches the trivial fixed point $m = \infty$. We want to compute the associated sum rule eq. (2.9), to leading order in $1/N$. We start by finding the expression of the trace of the stress tensor for this perturbation in terms of the α field. This is obtained by putting the theory on a curved space-time background and taking a Weyl variation with respect to the metric of both the effective action and the saddle-point equation [6]. Furthermore, for $2 < d < 4$ we let the cut-off go to infinity after one subtraction of the coupling. We obtain *,

$$\begin{aligned} \langle \Theta(x) \rangle &= \beta(m) \langle \alpha(x) \rangle_{\text{s.p.}} = 0 \\ \langle \Theta(x) \Theta(0) \rangle &= (\beta(m))^2 \langle \alpha(x) \alpha(0) \rangle_{\text{s.p.}}, \quad x^2 \neq 0, \end{aligned} \quad (4.7)$$

* Note that Θ vanishes classically for $d = 2$, but not at the quantum level (saddle point). We renormalize the theory by going to $d = 2 + \epsilon$, and find a non-vanishing beta function to leading order in $1/N$.

where

$$\beta(m) = m^{d-2} \sqrt{N} \frac{\Gamma(2 - (d/2))}{2^{d-1} \Gamma(d/2)}. \quad (4.8)$$

The leading contribution to the α propagator is obtained by expanding S_{eff} to second order in α

$$\langle \alpha(p) \alpha(-p) \rangle_{\text{s.p.}} = -2/B(p), \quad (4.9)$$

where $B(p)$ is the well-known bubble diagram

$$\begin{aligned} B(p) &= \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + m^2)((p-q)^2 + m^2)} = \int_{4m^2}^{\infty} \frac{d\mu^2}{\pi} \frac{\text{Im } B(p^2 = -\mu^2)}{p^2 + \mu^2} \\ &= \frac{m^{d-4} \Gamma((4-d)/2)}{2^d \pi^{d/2}} F\left(1, \frac{4-d}{2}; \frac{3}{2}; -\frac{p^2}{4m^2}\right) \end{aligned} \quad (4.10)$$

$$\text{Im } B(\mu^2) = \frac{1}{\Gamma((d-1)/2) 2^d \pi^{(d-3)/2}} \frac{1}{\mu} \left(\frac{\mu^2}{4} - m^2 \right)^{(d-3)/2} \theta(\mu^2 - 4m^2)$$

where $\theta(x)$ is the step function and F the hypergeometric function.

4.2. THE SUM RULE IN TWO DIMENSIONS

In two dimensions, the critical point is $g_0 = 0$, and there is only the symmetric phase, in agreement with Coleman's theorem on the absence of spontaneous symmetry breaking. Around the critical point, the theory contains $N-1$ weakly interacting bosons, asymptotically free [16], thus we can assign the value $c_{\text{UV}} = N-1$. The symmetric phase is massive and the IR Hilbert space contains N interacting massive scalar particles with purely elastic scattering, thus $c_{\text{IR}} = 0$. Therefore, the c -theorem sum rule is expected to give $\Delta c = N$ for the flow in the symmetric phase, to leading order in $1/N$.

In order to compute it, we need the spectral function $c(\mu) \propto \text{Im} \langle \Theta \Theta \rangle$, given by eqs. (4.7), (4.9). The imaginary part of the α -propagator is $\text{Im}(-2/B) = 2 \text{Im}(B)/|B|^2$, where $B(p)$ can be expressed in terms of logarithms for $d=2$. Putting all together, we obtain

$$\begin{aligned} c(\mu) &= (6/\pi^2 \mu^3) \text{Im} \langle \Theta(p) \Theta(-p) \rangle|_{p^2 + \mu^2 = 0} \\ &= \frac{6N}{\mu} \sqrt{1 - \frac{4m^2}{\mu^2}} \left[\pi^2 + \log^2 \left(\frac{1 + \sqrt{1 - 4m^2/\mu^2}}{1 - \sqrt{1 - 4m^2/\mu^2}} \right) \right]^{-1} \theta(\mu^2 - 4m^2). \end{aligned} \quad (4.11)$$

The sum rule is

$$\Delta c = \int_0^\infty d\mu \, c(\mu) = 3N \int_{-1}^1 \frac{dz}{1-z^2} \frac{z^2}{\pi^2 + \log^2((1+z)/(1-z))}. \quad (4.12)$$

After a change of variable, the integral can be computed in the complex plane, giving the expected result

$$\Delta c = N \quad (d = 2, N \rightarrow \infty). \quad (4.13)$$

This is a rather remarkable check of Zamolodchikov's theorem, owing to the non-perturbative character of this flow.

In this calculation we had no problems with IR singularities, which instead can appear in the perturbation expansion ^{*} in g_0^2 . Actually, the large- N expansion is better because it correctly reproduces the mass of the theory by resumming infinite perturbative diagrams. Let us however stress that IR singularities cannot, in general, appear in the c -theorem, unless the theory itself is sick and plagued with them. As remarked in sect. 2, our formulation of the c -theorem makes use of the states of the Hilbert space, which would be ill-defined in the presence of IR singularities.

4.3. THE SUM RULE IN MORE THAN TWO DIMENSIONS

The sum rule can also be computed for $2 < d \leq 4$ by using the previous formulae. Before that, let us find the value of the c -charge at the critical point of the sigma model. Having no intrinsic means to compute it, we have to relate it to the value of the gaussian free theory, $c = N$ in our conventions. We use a number of arguments known in the literature [16] to embed the sigma model into its linear realization, the $O(N)$ -symmetric $\lambda\varphi^4$ theory (fig. 6). The latter theory has a renormalized mass parameter m' , and the additional interaction strength λ , while the unique mass parameter of the non-linear sigma-model has to be thought as $m = m(m', \lambda)$, at least in a region close to the critical point. The flow in the sigma

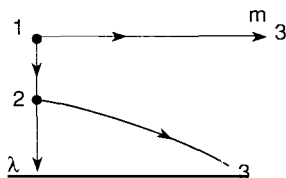


Fig. 6. Schematic picture of the RG flows in the linear and non-linear sigma-models.

^{*} It has been suggested that the c -theorem could fail because of them [27].

model in the symmetric phase $m \rightarrow \infty$, will correspond to the line drawn in the (m', λ) plane of fig. 6 (possibly leaving the plane for the large m). For $2 < d \leq 4$, the sigma-model critical point $m = 0$ is believed to fall into the universality class of the $O(N)$ -symmetric Wilson point of $\lambda\varphi^4$, discussed in subsect. 3.4, i.e. $(m' = 0, \lambda = \lambda^* = O((d-4)/N))$. The other fixed point of $\lambda\varphi^4$, the gaussian one, is close to it for large N , and the corresponding flow between them was previously shown to give $\Delta c = O(1)$, which is subleading for large N . Therefore, we can assign $c = N$ to the critical point of the sigma model for large N .

Actually, we are testing the c -charge along a chain of flows as in eq. (2.11), where the fixed points are: (1) the gaussian one, (2) the critical sigma-model, and (3) the trivial theory $c = 0$ (see fig. 6). The last one actually sits in two different points of coupling space, $(m' = \infty, \lambda = 0)$ and $(m = \infty)$, thus we are in the situation of non-perturbative flows of fig. 2.

Let us first discuss the sum rule in four dimensions, where the Wilson point merges with the gaussian one for any N . At $d = 4$ a new logarithmic singularity appears in the saddle-point equation eq. (4.3). According to the previous discussion (eqs. (4.7)), we regulate it by using dimensional regularization. Thus we find,

$$\begin{aligned}\Theta(x) &\sim \frac{\sqrt{N}}{4} \frac{m^{2-\epsilon}}{\epsilon} \alpha(x), \quad (d = 4 - \epsilon) \\ B(p) &\sim \frac{1}{\epsilon} \frac{m^{-\epsilon}}{2\pi^2} \left(1 + O\left(\epsilon \frac{p^2}{4m^2}\right) \right).\end{aligned}\tag{4.14}$$

The singularities $1/\epsilon^2$ cancel in the spectral density $c(\mu)$, which has a finite expression in any dimension, as it should. Finally,

$$\Delta c = N \int_0^\infty d\mu \frac{c(\mu)}{\mu^2} = N \int_{2m}^\infty d\mu \, 120 \frac{m^4}{\mu^5} \sqrt{1 - \frac{4m^2}{\mu^2}} = N, \quad (d = 4, N \rightarrow \infty) \tag{4.15}$$

Indeed, $c(\mu)$ has the same form as in the massive perturbation of the free four-dimensional theory, computed in ref. [6] (see eq. (3.31) therein). Therefore, the sigma-model approaches the free theory all along the massive flow, and the sum rule confirms the expectations on the triviality of the model, to leading order in $1/N$. Moreover, the test of additivity of the c -charge eq. (2.11) is trivial in this case.

As an example of the case $2 < d < 4$, we shall discuss $d = 3$, where eqs. (4.10) can be expressed in terms of elementary functions, and the final integral of $c(\mu)/\mu$ computed numerically. This was already done in our previous work [6], eq. (6.47), and we only quote the result,

$$\Delta c = N(0.5863 \dots), \quad (d = 3, N \rightarrow \infty).\tag{4.16}$$

Therefore, we do not find agreement with the expected result $\Delta c = N$. A possible explanation of this failure is, of course, that our candidate c -charge does not fulfil the additivity property, thus it is not a universal quantity uniquely associated to the fixed points. Indeed, in this case of non-perturbative flows, we do not have arguments in support of additivity, nor can we exclude coordinate singularities in the space of theories of fig. 6.

Another explanation could be that the c -theorem, though correct, is not easy to verify for non-perturbative flows. In particular, we cannot exclude mixing with irrelevant fields in the expansion of $\Theta = \beta^i \Phi_i$, due to our partial understanding of the critical field theory of the sigma model. Actually, the example of subsect. 3.4 showed that irrelevant fields can appear, when flowing off interacting critical theories. The fact that the sum rule (4.16) has a value of Δc lower than expected may be an indication that we are missing contributions to Θ .

5. Conclusions

In this paper, we have put forward a candidate for a monotonically decreasing function along RG trajectories in $d > 2$, which is the analogue of the Zamolodchikov c -function in two dimensions. The analysis of this c -function in the Ginsburg–Landau models using epsilon expansion as well as conformal perturbation theory displays all the nice properties expected from a c -theorem. Our c -function behaves as a height function in a perturbative domain of the space of theories.

The study of the sigma models is less clear. In two dimensions, the theorem works as it should, proving harmless the fears concerning infrared problems. In four dimensions, it confirms that the theory coalesces with the free massive one and there are, again, no problems. Nevertheless, in three dimensions, our computation of ref. [6] presents a result which is in disagreement with the theorem. Different ways out are sketched in sect. 4, which deserve further investigation.

There is an observation we want to emphasize. Physically meaningful theories, like QCD, behave at short and long distances as free theories *. Remarkably enough, it was found in ref. [6] that the c -charges associated to the spin 0 and 2 spectral densities are equal for free theories of spin 0 and 1/2. It is, then natural to conjecture that both c -charges are equal in general, that is $c^{(2)} = c^{(0)}$ for free theories. Since $c^{(2)}$ is well defined at fixed points, a general proof stating this equality for free theories would be enough to apply the c -theorem to the Standard Model and beyond.

As an example, it is easy to see that the long-distance Nambu–Goldstone realization of QCD in terms of pions is in agreement with the conjectured

* In fact, there are fewer non-trivial scale invariant theories in four dimensions, in contradistinction with many non-gaussian fixed points known to exist in less than four dimensions.

c -theorem. Adapting a previous example [3], we consider QCD with N_f flavors and N_c colours in four dimensions. Using the fact that $c_0 = 1$ for particles with spin-0 and $c_{1/2} = 6$ for spin-1/2, we look at the balance between the short- and long-distance realizations of QCD

$$N_f N_c c_{1/2} + (N_c^2 - 1)c_1 \geq (N_f^2 - 1)c_0,$$

for any value of c_1 , provided asymptotic freedom holds, i.e. $N_f < \frac{11}{2}N_c$. Note that we did not write the value of c_1 , because we cannot easily compute this number by a free massive perturbation, as done in ref. [6] for the spin 0 and 1/2 particles. The massless limit of the massive spin-1, or Proca, particle is not the massless spin-zero particle, because the number of degrees of freedom changes. Probably, one has to resort to the Higgs mechanism to give mass to a gauge field in a correct way.

To conclude, we would like to mention some lines to progress. The inclusion of additional symmetries (e.g. current algebra) remains to be done. Further analysis of the constraints imposed by the present form of the c -theorem in theories which go beyond the Standard Model also is left for the future.

We want to thank J.L. Cardy, P.H. Damgaard, G. Shore and A.B. Zamolodchikov for useful discussions. This work has been supported by CYCIT and the EEC Science Twinning grant SC1000337. X. V. acknowledges the support of an FPI grant.

Appendix A. Conformal perturbation theory in d dimensions

Conformal invariance fixes in any dimension the form of the two- and three-point correlators. This property can be used to set a perturbative expansion around conformal field theories which are non-gaussian (see also ref. [5]). Standard perturbation theory appears as a particular case of this conformal perturbation theory.

Let us consider then a quantum field theory, invariant under the conformal group, in an arbitrary dimension d . We denote by $\phi_i(x)$ a generic field and by Δ_i its dimension. Conformal invariance fixes the form of the two- and three-point functions to be

$$\begin{aligned} \langle \phi_i(x) \phi_j(y) \rangle_{\text{CFT}} &= \frac{\delta_{ij}}{|x - y|^{2\Delta_i}} \\ \langle \phi_i(x) \phi_j(y) \phi_k(z) \rangle_{\text{CFT}} &= \frac{C_{ijk}}{|x - y|^{\Delta_i + \Delta_j - \Delta_k} |y - z|^{\Delta_j + \Delta_k - \Delta_i} |x - z|^{\Delta_i + \Delta_k - \Delta_j}}, \end{aligned} \quad (\text{A.1})$$

where C_{ijk} are constant coefficients called structure constants.

A conformally invariant field theory can be perturbed with one of its operators (call it ϕ_p , with dimension Δ_p). A correlator in the perturbed theory is defined to be

$$\langle \phi_1(x_1) \dots \phi_N(x_N) \rangle = \frac{\left\langle \phi_1(x_1) \dots \phi_N(x_N) \exp\left(\lambda_0 \int d^d x \phi_p(x)\right) \right\rangle_{\text{CFT}}}{\left\langle \exp\left(\lambda_0 \int d^d x \phi_p(x)\right) \right\rangle_{\text{CFT}}}. \quad (\text{A.2})$$

The subscript CFT means that the vacuum expectation value has to be computed in the conformal theory. For two-point functions, at first order in λ_0 , eq. (A.1) allows one to write a general expression for any dimension of space-time,

$$\begin{aligned} \langle \phi(x) \phi(0) \rangle &= \langle \phi(x) \phi(0) \rangle_{\text{CFT}} + \lambda_0 \int d^d y \langle \phi(x) \phi(0) \phi_p(y) \rangle_{\text{CFT}} + \mathcal{O}(\lambda_0^2) \\ &= \frac{1}{(x^2)^\Delta} \left(1 + \lambda_0 C_{\phi\phi p} \frac{4A}{(d-\Delta)} |x|^{d-\Delta_p} + \mathcal{O}(\lambda_0^2) \right). \end{aligned} \quad (\text{A.3})$$

The constant A is given by

$$A = \frac{\pi^{d/2} \Gamma(\Delta_p - d/2) \Gamma^2(1 + (d - \Delta_p)/2)}{\Gamma^2(\Delta_p/2) \Gamma(1 + d - \Delta_p)}. \quad (\text{A.4})$$

Remark that, for $(d - \Delta_p) \rightarrow 0$, A goes to $\pi^{d/2}$. This is the case of slightly relevant perturbing fields. For these nearly marginal perturbations, expression (A.3) does not make sense and needs renormalization.

At this point, the procedure mimics the two-dimensional case [1,17], so we will skip the details, recalling only some important points. We define the renormalized field and the renormalized coupling constant as

$$\Phi(x, g) \equiv \frac{1}{\sqrt{Z}} \phi(x), \quad g \equiv Z_g \lambda_0. \quad (\text{A.5})$$

We then set wave function and coupling constant renormalizations at a scale κ to be

$$\langle \Phi_p(x, g) \Phi_p(0, g) \rangle|_{|x|=\kappa^{-1}} = \kappa^{2d}, \quad (\text{A.6})$$

$$\Theta(x) = V_d \beta(g) \Phi_p(x, g), \quad (\text{A.7})$$

Following the steps described in ref. [17], one can get the expression for the renormalized coupling constant in terms of the bare one

$$g = \kappa^{\Delta-d} \lambda_0 \left(1 + \lambda_0 \kappa^{\Delta-d} \frac{A}{d-\Delta} C_{\text{ppp}} \right). \quad (\text{A.8})$$

It is now convenient to rescale the coupling constant $g \rightarrow g/A$ to simplify our formulae. Next, we compute the beta function

$$\beta(g) = -(d - \Delta_p)g - C_{\text{ppp}}g^2, \quad (\text{A.9})$$

and the anomalous dimension for the new scaling dimensions for the perturbing field as well as for any other field

$$\begin{aligned} \Delta_p(g) &= \Delta_p - 2C_{\text{ppp}}g, \\ \Delta(g) &= \Delta - 2C_{\phi\phi p}g. \end{aligned} \quad (\text{A.10})$$

From eq. (A.9), we see that there is a new fixed point at an infinitesimal distance from the original conformal point,

$$g^* = -\frac{d - \Delta_p}{C_{\text{ppp}}}. \quad (\text{A.11})$$

At this point, the scaling dimensions are

$$\begin{aligned} \Delta_p^* &\equiv \Delta_p(g^*) = 2d - \Delta_p, \\ \Delta^* &\equiv \Delta(g^*) = \Delta + 2(d - \Delta_p) \frac{C_{\phi\phi p}}{C_{\text{ppp}}}. \end{aligned} \quad (\text{A.12})$$

Note that the IR scaling dimension of the perturbing field is insensitive to the value of the structure constant.

Using eq. (A.9), one can also write the Callan–Symanzik equation for the two-point correlators,

$$\left(2|x| \frac{\partial}{\partial |x|} + 2\Delta_p(g) + \beta(g) \frac{\partial}{\partial g} \right) \langle \Phi_p(x, g) \Phi_p(0, g) \rangle = 0, \quad (\text{A.13})$$

and find a renormalization group improved two-point function for the perturbing field as the solution of (A.13) that fulfills condition (A.6). Proceeding thus, one has

$$\langle \Phi_p(x, g) \Phi_p(0, g) \rangle = \frac{\kappa^{2d}}{|\kappa x|^{2\Delta_p}} \frac{1}{\left(1 - g C_{\text{ppp}} \frac{(|\kappa x|^{d-\Delta_p} - 1)}{d - \Delta_p} \right)^4}. \quad (\text{A.14})$$

With this expression, Δc can be computed, using either of the following forms of the sum rule

$$\Delta c = \frac{d+1}{dV_d} \int_{|x|>\epsilon} d^d x |x|^d \langle \Theta(x) \Theta(0) \rangle = \int_0^\infty d\mu \frac{c(\mu)}{\mu^{d-2}}, \quad (\text{A.15})$$

where the spin-0 part of the spectral representation is

$$c(\mu) = \frac{2^d \Gamma(d)(d+1)}{\pi V_d} \frac{1}{\mu^3} \text{Im}(\langle \Theta(p) \Theta(-p) \rangle) |_{p^2 = -\mu^2}. \quad (\text{A.16})$$

In any case, what we find is

$$\Delta c = \frac{8}{3} \frac{d+1}{d} V_d^2 \frac{(d - \Delta_p)^3}{A^2 C_{\text{ppp}}^2}. \quad (\text{A.17})$$

which is the change of the c -charge between the UV and IR fixed points.

In general, the perturbing field does not close an algebra by itself in the sense of the operator product expansion. This causes mixing with other fields in the theory. It is then necessary to consider a system of beta functions

$$\beta_i = -(d - \Delta_i) g_i - \sum_{j,k} C_{ijk} g_j g_k, \quad (\text{A.18})$$

where $\{g_i\}$ span the coupling space. The c -function is then

$$c = -\frac{1}{2} \sum_i (d - \Delta_i) g_i^2 - \frac{1}{3} \sum_{i,j,k} C_{ijk} g_i g_j g_k, \quad (\text{A.19})$$

which contains the most general case. In practice, the system (A.18) may be simpler and solvable so that explicit expressions for the variation of c can be obtained, as in subsect. 3.5.

A.1. EXAMPLE 1. TWO-DIMENSIONAL CONFORMAL FIELD THEORIES

One particular set of two-dimensional conformal field theories is the unitary minimal series, whose central charges are given by the formula $c = 1 - 6/m(m+1)$, $m \geq 3$, where m is an integer. As discussed in ref. [1], perturbing one such minimal model with its least relevant field, if m is large, we obtain the next minimal model at $c(m-1)$. We can have an inkling of this fact recalling that

$$\begin{aligned} d - \Delta_p &= \frac{4}{m+1} = \frac{4}{m} + \mathcal{O}\left(\frac{1}{m^2}\right), \\ C_{\text{ppp}} &= \frac{4}{\sqrt{3}} + \mathcal{O}\left(\frac{1}{m}\right). \end{aligned} \quad (\text{A.20})$$

Applying (A.17), we correctly obtain

$$\Delta c = \frac{12}{m^3} + O\left(\frac{1}{m^4}\right) = c(m) - c(m-1), \quad (\text{A.21})$$

which is a check for our formulae.

A.2. EXAMPLE 2. THE ϵ -EXPANSION

The ϵ -expansion can be thought as an example of conformal perturbation theory. Recall that it consists in perturbing the massless free theory with $\lambda\varphi^4$ theory in $4 - \epsilon$ dimensions. In order to apply the previous formulae, we recall that the propagator of the scalar field is

$$\langle \varphi(x)\varphi(0) \rangle = \frac{1}{V_d(d-2)} \frac{1}{|x|^{d-2}}. \quad (\text{A.22})$$

To obtain proper conformal fields, operators have to be normalized. For instance,

$$\phi(x) = \sqrt{V_d(d-2)} \varphi(x).$$

As a conformal operator, φ^4 has dimension $\Delta_p = 4 - 2\epsilon$, so that $d - \Delta_p = \epsilon$. We see then that the operator is slightly relevant if ϵ is small enough. Other conformal features are

$$C_{\text{ppp}} = 6^{3/2}, \quad C_{\varphi\varphi\text{p}} = 0 \quad (\text{A.23})$$

(from this last equation we see that the two-point function $\langle \varphi\varphi \rangle$ gets no correction at the first order). This is just a particular case of the general conformal perturbation expansion and it is elaborated in subsect. 3.4.

References

- [1] A.B. Zamolodchikov, JETP Lett. 43 (1986) 730; Sov. J. Nucl. Phys. 46 (1987) 1090
- [2] D.A. Kastor, E.J. Martinec and S.H. Shenker, Nucl. Phys. B316 (1989) 590
- [3] J.L. Cardy, Phys. Lett. B215 (1988) 749
- [4] H. Osborn, Phys. Lett. B222 (1989) 97;
I. Jack and H. Osborn, Nucl. Phys. B343 (1990) 647
- [5] N.E. Mavromatos, J.L. Miramontes and J.L. Sánchez de Santos, Phys. Rev. D40 (1989) 535
- [6] A. Cappelli, D. Friedan and J.I. Latorre, Nucl. Phys. B352 (1991) 616
- [7] G. Shore, Phys. Lett. B253 (1991) 380; B256 (1991) 407
- [8] P.H. Damgaard, Stability and Instability of the renormalization group flows, preprint CERN-TH-6073-91, Proc. 2nd Conf. on the renormalization group, Dubna, September 1991
- [9] A.W. Ludwig and J.L. Cardy, Nucl. Phys. B285 (1987) 687
- [10] C. Athorne and I.D. Lawrie, Nucl. Phys. B265 (1985) 577
- [11] K.G. Wilson and J. Kogut, Phys. Rep. 12 (1974) 75

- [12] D.J. Wallace and R.K.P. Zia, *Ann. Phys. (N.Y.)* 92 (1975) 142
- [13] L.N. Lipatov, *Sov. Phys. JETP* 44 (1976) 1055;
P.S. Howe and P.C. West, *Phys. Lett.* B223 (1989) 371
- [14] E. Brézin, J.C. Le Guillou and J. Zinn-Justin, *in* Phase transitions and critical phenomena, Vol. 6, ed. C. Domb and M.S. Green (Academic Press, New York, 1976)
- [15] A.B. Zamolodchikov, *Sov. J. Nucl. Phys.* 44 (1986) 529
- [16] J. Zinn-Justin, *Quantum field theory and critical phenomena* (Clarendon, Oxford, 1989);
E. Brézin and J. Zinn-Justin, *Phys. Rev.* B14 (1976) 3110
- [17] A. Cappelli and J.I. Latorre, *Nucl. Phys.* B340 (1990) 659
- [18] D.Z. Freedman, J.I. Latorre and X. Vilasís, *Mod. Phys. Lett.* A6 (1991) 531
- [19] J.L. Cardy, *Phys. Rev. Lett.* 60 (1988) 2709
- [20] J.L. Cardy, *Nucl. Phys.* B290 (1987) 355; *Phys. Rev. Lett.* 65 (1990) 1443;
A. Cappelli and A. Coste, *Nucl. Phys.* B314 (1989) 707
- [21] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Nucl. Phys.* B241 (1984) 333
- [22] C. Vafa, *Phys. Lett.* B212 (1988) 28
- [23] M. Lässig, *Nucl. Phys.* B334 (1990) 652
- [24] C. Itzykson and J.-M. Drouffe, *Statistical field theory* (Cambridge University Press, Cambridge, 1989)
- [25] G. Felder, *Commun. Math. Phys.* 111 (1987) 101
- [26] B. Rosenstein, B.J. Warr and S.H. Park, *Nucl. Phys.* B336 (1990) 435
- [27] G. Curci and G. Paffuti, *Nucl. Phys.* B312 (1989) 227