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Modular invariant partition functions in the quantum Hall effect

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Abstract

We study the partition function for the low-energy edge excitations of the incompressible electron fluid. On an annular geometry, these excitations have opposite chiralities on the two edges; thus, the partition function takes the standard form of rational conformal field theories. In particular, it is invariant under modular transformations of the toroidal geometry made by the angular variable and the compact Euclidean time. The Jain series of plateaus have been described by two types of edge theories: the minimal models of the $W_{1+\infty}$ algebra of quantum area-preserving diffeomorphisms, and their non-minimal version, the theories with $\widehat{U(1)} \times \widehat{SU(m)}_1$ affine algebra. We find modular invariant partition functions for the latter models. Moreover, we relate the Wen topological order to the modular transformations and the Verlinde fusion algebra. We find new, non-diagonal modular invariants which describe edge theories with extended symmetry algebra; their Hall conductivities match the experimental values beyond the Jain series.
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To Claude Itzykson

1. Introduction

At the plateaus of the quantum Hall effect [1], the electrons form an incompressible fluid [2], in which $(2+1)$-dimensional density waves have a high gap. At energies below this gap, there exist $(1+1)$-dimensional, chiral excitations at the edge of the sample, which can be described by a low-energy effective field theory [3]. This is the

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$(1+1)$-dimensional conformal theory [4] on the edge [3,5], which is equivalent to the topological $(2+1)$-dimensional Chern-Simons theory [3,6,7]. For example, the Laughlin plateaus with Hall conductivity $\sigma_H = (e^2/h)\nu$, $\nu = 1, 1/3, 1/5, \ldots$ have been successfully described by the Abelian Chern-Simons theory, or by the corresponding conformal field theory with $\widehat{U(1)}$ current algebra and Virasoro central charge $c = 1$. The spectrum predicted by these theories has been confirmed by recent experiments [8], such as the resonant tunneling experiment [9] analysed in Ref. [10].

The quantum numbers of edge excitations always take rational values; this suggests [6] that the conformal field theories describing them should belong to the special, well-understood class of *rational* conformal field theories (RCFT) [11]. By definition, a RCFT contains a *finite* number of representations of the (chiral) symmetry algebra, which is the Virasoro algebra extended by other currents in the theory. These representations are encoded in the partition function [12] of the Euclidean theory defined on the geometry of a space-time torus $S^1 \times S^1$. This gives a precise inventory of all the states in the Hilbert space and serves as a *definition* of the RCFT. Clearly, each representation of the extended symmetry algebra describes a sector of the Hilbert space which contains infinite states. In the literature, it was shown that the rational spectrum follows from the extended symmetry and, moreover, from the invariance of the partition function under modular transformations of the periods of the torus [12,13], which span the discrete group $\Gamma = SL(2,\mathbb{Z})/\mathbb{Z}_2$. Furthermore, Verlinde [14,11] has shown the relation between modular invariance and the *fusion rules* of the extended symmetry algebra, which are the selection rules for making composite states. Moreover, Witten [15,16] has explained that any RCFT is associated to a Chern-Simons theory, and that the torus partition function in the former theory corresponds to a path-integral amplitude for the latter theory on the manifold $S^1 \times S^1 \times \mathbb{R}$, where $\mathbb{R}$ is the time axis.

In the quantum Hall effect, a complete description of the edge excitations similarly requires the construction of their partition function. In this paper, we consider a spatial annulus with Euclidean compact time: this space-time manifold is topologically $\mathcal{M} = S^1 \times S^1 \times I$, because the radial coordinate runs over the finite interval I and the angular coordinate and Euclidean time are both compact. We thus find the partition function of the conformal field theory on the edge(s) $\partial\mathcal{M} = S^1 \times S^1$, i.e. a space-time torus. Due to the Witten relation, this is also a path integral of the Chern-Simons theory on the manifold $\mathcal{M}$, modulo a convenient renaming of the radial and Euclidean time coordinates.

This annulus partition function takes the standard sesquilinear form [12] in terms of the characters of the chiral and antichiral algebras, which pertain to the inner and outer edges, respectively. Moreover, it is invariant under transformations of the modular parameter τ of the space-time torus $S^1 \times S^1$. Actually, the presence of fermionic excitations in the quantum Hall effect implies the invariance under the $\Gamma(2)$ subgroup of the modular group, as in the Neveu-Schwarz sector of super-symmetric conformal theories [4]. The charge spectrum is constrained by additional modular conditions, which have analogs in the Chern-Simons quantization on $\mathcal{M}$ [16].

In Section 2, we illustrate the derivation of modular invariant partition functions in the simpler case of $\widehat{U(1)}$ theories. We obtain the partition functions for the Laughlin plateaus $\nu = 1/p$, p odd, starting from the raw conformal field theory data of the characters of the affine Abelian algebra, and then imposing the modular conditions. We find that the maximal chiral algebra of the RCFT is the extension of $\widehat{U(1)}$ by a current of Virasoro dimension $h = p/2$, and the Verlinde fusion rules are the Abelian group $\mathbb{Z}_p$ (addition modulo p). This is an *odd* variant of the free boson theory compactified on a rational torus [4]. Note that these partition functions were also found in Refs. [17,18] by using physical arguments, but their modular properties were not analysed.

A nice property of the annulus partition function is that it encodes the Wen *topological order* [19], which is the degeneracy of the quantum Hall ground state on the (ideal) compact surface Σ_g of genus g. This degeneracy is accounted for by the effective Chern-Simons theory, where it corresponds [16] to the dimension of the Hilbert space of topologically non-trivial gauge fields on $\Sigma_g \times \mathbb{R}$. This dimension can be readily computed from the RCFT fusion rules and the S modular transformation, by using the Verlinde formulas and the Witten relation, suitably extended to manifolds with boundaries. We show that the Wen topological order can also be defined for the annulus geometry, and that it always takes the same value of the toroidal geometry Σ_1. Moreover, the Verlinde formulas immediately yield the topological order in the presence of static impurities in the Hall sample, a quantity which would otherwise need the explicit calculation of Chern-Simons amplitudes with Wilson lines [19]. For Abelian theories, we easily confirm that the topological order is independent of the impurities.

The Hall effect of spin-polarized electrons at the stable plateaus $\nu = m/(ms \pm 1)$, with $m = 2, 3, \ldots$ and s even, is well understood in terms of the Jain trial wave functions [20]. Two specific types of edge conformal field theories have been proposed. The first ones are characterized by the dynamical symmetry of the incompressible fluid under area-preserving diffeomorphisms of the plane [21,22], which implies [23] the quantum algebra $W_{1+\infty}$ [24]. Actually, the Jain plateaus are in one-to-one correspondence with the $W_{1+\infty}$ *minimal models* [25], which are $c = m$ conformal field theories made by fully degenerate $W_{1+\infty}$ representations [26]. The second proposal [27,3,7,28] is based on the multi-component Abelian Chern-Simons theories, which correspond to RCFTs with $\widehat{U(1)}^{\otimes m}$ affine algebras. In particular, the Jain series were shown to correspond to the special cases for which the $\widehat{U(1)}^{\otimes m}$ algebra extends to the $\widehat{U(1)} \times \widehat{SU(m)}_1$ one [27,7].

In Section 3, we recall some properties of these two classes of conformal field theories. The $\widehat{U(1)} \times \widehat{SU(m)}_1$ theory is a non-minimal version of the $W_{1+\infty}$ theory, which displays the same spectrum of charge and fractional spin of excitations, but with higher multiplicities [25]. We explain that the non-minimal and minimal theories have full and "hidden" $SU(m)$ symmetry, respectively. For $m = 2$, we discuss in detail the reduction of degrees of freedom connecting the former to the latter,[1] and the different properties of excitations in the two theories. In the literature, the non-minimal theory was first introduced and is more widely accepted, although the physical origin of the $SU(m)$ symmetry at the Jain plateaus is not well understood.

[1] This is the so-called Hamiltonian reduction [29].

In Section 4, we construct the partition functions for the $\widehat{U(1)} \times \widehat{SU(m)}_1$ theories. The simple *left-right diagonal* modular invariant is found for any $m = 2, 3, \ldots$; actually, it exists for any $\widehat{U(1)}^{\otimes m}$ theory, as also proposed in Refs. [17,18]. The spectrum of edge excitations of this partition function matches those described in earlier studies [27,3], with, in addition, a pairing of excitations between the two edges. We easily compute the topological order from the modular transformations and recover the result of previous explicit Chern–Simons calculations [19].

Next, we show that the $W_{1+\infty}$ minimal models do not possess a modular invariant partition function, which implies that they are not RCFTs. Nevertheless, they are consistent theories because their representations form a closed set under the fusion rules [25]. If the minimal models are physical, i.e. the Jain fluids do not realize the full $SU(m)$ symmetry, then their partition functions should satisfy a weaker condition than modular invariance. Here, we cannot develop this issue any further,[2] besides showing that it is not correct to simply dispose of modular invariance. Actually, in Appendix B, we construct the modular variant partition functions fulfilling the remaining building and self-consistency criteria, and we obtain too many solutions whose Hall conductivities span all rational values, in contradiction with the few experimental points.

In Section 4.4, we continue our analysis of the non-minimal rational theories based on the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ affine algebras. We search for left–right *non-diagonal* solutions to the modular invariance conditions, which define new RCFTs. Actually, non-diagonal partition functions exist generically in RCFT and have been much investigated in the past, revealing many connections with deep aspects of mathematics [13]. There are two possible mechanisms which produce them [11]: (i) the extended chiral algebra of the RCFT can be larger than those of diagonal invariants; (ii) there is an automorphism of the fusion algebra, which can be used to pair non-trivially the left and right representations.

We do find non-diagonal partition functions in our problem, which correspond to those found by Itzykson in the analysis of the $\widehat{SU(m)}_1$ theories [31].[3] For m containing a square factor, $m = a^2 m'$, we find partition functions with an extension of the $\widehat{SU(m)}_1$ symmetry and a reduced set of excitations with respect to the diagonal Jain theories. These new edge theories can describe the experimental pattern beyond the Jain series. Moreover, there are modular invariants with a twist between charged and neutral weights, which is due to an automorphism of the $\widehat{SU(m)}_1$ fusion rules. These invariants are not experimentally relevant, because their filling fractions have unobservable large denominators.

The non-diagonal partition functions with extensions of the $\widehat{U(1)} \otimes \widehat{SU(a^2m')}$ symmetry span again the Jain filling fractions, with $m \to m'$, and further series of fractions which include $\nu = 2/3, 4/5, 6/7, 8/9, \ldots, 4/11, 4/13$ (only) for small values of the denominator. In Section 4.5 we present a consistent interpretation of almost all the experimental data [33], which only contains these filling fractions and their (less stable)

[2] For a proposal, see Ref. [30].

[3] The general solution was found in Ref. [32].

"charge-conjugated" partners $\nu \to (1-\nu)$. Although the few points beyond the Jain series are not enough for a complete check, our analysis suggests that the non-diagonal RCFTs could be physically relevant. They could be an economic alternative to the higher orders of the Jain hierarchy of wave functions [20] and similar constructions, which generically predict too many unobserved filling fractions beyond $\nu = 4/11, 4/13$.

In the conclusions, we remark that the modular invariant partition function and the associated RCFT calculus can be very useful for understanding the so-called non-Abelian edge theories, which have been discussed in Refs. [6,34,35,17]. The partition functions for these theories have been recently proposed in Ref. [18], but their modular invariance was not analysed. Finally, in Appendix A, we collect some useful formulas for the theta functions occurring in the conformal characters.

2. Partition functions for the $\widehat{U(1)}$ theories and the Laughlin plateaus

Let us consider the annulus geometry $\mathrm{I} \times S^1 \times S^1$, with coordinates (r, φ, t_E), for the plane and the Euclidean compact time, where $r \in \mathrm{I} = [R_L, R_R]$ and $t_E = -it \equiv t_E + \beta$. The electrons are supposed to form an incompressible fluid in the bulk of the annulus, with gapful quasi-particle excitations. We want to consider the partition function for the excitations below the gap, which can be created in a conduction experiment by attaching wires to the two edges. These excitations are chiral and anti-chiral waves on the outer (R) and inner (L) edges, respectively. Their Hilbert space is made of pairs of representations of the $W_{1+\infty}$ algebra of quantum area-preserving diffeomorphisms of the incompressible fluid. [4] In this section, we consider the theories made by the simplest $W_{1+\infty}$ representations, which are equivalent to those of the $\widehat{U(1)}$ affine algebra and have a Virasoro central charge c equal to one [25].

Previous investigations [3,7,5] analysed the excitations of a single edge, which are described by *chiral* CFTs. For these theories, the Virasoro eigenvalue L_0 gives the fractional spin J which is also one-half of the fractional statistics θ/π of the excitations of the incompressible fluid [37]. The following spectra were obtained for J, the filling fraction ν and the charge Q:

$$\nu = \frac{1}{p}, \qquad Q = \frac{n}{p}, \qquad J = L_0 = \frac{n^2}{2p}, \qquad n \in \mathbb{Z},\ p = 1,3,5,\ldots \tag{2.1}$$

Therefore, these edge theories describe the quantum Hall effect at the Laughlin plateaus. Each value of (Q, L_0) is the weight of a highest-weight representation of the $\widehat{U(1)}$ algebra, which contains a tower of neutral excitations with quantum numbers (Q, L_0+k), $k > 0$ integer (the descendant states) [4].

Here, we shall derive independently the chiral-antichiral spectrum on the annulus geometry, and find agreement with (2.1) for each of the two edges. We shall use the

[4] See Refs. [5,36,25] for an introduction to the $W_{1+\infty}$ symmetry in the quantum Hall effect.

conformal field theory data, impose the modular invariance of the partition function and some other physical conditions. The partition function [12,13] is defined by

$$Z(\tau,\zeta) = \mathcal{K}\,\mathrm{Tr}\left[e^{i2\pi\left(\tau(L_0^L-c/24)-\bar\tau(L_0^R-c/24)+\zeta Q^L+\bar\zeta Q^R\right)}\right], \qquad (2.2)$$

where the trace is over the states of the Hilbert space, $\mathcal{K}$ is a normalization to be described later and (τ,ζ) are complex numbers. We recognize (2.2) as a grand-canonical partition function, with a heat bath and a particle reservoir: the operators

$$H_{\mathrm{CFT}} = \frac{1}{R}\left(L_0^R + L_0^L - \frac{c}{12}\right) + V_o\left(Q^L - Q^R\right) + \text{const.}, \qquad J = L_0^L - L_0^R, \qquad (2.3)$$

measure the energy and spin of the excitations, respectively. The real and imaginary parts of (τ,ζ) are related to the inverse temperature $\beta = 1/k_BT = 2\pi R\,\mathrm{Im}\,\tau > 0$, the "torsion" $\mathrm{Re}\,\tau$, the chemical potential $\mu/k_BT = 2\pi\,\mathrm{Re}\,\zeta$ and the electric potential between the edges $V_0/k_BT = 2\pi\,\mathrm{Im}\,\zeta$. Given that the Virasoro dimension is roughly the square of the charge, the partition sum is convergent for $\mathrm{Im}\,\tau > 0$ and $\zeta \in \mathbb{C}$. We have naturally chosen a symmetric Hamiltonian for the two edges, by adjusting the velocities of propagation of excitations, $(v_L/R_L = v_R/R_R \equiv 1/R)$. The energy spectrum given by H_{CFT} is not completely realistic, because it does not include the logarithmic correction due to the bulk Coulomb interaction among the electrons. However, we know that this correction would not change the form of the Hilbert space [36]; one can use the simpler H_{CFT} to construct a "kinematical" partition function which describes the universality class of the low-energy excitations, namely that is an inventory of their quantum numbers and abundances.

As usual, we can divide the trace in (2.2) into a sum over pairs of highest-weight $\widehat{U(1)}$ representations and the sums over the states within each representation. The latter give rise to the $\widehat{U(1)}$ characters [4]

$$\mathrm{Ch}(Q,L_0) = \mathrm{Tr}|_{\widehat{U(1)}}\left[e^{i2\pi(\tau(L_0-c/24)+\zeta Q)}\right] = \frac{q^{L_0}w^Q}{\eta(q)}, \qquad (2.4)$$

where η is the Dedekind function,

$$\eta(q) = q^{1/24}\prod_{k=1}^{\infty}\left(1-q^k\right), \qquad q = e^{i2\pi\tau}, \qquad w = e^{i2\pi\zeta}. \qquad (2.5)$$

Any conformal field theory with $c \geqslant 1$ contains an infinity of Virasoro (and $\widehat{U(1)}$) representations [12]; therefore, we must further regroup the $\widehat{U(1)}$ characters into characters χ of an extended algebra in order to get the finite-dimensional expansion

$$Z = \sum_{\lambda,\bar\lambda=1}^{N} \mathcal{N}_{\lambda,\bar\lambda}\chi_\lambda\bar\chi_{\bar\lambda}, \qquad (2.6)$$

which characterizes a rational conformal field theory [11]. The coefficients $\mathcal{N}_{\lambda,\bar\lambda}$ are unknown positive integers known as the *multiplicities* of the excitations. The matrix $\mathcal{N}$, and the parameters specifying the (Q,L_0) spectrum are the unknown quantities to be determined by the following conditions.

2.1. Modular invariance conditions

The partition function describes the pairing of excitations on the two edges to form physical excitations of the entire sample, which can be measured in a conduction experiment. These can only be electron-like and should have integer or half-integer spin,

$$T^2 : \quad Z\left(\tau+2,\zeta\right) \equiv \mathrm{Tr}\left[\ldots \mathrm{e}^{i2\pi 2\left(L_0^L - L_0^R\right)}\right] = Z\left(\tau,\zeta\right). \tag{2.7}$$

In (2.2), $Z(\tau)$ is the partition function of a $(1+1)$-dimensional conformal field theory, where τ is the modular parameter of a space-time torus. On the other hand, (2.2) is also the path-integral amplitude of the Abelian Chern–Simons theory on the manifold $\mathcal{M} = S^1 \times S^1 \times \mathrm{I}$. Actually, by exchanging the Euclidean time with one space coordinate, such that $t_E \in \mathrm{I}$, this is also an amplitude for compact space, which has been considered by Witten and other authors [15,16]. They have shown that the Hilbert space of the Chern–Simons theory on a space torus is described by a pair of $\widehat{U(1)}$ affine algebras with opposite chiralities, which is precisely our setting. In both space-space and Euclidean space-time tori, we can exchange the two periods by a corresponding renaming of the parameters [12],

$$S : \quad Z\left(-\frac{1}{\tau}, -\frac{\zeta}{\tau}\right) = Z\left(\tau,\zeta\right). \tag{2.8}$$

The transformations ST^2S and S generate the subgroup $\Gamma(2)$ of the modular group $\Gamma = SL(2,\mathbb{Z})/\mathbb{Z}_2$ of rational transformations $\tau \to (a\tau+b)/(c\tau+d)$, $a,b,c,d \in \mathbb{Z}$, which are subjected to the conditions (a,d) odd and (b,c) even. Thus, Z is invariant under the modular subgroup $\Gamma(2)$, due to the presence of one-fermion states, as it occurs in the Neveu–Schwarz sector of the supersymmetric theories [4]. [5]

We now discuss two further conditions concerning the charge spectrum. In general, the incompressible fluid contains anyons, which are vortices in the bulk of the fluid, spilling a fractional charge to the edge. They have a finite gap $O(e^2/\ell) = O(e^2\sqrt{eB}) \gg eV_o$, due to the Coulomb interaction [2], and cannot be generated in a conduction experiment. They can be produced by tuning the magnetic field away from the center of the plateaus, becoming subsequently pinned down by impurities and turning into static, fractional charges [1]. The edge excitations in presence of them are described by the expectation value of the Wilson lines in the Chern–Simons theory [15], while the on-centered situation without bulk anyons is described by the partition function (2.2). In the latter case, the edge excitations should have total integer charge, $Q^L + Q^R \in \mathbb{Z}$, which is the number of electrons carried to the edges by the wires. This condition is enforced by

$$U : \quad Z\left(\tau,\zeta+1\right) \equiv \mathrm{Tr}\left[\ldots \mathrm{e}^{i2\pi\left(Q^R + Q^L\right)}\right] = Z\left(\tau,\zeta\right). \tag{2.9}$$

Owing to this condition, there can be fractionally charged excitations at one edge, but these should pair with complementary ones on the other boundary. Consider, for

[5] Actually, S and T^2 generate a slightly larger subgroup of Γ; see Appendix A.

example, $\nu = 1/3$; we can imagine to drop in an electron, which splits into the pair $(Q^L, Q^R) = (1/3, 2/3)$, or another one among $(0,1), (2/3,1/3), (1,0)$. These different splittings should all be realizable by appropriate tuning of the potential V_0 in (2.3). Their equivalence gives a symmetry of the partition function; the shift among them is called *spectral flow* [5,25]. This flow has been discussed by Laughlin in the thought experiment defining the fractional charge on the annulus [2]: the addition of a quantum of flux through the center of the annulus is a symmetry of the gauge-invariant Hamiltonian but causes a flow of all the quantum states among themselves. We can simulate this flow by requiring invariance of the partition functions under a shift of the electric potential by a corresponding amount (in our notations $e = c = \hbar = 1$) $V_0 \to V_0 + 1/R$, namely $\zeta \to \zeta + \tau$,

$$V: \quad Z(\tau, \zeta + \tau) = Z(\tau, \zeta). \tag{2.10}$$

Note that the transport of an elementary fractional charge between the two edges is related to the Hall conductivity, thus the spectral flow also determines the Hall current.

The conditions (T^2, U, V) have been motivated by specific properties of the quantum Hall effect. However, analogous conditions are found in the canonical quantization of the Abelian Chern-Simons theory on the space torus [16]: the Chern-Simons field $A_i, i = 1,2$ is a flat connection, which has two non-trivial degrees of freedom on the torus, $(\theta_1(t), \theta_2(t))$. One identifies our variable $\zeta \sim \theta_1 + \tau\theta_2$; furthermore, the conditions (U, V) correspond to the Gauss law which enforces gauge invariance on physical states.

We now solve the conditions (T^2, S, U, V) for the $c = 1$ theory. Owing to the condition U, we can first collect left $\widehat{U(1)}$ representations which have integer spaced charges $Q^L = \lambda/p + \mathbb{Z}$, p integer, and then combine them with the corresponding right representations. The Virasoro dimensions are given by Eq. (2.1), without loss of generality; the charge unit $1/p$ is free and related to the normalization of ζ. The sums of $\widehat{U(1)}$ characters having integer spaced charge should give the characters of the extended algebra χ_λ (2.6), which carry a finite-dimensional representation of the modular group [11]. These are known to be the theta functions with rational characteristics [6] $\Theta\left[{a \atop b}\right]$,

$$\begin{aligned}\chi_\lambda &= \exp\left(-\frac{\pi}{p}\frac{(\mathrm{Im}\,\zeta)^2}{\mathrm{Im}\,\tau}\right)\frac{1}{\eta}\Theta\begin{bmatrix}\lambda/p\\0\end{bmatrix}(\zeta|p\tau)\\ &= \exp\left(-\frac{\pi}{p}\frac{(\mathrm{Im}\,\zeta)^2}{\mathrm{Im}\,\tau}\right)\frac{1}{\eta}\sum_{k\in\mathbb{Z}}\exp\left(i2\pi\left(\tau\frac{(pk+\lambda)^2}{2p}+\zeta\left(\frac{\lambda}{p}+k\right)\right)\right),\end{aligned} \tag{2.11}$$

which carry charge $Q^L = \lambda/p + \mathbb{Z}$, $\lambda = 1, 2, \dots, p$. The prefactor will be explained later. The transformations T^2, S, U, V of these generalized characters are

$$T^2: \quad \chi_\lambda(\tau+2, \zeta) = \exp\left(i2\pi\left(\frac{\lambda^2}{p} - \frac{1}{12}\right)\right)\chi_\lambda(\tau, \zeta), \tag{2.12}$$

[6] See Appendix A for more details on modular functions and transformations.

$$S: \quad \chi_\lambda\left(-\frac{1}{\tau}, -\frac{\zeta}{\tau}\right) = \frac{\exp\left(i\frac{\pi}{p}\mathrm{Re}\frac{\zeta^2}{\tau}\right)}{\sqrt{p}} \sum_{\lambda'=0}^{p-1} \exp\left(i2\pi\frac{\lambda\lambda'}{p}\right) \chi_{\lambda'}(\tau,\zeta), \tag{2.13}$$

$$U: \quad \chi_\lambda(\tau, \zeta + 1) = \exp\left(i2\pi\lambda/p\right) \chi_\lambda(\tau,\zeta), \tag{2.14}$$

$$V: \quad \chi_\lambda(\tau, \zeta + \tau) = \exp\left(-i\frac{2\pi}{p}\left(\mathrm{Re}\,\zeta + \mathrm{Re}\frac{\tau}{2}\right)\right) \chi_{\lambda+1}(\tau,\zeta). \tag{2.15}$$

These transformations show that the characters χ_λ carry a unitary representation of the modular group $\Gamma(2)$, which is projective for $\zeta \neq 0$ (the composition law is verified up to a phase).

The corresponding sums of the $\widehat{U(1)}$ representations for the other edge are given by $\bar{\chi}_{\bar{\lambda}}$ carrying charge $Q^R = -\bar{\lambda}/p + \mathbb{Z}$. The U condition (2.9) is applied to Z in the form (2.6) and it requires that left and right charges satisfy $\lambda = \bar{\lambda}$ mod p. The unique[7] solution is $\lambda = \bar{\lambda}$, which also satisfies the (T^2, S, V) conditions, leading to the invariant

$$Z = \sum_{\lambda=1}^{p} \chi_\lambda \bar{\chi}_\lambda. \tag{2.16}$$

This is the annulus partition function for the $c = 1$ edge theories we were after. It yields the multiplicities of the representations $\mathcal{N}_{\lambda,\bar{\lambda}} = \delta^{(p)}_{\lambda,\bar{\lambda}}$, and still depends on two free parameters, p and ζ, which are determined by further physical conditions. These have been discussed in Refs. [28,23,25] and are simply reformulated in the present context:

(i) The normalization of ζ, i.e. of the charge unit, is determined self-consistently as follows. Among the transformations (2.12)-(2.15), V is the only one sensitive to rescalings $\zeta \to a\zeta$, with a integer: we fix $a = 1$ by requiring that the minimal spectral flow $\zeta \to \zeta + \tau$ carries the minimal amount of fractional charge $(1/p)$ from one edge to the other, $\chi_\lambda(\zeta + \tau) \propto \chi_{\lambda+1}(\zeta)$. This corresponds to the definition of the (fractional) charge in the Laughlin thought experiment [2]. Moreover, the amount of displaced charge per unit of flux is a measure of the Hall conductivity: we conclude that (2.16) describes the Hall effect at filling fractions $\nu = 1/p$.

(ii) Within the spectrum of charges and fractional spins of Z (2.16), there should be electron states on each edge which have unit charge and odd statistics, $2J = 1$ mod 2. From (2.1) one can see that this requirement fixes p to be an odd integer. Therefore, these partition functions describe the Laughlin plateaus $\nu = 1, 1/3, 1/5, \ldots$

(iii) One can also verify that all the excitations have integer monodromies with respect to the electrons,

$$J[n_e] + J[n] - J[n_e + n] \in \mathbb{Z}, \tag{2.17}$$

where $(n_e = p, n)$ denote the integer labels in (2.1) for the electron and a generic excitation, respectively.

[7] This will be proven in Section 4.3.

Let us add more comments about the main result of this section, Eqs. (2.16) and (2.11). The non-holomorphic prefactor added to the characters in (2.11) is the constant term in the Hamiltonian (2.3), appropriately tuned to have the spectrum

$$E_{n_L,n_R} = \frac{1}{R}\frac{1}{2p}\left[(n_L + RV_0)^2 + (n_R - RV_0)^2\right], \tag{2.18}$$

whose minimum is independent of the value of V_0. Actually, this is necessary for the invariance of Z under the spectral flow, and it amounts to adding a capacitive energy to the edges equal to $E_c = RV_0^2/2p$. Note that this prefactor also appears in the quantization of the Chern-Simons theory on the space torus, in the measure for the wave-functions inner product [16].

A more explicit form of the partition function (2.16) is

$$Z = \frac{\exp\left(-\frac{2\pi(\mathrm{Im}\zeta)^2}{p\,\mathrm{Im}\,\tau}\right)}{|\eta|^2}\sum_{\lambda=0}^{p-1}\sum_{k,\bar{k}\in\mathbb{Z}} q^{(pk+\lambda)^2/2p}\bar{q}^{(p\bar{k}+\lambda)^2/2p} w^{(pk+\lambda)/p}\bar{w}^{(p\bar{k}+\lambda)/p}. \tag{2.19}$$

This expression is *not* the same as the well-known partition function of a real bosonic field compactified on a rational circle of radius $R_c = r/2s$ [4],

$$Z_B = \frac{1}{|\eta|^2}\sum_{n,m\in\mathbb{Z}} q^{\frac{1}{2}((n/2R_c)+mR_c)^2}\bar{q}^{\frac{1}{2}((n/2R_c)-mR_c)^2}. \tag{2.20}$$

Indeed, after setting $\zeta = 0$, Z and Z_B would be equal for $R_c = 1/p$, which would require p *even*. Therefore, the RCFT of the Laughlin plateaus is an *odd* modification of the rational compactified boson, which is invariant under the subgroup $\Gamma(2)$ of the modular group only.

In conclusion, the modular invariance conditions (S, T^2, U, V) and a few physical requirements determine uniquely the form of the annulus partition functions for the $\widehat{U(1)}$ theories: their filling fractions identify the Laughlin plateaus and their spectra match the previous results. We remark that these partition functions, for $\zeta = 0$, have been also found in Refs. [17,18] by more direct physical arguments, but their modular invariance was not verified. Our approach is rather different, because we use modular invariance as a self-consistent sufficient condition for finding the partition functions from the raw conformal field theory data.

2.2. *Modular invariance and the fusion algebra*

We have shown that the partition function for the Laughlin fluids (2.19) on the annulus takes the standard form of rational conformal field theories (RCFT) [11] and is modular invariant (S invariant, in particular). This implies that we can carry over to the quantum Hall effect some interesting properties of the RCFT [6], due to Verlinde [14], Witten [15] and others [11], relating the fusion rules, the modular transformation S and the dimension of Hilbert spaces of the Chern-Simons theories. In particular, we shall use the fusion rules to compute the Wen "topological order" [19] of the quantum Hall effect in any geometry and in the presence of impurities.

Let us first recall the Verlinde theory relating the fusion rules to the S transformation [14,11]. The generalized characters χ_λ of the RCFT (2.11) correspond to the extension of the $\widehat{U(1)}$ algebra by a chiral current J_p. This is a $\widehat{U(1)}$ representation which enters in the vacuum representation of the extended algebra, and then it can be identified from the first non-trivial term in the expansion of χ_0 into $\widehat{U(1)}$ characters: J_p has half-integer dimension $L_0 = p/2$ and unit charge. [8] The highest-weight representations of the RCFT algebra correspond to generalized primary fields ϕ_λ, whose charge is defined modulo one (the charge of J_p); namely, they collect all chiral excitations with the same fractional charge.

The fusion rules [4] are the selection rules for the operator-product expansion (in simple terms, the making of composite excitations), and are an associative Abelian algebra of the fields. The $\widehat{U(1)}$ fusion rules are simply given by charge conservation; thus, those of the extended algebra are given by the additive group $\mathbb{Z}_p$ of integers modulo p,

$$\phi_i \cdot \phi_j \equiv N_{ij}^k \phi_k, \qquad N_{ij}^k = \delta^{(p)}_{i+j,k}, \quad i,j,k \in \mathbb{Z}_p. \tag{2.21}$$

One can use the conjugacy matrix $C_{ij} = N_{ij}^0$ as a metric to raise and lower indices: then $N_{ijk} = \delta^{(p)}_{i+j+k,0}$ is a completely symmetric tensor. The $N \times N$ matrices $(N_i)_j^k = N_{ij}^k$ give a representation of the fusion algebra which decomposes into N one-dimensional irreducible ones, represented by the eigenvalues $\lambda_i^{(n)}$, $n = 0, \dots, p-1$ of the matrix N_i. Verlinde has shown that these eigenvalues are related to the matrix elements of the modular transformation S_k^n, as follows:

$$\lambda_i^{(n)} = \frac{S_i^n}{S_0^n} \qquad \text{and} \qquad N_i = S\Lambda_i S^\dagger, \quad (\Lambda_i)_n^k = \delta_n^k \lambda_i^{(n)}. \tag{2.22}$$

One immediately verifies these formulae for the case of the annulus partition function (2.16) by using the representation (2.13) for $\zeta = 0$: $S_k^n = \exp(i2\pi kn/p)/\sqrt{p}$. Hereafter we consider the specialized partition functions $Z(\tau, \zeta = 0)$, which have been analysed by Verlinde.

A general property of observables in conformal field theory is that they decompose into a finite sum of chiral–antichiral terms, called conformal blocks, which are determined by representation-theoretic properties [4]. For example, Z decomposes into pairs of characters. The space of conformal blocks $\mathcal{H}$ for the n-point functions on a genus g Riemann surface Σ_g is a vector bundle, whose vectors are labelled by the weights of the representations of the symmetry algebra [11,14]. A nice application of the fusion rules is to compute the dimension of this space. Clearly, a representation index i_k is associated to each of the n points, or "punctures" on Σ_g. The coefficients N_{ij}^k represent the three-point function on the sphere Σ_0; we can associate to it a "dual" φ^3 vertex made by three oriented lines joined to a point. Higher genus surfaces can be built by sewing

[8] Similar to the supersymmetric partner of the stress tensor T_F $(L_0 = 3/2)$ [4].

three-punctured spheres: for example, a handle is made by joining two punctures and summing over representations with the "propagator" δ_i^k or C_{ik}, according to the indices of the punctures, i.e. to the line orientations in the φ^3 graph. There are many ways to associate a φ^3 graph to a Riemann surface, which are related by the duality symmetry, a fundamental property of the fusion rules (2.21). For example, let us compute the number of blocks occurring in the one-point functions on the torus $\langle\phi_i\rangle_T$ for the Abelian $\mathbb{Z}_p$ fusion rules,

$$\dim \mathcal{H}\left(\Sigma_1; (P,i)\right) = \sum_{k=1}^{p} N_{ik}^k = \sum_{k=1}^{p} \delta_{i,0}^{(p)} = p\delta_{i,0}^{(p)}. \tag{2.23}$$

Namely, all torus one-point functions vanish (by charge conservation) and the torus partition function $\langle\mathbf{1}\rangle_T$ decomposes into p blocks. Note that the same number of terms was found for the Z (2.16) computed before.

The general formula for the dimension of the space of conformal blocks on a genus g surface with n punctures was found in Ref. [11]; owing to the relations (2.22), this can be expressed in terms of the S matrix elements, as follows:

$$\dim \mathcal{H}\left(\Sigma_g; (P_1,i_1),\ldots,(P_n,i_n)\right) = \sum_k \left(\frac{1}{S_0^k}\right)^{2(g-1)} \frac{S_{i_1}^k}{S_0^k}\cdots\frac{S_{i_n}^k}{S_0^k}. \tag{2.24}$$

We now recall the two Witten correspondences [15,16] between the RCFT based on the affine Lie algebra $\widehat{G}$ (the Wess–Zumino–Witten model) and the Chern–Simons theory with gauge group G canonically quantized on a manifold $\mathcal{M} = \Sigma \times \mathbb{R}$, where $\mathbb{R}$ is the time axis.

(i) If Σ has a boundary, for example a disk, the quantization of the Chern–Simons gauge field gives rise to the Wess–Zumino–Witten model on $\partial\mathcal{M}$. This is the familiar relation for the CFT description of edge excitations in the quantum Hall effect [3]. On any boundary circle, we have the spectrum of edge excitations described by an affine highest-weight representation; its weight is free, but the global conservations laws on the surface give a constraint.

(ii) If Σ_g is compact, the Chern–Simons gauge field is a non-trivial flat connection which has a finite number of degrees of freedom. The finite-dimensional Hilbert space is equivalent to the space of conformal blocks $\mathcal{H}(\Sigma_g)$ of the Euclidean RCFT defined on Σ_g: the n-point conformal block of the RCFT corresponds to the Chern–Simons wave functional for n time-like Wilson lines piercing the surface.

From the second relation, we conclude that the formula (2.24) gives the dimension of the Chern–Simons Hilbert space on Σ_g with n Wilson lines. Furthermore, the surfaces $\Sigma_{g,k}$ with k boundaries are relevant for the quantum Hall effect, and we need to extend the formula (2.24) to them. Concerning the counting of states, a disk is equivalent to a puncture whose quantum numbers are not fixed a priori. This fits with the simple-minded observation that, topologically, one can introduce a boundary on a closed surface by removing a disk from it. Therefore, the dimension formula for $\mathcal{H}(\Sigma_{g,k})$ with boundaries is obtained from those for $\mathcal{H}(\Sigma_g)$ in (2.24), by adding k punctures, and by summing over the corresponding free indices $\{i_1,\ldots,i_k\}$.

2.3. The topological order and its independence of impurities

Wen has introduced and stressed the concept of topological order in the quantum Hall effect [19]; by this he means a number of properties which can be summarized by saying that the incompressible fluid ground state is not a single quantum state but a topological field theory, which is actually the Chern-Simons theory. Here, the topological order will specifically mean the *number* of degenerate ground states on a closed surface Σ_g. This degeneracy is hard to derive from the microscopic dynamics of the electrons, but it is indeed present in the trial wave functions and in numerical spectra on a toroidal space [38].

From the previous discussion, we understand that these degenerate ground states are described by the Hilbert space $\mathcal{H}(\Sigma_g)$ of the Chern-Simons theory. Therefore, we can compute the topological order by using the previous counting formula (2.24). For the torus, we find

$$\dim\mathcal{H}\,(\Sigma_1) = \sum_{k=1}^{N} 1 = N, \quad \text{for any RCFT}, \tag{2.25}$$

where N is the dimension of the representation of the S transformation, i.e. the number of representations of the (maximally) extended symmetry algebra of the RCFT.

The same topological order can be computed from the annulus partition function introduced before. Actually, consider the dimension formula for the Chern-Simons theory on $\mathrm{I}\times S^1\times S^1$. The compact time evolution is not important, so this is equivalent to $\mathrm{I}\times S^1\times\mathbb{R}$. The annulus surface is topologically equivalent to the sphere with two disks removed, i.e. with two punctures. We use again the dimension formula (2.24) adding the boundaries, and we find

$$\begin{aligned}\dim\mathcal{H}\left(\Sigma_{0,2}\right) &= \sum_{i,j=1}^{N}\dim\mathcal{H}\left(\Sigma_0,(P,i),(P',j)\right)\\ &= \sum_{i,j} N_{ij,0} = \sum_{i,j} C_{i,j} = N, \quad \text{for any RCFT}.\end{aligned} \tag{2.26}$$

From this and previous results, we conclude that the annulus partition function in Section 2.1 gives a correct description of the standard RCFT properties and it encodes the Wen topological order, previously defined for the quantum Hall effect on compact spaces [19].

In the literature of the quantum Hall effect, both Witten relations have been considered, resulting into a "bulk" description of the incompressible Hall fluids by an Euclidean RCFT [39] and an "edge" RCFT description, as in the present paper. The two descriptions and conformal field theories are equivalent [6]: the second one has a more immediate physical meaning and is often simpler. It is rather useful that the important concept of topological order is shown by observables of the edge conformal field theory.

The topological order on higher genus surfaces can be similarly computed from Eq. (2.24): one needs the specific form of the S transformation and the result is characteristic of each theory. For the Abelian Laughlin fluids, we use the $\mathbb{Z}_p$ S matrix (2.13) and obtain

$$\dim \mathcal{H}\left(\Sigma_g\right) = \sum_{k=1}^{p} p^{g-1} = p^g, \qquad \nu = \frac{1}{p},\ p = 1,3,\ldots \tag{2.27}$$

This agrees with the explicit calculation in the Abelian Chern–Simons theory [19].

Another interesting application of the Verlinde calculus is to compute the topological order in the presence of impurities. Let us again consider the quantum Hall effect on the annulus, and assume there exist n impurities in the bulk, i.e. n anyons. These are static and do not contribute to the Hall conduction [1]; as discussed before, they modify the ground state for the edge excitations, which are no longer described by the partition function (2.6). For example, the charge balance between the edges (2.9) is modified to $Q^L + Q^R + Q_{\text{bulk}} \in \mathbb{Z}$. These edge excitations are described by the Chern–Simons expectation value of n time-like Wilson lines piercing the annulus [16]. Contrary to the partition function, this expectation value is not modular invariant because the Wilson lines select a time direction. Nevertheless, we can easily find the corresponding topological order by using again (2.24). The n punctures on the annulus correspond to $n+2$ punctures on the sphere, with sum over the indices of the two additional ones; for the Laughlin plateaus, we use again the $\mathbb{Z}_p$ S transformation and find

$$\dim \mathcal{H}\left(\Sigma_{0,2}; (P_1, i_1), \ldots, (P_n, i_n)\right) = \sum_{i,j=1}^{p} \delta^{(p)}_{i+j+i_1+\ldots+i_n,0} = p, \qquad \nu = \frac{1}{p}. \tag{2.28}$$

This shows that the topological order on the annulus is independent of impurities, for Abelian fluids. Similarly, one verifies that the topological order on Σ_g is also independent of impurities[9] in agreement with the result of Ref. [19] in the Chern–Simons approach. The independence of impurities is a rather important result; it shows that the topological order is robust, since it holds in generic experimental situations. On the other hand, the partition function is modular invariant only in the absence of impurities, although it encodes many universal properties which are independent of them.

3. $W_{1+\infty}$ minimal and non-minimal theories

In this section, we recall some properties of the conformal field theories which have been proposed to describe the Jain series of stable plateaus at $\nu = m/(ms \pm 1)$, $m = 2,3,\ldots$ and $s > 0$ an even integer. These are of two types:

[9] The added anyons must have total vanishing charge on a compact surface.

(i) the minimal models of the $W_{1+\infty}$ algebra [25];
(ii) their non-minimal version, the theories with $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ affine algebra [27,3,7,28].

Both types of theories possess the $W_{1+\infty}$ dynamical symmetry of area-preserving diffeomorphisms, which expresses at the quantum level the semiclassical property of incompressibility of the electron fluid at the plateaus [21–23]. These theories have been constructed in detail in Ref. [25] by using the representation theory of the $W_{1+\infty}$ algebra developed in Refs. [24,26].

Historically, the non-minimal theories were first introduced as a multi-component generalization of the Abelian theory of the previous section [27,3,7]. Actually, the m-dimensional generalization of the Abelian spectrum (2.1) is given by

$$
\begin{aligned}
Q &= \sum_{i,j=1}^{m} K_{ij}^{-1} n_j, \quad n_i \in \mathbb{Z}, \\
J &= \frac{1}{2} \sum_{i,j=1}^{m} n_i K_{ij}^{-1} n_j, \\
\nu &= \sum_{i,j=1}^{m} K_{ij}^{-1}, \qquad c = m,
\end{aligned}
\tag{3.1}
$$

where K_{ij} is an arbitrary symmetric matrix of couplings, with integer elements, odd on the diagonal. The derivation of (3.1) goes as follows: one assumes that the electron fluid has m independent layers – naively, one could make an analogy with the integer Hall effect, with m filled Landau levels, each one having an edge described by an Abelian current, altogether yielding the $\widehat{U(1)}^{\otimes m}$ affine algebra [4]. Its highest-weight representations are labelled by a vector of (mathematical) charges r_a, $a = 1, \dots, m$, which spans an m-dimensional lattice $\boldsymbol{r} = \sum_i \boldsymbol{v}^{(i)} n_i$, $n_i \in \mathbb{Z}$ as required by the closure of the fusion rules, given by the addition of charge vectors [25]. The physical charge is a linear functional of $\boldsymbol{r}$ and the Virasoro dimension is a quadratic form, both parametrized by the metric [10] of the lattice $K_{ij}^{-1} \sim \boldsymbol{v}^{(i)} \cdot \boldsymbol{v}^{(j)}$. This matrix K is further constrained to take integer values by the requirement that the spectrum contains m electron-like excitations [28]. Therefore, each m-plet of integers n_i in (3.1) gives a highest-weight state of the $\widehat{U(1)}^{\otimes m}$ algebra, which is physically interpreted as an anyon excitation in the multi-layered electron fluid.

The spectrum (3.1) is very general, due to the many free parameters in the K matrix. Actually, these can be chosen to comply with the results of all the known hierarchical constructions of wave functions [27,3,28]. In particular, the Jain hierarchy [20] was shown to correspond to the matrices $K_{ij} = \pm\delta_{ij} + sC_{ij}$, where $s > 0$ is an even integer and $C_{ij} = 1$, $\forall i, j = 1, \dots, m$. The corresponding spectrum is

$$
\nu = \frac{m}{ms \pm 1}, \quad s > 0 \text{ even integer}, \qquad c = m,
$$

[10] The Lorentzian signature corresponds to excitations with mixed chiralities [3,25].

$$Q = \frac{1}{ms \pm 1} \sum_{i=1}^{m} n_i,$$
$$J = \pm \frac{1}{2} \left(\sum_{i=1}^{m} n_i^2 - \frac{s}{ms \pm 1} \left(\sum_{i=1}^{m} n_i \right)^2 \right). \tag{3.2}$$

This spectrum is rather peculiar because it contains $m(m-1)$ *neutral* states with unit Virasoro dimension $(Q, L_0) = (0, 1)$. By using an explicit free bosonic field construction, one can show [27,7] that these are chiral currents $J_{\boldsymbol{\beta}}$, where $\boldsymbol{\beta}$ is a root of $SU(m)$ [40], which generate the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ affine algebra, together with the m original $\widehat{U(1)}$ currents for the Cartan subalgebra [4]. The $\widehat{SU(m)}_1$ affine algebra is a RCFT with m highest-weight representations, each one resumming a sector of the $\widehat{U(1)}^{\otimes m}$ spectrum (3.2).

The minimal models were obtained from the analysis of the $W_{1+\infty}$ symmetry [25]. The Laughlin semiclassical picture of the incompressible fluid of electrons was assumed. At constant electron density and number, all the configurations of the incompressible fluid have the same area; thus, they can be mapped into each other by area-preserving diffeomorphisms of the plane, obeying the classical w_∞ algebra [21,22]. The infinitesimal deformations are the edge excitations. Their quantization leads to a conformal field theory with quantum algebra $W_{1+\infty}$, which contains $\widehat{U(1)}$ and Virasoro as subalgebras [23]. We refer to our previous works for the description of this semiclassical picture and the definition of the $W_{1+\infty}$ algebra, and we recall here the main properties of these theories [25].

The $W_{1+\infty}$ theories can be constructed by assembling highest-weight representations of the $W_{1+\infty}$ algebra which are closed under their fusion rules. The unitary representations have integer central charge ($c = m$) and can be of two types: *generic* or *degenerate* [24,26]. The *generic* representations are one-to-one equivalent to those of the Abelian affine algebra $\widehat{U(1)}^{\otimes m}$ with charge vectors r_a satisfying $(r_a - r_b) \notin \mathbb{Z}$, $\forall a \neq b$. Clearly, the $W_{1+\infty}$ theories made of generic representations correspond to the generic hierarchical Abelian theories in (3.1). On the other hand, the (fully) degenerate $W_{1+\infty}$ representations are not equivalent to $\widehat{U(1)}^{\otimes m}$ representations, they rather are contained into them. They are labelled by a charge vector $\boldsymbol{r}$ satisfying $(r_a - r_b) \in \mathbb{Z}$, $\forall a \neq b$: an Abelian representation with the same charge vector decomposes into many degenerate $W_{1+\infty}$ representations with vectors $\boldsymbol{r}+$ (integers). One can construct edge theories which are made of degenerate representations only, which are called the $W_{1+\infty}$ *minimal models* [25]. The main result is that they have a one-to-one relation with the Jain series of the filling fractions and yield the same spectrum (3.2) of the charge and fractional spins of the Abelian theories. Basically, the $W_{1+\infty}$ degeneracy conditions on the vector $\mathbf{r}$ select uniquely the lattices corresponding to the Jain K matrices.

The minimal models are *reductions* of the $\widehat{U(1)}^{\otimes m}$ theories with the corresponding spectrum, because the $W_{1+\infty}$ representations are related to the Abelian ones by projecting out an infinity of edge excitations. Moreover, the multiplicities of the excitations are reduced by the additional condition $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_m$ in the spectrum (3.2). This reduction of degrees of freedom implies important differences between the $W_{1+\infty}$ minimal and Abelian theories [25].

(i) There is a *single*, as opposed to m, independent Abelian current, and, therefore, a single (fractionally) charged elementary excitation; the neutral excitations cannot be associated to $(m-1)$ independent edges, because they are described by the degenerate $W_{1+\infty}$ representations, which are "smaller" than the corresponding Abelian representations.

(ii) The neutral excitations have a $SU(m)$ (not $\widehat{SU(m)}_1$) associated "isospin" quantum number, given by the highest weight,

$$\Lambda = \sum_{i=1}^{m-1} \Lambda^{(i)} \left(n_i - n_{i+1}\right) , \tag{3.3}$$

where $\Lambda^{(i)}$ are the fundamental weights of $SU(m)$ [40] and n_i are the integers in Eq. (3.2). Actually, the degenerate $W_{1+\infty}$ representations are equivalent to representations of the $\widehat{U(1)} \otimes \mathcal{W}_m$ algebra, where $\mathcal{W}_m$ is the Zamolodchikov–Fateev–Lykyanov algebra at $c = m-1$ [41]. The fusion rules of this algebra are isomorphic to the tensor product of representations of the $SU(m)$ Lie algebra; thus the neutral excitations in these theories are quark-like and their statistics is non-Abelian. For example, the edge excitation corresponding to the electron is a composite state, carrying both electric charge and the isospin of the fundamental representation $\{\mathbf{m}\}$ of $SU(m)$.

(iii) The number of particle–hole excitations above the ground state is smaller in the $W_{1+\infty}$ minimal models than in the Abelian ones, due to the corresponding inclusion of highest-weight representations.

These results show that the edge theories of the Jain plateaus possess rather non-trivial algebraic properties. The algebra inclusions are

$$\widehat{U(1)} \otimes \widehat{SU(m)}_1 \supset \widehat{U(1)}^{\otimes m} \supset \widehat{U(1)} \otimes \mathcal{W}_m . \tag{3.4}$$

The $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ theories are non-minimal realizations of the $W_{1+\infty}$ symmetry, which are possible precisely at the Jain filling fractions. More detailed properties can be better understood by an explicit discussion of the simplest case $m = 2$.

3.1. The $c = 2$ case

The $W_{1+\infty}$ minimal models are made of representations of the $\widehat{U(1)} \otimes \mathrm{Vir}$ algebra, because $\mathcal{W}_2$ is simply the $c = 1$ Virasoro algebra. The reduction of states from the non-minimal to the minimal theories can be derived from the characters [4] of the corresponding representations [25]. Neglecting the common $\widehat{U(1)}$ factor, we compare the characters of the $\widehat{SU(2)}_1$, $\widehat{U(1)}$ and Virasoro representations. The charge vector of the $c = 2$ degenerate $W_{1+\infty}$ representation is $\boldsymbol{r} = \{r+n, r\}$, with $r \in \mathbb{R}$ and n a positive integer; the $c = 1$ Virasoro part has the weight $L_0 = n^2/4$ and the character

$$\chi^{\mathrm{Vir}}_{n^2/4} = q^{n^2/4}\left(1 - q^{n+1}\right)/\eta(q). \tag{3.5}$$

These Virasoro representations have associated an $SU(2)$ isospin quantum number $s = n/2 > 0$, because their fusion rules are equivalent to the addition of (total) spins $\{s+s'\} \oplus \{s+s'-1\} \oplus \ldots \oplus \{|s-s'|\}$. The character of the $\widehat{U(1)}$ algebra with the same weight L_0 is

$$\chi^{\widehat{U(1)}}_{n^2/4} = q^{n^2/4}/\eta(q). \tag{3.6}$$

Note that the Virasoro and $\widehat{U(1)}$ characters differ by a negative term in the numerator, which cancels part of the power expansion of $\eta(q)$. Thus, the number of states above the highest-weight state is lower in the Virasoro than in the $\widehat{U(1)}$ representations. The missing states are called *null states* of the Virasoro representation, which is said to be *degenerate* [4].

The $\widehat{SU(2)}_1$ algebra has two representations of "spin" $\sigma = 0, 1/2$, whose characters are [13]

$$\begin{aligned}
\chi^{\widehat{SU(2)}_1}_{\sigma=0} &= \frac{1}{\eta(q)^3}\sum_{n\in\mathbb{Z}} (6n+1)\, q^{(6n+1)^2/12} = \sum_{n\in\mathbb{Z}} q^{n^2}/\eta(q), \\
\chi^{\widehat{SU(2)}_1}_{\sigma=1/2} &= \frac{1}{\eta(q)^3}\sum_{n\in\mathbb{Z}} (6n+2)\, q^{(6n+2)^2/12} = \sum_{n\in\mathbb{Z}} q^{(2n+1)^2/4}/\eta(q).
\end{aligned} \tag{3.7}$$

From these characters, we obtain the decompositions[11] of the representations

$$\begin{aligned}
\{\sigma=0\}_{\widehat{SU(2)}_1} &= \sum_{n\in\mathbb{Z}\ \text{even}} \{Q=n\}_{\widehat{U(1)}} = \sum_{k=0}^{\infty} (2s+1)\,\{s=k\}_{\mathrm{Vir}}, \\
\left\{\sigma=\frac{1}{2}\right\}_{\widehat{SU(2)}_1} &= \sum_{n\in\mathbb{Z}\ \text{odd}} \{Q=n\}_{\widehat{U(1)}} = \sum_{k=0}^{\infty} (2s+1)\left\{s=k+\frac{1}{2}\right\}_{\mathrm{Vir}}.
\end{aligned} \tag{3.8}$$

These results show that:

(i) The $\widehat{SU(2)}_1$ "spin" σ is only a spin parity. The $\widehat{U(1)}$ charge Q is not a spin quantum number because it is additive. The Virasoro weight $s = n/2$ composes instead as a good spin quantum number.

(ii) The non-minimal $\widehat{U(1)} \otimes \widehat{SU(2)}_1$ theory has one representation for each $\sigma = 0, 1/2$ value (see the next section); Eq. (3.8) shows that this theory contains the correct number of $W_{1+\infty}$ representations for making spin multiplets. Therefore, the non-minimal theory realizes an incompressible quantum fluid with full $SU(2)$ symmetry.

(iii) The minimal $W_{1+\infty}$ models contains just one representation for each value of the spin s and does not have the full $SU(2)$ symmetry, although each $W_{1+\infty}$ excitation has associated a spin weight; this is a "hidden" symmetry. For $s = 1$ in particular, there is only J_+ out of the three currents $\{J_+, J_0, J_-\}$. These features extend to

[11] A similar discussion is found in Ref. [42].

generic m: the partial reduction of the $SU(m)$ symmetry in the $\mathcal{W}_m$ theories can be understood [43] in the framework of the Hamiltonian reduction of Ref. [29].

Given these algebraic properties, the physical motivations for preferring either the minimal or the non-minimal $W_{1+\infty}$ theories should come from the understanding of the $SU(m)$ symmetry of the Jain plateaus in the neutral edge spectrum. At present, we do not have sufficient numerical and experimental evidence in favour of either hidden or manifest symmetry; on the theoretical side, a fundamental derivation of the Jain approach is also missing. Hereafter, we collect the pieces of evidence we are aware of.

(i) The minimal models (hidden $SU(m)$ symmetry) realize the simplest consistent theory allowed by the $W_{1+\infty}$ symmetry and its fusion rules. There are no additional conserved charges besides the electric, a rather economic and, thus, appealing feature. The reduced number of excitations with respect to the generic hierarchical Abelian theories could be related to the stronger stability of the Jain plateaus [25].

(ii) On the other hand, the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ symmetry could be motivated as follows. In the Jain approach, the $\nu = m/(ms \pm 1)$ and the $\nu = m$ plateaus are qualitatively similar [20]. One can consider the naive model of N *non-interacting* electrons filling m Landau levels on a disk: the (particle–hole) edge excitations of each filled level gives rise, for $N \to \infty$, to an independent copy of the $\nu = 1$ edge, which is described by a Weyl fermion $c = 1$ theory [5]. These m Weyl fermions have naturally associated the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ symmetry of unitary rotations in spinor space.

(iii) This hypothesis of many independent and equivalent edges is supported by a numerical simulation at $\nu = 2/5$ $(m = 2)$ [44]. The spectrum of the microscopic Hamiltonian for few electrons was found in a two Landau-level Hilbert space, as a function of the cyclotron energy ω_c. The incompressible fluid ground state was shown to remain in the same universality class all along the range from zero to a large value ω_c. The vanishing limit can be considered as a model for the independent edges, while the infinite limit is closer to the experimental setting of large magnetic fields.

(iv) However, these effective m Landau levels are certainly not equivalent in the microscopic theory [45]: there are cyclotron gaps among them, which imply a band structure of the quasi-particle excitations; moreover, they have different wave functions even for $\omega_c = 0$. We do not yet know an explicit derivation of the edge excitations from a realistic microscopic theory of interacting electrons.

Finally, we describe a different experimental situation in which the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ theories and their $SU(2)$ symmetry is fully justified. This is the spin-singlet quantum Hall effect found at $\nu = 2/3$ in samples of low electron density [46]. Since the same filling fraction is achieved for smaller magnetic fields, the Zeeman energy is reduced and the electrons may anti-align. Experimental evidence was found for a phase transition between a spin-singlet and the usual spin-polarized ground state at $\nu = 2/3$, when the magnetic field is tilted with respect to the vertical of the plane. According to Ref. [47], the new universality class of the spin-singlet incompressible fluid is still described by the $\widehat{U(1)} \otimes \widehat{SU(2)}_1$ theory (3.2) of the spin-polarized phase, where the two fluids are now interpreted as spin states. Above the spin-singlet ground state, the edge excitations carry a spin, which can be naturally identified with the isospin number s of degenerate $W_{1+\infty}$ representations. Actually, they occur in the right multiplicity and give the elementary, i.e. irreducible, excitations of the incompressible fluid. Therefore, the degenerate $W_{1+\infty}$ representations are a useful basis for making explicit the $SU(m)$ symmetry of the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ theories.

4. Partition functions for hierarchical plateaus

4.1. Diagonal modular invariant for $\widehat{U(1)}^{\otimes m}$ theories

We now focus our attention on minimal and non-minimal $W_{1+\infty}$ theories with positive integer central charge $c = m = 2, 3, \ldots$. We first consider the $\widehat{U(1)}^{\otimes m}$ theories, describing generic hierarchical plateaus, whose one-edge spectrum (3.1) is parametrized by an integer symmetric K matrix. We shall now find the corresponding annulus partition function. We take a lattice of generic $\widehat{U(1)}^{\otimes m}$ representations which is closed under the fusion rules. Their weights are, in m-dimensional vector notation,

$$Q = t^T K^{-1} n, \qquad L_0 = \frac{1}{2} n^T K^{-1} n, \tag{4.1}$$

where K is a real symmetric positive-definite matrix and $\boldsymbol{t}$ is a real vector of charge units. We look for partition functions of the form Eq. (2.6) properly generalized to the multi-component case, which are solutions of the conditions (T^2, S, U, V) in Eqs. (2.7)-(2.10). As in the $c = 1$ case, we first consider the U condition by collecting the sectors with integer spaced charges in the spectrum (4.1). These are given by $\boldsymbol{n} = K\ell + \lambda$, $\boldsymbol{\ell} \in \mathbb{Z}^m$ provided that $\boldsymbol{t}$ has integer components. If the matrix K is also integer, there is a finite number of $\boldsymbol{\lambda}$ values (the sectors of the RCFT), belonging to the quotient of the $\boldsymbol{n}$ lattice by the $\boldsymbol{\ell}$ lattice:

$$\boldsymbol{\lambda} \in \frac{\mathbb{Z}^m}{K\mathbb{Z}^m}. \tag{4.2}$$

As in Section 2, the $\widehat{U(1)}$ characters in each sector sum up into m-dimensional theta functions (see Eq. (2.11)),

$$\begin{aligned}\chi_{\boldsymbol{\lambda}} = {} & \exp\left(-\pi t^T K^{-1} t \frac{(\operatorname{Im}\zeta)^2}{\operatorname{Im}\tau}\right) \frac{1}{\eta(q)^m} \\ & \times \sum_{\ell \in \mathbb{Z}^m} \exp\left(i2\pi\left\{\frac{\tau}{2}(K\ell+\lambda)^T K^{-1}(K\ell+\lambda) + \zeta t^T\left(\ell + K^{-1}\lambda\right)\right\}\right).\end{aligned} \tag{4.3}$$

Their T^2, S, U, V transformations read

$$T^2: \quad \chi_{\boldsymbol{\lambda}}(\tau+2,\zeta) = \exp\left(i2\pi\left(\lambda^T K^{-1}\lambda - \frac{m}{12}\right)\right)\chi_{\boldsymbol{\lambda}}(\tau,\zeta), \tag{4.4}$$

$$S:\quad \chi_{\boldsymbol{\lambda}}\left(-\frac{1}{\tau},-\frac{\zeta}{\tau}\right)=\frac{\exp\left(i\pi t^T K^{-1} t\,\mathrm{Re}\frac{\zeta^2}{\tau}\right)}{\sqrt{\det K}} \times \sum_{\boldsymbol{\lambda}'\in\mathbb{Z}^m/K\mathbb{Z}^m} \exp\left(i2\pi\boldsymbol{\lambda}^T K^{-1}\boldsymbol{\lambda}'\right)\chi_{\boldsymbol{\lambda}'}(\tau,\zeta)\,, \tag{4.5}$$

$$U:\quad \chi_{\boldsymbol{\lambda}}(\tau,\zeta+1)=\exp\left(i2\pi t^T K^{-1}\boldsymbol{\lambda}\right)\chi_{\boldsymbol{\lambda}}(\tau,\zeta)\,, \tag{4.6}$$

$$V:\quad \chi_{\boldsymbol{\lambda}}(\tau,\zeta+\tau)=\exp\left(-i2\pi t^T K^{-1} t\left(\mathrm{Re}\,\zeta+\mathrm{Re}\frac{\tau}{2}\right)\right)\chi_{\boldsymbol{\lambda}+t}(\tau,\zeta)\,. \tag{4.7}$$

Therefore, they carry a finite-dimensional unitary projective representation of the modular group $\Gamma(2)$. According to the discussion in Section 2, the dimension of this representation gives the Wen topological order (see Eq. (2.26)),

$$\dim\mathcal{H}(\Sigma_1)=|\det K|. \tag{4.8}$$

Some physical conditions further constrain the parameters t and K. By extending the $c=1$ analysis of Section 2, one requires m electron states in the chiral spectrum, having unit charge and odd statistics, and integer statistics relative to all the states. These conditions imply, in a suitable basis, that $t_i=1$ and K_{ii} is odd for $i=1,\dots,m$. Moreover, the Hall current $\nu=t^T K^{-1} t$ is obtained from the transport of the minimal fractional charge $\boldsymbol{\lambda}=t$ between the two edges. The U invariance of the partition function, written as a sesquilinear form of the characters (2.6), implies the equation $t^T K^{-1}(\lambda-\bar{\lambda})\in\mathbb{Z}$ for the left and right weights. Its solutions depend on the specific form of K; here, we shall only discuss the general *diagonal* solution, $\boldsymbol{\lambda}=\bar{\boldsymbol{\lambda}}$. This is also a solution of the other (T^2,S,V) conditions. The modular invariant partition function is thus a diagonal quadratic form of the generalized characters (4.3),

$$Z=\sum_{\boldsymbol{\lambda}\in\mathbb{Z}^m/K\mathbb{Z}^m}\chi_{\boldsymbol{\lambda}}\bar{\chi}_{\boldsymbol{\lambda}}. \tag{4.9}$$

This partition function contains the chiral spectrum (3.1) for each edge, in the case of positive-definite[12] K. Moreover, it describes pairs of left–right excitations with integer total charge. Note that it is again an odd variant of the many-component boson compactified on a rational torus [4]. Some results on these partition functions were also found in Refs. [17,18], but their modular invariance was not fully analysed.

4.2. Diagonal invariant for the $\widehat{U(1)}\otimes\widehat{SU(m)}_1$ theories

Next, we describe more specific results for the non-minimal and minimal $W_{1+\infty}$ models describing the Jain plateaus. First we rederive the diagonal modular invariant (4.9) in a simpler two-dimensional basis with explicit $\widehat{SU(m)}_1$ symmetry. In this basis, we make contact with Itzykson's analysis of $\widehat{SU(m)}_1$ modular invariants [31]: although our setting is rather different, this analysis will be used to find many non-trivial $\widehat{U(1)}\otimes\widehat{SU(m)}_1$ invariants, which define new rational CFTs.

[12] The case of mixed chiralities will be described later.

The $\widehat{SU(m)}_1$ representations and characters were described in Ref. [31]: there are m highest-weight representations, corresponding to completely antisymmetric tensor representations of the $SU(m)$ Lie algebra, which can be labelled by $\alpha = 1, \ldots, m$. This is an additive quantum number modulo m, the so-called m-ality. Therefore, the $\widehat{SU(m)}_1$ fusion rules are isomorphic to the $\mathbb{Z}_m$ group. We shall assume that the $\widehat{SU(m)}_1$ excitations do not carry directly a charge quantum number; thus, we shall need their characters for $\zeta = 0$, which satisfy

$$\chi_{\alpha}^{\widehat{SU(m)}_1}(\tau, 0) = \chi_{m-\alpha}^{\widehat{SU(m)}_1}(\tau, 0) = \chi_{m+\alpha}^{\widehat{SU(m)}_1}(\tau, 0); \tag{4.10}$$

the Virasoro dimension of the highest-weight states is

$$L_0 = \frac{\alpha(m-\alpha)}{2m}, \qquad \alpha = 0, \ldots, m-1. \tag{4.11}$$

As shown for the $m = 2$ case in Section 3 (Eq. (3.8)), these characters resum an infinity of degenerate $W_{1+\infty}$ representations, which have dimensions $L_0 + \mathbb{Z}$, and $SU(m)$ weights of the same m-ality α. The modular transformations are [31]

$$\begin{aligned} T^2: &\quad \chi_{\alpha}^{\widehat{SU(m)}_1}(\tau+2) = \exp\left(i2\pi\left(\frac{\alpha(m-\alpha)}{m} - \frac{m}{12}\right)\right) \chi_{\alpha}^{\widehat{SU(m)}_1}(\tau), \\ S: &\quad \chi_{\alpha}^{\widehat{SU(m)}_1}\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{m}} \sum_{\alpha'=1}^{m} \exp\left(-i2\pi \frac{\alpha\alpha'}{m}\right) \chi_{\alpha'}^{\widehat{SU(m)}_1}(\tau), \end{aligned} \tag{4.12}$$

while U, V do not act on neutral states.

Furthermore, the $\widehat{U(1)}$ factor contained in the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ theories can be described by the finite set of $\mathbb{Z}_p$ characters (2.11) of Section 2, with free parameter $p \in \mathbb{Z}$. They will be denoted as $\chi^{\widehat{U(1)}}{}_{\lambda}(\tau, \zeta)$, with $\lambda = 1, \ldots, p$. In summary, the modular invariance problem has been cast into a two-dimensional basis, where the S transformation acts as the $\mathbb{Z}_p \times \mathbb{Z}_m$ discrete Fourier transform, plus additional physical conditions. Clearly, we should discard the solutions with decoupled $\widehat{U(1)}$ and $\widehat{SU(m)}_1$ sectors, because these are simple superpositions of a Laughlin fluid and an independent neutral fluid.

The Jain spectrum (3.2) possesses the $\widehat{SU(m)}_1$ symmetry [27] and can be rewritten in this basis. Upon substitution of

$$\begin{aligned} n_1 &= l + \sum_{i=2}^{m} k_i \pm \alpha, \\ n_i &= l - k_i, \quad i = 2, \ldots, m, \end{aligned} \tag{4.13}$$

where $l, k_i \in \mathbb{Z}$ and α is the $\widehat{SU(m)}_1$ weight, the Jain spectrum becomes

$$\begin{aligned} \nu = \frac{m}{ms \pm 1}: \quad & Q = \frac{ml \pm \alpha}{ms \pm 1}, \\ & J = \frac{(ml \pm \alpha)^2}{2m(ms \pm 1)} \pm \frac{\alpha(m-\alpha)}{2m} + r, \quad r \in \mathbb{Z}. \end{aligned} \tag{4.14}$$

Consider now these formulas with the upper signs, the other choice will be discussed later. One recognizes the $\widehat{U(1)}$ and $\widehat{SU(m)}_1$ contributions to L_0 (J), which identify $p = m\hat{p}$, $\hat{p} = (ms+1)$ and $\lambda = ml + \alpha \bmod (m\hat{p})$; note that $\hat{p} = 1 \bmod m$ and $\lambda = \alpha \bmod m$, and that $\hat{p}, m$ are coprime numbers $(\hat{p}, m) = 1$. Consider the m-term linear combinations of the $(m^2\hat{p})$ tensor characters

$$\theta_\lambda(\tau,\zeta) = \sum_{\alpha=1}^{m} \chi^{\widehat{U(1)}}_{\lambda+\alpha\hat{p}}(\tau, m\zeta)\, \chi^{\widehat{SU(m)}_1}_{\lambda+\alpha\hat{p} \bmod m}(\tau,0), \qquad \lambda = 1,\dots,m\hat{p}. \tag{4.15}$$

They satisfy $\theta_{\lambda+\hat{p}} = \theta_\lambda$, due to $\hat{p} = 1 \bmod m$; thus, there are $\hat{p}$ independent ones, which can be chosen to be θ_{ma}, $a = 1,\dots,\hat{p}$. One can check that they carry a representation of the modular transformations (T^2, S, U, V), in particular $S_{ab} \propto \exp(i2\pi mab/\hat{p})/\sqrt{\hat{p}}$; its dimension matches the Wen topological order $\det(1+sC) = \hat{p}$. The unique solution of the U condition in the basis θ_λ is given by the diagonal pairing of left and right characters,

$$Z = \sum_{a=1}^{\hat{p}} \theta_{ma}\bar{\theta}_{ma} = \sum_{a=1}^{\hat{p}} \left(\sum_{\alpha=1}^{m} \chi^{\widehat{U(1)}}_{ma+\alpha\hat{p}} \chi^{\widehat{SU(m)}_1}_{\alpha} \right) \left(\sum_{\beta=1}^{m} \bar{\chi}^{\widehat{U(1)}}_{ma+\beta\hat{p}} \bar{\chi}^{\widehat{SU(m)}_1}_{\beta} \right). \tag{4.16}$$

One can check explicitly that the characters θ_{ma} are equal to the χ_λ in (4.3) for $K = 1+sC$, once the corresponding $\widehat{U(1)}$ charges are identified; the partition function (4.16) is equal to (4.9), and the spectrum of charges and fractional spins reproduces (3.2), with upper signs, for each chirality. The V transformation is

$$V: \quad \theta_{ma}(\tau,\zeta+1) = \exp\left(-i2\pi\frac{m}{\hat{p}}\left(\mathrm{Re}\,\zeta + \mathrm{Re}\frac{\tau}{2}\right)\right)\theta_{ma+m}(\tau,\zeta). \tag{4.17}$$

This shows that the minimal transport of charge between the two edges is m times the elementary fractional charge; this is the smallest spectral flow among the states contained in (4.16) which keeps α constant, namely which conserves the $\widehat{SU(m)}_1$ quantum number carried by the neutral excitations. The Hall current is thus $\nu = m/\hat{p}$.

Finally, it remains to check the existence of *one* electron excitation for each edge, which is described by the states (a,α) in Z (4.16) satisfying

$$\begin{aligned} Q &= \frac{m(a+k\hat{p})+\alpha\hat{p}}{\hat{p}} = 1, \quad k \in \mathbb{Z}, \\ 2J &= \frac{(m(a+k\hat{p})+\alpha\hat{p})^2}{m\hat{p}} + \frac{\alpha(m-\alpha)}{m} = 1, \quad \bmod 2, \end{aligned} \tag{4.18}$$

with $\hat{p} = 1+ms$. This is solved by $\alpha = 1$ and $a = 0$, provided that s is even. Moreover, the condition (2.17) is also satisfied, such that the electron ($\alpha = 1$) is local, i.e. it has integer monodromy with all excitations. We thus found the partition functions (4.16) of the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ theories at the Jain plateaus; they describe rational conformal field theories with an extension of the affine algebra $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ and fusion algebra $\mathbb{Z}_{\hat{p}}$.

We now find the partition function for the case of mixed chiralities in the one-edge spectrum (3.2), at $\nu = m/(ms-1)$. The fractional spin is not positive definite and cannot be identified with the Virasoro dimension. Moreover, the spectrum is not decomposable into chiral and anti-chiral independent parts: $K \neq K^R - K^L$, with K^L, K^R positive definite in the notation of (4.1). Nevertheless, a simple modification of the characters (4.15) switches the chirality of the neutral excitations,

$$\theta_\lambda^{(-)} = \sum_{\alpha=1}^{m} \chi_{\lambda+\alpha\hat{p}}^{\widehat{U(1)}} \bar{\chi}_{\lambda+\alpha\hat{p} \bmod m}^{\widehat{SU(m)}_1}, \qquad \lambda = 1, \ldots, m\hat{p} = m(ms-1). \tag{4.19}$$

This gives again a representation of the modular group for $\hat{p} = ms - 1 > 0$. The U condition implies the partition function

$$Z = \sum_{a=1}^{\hat{p}} \left(\sum_{\alpha=1}^{m} \chi_{ma+\alpha\hat{p}}^{\widehat{U(1)}} \bar{\chi}_{\alpha}^{\widehat{SU(m)}_1} \right) \left(\sum_{\beta=1}^{m} \bar{\chi}_{ma+\beta\hat{p}}^{\widehat{U(1)}} \chi_{\beta}^{\widehat{SU(m)}_1} \right). \tag{4.20}$$

which satisfies all the conditions and reproduces the spectrum (3.2) for $\nu = m/(ms-1)$. Note that the spectrum of $(L_0^R + L_0^L)$ is still positive definite and ensures the convergence of the sum. This partition function combines representations of the two chiralities in a non-trivial way, thus it difficult to assign excitations to either the inner or the outer edge. From the physical point of view, we are already used to non-local effects in the incompressible fluid, like fractional statistics. Moreover, this spectrum has been partially confirmed by experiments. From the theoretical point of view, this was not easily accounted for by the chiral theories of one edge, because it is not left-right decomposable [28,25]. Instead, it is naturally described by annulus partitions functions which are not left-right diagonal.

4.3. Minimal $W_{1+\infty}$ models

As discussed in Section 3, the partition functions (4.16) and (4.20) can be decomposed into degenerate $W_{1+\infty}$ representations, owing to the algebra inclusions (3.4). For $m = 2$, we can use the relations among the characters (3.7) and find that the $W_{1+\infty}$ representation of spin ℓ occurs with multiplicity $(2\ell + 1)$. Therefore, these partition functions describe non-minimal $W_{1+\infty}$ theories which possess a full $SU(2)$ symmetry.

Actually, these seem to be the *unique* RCFT with $W_{1+\infty}$ symmetry. Indeed, the $(m = 2)$ $W_{1+\infty}$ characters yield a linear finite-dimensional representation of the S transformation only if they are summed up with multiplicity $(2\ell + 1)$. Here is the argument. The non-trivial part of the character is the $\mathcal{W}_2$, i.e. Virasoro, part (3.6), which we rewrite as

$$\chi_d^{\mathrm{Vir}} = \frac{1}{\eta} \left(q^{(d-1)^2/4} - q^{(d+1)^2/4} \right) = -\chi_{-d}^{\mathrm{Vir}}, \qquad d = 2\ell + 1 \geqslant 1. \tag{4.21}$$

Consider the sum of these characters with generic multiplicities N_d, extended odd for $d < 0$ for convenience,

$$\chi[N] = \sum_{d=1}^{\infty} N_d \chi_d^{\mathrm{Vir}} = \sum_{k \in \mathbb{Z}} N_k \frac{q^{(k-1)^2/4}}{\eta}, \qquad N_d = -N_{-d} > 0 \quad \text{for } d > 0. \tag{4.22}$$

It is rather easy to compute the S transformation of $\chi[N]$ by using the Poisson formula (Appendix A), and check that it reproduces itself, up to phases, only for the multiplicity $N_d = d$ leading to the $\widehat{SU(2)}_1$ characters (3.7).

Therefore, we cannot form a modular invariant partition function of the RCFT type (2.6) for the $W_{1+\infty}$ minimal models [25], which contain $W_{1+\infty}$ representations with multiplicities $N_d = 1$, for $d \geqslant 1$: the $W_{1+\infty}$ minimal models are *non-rational* conformal field theories. Nevertheless, they are consistent CFTs, because the fusion rules close on the degenerate $W_{1+\infty}$ representations belonging to the $U(m)$ weight lattice (3.2) [25].

One can tentatively disregard the S modular invariance and build minimal $W_{1+\infty}$ partition functions, subjected to the remaining conditions (T^2, U, V) and the existence of the electron state. However, it turns out that these building criteria are not sufficient enough, because they allow a very large class of consistent models. Their fusion rules can close on a subset of degenerate $W_{1+\infty}$ representations with $SU(m)$ weights of given m-alities $\alpha = n\delta$, $n = 1, \ldots, m/\delta$, where δ divides m and satisfies $(\delta, m/\delta) = 1$. These solutions are discussed in Appendix B and were not considered in Ref. [25]. They possess Hall conductivities which span almost any fraction $\nu = n/d$ and are thus phenomenologically unrealistic. Following the discussion of Section 3, we conclude that:

(i) If the minimal $W_{1+\infty}$ theories are realized at the Jain plateaus (no full $SU(m)$ symmetry of the edge excitations), then another building requirement, replacing modular invariance, is necessary to construct their partition functions. These non-rational conformal theories are not well understood in the literature and are not further discussed in this paper.

(ii) If the non-minimal, $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ theories are realized (full $SU(m)$ symmetry), then their partition functions can be found in this paper.

4.4. *Non-diagonal* $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ *invariants*

We now return to the analysis of non-minimal theories based on the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ algebra and look for other solutions to the modular invariance conditions (S, T^2, U, V) besides the diagonal invariant (4.16). The general theory [14,11] states that there are two classes of modular invariants:

(i) There are invariants describing extensions of the symmetry algebra by some dimension-one field in the theory. For example, the θ_λ in (4.17) are characters of an extended algebra representations which finitely decompose into those of $\widehat{U(1)} \otimes \widehat{SU(m)}_1$. The partition function (4.16) is diagonal in the extended basis (θ_λ), $Z = \sum_\lambda |\theta_\lambda|^2$, but actually non-diagonal in the original basis $(\chi^{\widehat{U(1)}}{}_\lambda \, \chi^{\widehat{SU(m)}_1}{}_\alpha)$.

(ii) Once the algebra is maximally extended, the non-diagonal invariants are necessarily of the form $Z = \sum \chi_\lambda \bar{\chi}_{\pi(\lambda)}$, where $\lambda \to \pi(\lambda)$ is a permutation of λ and an automorphism of the fusion rules, $N_{ijk} = N_{\pi(i),\pi(j),\pi(k)}$. Indeed, these Z are invariant under the S transformation, $[\pi, S] = 0$, due to the Verlinde relation between fusion rules and the S transformation (2.22); moreover, $[\pi, T] = 0$ by hypothesis.

The modular invariance problem in this paper is slightly non-standard, because there are additional conditions U, V concerning the $\widehat{U(1)}$ charge, and some physical conditions. We shall find that there are no algebra extensions of the $\widehat{U(1)}$ algebra beyond that of Section 2, while they exist in the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ algebra. Moreover, the automorphisms involving the $U(1)$ charge are forbidden by the U condition, while they are possible among the neutral $\widehat{SU(m)}_1$ weights.

In order to proceed, we need to be more specific about the general form of the invariants $Z = \chi \cdot \mathcal{N} \cdot \bar{\chi}$, characterized by matrices $\mathcal{N}_{\lambda\bar{\lambda}}$, which commute with the modular transformations (T^2, S, U, V). Let us first discuss the $\widehat{U(1)}$ theory of Section 2 and show that the unique solution is given by the diagonal partition function (2.16), namely $\mathcal{N}_{\lambda\bar{\lambda}} = \delta_{\lambda,\bar{\lambda}}$, where λ labels the $\mathbb{Z}_p$ extended $\widehat{U(1)}$ characters in (2.11). The $\mathcal{N}$ matrices commuting with the $c = 1$ transformations (T^2, S) in (2.12)–(2.15) can be found by generalizing the method of Refs. [13,31] – the present case only differs by some factors of two. The commutation with T^2 implies

$$\mathcal{N}_{\lambda,\bar{\lambda}} \neq 0 \quad \text{for } \lambda^2 = \bar{\lambda}^2 \mod p. \tag{4.23}$$

Following Ref. [13] closely, we consider a decomposition into factors $p = rs$, where $\delta = (r, s)$ is their common factor, and we rewrite (4.23) as the system

$$\begin{cases} \lambda = \dfrac{\delta}{2}\left(\dfrac{r}{\delta}\rho + \dfrac{s}{\delta}\sigma\right), \\[2ex] \bar{\lambda} = \dfrac{\delta}{2}\left(\dfrac{r}{\delta}\rho - \dfrac{s}{\delta}\sigma\right). \end{cases} \tag{4.24}$$

The solutions are pairs $(\lambda, \bar{\lambda})$ multiples of δ $(\delta/2)$ for δ odd (even), respectively, which can be written as follows. Consider first δ odd. Introduce ω mod $2p/\delta^2$ by the system

$$\begin{cases} 1 = \dfrac{r}{\delta}R - \dfrac{s}{\delta}S, \quad (R, S) \quad \mod \left(\dfrac{s}{\delta}, \dfrac{r}{\delta}\right), \\[2ex] \omega = \dfrac{r}{\delta}R + \dfrac{s}{\delta}S, \quad \omega \quad \mod 2p/\delta^2. \end{cases} \tag{4.25}$$

This satisfies $\omega^2 = 1 \mod 4p/\delta^2$; we actually need ω mod p/δ^2. For δ odd, the solutions $\lambda, \bar{\lambda}$ of (4.24) can be shown to satisfy

$$\lambda = \omega\bar{\lambda} + \xi\frac{p}{\delta} \mod p, \qquad \xi = 1, \dots, \delta \quad \text{for} \delta|\lambda, \delta|\bar{\lambda}. \tag{4.26}$$

These are all the solutions of (4.23) corresponding to the factorization $p = rs$, with $\delta = (r, s)$ odd. The multiplicative relation (4.26) is useful to find a corresponding solution of the S condition $\mathcal{N}S = S\mathcal{N}$,

$$\mathcal{N}^{(\delta,\omega)}_{\lambda\bar{\lambda}} = \begin{cases} \displaystyle\sum_{\xi=1}^{\delta} \delta^{(p)}_{\lambda,\omega\bar{\lambda}+\xi p/\delta}, & \text{if } \delta|\lambda, \delta|\bar{\lambda}, \\ 0, & \text{otherwise.} \end{cases} \quad \left(\omega^2 = 1 \mod \frac{p}{\delta^2}\right). \tag{4.27}$$

Therefore, there is a solution $\mathcal{N}^{(\delta,\omega)}$ to the S and T^2 conditions for each pair (δ,ω), if δ is odd.

Consider now δ even: define $\delta = 2\delta'$ and $p = 4\delta'^2 k$. From Eq. (4.24), λ and $\bar{\lambda}$ are multiples of δ'. Moreover, λ/δ' and $\bar{\lambda}/\delta'$ have the same parity mod two. If λ and $\bar{\lambda}$ are also multiples of δ, then they satisfy (4.26) as before and give the solution (4.26) and (4.27). If, otherwise, λ/δ' and $\bar{\lambda}/\delta'$ are both odd, we must generalize the argument to the larger period p/δ'. Consider again ω mod $2p/\delta^2 = p/2\delta'^2$ in Eq. (4.25), and introduce $\omega_1 = \omega$ and $\omega_2 = \omega + p/2\delta'^2$ mod p/δ'^2, which satisfy $\omega_1^2 = \omega_2^2 = 1$ mod p/δ'^2. By repeating the argument, the solutions $(\lambda,\bar{\lambda})$ of (4.24), with $(\lambda/\delta', \bar{\lambda}/\delta')$ odd, are shown again to satisfy Eq. (4.26) with $\delta \to \delta'$ and $\omega \to \omega_1$ *or* $\omega \to \omega_2$. These, in turn, yield two solutions $\mathcal{N}^{(\delta',\omega_1)}$, $\mathcal{N}^{(\delta',\omega_1)}$ of the form (4.27). Conversely, all solutions of (4.27) with (δ,ω), (δ',ω_1) and (δ',ω_2) are manifestly solutions of (4.23). In conclusion, we found one (three) solutions for δ odd (even), respectively; however, they can be linearly dependent. In Ref. [32], it has been shown that these are the general solution of the (S,T^2) conditions for the $\widehat{U(1)}$ theories.

These solutions show explicitly the two general classes of invariants discussed before: for $\delta \neq 1$ and $\omega = \pm 1$ the partition functions (4.27) are diagonal sums of the generalized characters

$$\Theta_{\alpha'} = \sum_{\xi=1}^{\delta} \chi_{\delta\alpha' + \xi p/\delta}, \quad \alpha' = 1,\dots,p/\delta^2, \tag{4.28}$$

for representations of an extended chiral algebra, which transform by $S_{\alpha'}^{\beta'} = \exp(-2i\pi \times \alpha'\beta'\delta^2/p)$. Moreover, solutions (4.27) with $\omega \neq \pm 1$ and $\delta = 1$ pair non-trivially the left and right algebras by an automorphism of the fusion rules.

However, these solutions do not pass the other conditions (U,V) or they are equivalent to the $\widehat{U(1)}$ diagonal solution of Section 2. The sums (4.28) of the $\widehat{U(1)}$ characters (2.11) satisfy the identity $\Theta_n(p = \delta^2 p') \propto \chi_n^{\widehat{U(1)}}(p')$; this implies that the extended algebra solutions $Z = \sum_\alpha |\Theta_\alpha|^2$ are actually identical to the diagonal solutions (2.16) for another p value, which is considered anyhow. Moreover, the solutions with automorphisms ω are forbidden by the U condition (2.9), which implies $\lambda = \bar{\lambda}$ mod p, i.e. $\omega = 1$ mod p/δ^2 for any δ.

We now find the non-trivial invariants for the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ algebra. This problem is two-dimensional: the conditions S, T^2 for the pure $\widehat{U(1)}$ part were discussed before, while those of the $\widehat{SU(m)}_1$ part have the same solutions (4.25)-(4.27), with λ replaced by α and p by m. Actually, S is a $\mathbb{Z}_m$ discrete Fourier transform and T^2 implies $\alpha^2 = \bar{\alpha}^2$ mod m, according to Eq. (4.12). Therefore, we know the general solutions for each of the one-dimensional sub-problems. Here, we shall not produce a complete solution of the two-dimensional problem, but present some two-dimensional coupled invariants which are suggested by the one-dimensional solutions.[13]

[13] The general solution of the $\widehat{U(1)}^{\otimes m}$ problem has been found in Ref. [32].

Let us first consider the new invariants associated to extensions of the $\widehat{SU(m)}_1$ algebra for $m = \delta^2 m'$. For $\hat{p} = 1 + m's$, the following invariants generalize (4.16):

$$Z = \sum_{a=1}^{\hat{p}} \left(\sum_{\alpha'=1}^{m'} \chi^{\widehat{U(1)}}_{m'a+\alpha'\hat{p}} \Theta^{\widehat{SU(m)}_1}_{\alpha'} \right) \left(\sum_{\beta'=1}^{m'} \bar{\chi}^{\widehat{U(1)}}_{m'a+\beta'\hat{p}} \bar{\Theta}^{\widehat{SU(m)}_1}_{\beta'} \right),$$

$$\Theta^{\widehat{SU(m)}_1}_{\alpha'} = \sum_{\xi=1}^{\delta} \chi^{\widehat{SU(m)}_1}_{\delta(\alpha'+\xi m')} \quad (m = \delta^2 m'). \tag{4.29}$$

These extended characters resum $\widehat{SU(m)}_1$ representations with Virasoro dimensions differing by integers or half-integers. The electron conditions are sensitive to half-integer spins: in analogy with (4.18), the electron is described by the extended weights (a, α') satisfying

$$\begin{aligned} Q &= \frac{m'a}{\hat{p}} + \alpha' = 1, \qquad \hat{p} = 1 + m's \qquad \rightarrow \qquad a = 0, \quad \alpha' = 1, \\ 2J &= \frac{(m'a + \alpha'\hat{p})^2}{m'\hat{p}} + \frac{(\alpha' + m'\xi)(\delta m' - \alpha' - m'\xi)}{m'} \\ &= s + \delta + m'\xi(\delta - \xi) = 1 \mod 2, \quad \forall \xi = 1, \ldots, \delta. \end{aligned} \tag{4.30}$$

For $m = \delta^2 m'$ and δ odd, the solutions are s even and any m'. These edge theories span again the Jain filling fractions $\nu = m'/(m's+1)$; their neutral spectrum is more involved than in the theories (4.16), and is described by extensions of the $\widehat{U(1)} \otimes \widehat{SU(\delta^2 m')}_1$ algebra, with $\delta = 1, 3, \ldots$ Furthermore, the solutions to (4.30) with δ even are given by s odd and m' even; these edge theories exist for other filling fractions,[14]

$$\nu = \frac{m'}{m's \pm 1}, \quad m' \text{ even}, \ s \text{ odd}, \tag{4.31}$$

The first few values of $0 < \nu < 1$ with small denominator are $\nu = 2/3, 4/5, 6/7, \ldots, 2/7, 4/13, \ldots, 2/5, 4/11, \ldots$. Note that the condition (2.17) of locality of the electron state is fulfilled by all these solutions.

Next, we discuss the invariants including non-trivial twists $\omega \neq \pm 1$ (4.25). According to the previous discussion, these cannot occur in the $\widehat{U(1)}$ spectrum, due to the U condition, but can be introduced for the neutral weights. The twisted versions of the invariants (4.29) exist at the same values $\hat{p} = 1 + m's$ and $m = \delta^2 m'$,

$$Z = \sum_{a=1}^{\hat{p}} \left| \sum_{\alpha'=1}^{m'} \chi^{\widehat{U(1)}}_{m'a+\alpha'\hat{p}} \sum_{\xi=1}^{\delta} \chi^{\widehat{SU(m)}_1}_{\delta(\omega\alpha'+\xi m')} \right|^2. \tag{4.32}$$

This invariant is again diagonal with respect to the (maximally) extended chiral algebra, thus ω is not an automorphism of its fusion rules, i.e. this is not an example of the

[14] We include the analogous solution (4.20) with mixed chiralities.

automorphism invariants discussed before [11]; actually, the twist is between the $\widehat{U(1)}$ and $\widehat{SU(m)}_1$ weights entering the extended algebra representations. Another type of invariant with, e.g., the ω twist for the right $\widehat{SU(m)}_1$ characters only, does not satisfy the reality condition $\bar{Z} = Z$ and must be discarded.

The values of ω mod m' for the twisted invariant (4.32) are found in (4.25) for any factorization $m = ab$ with $\rho = (a,b)$: for any ρ, there is one solution by identifying $\delta = \rho$ in (4.32); for ρ even, there can be two further solutions ω_1, ω_2 with $\delta = \rho/2$. The filling fractions take either the Jain values (3.2) or the new ones (4.31), according to the parity of s, which is determined by the following electron conditions, analogous to (4.30):

$$s + \frac{1-\omega^2}{m'} + \omega\delta + m'\xi\,(\delta - \xi) = 1 \mod 2, \quad \forall \xi = 1, \dots, m'. \tag{4.33}$$

For factorizations $m = ab$ with $\rho = (a,b) = \delta$ odd, the solutions are s even and any m' (independently of ω), leading to the Jain filling fractions; for $\rho = (a,b) = \delta$ even, they are the new ones (4.31). The two further solutions for $\rho = 2\delta$, with ω_1, ω_2 and $m' = 4k$ give the following cases: if k is odd, Eq. (4.33) reduces to $s + \delta = 1$ mod 2, independent of ω; if k is even, it remains ω dependent and s takes opposite parities for the two ω_i values. Note that the electron states obtained by these solutions are always local with respect to the other excitations.

These solutions with non-trivial $\omega \neq \pm 1$ mod m' are only possible for relatively large m', which correspond to filling fractions with large denominators that are not phenomenologically relevant. Actually, the lowest non-trivial values of $\omega \neq \pm 1$, m' and ν are

(i) $\omega = 3$ mod $m = m' = 8$, corresponding to filling fractions $\nu = 8/(8s \pm 1)$, s odd, i.e. $\nu = 8/9, 8/23, 8/25, \dots$

(ii) $\omega = 5$ mod $m = m' = 12$ corresponding to $\nu = 12/(12s \pm 1)$, s even, i.e. $\nu = 12/23, 12/25, \dots$

In conclusion, we cannot prove here that the modular invariants (4.16), (4.29) and (4.32) form the complete solution of the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ problem; there might exist additional exceptional solutions, i.e. sporadic $(\hat{p}, m)$ values not falling into sequences (see Ref. [32]). All the partition functions found in this section define new RCFTs with non-trivial extensions of the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ algebra. We finally remark that the many new invariants (4.29) and (4.32) possess a rather limited set of filling fractions: this feature has some phenomenological appeal, as discussed in the following section.

4.5. Comparison with the experiments: higher order hierarchical pattern

The experimental values of the filling fraction $0 < \nu = n/d < 1$ are described in Fig. 1, where they are ordered on the vertical axis according to the value of their denominator [33]. Actually, within the mean field theory [48], the size of the gap of the incompressible fluid is of the order $O(1/d)$, thus stability decreases towards the bottom of the figure. These fractions are usually determined by the minima of the

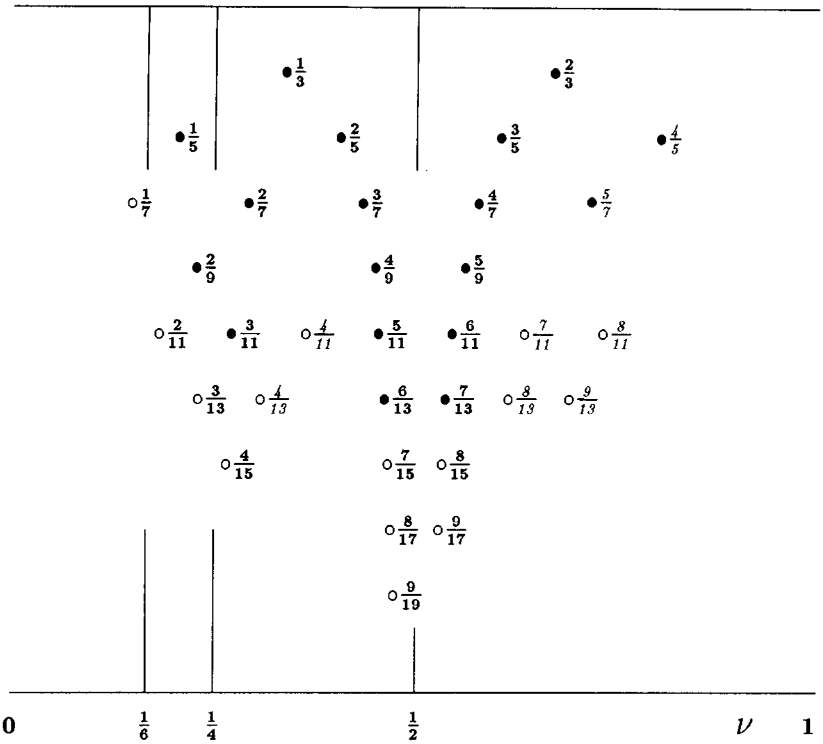

Fig. 1. Experimentally observed plateaus: their Hall conductivity $\sigma_H = (e^2/h)\nu$ is displayed in units of (e^2/h). The marks (•) denote stable (i.e. large) plateaus, which have been seen in several experiments; the marks (○) denote less developed plateaus and plateaus found in one experiment only. Plateaus belonging to the Jain main series have their filling fraction in bold typeface, the other ones are in italics. Coexisting fluids at the same filling fraction have been found at $\nu = 2/3, 2/5, 3/5, 5/7$.

longitudinal resistivity and the intersect of the plateau in the Hall resistivity with the classical curve $\rho \propto 1/B$. This is not a very accurate measure, because large, stable plateaus might hide closer, weaker ones, as for example near $\nu = 1$. Moreover, the data are now a few years old, because the recent experiments have focussed on the dynamical properties of the better understood Jain plateaus. The fractions denoted by an open dot are weak or may have been found under special conditions in one experiment only. Clearly, more than one edge theory can exist for the same filling fraction, and a phase transition among them may occur as some parameter is tuned at a fixed ratio of density to magnetic field. Transitions between different incompressible fluids have been observed at $\nu = 2/3, 2/5, 3/5, 5/7$ [46]; more might be found as the engineering of samples improves.

The points belonging to the Jain series (4.14) $\nu = m/(ms \pm 1)$, with $s = 2,4,6$ are well established and their spectra of $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ edge excitations have been confirmed in part [8,9]. These are denoted by filling fractions in bold typeface. Their annulus partition functions is given by Eqs. (4.16) and (4.20).

A very interesting problem is to identify the pattern of the remaining fractions, which are written in italics in Fig. 1. Previous approaches were based on the iteration of the Jain hierarchical wave-function construction [20], or by the classification of $\widehat{U(1)}^{\otimes m}$ edge theories associated to integral lattices of low dimension [28]. A generic feature of these approaches is that they predict too many unobserved points within the same hypotheses which describe the few observed ones. Here, we would like to show that these remaining experimental points can be consistently described by the new rational conformal field theories corresponding to the partition functions (4.29) with non-trivial extensions of the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ symmetry.

We shall also use the phenomenological hypothesis of "charge conjugation" of the incompressible fluid, which states that to any fraction $0 < \nu < 1/2$ there is a corresponding value $1/2 < (1-\nu) < 1$, which may be weaker [1]. A droplet of charged conjugate fluid can be thought as a ($\nu = 1$) fluid with a (ν) fluid of "holes" carved in it. Therefore, it has an additional, decoupled, $\nu = 1$ charged edge excitation with chirality opposite to the rest. Let us check this hypothesis. The $\nu = m/(2m-1)$ Jain plateaus have associated a second edge theory, which is the conjugate of the $\nu = m/(2m+1)$ one; as said before, many theories for the same filling fraction do not cause a problem, because the most stable (simplest) is observed in generic experimental conditions. The conjugates of the observed points of the $\nu = m/(4m \pm 1)$ series exist in part: $7/9$ does not exist but would be expected, because $2/9$ is stable; less relevant is the absence of $10/13$ and $11/15$, because $3/13$ and $4/15$ are themselves weak; the same can be said about the conjugates of the third Jain series, $1/7$ and $2/11$. The remaining filling fractions are beyond the Jain series and occur in the conjugate pairs $(4/11, 7/11)$ and $(4/13, 9/13)$ plus $8/13$, which does not fit this pattern because $5/13$ does not exist. In conclusion, the charge-conjugation hypothesis is reasonable, with problems in the missing $7/9$ and the unexplained $8/13$.

Within this analysis, there are only two conjugate pairs of points outside the Jain series, $(4/11, 7/11)$ and $(4/13, 9/13)$. Although this is only one of the possible analyses of the data, we find it rather economic and appealing. The non-diagonal partition functions (4.29), corresponding to RCFTs with non-trivial extensions of the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ symmetry, display the filling fractions (4.31): besides repetitions of the Jain points and their conjugates, the values of smaller denominator are $\nu = 4/11$ and $4/13$, which indeed match the experimental analysis. The small number of points does not allow for a definite identification of the new pattern; nevertheless, it is clear that these new RCFTs are not in manifest disagreement with the experiments. In particular, they do not predict many unobserved filling fractions, as in the existing hierarchical constructions.

Previous experience in statistical mechanics of critical phenomena has shown that diagonal and non-diagonal modular invariants yield equally good universality classes [13]. In the quantum Hall effect, further work will be necessary for understanding the $(2+1)$-dimensional incompressible fluids associated to the non-diagonal edge theories: in particular, it would be interesting to find a modification of the Jain wave-function description [27,20], which incorporates the higher symmetry and the reduced set of neutral excitations.

5. Conclusions

In this paper, we have defined and found the partition functions for the edge excitations of the quantum Hall effect in the annulus geometry. We have shown their modular invariance and we have applied the properties of rational conformal theories to this physical problem; in particular, the topological order (2.26) and (2.25) has been identified with the dimension of the representation of the modular group. In Section 4, we have shown that the $\widehat{U(1)} \otimes \widehat{SU(m)}_1$ edge theories can describe the plateaus of the Jain series as well as those beyond them, which have been characterized by an extended symmetry algebra and the non-trivial partition functions (4.29). We also found that the $W_{1+\infty}$ minimal models are not rational conformal field theories: the correct definition of their partition functions remains an open problem [43].

The partition function and the related RCFT properties are also useful for understanding the non-Abelian edge theories which are relevant for the Hall effect in two-layer samples (e.g. the Pfaffian state) and with spinful electrons (e.g. the Haldane–Rezayi state) [6,34]. Actually, the annulus partition functions for these edge theories have already been proposed in Ref. [18], although their modular invariance has not been verified explicitly. Assuming that this holds, we can read the value of the topological order (2.26) from the number of terms in the expansion of the partition function into characters: this is $(3m)$ for the Pfaffian theory at $\nu = 1/m$, $m = 2, 4, \ldots$, and $(4m)$ for the Haldane–Rezayi theory at $\nu = 1/m + 2$ (we consider here the simplest case $m = 2$). Actually, the value (6) for the Pfaffian theory matches the number of degenerate ground states found by the numerical analysis of the spectrum on the torus [35], in agreement with our analysis of Section 2. On the other hand, the value (8) for the Haldane–Rezayi theory does not match the degeneracy (10) of the trail ground-state wave function on a torus found in Ref. [17]; nevertheless, this is a special, non-unitary conformal theory which requires further studies. In conclusion, we would like to stress the importance of the partition function for charting the Hilbert spaces of these theories, which are more involved than the previous Abelian theories, as well as for exposing their symmetries.

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Appendix A. Modular forms and functions

In this appendix, we briefly review some of the properties of the modular functions of interest to us. We consider functions on a torus, which can be realized as a fundamental domain (or unit cell) of the quotient of the complex plane by a lattice Λ of translations generated by two independent periods w_1 and w_2. Any other pair of periods ω_1' and ω_2' would generate the same lattice provided the relation between this and the original pair of periods is both ways linear with integral coefficients and unit determinant. The torus is characterized by a single variable $\tau = \omega_2/\omega_1$, after rotation and dilatation invariance are considered. The above transformations act on τ as

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1. \tag{A.1}$$

Note that changing the sign of every coefficient yields the same transformation. The group of these transformations is known as the *modular group* $\Gamma \equiv PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$, which is generated by two transformations $T:\ \tau \to \tau + 1$ and $S:\ \tau \to -1/\tau$, satisfying the relations $S^2 = (ST)^3 = 1$ [49]. Moreover, by a modular transformation it is always possible to take τ to belong to the fundamental domain,

$$\mathcal{F} = \left\{ -\frac{1}{2} < \mathrm{Re}\,\tau \leqslant \frac{1}{2}, |\tau|^2 \geqslant 1\ (\mathrm{Re}\,\tau \geqslant 0), |\tau|^2 > 1\ (\mathrm{Re}\,\tau < 0) \right\}. \tag{A.2}$$

The fermionic excitations present in the quantum Hall effect only allow invariance under the S and T^2 transformations. They generate the subgroup $\Gamma_\theta \subset \Gamma$ which is isomorphic to $\Gamma^0(2) = T\Gamma_\theta T^{-1} = \{(a,b,c,d) \in \Gamma|\ b = 0 \bmod 2\}$. A smaller, normal, subgroup is $\Gamma(2) = \{(a,b,c,d) \in \Gamma|\ a = c = 1,\ b = d = 0 \bmod 2\}$, which is generated by T^2 and ST^2S.

In the study of the functions on the torus, one naturally encounters the modular forms $F(z)$, which transform under (A.1) as

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon\,(c\tau + d)^\beta\, F(\tau), \tag{A.3}$$

where ε is a phase and β is the weight of the modular form. A modular function has weight $\beta = 0$. The simplest example of a modular form is the Dedekind function,

$$\eta\,(\tau) = q^{1/24} \prod_{k=1}^{\infty} \left(1 - q^k\right) = q^{1/24} \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n(3n+1)}, \qquad q = \mathrm{e}^{i2\pi\tau}. \tag{A.4}$$

The last equality is known as Euler's pentagonal identity, and it is a consequence of Jacobi's triple product identity [4],

$$\prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-1/2} w)(1 + q^{n-1/2} w^{-1}) = \sum_{n\in\mathbb{Z}} q^{n^2/2} w^n, \tag{A.5}$$

after replacing in (A.5) $q \to q^3$ and $w \to -q^{-1/2}$. Under the two generators $T:\ \tau \to \tau + 1$ and $S:\ \tau \to -1/\tau$ of the modular group, the transformations laws of $\eta(\tau)$ are

$$T: \quad \eta(\tau+1) = e^{2i\pi/24}\eta(\tau), \tag{A.6}$$

$$S: \quad \eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau). \tag{A.7}$$

The proof of Eq. (A.6) is straightforward, and that of Eq. (A.7) follows from the application of Poisson's resummation formula,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{p\in\mathbb{Z}} \int_{-\infty}^{+\infty} dx\, f(x) e^{2i\pi px} \tag{A.8}$$

to the r.h.s. of Eq. (A.4). Under a general transformation (A.1) we therefore have

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon_A\,(c\tau+d)^{1/2}\eta(\tau), \tag{A.9}$$

where ε_A is a 24th root of unity. Thus, the Dedekind is a modular form of weight $1/2$.

Another important example of a modular form considered in Section 2 is the theta function with characteristics a and b, which is a map $\mathcal{F}\times\mathbb{C}\to\mathbb{C}$ defined by

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(\zeta|\tau) = \sum_{n\in\mathbb{Z}} \exp\left(i\pi\tau(n+a)^2 + i2\pi(n+a)(\zeta+b)\right). \tag{A.10}$$

The case of interest to Eq. (2.11) is for $a = \lambda/p$ and $b = 0$, with $\lambda = 1,2,\ldots,p$. The transformation properties of (2.11), Eqs. (2.12)-(2.15), follow easily from its definition. The only non-trivial calculation regards the S transformation, which can be done following the example of the Dedekind function. It is also easy to verify that (A.10) is a modular form of weight $1/2$. It follows that the quotient of the theta function (A.10) by the Dedekind function (A.4) is a modular function. Multi-dimensional generalizations of the results of this appendix are straightforward, and lead to the character (4.3) and transformation properties (4.4)-(4.7).

Appendix B. Minimal model non-invariant partition functions

In Section 4 we have considered the $\widehat{U(1)}\times\widehat{SU(m)}_1$ theories, and found that their characters (2.11) and (4.10) give a unitary representation of the modular transformations T^2, S, U and V, see Eqs. (4.12) and (4.17). Using these characters, we have built partition functions which are modular invariant, see Eqs. (4.16), (4.29) and (4.32). Moreover, we have shown that the characters of the degenerate $W_{1+\infty}$ representations do not carry a representation of the S transformation, unless they are summed with multiplicities greater than one, which actually build up the corresponding $\widehat{U(1)}\times\widehat{SU(m)}_1$ characters. Therefore, the $W_{1+\infty}$ minimal models, defined in Ref. [25] as a collection of $W_{1+\infty}$ representations, each counted only once, do not possess a modular invariant partition function of the standard type of rational conformal field theories (2.6). We do not presently know whether an alternative, non-rational partition function can be defined.

In this appendix, we report a simple exercise which shows that the S invariance is rather important and cannot simply be removed from the set of self-consistency building criteria. We relax the S condition and find that many partition functions of $W_{1+\infty}$ minimal models satisfy the remaining conditions (T^2, U, V), the closure of the fusion rules and the electron conditions. The minimal $W_{1+\infty}$ characters are of the form $\chi^{\widehat{U(1)}}{}_{\lambda}\,\chi^{W}{}_{\alpha}$, where the $\widehat{U(1)}$ part is standard (Eq. (2.11)) and $\chi^{W}{}_{\alpha}$ sums one copy of all the $\mathcal{W}_m$ representations with $SU(m)$ weight of given m-ality $\alpha = 1, \dots, m$ [41]. The m-ality is additive mod m, thus the fusion rules are closed for theories having all values of α, or a subset $\alpha = \delta, 2\delta, \dots, m/\delta$, where $\delta | m$. For simplicity, we consider left–right diagonal partition functions, of the type (4.16), for which the U condition is satisfied. Thus, these are determined once their chiral spectrum is found; in general, this is of the form (4.14),

$$\begin{aligned} Q &= \frac{ml+\alpha}{\hat{p}}, \qquad \nu = \frac{m}{\hat{p}}, \\ J &= \frac{(ml+\alpha)^2}{2m\hat{p}} + \frac{\alpha(m-\alpha)}{2m} + r, \quad r \in \mathbb{Z}, \end{aligned} \tag{B.1}$$

with $\hat{p}$ free. The value of ν is determined by the V condition as shown later.

Starting from Eq. (B.1), one imposes the existence of an excitation with the quantum numbers of the electron, namely $Q = 1$ and $2J$ an odd integer, which leads to

$$\begin{aligned} m\hat{l} + \hat{\alpha} &= \hat{p}, \\ \hat{l} + \hat{\alpha} - \frac{\hat{\alpha}(\hat{\alpha}-1)}{m} &= 1 \bmod 2, \end{aligned} \tag{B.2}$$

where $\hat{\alpha}$ and $\hat{l}$ are the labels of the excitation corresponding to the electron. The condition of locality (integrability of the relative statistics) between an arbitrary excitation in each sector α and the electron in the $\hat{\alpha}$ sector is (see Eq. (2.17))

$$\alpha(\hat{\alpha} - 1) = 0 \bmod m. \tag{B.3}$$

Moreover, the T^2 condition is satisfied by the solutions of (B.2) and (B.3). Note that the full spectrum of $W_{1+\infty}$ representations $\alpha = 1, \dots, m$ requires $\hat{\alpha} = 1$. As explained in Section 4.2, this is also the condition that arises upon imposing the S representation. However, there are other solutions for $\hat{\alpha} \neq 1$, which involve the subsectors $\alpha = a\delta$, with $\delta | m$. These additional solutions were not considered in our earlier discussion of the $W_{1+\infty}$ minimal models [25].

Consider first the general solution to Eq. (B.2), which can be rewritten as

$$\begin{aligned} &\hat{\alpha}(\hat{\alpha}-1) = 0 \bmod m, \\ &\hat{l} = \frac{\hat{\alpha}(\hat{\alpha}-1)}{m} - \hat{\alpha} \bmod 1, \qquad \hat{p} = m\hat{l} + \hat{\alpha}. \end{aligned} \tag{B.4}$$

The solutions of $\hat{\alpha}(\hat{\alpha} - 1) = 0 \bmod m$ can be found by considering each divisor δ of $m = \delta\rho$,

$$\hat{\alpha} = a\delta, \qquad \hat{\alpha} - 1 = b\frac{m}{\delta} \qquad \longrightarrow \qquad a\delta - b\frac{m}{\delta} = 1. \tag{B.5}$$

A solution (a,b) is found whenever δ and ρ are coprime integers. The corresponding $W_{1+\infty}$ theory contains the subset of representations $\alpha = n\delta$, $n = 1,\dots,\rho$, which are a solution to Eq. (B.3) and are, therefore, closed under the fusion rules. Their spectrum is given by (B.1) for $\hat{p} = m\hat{l} + \hat{\alpha}$, where $\hat{l} = ab - a\delta$ mod 1. Clearly, a common factor δ cancels in the formula for Q and ν; the V condition is satisfied as in Eq. (4.17).

In conclusion, there is a solution for any factorization of $m = \delta{\cdot}m/\delta$ with $(\delta, m/\delta) = 1$. The following solutions exist for any m:

(i) $\delta = 1$, with solution $a = 1$ and $b = 0$, so that $\hat{\alpha} = 1$ (this coincides with the Jain series).

(ii) $\delta = m$, with solution $a = 1$ and $b = m - 1$, so that $\hat{\alpha} = 0$ (this gives a Laughlin decoupled fluid).

The simplest solutions with non-trivial factorization of m, exist for $m = 6$ with $\delta = 3$ and $\delta = 2$. In the first case, $\hat{\alpha} = 3$ and one has a consistent theory with $SU(6)$ sectors $\alpha = 0, 3$ and filling fraction $\nu = 2/(2s+3)$, s even. In the second case, $\hat{\alpha} = 4$ and the sectors in the theory are $\alpha = 0, 2, 4$ and the filling fraction is $\nu = 3/(3s+5)$, s even. The number of $(\hat{\alpha} \neq 1)$ solutions to Eq. (B.5) grows rapidly with increasing m, and they yield almost any filling fraction $\nu = n/d$. This feature makes these S-variant partition functions inconsistent with the phenomenological pattern.

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