Large N limit in the quantum Hall effect

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The Laughlin states for N interacting electrons at the plateaus of the fractional Hall effect are studied in the thermodynamic limit of large N. It is shown that this limit leads to the semiclassical regime for these states, thereby relating their stability to their semiclassical nature. The equivalent problem of two-dimensional plasmas is solved analytically, to leading order for $N \to \infty$, by the saddle-point approximation – a two-dimensional extension of the method used in random matrix models of quantum gravity and gauge theories. To leading order, the Laughlin states describe classical droplets of fluids with uniform density and sharp boundaries, as expected from the Laughlin "plasma analogy". In this limit, the dynamical W_∞-symmetry of the quantum Hall states expresses the kinematics of the area-preserving deformations of incompressible liquid droplets.

1. Introduction

The current theory for the explanation of the plateaus in the fractional quantum Hall effect [1] is based on the seminal work of Laughlin [2]. The main idea is the existence of *incompressible quantum fluids* at specific rational values of the electron density. These are very stable, macroscopic quantum states with *uniform* density $\rho(x) = \nu eB/hc = \text{const.}$, $\nu = 1/m$, $m = 1, 3, 5, ...$, which possess an *energy gap* (here B is the external magnetic field). Incompressibility accounts for the lack of low-lying conduction modes, which causes the longitudinal conductivity σ_{xx} to vanish, while the overall rigid motion of the *uniform droplet* of fluid gives the rational values of the Hall conductivity $\sigma_{xy} = \nu e^2/h$.

In Laughlin's theory, the incompressible quantum fluid configurations of N interacting electrons are described by the wave functions

$$\psi_m(z_1, ..., z_N) = \mathscr{C} \prod_{i<j=1}^{N} (z_i - z_j)^m$$

$$\times \exp\left(-\frac{1}{2l^2} \sum_{i=1}^{N} |z_i|^2\right), \quad m = 1, 3, 5, ..., \quad (1)$$

where $l = \sqrt{2\hbar c/eB}$ is the magnetic length, $\mathscr{C} = l^{-mN(N-1)/2}$, and $\{z_i, \bar{z}_i\}$ are the coordinates of the particles in complex notation. The many-body properties of these states are described by expectation values of the form

$$\langle \mathcal{O} \rangle = \frac{1}{Z_m} \int \prod_{i=1}^{N} \frac{d^2 z_i}{\pi l^2} \, \mathcal{O}[z_i] \, |\psi_m|^2, \quad (2)$$

where

$$Z_m = \int \prod_{i=1}^{N} \frac{d^2 z_i}{\pi l^2}$$

$$\times \exp\left(-\sum_{i=1}^{N} \frac{|z_i|^2}{l^2} + m \sum_{i<j=1}^{N} \log \frac{|z_i - z_j|^2}{l^2}\right). \quad (3)$$

These are interpreted as averages in the reduced statistical problem of a *two-dimensional one-component plasma* [3], characterized by the effective tempera-

ture $T \propto 1/m$. This "*plasma analogy*" has been an important source of physical insight. For example, the property $\rho(x) = $ const. of the electron ground states has been deduced [2] from the fact that the plasma is a *liquid* at high effective temperatures, $m \ll 70$ [4]. Debye screening, translational invariance of the liquid and numerical results on correlation functions have been used to derive the ground state energy [5], and the properties of quasi-particle excitations (like their fractional statistics [6] and their energy gaps [2]).

A challenging problem is to devise a reliable analytic method for computing the plasma partition function (3). Similar expressions for *one-dimensional* plasmas are found in the theory of random matrices [7]. Actually, Laughlin's plasma (3) has been recently reformulated as a *two-dimensional* matrix model [8]. Given that matrix models can be solved with the $1/N$-expansion technique [9], we are lead to conclude that the same technique might be successfully applied to Laughlin's theory. it is the purpose of this letter to set the frame for such a large N expansion in Laughlin's theory.

We derive the leading term by the *saddle-point approximation* of the plasma partition function (3). This correctly describes the *semiclassical* incompressible fluid state of the Laughlin plasma analogy. Our main physical point is to show explicitly that, for the Laughlin states, *the thermodynamic limit $N \to \infty$ implies the semiclassical limit $\hbar \to 0$, and vice versa*. While this equivalence between limits is well known in gauge theories and matrix models, it acquires a different physical status in our problem, because N is not a tunable parameter but, rather, it is naturally forced to take large values. Therefore, the stability of the Laughlin fluid follows from its prominent semiclassical nature [#1].

In the second part of this letter, we discuss the W_∞-symmetry underlying both Laughlin's incompressible fluids and the $c=1$ matrix models. This symmetry has been explicitly shown to account for the incompressibility of the $\nu = 1$ Hall ground state [11] (similar arguments apply also to the $\nu = 1/m$ Laughlin states [11]). Here we show that, in the large N limit, the W_∞ transformations reduce to the classical

deformations of liquid droplets which preserve the area. These deformations, called *area-preserving diffeomorphisms*, satisfy the w_∞-algebra [12] [#2]. This classical droplet picture was already developed for the $c=1$ matrix models of string theory in refs. [14,15], where the relation to the Landau level problem has also been recognized. By identifying the $\hbar \to 0$ and the $N \to \infty$ limits, we show that this picture nicely fits into Laughlin's plasma analogy.

2. Saddle-point approximation

Let us start by recalling some known facts about the plasma partition function Z_m in eq. (3). For $m = 1$, it can be computed exactly by using the orthogonality of the first Landau level wave functions,

$$\varphi_k(z, \bar{z}) = \frac{1}{l\sqrt{\pi k!}} \left(\frac{z}{l}\right)^k \exp\left(-\frac{|z|^2}{2l^2}\right), \qquad (4)$$

where k is the angular momentum eigenvalue. The result is

$$\log Z_1 = \log\left(N! \prod_{n=0}^{N-1} n!\right)$$

$$= \tfrac{1}{2}N^2(\log N - \tfrac{3}{2}) + \log N! + \tfrac{1}{2}N \log 2\pi$$

$$- \tfrac{1}{12} \log N + O(1). \qquad (5)$$

We are interested in the observable one-particle density:

$$\rho_1(x) \equiv \langle \Omega_1 | \Psi^\dagger(x) \Psi(x) | \Omega_1 \rangle, \qquad (6)$$

where $|\Omega_1\rangle$ is the $\nu = 1$ ground state and Ψ the field operator in fermionic Fock space,

$$\Psi(z, \bar{z}) = \sum_{k=0}^{\infty} F_k \varphi_k(z, \bar{z}), \quad \{F_k, F_l^\dagger\} = \delta_{k,l}. \qquad (7)$$

The density is easily computed:

$$\rho_1(z, \bar{z}) = \frac{1}{l^2\pi} \exp\left(-\frac{r^2}{l^2}\right) \sum_{k=0}^{N-1} \frac{1}{k!} \left(\frac{r}{l}\right)^{2k}, \quad r \equiv |z|, \qquad (8)$$

and is plotted in fig. 1 for $N = 50$. It is constant for $r \ll l\sqrt{N}$, and drops rapidly to zero around $r \simeq l\sqrt{N}$.

[#1] A similar conclusion was reached in ref. [10] by a functional approach.

[#2] See also ref. [13] for a related discussion of classical incompressible fluids.
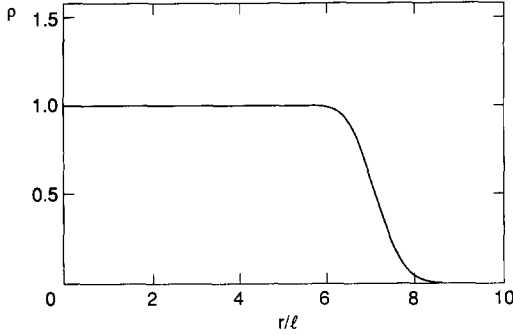
Fig. 1. The density profile in units of $1/\pi l^2$ for the first Landau level filled up to $L = 50$ as a function of r/l.

This is the density profile of a *quantum* droplet of incompressible fluid. The fluid character is reflected by the uniform value in the interior, and incompressibility follows from the gap for density fluctuations (the cyclotron energy). The quantum behaviour is apparent from the *smooth* boundary, where the occupation probability is neither zero nor one. In contrast, the density of a classical droplet of liquid has a *sharp* boundary.

Actually, *the large N limit of the quantum density* (8) *is a classical density.* Let us introduce the rescaled coordinate

$$w = \frac{z}{\sqrt{N}}, \qquad (9)$$

and the rescaled density $\rho_1(w) \equiv \rho_1(|z| = \sqrt{N}|w|)$, satisfying $\int d^2 w\, \rho_1(w) = 1$. The rescaled density possesses a finite large N limit,

$$\lim_{N \to \infty} \rho_1(|z| = \sqrt{N}|w|) = \frac{1}{\pi l^2} \Theta\left(1 - \frac{|w|^2}{l^2}\right), \qquad (10)$$

where Θ is the step function. The sharpness of the boundary is a first indication of the classical nature of the $N \to \infty$ limit. In eq. (10), l^2 has to be understood as the classical parameter setting the scale for the electron density through

$$\frac{N}{A} = \nu \frac{B}{\Phi_0} = \nu \frac{B}{hc/e} = \nu \frac{1}{\pi l^2}, \qquad (11)$$

for uniform filling ν and given area A of the sample.

Eq. (11) implies that the naive semiclassical limit $\hbar \to 0$, with all the other parameters fixed, cannot be taken in our problem. For a given external magnetic field $\boldsymbol{B}$ and type of fluid characterized by ν, the limit $\hbar \to 0$ enforces $N \to \infty$ if the system is to have a finite macroscopic area A. This is another general argument for the advertised equivalence of the $N \to \infty$ and $\hbar \to 0$ limits.

No analytic expression is known for the densities ρ_m, $m = 3, 5, ...,$ corresponding to the Laughlin states (1). Numerical studies [16,17] for a large number of particles (up to $N = 200$) show a constant density $\rho_m = 1/m\pi l^2$ for $r \ll l\sqrt{mN}$, followed by an upward bump near the boundary region, where ρ_m drops rapidly to zero. The fact that this bump does not seem to decrease rapidly when the particle number N is increased up to $N = 200$ has led to authors of ref. [17] to question the fact that Laughlin's wave function describes a uniform quantum fluid. Actually, we are now going to show that any potential non-uniformity must disappear in the thermodynamic limit. Indeed, the limiting large N form of the densities ρ_m will be shown analytically to be

$$\lim_{N \to \infty} \rho_m(|z| = \sqrt{N}|w|) = \frac{1}{m\pi l^2} \Theta\left(m - \frac{|w|^2}{l^2}\right). \qquad (12)$$

These describe again classical droplets of incompressible fluid; therefore, the bump seen in ref. [17] has to disappear for $N \to \infty$.

This result can be obtained by extending the saddle-point technique of Brezin, Itzykson, Parisi and Zuber [9] to Z_m in eq. (3) [3]. We first rewrite

$$Z_m = \int \prod_{i=1}^{N} \frac{d^2 z_i}{\pi l^2} \exp(-N H_m[w]),$$

$$H_m[w] = \sum_{i=1}^{N} \frac{|w_i|^2}{l^2} - \frac{m}{N} \sum_{i<j=1}^{N} \log \frac{|w_i - w_j|^2}{l^2}. \qquad (13)$$

For large N, the particles are driven into a saddle-point configuration $\{w_i = w_i^0\}$, determined by equation

$$\bar{w}_i = \frac{m}{N} l^2 \sum_{j, j \neq i} \frac{1}{w_i - w_j}. \qquad (14)$$

By considering a lattice decomposition of the plane, we can replace the sum over particles by the sum over cells times the characteristic function for cell occupation. For $N \to \infty$, the latter becomes a continuous

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[3] See also ref. [18] for a similar approach.

distribution for the rescaled variable w, which equals the electron density $\rho_m(w)$ to leading order. Therefore, we perform the replacement

$$\sum_i \rightarrow \int d^2z\, \rho_m(z) = N \int d^2w\, \rho_m(w) \ . \tag{15}$$

The saddle-point equation (14) becomes the integral equation [#4]

$$\bar{w} = ml^2 \int d^2w' \frac{\rho_m(w')}{w - w'} \ , \tag{16}$$

whose solutions are subjected to the normalization

$$\int d^2w\, \rho_m(w) = 1 \ . \tag{17}$$

The double integration makes the integral equation (16) more involved than the corresponding one for one-dimensional matrix models [9] – no general solution is known to us. However, it is easy to check that the rotational invariant density (12) is a solution [#5]. Furthermore, the following argument shows that this solution is stable, i.e., it is a local minimum of H_m in eq. (13). By scaling the variable $w = \sqrt{m}\, q$, one obtains the identity

$$Z_m = \exp\left[\tfrac{1}{2}N(N-1)m\log m\right](Nm)^N$$

$$\times \int \prod_i \frac{d^2q_i}{\pi l^2} \exp(-mNH_1[q]) \ , \tag{18}$$

which can be approximated semiclassically to second order, yielding

$$Z_m \sim \exp(-NH_m[w^0])N^N$$

$$\times \int \prod_i \frac{d^2\delta w_i}{\pi l^2} \exp\left(-\tfrac{1}{2}N \sum_{i,j} \delta v_i \frac{\partial^2 H_1[w^0]}{\partial v_i\, \partial v_j} \delta v_j\right), \tag{19}$$

where $v_i = (w_i, \bar{w}_i)$. This equation shows that quadratic fluctuations are *independent of m*. Moreover, for $m = 1$ the saddle-point solution is clearly quadratically stable, since it agrees with the exact solution (8), (10). Therefore, the solution (12) is stable for any m.

The value of the Hamiltonian at these saddle points

can also be easily computed in the continuum approximation and it reads

$$\log Z_m = \tfrac{1}{2}N^2m(\log Nm - \tfrac{3}{2}) + \log N! + \dots , \tag{20}$$

which also matches smoothly the exact value for $m = 1$ in eq. (5) to leading order.

The occurrence of other stable solutions of lower "energy" than (20) for $m = 3, 5, \dots$, is not excluded by our analysis, but it is very unlikely for small values of m. Numerical simulations of the two-dimensional plasma [4] indicate a phase transition to a Wigner crystal only at a large value $m_{\text{crit}} \sim 70$. It would be very interesting to find the semiclassical solution with broken rotational invariance corresponding to the Wigner crystal. Let us also remark that the one-dimensional plasma of matrix models has only one phase [7].

In conclusion, for $N \rightarrow \infty$ we confirm that the Laughlin wave functions (1) describe droplets of uniform fluid with densities $\rho_m = 1/m\pi l^2$ and sharp boundaries. The interesting structure at the boundary of the droplets found by the numerical calculations [17] might be related to *edge excitations* [19], which are subleading $O(1/N)$ boundary effects [20].

3. Observables to leading order

Some physical information on quasi-particle excitations can be obtained by evaluating observables (2) within the saddle-point approximation. More precisely, all Laughlin's results based on the plasma analogy can be rephrased in this approximation. Following the review article [6] we can, for instance, verify the normalization of the wave function for one quasi-hole at the point z,

$$\psi_{\text{QH}}(z; z_1, \dots, z_N)$$

$$= \exp\left(-\frac{|z|^2}{2ml^2}\right) \prod_{i=1}^{N} (z - z_i)\psi_m(z_1, \dots, z_N)$$

$$\equiv S[z, z_i]\psi_m(z_1, \dots, z_N) \ , \tag{21}$$

with ψ_m the Laughlin wave function (1). Let us compute $\|\psi_{\text{QH}}\|^2$, i.e. eq. (2) for the operator $\mathcal{O} = S^\dagger S$. Repeating the previous steps for the modified plasma with one added charge, we find that the saddle-point equation (16) is not modified to leading order, and that the saddle-point value of this observable is in-

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[#4] The point $w = w'$ excluded in the sum (14) causes no harm in the following integral for continuous ρ functions.

[#5] To see this, it is easier to perform first the angular integration using the theorem of residues.

deed unity. Actually, the saddle-point solution (12) is not modified by the inclusion of any finite number of charges.

The wave function of two quasi-holes [6] can be treated similarly. One finds that the correlation among them, showing their fractional statistics, is a subleading effect. This agrees with the result of the theory of edge excitations [19], in which it has been shown that quasi-particle correlations are of order $O(1/N)$ [20]. As expected, quasi-particles, which are quantum effects, are not seen to leading large N order.

4. Droplet picture and w_∞-symmetry

So far, we have been considering the large N limit of the density from a computational point of view. Now, we would like to discuss the *geometrical* interpretation of this limit.

Before doing this, let us recall the W_∞ dynamical *quantum* symmetry of the $\nu = 1$ ground state recently found in ref. [11]. There, we constructed operators $\mathscr{L}_{n,m}$, living at the first Landau level,

$$\mathscr{L}_{n,m} \equiv \sum_{i=1}^{N} (b_i^\dagger)^{n+1} b_i^{m+1}, \quad n, m \geqslant -1, \quad (22)$$

where b_i, $b_i^\dagger$ are the harmonic oscillators for angular momentum J excitations,

$$b_i = \frac{\bar{z}_i}{2l} + l\partial_i, \quad b_i^\dagger = \frac{z_i}{2l} - l\bar{\partial}_i, \quad [b_i, b_j^\dagger] = \delta_{ij},$$

$$J = \sum_i b_i^\dagger b_i. \quad (23)$$

The $\mathscr{L}_{n,m}$ satisfy the W_∞ algebra [#6]

$$[\mathscr{L}_{n,m}, \mathscr{L}_{k,l}]$$
$$= \left(\sum_{s=0}^{\text{Min}(m,k)} \frac{(m+1)!(k+1)!}{(m-s)!(k-s)!(s+1)!} \mathscr{L}_{n+k-s,m+l-s} \right)$$
$$- (m \leftrightarrow l, n \leftrightarrow k)$$
$$= [(m+1)(k+1) - (n+1)(l+1)]\mathscr{L}_{n+k,m+l}$$
$$+ (...)\mathscr{L}_{n+k-1,m+l-1} + \cdots. \quad (24)$$

In particular, the angular momentum is $J = \mathscr{L}_{00}$. The subalgebra

[#6] More precisely, this is a $W_{1+\infty}$ algebra.

$$[\mathscr{L}_{00}, \mathscr{L}_{n,m}] = (n-m)\mathscr{L}_{n,m} \quad (25)$$

shows that the $\mathscr{L}_{n,m}$ are raising ($n > m$) and lowering ($n < m$) operators for angular momentum.

The quantum symmetry of the $\nu = 1$ ground state is encoded in the invariance of the ground state under an infinite (for $N \to \infty$) set of these transformations, i.e., the highest-weight conditions [11]

$$\mathscr{L}_{n,m}\psi_1(z_1, ..., z_N) = 0, \quad n < m. \quad (26)$$

These are interpreted as the algebraic conditions of incompressibility, since eq. (26) means that all transitions lowering the angular momentum of the ground state, i.e., *compressions*, are impossible.

Next, we discuss the corresponding classical picture. As shown before, when $N \to \infty$, the quantum density of the Laughlin states at $\nu = 1/m$ reduces to the profile of a classical droplet of liquid with uniform density $\rho_m = 1/m\pi l^2$. Consider now a deformation of this droplet. The density cannot change locally, due to incompressibility. Moreover, its space integral gives the particle number N, and is, therefore, constant. Thus, the *area* occupied by the droplet stays constant, i.e., deformations can only produce droplets of the same area and different shapes. Therefore, the different configurations of the incompressible fluid are related by *area-preserving diffeomorphisms*, whose generators satisfy the w_∞ algebra [12]. In the following, we give an explicit derivation of the action of W_∞ on the observable density and its classical limit w_∞, thereby confirming this picture.

To this end we use a different basis for the generators of W_∞. By using this new basis, we will stress the analogy to the parallel discussion of the $\hbar \to 0$ limit of the $c = 1$ matrix model in $(1 + 1)$ dimensions, as formulated in refs. [14,15], which possesses the same W_∞-symmetry. The basic reason for this analogy is that the Hilbert spaces of the two systems are isomorphic. In the matrix model, one considers the Hamiltonian $H = \frac{1}{2}(p^2 - x^2)$ yielding a real representation of the harmonic Fock space. In the first Landau level, we have the holomorphic representation (23), where angular momentum plays the role of the Hamiltonian. As is well known, in Bargmann space the two coordinates $(z, \bar{z})$ become conjugate variables of a $(1 + 1)$-dimensional phase space [21].

In analogy with ref. [14], we consider the Wigner phase-space distribution:

$$W(k, \bar{k}) \equiv \int d^2z \, \Psi^\dagger(z) \exp[\tfrac{1}{2}il(\bar{k}b^\dagger + kb)] \Psi(z)$$

$$= \exp(\tfrac{1}{8}l^2 k\bar{k})$$

$$\times \int d^2z \, \Psi^\dagger(z) \exp[\tfrac{1}{2}i(\bar{k}z + k\bar{z})] \Psi(z). \qquad (27)$$

This is the generating function for the operators [#7] $\mathcal{L}_{n,m}$ in eq. (22),

$$W(k, \bar{k}) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} (\tfrac{1}{2}il\bar{k})^n (\tfrac{1}{2}ilk)^m : \mathcal{L}_{n-1,m-1}: \, , \qquad (28)$$

where :: denotes Weyl normal ordering [14], e.g., $:(b^\dagger)^2 b: \equiv \tfrac{1}{3}[(b^\dagger)^2 b + b^\dagger b b^\dagger + b(b^\dagger)^2]$. This shows that the generating functions $W(k, \bar{k})$ form a different basis for the W_∞ generators. In terms of these new generators, the W_∞ algebra (24) acquires the compact form

$$[W(k, \bar{k}), W(p, \bar{p})]$$

$$= 2 \sinh(\tfrac{1}{8}l^2(p\bar{k} - \bar{p}k)) W(p+k, \bar{p}+\bar{k}). \qquad (29)$$

This result is analogous to the $(1+1)$-dimensional result of ref. [14], owing to the previously noticed mapping between Hilbert spaces.

The Fourier transform of $W(k, \bar{k})$ is the one-particle density *operator* of eq. (6),

$$\rho(z, \bar{z}) = \int \frac{d^2k}{(2\pi)^2}$$

$$\times \exp[-\tfrac{1}{2}i(k\bar{z} + \bar{k}z)] W(k, \bar{k}) \exp(-\tfrac{1}{8}l^2 k\bar{k})$$

$$= \Psi^\dagger(z) \Psi(z), \qquad (30)$$

apart from a normal-ordering factor. We recall that, originally [22], Wigner distributions were introduced as the quantum analogues of phase-space distributions of classical statistical mechanics. This is because quantum expectation values can be written as classical averages in phase space with these distributions. In our Landau level problem, the phase space is given by $(z, \bar{z})$; thus a phase-space distribution is actually a two-dimensional space distribution, as in (30).

[#7] In eq. (27), Ψ is the field operator (7); the $\mathcal{L}_{n,m}$ appearing hereafter are, therefore, expressed in the second quantized formulation [11].

The action of a W_∞ transformation generated by $W(k, \bar{k})$, with infinitesimal parameter $\epsilon(k, \bar{k})$, on the ground-state density (6) is given by

$$\delta_\epsilon \rho_1(z, \bar{z}) = \int d^2k \, \epsilon(k, \bar{k}) \delta_{k,\bar{k}} \rho_1(z, \bar{z}), \qquad (31)$$

where

$$\delta_{k,\bar{k}} \rho_1(z, \bar{z}) \equiv i\langle \Omega_1 | [\rho_1, W(k, \bar{k})] | \Omega_1 \rangle$$

$$= i \exp(\tfrac{1}{8}l^2 k\bar{k}) \left[\exp\left(\tfrac{1}{2}il^2 k \frac{\partial}{\partial z} \right) - \exp\left(\tfrac{1}{2}il^2 \bar{k} \frac{\partial}{\partial \bar{z}} \right) \right]$$

$$\times \exp[\tfrac{1}{2}i(k\bar{z} + \bar{k}z)] \rho_1(z, \bar{z}). \qquad (32)$$

This expression becomes more transparent in the large N limit, which is achieved, as before, by introducing the rescaled coordinate $w = z/\sqrt{N}$, and correspondingly, the rescaled momentum $\kappa = k\sqrt{N}$. After rescaling, eq. (32) reads

$$\delta_{\kappa,\bar{\kappa}} \rho_1(w, \bar{w}) = i\langle \Omega_1 | \left[\rho_1, W\left(\frac{\kappa}{\sqrt{N}}, \frac{\bar{\kappa}}{\sqrt{N}} \right) \right] | \Omega_1 \rangle$$

$$= i \exp\left(\frac{l^2}{N} \frac{\kappa\bar{\kappa}}{8} \right) \left[\exp\left(\frac{il^2}{2N} \kappa \frac{\partial}{\partial w} \right) - \exp\left(\frac{il^2}{2N} \bar{\kappa} \frac{\partial}{\partial \bar{w}} \right) \right]$$

$$\times \exp[\tfrac{1}{2}i(\kappa\bar{w} + \bar{\kappa}w)] \rho_1(w, \bar{w}), \qquad (33)$$

where we recall that $l^2/N = 2\hbar c/eBN$. In the large N limit, eq. (33) reduces to

$$\delta^{(cl)}_{\kappa,\bar{\kappa}} \rho_1(w, \bar{w}) = \{\rho_1(w, \bar{w}), -\exp[\tfrac{1}{2}i(\kappa\bar{w} + \bar{\kappa}w)]\}_{PB}, \qquad (34)$$

where $\delta^{(cl)}_{\kappa,\bar{\kappa}} \equiv (N/\hbar)\delta_{\kappa,\bar{\kappa}}$, and $\{f, g\}_{PB} \equiv -(i/\mathscr{B}) \times [(\partial f/\partial w)(\partial g/\partial \bar{w}) - (\partial f/\partial \bar{w})(\partial g/\partial w)]$, with $\mathscr{B} \equiv eB/2c$, denotes the correct Poisson bracket with respect to the classical variables w and $\bar{w}$. This equation shows that, to leading large N order, $W(\kappa/\sqrt{N}, \bar{\kappa}/\sqrt{N})$ is the generating function of canonical, and therefore area-preserving, transformations in the two-dimensional phase space $(w, \bar{w})$. Eq. (34) makes it manifest that a W_∞-transformation acting on the density, in the *large N limit*, is of a *classical nature*. As we now show, in this limit the quantum W_∞ algebra reduces to the classical algebra of area-preserving diffeomorphisms w_∞. Our result matches the $\hbar \to 0$ limit of refs. [14,15], thereby further confirming that $N \to \infty$ is a semiclassical limit.

We now verify the w_∞ algebra for the classical analogous of the operators $W(k, \bar{k})$ and $\mathcal{L}_{n,m}$ acting by

Poisson brackets on functions of the holomorphic phase space. To this end, we perform the coordinate and momentum rescalings as before, and we use the correspondence between operators and classical functions, $NW(\kappa/\sqrt{N}, \bar{\kappa}/\sqrt{N}) \to W^{(cl)}(\kappa, \bar{\kappa})$ and $[,] \to i\hbar\{ , \}_{PB}$. The classical limit of eq. (29) reads

$$\{W^{(cl)}(\kappa, \bar{\kappa}), W^{(cl)}(\lambda, \bar{\lambda})\}_{PB}$$
$$= -\frac{i}{4\mathscr{B}}(\lambda\bar{\kappa} - \bar{\lambda}\kappa)W^{(cl)}(\kappa+\lambda, \bar{\kappa}+\bar{\lambda}) , \qquad (35)$$

which verifies the identification $W^{(cl)}(\kappa, \bar{\kappa}) = -\exp[\frac{1}{2}i(\kappa\bar{w}+\bar{\kappa}w)]$ from eq. (34). This, in turn, identifies the classical limit of the operators $\mathscr{L}_{n,m}$ through eq. (28),

$$l^2\left(\frac{l}{\sqrt{N}}\right)^{n+m}\mathscr{L}_{n,m} \to \mathscr{L}^{(cl)}_{n,m} = -w^{n+1}\bar{w}^{m+1} . \qquad (36)$$

Their classical algebra is

$$\{\mathscr{L}^{(cl)}_{n,m}, \mathscr{L}^{(cl)}_{k,l}\}_{PB} = -\frac{i}{\mathscr{B}}$$
$$\times [(m+1)(k+1) - (n+1)(l+1)]\mathscr{L}^{(cl)}_{n+k,m+l}, \qquad (37)$$

which agrees with the classical limit of eq. (24). Eqs. (35) and (37) are equivalent forms for the w_∞ algebra of area-preserving diffeomorphisms. Note that the classical functions $\mathscr{L}^{(cl)}_{n,m}$ and $W^{(cl)}$ and their algebras (35) and (37) can also be derived directly from the canonical treatment of the classical theory describing the dynamics of the first Landau level, the "topological quantum mechanics" of ref. [23] (see also ref. [11]).

Having identified the classical $\mathscr{L}^{(cl)}_{n,m}$, we can evaluate their action on the classical density (10) by using eq. (34):

$$\delta^{(cl)}_{n,m}\rho_1(w, \bar{w}) = \{\rho_1(w, \bar{w}), w^{n+1}\bar{w}^{m+1}\}_{PB}$$
$$= i(n-m)\frac{l^{n+m-2}}{\pi\mathscr{B}}\exp[i(n-m)\theta]\delta\left(1 - \frac{|w|^2}{l^2}\right), \qquad (38)$$

where θ is the angular variable on the circle delimiting the classical droplet. These variations correspond to *density waves localized on the one-dimensional sharp boundary of the classical droplet*. The quantization of these edge waves leads to a $(1+1)$-dimensional $c = 1$ conformal field theory [19,20].

So far, our explicit derivation of the classical w_∞ algebra referred only to the $\nu = 1$ ground state; however, it can be easily extended to the Laughlin states at $\nu = 1/m$. Indeed, their classical densities ρ_m are related to ρ_1 by the simple scaling $\rho_m(w) = (1/m)\rho_1(w/\sqrt{m})$. Therefore, we can apply the same scaling in the classical phase space to obtain the corresponding w_∞ action (34)–(37) for $\nu = 1/m$. We can also argue that the dynamical quantum symmetry of the Laughlin states must be the quantum extension W_∞. Actually, the analogues of the symmetry conditions (26) were found in ref. [11]. However, in going from w_∞ to W_∞, one needs the explicit form of the higher order corrections $O(1/N^k)$, $k > 0$ in the algebra W_∞, as in eq. (24). These are not unique and their correct form is not known to us at present for $\nu = 1/m$.

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