Thermal transport in chiral conformal theories and hierarchical quantum Hall states

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Abstract

Chiral conformal field theories are characterized by a ground-state current at finite temperature, that could be observed, e.g., in the edge excitations of the quantum Hall effect. We show that the corresponding thermal conductance is directly proportional to the gravitational anomaly of the conformal theory, upon extending the well-known relation between specific heat and conformal anomaly. The thermal current could signal the elusive neutral edge modes that are expected in the hierarchical Hall states. We then compute the thermal conductance for the Abelian multi-component theory and the $W_{1+\infty}$ minimal model, two conformal theories that are good candidates for describing the hierarchical states. Their conductances agree to leading order but differ in the first, universal finite-size correction, that could be used as a selective experimental signature. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

The Jain hierarchical states [1] are an interesting class of plateaus in the fractional quantum Hall effect [2] that still requires further theoretical and experimental investigation. A characteristic new feature with respect to the Laughlin's plateaus, is the presence of edge excitations with neutral propagating modes that cannot be directly measured in

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the conduction experiments. In this paper, we elaborate the proposal of Ref. [3] of testing these neutral excitations by means of their thermal current.

We first use the conformal field theory (CFT) description of the edge states to obtain a general relation between the thermal conductance and the gravitational anomaly [6] of the chiral conformal theory. We start by recalling the well-known CFT result expressing the specific heat in terms of the conformal anomaly c [4]:

$$c_V = \frac{\partial \langle \mathcal{E} \rangle_T}{\partial T} = \frac{\pi k_B^2 T}{3v} c, \qquad T \to 0. \tag{1.1}$$

The proof of this formula goes as follows: the thermal field theory is defined on the cylinder geometry made by the periodic Euclidean time and the unbounded space; in this geometry, the Casimir effect implies a non-vanishing ground-state energy $\langle \mathcal{E} \rangle_T$, that is given by the expectation value of the stress tensor and is proportional to the conformal anomaly c [5].

The edge excitations of the quantum Hall effect are described by more general CFTs with unbalanced chiral and antichiral modes that propagate in opposite directions and are characterized by different central charges c and $\bar{c}$, respectively. In these chiral theories, the expectation values of the energy $\langle \mathcal{E} \rangle_T$ and the momentum $\langle \mathcal{P} \rangle_T$ are tied together, such that the Casimir effect is associated to a thermal current $J_Q \propto \langle \mathcal{P} \rangle_T$. The corresponding thermal Hall conductance K can be found in full generality by adapting the derivation of (1.1) (see Section 2); it reads:

$$K \equiv \frac{\partial J_Q}{\partial T} = \frac{\pi k_B^2 T}{6} (c - \bar{c}). \tag{1.2}$$

In this equation, the difference of central charges parametrizes the two-dimensional (pure) gravitational anomaly [6] of the chiral conformal theory describing the single edge (e.g., $c = 1$, $\bar{c} = 0$ for the edges of the Laughlin plateaus). Eq. (1.2) generalizes the results of Ref. [3].

The formula (1.2) for the thermal conductance allows for the direct determination of the central charges of the edge states; clearly, the measurement of K requires a high-precision experiment that has not yet been realized, to our knowledge [7]; nevertheless, it should be feasible in principle [3]. On the other hand, the specific heat of edge states (proportional to $(c + \bar{c})/2$, in general) cannot be measured because it is masked by the overwhelming contribution of the lattice phonons [2].

In a finite sample, of typical linear extension R, there are finite-size corrections to the leading ($R \to \infty$) results (1.2) and (1.1). The exact thermal averages are then obtained by differentiation of the partition function; the latter can be best computed on the geometry of the annulus, that involves an inner and an outer edge. Actually, the annulus partition functions enjoy the property of invariance (or, more generally, of covariance) under modular transformations of the periodic time and angular coordinates [8]. Once the expression of the partition function is known, one can determine the finite-size corrections to (1.2): in particular the first, universal, term $O(1/R)$ is of some interest as described hereafter.

The edge excitations of the hierarchical Hall states have been mainly described by two classes of conformal field theories: the Abelian theories [9], and the $W_{1+\infty}$ minimal models [10]. Both theories are suitable generalizations of the simple scalar field theory that

describes Laughlin's plateaus (chiral Luttinger liquid) [11]. The first class considers several copies of the scalar theory, whereas the second class exploits the physical picture of the droplet of incompressible fluid [12], that is characterized by the $W_{1+\infty}$ symmetry [13,14]. Remarkably, these two classes of theories possess the same spectra of charged excitations, but differ in the neutral sector.

Therefore, the thermal conductances are expected to be different in the two theories and could provide a significant experimental test. Actually, the conductances are equal to leading order (1.2), because the two theories share the same central charges. From the known expressions of their partition functions [8], we nevertheless obtain different first-order $O(1/R)$ corrections: they are vanishing in the Abelian theories, and non-vanishing in the $W_{1+\infty}$ minimal models. The derivation makes it apparent that power-law $O(1/R^k)$ corrections are absent for all rational CFT, namely the class of CFTs with modular invariant partition functions.

The outline of the paper is the following. In Section 2, we describe the thermal transport in CFT and derive the general formula (1.2); we then digress on the relation between the anomalies, chiral and gravitational, and the corresponding out-of-equilibrium processes of the Hall and thermal currents, respectively. These are nice examples for the general picture of non-equilibrium dynamics envisaged in Ref. [15]. In Section 3, we introduce the annulus partition function and obtain the finite-size corrections to (1.2), using the well-known $c = 1$ CFT of the Laughlin plateaus as an example; we then extend the finite-size analysis to the multi-component Abelian CFT of the hierarchical states. In Section 4, we analyze the $W_{1+\infty}$ minimal model and find the $O(1/R)$ correction stemming from the lack of modular invariance of its partition function. In the conclusions, we estimate the experimental precision that is necessary for measuring these finite-size differences.

2. Thermal conductance in conformal field theory

2.1. Earlier results

We start by recalling the results of Ref. [3] for the thermal transport of the edge excitations that are described by m independent $(1 + 1)$-dimensional scalar fields. The thermal current J_Q was defined as:

$$J_Q = \sum_{i=1}^{m} \eta_i v_i \varepsilon_i, \tag{2.1}$$

where v_i and ε_i are, respectively, the velocity and the average energy density of the ith mode, and the chiralities $\eta_i = \pm 1$ indicate the sense of propagation. The thermal conductance was given by:

$$K = \frac{\partial J_Q}{\partial T} = \sum_{i=1}^{m} \eta_i v_i \frac{\partial \varepsilon_i}{\partial T}. \tag{2.2}$$

In Ref. [3], the energy densities ε_i were obtained by assuming linear dispersion relations[2] $E_i(k) = v_i k$, $k > 0$, being the momentum of chiral excitations, and by thermal averaging with independent Bose distribution functions at temperature T. The result was, in the thermodynamic limit,

$$\varepsilon_i = \frac{1}{v_i} \frac{\pi k_B^2 T^2}{12}, \tag{2.3}$$

and led to the thermal conductance:

$$K = \frac{\pi k_B^2 T}{6} \sum_{i=1}^{m} \eta_i. \tag{2.4}$$

One interesting remark of Ref. [3] was that this result only depends on the (topological invariant) sum of chiralities, that can vanish or even be negative if the edge contains a number of neutral modes propagating oppositely to the electric current. Actually, the multi-component scalar theory [9] for the hierarchical plateaus at filling fraction $\nu = m/(pm + 1)$ (respectively, $\nu = m/(pm - 1)$), with $m = 2, 3, \ldots$ and $p = 2, 4, \ldots$, predicts one charged mode and $(m - 1)$ neutral ones, whose chiralities are all equal (respectively, all opposite) to that of the charged mode. For example, the thermal conductance was predicted to vanish for $\nu = 2/3$ and be negative for $3/5$.

2.2. General formula

In the following, we would like to generalize the previous results using the CFT methods [5]. In Euclidean coordinates $z = v\tau + ix$, the local energy $\mathcal{E}(x, \tau)$ and momentum $\mathcal{P}(x, \tau)$ densities, and their continuity equation, are expressed in terms of the components of the stress tensor $\mathcal{T}(z)$ and $\overline{\mathcal{T}}(\bar{z})$ as follows:

$$\frac{\partial}{\partial \bar{z}} \mathcal{T} = \frac{\partial}{\partial z} \overline{\mathcal{T}} = 0, \qquad \mathcal{E} = \frac{v}{2\pi}(\mathcal{T} + \overline{\mathcal{T}}), \qquad \mathcal{P} = \frac{v^2}{2\pi}(\mathcal{T} - \overline{\mathcal{T}}). \tag{2.5}$$

The (steady) thermal current is defined as the ground-state value of the *momentum density* in the thermal field theory:

$$J_Q \equiv \langle \mathcal{P} \rangle_T = \frac{v^2}{2\pi} \langle \mathcal{T} - \overline{\mathcal{T}} \rangle_T; \tag{2.6}$$

thus, it is proportional to the difference of the two chiral components of the stress tensor and is non-vanishing for chiral conformal theories.[3]

In the quantum Hall effect, this analysis applies to the single edge, that is typically described by a chiral CFT (e.g., $\overline{\mathcal{T}} = 0$ for the Laughlin states); nevertheless, any Hall device, like the bar or the annulus (Fig. 1), always involves two edges with conjugate chiralities; therefore, the total thermal current vanishes in thermal equilibrium. Let us focus on the annulus geometry, characterized by the space-independent currents $J_Q^{(1)}(T)$

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[2] Hereafter, we set $\hbar = c = 1$.
[3] Persistent charge currents along the edge have been discussed in Ref. [16].
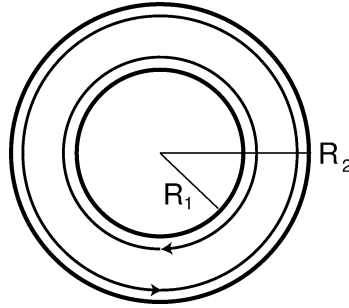
Fig. 1. Annulus geometry with inner and outer radii R_1 and R_2, respectively. The arrows indicate the sense of propagation of the edge excitations.

and $J_Q^{(2)}(T)$, with indices numbering the edges, that satisfy $J_Q^{(1)}(T) + J_Q^{(2)}(T) = 0$. The non-equilibrium current can be described as follows: we homogeneously heat up one of the edges, say the inner one (1), and find, within the linear response:

$$\Delta J_Q = J_Q^{(1)}(T + \Delta T) + J_Q^{(2)}(T) \sim \frac{\partial J_Q^{(1)}}{\partial T} \Delta T. \tag{2.7}$$

Therefore, the thermal Hall conductance is defined by

$$K \equiv \frac{\partial J_Q^{(1)}}{\partial T} = \frac{v^2}{2\pi} \frac{\partial}{\partial T} \langle \mathcal{T} - \overline{\mathcal{T}} \rangle_T^{(1)}, \tag{2.8}$$

and it involves the stress-tensor components of the single-edge CFT. This formula is consistent with the earlier definition (2.2), but it is also an exact result that can be used for computing the finite-size corrections. In the annulus geometry, the thermal current is longitudinal, i.e., orthogonal to the temperature gradient, due to the presence of the external magnetic field (this thermal analog of the Hall conduction is called the Leduc–Righi effect in magneto-hydrodynamics [17]).

We now proceed to compute the thermal conductance (2.8) to the leading ($R \to \infty$) order; disregarding the space periodicity, the thermal field theory is defined on the cylinder made by the unbounded space and the periodic Euclidean time, with period $\beta = 1/k_B T$. The thermal average of the stress tensor is obtained by mapping the cylinder to the two-dimensional Euclidean plane with the help of the conformal transformation:

$$z(w) = \exp\left[\frac{i2\pi(v\tau + ix)}{v\beta}\right] \quad \longleftrightarrow \quad w = v\tau + ix. \tag{2.9}$$

The transformation law of the stress tensor involves an anomalous term given by the Schwarzian derivative $\{z, w\}$ times the Virasoro central charge [5]:

$$\begin{aligned}
\langle \mathcal{T}(w) \rangle_{\text{cyl}} &= \langle \mathcal{T}(z) \rangle_{\text{plane}} \left(\frac{dz}{dw}\right)^2 + \frac{c}{12}\{z, w\} \\
&= 0 + \frac{c}{12}\left[\frac{z'''(w)}{z'(w)} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2\right] = \frac{\pi^2 c}{6v^2\beta^2}.
\end{aligned} \tag{2.10}$$

Using Eq. (2.6), the thermal current is found to be a translation-invariant constant, that is equal to the space average of the momentum density, i.e., the zero mode on the cylinder $\left(L_{-1} - \overline{L}_{-1}\right)_{\text{cyl}}$:

$$
J_Q = \frac{v^2}{M}(L_{-1} - \overline{L}_{-1})_{\text{cyl}} = \frac{v^2}{M} \int_{-iM/2}^{iM/2} \frac{dw}{2\pi i} \left(\mathcal{T}(w) - \overline{\mathcal{T}(w)}\right) = \frac{\pi}{12} k_B^2 T^2 (c - \bar{c}).
$$
(2.11)

In this equation, we temporary set a cutoff M in the unbounded spatial direction along the cylinder axis. Upon differentiation w.r.t. the temperature, we obtain the general formula for the thermal Hall conductance that was anticipated in the introduction (1.2):

$$
K = \frac{\pi k_B^2 T}{6}(c - \bar{c}).
$$
(2.12)

Some comments are in order:

(i) The Virasoro central charges $(c, \bar{c})$ parametrize both the conformal and gravitational anomalies, which can be traded into one another by changing the renormalization scheme; if $c = \bar{c}$, the gravitational anomaly can be set to zero, which is the usual case for statistical mechanics models; if $c \neq \bar{c}$, the minimal gravitational anomaly is proportional to $(c - \bar{c})$. The CFT of one Hall edge is necessarily chiral and anomalous.

(ii) the corresponding result for the specific heat is:

$$
c_V = \frac{\partial \langle \mathcal{E} \rangle_T}{\partial T} = \frac{\pi k_B^2 T}{6v}(c + \bar{c}),
$$
(2.13)

in agreement with [4]. Although this is a well-known formula that has been widely used in statistical mechanics, the corresponding thermal conductivity was not discussed before, since chiral CFTs are rather unusual in this domain.

(iii) The general expression reproduces the earlier result (2.4) for the multi-component scalar field theory [3], since each field contributes to c or $\bar{c}$ by one unit. Note also that the dependence on the velocity cancels out in the final result of the thermal conductance, that is fully universal; in case of independent velocities for chiral and antichiral modes, v and $\bar{v}$, respectively, one should write an independent map to the cylinder for each chiral part, involving w as above and $\bar{w} = \bar{v}\tau + ix$.

2.3. Anomalies and non-equilibrium processes

We have seen that the Hall and thermal currents have some striking similarities: both are orthogonal to the magnetic field and the applied force, the in-plane electric field or the temperature gradient, respectively. Both currents correspond to an out-of-equilibrium steady motion that is dissipationless due the orthogonality to the force. Another fact is that these currents are associated to the two anomalies of the chiral CFT: the chiral and gravitational ones, respectively. It is interesting to discuss the physical mechanisms underlying the two flows, that are tractable cases of non-equilibrium dynamics.

The relation of the Hall current to the chiral anomaly of the edge CFT is well understood [18]: the mechanism is that of the "spectral flow", in which the leak of electric charge Q out of (and orthogonal to) the edge is caused by the electrons that are pulled out of the Dirac sea by the applied tangential electric field E. Indeed, the chiral anomaly equation:

$$\partial_{\bar{z}} J = \frac{\kappa e^2}{2\pi} F, \qquad F_{ij} = \varepsilon_{ij} F = \partial_i A_j - \partial_j A_i, \quad i = \hat{t}, \hat{x}, \tag{2.14}$$

with x the tangential coordinate, can be integrated over one edge of the annulus,

$$\frac{\partial Q^{(1)}}{\partial t} = \frac{\kappa e^2}{2\pi} \int dx \, E_{\hat{x}}, \tag{2.15}$$

and can be recognized as the radial Hall current $J_{\hat{r}} = \sigma_H E_{\hat{x}}$ in $2 + 1$ dimensions; the Hall conductivity, $\sigma_H = e^2 \nu / 2\pi$, is parametrized by $\nu = \kappa$, a combination of the charge unit and the coupling constant of the CFT. This is clearly a steady out-of-equilibrium process, where the charge is not conserved, but the flux of the corresponding current does. Such states have been called "flux states" by Polyakov [15], which has stressed the relevance of anomalies in quantum field theory for modeling non-equilibrium processes. The quantum Hall effect provides two neat examples of this picture, that are not hampered by the difficulties of describing the dissipation.

The gravitational anomaly amounts to the non-conservation of the stress tensor in a gravitational background:

$$\nabla^z T_{zz} = -\frac{c}{24} \nabla_z \mathcal{R}, \tag{2.16}$$

where ∇_z is the covariant derivative and $\mathcal{R}$ the scalar curvature of the background metric. Upon integration of this equation, one obtains the anomalous transformation of the stress-tensor involving the Schwarzian derivative (2.10) [5], that yields the Casimir effect and the thermal current. In less technical terms, the mechanism can be explained as follows: on the cylinder geometry of the thermal field theory, the constant, steady thermal flow implies that energy is conserved locally but not globally; there exist a source and a drain at space infinity, $x = \pm\infty$, where the curvature is non-vanishing and the conformal mapping (2.9) is singular, in agreement with the anomaly equation (2.16). On the annulus, each edge is closed, the points at space infinity are identified and there is no non-conservation: the total thermal current is non-vanishing when the two edges have different temperatures (different backgrounds).

In the case of turbulence [15], the energy flux is constant in momentum, rather than in space, and the singular points at infinity correspond to the infrared (respectively, ultraviolet) limits, where energy is injected (respectively, dissipated). It is possible that chiral conformal theories defined on other gravitational backgrounds may yield further models of out-of-equilibrium processes.

3. Annulus partition function and finite-size corrections

3.1. The Laughlin states

Now we would like to evaluate the finite-size corrections to the infinite-volume result (1.2); we shall obtain them by differentiation of the partition function, that completely accounts for the properties of the spectrum including the finite-size effects. As an example, we shall discuss the simplest CFT with central charge $c = 1$, that describes the Laughlin states.

We consider again the annulus geometry (Fig. 1), whose partition function can be computed using the data of the representation theory of the Virasoro algebra; moreover, this partition function obeys the powerful constraints of modular invariance (covariance) [8]. In the course of the argument, it will become apparent that the first finite-size correction is universal and equal to that of the bar geometry.

The partition function is defined by the trace over the Hilbert space of the Boltzmann weight, that can be expressed in terms of the Virasoro generators [5]: one employs the map from the z-plane to the u-cylinder with space period $2\pi R$:

$$z = \exp\left(\frac{u}{R}\right) \quad \longleftrightarrow \quad u = v\tau + ix. \tag{3.1}$$

Proceeding as in the previous section, we find that the total energy–momentum on the cylinder is transformed as follows [5]:

$$(L_{-1})_{\text{cyl}} = \frac{1}{R}\left[(L_0)_{\text{plane}} - \frac{c}{24}\right], \tag{3.2}$$

and similarly for the other chirality. The eigenvalues of the $L_0, \overline{L}_0$ operators in the plane are determined by the representation theory of the relevant chiral algebras.

The annulus partition function, defined, e.g., in Ref. [8], describes the full physical system involving both the inner and outer edges, labeled by the indices $\kappa = 1, 2$, and having conjugate chiralities. The Hamiltonian on one edge, say (1), is: $H_1 = (v_1/R_1)(L_0^{(1)} - c_1/24) + (\bar{v}_1/R_1)(\overline{L}_0^{(1)} - \bar{c}_1/24)$; on the other edge, it is the conjugate expression ($c_2 = \bar{c}_1$, $\bar{c}_2 = c_1$) parametrized by $\{v_2, \bar{v}_2, R_2\}$. It is natural to assume the equilibration between the edges, i.e., $v_1/R_1 = v_2/R_2$. The full theory is characterized by the total Virasoro operators, $L_0 = L_0^{(1)} + L_0^{(2)}$ and its conjugate, and by the total central charge $c = c_1 + c_2 = \bar{c}$. The partition function takes the form

$$Z(\tau, \bar{\tau}) = \text{Tr}\left[q^{L_0 - c/24} \bar{q}^{\overline{L}_0 - c/24}\right], \tag{3.3}$$

where the trace extends over all the states in the Hilbert space. The parameters $q = \exp(2\pi i\tau)$ and $\bar{q} = \exp(-2\pi i\bar{\tau})$, with $\tau = (v/2\pi R)(\gamma + i\beta)$ and $\bar{\tau} = (\bar{v}/2\pi R)(\gamma - i\beta)$ encode the dependence on the temperature, $\beta = 1/k_BT$, and the "torsion" γ. In general, the partition function would also contain the dependence on the electric potential, conjugate to the charge of the excitations [8], but this is discarded here.

The edge states of the Laughlin plateaus at $\nu = 1/p$, $p = 1, 3, 5, \ldots$, are described by the CFT of the chiral boson field compactified on a circle of radius $r^2 = 1/p$ [11,18]; its

central charge $c = 1$ accounts for a single chiral propagating mode per edge ($c_1 = \bar{c}_2 = 1$, $c_2 = \bar{c}_1 = 0$). The partition function is given by [8]:

$$Z(\tau, \bar{\tau}) = \sum_{\lambda=1}^{p} \chi_\lambda(\tau)\overline{\chi_\lambda(\tau)},$$

$$\chi_\lambda(\tau) = \frac{1}{\eta(\tau)} \sum_{k=-\infty}^{\infty} q^{(pk+\lambda)^2/2p}, \qquad \eta(\tau) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k), \tag{3.4}$$

where each χ_λ is a sum of characters of the Abelian current algebra $\widehat{U(1)}$ and $\eta(\tau)$ is the Dedekind function.

In order to compute the thermal current on one edge, say (1), we need to split the statistical sum into parts pertaining to each edge. These are given by the "chiral partition functions" $\chi_\lambda(\tau)$ in (3.4): actually, these sums involve states of a single edge and depend on the parameter $\lambda = 1, \ldots, p$ specifying the sector of fractional charge $Q = \lambda/p + \text{integer}$ (see also Ref. [16]). As shown by Eq. (3.4), the chiral partition functions on the two edges are only coupled through the integrality condition on the total charge.

The (constant) thermal current (2.6), (2.11) can be rewritten:

$$J_Q = v(\varepsilon_1 - \bar{\varepsilon}_1), \tag{3.5}$$

in terms of the average chiral energy densities on the first edge; in the chiral boson theory, $\bar{\varepsilon}_1 = 0$ and

$$\varepsilon_1(\lambda) = -\frac{i}{2\pi R} \frac{\partial \log \chi_\lambda}{\partial \gamma}\bigg|_{\gamma=0}, \tag{3.6}$$

that may depend on the topological sector. Note that differentiation w.r.t. β would not take the chirality sign of (3.5) into account.

Next, we evaluate this quantity in the low-temperature, $\beta \to \infty$, and large-size limit, $v\beta \lesssim R$, i.e., small $x = v\beta/R$, keeping the first finite-size correction. The sum in the numerator of χ_λ (3.4) can be approximated by a continuous Gaussian integral, that is actually λ-independent; in the denominator, the derivative of the Dedekind function is a sum that is also approximated by an integral plus the first finite-size correction given by the Euler–Maclaurin formula. The result is:

$$\begin{aligned} J_Q &= \frac{v^2}{2\pi R^2}\left[\left(\frac{1}{2x} + O(1)\right) + \left(\frac{\pi^2}{6x^2} - \frac{1}{2x} + O(x)\right)\right] \\ &= \frac{\pi}{12\beta^2}\left(1 + O\left(\frac{\beta^2 v^2}{R^2}\right)\right) \qquad\qquad \left(x = \frac{v\beta}{R}\right). \end{aligned} \tag{3.7}$$

In the first line of this equation, the first (respectively, second) parenthesis contains the contribution of the numerator (respectively, denominator) of χ_λ. We thus reproduce the general result Eq. (2.11) to leading order in the finite-size expansion; note the cancellation of the first correction between numerator and denominator of χ_λ.

As is well known [5], the partition functions of the rational CFTs, such as (3.4), are invariant under modular transformations of the torus made by the periodic space and

compact time [8]; moreover, the chiral parts χ_λ transform linearly among themselves [5]. In particular, for the S modular transformation, $\tau \to -1/\tau$, one finds:

$$Z(-1/\tau, -1/\bar{\tau}) = Z(\tau, \bar{\tau}),$$

$$\chi_\lambda(-1/\tau) = \frac{1}{\sqrt{p}} \sum_{\lambda'=1}^{p} \exp\left(i2\pi \frac{\lambda\lambda'}{p}\right) \chi_{\lambda'}(\tau). \tag{3.8}$$

One can use this transformation to simplify the calculation of (3.6), by mapping $|q| \lesssim 1$ to $|\tilde{q}| = \exp(-2\pi/\operatorname{Im}\tau) \ll 1$, such that the sum (product) in $\chi(\tilde{q})_\lambda$ can be approximated by the leading term. One easily obtains:

$$J_Q = \frac{\pi}{12\beta^2} \left[1 + O\left(\exp\left(-\frac{2\pi^2 R}{v\beta} \frac{1}{p}\right) \right) \right]. \tag{3.9}$$

We remark that: (i) the leading term arises from the prefactor $\tilde{q}^{-1/24}$ of the Dedekind function in the characters (3.4); (ii) there are no power-law $O((v\beta/R)^k)$ finite-size corrections.[4] The dominant term in the partition function for $|q| \to 0$ has actually the general form $q^{-c/24}$ involving the Virasoro central charge, thus proving again the general formula for the thermal conductance (1.2). Furthermore, the derivation using the modular transformation can be extended to any rational CFT, that possesses a modular invariant partition function and a finite basis of characters linearly transforming among themselves. We conclude that rational CFTs do not display power-law finite-size corrections to the thermal conductivity in (1.2).

3.2. Abelian conformal theories for the hierarchy

We now consider the multi-component generalizations of the scalar theory (also called multi-component Abelian CFT), that could describe the hierarchical Hall states with $v = m/(mp \pm 1)$: for the plateaus at $v = 2/(2p \pm 1)$, $p = 2, 4, \ldots$, such CFTs have central charge $c = 2$, i.e., two propagating modes per edge, one charged and one neutral [9]. Although the chirality of the charged excitations is fixed by the external magnetic field, neutral excitations can move in either direction (see Fig. 2), leading to $(c_1, \bar{c}_1) = (2, 0)$ or $(1, 1)$ in the previous notation. These theories are rational CFTs and their Abelian
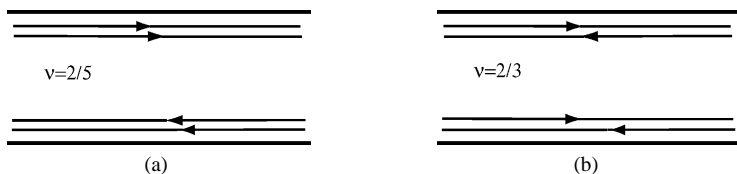

Fig. 2. Neutral and charged modes in the plateaus with $v = 2/5$ and $v = 2/3$, having equal and opposite senses of propagation, respectively.

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[4] One can check that the power-law corrections also vanish in the earlier expansion (3.7), but note that the Euler–Maclaurin formula fails to reproduce the non-analytic terms in (3.9).

current algebra is extended to $\widehat{U(1)} \times \widehat{SU(2)}_1$; the modular invariant partition function is again given by a finite sum of the chiral parts for each edge, that are themselves sums of representation-theory characters, as follows [8]:

$$Z^{(\pm)} = \sum_{a=1}^{\hat{p}} \theta_a^{(\pm)} \bar{\theta}_a^{(\pm)}, \tag{3.10}$$

where '+' (respectively, '−') indicates propagation of the neutral mode parallel (respectively, antiparallel) to the charged one. The chiral partition functions are:

$$\theta_a^{(+)} = \sum_{\alpha=0}^{1} \chi_{2a+\alpha\hat{p}}^{\widehat{U(1)}} \chi_{\alpha}^{\widehat{SU(2)}_1}, \quad a = 1, 2, \ldots, \hat{p},$$

$$\theta_a^{(-)} = \sum_{\alpha=0}^{1} \chi_{2a+\alpha\hat{p}}^{\widehat{U(1)}} \bar{\chi}_{\alpha}^{\widehat{SU(2)}_1}, \tag{3.11}$$

where the characters are given by,

$$\chi_{\lambda}^{\widehat{U(1)}}(\tau) = \frac{1}{\eta(\tau)} \sum_{k=-\infty}^{\infty} q^{(2\hat{p}k+\lambda)^2/4\hat{p}},$$

$$\chi_{\alpha}^{\widehat{SU(2)}_1}(\tau) = \frac{1}{\eta(\tau)} \sum_{k=-\infty}^{\infty} q^{(2k+\alpha)^2/4}. \tag{3.12}$$

The $\widehat{U(1)}$ and $\widehat{SU(2)}_1$ characters describe the charged (c) and neutral (n) modes, respectively; $\alpha = 0, 1$ is the $\widehat{SU(2)}_1$ isospin parity, $\hat{p} = 2/\nu = 2p \pm 1$ and $\lambda = 2a + \alpha\hat{p}$ counts the units of fractional charge. The two modes can have different velocities, which are accounted for by redefining the q parameters in the corresponding characters: $q \rightarrow q_j = \exp(-(\beta - i\gamma_j)v_j/R)$, with $j = c, n$. Note that independent rescalings are possible because the Hamiltonian is factorized: $H = H_c + H_n$ [19].

In order to compute the thermal current on one edge (3.5), $J_Q = v_c\varepsilon_c \pm v_n\varepsilon_n$, we vary the chiral partition functions (3.11) as follows:

$$-\frac{i}{2\pi R} \frac{\partial \log \theta_{\lambda}^{(\pm)}}{\partial \gamma_c}\bigg|_{\gamma=0} = \varepsilon_c, \qquad -\frac{i}{2\pi R} \frac{\partial \log \theta_{\lambda}^{(\pm)}}{\partial \gamma_n}\bigg|_{\gamma=0} = \pm\varepsilon_n. \tag{3.13}$$

The computation of (3.13) to leading order in (β/R) can be done as before, by performing a modular transformation: the characters in (3.12) and chiral partition functions $\theta_a^{(\pm)}$ (3.11) undergo finite Fourier transforms as in (3.8) [8]; by expanding the resulting sums, the leading term reproduces the thermal conductance in agreement with (1.2) and the subleading terms are not polynomial in $1/R$. We can actually give the general result for all the hierarchical plateaus, as described by the multi-component scalar theories (whose modular invariant partition functions can be found in Ref. [8]). These correspond to the $c = m$ CFTs with current algebra $\widehat{U(1)} \times \widehat{SU(m)}_1$, the second factor pertaining to the $(m-1)$ neutral modes with equal chirality, namely $(c_1, \bar{c}_1) = (m, 0)$ or $(1, m-1)$. The

result is:

$$K_{\text{Abelian}} = \frac{\pi k_B^2 T}{6} \big[1 \pm (m-1) \big], \quad \text{for} \quad \nu = \frac{m}{mp \pm 1}. \tag{3.14}$$

4. $W_{1+\infty}$ minimal models for the hierarchy

We now focus our attention to another CFTs for the hierarchical edge states, the $W_{1+\infty}$ minimal models, which have been introduced in Ref. [10] and further analyzed in the Refs. [8,19]. These models have been derived from the requirement of the $W_{1+\infty}$ symmetry, which is the characteristic symmetry of the incompressible fluids under area-preserving reparametrizations of the spatial coordinates [13,14]. This symmetry can be naturally implemented in the CFTs describing the edge excitations of the Hall droplet. Another requirement is the minimality of the set of excitations, which translates into irreducibility of the $W_{1+\infty}$ representations [20]. These two ingredients uniquely determine the $W_{1+\infty}$ minimal models, that are actually in one-to-one correspondence with the Jain hierarchical plateaus [10].

The relation between the minimal and Abelian theories has been discussed in Ref. [19]: each minimal model can be obtained by projecting some neutral states out of the Abelian theory for the same plateau. In the $c = 2$ case, the projection has been done very explicitly: a term was added to the Abelian Hamiltonian, parametrized by a positive coupling ω, as follows:

$$H = \frac{1}{R} \left(v_c L_0^{(c)} + v_n L_0^{(n)} - \frac{1}{12} \right) + \omega J_0^+ J_0^-. \tag{4.1}$$

This defines a theory interpolating between the Abelian ($\omega = 0$) and minimal ($\omega = \infty$) model. Actually, the neutral states in the Abelian theory carry a $SU(2)$ isospin label, besides the Virasoro dimension, that implies degenerate multiplets: there are $(2s + 1)$ Virasoro representations with conformal dimension $h = s^2$. The additional term in the Hamiltonian (4.1) assigns a large weight $O(\omega)$ to all the states in the multiplets except the highest weight state $|s, -s\rangle$ that satisfies $J_0^- |s, -s\rangle = 0$. As a result, the $W_{1+\infty}$ minimal models have no multiplicities for the Virasoro representations and the $SU(2)$ symmetry is broken.

The parameter ω has the dimension of a mass, thus the term $J_0^+ J_0^-$ is a relevant perturbation that also breaks conformal invariance up to the infrared limit ($\omega = \infty$). This renormalization-group flow takes place in the same phase of the system, because the ground-state ($s = 0$) is not affected by the projection. The Hamiltonian (4.1) naturally suggests the physical relevance of the $W_{1+\infty}$ minimal models: since the CFTs are effective low-energy, long-distance descriptions of the edge dynamics, the farthest infrared fixed-point is physically relevant; in other words, $\omega \to \infty$ is naturally reached without fine-tuning (if switched on). Another indication of the relevance of the $W_{1+\infty}$ minimal models comes from the numerical study of ten electrons in the first Landau level, that has been done in Ref. [21] for the disk geometry: the low-energy levels can be consistently interpreted by the edge excitations of the $W_{1+\infty}$ minimal models, because the isospin multiplicities are not observed (up to the finite-size uncertainties).

The partition functions of the minimal models have the same structure as those of the Abelian theories (3.10):

$$Z^{(\pm)} = \sum_{a=1}^{\hat{p}} \Theta_a^{(\pm)} \overline{\Theta}_a^{(\pm)}, \tag{4.2}$$

but the characters for the neutral sector are different. The expressions for the $c = m = 2$ case are [8]:

$$\Theta^{(+)} = \sum_{\alpha=0}^{1} \chi_{2a+\alpha\hat{p}}^{\widehat{U(1)}} \chi_\alpha^{(n)}, \qquad \Theta^{(-)} = \sum_{\alpha=0}^{1} \chi_{2a+\alpha\hat{p}}^{\widehat{U(1)}} \bar{\chi}_\alpha^{(n)}, \tag{4.3}$$

where the $\chi_\alpha^{(n)}$ are sums of the characters χ_h^{Vir} of the $c = 1$ Virasoro degenerate representations with conformal weights $h = n^2/4$, $n = 0, 1, 2, \ldots,$ [5],

$$\chi_\alpha^{(n)} = \sum_{\ell=0}^{\infty} \chi_{(2\ell+\alpha)^2/4}^{\text{Vir}} = \frac{q^{\alpha/4}}{\eta(\tau)}. \tag{4.4}$$

The projection $\omega \to \infty$ in the Hamiltonian (4.1) can be seen at the level of partition functions: the $\widehat{SU(2)}_1$ characters $\chi_\alpha^{\widehat{SU(2)}_1}$ of the Abelian theory (3.12) can be continuously connected to the $\chi_\alpha^{(n)}$ of the minimal theory (4.4) by varying ω from zero to infinity. The resulting partition function (4.2) is not modular invariant, namely the $W_{1+\infty}$ minimal models are not rational CFTs. The reason for this fact can again be traced back to the Hamiltonian (4.1): the ω-perturbation is actually made by the non-local operator $J_0^+ J_0^-$, thus the general arguments for modular invariance in local field theory do not apply. Note that the partition function (4.2) is uniquely determined by the Hamiltonian definition of the minimal models just outlined [19].

We now proceed to compute the thermal conductance in the $c = 2$ $W_{1+\infty}$ minimal models: using the definitions (3.13), we evaluate the average energies ε_c and ε_n following the same steps as in the Abelian case. The S modular transformation is still useful for expanding the non-covariant neutral characters $\chi_\alpha^{(n)}$ in (4.4): we need the transformation of the Dedekind function,

$$\eta(\tau) = (-i\tau)^{-1/2} \eta(-1/\tau). \tag{4.5}$$

The result is (including the general case $c = m$):

$$K_{\text{Minimal}} = \frac{\pi k_B^2 T}{6} \left[1 \pm (m-1)\right] \mp \frac{k_B v_n}{4\pi R}(m-1) \quad \text{for} \quad \nu = \frac{m}{mp \pm 1}. \tag{4.6}$$

This conductance contains an additional term w.r.t. the Abelian result (3.14), namely the first finite-size correction $O(v\beta/R)$. One can check that it originates from the prefactor $(\text{Im}\,\tau)^{-1/2}$ in the modular transformation of η (4.5).

In summary, we have found that the thermal conductance in the $W_{1+\infty}$ minimal models (4.6) displays a finite-size correction to the general result (1.2) that is absent for all rational CFTs including the Abelian theories of the hierarchical states. Such correction is a consequence of the lack of modular invariance in the minimal partition functions, but can

also be obtained by straightforward expansion in $v\beta/R$, as described for the $c = 1$ theory in Eq. (3.7); the latter derivation would make apparent that the correction arises from the different state counting in the minimal models, which is due to the projection of infinite states out of the Abelian theory. It is also clear that this finite-size correction in K_{Minimal} is universal and independent of the shape of the sample.

5. Conclusions

In this paper, we have found a general expression for the thermal Hall conductance in terms of the gravitational anomaly of the chiral CFT describing each edge. We have also obtained the finite-size expansion for the thermal conductance and computed the first universal, shape-independent correction. This was found to be different for two candidate conformal theories of the hierarchical Hall states: the multi-component Abelian theory has a vanishing correction, as any rational CFT, while the $W_{1+\infty}$ minimal model has it non-vanishing.

The relative size of this correction w.r.t. the leading term $|(K_{\text{Minimal}} - K_{\text{Abelian}})/K|$ is equal to $x/2\pi \simeq \varepsilon_{\text{edge}}/\varepsilon_{\text{therm}}$, with $x = v_n\beta/R$. We estimate: $|\Delta K/K| \simeq 0.09$ and $x \simeq 0.5$ for Hall samples of size $2\pi R \simeq 0.3$ cm and temperature $T \simeq 50$ mK, with $v_n \simeq c/1000$, assuming comparable Fermi velocities for neutral and charged modes [22]. Although it is a small effect, it may be measurable in future experiments of the type proposed in [3], in the favorable cases, like $\nu = 2/3$, where the leading term in the thermal conductance is predicted to vanish. Such a difference could support either theoretical proposal for the hierarchical states. Let us add that the CFT prediction for the thermal conductivity presented here is independent of the impurity interactions, as discussed in Ref. [3]. Note also that further differences between the two CFTs for the hierarchical theories involve the quantum statistics and require the measurement of four-point functions.

Finally, the thermal Hall conductance can also be computed [23] for further conformal theories of the hierarchical states [24] and for the paired Hall states [25].

Acknowledgements

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