# Lecture 3: Temporal models

April 12, 2016

# Why start with time?

1. What date is closer to August 8th
   1. August 6th?
   2. August 10th?
2. Time is 1-D
3. Time has a direction
4. Simplest case for considering autocorrelation
5. Still – it is one component of spatio-temporal processes

**Syllabus**

- White noise and random walk process
- Autoregressive (AR) structure
- Moving average (MA) structure
- State-space models

**Definitions**

- Discrete time sequence of real-valued random variables $Y_t$

  $\{Y_t : t \in 0,1,\ldots\}$

- Mean function

  $\mu_t = \mathrm{E}(\mathrm{Y}_t)$

- Autocovariance function

  $C(t,r) = \mathrm{cov}(Y_t, Y_r) = E\{(Y_t - \mu_t)(Y_r - \mu_r)\}$

- Autocorrelation function

$$\rho(r,t) = \frac{\mathrm{cov}(Y_t, Y_r)}{\sqrt{\mathrm{var}(Y_t)\mathrm{var}(Y_r)}}$$

## Stationarity

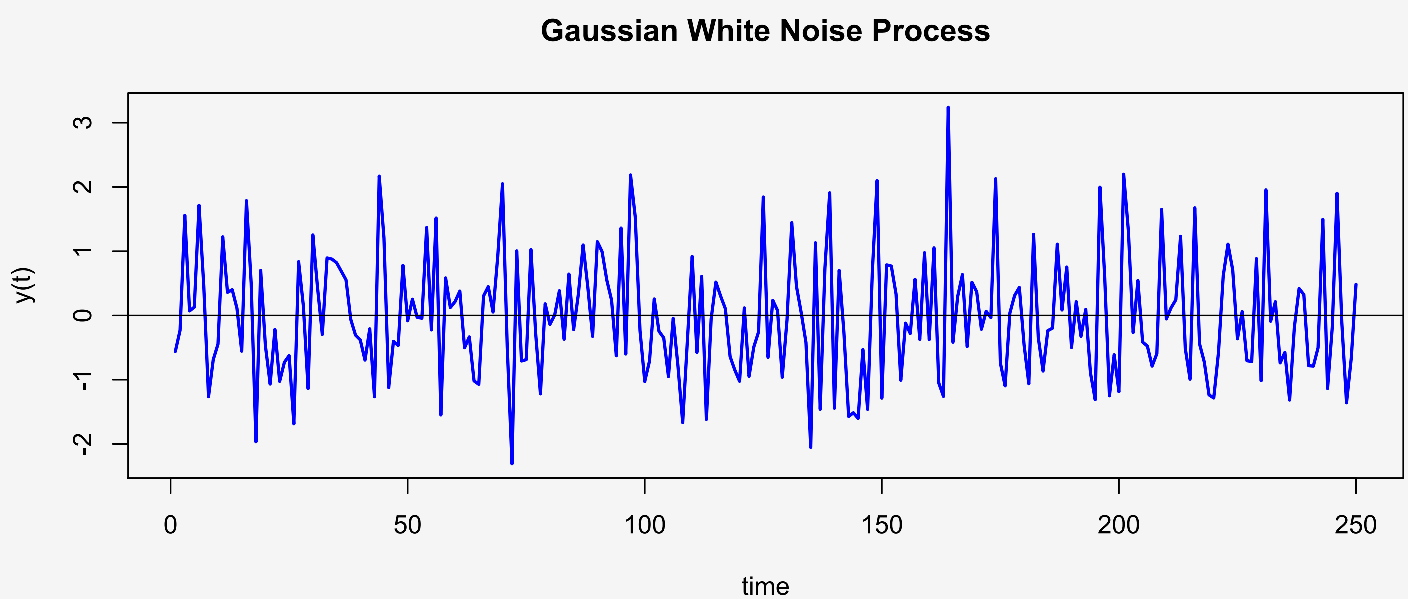
- Strong stationarity: time series model for $Y_t$ is stationary if all statistics remain unchanged after time shifts, i.e. remain unchanged for all $Y_{t0+t}$ for all possible $t_0$ (all moments are invariant)

- Weak stationarity: time series model for $Y_t$ is stationary if the mean function is constant and auto-covariance is a function of difference of arguments (only mean and variance are invariant)

## White noise process

$$W_t \sim Normal(0, \sigma^2)$$

- Why “white noise”?
  - The spectral density function is equal for all frequencies, thus an equal mixture of all colors, which is equivalent to white light

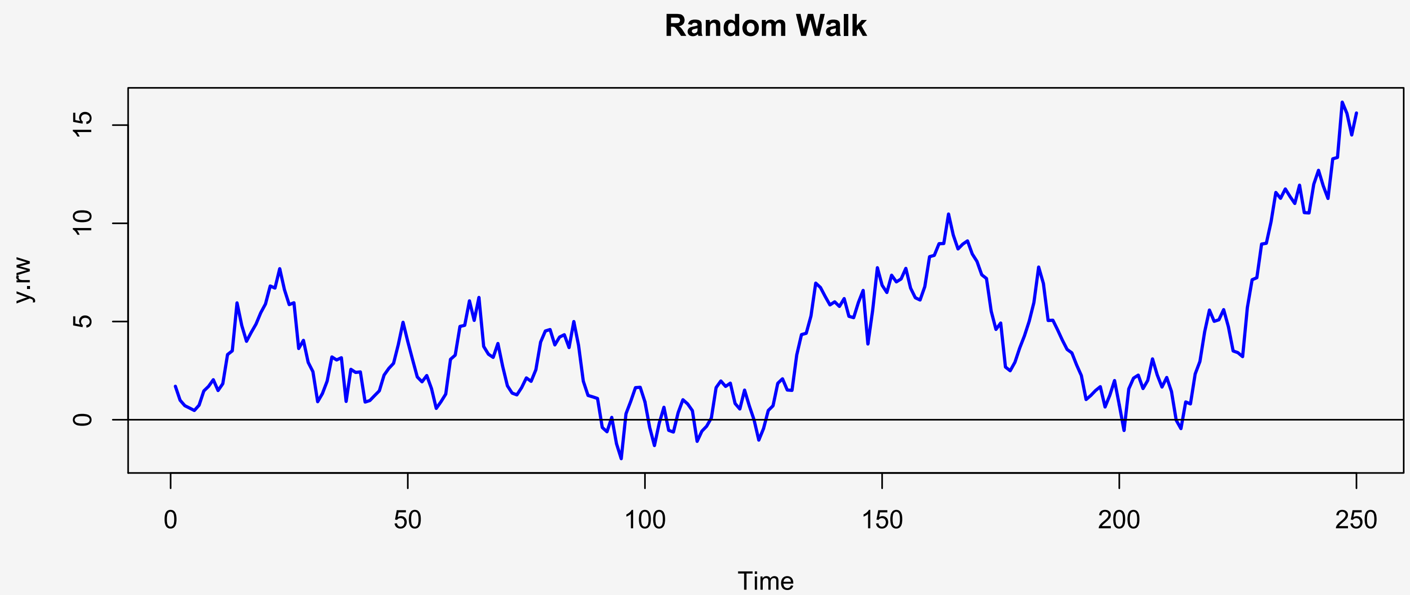


## Random walk process

$$Y_t = Y_{t-1} + W_t$$

$$W_t \sim Normal(0, \sigma^2)$$

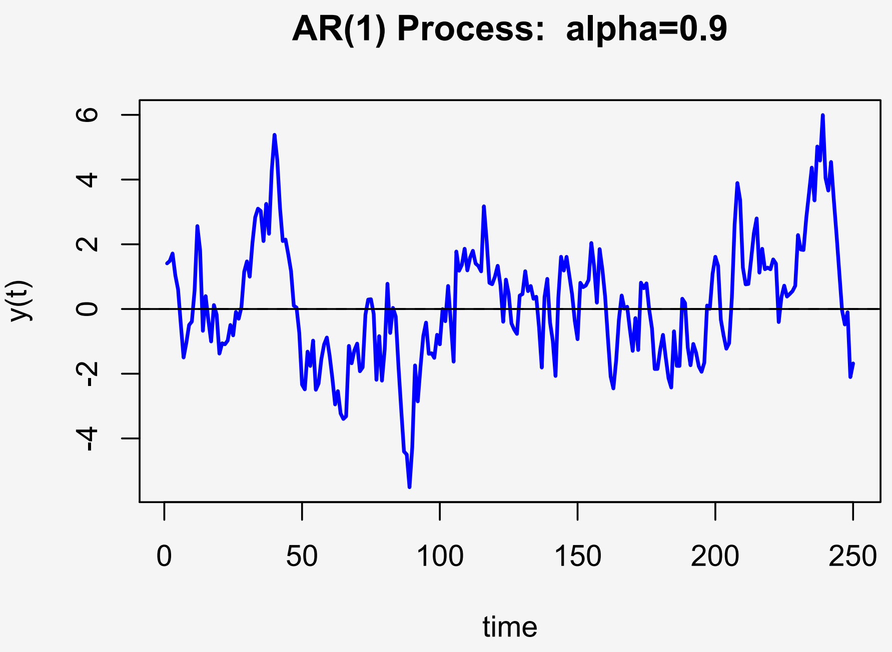
- Random walk incorporates stochastic innovation to the process at each time step

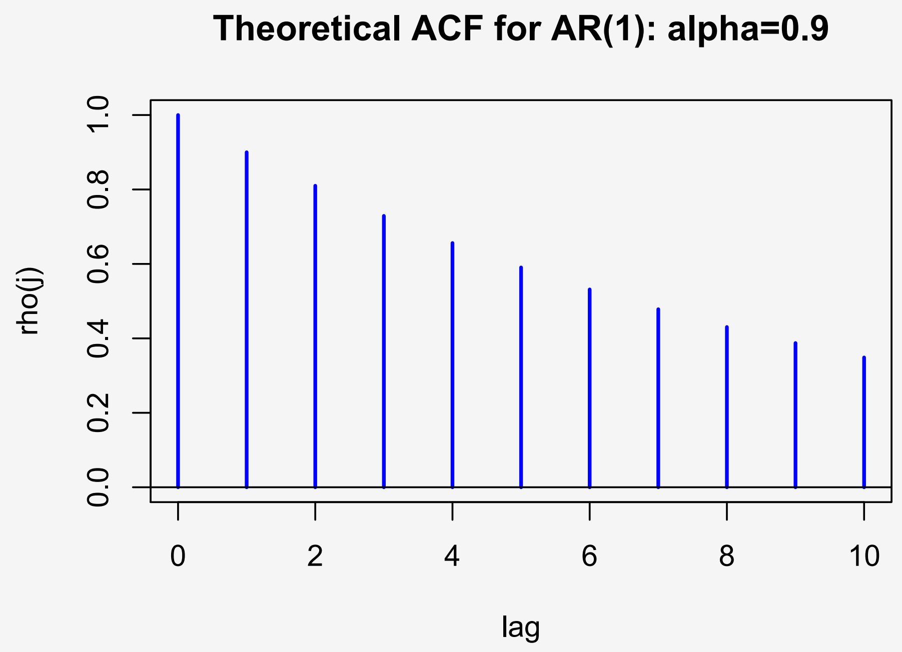


# Autoregressive process

$$Y_t = \alpha_1 Y_{t-1} + \ldots \alpha_p Y_{t-p} + W_t$$

$$W_t \sim Normal(0, \sigma^2)$$

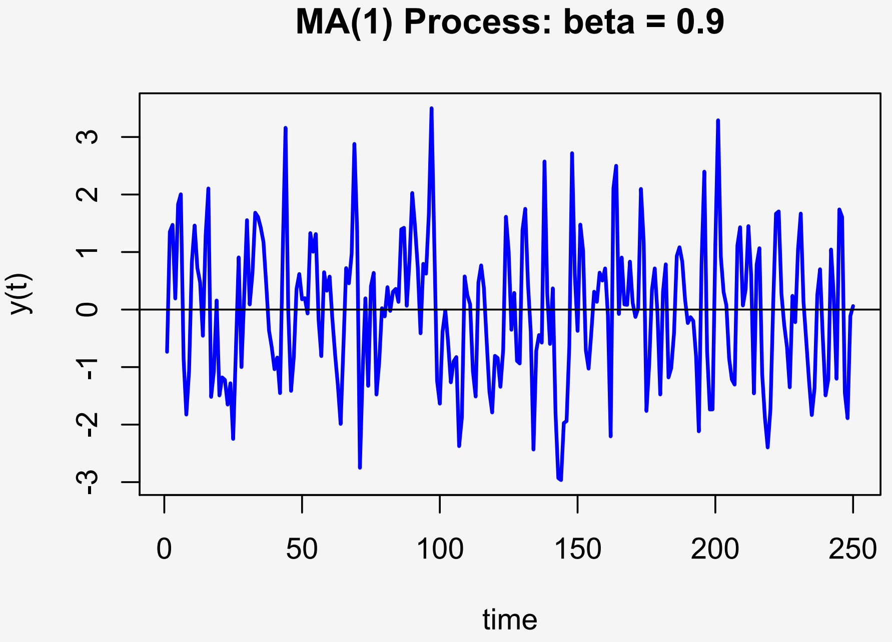


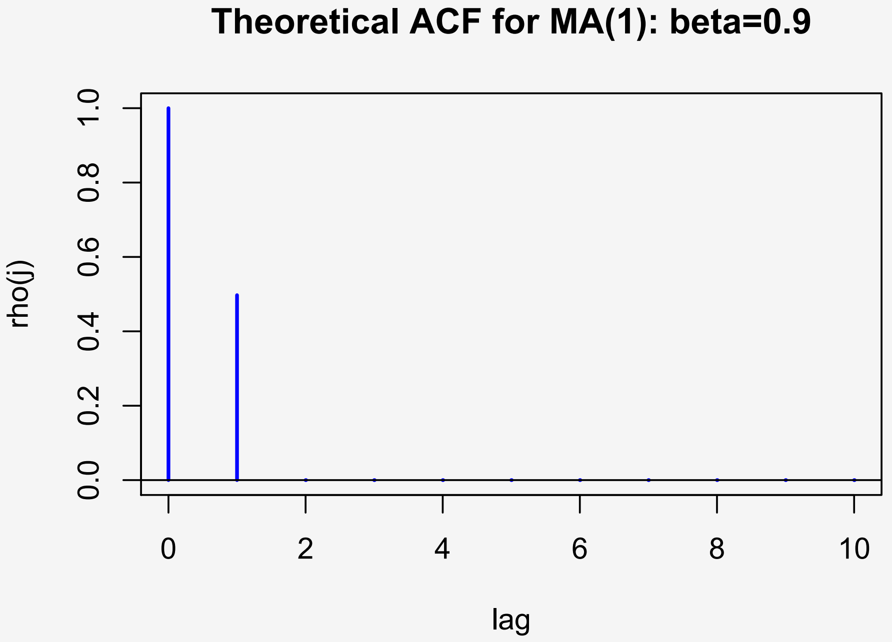



# Moving Average process

$$Y_t = W_t + \beta_1 W_{t-1} + \dots \beta_q W_{t-q}$$

$$W_t \sim Normal(0, \sigma^2)$$



**Relationship of AR and MA processes**

- AR as an infinite sequence of MA

$$Y_t = \alpha Y_{t-1} + W_t$$
$$Y_t = \alpha(\alpha Y_{t-2} + W_{t-1}) + W_t$$
$$Y_t = W_t + \alpha W_{t-1} + \alpha^2 W_{t-2} + \ldots$$
$$Y_t = \sum_{k=0}^{\infty} \alpha^k W_{t-k}$$
$$\mathrm{var}(Y_t) = \sigma_w^2 (1 + \alpha^2 + \alpha^4 + \ldots) = \frac{\sigma_w^2}{(1-\alpha^2)}$$

- MA as an infinite sequence of AR

$$Y_t = W_t + \beta W_{t-1}$$
$$W_t = Y_t - \beta W_{t-1}$$
$$W_t = Y_t - \beta(Y_{t-1} - \beta W_{t-2})$$
$$W_t = \sum_{k=0}^{\infty} \beta^k Y_{t-k}$$

- The ARMA($p$,$q$) model

$$Y_t = \alpha_1 Y_{t-1} + \ldots + \alpha_p Y_{t-p} + W_t + \beta_1 W_{t-1} + \ldots \beta_q W_{t-q}$$

## Extensions 1 – models for the process Y

- Non-linear time-series models

$$Y_t = \alpha_t Y_{t-1} + \epsilon_t$$
$$\epsilon_t \sim Normal(0, \sigma_\epsilon^2)$$
$$\alpha_t = b_0 + b_1 X_t + \gamma_t$$
$$\gamma_t \sim Normal(0, \sigma_\gamma^2)$$

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$$\alpha_t = b_0 + b_1 \alpha_{t-1} + \gamma_t$$
$$\gamma_t \sim Normal(0, \sigma_\gamma^2)$$

- Multivariate time series (Vector Autoregressive Process)

$$\mathbf{Y}_t = \mathbf{M}_1 \mathbf{Y}_{t-1} + \mathbf{M}_2 \mathbf{Y}_{t-2} + ... + \mathbf{M}_p \mathbf{Y}_{t-p} + \mathbf{W}_t$$
$$\mathbf{W}_t \sim MVNormal(\mathbf{0}, \mathbf{Q})$$

## Extensions 2 – models for the variance of the process

- Model the variance as changing over time – generalized autoregressive conditional heteroskedasticity (GARCH)

$$\sigma^2_{\varepsilon,t} = a_0 + \sum_{i=1}^{p} a_i Y_{t-i} + \sum_{j=1}^{q} b\sigma^2_{\varepsilon,t-j}$$

- Stochastic Volatility models – variance changing over time stochastically

$$\log(\sigma_{\varepsilon}{}^2) = b_0 + b_1 \log(\sigma_{\varepsilon,t-1}{}^2) + v_t$$

$$v_t \sim Normal(0, \sigma^2{}_v)$$

## Stationarity assumptions

- Need to have a stationary series, thus need to remove any trend in the data and have homogeneous variance
  - Model for the mean or differencing to remove any trend
  - Variance stabilization transformations
    - log(), sqrt()
- MA series is stationary
- AR series is stationary if $|\alpha| < 1$
- Unit-root tests or Dickey-Fuller tests for stationarity
  - In an AR(1) model, test for whether $\alpha < 1$ versus $\alpha = 1$ (random walk)

## Fitting models to data

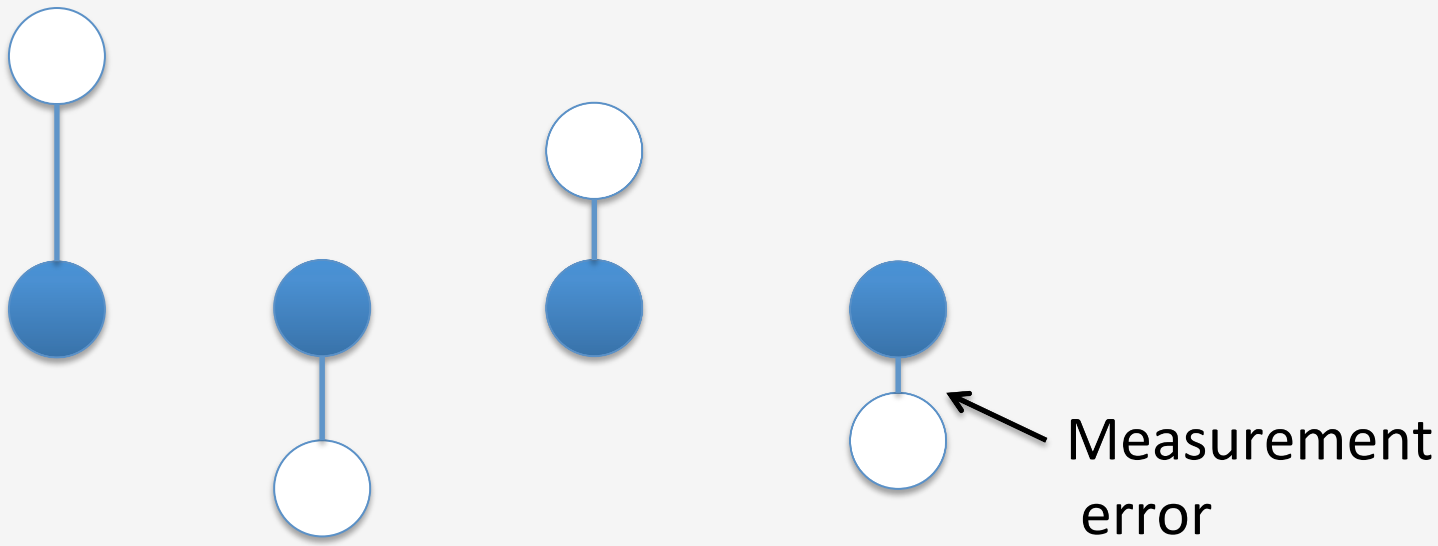
- Transform to obtain a stationary time series
  - Remove trends and seasonal components
  - Differentiate (if needed) – ARIMA or Box-Jenkins model
  - Test for stationarity
- Determine order of AR and MA processes
  - Compute partial auto-correlation function to determine order of model for AR,
  - partial auto-correlation of lag *k* is the autocorrelation between $z_t$ and $z_{t+k}$ with the linear dependence of $z_t$ on $z_{t+1}$ through $z_{t+k-1}$ removed
  - Compute auto-correlation function to determine order of MA model
  - Estimate auto-regression and variance using Yule-Walker (AR) or maximum likelihood (AR, MA, or ARMA)
- Compute residuals and test for white noise

## Hierarchical modeling of time-series

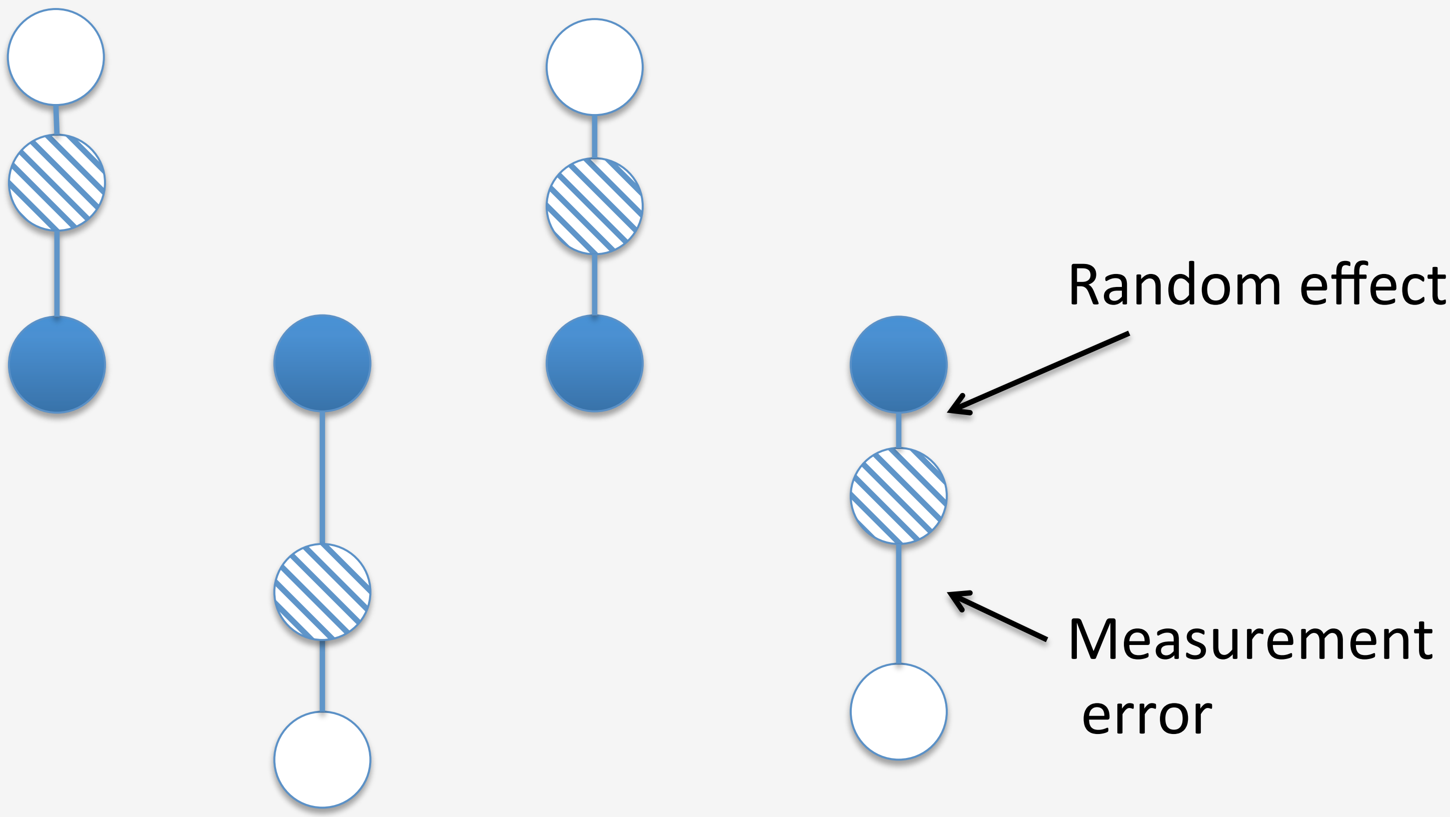
- Conditional description of joint distribution of data, process, and parameters
- Data model
  - data|process, parameters
- Process model that incorporates temporal dynamics
  - Process|parameters
- Parameters
  - parameters|hyperpriors   for a Bayesian Hierarchical Model

Cressie and Wikle (2011) Statistics for Spatiotemporal Data

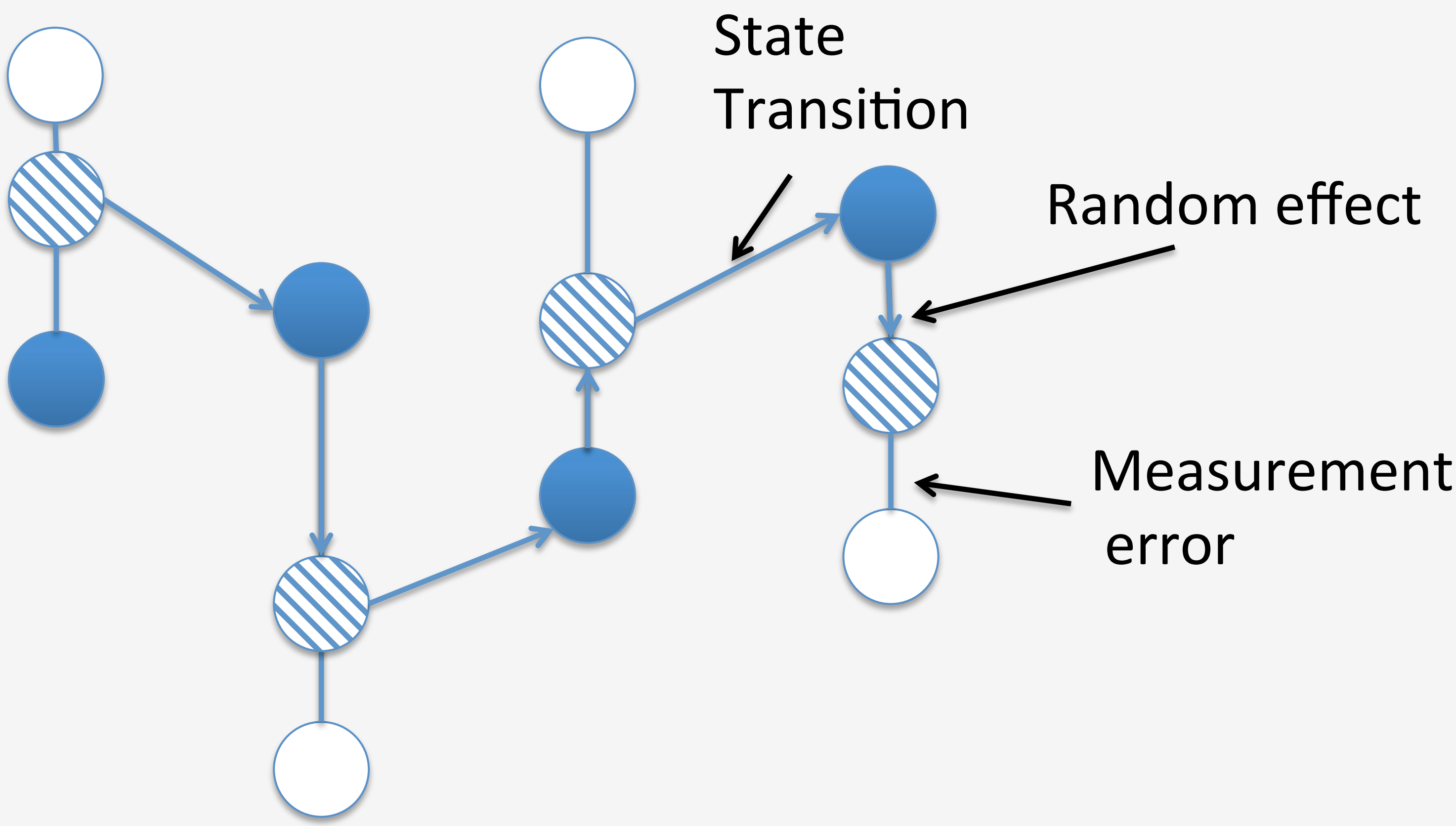
## Generalized linear models and linear models



# Hierarchical models with random effects



## State-space model



# *Temporal models*

**State-space model example - Dynamic Linear Model**

$$S_t | S_{t-1} \sim N(\lambda S_{t-1}, \sigma_S^2)$$

$$y_t | S_t \sim N(S_t, \sigma_y^2)$$

- The state $S_t$ is a stochastic function of the state the previous time step and the population growth rate $\lambda$ and the level of process noise $\sigma_S$
- The observation process is a function of the measurement error $\sigma_y$

# *Temporal models*