Fourier series – trigonometric form

Given $f(t)$ periodic with period T: $f(t+T) = f(t)$

The following decomposition into harmonic functions is possible

$$f(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos\left(k \frac{2\pi}{T} t\right) + b_k \sin\left(k \frac{2\pi}{T} t\right)\right)$$

with:

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt \quad a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\left(k\frac{2\pi}{T}t\right)dt \quad ; a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\left(k\frac{2\pi}{T}t\right)dt \quad k = 1, 2, \ldots$$

or:

$$f(t) = F_0 + \sum_{k=1}^{\infty} F_k \cos\left(k\frac{2\pi}{T}t + \varphi_k\right)$$

With:

$$F_0 = a_0 \quad ; F_k = \sqrt{a_k^2 + b_k^2} \quad ; \varphi_k = tan^{-1}\left(\frac{-b_k}{a_k}\right)$$

Fourier series – complex form

Recalling the exponential form of the cosine and sine function, the series can be rewritten as:

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \frac{e^{i\left(k\frac{2\pi}{T}t\right)}+e^{-i\left(k\frac{2\pi}{T}t\right)}}{2} + b_k \frac{e^{i\left(k\frac{2\pi}{T}t\right)}-e^{-i\left(k\frac{2\pi}{T}t\right)}}{2i}\right)=$$

$$= a_0 + \sum_{k=1}^{\infty}\left(\frac{a_k-ib_k}{2}e^{i\left(k\frac{2\pi}{T}t\right)} + \frac{a_k+ib_k}{2}e^{-i\left(k\frac{2\pi}{T}t\right)}\right)$$

We set:

$$f_0 = a_0 \quad ; \quad f_k = \frac{a_k-ib_k}{2} \quad ; f_{-k} = \frac{a_k+ib_k}{2}$$

And get the «exponential» or «complex» form of the Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{i\left(k\frac{2\pi}{T}t\right)}$$

Noting that f_k and f_{-k} are complex conjugates, we recognise:

$$f(t) = F_0 + \sum_{k=1}^{\infty} F_k \cos\left(k\frac{2\pi}{T}t + \varphi_k\right)$$

With:

$$F_0 = f_0 \quad ; \quad F_k = 2|f_k| \quad ; \varphi_k = \angle f_k$$

Response to periodic excitation

We use a Fourier series decomposition of the input $f(t) =$ in the form:

$$f(t) = F_0 + \sum_{k=1}^{\infty} F_k \cos\left(k\frac{2\pi}{T}t + \varphi_k\right)$$

Owing to the principle of superimposition we get the response of the system $x(t)$ as:

$$x(t) = X_0 + \sum_{k=1}^{\infty} X_k \cos\left(k\frac{2\pi}{T}t + \psi_k\right)$$

with:

$$X_0 = G(i0)F_0 \quad ; \quad X_k = |G(i\Omega_k)|F_k \quad ; \quad \psi_k = \angle G(i\Omega_k) + \varphi_k \ ; \Omega_k = k\frac{2\pi}{T}$$

which is equivalent to:

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{i\left(k\frac{2\pi}{T}t\right)}$$

With:

$$x_k = G(i\Omega_k)\, f_k$$

Impulsive excitation – Fourier transform

We now consider an impulsive excitation, in the form of a non-periodic, absolutely integral function of time $f(t)$

It is shown in the book by Meirovitch that, considering the exponential form of Fourier series for a periodic function of time:

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{i\left(k\frac{2\pi}{T}t\right)}$$

and letting the period T of the periodic function increase without bound, the two following interal pairs are obtained:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\Omega)e^{i\Omega t}d\Omega$$

$$F(\Omega) = \int_{-\infty}^{\infty} f(t)e^{-i\Omega t}dt$$

where $F(\Omega)$ is known as the «Fourier transform» of $f(t)$

Response to impulsive excitation

For a linear SISO system, represented by its frequency response function $G(i\Omega)$ the response to a non-periodic, impulsive excitation $f(t)$ is obtained as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{i\Omega t}d\Omega$$

with:

$$X(\Omega) = G(i\Omega)F(\Omega)$$

However, the integrals defining the Fourier transform of the input force and the inverse Fourier transform of the output motion are difficult to be obtained unless very simple cases are considered (e.g. single dof system excited by a simple rectangular pulse) or even not exixting in case of measured input signals (e.g. earthquake). Therefore, a discrete approximation of the FT, known as the «Discrete Fourier Transform» (DFT) shall be used.

The Discrete Fourier transform

We consider a discrete set of N samples of the input at equi-spaced times $t=0, dt, 2dt, \ldots, (N\text{-}1)\Delta t$, which refers to a finite duration in time $T=N\Delta t.$ The Fourier transform of the input defined as:

$$F(\Omega) = \int_{-\infty}^{\infty} f(t) e^{-i\Omega t} dt$$

Can be approximated as:

$$\bar{F}_k = \sum_{j=0}^{N-1} f_j e^{-i\left(k\frac{2\pi}{T}\right)t_j} = \sum_{j=0}^{N-1} f_j e^{-i\frac{2\pi k j}{N}} \quad k = 0,1,\ldots,N-1$$

Note that since $2\pi\frac{N-k}{N} = 2\pi - 2\pi\frac{k}{N}$, it follows that $\bar{F}_{N-k} = conj(\bar{F}_k)$. The N samples of the input can be obtained from the coefficients $\bar{F}_k$ according to the following approximation of the inverse Fourier transform:

$$f_j = \frac{1}{N}\sum_{k=0}^{N-1} \bar{F}_k e^{i\frac{2\pi k j}{N}} \quad j = 0,1,\ldots,N-1$$

The above two expressions represent the direct and inverse DFT of the sampled input signal. Note that if the above inverse DFT is used to obtain the value of the input at any discrete $t_j = jdt$ with $j<0$ or $j>N$-1 the sequence of the sampled values is repeated periodically. Hence, the use of a discrete set of data to describe the time-dependent excitation ultimately results in considering the excitation as periodic with period T.

Relationship between the Fourier series and the DFT

Let us assume function $f(t)$ is a periodic function of time with period T which is known only in the form of N sampled values f_j. The coefficients of the Fourier series in complex form can be approximated as:

$$f_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-i\left(k\frac{2\pi}{T}\right)t_j} dt \cong \frac{1}{T}\sum_{j=0}^{N-1} f_j e^{-i\left(k\frac{2\pi}{T}\right)j\Delta t}\Delta t = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-i\frac{2\pi kj}{N}} = \frac{1}{N}\bar{F}_k \qquad k = 0,1,\dots,N-1$$

Thus, the DFT can also be used to derive the coefficients of the Fourier series.

The highest harmonics in the Fourier series that can be obtained from the above relationship corresponds to $k = \frac{N}{2} - 1$.

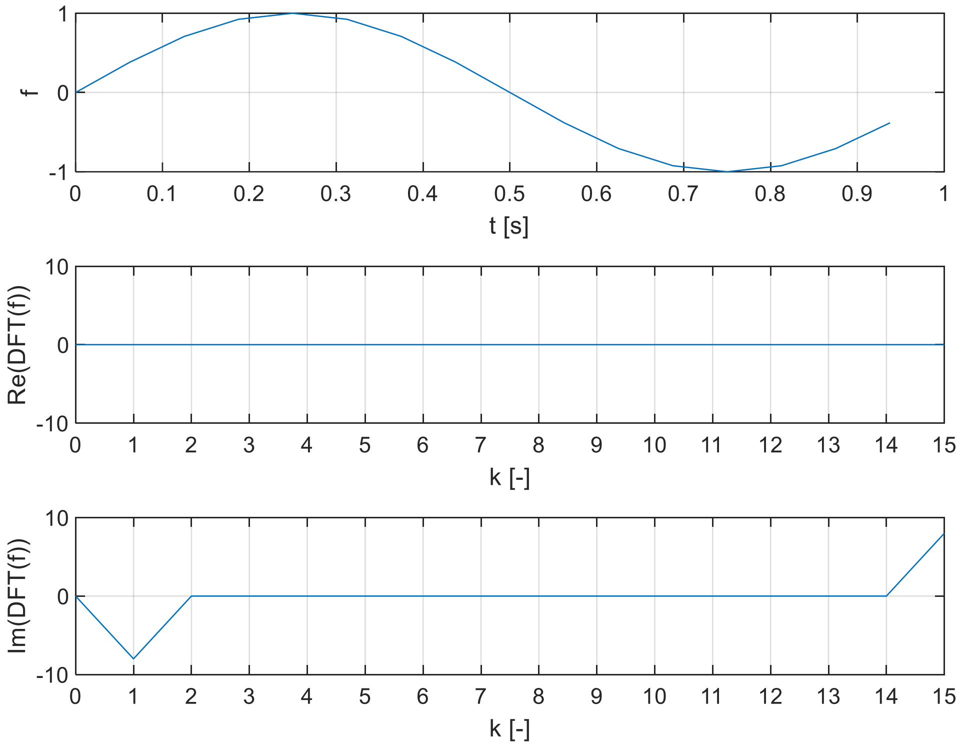
Note that :

$$\Omega_{\frac{N}{2}} = \frac{N}{2}\frac{2\pi}{T} = 2\pi\frac{1}{2}\frac{1}{\Delta t} = 2\pi\frac{f_s}{2}$$

with f_s the sampling frequency of the discrete series of values, which is consistent with Shannon's theorem about sampling of analog signals (analog to digital conversion /ADC).

Relationship between the Fourier series and the DFT

Example: DFT of a single period of a sinusoidal signal having frequency 1Hz, sampled at 16 Hz ($\Delta t=0.0625$ s), T=1s. The only non-zero coefficients of the DFT are for k=1 and k=N (both corresponding to the frequency f=1Hz). These two coefficients are complex conjugates, as expected, and are purely immaginary sue to the fact that the signal has sinusoidal shape (what would happen if the cos function was considered intead of the sin function ?)



Response to impulsive excitation with the DFT

For a linear SISO system, represented by its frequency response function $G(i\Omega)$ the response to a non-periodic, impulsive excitation represented by a discrete set of N samples of the input at equi-spaced times $t=0, \Delta t, 2\Delta t, \ldots, (N\text{-}1)\Delta t$ can be obtained using the DFT and inverse DFT in the form of N samples of the time histoy of the output at the same times:

$$x_j = \frac{1}{N}\sum_{k=0}^{N-1} \bar{X}_k e^{i\frac{2\pi kj}{N}} \quad j = 0,1,\ldots,N-1$$

with:

$$\bar{X}_0 = G(i0)\bar{F}_0 \;\; ; \;\; \bar{X}_k = G(i\Omega_k)\bar{F}_k \;\; ; \;\; \bar{X}_{N-k} = conj(\bar{X}_k) \;\; ; \;\; k = 1,\ldots,\frac{N}{2}-1 \;\; ; \;\; \Omega_k = 2\pi\frac{k}{T}$$

Alternatively, the response can be obtained as one single period of the system's response to a fictitious periodic input defined by the same N discrete values over one period, using the equivalence between the coefficients of the DFT and those of the Fourier series introduced in the previous slide.