1) ESAME 12/07/2012

$G=(x,0);\ B=(x+l\cos\theta,\ l\sin\theta);\ A=(x-l\cos\theta,\ -l\sin\theta)$

1.

$$U=-mg\overset{\nearrow 0}{y_G}-\frac{1}{2}K|OA|^2+\int \underline{M}\cdot\underline{k}\,d\theta+\text{cost}=$$

$$=-\frac{1}{2}k\left(x^2-2lx\cos\theta+l^2\cos^2\theta+l^2\sin^2\theta\right)-\int M\,\underline{k}\cdot\underline{k}\,d\theta+\text{cost}=$$

$$=-\frac{1}{2}kx^2+klx\cos\theta-M\theta+\text{cost}$$

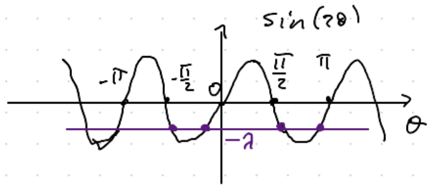
$$Q_\theta=\frac{\partial U}{\partial\theta}=-klx\sin\theta-M\ ;\qquad Q_x=\frac{\partial U}{\partial x}=-kx+kl\cos\theta$$

Equilibrio: $Q_\theta=Q_x=0$; dalla seconda $x=l\cos\theta$

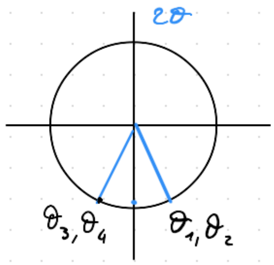
Nella prima: $-kl^2\sin\theta\cos\theta-M=0 \iff \frac{1}{2}kl^2\sin(2\theta)=-M$

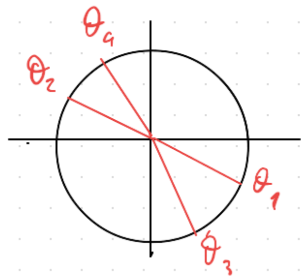
Definiamo $\lambda=\frac{2M}{kl^2}$; cercare $\sin(2\theta)=-\lambda$

Se $\theta\in(-\pi,\pi]$, $2\theta\in(-2\pi,2\pi]$



$$\begin{cases}2\theta_1=\arcsin(-\lambda)=-\arcsin\lambda\\ 2\theta_2=2\pi-\arcsin\lambda\\ 2\theta_3=-\pi+\arcsin\lambda\\ 2\theta_4=\pi+\arcsin\lambda\end{cases}$$





$$\begin{cases}\theta_1=-\frac{1}{2}\arcsin\lambda\\ \theta_2=\pi-\frac{1}{2}\arcsin\lambda\\ \theta_3=-\frac{\pi}{2}+\frac{1}{2}\arcsin\lambda\\ \theta_4=\frac{\pi}{2}+\frac{1}{2}\arcsin\lambda\end{cases}\qquad \begin{cases}\sin\theta_1=-\sqrt{\frac{1-\sqrt{1-\lambda^2}}{2}}\\ \sin\theta_2=\sqrt{\frac{1-\sqrt{1-\lambda^2}}{2}}\\ \sin\theta_3=-\sqrt{\frac{1+\sqrt{1-\lambda^2}}{2}}\\ \sin\theta_4=\sqrt{\frac{1+\sqrt{1-\lambda^2}}{2}}\end{cases}\qquad \begin{cases}\cos\theta_1=\sqrt{\frac{1+\sqrt{1-\lambda^2}}{2}}\\ \cos\theta_2=-\sqrt{\frac{1+\sqrt{1-\lambda^2}}{2}}\\ \cos\theta_3=\sqrt{\frac{1-\sqrt{1-\lambda^2}}{2}}\\ \cos\theta_4=-\sqrt{\frac{1-\sqrt{1-\lambda^2}}{2}}\end{cases}$$

$$\sin\theta=\pm\sqrt{\frac{1-\cos 2\theta}{2}}\ ;\quad \cos\theta=\pm\sqrt{\frac{1+\cos 2\theta}{2}}\ ;\quad \cos 2\theta=\pm\sqrt{1-\lambda^2}$$

$$(\theta_1, x_1) = \left(-\frac{1}{2}\arcsin\lambda,\ \ell\sqrt{\frac{1+\sqrt{1-\lambda^2}}{2}}\right)$$

$$(\theta_2, x_2) = \left(\pi - \frac{1}{2}\arcsin\lambda,\ -\ell\sqrt{\frac{1+\sqrt{1-\lambda^2}}{2}}\right)$$

$$(\theta_3, x_3) = \left(-\frac{\pi}{2} + \frac{1}{2}\arcsin\lambda,\ \ell\sqrt{\frac{1-\sqrt{1-\lambda^2}}{2}}\right)$$

$$(\theta_4, x_4) = \left(\frac{\pi}{2} + \frac{1}{2}\arcsin\lambda,\ -\ell\sqrt{\frac{1-\sqrt{1-\lambda^2}}{2}}\right)$$

Se $\lambda \leq 1$ esistono configurazioni di equilibrio, altrimenti no.

Se $\lambda = 1$, la 1 coincide con la 3 e la 2 con la 4.

Stabilità: $\frac{\partial^2 U}{\partial \theta^2} = -k\ell x\cos\theta$;

$$\frac{\partial^2 U}{\partial x \partial \theta} = \frac{\partial^2 U}{\partial \theta \partial x} = -k\ell\sin\theta$$

$$\frac{\partial^2 U}{\partial x^2} = -k$$

$$\underline{\underline{H}}(\theta, x) = \begin{pmatrix} -k\ell x\cos\theta & -k\ell\sin\theta \\ -k\ell\sin\theta & -k \end{pmatrix}$$

Sulle configurazioni di equilibrio $x = \ell\cos\theta$

$$\underline{\underline{H}}(\theta_{eq}, x_{eq}) = \begin{pmatrix} -k\ell^2\cos^2\theta & -k\ell\sin\theta \\ -k\ell\sin\theta & -k \end{pmatrix} \qquad \mathrm{Tr}\,\underline{\underline{H}}_{eq} = -k\ell^2\cos^2\theta - k < 0$$

$$\det \underline{\underline{H}}(\theta_{eq}, x_{eq}) = k^2\ell^2\cos^2\theta - k^2\ell^2\sin^2\theta = k^2\ell^2\cos(2\theta)$$

Ma:

$\cos(2\theta_1) = \cos(2\theta_2) > 0$ e $\cos(2\theta_3) = \cos(2\theta_4) < 0$ $(\lambda < 1)$

Quindi, se $\lambda < 1$, (θ_1, x_1) e (θ_2, x_2) sono stabili

(θ_3, x_3) e (θ_4, x_4) sono instabili

2. Energia cinetica

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2 \qquad \underline{\omega} = \omega \underline{k} = \dot{\vartheta} \underline{k} \; ; \; \underline{v}_G = \dot{x} \, \underline{i}$$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \frac{1}{12} m (2l)^2 \dot{\vartheta}^2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{6} m l^2 \dot{\vartheta}^2 = \frac{1}{6} m l^2 \dot{\vartheta}^2 + \frac{1}{2} m \dot{x}^2$$

momenti cinetici

$$p_\vartheta = \frac{\partial T}{\partial \dot{\vartheta}} = \frac{1}{3} m l^2 \dot{\vartheta} \; ; \quad p_x = \frac{\partial T}{\partial \dot{x}} = m \dot{x} \; ; \quad \left(\frac{\partial T}{\partial \vartheta} = \frac{\partial T}{\partial x} = 0 \right)$$

3. Equazioni di Lagrange $\qquad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_h} - \frac{\partial T}{\partial q_h} = Q_h$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\vartheta}} = \frac{1}{3} m l^2 \ddot{\vartheta}$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{x}} = m \ddot{x}$$

$$\begin{cases} \frac{1}{3} m l^2 \ddot{\vartheta} = -Klx \sin\vartheta - M \\ m \ddot{x} = -Kx + Kl\cos\vartheta \end{cases}$$

4. $x_0 = l \; , \quad \vartheta_0 = \frac{\pi}{2}$

$$\underline{a}_G(0) = \ddot{x}(0) \, \underline{i}$$

Dalla 2^a eq. di Lagrange:

$$m \ddot{x}(0) = -Kx_0 + Kl\cos\vartheta_0 = -Kl$$

$$\Rightarrow \underline{a}_G(0) = \ddot{x}(0) \, \underline{i} = -\frac{Kl}{m} \, \underline{i}$$

5. Piccoli moti: $\quad \underline{\eta} = \underline{q} - \underline{q}_e$

$$\underline{\underline{A}}(\underline{q}_e) \ddot{\underline{\eta}} - \underline{\underline{H}}(\underline{q}_e) \underline{\eta} = \underline{0}$$

dove $\underline{\underline{A}}(\underline{q})$ è la matrice di massa.

Ricordiamo che a vincoli fissi $\quad T = \frac{1}{2} \underline{\underline{A}}(\underline{q}) \dot{\underline{q}} \cdot \dot{\underline{q}}$

Nel nostro caso: $\underline{\underline{A}}(\underline{q}) = \underline{\underline{A}}(\vartheta, x) = \begin{pmatrix} \frac{1}{3} m l^2 & 0 \\ 0 & m \end{pmatrix} = \underline{\underline{A}}(\vartheta_{eq}, x_{eq})$

Scelgo $(\vartheta_{eq}, x_{eq}) = (\vartheta_1, x_1)$

$$\eta_1 = \vartheta - \vartheta_1 = \vartheta + \frac{1}{2} \arcsin\lambda$$

$$\eta_2 = x - x_1 = x - l \sqrt{\frac{1 + \sqrt{1 - \lambda^2}}{2}}$$

$$\underline{\underline{H}}(\vartheta_{eq}, x_{eq}) = \begin{pmatrix} -Kl \, x_{eq} \cos\vartheta_{eq} & -Kl\sin\vartheta_{eq} \\ -Kl\sin\vartheta_{eq} & -K \end{pmatrix}$$

$$\begin{cases} \frac{1}{3} m l^2 \ddot{\eta}_1 + Kl x_{eq} \cos\vartheta_{eq} \, \eta_1 + Kl\sin\vartheta_{eq} \, \eta_2 = 0 \\ m \ddot{\eta}_2 + Kl\sin\vartheta_{eq} \, \eta_1 + K \eta_2 = 0 \end{cases}$$

quello che ho scritto $\rightarrow$

$$\begin{cases} A_{11} \ddot{\eta}_1 + A_{12} \ddot{\eta}_2 - H_{11} \eta_1 - H_{12} \eta_2 = 0 \\ A_{21} \ddot{\eta}_1 + A_{22} \ddot{\eta}_2 - H_{21} \eta_1 - H_{22} \eta_2 = 0 \end{cases}$$

$$\begin{cases} \frac{1}{3} m \cancel{l^2} \ddot{\eta}_1 + K \cancel{l^2} \frac{1+\sqrt{1-\lambda^2}}{2} \eta_1 - K \cancel{l} \sqrt{\frac{1-\sqrt{1-\lambda^2}}{2}} \eta_2 = 0 \\ m \ddot{\eta}_2 - K l \sqrt{\frac{1-\sqrt{1-\lambda^2}}{2}} \eta_1 + K \eta_2 = 0 \end{cases}$$

$$\begin{cases} \frac{1}{3} m l \ddot{\theta} + k l \frac{1+\sqrt{1-\lambda^2}}{2} \left(\theta + \frac{1}{2} \arcsin \lambda\right) - K \sqrt{\frac{1-\sqrt{1-\lambda^2}}{2}} \left(x - l \sqrt{\frac{1+\sqrt{1-\lambda^2}}{2}}\right) = 0 \\ m \ddot{x} - K l \sqrt{\frac{1-\sqrt{1-\lambda^2}}{2}} \left(\theta + \frac{1}{2} \arcsin \lambda\right) + K \left(x - l \sqrt{\frac{1+\sqrt{1-\lambda^2}}{2}}\right) = 0 \end{cases}$$

Per una volta linearizziamo "a mano".

Dalle equazioni di Lagrange: $\begin{cases} \frac{1}{3} m l^2 \ddot{\theta} + K l x \sin\theta + M = 0 \\ m \ddot{x} + K x - K l \cos\theta = 0 \end{cases}$

Definisco $\begin{cases} \eta_1 = \theta - \theta_{eq} \\ \eta_2 = x - x_{eq} \end{cases}$; $\begin{cases} \theta = \theta_{eq} + \eta_1 \\ x = x_{eq} + \eta_2 \end{cases}$; se $\underline{\eta} = \varepsilon \underline{\varrho}$ $\begin{cases} \theta = \theta_{eq} + \varepsilon \varrho_1 \\ x = x_{eq} + \varepsilon \varrho_2 \end{cases}$

$\ddot{\theta} = \ddot{\eta}_1 = \varepsilon \ddot{\varrho}_1$ e $\ddot{x} = \ddot{\eta}_2 = \varepsilon \ddot{\varrho}_2$

Eq. di Lagrange $\begin{cases} \frac{1}{3} m l^2 \varepsilon \ddot{\varrho}_1 + K l (x_{eq} + \varepsilon \varrho_2) \sin(\theta_{eq} + \varepsilon \varrho_1) + M = 0 \\ m \varepsilon \ddot{\varrho}_2 + K (x_{eq} + \varepsilon \varrho_2) - K l \cos(\theta_{eq} + \varepsilon \varrho_1) = 0 \end{cases}$

$$\sin(\theta_{eq} + \varepsilon \varrho_1) = \sin\theta_{eq} \cos(\varepsilon \varrho_1) + \cos\theta_{eq} \sin(\varepsilon \varrho_1) = \sin\theta_{eq} + \cos\theta_{eq} \varepsilon \varrho_1 + O(\varepsilon^2)$$
$$\cos(\theta_{eq} + \varepsilon \varrho_1) = \cos\theta_{eq} \cos(\varepsilon \varrho_1) - \sin\theta_{eq} \sin(\varepsilon \varrho_1) = \cos\theta_{eq} - \sin\theta_{eq} \varepsilon \varrho_1 + O(\varepsilon^2)$$

$$\begin{cases} \frac{1}{3} m l^2 \varepsilon \ddot{\varrho}_1 + K l (x_{eq} + \varepsilon \varrho_2)(\sin\theta_{eq} + \cos\theta_{eq} \varepsilon \varrho_1) + M = 0 \\ m \varepsilon \ddot{\varrho}_2 + K (x_{eq} + \varepsilon \varrho_2) - K l \cos\theta_{eq} + K l \sin\theta_{eq} \varepsilon \varrho_1 = 0 \end{cases}$$

All'ordine ε

Trascurando $O(\varepsilon^2)$

$$\begin{cases} \frac{1}{3} m l^2 \varepsilon \ddot{\varrho}_1 + K l x_{eq} \sin\theta_{eq} + K l x_{eq} \cos\theta_{eq} \varepsilon \varrho_1 + K l \sin\theta_{eq} \varepsilon \varrho_2 + M = 0 \\ m \varepsilon \ddot{\varrho}_2 - K l \cos\theta_{eq} + K l \sin\theta_{eq} \varepsilon \varrho_1 + K \varepsilon \varrho_2 + K x_{eq} = 0 \end{cases}$$

Notare che:

$$K l x_{eq} \sin\theta_{eq} + M = K l^2 \cos\theta_{eq} \sin\theta_{eq} + M = \frac{1}{2} K l^2 \sin(2\theta_{eq}) + M = -\frac{1}{2} \lambda K l^2 + M = 0$$
$$-K l \cos\theta_{eq} + K x_{eq} = 0 \qquad (x_{eq} = l \cos\theta_{eq})$$

$$\begin{cases} \frac{1}{3} m l^2 \ddot{\eta}_1 + K l x_{eq} \cos\theta_{eq} \eta_1 + K l \sin\theta_{eq} \eta_2 = 0 \\ m \ddot{\eta}_2 + K l \sin\theta_{eq} \eta_1 + K \eta_2 = 0 \end{cases}$$

Risostituisco $\underline{\eta} = \varepsilon \underline{\varrho}$

6) 1^a eq. card.

$$\underline{\phi}_G - mg\underline{j} + K(O-A) = \underline{0}$$

$$\phi_{Gx}\underline{i} + \phi_{Gy}\underline{j} - mg\underline{j} + K(\underbrace{-x_{eq} + l\cos\theta_{eq}}_{=0})\underline{i} + Kl\sin\theta_{eq}\underline{j} = \underline{0}$$

$$\underline{\phi}_G = \phi_{Gy}\underline{j} = mg\underline{j} - Kl\sin\theta_{eq}\underline{j} \qquad \begin{cases}\phi_{Gx} = 0 \\ \phi_{Gy} = mg - Kl\sin\theta_{eq}\end{cases}$$

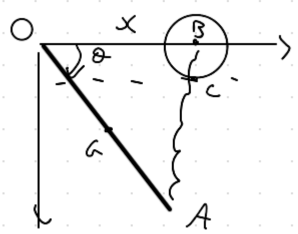
Se $(\theta_{eq}, x_{eq}) = (\theta_1, x_1)$

$$\underline{\phi}_G = mg\underline{j} + Kl\sqrt{\frac{1-\sqrt{1-\lambda^2}}{2}}\,\underline{j} = \left(0, mg + Kl\sqrt{\frac{1-\sqrt{1-\lambda^2}}{2}}\right)$$

Non ho bisogno della 2^a eq. cardinale

$A = (2l\cos\theta, 2l\sin\theta) \xrightarrow{(\theta_1,x_1)} (0, 2l)$

$B = (x, 0) \xrightarrow{(\theta_1,x_1)} (0, 0)$



$K = \frac{mg}{8l} \Rightarrow \lambda = \frac{mg}{4Kl} = \frac{mg}{4l}\frac{8l}{ml} = 2 > 1$

$(\theta_1, x_1) = (\frac{\pi}{2}, 0)$ stabile se $\lambda > 1$

d) $\underline{\phi}_O + mg\underline{j} + K(B-A) = \underline{0}$ $\qquad 1^a$ eq. cardinale per l'asta

$$\phi_{Ox}\underline{i} + \phi_{Oy}\underline{j} + mg\underline{j} - 2Kl\underline{j} = \underline{0}$$

$$\phi_{Ox}\underline{i} + \phi_{Oy}\underline{j} + mg\underline{j} - 2\frac{mg}{8l}l\underline{j} = \underline{0} \iff \phi_{Ox}\underline{i} + \phi_{Oy}\underline{j} + \frac{3}{4}mg\underline{j} = \underline{0}$$

$$\underline{\phi}_O = (0, -\tfrac{3}{4}mg)$$

$\underline{\phi}_C + mg\underline{j} + K(A-B) = \underline{0}$ $\qquad 1^a$ eq. cardinale per il disco

$$\phi_{Cx}\underline{i} + \phi_{Cy}\underline{j} + mg\underline{j} + \frac{mg}{4}\underline{j} = \underline{0} \Rightarrow \underline{\phi}_C = (0, -\tfrac{5}{4}mg)$$