CHAPTER 8

Lagrangian Mechanics

There is a very effective and elegant way for obtaining equations of motion for mechanical systems under certain conditions; it consists of a variational method, whose application has led to countless advances in science. The method is due to Joseph-Louis Lagrange, born Turin in 1736. Before we introduce Lagrange's mechanics, we need some preliminary notions. Let us begin introducing a reference frame that translates in such a way that a mass point remains at rest in its origin. In such a frame, the point perceives the effect of a drag force:

$$\vec{F}_\tau = -m\vec{a} \tag{8.1}$$

that is called *force of inertia* and, including the effect of possible constraints, the fundamental law of relative dynamics can be written as:

$$\underbrace{\left(\vec{F} + \vec{F}_\tau\right)}_{\substack{sum\ of \\ active \\ forces}} + \underbrace{\vec{\Phi}}_{\substack{sum\ of \\ constraining \\ forces}} = \underbrace{\left(\vec{F} - m\vec{a}\right)}_{\substack{lost \\ force}} + \vec{\Phi} = \mathbf{0} \tag{8.2}$$

which is to say that constraints balance *lost forces* at all times. This is generalized by the following:

D'Alembert principle: *given a system of mass points, one switches from equilibrium equations to dynamical equations by adding the forces of inertia.*

Suppose we have holonomic, ideal bilateral constraints, and generalized coordinates $(q_1, \cdots, q_N)$, so that:

$$\delta P_i = \sum_{k=1}^{N} \frac{\partial (P_i - O)}{\partial q_k} \delta q_k$$

Given a system of *n* point masses subjected to ideal bilateral constraints, which do no virtual work, and using $\vec{F}_i - m_i\vec{a}_i = -\vec{\Phi}_i$, for all the points, we can write:

$$\sum_{i=1}^{n} \left(\vec{F}_i - m_i\vec{a}_i\right) \cdot \delta P_i = 0$$

which implies:

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \left(\vec{F}_i - m_i\vec{a}_i\right) \cdot \delta P_i = \sum_{i=1}^{n} \left(\vec{F}_i - m_i\vec{a}_i\right) \cdot \sum_{k=1}^{N} \frac{\partial (P_i - O)}{\partial q_k} \delta q_k \\ &= \sum_{k=1}^{N} \left[\sum_{i=1}^{n} \left(\vec{F}_i - m_i\vec{a}_i\right) \cdot \frac{\partial (P_i - O)}{\partial q_k} \right] \delta q_k = \sum_{k=1}^{N} (Q_k - \tau_k) \delta q_k \end{aligned} \tag{8.3}$$

for all virtual displacements $(\delta q_1, \cdots, \delta q_N)$. Here, we have introduced the *Lagrangian components* of the active forces, and the Lagrangian components of the opposite of the forces of inertia:

$$\underbrace{Q_k = \sum_{i=1}^{n} \vec{F}_i \cdot \frac{\partial (P_i - O)}{\partial q_k}}_{\substack{\textit{Lagrangian component} \\ \textit{of active forces}}}; \qquad \underbrace{\tau_k = \sum_{i=1}^{n} m_i\vec{a}_i \cdot \frac{\partial (P_i - O)}{\partial q_k}}_{\substack{\textit{Lagrangian component of the} \\ \textit{opposite of the forces of inertia}}} \tag{8.4}$$

This is most useful because the δq_k are independent by definition, while the δP_i need not be.

THEOREM 8.1: *The equations of dynamics are equivalent to*

$$Q_k = \tau_k, \;\; k = 1, \cdots, N \tag{8.5}$$

Proof: if $Q_k = \tau_k \;\forall k$, each addend in Eq.(8.3) vanishes. Then the equations of dynamics are satisfied. This means that we have a sufficient condition. Take now $\delta q_j \neq 0$ for one j and $\delta q_k = 0$ for $k \neq j$; then must satisfy $(Q_j - \tau_j)\delta q_j = 0$, which implies $Q_j = \tau_j$. This can be done for all j, hence Eq.(8.5) is also a necessary condition.

Note: Q_k contains only active forces! The constraints are hidden within the Lagrangian coordinates, which enormously simplifies the calculation of the motion.

THEOREM 8.2: *Let T be the kinetic energy of a holonomic system of generalized coordinates* $(q_1, \cdots, q_N)$. *Then, the following holds:*

$$\boxed{\tau_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}} \quad k = 1, \cdots, N \tag{8.6}$$

where the right hand side is called Lagrange binomial.

Proof: Starting from the definition, we have

$$\tau_k = \sum_{i=1}^{n} m_i \vec{a}_i \cdot \frac{\partial (P_i - O)}{\partial q_k} = \sum_{i=1}^{n} m_i \frac{d\vec{v}_i}{dt} \cdot \frac{\partial (P_i - O)}{\partial q_k}$$

$$= \sum_{i=1}^{n} m_i \frac{d}{dt}\left(\vec{v}_i \cdot \frac{\partial (P_i - O)}{\partial q_k}\right) - \sum_{i=1}^{n} m_i \vec{v}_i \cdot \frac{d}{dt}\frac{\partial (P_i - O)}{\partial q_k} \tag{8.7}$$

Considering that $P_i(t) = P_i(q_1(t), \cdots, q_N(t); t)$, we have:

$$\vec{v}_i = \frac{d(P_i - O)}{dt} = \sum_{j=1}^{N} \frac{\partial (P_i - O)}{\partial q_j}\dot{q}_j + \frac{\partial (P_i - O)}{\partial t} \;, \qquad i = 1, \ldots n \tag{8.8}$$

which means that $\vec{v}_i$ depends linearly on the $\dot{q}_k$ and, differentiating Eq.(8.8), that:

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_k} = \frac{\partial (P_i - O)}{\partial q_k}$$

Moreover, differentiating with respect to q_k yields:

$$\frac{\partial \vec{v}_i}{\partial q_k} = \sum_{j=1}^{N} \frac{\partial^2 (P_i - O)}{\partial q_k \partial q_j}\dot{q}_j + \frac{\partial^2 (P_i - O)}{\partial q_k \partial t} = \frac{d}{dt}\frac{\partial (P_i - O)}{\partial q_k} \tag{8.9}$$

if the order of derivatives can be exchanged. Then, substituting in Eq.(8.7), we obtain:

$$\tau_k = \sum_{i=1}^{n} m_i \frac{d}{dt}\left(\vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_k}\right) - \sum_{i=1}^{n} m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_k} = \frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^{n} m_i \frac{\partial {v_i}^2}{\partial \dot{q}_k}\right) - \frac{1}{2}\sum_{i=1}^{n} m_i \frac{\partial {v_i}^2}{\partial q_k}$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}$$

where ${v_i}^2 = \vec{v}_i \cdot \vec{v}_i$. This completes the proof.

Substituting in Eq.(8.5), this yields:

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k} \quad k = 1, \cdots, N \tag{8.10}$$

which are known as the **Lagrange Equations**. The nature of the forces does not matter, here, but constraints must be ideal ($\delta L^{(v)} \geq 0$) and bilateral (all virtual displacements and velocities reversible). For holonomic systems, the kinetic energy may be written as:

$$T = \underbrace{\frac{1}{2}\sum_{j,k=1}^{N} a_{jk}\dot{q}_j\dot{q}_k}_{T_2} + \underbrace{\sum_{k=1}^{N} b_k\dot{q}_k}_{T_1} + \underbrace{C}_{T_0} \tag{8.11}$$

Indeed, $\forall P_i,\ i = 1, \cdots, n,$ Eq.(8.8) holds, and we have to compute v_i^2. Some simple calculation shows that Eq.(8.11) holds setting:

$$a_{hk} = \sum_{i=1}^{n} m_i \frac{\partial (P_i - O)}{\partial q_k} \cdot \frac{\partial (P_i - O)}{\partial q_h}; \tag{8.12}$$

$$b_k = \frac{1}{2}\sum_{i=1}^{n} m_i \frac{\partial (P_i - O)}{\partial q_k} \cdot \frac{\partial (P_i - O)}{\partial t}; \tag{8.13}$$

$$C = \frac{1}{2}\sum_{i=1}^{n} m_i \frac{\partial (P_i - O)}{\partial t} \cdot \frac{\partial (P_i - O)}{\partial t} \tag{8.14}$$

The $N \times N$ matrix $A = (a_{hk})_{N\times N}$, the $N \times 1$ vector $b = (b_k)_1^N$, and the scalar C depend on the coordinates, in general. Equation (8.11) may also be written as:

$$T = \tfrac{1}{2}\dot{q}^T A\dot{q} + b^T\dot{q} + c \tag{8.15}$$

which in general depends on both generalized coordinates and velocities. For fixed constraints, one finds $T_1 = T_0 = 0$, because the partial time derivative of $(P_i - O)$ vanishes, since $P_i = P_i(q_1, \dots, q_N)$, and it does not depend explicitly on time.

THEOREM 8.3: *For a holonomic system under ideal bilateral constraints, the initial value problem*

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_k}\right) - \dfrac{\partial T}{\partial q_k} = Q_k(q, \dot{q}, t) \\ q_k(t_0) = q_{k0} \end{cases} \quad k = 1, \cdots, N \tag{8.16}$$

has a unique solution describing the dynamics in the interval $[t_0, t_1]$, *for some* $t_1 > t_0$, *if all* Q_k *are Lipschitz in* q *and* $\dot{q}$.

Proof: first, note that Eq.(8.11) implies:

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_{j=1}^{N} a_{jk}\dot{q}_j, \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_{j=1}^{N}\left[a_{jk}\ddot{q}_j + \frac{da_{jk}}{dt}\dot{q}_j\right]$$

Then, it suffices to write the problem as:

$$A\ddot{q} = F(q, \dot{q}, t) + Q(q, \dot{q}, t) \quad \text{with} \quad F_h = \tfrac{\partial T}{\partial q_h} - \tfrac{da_{jk}}{dt}\dot{q}_j$$

where A is invertible and can be diagonalized because it is symmetric and positive definite: its eigenvalues are all strictly positive. This comes from the fact that the scalar product (of velocities) commutes, and from the fact that the kinetic energy is always positive, except when ***all*** velocities vanish, in which case it vanishes too. Then, one can write:

$$\ddot{q} = A^{-1}F + A^{-1}Q$$

Suppose active forces are conservative, hence they only depend on configurations $(q_1, \cdots, q_N)$, not on velocities, and the generalized forces derive from a potential (not necessarily potential energy): $Q_k = \frac{\partial U}{\partial q_k}$. Substituting in Eq.(8.10), this yields:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} - \frac{\partial U}{\partial q_k} = 0$$

which can also be written as:

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_k} + \frac{\partial U}{\partial \dot{q}_k}\right] - \frac{\partial}{\partial q_k}(T + U) = \frac{d}{dt}\left(\frac{\partial (T+U)}{\partial \dot{q}_k}\right) - \frac{\partial (T+U)}{\partial q_k}$$

Therefore, introducing the *Lagrangian* of the system as:

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) + U(q, t) \tag{8.17}$$

the Lagrange equations Eq.(8.10) take the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0; \quad k = 1, \cdots, N \tag{8.18}$$

For non-conservative active forces, the Lagrangian formalism still works, and one writes:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k^{(nc)}$$

where $Q_k^{(nc)}$ is the Lagrangian component of the active non-conservative forces.

Remark: the Lagrangian components of the forces (active or not) are not necessarily forces. The conservative forces are derivatives of the potential with respect to the cartesian coordinates.

Ex1. Consider an inextensible rope of length L_{rope} and negligible mass, which does not slip on the disk of center F. As coordinates, let us take y_C, y_P. Note that $\hat{\imath}_3 = \hat{\imath}_1 \wedge \hat{\imath}_2$ enters this page sheet, so the angular velocities can be written as $-\dot{\vartheta}\hat{\imath}_3$ and $-\dot{\varphi}\hat{\imath}_3$. The travelling disk has radius *r*, the other disk has radius *R*. Because the length of the rope is fixed, we have:

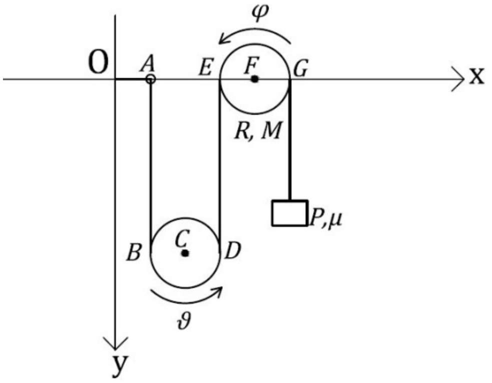


Figure 8.1. *pulleys and an inextensible rope*

$$|\overrightarrow{AB}| + |\overrightarrow{DB}| + |\overrightarrow{DE}| + |\overrightarrow{EG}| + |\overrightarrow{GP}| = const$$

i.e.

$$y_B + r\pi + y_B + R\pi + y_P = L_{rope}$$

Then,

$$2\dot{y}_B + \dot{y}_P = 0 \tag{8.19}$$

Note, if B is seen as the geometric contact of rope and pulley, it belongs to neither of them. The point B of the rope, on the other hand, remains at fixed distance from A along the rope, which implies $\vec{v}_B = \vec{v}_A = \mathbf{0}$. Similarly, one has:

$$\vec{v}_E = \vec{v}_D, \ \vec{v}_G = \vec{v}_P = \dot{y}_P\hat{\imath}_2$$

For the geometric point of contact at B, we have: $(B \,\dot{-}\, O) = \dot{y}_B\hat{\imath}_2$

while the point B belonging to the disk is at rest because it does not slip on the rope and the rope is at rest there. Also, we have:

$$\dot{y}_B = \dot{y}_C = \dot{y}_D^{contact} \neq \dot{y}_D^{disk} = -2\, r\, \dot{\vartheta}$$

where we stressed that also D can be seen both as a geometric contact point and as a point belonging to the disk. Observe that

$$(C - A) = r\hat{\imath}_1 + y_B\hat{\imath}_1 \quad \text{which implies} \quad \vec{v}_c = \dot{y}_B\hat{\imath}_2$$

If there is enough friction on the disk of center F, the rope does not slide on it, then:

$$\vec{v}_G^{rope} = \vec{v}_P = \dot{y}_P\hat{\imath}_2$$

The fundamental equation of kinematics applied to G gives

$$\vec{v}_G^{disk} = \underbrace{\vec{v}_F}_{\mathbf{0}} + \underbrace{\vec{\omega}}_{-\dot{\varphi}\hat{\imath}_3} \wedge \underbrace{(G - F)}_{R\hat{\imath}_1} = -R\dot{\varphi}\hat{\imath}_2$$

Thus, provided the two objects at G do not slip, we have:

$$R\dot{\varphi} = \dot{y}_P = -2\dot{y}_B$$

because of Eq.(8.19). Analogously, we have $r\dot{\vartheta} = -\dot{y}_C$, since the rope does not slip on the disk, and the height of B and C is the same. This means that $-\dot{y}_C = r\dot{\vartheta} = -\frac{1}{2}R\dot{\varphi} = \frac{1}{2}\dot{y}_P$, and there is only one free variable. Choose, *e.g.* the contact point B, and recall that

$$T = \frac{1}{2}m{v_G}^2 + \frac{1}{2}I_G\omega^2 \tag{8.20}$$

for a rigid body. This implies:

$$\begin{aligned} T &= \frac{1}{2}m{v_C}^2 + \frac{1}{4}mr^2\dot{\vartheta}^2 + \frac{1}{4}MR^2\dot{\varphi}^2 + \frac{1}{2}\mu{v_P}^2 \\ &= \frac{1}{2}m{\dot{y}_B}^2 + \frac{1}{4}mr^2\left(\frac{\dot{y}_B}{r}\right)^2 + \frac{1}{4}MR^2\left(\frac{2\dot{y}_B}{R}\right)^2 + \frac{1}{2}\mu(2\dot{y}_B)^2 \\ &= \frac{1}{2}m{\dot{y}_B}^2 + \frac{1}{4}m{\dot{y}_B}^2 + M{\dot{y}_B}^2 + 2\mu{\dot{y}_B}^2 = \left(\frac{3}{4}m + M + 2\mu\right){\dot{y}_B}^2 \end{aligned}$$

At the same time

$$U = mgy_C + Mgy_F + \mu gy_P$$

where $y_C = y_B$. Furthermore, $y_P = L_{rope} - \pi r - \pi R - 2y_B = const - 2y_B$, hence

$$U = mgy_B - 2\mu gy_B + const$$

Therefore, we can now write the Lagrangian as:

$$L = T + U = \left(\frac{3}{4}m + M + 2\mu\right){\dot{y}_B}^2 + (m - 2\mu)gy_B + const$$

where the constant is irrelevant, since only derivatives of L are considered. Now, set $q = y_B$ and derive the equations of motion:

$$\frac{\partial L}{\partial \dot{q}} = 2\left(\frac{3}{4}m + M + 2\mu\right)\dot{q} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \left(\frac{3}{2}m + 2M + 4\mu\right)\ddot{q}$$

$$\frac{\partial L}{\partial q} = (m - 2\mu)g$$

hence:

$$\left(\frac{3}{2}m + 2M + 4\mu\right)\ddot{q} - (m - 2\mu)g = 0 \quad \text{or} \quad \ddot{q} = \frac{2(m - 2\mu)g}{3m + 4M + 8\mu}$$

$$\dot{q} = \frac{2(m - 2\mu)g}{3m + 4M + 8\mu}t + C_1$$

$$q = \frac{(m-2\mu)g}{3m+4M+8\mu}t^2 + C_1 t + C_2$$

where the constants can be determined by the initial conditions $q(0)$ and $\dot{q}(0)$.

LAGRANGE PRIME INTEGRALS: in cases in which the Lagrangian L does not explicitly depend on one of the q_k, *i.e.* $\partial L/\partial q_k = 0$, we have:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \quad i.e. \quad \frac{\partial L}{\partial \dot{q}_k} = const$$

The conserved quantity

$$\boxed{p_k = \frac{\partial L}{\partial \dot{q}_k}} \tag{8.21}$$

called *kinematic moment*, is called *cyclic* or *ignorable.*

Ex2: take a point *P* of mass *m* in a potential that does not depend on *x*: $U(P) = U(y,z)$. In this case, we have:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(y,z), \qquad \text{hence} \quad \frac{\partial L}{\partial x} = 0 \Rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} = const$$

Indeed

$$F_x = \frac{\partial U}{\partial x} = 0.$$

Ex3: Rigid body *B* with center of mass *G*. Let *O* be the origin of the fixed reference frame:

$$(G - O) = (x_G, y_G, z_G)\,.$$

Let $(\vartheta, \psi, \varphi)$ be the Euler angles, and $(\hat{e}_{G1}, \hat{e}_{G2}, \hat{e}_{G3})$ be a rest frame for *B*, precisely a central inertia frame. Then we can write

$$\vec{\omega} = \left(\dot{\vartheta}\cos\varphi + \dot{\psi}\sin\vartheta\sin\varphi\right)\hat{e}_{G1} - \left(\dot{\vartheta}\sin\varphi - \dot{\psi}\sin\vartheta\cos\varphi\right)\hat{e}_{G2} + \left(\dot{\varphi} + \dot{\psi}\cos\vartheta\right)\hat{e}_{G3}$$

and

$$\begin{aligned} T = {} & \frac{1}{2}m(\dot{x}_G + \dot{y}_G + \dot{z}_G) \\ & + \frac{1}{2}\Big[I_{G1}\left(\dot{\vartheta}\cos\varphi + \dot{\psi}\sin\vartheta\sin\varphi\right)^2 + I_{G2}\left(\dot{\vartheta}\sin\varphi - \dot{\psi}\sin\vartheta\cos\varphi\right)^2 \\ & + I_{G3}\left(\dot{\varphi} + \dot{\psi}\cos\vartheta\right)^2\Big] \end{aligned}$$

Provided U does not depend on $\dot{\psi}$, we have:

$$\begin{aligned} p_\psi = \frac{\partial L}{\partial \dot{\psi}} = \frac{\partial T}{\partial \dot{\psi}} = {} & \frac{1}{2}I_{G1}2\left(\dot{\vartheta}\cos\varphi + \dot{\psi}\sin\vartheta\sin\varphi\right)\sin\vartheta\sin\varphi \\ & + \frac{1}{2}I_{G2}2\left(\dot{\vartheta}\sin\varphi - \dot{\psi}\sin\vartheta\cos\varphi\right)(-\sin\vartheta\cos\varphi) \\ & + \frac{1}{2}I_{G3}2\left(\dot{\varphi} - \dot{\psi}\cos\vartheta\right)\cos\vartheta = \vec{K}_G \cdot \hat{\imath}_3 \end{aligned}$$

as it appears after some tedious algebra, while for φ we have:

$$p_\varphi = \frac{\partial T}{\partial \dot{\varphi}} = \vec{K}_G \cdot \hat{e}_3$$

DEFINITION 8.1: Given a holonomic system, the function

$$H(q, \dot{q}, t) = \sum_{k=1}^{N} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \tag{8.22}$$

is called *Hamiltonian* of the system.

Proposition 8.1: *if forces are conservative, the Hamiltonian and the Lagrangian of a given system obey:*

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \tag{8.23}$$

Proof: observe that

$$\frac{dH}{dt} = \sum_{k=1}^{N} \left[\ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \dot{q}_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \right] - \sum_{k=1}^{N} \left[\frac{\partial L}{\partial q_k} \dot{q}_k + \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k \right] - \frac{\partial L}{\partial t}$$

$$= \sum_{k=1}^{N} \dot{q}_k \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} \right] - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}$$

Indeed, we do not admit non-conservative forces here, hence the square brackets in the second line of the above equation all vanish, as stated by Eq.(8.18). Therefore, the Hamiltonian H is preserved in time if L does not depend explicitly on time, which is the case if constraints and active forces, that are required to be conservative, do not depend on time.

THEOREM 8.4: *For time independent constraints and conservative forces one may write:*

$$H = T - U = T + V = E$$

Proof: recall first that a function f is called homogeneous of degree α if

$$f(\lambda x_1, \cdots, \lambda x_m) = \lambda^\alpha f(x_1, \cdots, x_m), \quad \forall \lambda \in \mathbb{R}^+ \tag{8.24}$$

Such functions obey the Euler theorem, that states:

$$\sum_{i=1}^{m} x_i \frac{\partial f}{\partial x_i} = \alpha f \tag{8.25}$$

Then, writing the kinetic energy as $T = T_2 + T_1 + T_0$, cf. Eq.(8.11) or Eq.(8.15), we find that T_2 is homogeneous of degree 2, T_1 is of degree 1 and T_0 of degree 0. Therefore, we can write:

$$\sum_{k=1}^{N} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} = \sum_{k=1}^{N} \dot{q}_k \frac{\partial (T+U)}{\partial \dot{q}_k} = \sum_{k=1}^{N} \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = \sum_{k=1}^{N} \dot{q}_k \frac{\partial}{\partial \dot{q}_k} (T_2 + T_1 + T_0) = 2T_2 + T_1 + 0$$

where the second equality derives from the fact that U does not depend on any $\dot{q}_k$. Consequently,

$$H = \sum_{k=1}^{N} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = 2T_2 + T_1 - (T_2 + T_1 + T_0 + U) = T_2 - T_1 + U$$

However, for fixed constraints, $T_1 = T_0 = 0$, we obtain $H = T - U$.

It appears that equations of motion are generally quite complex and can only be solved numerically. Nevertheless, there are limiting situations, in which solutions can be approximated by simple harmonic oscillations. This is the case of small oscillations about a stable equilibrium configuration. This important subject is investigated below.

Theorem 8.5: *Consider a holonomic, conservative system with time independent constraints. Let U be continuous in q and let q_0 be an equilibrium. If $U(q_0)$ is an isolated local maximum of U, then q_0 is Lyapunov stable.*

DEFINITION 8.2: the configuration $P_0 = (P_{01}, \cdots, P_{0n})$ is called Lyapunov stable if $\forall \varepsilon, \varepsilon' > 0$, $\exists \delta, \delta' > 0$ such that:

$$\begin{cases} |P_{0i} - P_i(t_0)| < \delta \quad \forall i \\ |\vec{v}_i(t_0)| < \delta' \quad \forall i \end{cases} \quad \text{implies} \quad \begin{cases} |P_{0i} - P_i(t)| < \varepsilon \quad \forall i \\ \lceil \vec{v}_i(t) \rceil < \varepsilon' \quad \forall i \end{cases} \quad \forall t \geq t_0 \tag{8.26}$$

Consider a holonomic system under time independent constraints. Suppose there is only one degree of freedom and let the active forces be conservative. In the case q^0 is an equilibrium, the following holds

$$\left.\frac{\partial U}{\partial q}\right|_{q^0} = 0$$

If q^0 is Lyapunov stable, starting close to q^0 we always remain close q^0, hence we can write

$$q(t) = q^0 + \varepsilon \eta(t) \tag{8.27}$$

where ε is small and η is bounded. In this case,

$$T(q, \dot{q}) = \frac{1}{2} a(q) \dot{q}^2$$

hence

$$L(q, \dot{q}) = \frac{1}{2} a(q) \dot{q}^2 + U(q)$$

Then, we can write

$$\frac{\partial L}{\partial \dot{q}} = a(q)\dot{q}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial a}{\partial q} \dot{q}\, \dot{q} + a(q)\ddot{q} = \frac{\partial a}{\partial q} \dot{q}^2 + a(q)\ddot{q}$$

and

$$\frac{\partial L}{\partial q} = \frac{1}{2} \frac{\partial a}{\partial q} \dot{q}^2 + \frac{\partial U}{\partial q}$$

which implies

$$a(q)\ddot{q} + \frac{1}{2} a'(q) \dot{q}^2 - U'(q) = 0$$

where the symbol $'$ denotes derivative with respect to the coordinate q. We can also write:

$$\dot{q}(t) = \frac{d}{dt} q^0 + \varepsilon \dot{\eta}(t) = \varepsilon \dot{\eta}(t) \quad \text{which implies} \quad \boxed{\ddot{q} = \varepsilon \ddot{\eta}}$$

And then

$$a\big(q^0 + \varepsilon \eta(t)\big) \varepsilon \ddot{\eta} - \frac{1}{2} a'\big(q^0 + \varepsilon \eta(t)\big) \varepsilon^2 \dot{\eta}^2 - U'\big(q^0 + \varepsilon \eta(t)\big) = 0 \tag{8.28}$$

So far, calculations are exact. Now, observe that $dq = d(q^0 + \varepsilon \eta) = d(\varepsilon \eta)$ so that

$$\left.\frac{\partial a}{\partial (\varepsilon \eta)}\right|_{\varepsilon \eta = 0} = \left.\frac{\partial a}{\partial q}\right|_{q^0} = a'(q^0)$$

Hence, expanding as a Taylor series in the variable $\varepsilon \eta$ about $\varepsilon \eta = 0$, we obtain:

$$a(q^0 + \varepsilon \eta) \varepsilon \ddot{\eta} = [a(q^0) + a'(q^0) \varepsilon \eta + O(\varepsilon^2)] \varepsilon \ddot{\eta} = \varepsilon a(q^0) \ddot{\eta} + \varepsilon^2 a'(q^0) \eta \ddot{\eta} + o(\varepsilon^2) \tag{8.29}$$

Analogously, expanding a' and U', we obtain:

$$\frac{1}{2}a'(q)\dot{q}^2 = \frac{1}{2}a'(q^0+\varepsilon\eta)\varepsilon^2\dot{\eta}^2 = \frac{1}{2}[a'(q^0)+a''(q^0)\varepsilon\eta]\varepsilon^2\dot{\eta}^2 = \frac{1}{2}a'(q^0)\varepsilon^2\dot{\eta}^2 + o(\varepsilon^2) \qquad (8.30)$$

and

$$U'\big(q^0+\varepsilon\eta(t)\big) = U'(q^0) + U''(q^0)\,\varepsilon\eta + \frac{1}{2}U'''(q^0)\varepsilon^2\eta^2 + o(\varepsilon^2) \qquad (8.31)$$

where $U'(q^0)=0$ because q^0 is an equilibrium configuration. Therefore, to first order in ε we have:

$$\varepsilon a(q^0)\ddot{\eta} - 0 - U''(q^0)\varepsilon\eta = 0$$

This then implies:

$$\ddot{\eta} = \frac{U''(q^0)}{a(q^0)}\eta \qquad (8.32)$$

Given that $a(q^0)>0,$ because $T\geq 0$, and that $U''(q^0)<0$, since q^0 is assumed to be stable, we have oscillations with frequency

$$\omega = \sqrt{-\frac{U''(q^0)}{a(q^0)}}$$

and we can write:

$$\eta(t) = \eta(0)\cos\omega t + \frac{1}{\omega}\dot{\eta}(0)\sin\omega t \qquad (8.33)$$

Had we had $U''(q_0)=0$, we would have obtained:

$$\eta(t) = \eta(0) + \dot{\eta}(0)t$$

which corresponds to an unstable situation, while $U''(q_0)>0$ implies

$$\eta(t) = \eta(0)\cosh\omega t + \frac{1}{\omega}\dot{\eta}(0)\sinh\omega t$$

which is a strongly unstable situation. In the case of Eq.(8.33), one speaks of <u>*small oscillations*</u> because the motion takes the form:

$$q(t) = q^0 + \varepsilon\eta(t); \quad \dot{q}(t) = \varepsilon\dot{\eta}(t) \quad \text{with small } \varepsilon$$

If we have more than one degree of freedom, we write:

$$q_k(t) = q_k^0 + \varepsilon\eta_k(t), \quad k = 1,\cdots,N \qquad (8.34)$$

For time independent constraints, we have:

$$T = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(q)\dot{q}_i\dot{q}_j = \frac{\varepsilon^2}{2}\sum_{i,j=1}^{N} a_{ij}(q^0+\varepsilon\eta)\dot{\eta}_i\dot{\eta}_j = \frac{\varepsilon^2}{2}\dot{\eta}^T A(q^0+\varepsilon\eta)\dot{\eta} \qquad (8.35)$$

because $\dot{q}_k = \dot{q}_k^0 + \varepsilon\dot{\eta}_k = \varepsilon\dot{\eta}_k$. Analogously to the case N=1, observe that:

$$\left.\frac{\partial U}{\partial(\varepsilon\eta_k)}\right|_{\varepsilon\eta_k=0} = \left.\frac{\partial U}{\partial q_k}\right|_{q^0} \quad \text{and} \quad \left.\frac{\partial a_{ij}}{\partial(\varepsilon\eta_k)}\right|_{\varepsilon\eta_k=0} = \left.\frac{\partial a_{ij}}{\partial q_k}\right|_{q^0} \quad \text{for} \quad k = 1,\ldots,N$$

Then, considering that the first derivatives of U vanish in the equilibrium point, we can write:

$$U(q^0+\varepsilon\eta) = U\,(q^0) + \sum_{j=1}^{N}\left.\frac{\partial U}{\partial q_j}\right|_{q^0}\varepsilon\eta_j + \frac{1}{2}\sum_{i,j=1}^{N}\left.\frac{\partial^2 U}{\partial q_i\partial q_j}\right|_{q^0}\varepsilon^2\eta_i\eta_j + o(\varepsilon^2)$$

$$= U\,(q^0) + \frac{\varepsilon^2}{2}\sum_{i,j=1}^{N} b_{ij}^0\eta_i\eta_j + o(\varepsilon^2) = U\,(q^0) + \frac{\varepsilon^2}{2}\eta^T B\eta + o(\varepsilon^2) \qquad (8.36)$$

where we have introduced the Hessian matrix:

$$B = (b_{ij}^0) = \left(\frac{\partial^2 U}{\partial q_i \partial q_j}\bigg|_{q^0} \right) \tag{8.37}$$

For the $a_{ij}(q)$ that appears in the definition of T, Eq.(8.11), we have:

$$a_{ij}(q^0 + \varepsilon\eta) = a_{ij}(q^0) + \sum_{k=1}^{N} \frac{\partial a_{ij}}{\partial q_k}\bigg|_{q^0} \varepsilon\eta_k + o(\varepsilon) = a_{ij}(q^0) + \sum_{k=1}^{N} \frac{\partial a_{ij}}{\partial q_k}\bigg|_{q^0} \varepsilon\eta_k + o(\varepsilon).$$

This implies:

$$\frac{\varepsilon^2}{2} \sum_{i,j=1}^{N} a_{ij}(q^0 + \varepsilon\eta)\dot{\eta}_i\dot{\eta}_j = \frac{\varepsilon^2}{2} \sum_{i,j=1}^{N} \left[a_{ij}(q^0) + \sum_{k=1}^{N} \frac{\partial a_{ij}}{\partial q_k}\bigg|_{q^0} \varepsilon\eta_k + o(\varepsilon) \right] \dot{\eta}_i\dot{\eta}_j$$

Therefore, the Lagrangian looks like:

$$L(\eta, \dot{\eta}) = \frac{\varepsilon^2}{2}\dot{\eta}^T A(q^0)\dot{\eta} + \frac{\varepsilon^2}{2}\eta^T B\eta + o(\varepsilon^2) \tag{8.38}$$

where we neglected the irrelevant constant $U\,(q^0)$. For the Lagrange equations, neglecting the $o(\varepsilon^2)$ term, we now obtain:

$$\frac{\partial L}{\partial \dot{\eta}_k} = \frac{\varepsilon^2}{2}\left[\sum_{j=1}^{N} a_{kj}(q^0)\dot{\eta}_j + \sum_{i=1}^{N} a_{ik}(q^0)\dot{\eta}_i \right] = \varepsilon^2 \sum_{j=1}^{N} a_{jk}(q^0)\dot{\eta}_j \tag{8.39}$$

because $a_{kj} = a_{jk}$, and then:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_k}\right) = \varepsilon^2 \sum_{j=1}^{N} a_{jk}(q^0)\ddot{\eta}_j \;; \qquad \frac{\partial L}{\partial \eta_k} = \frac{\varepsilon^2}{2} \sum_{i,j=1}^{N} \frac{\partial a_{ij}}{\partial \eta_k}\dot{\eta}_i\dot{\eta}_j + \frac{\partial U}{\partial \eta_k}$$

Therefore, the Lagrange equations take the form:

$$\varepsilon^2 \sum_{j=1}^{N} a_{jk}(q^0)\ddot{\eta}_j = \varepsilon^2 \sum_{j=1}^{N} b_{jk}^0 \eta_j$$

i.e.

$$A^0\ddot{\eta} = B^0\eta \tag{8.40}$$

We now look for solutions of form $\eta = \eta_0 e^{\lambda t}$, with $\eta_0 \in \mathbb{R}^N, \lambda \in \mathbb{C}$. Then, we can write:

$$\dot{\eta} = \eta_0 \lambda e^{\lambda t}; \qquad \ddot{\eta} = \eta_0 \lambda^2 e^{\lambda t}\;; \qquad \lambda^2 A^0 \eta_0 e^{\lambda t} = B^0 \eta_0 e^{\lambda t}$$

Hence:

$$(B^0 - \lambda^2 A^0)\eta_0 = \mathbf{0} \tag{8.41}$$

Equation (8.41) has solutions $\neq 0$ if and only if $\det(B^0 - \lambda^2 A^0) = 0$. Theorems of algebra then prove that in our case of symmetric matrices, one has $\lambda^2 \in \mathbb{R}$ and the eigenvectors of B^0 with respect to A^0 are a basis in $\mathbb{R}^3$. Suppose $\lambda^2 < 0$ and η_0 is one corresponding eigenvector. Then

$$\eta_+(t) = \eta_0 e^{i\sqrt{-\lambda^2}t}, \qquad \eta_-(t) = \eta_0 e^{-i\sqrt{-\lambda^2}t}$$

are two independent solutions of Eq.(8.40). Therefore, taking $\omega = \sqrt{-\lambda^2}$

$$\eta(t) = \eta_0(C_1 \cos\omega t + C_2 \sin\omega t) \tag{8.42}$$

is a solution of Eq.(8.40). Such an η is called *normal mode* of oscillation.

In the case in which $\lambda^2 = 0$, one has:

$$\eta(t) = \eta_0(C_1 + C_2 t)$$

which is called linear normal mode. If $\lambda^2 > 0$, one has:

$$\eta(t) = \eta_0\left(C_1 e^{\lambda t} + C_2 e^{-\lambda t}\right)$$

which is called hyperbolic normal mode. So $\lambda^2 > 0$ violates the assumption that oscillations about q^0 remain small, and the approximation turns immediately inaccurate. $\lambda^2 = 0$ is instead uncertain. Repeating the reasoning for the remaining eigenvalues of B^0 with respect to A^0, the general solution is expressed by:

$$\eta(t) = \sum_{k=1}^{N_1} \eta_k^{(s)} \left(A_k \cos\omega_k t + \frac{B_k}{\omega_k}\sin\omega_k t\right) + \sum_{k=N_1+1}^{N_2} \eta_k^{(0)}(C_k + D_k t) +$$
$$+ \sum_{k=N_2+1}^{N} \eta_k^{(u)} \left(E_k \cosh\lambda_k t + \frac{F_k}{\lambda_k}\sinh\lambda_k t\right) \tag{8.43}$$

where the $2N$ constants are determined by the initial conditions. However, this violates the assumption that the motion remain close to the equilibrium point, hence (8.43) holds for a short time only.

If the initial condition is parallel to one normal mode, the motion will forever remain that normal mode. If all the $2N$ eigenvalues ${\lambda_k}^2$ are negative, the small oscillations hypothesis is valid at all times.

Proposition 8.2: B^0 *has as many positive, vanishing and negative eigenvalues as many positive, vanishing and negative eigenvalues with respect to* A^0.

Therefore, the Lyapunov stability only depends on *U*, and not on *T*.

Ex4: consider a bar of length $2l$ and mass $2m$ attached to a torsional spring, and let a disk roll without slipping on it. Suppose the disk has mass *M* and radius *R*. This system has two degrees of freedom. For instance, we may take ϑ, s. Let φ be the rotation angle of the disk. We can write:

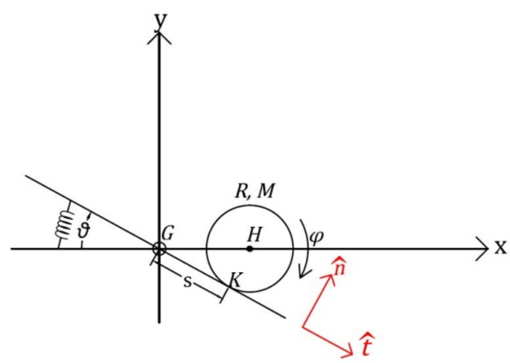


Figure 8.2. *disk rolling on an oscillating bar*

$$\vec{\omega}^{(bar)} = \vec{\omega}^{(b)} = -\dot{\vartheta}\hat{\imath}_3\,; \qquad \vec{\omega}^{(disk)} = \vec{\omega}^{(d)} = -\dot{\varphi}\hat{\imath}_3$$

Let us also introduce the unit vectors

$$\hat{t} = \cos\vartheta\,\hat{\imath}_1 - \sin\vartheta\,\hat{\imath}_2; \qquad \hat{n} = \sin\vartheta\,\hat{\imath}_1 + \cos\vartheta\,\hat{\imath}_2$$

at rest with the bar. The Poisson formula yields

$$\begin{cases} \dot{\hat{t}} = \vec{\omega}^{(b)} \wedge \hat{t} = -\dot{\vartheta}\hat{n} \\ \dot{\hat{n}} = \vec{\omega}^{(b)} \wedge \hat{n} = \dot{\vartheta}\hat{t} \end{cases}$$

and we can write

$$(H - G) = s\hat{t} + R\hat{n}\,; \quad (K - G) = s\hat{t}$$

from which we deduce:

$$\vec{v}_H = \dot{s}\hat{t} + s\,\dot{\hat{t}} + R\,\dot{\hat{n}} = \dot{s}\hat{t} - s\dot{\vartheta}\hat{n} + R\dot{\vartheta}\hat{t} = \left(\dot{s} + R\dot{\vartheta}\right)\hat{t} - s\dot{\vartheta}\hat{n}$$

$$\begin{aligned}{\vec{v}_K}^{(d)} &= \vec{v}_H + \vec{\omega}^{(d)} \wedge (K - H) = \left(\dot{s} + R\dot{\vartheta}\right)\hat{t} - s\dot{\vartheta}\hat{n} - \dot{\varphi}\hat{\imath}_3 \wedge (-R\hat{n}) \\ &= \left(\dot{s} + R\dot{\vartheta}\right)\hat{t} - s\dot{\vartheta}\hat{n} + \dot{\varphi}R(-\hat{t}) = \left(\dot{s} + R\dot{\vartheta} - \dot{\varphi}R\right)\hat{t} - s\dot{\vartheta}\hat{n}\end{aligned}$$

$${\vec{v}_K}^{(b)} = s\,\dot{\hat{t}} = -s\dot{\vartheta}\hat{n}$$

where by $\vec{v}_K^{\,(b)}$ we mean the material point of the bar, and by $\vec{v}_K^{\,(d)}$ we mean the material point of the disk (not just the geometric point of contact). The constraint of no slip at K implies:

$$-s\dot{\vartheta}\hat{n} = \left(\dot{s} + R\dot{\vartheta} - \dot{\varphi}R\right)\hat{t} - s\dot{\vartheta}\hat{n} \quad \Rightarrow \quad \dot{\varphi}R = \dot{s} + R\dot{\vartheta}$$

Moreover, one can write:

$$\begin{aligned} U(s,\vartheta) &= -\frac{1}{2}k\vartheta^2 - mgy_G - Mgy_H = -\frac{1}{2}k\vartheta^2 - Mg(-s\sin\vartheta + R\cos\vartheta) \\ &= Mg(s\sin\vartheta - R\cos\vartheta) - \frac{1}{2}k\vartheta^2 \end{aligned}$$

$$\frac{\partial U}{\partial s} = Mg\sin\vartheta\,; \qquad \frac{\partial U}{\partial \vartheta} = -k\vartheta - Mg(-s\cos\vartheta - R\sin\vartheta) = Mg(s\cos\vartheta + R\sin\vartheta) - k\vartheta$$

which vanishes for

$$\vartheta^* = 0,\pi \ \ \text{obtained from} \ \ \frac{\partial U}{\partial s} = 0, \ \ \text{and} \ \ s^* = 0 \ \ \text{which gives} \ \ \left.\frac{\partial U}{\partial \vartheta}\right|_{s^*,\vartheta^*=0} = 0$$

Then, let us consider $q^* = (0,0)$. For the small oscillations, one has:

$$H(s,\vartheta) = \begin{pmatrix} 0 & Mg\cos\vartheta \\ Mg\cos\vartheta & Mg(R\cos\vartheta - s\sin\vartheta) - k \end{pmatrix}$$

$$B^0 = H(0,0) = \begin{pmatrix} 0 & Mg \\ Mg & MgR - k \end{pmatrix}$$

hence $\det B^0 = -M^2g^2 < 0$ which means that one eigenvalue is positive and the other is negative, *i.e.* that q^* is an *unstable* equilibrium. Indeed, moving the disk away from 0 tilts the bar and the disk will then move further and further away.

Analogous situation arises for $\vartheta^* = \pi$, in which case $s^* = -\pi k/Mg$, which corresponds to disk upside down, in the positive x coordinates, in order to balance the action of the spring with an opposite torque generated by the constraint reaction in the origin of the axes, and the disk weight. Again, the equilibrium configuration is unstable. Indeed, moving the disk away from the position corresponding to the exact arm of this torque makes the bar tilt, in one or the other direction, and the disk runs away.

To write the Lagrange equations, we need the kinetic energy, which is:

$$T = \frac{1}{2}\frac{1}{12}2m(2l)^2\,\dot{\vartheta}^2 + \frac{1}{2}Mv_H^2 + \frac{1}{2}MR^2\,\dot{\varphi}^2 = \frac{3M}{4}\dot{s}^2 + \frac{3MR}{2}\dot{s}\dot{\vartheta} + \left(\frac{ml^2}{3} + \frac{3MR^2}{4} + \frac{Ms^2}{2}\right)\dot{\vartheta}^2$$

In (0,0), this can be written as

$$T = \frac{1}{2}\dot{q}^T A\dot{q} = \frac{1}{2}\left[\frac{3}{2}M\dot{s}^2 + 3MR\dot{s}\dot{\vartheta} + \left(\frac{2ml^2}{3} + \frac{3MR^2}{2}\right)\dot{\vartheta}^2\right] \ ; \ \ A^0 = \begin{pmatrix} 3M/2 & 3MR/2 \\ 3MR/2 & \dfrac{2ml^2}{3} + \dfrac{2MR^2}{2} \end{pmatrix}$$

But small oscillations do not persist, so, we must rely on the full Lagrange equations. We have:

$$L = \frac{3M}{4}\dot{s}^2 + \frac{3MR}{2}\dot{s}\dot{\vartheta} + \left(\frac{ml^2}{3} + \frac{3MR^2}{4} + \frac{Ms^2}{2}\right)\dot{\vartheta}^2 + Mg(s\sin\vartheta - R\cos\vartheta) - \frac{1}{2}k\vartheta^2$$

hence

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = \frac{d}{dt}\left(\frac{3M}{2}\dot{s} + \frac{3MR}{2}\dot{\vartheta}\right) = \frac{3M}{2}\ddot{s} + \frac{3MR}{2}\ddot{\vartheta} - Ms\dot{\vartheta}^2 - Mg\sin\vartheta = 0$$

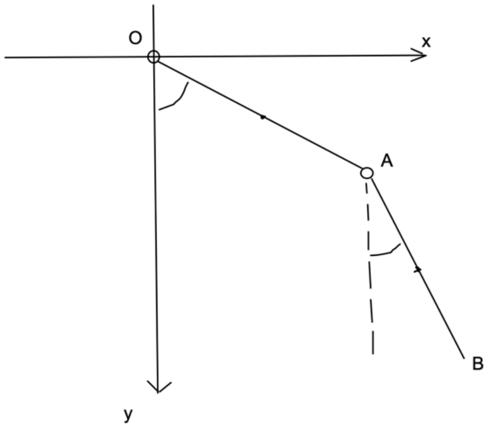
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vartheta}}\right) - \frac{\partial L}{\partial \vartheta} = \frac{d}{dt}\left[\frac{3MR}{2}\dot{s} + 2\left(\frac{ml^2}{3} + \frac{3MR^2}{4} + \frac{Ms^2}{2}\right)\dot{\vartheta}\right] - Mg(s\cos\vartheta + R\sin\vartheta) + k\vartheta = 0$$

which yields

$$\frac{3MR}{2}\ddot{s}+2\left[ms\dot{s}\dot{\vartheta}+\left(\frac{ml^2}{3}+\frac{3MR^2}{4}+\frac{Ms^2}{2}\right)\ddot{\vartheta}\right]-Mg(s\cos\vartheta+R\sin\vartheta)+k\vartheta=0$$

Hard to solve by hand, easy for a computer.

Ex5: consider two homogeneous rigid bars of length $2l$ and mass m, the first extremes O and A, and the second of extremes A and B. The system has two degrees of freedom, e.g. the angles of the bars with the vertical directions ϑ (the angle of the first bar with the vertical direction) and φ (the angle of the second bar with with the vertical direction).



Consider the vectors

$$(A-O)=2l(\sin\vartheta\,\hat{\imath}_1+\cos\vartheta\,\hat{\imath}_2),\quad (B-O)=2l[(\sin\vartheta+\sin\varphi)\hat{\imath}_1+(\cos\vartheta+\cos\varphi)\hat{\imath}_2]$$

that identify the ends of the bars. Because the bars are homogeneous, the centers of mass are given by:

$$(G_{OA}-O)=l(\sin\vartheta\,\hat{\imath}_1+\cos\vartheta\,\hat{\imath}_2),(G_{AB}-O)=l[(2\sin\vartheta+\sin\varphi)\hat{\imath}_1+(2\cos\vartheta+\cos\varphi)\hat{\imath}_2]$$

where G_{OA} is the center of mass of the first bar, and G_{AB} that of the second bar. We may find the equilibrium configurations using the principle of virtual works, which yields:

$$\begin{aligned}\delta L^{(a)}&=mg\hat{\imath}_2\cdot\delta G_{OA}+mg\hat{\imath}_2\cdot\delta G_{AB}=mg\hat{\imath}_2\cdot(l\cos\vartheta\,\delta\vartheta\hat{\imath}_1-l\sin\vartheta\,\delta\vartheta\hat{\imath}_2)+\\&+mg\hat{\imath}_2\cdot l[(2\cos\vartheta\,\delta\vartheta+\cos\varphi\,\delta\varphi)\hat{\imath}_1-(2\sin\vartheta\,\delta\vartheta+\sin\varphi\,\delta\varphi)\hat{\imath}_2]=-mgl\sin\vartheta\,\delta\vartheta+\\&-mgl(2\sin\vartheta\,\delta\vartheta+\sin\varphi\,\delta\varphi)=-3mgl\sin\vartheta\,\delta\vartheta-mgl\sin\varphi\,\delta\varphi=Q_\vartheta^{(a)}\delta\vartheta+Q_\varphi^{(a)}\delta\varphi\end{aligned}$$

where the variation of G_{OA}, for instance, is given by $\delta(G_{OA}-O)=G_{OA}(\vartheta+\delta\vartheta)-G_{OA}(\vartheta)$, and similarly for the variation of G_{AB}, which depends also on φ. Points of equilibrium are obtained setting active forces to 0:

$$Q_\vartheta^{(a)}=0\Rightarrow\sin\vartheta=0\Rightarrow\vartheta=0,\ \vartheta=\pi;\quad Q_\varphi^{(a)}=0\Rightarrow\sin\varphi=0\Rightarrow\varphi=0,\ \varphi=\pi$$

Therefore there are 4 points of equilibrium:

$$(\vartheta_1,\varphi_1)=(0,0),\quad(\vartheta_2,\varphi_2)=(0,\pi),\quad(\vartheta_3,\varphi_3)=(\pi,0),\quad(\vartheta_4,\varphi_4)=(\pi,\pi)$$

To check their stability, consider that

$$U(\vartheta,\varphi)=mgl\cos\vartheta+mgl(2\cos\vartheta+\cos\varphi)$$

so that the Hessian matrix takes the form:

$$H(\vartheta,\varphi)=\begin{pmatrix}-3mgl\cos\vartheta & 0\\ 0 & -mgl\cos\varphi\end{pmatrix}$$

The matrix is diagonal, so we can immediately see that it is negative-definite if and only if $cos\,\vartheta > 0$ and $cos\,\varphi > 0$ which means $\vartheta, \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and so we can say that the only stable position is (ϑ_1, φ_1).

Now, we need to calculate kinetic energy. Clearly, $\vec{\omega}_{OA} = -\dot{\vartheta}\hat{\imath}_3$, $\vec{\omega}_{AB} = -\dot{\varphi}\hat{\imath}_3$, and then the total kinetic energy, can be obtained as the sum of the kinetic energies of the two bars. For the first, we observe that it rotates about the fixed point *O*, which (using Huygens-Steiner theorem, for the moment of inertia with respect to *O*, when the moment of inertia about its center of mass is known) yields:

$$T_{OA} = \frac{1}{2} I_{O,z} \omega_{OA,z}^2 = \frac{1}{2}\left(\frac{1}{3}ml^2 + ml^2\right)\dot{\vartheta}^2 = \frac{2}{3}ml^2\dot{\vartheta}^2$$

For the second bar we have instead:

$$T_{AB} = \frac{1}{2}m\left|\vec{v}_{G_{AB}}\right|^2 + \frac{1}{2}I_{G_{AB},z}\omega_{AB,z}^2$$

where:

$$\vec{v}_{G_{AB}} = l\left[\left(2\,cos\,\vartheta\dot{\vartheta} + cos\,\varphi\dot{\varphi}\right)\hat{\imath}_1 - \left(2\,sin\,\vartheta\dot{\vartheta} + sin\,\varphi\dot{\varphi}\right)\hat{\imath}_2\right]$$

hence

$$\begin{aligned}\left|\vec{v}_{G_{AB}}\right|^2 &= l^2\big(4cos^2\vartheta\dot{\vartheta}^2 + cos^2\,\vartheta\dot{\vartheta}^2 + 4\,cos\,\vartheta\,cos\,\varphi\dot{\vartheta}\dot{\varphi} + 4\,sin^2\,\vartheta\dot{\vartheta}^2 + sin^2\,\varphi\dot{\varphi}^2 \\ &\quad + 4\,sin\,\vartheta\,sin\,\varphi\dot{\vartheta}\,\dot{\varphi}\big) = l^2\left(4\dot{\vartheta}^2 + \dot{\varphi}^2 + 4\,cos(\vartheta - \varphi)\,\dot{\vartheta}\dot{\varphi}\right)\end{aligned}$$

which leads to:

$$\begin{aligned}T_{AB} &= \frac{1}{2}ml^2\left(4\dot{\vartheta}^2 + \dot{\varphi}^2 + 4\,cos(\vartheta - \varphi)\,\dot{\vartheta}\dot{\varphi}\right) + \frac{1}{2}\cdot\frac{1}{3}ml^2\dot{\varphi}^2 = \\ &= 2ml^2\dot{\vartheta}^2 + \frac{2}{3}ml^2\dot{\varphi}^2 + 2ml^2\,cos(\vartheta - \varphi)\,\dot{\vartheta}\dot{\varphi}\end{aligned}$$

Finally, the total kinetic energy is expressed by:

$$\begin{aligned}T = T_{OA} + T_{AB} &= \frac{8}{3}ml^2\dot{\vartheta}^2 + \frac{2}{3}ml^2\dot{\varphi}^2 + 2ml^2\,cos(\vartheta - \varphi)\,\dot{\vartheta}\dot{\varphi} = \\ &= \frac{1}{2}\left(\frac{16}{3}ml^2\dot{\vartheta}^2 + \frac{4}{3}ml^2\dot{\varphi}^2 + 2\cdot 2ml^2\,cos(\vartheta - \varphi)\,\dot{\vartheta}\dot{\varphi}\right)\end{aligned}$$

where

$$A(\vartheta, \varphi) = \begin{pmatrix} \frac{16}{3}ml^2 & 2ml^2\,cos(\vartheta - \varphi) \\ 2ml^2\,cos(\vartheta - \varphi) & \frac{4}{3}ml^2 \end{pmatrix}$$

To describe the small oscillations around the stable equilibrium, we use Eq.(8.40), which requires *A* and *H* in $(0,0)$. This amounts to:

$$A(0,0) = \begin{pmatrix} \frac{16}{3}ml^2 & 2ml^2 \\ 2ml^2 & \frac{4}{3}ml^2 \end{pmatrix}, \quad H(0,0) = \begin{pmatrix} -3mgl & 0 \\ 0 & -mgl \end{pmatrix}$$

so that we have:

$$A(0,0)\ddot{\eta} = H(0,0)\eta \;\Rightarrow\; \begin{cases} \frac{16}{3}ml^2\ddot{\eta}_1 + 2ml^2\ddot{\eta}_2 = -3mgl\eta_1 \\ 2ml^2\ddot{\eta}_1 + \frac{4}{3}ml^2\ddot{\eta}_2 = -mgl\eta_2 \end{cases}$$