VETTORI APPLICATI

1) $\begin{cases} P_1 = (5,-2), & \underline{v}_1 = \underline{i} + \underline{j} \\ P_2 = (3,0), & \underline{v}_2 = 3\underline{i} - 4\underline{j} \\ P_3 = (1,-3), & \underline{v}_3 = -2\underline{i} + 6\underline{j} \end{cases}$

a) $\underline{R} = \sum_i \underline{v}_i = \underline{i} + \underline{j} + 3\underline{i} - 4\underline{j} - 2\underline{i} + 6\underline{j} = 2\underline{i} + 3\underline{j}$ (risultante)

$\underline{M}_0 = \sum_i (P_i - O) \wedge \underline{v}_i =$

$= (5\underline{i} - 2\underline{j}) \wedge (\underline{i} + \underline{j}) + 3\underline{i} \wedge (3\underline{i} - 4\underline{j}) + (\underline{i} - 3\underline{j}) \wedge (-2\underline{i} + 6\underline{j}) =$

$= 5\underline{k} + 2\underline{k} - 12\underline{k} + 6\underline{k} - 6\underline{k} = -5\underline{k}$ (Notare che $\underline{R} \cdot \underline{M}_0 = 0$)

b) Asse centrale:

I° metodo) $OP(\lambda) = \dfrac{\underline{R} \wedge \underline{M}_0}{R^2} + \lambda \underline{R}$ $(\lambda \in \mathbb{R})$

$x\underline{i} + y\underline{j} + z\underline{k} = \frac{1}{13}\left((2\underline{i} + 3\underline{j}) \wedge (-5\underline{k})\right) + \lambda(2\underline{i} + 3\underline{j}) =$ $R = \sqrt{4+9} = \sqrt{13}$

$= \frac{1}{13}(10\underline{j} - 15\underline{i}) + \lambda(2\underline{i} + 3\underline{j}) = \left(-\frac{15}{13} + 2\lambda\right)\underline{i} + \left(\frac{10}{13} + 3\lambda\right)\underline{j}$

$\begin{cases} x = -\frac{15}{13} + 2\lambda \\ y = \frac{10}{13} + 3\lambda \\ z = 0 \end{cases}$ $(\lambda \in \mathbb{R})$ retta: asse centrale

II° metodo) In generale: $\underline{M}_0 = \underline{M}_P + (P - O) \wedge \underline{R}$

In questo caso $\underline{M}_P = \underline{0}$ $(\underline{R} \cdot \underline{M}_P = 0)$ se $P \in$ asse centrale.

$(P - O) \wedge \underline{R} = \underline{M}_0$

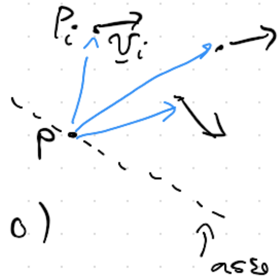
$(x\underline{i} + y\underline{j} + z\underline{k}) \wedge (2\underline{i} + 3\underline{j}) = -5\underline{k}$

$3x\underline{k} - 2y\underline{k} + 2z\underline{j} - 3z\underline{i} = -5\underline{k}$

$\begin{cases} 3x - 2y = -5 \\ z = 0 \end{cases}$ Retta nel piano di eq. cartesiana $3x - 2y + 5 = 0$

III° mul.c(1)

Per i punti $P(x,y,z)$ dell'asse centrale deve sempre valere che il momento risultante è parallelo a $\underline{R}$.



$$\underbrace{\sum_i (P_i - P) \wedge \underline{v}_i}_{\underline{M}_P} = \lambda \underline{R} \qquad (\lambda \in \mathbb{R})$$

$$// \quad \underline{R}$$

$[(5-x)\underline{i} + (-2-y)\underline{j}] \wedge (\underline{i}+\underline{j}) +$ (sto già assumendo $z=0$)

$[(3-y)\underline{i} - y\underline{j}] \wedge (3\underline{i} - 4\underline{j}) +$

$[(1-x)\underline{i} + (-3-y)\underline{j}] \wedge (-2\underline{i} + 6\underline{j}) = \lambda (2\underline{i} + 3\underline{j})$

...

$(-5-3x+2y)\underline{k} = \underline{0}$

Se non avessi assunto $z=0$ da subito:

$2z\underline{i} - 3z\underline{j} + (-5-3x+2y)\underline{k} = \lambda(2\underline{i} + 3\underline{j})$

$\hookrightarrow z=0$

2) $\begin{cases} P_1 = (1,0,0), & \underline{v}_1 = 3\underline{i} + 2\underline{j} - 5\underline{k} \\ P_2 = (0,1,1), & \underline{v}_2 = -\frac{3}{2}\underline{i} - \underline{j} + \frac{5}{2}\underline{k} \\ P_3 = (0,0,1), & \underline{v}_3 = 6\underline{i} + 4\underline{j} - 10\underline{k} \end{cases}$

a) $\underline{v}_2 = -\frac{1}{2}\underline{v}_1$; $\underline{v}_3 = 2\underline{v}_1$

b) Quando i vettori sono paralleli: $\underline{v}_i = v_i \underline{u}$

$\underline{R} = \sum_i \underline{v}_i = (\sum_i v_i)\underline{u} = R\underline{u} \qquad R = \sum_i v_i$

$\underline{M}_O = \sum_i OP_i \wedge \underline{v}_i = \sum_i OP_i \wedge v_i \underline{u} = \sum_i v_i (OP_i \wedge \underline{u})$

$\underline{M}_O \perp \underline{R}$

Si dimostra che i punti dell'asse centrale in questo caso soddisfano: $OP(\lambda) = \frac{1}{R} \sum_i v_i OP_i + \lambda R \underline{u}$

Il centro del sistema <u>non</u> dipende da $\underline{u}$, ma solo dai punti P_i e componenti v_i.
Allora lo trovo imponendo $\lambda = 0$ nella precedente espressione: $OC = \frac{1}{R} \sum_i v_i \, OP_i$

Calcolo $\underline{u}$, direzione di $\underline{v}_1$ (e $\underline{v}_2$ e $\underline{v}_3$)

$$\underline{u} = \frac{\underline{v}_1}{|\underline{v}_1|} = \frac{1}{\sqrt{9+4+25}}(3,2,-5) = \frac{1}{\sqrt{38}}(3\underline{i}+2\underline{j}-5\underline{k})$$

$$v_1 = \sqrt{38} \quad ; \quad v_2 = -\frac{1}{2}\sqrt{38} \quad ; \quad v_3 = 2\sqrt{38}$$

$$R = v_1 + v_2 + v_3 = \left(1 - \frac{1}{2} + 2\right)\sqrt{38} = \frac{5}{2}\sqrt{38}$$

$$OC = \frac{2}{5\sqrt{38}}\left(\underbrace{\sqrt{38}}_{v_1}\underbrace{(1,0,0)}_{P_1} \underbrace{-\frac{1}{2}\sqrt{38}}_{v_2}\underbrace{(0,1,1)}_{P_2} \underbrace{+2\sqrt{38}}_{v_3}\underbrace{(0,0,1)}_{P_3}\right) =$$

$$OC = \frac{2}{5}\left((1,0,0) - \frac{1}{2}(0,1,1) + 2(0,0,1)\right) =$$

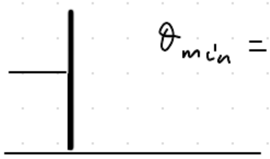
$$\frac{2}{5}\left(1, -\frac{1}{2}, -\frac{1}{2}+2\right) = \frac{2}{5}\left(1, -\frac{1}{2}, \frac{3}{2}\right) = \left(\frac{2}{5}, -\frac{1}{5}, \frac{3}{5}\right)$$

Centro del sistema ha coordinate $C = \left(\frac{2}{5}, -\frac{1}{5}, \frac{3}{5}\right)$

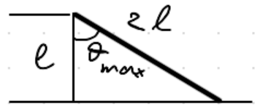
EQUILIBRIO, STATICA, STABILITA'

3)

a)

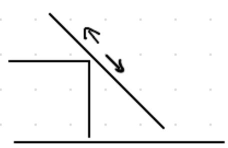


$\theta_{min} = 0$



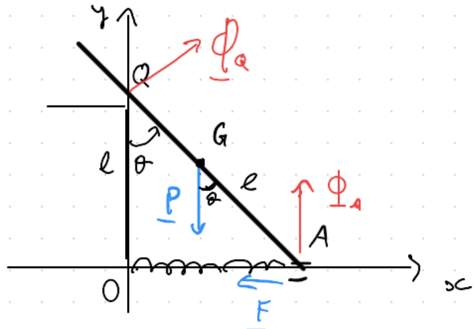
$$2\ell \cos\theta_{max} = \ell$$
$$\cos\theta_{max} = \frac{1}{2}$$
$$\theta_{max} = \frac{\pi}{3}$$

$$0 \le \theta \le \frac{\pi}{3}$$



b) I)

$$\delta L^{(a)} = \sum_{i=1}^{n} \underline{F}_i \cdot \delta \underline{P}_i = \sum_{h=1}^{N} \underbrace{Q_h}_{\text{Componenti generalizzate}} \delta q_h$$



$$\delta L^{(a)} = \underline{P} \cdot \delta G + \underline{F} \cdot \delta A$$

$$\delta G = \ell\left(\frac{1}{\cos^2\theta} - \cos\theta\right)\delta\theta\, \underline{i} - \ell \sin\theta\, \delta\theta\, \underline{j}$$

$$\delta A = \frac{\ell}{\cos^2\theta}\delta\theta\ \underline{i}$$

$A = (\ell \tan\theta, 0)$

$G = (\ell\tan\theta - \ell\sin\theta, \ell\cos\theta)$

$Q = (0, \ell)$

$\underline{P} = -mg\underline{j}$;

$\underline{F} = k(O - A) = -k\ell\tan\theta\, \underline{i}$

$$\delta L^{(a)} = (-mg\underline{j}) \cdot \ell\left[\left(\frac{1}{\cos^2\theta} - \cos\theta\right)\delta\theta\, \underline{i} - \sin\theta\,\delta\theta\,\underline{j}\right] + $$
$$- k\ell\tan\theta\, \underline{i} \cdot \frac{\ell}{\cos^2\theta}\delta\theta\,\underline{i} =$$
$$= mg\ell\sin\theta\,\delta\theta - k\ell^2\frac{\sin\theta}{\cos^3\theta}\delta\theta = \underbrace{mg\ell\sin\theta\left(1 - \frac{\lambda}{\cos^3\theta}\right)}_{Q_\theta}\delta\theta$$

$$\lambda = \frac{k\ell}{mg}$$

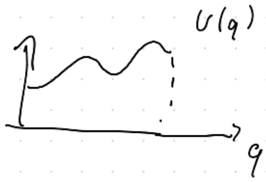
II) $U = -mg\,y_G - \frac{1}{2}k|OA|^2 + cost$ $y_G = \ell\cos\theta$

$OA = \ell\tan\theta$

$$U = -mg\ell\cos\theta - \frac{1}{2}k\ell^2\tan^2\theta$$

$$Q_\theta = U' = mg\ell\sin\theta - k\ell^2\frac{\tan\theta}{\cos^2\theta} = mg\ell\sin\theta - k\ell^2\frac{\sin\theta}{\cos^3\theta} = mg\ell\sin\theta\left(1 - \frac{\lambda}{\cos^3\theta}\right)$$

In generale $Q_h = \dfrac{\partial U}{\partial q_h}$



c) Ordinario: $\theta \in (0, \frac{\pi}{3})$

Equilibrio se $Q_\theta = U'(\theta) = 0$

$$mg\ell\sin\theta\left(1 - \frac{\lambda}{\cos^3\theta}\right) = 0 \qquad \text{se } \theta \in \left(0, \frac{\pi}{3}\right) \Rightarrow \frac{\lambda}{\cos^3\theta} = 1$$

$\cos^3\theta = \lambda$. Non esiste se $\lambda > 1$ (e se $\lambda = 1$ è di confine)

So anche che se $\theta \in (0, \frac{\pi}{3})$ allora $\frac{1}{2} < \cos\theta < 1$ $\boxed{\cos\theta_{eq} = \sqrt[3]{\lambda}}$

Quindi perchè esista come equilibrio ordinario deve valere: $\frac{1}{8} < \lambda < 1$

$\theta \in \{0, \frac{\pi}{3}\}$

Se $\theta = \frac{\pi}{3}$, $Q_\theta = U' = mgl \sin\frac{\pi}{3}\left(1 - \frac{\lambda}{\cos^3\frac{\pi}{3}}\right) = mgl\frac{\sqrt{3}}{2}(1-8\lambda)$

Equilibrio se $Q_\theta \delta\theta \le 0$. Qui $\delta\theta < 0$

Per l'equilibrio deve valere $Q_\theta \ge 0$, cioè $1-8\lambda \ge 0$

$\frac{\pi}{3}$ (di confine) è di equilibrio se $\lambda \le \frac{1}{8}$

Se $\theta = 0$, $Q_\theta = U' = mgl \cdot \overset{\sin\theta}{0}(1-\lambda) = 0$

È di equilibrio, essendo sempre $Q_\theta = 0$

d) Ordinarie: $\theta \in (0, \frac{\pi}{3})$

Stabilità

$$Q' = U'' = mgl\cos\theta\left(1 - \frac{\lambda}{\cos^3\theta}\right) - 3mgl\lambda\frac{\sin^2\theta}{\cos^4\theta}$$

La configurazione di equilibrio ordinaria è definita da:

$\lambda = \cos^3\theta_{eq} \longrightarrow \quad Q' = U'' = -3mgl\lambda\frac{\sin^2\theta_{eq}}{\cos^4\theta_{eq}}$

$\left(1 - \frac{\lambda}{\cos^3\theta_{eq}}\right) = 0$ avendo escluso $\theta = 0$ vale $Q' = U'' < 0$

Quando esiste, θ_{eq} (ordinaria) è sempre stabile.

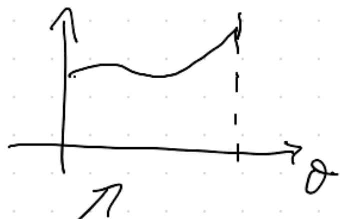
Confine:

Se $\theta = \frac{\pi}{3}$, $Q' = U'' = \frac{mgl}{2}(1-8\lambda) - 3mgl\lambda\frac{\frac{3}{4}}{\frac{1}{16}} =$

$= \frac{mgl}{2}(1-8\lambda) - 36mgl\lambda = \frac{mgl}{2}(1-8\lambda-72\lambda) = \frac{mgl}{2}(1-80\lambda)$

Ricordiamo che esiste se $\lambda \le \frac{1}{8}$

$$Q' = U'' = \frac{mgl}{2}\left((1-80\lambda)\right)$$



Distinguiamo

Se $\lambda < \frac{1}{8}$ è crescente e quindi stabile

Se $\lambda = \frac{1}{8}$, $Q' = U'' < 0$

Max locale

Sempre stabile