CHAPTER 2

Rigid Body Kinematics

In 3-dimensional (3D) space, we need a 3D reference frame, let it be denoted by T and let its origin be the point O. This is expressed by a basis made of 3 orthonormal vectors, say $\hat{\imath}, \hat{\jmath}, \hat{k}$ and by the point O. An observer is assumed to be at rest with respect to T. In such a frame, the origin is expressed by $O = 0\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k} = (0,0,0)$ and any other point by

$$P = \ x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k} = (x, y, x)$$

The vector joining O to P, as represented *e.g.* in red in Figure 2.1, is also denoted by

$$\overrightarrow{OP} = (P - O) = (x - 0, y - 0, x - 0)$$

A point $P \in \mathbb{R}^3$ is now thought to move according to a certain law, which gives the travelled distance at each time $t \in [t_1, t_2]$; the law is expressed as follows:

$$S^t : T \to T \quad t \in [t_1, t_2] \tag{2.1}$$

where S^t is the operator that transforms the point $P \in \mathbb{R}^3$ into the point $S^t P \in \mathbb{R}^3$, that is reached after a time t, starting from P. This notation may be replaced by $P(t)$ where it is understood that $P(0)$ is the initial point. Projecting the point on the three axes of the reference frame T, one may write the coordinates of the point, or of the vector that joins the origin O to the point $P(t)$ as follows:

$$\big(x(t), y(t), z(t)\big) = \vec{r}(t) \tag{2.2}$$

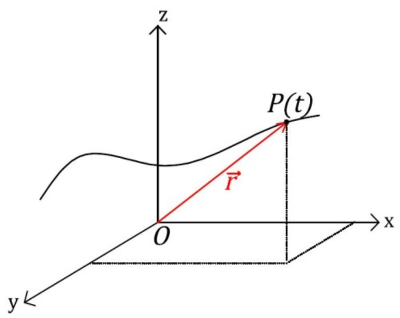

Figure 2.1. *Trajectory of point P in a 3-dimensional space.*

Equivalent notations are given by the following equalities:

$$\vec{r}(t) = \overrightarrow{OP}(t) = (P - O)(t) \tag{2.3}$$

where

$$\vec{r} = (x, y, z) = (x - 0, y - 0, z - 0) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k} \tag{2.4}$$

The set $\gamma = \{P(t)\}_{t \in [t_1, t_2]}$ is called *trajectory*. It is commonly assumed that a trajectory is at least piecewise differentiable, because mechanical laws are differential equations. Then, we can introduce the *velocity* of P :

$$\begin{aligned}\vec{v}_P(t) = \frac{dP}{dt} = \frac{d}{dt}(P - O) = \dot{P}(t) &\coloneqq \dot{\vec{r}}\,(t) \\ &= \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath} + \dot{z}(t)\hat{k}\end{aligned} \tag{2.5}$$

Analogously, the *acceleration* of P is defined by

$$\vec{a}_P(t) = \frac{d^2P}{dt^2} = \ddot{P}(t) = \ddot{\vec{r}}\,(t) = \ddot{x}(t)\hat{\imath} + \ddot{y}(t)\hat{\jmath} + \ddot{z}(t)\hat{k} \tag{2.6}$$

If a Taylor expansion is possible, for small Δt one can write:

$$\Delta P = (P(t + \Delta t) - O) - (P(t) - O) = P(t + \Delta t) - P(t) = \vec{v}(t)\Delta t + \frac{1}{2}\vec{a}(t)\Delta t^2 + o(\Delta t^2) \tag{2.7}$$

where $o(\Delta t^2)$ is negligible. Now, ΔP becomes tangent to the trajectory for $\Delta t \to 0$, as $P(t + \Delta t)$ and $P(t)$ belong to it and tend to a single point. Denote by

$$dP = \vec{v}dt \tag{2.8}$$

the *infinitesimal displacement*; its magnitude, denoted by ds, obeys

$$ds \coloneqq |dP| = |\vec{v}(t)|dt = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}dt \tag{2.9}$$

and it represents the length of the elementary trajectory arc. Clearly, $\vec{v}$ is tangent to the trajectory, because dP is. One may then introduce the time law:

$$s(t) := \int_{t_1}^{t} |\vec{v}(t')| dt' \tag{2.10}$$

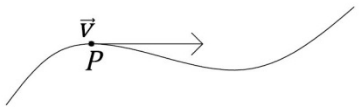

Figure 2.2. *The velocity vector along a trajectory*

and s is then called *curvilinear coordinate*. If in a time interval $\vec{v}(t) \neq \mathbf{0}$, we have $\frac{ds}{dt} = |\vec{v}(t)| > 0$ hence the function $s = s(t)$ can be inverted, so that $t = t(s)$ can also be obtained. In this case, There is a bijective correspondence between the point P and the curvilinear coordinate s, and we can introduce $\tilde{P}(s) = P\big(t(s)\big)$. Moreover, the inverse function derivation rule yields:

$$\frac{dt}{ds} = \left(\frac{ds}{dt}\right)^{-1} \; ; \quad \left|\frac{dt}{ds}\right| = \frac{1}{|\vec{v}|} \tag{2.11}$$

The unitary vector tangent to a trajectory (cf. Eq.(2.7) above) is given by:

$$\hat{t}(s) := \frac{d\tilde{P}}{ds} = \frac{dP}{dt}\frac{dt}{ds} = \vec{v}\frac{1}{|\vec{v}|} \tag{2.12}$$

Moreover, consider that

$$\frac{d}{dt}\left(\frac{dP}{dt}\frac{dt}{ds}\right)\frac{dt}{ds} = \frac{d^2P}{dt^2}\left(\frac{dt}{ds}\right)^2 = \frac{\vec{a}}{|\vec{v}|^2}$$

which implies:

$$\hat{t} \cdot \frac{d\hat{t}}{ds} = \frac{\vec{v}}{|\vec{v}|} \cdot \frac{d}{ds}\left[\frac{dP}{dt}\frac{dt}{ds}\right] = \frac{\vec{v}}{|\vec{v}|} \cdot \frac{d}{dt}\left[\frac{dP}{dt}\frac{dt}{ds}\right]\frac{dt}{ds} = \frac{\vec{v}}{|\vec{v}|} \cdot \frac{\vec{a}}{|\vec{v}|^2} = 0 \tag{2.13}$$

Indeed, $\hat{t}$ has constant (unitary) length, so that one can write:

$$\frac{d}{ds}(\hat{t} \cdot \hat{t}) = \frac{d}{ds}|\hat{t}|^2 = \frac{d}{ds}1 = 0$$

and, at the same time,

$$\frac{d}{ds}(\hat{t} \cdot \hat{t}) = \frac{d\hat{t}}{ds} \cdot \hat{t} + \hat{t} \cdot \frac{d\hat{t}}{ds} = 2\frac{d\hat{t}}{ds} \cdot \hat{t} \Rightarrow \boxed{\hat{t} \cdot \frac{d\hat{t}}{ds} = 0} \tag{2.14}$$

Therefore, $\vec{v} \perp \vec{a}$: velocity and acceleration are orthogonal.

As a matter of fact $d\hat{t}/ds$, is the rate of variation of the direction of the trajectory, hence it must be perpendicular to the trajectory. This produces a *curvature* in the trajectory which we quantify by $c = |d\hat{t}/ds|$, and a *radius of curvature* $\rho = 1/c$. Now, let us introduce

$$\hat{n} = \frac{1}{c}\frac{d\hat{t}}{ds} := principal\ normal \tag{2.15}$$

$$\hat{b} = \hat{t} \wedge \hat{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ t_x & t_y & t_z \\ n_x & n_y & n_z \end{vmatrix} := binormal \tag{2.16}$$

and the $(\hat{t}, \hat{n}, \hat{b})$ which is called $intrinsic\ triple$. Then, we have

$$\vec{v} = \frac{dP}{dt} = \frac{d\tilde{P}}{ds}\frac{ds}{dt} = \dot{s}\hat{t} \tag{2.17}$$

Considering that $d\hat{t}/ds = c\hat{n}$ we also have

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{s}\hat{t}) = \ddot{s}\hat{t} + \dot{s}\frac{d\hat{t}}{dt} = \ddot{s}\hat{t} + \dot{s}\frac{d\hat{t}}{ds}\frac{ds}{dt} = \ddot{s}\hat{t} + \dot{s}c\hat{n}\dot{s}$$

which is to say:

$$\vec{a} = \ddot{s}\hat{t} + \dot{s}^2 c\hat{n} = \ddot{s}\hat{t} + \frac{\dot{s}^2}{\rho}\ \hat{n} \tag{2.18}$$

This shows that, in general, $\vec{a}$ has a component tangent and one normal to the trajectory. The first affects the speed, the other the direction of motion.

When the trajectory remains on a given plane, we speak of planar motion, and then it is often better to use polar coordinates (r, ϑ) instead of *(x, y)*. In that case, one writes:

$$\overrightarrow{OP} = (P - O) = r\cos\vartheta\,\hat{\imath} + r\sin\vartheta\,\hat{\jmath} \tag{2.19}$$

$$r = |\overrightarrow{OP}| = |P - O| \tag{2.20}$$

Introducing the unit vectors in radial and transverse directions:

$$\hat{e}_r \coloneqq \cos\vartheta\,\hat{\imath} + \sin\vartheta\,\hat{\jmath} \qquad \text{RADIAL} \tag{2.21}$$

$$\hat{e}_\vartheta \coloneqq -\sin\vartheta\,\hat{\imath} + \cos\vartheta\,\hat{\jmath} \qquad \text{TRANSVERSE} \tag{2.22}$$

one can then write:

$$\hat{e}_\vartheta = \frac{d\hat{e}_r}{d\vartheta}; \qquad \hat{e}_r = -\frac{d\hat{e}_\vartheta}{d\vartheta}$$

and

$$(P - O) = r\hat{e}_r \qquad \text{which implies}$$

$$\vec{v} = \frac{d(P - O)}{dt} = \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{dt} = \dot{r}\hat{e}_r + r\left(\frac{d\hat{e}_r}{d\vartheta}\frac{d\vartheta}{dt}\right)$$

and leads to

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\vartheta}\hat{e}_\vartheta = v_r\hat{e}_r + v_\vartheta\hat{e}_\vartheta\ ; \quad v_r \coloneqq \dot{r}; \quad v_\vartheta = r\dot{\vartheta} \tag{2.23}$$

Differentiating one more time, one analogously gets:

$$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}\frac{d\hat{e}_r}{dt} + \dot{r}\dot{\vartheta}\hat{e}_\vartheta + r\ddot{\vartheta}\hat{e}_\vartheta + r\dot{\vartheta}\frac{d\hat{e}_\vartheta}{dt} = \ddot{r}\hat{e}_r + \dot{r}\dot{\vartheta}\hat{e}_\vartheta + \dot{r}\dot{\vartheta}\hat{e}_\vartheta + r\ddot{\vartheta}\hat{e}_\vartheta + r\dot{\vartheta}\dot{\vartheta}(-\hat{e}_r)$$

i.e.

$$\boxed{\begin{array}{l}\vec{a} = \left(\ddot{r} - r\dot{\vartheta}^2\right)\hat{e}_r + \left(r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta}\right)\hat{e}_\vartheta = a_r\hat{e}_r + a_\vartheta\hat{e}_\vartheta \\ with \qquad a_r \coloneqq \ddot{r} - r\dot{\vartheta}^2;\ a_\vartheta \coloneqq r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta}\end{array}} \tag{2.24}$$

2.1 Rigid body kinematics

A system made of a discrete set of material points, or a continuum, $\mathcal{B}$ is called *rigid* if the distance between any two of its points is constant in time. Four points of such rigid systems suffice to introduce a reference frame, in which the object is at rest.

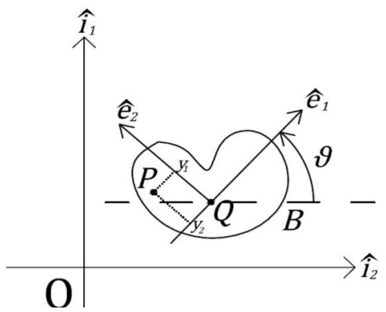

Figure 2.3. *Representation of planar motion using polar coordinates*

PLANAR RIGID MOTION: if the object $\mathcal{B}$ is rigid and the velocities of its points remain parallel to a plane π, we have a *planar* rigid motion. Taking two reference frames with coinciding third axes, $\hat{e}_3 = \hat{\imath}_3$ (in the sense of equivalence classes), perpendicular to π, the motion of $\mathcal{B}$ with respect to the first reference frame is fully characterized by one point Q and by two vectors $\hat{e}_1, \hat{e}_2$ at rest with $\mathcal{B}$:

$$\begin{cases}\hat{e}_1 = \alpha_{11}\hat{\imath}_1 + \alpha_{12}\hat{\imath}_2 \\ \hat{e}_2 = \alpha_{21}\hat{\imath}_1 + \alpha_{22}\hat{\imath}_2\end{cases} \tag{2.25}$$

where we may consider fixed the vectors $\hat{\imath}_2$ and $\hat{\imath}_3$ while $\hat{e}_2$ and $\hat{e}_3$ move in space with $\mathcal{B}$. Here the coefficients must obey:

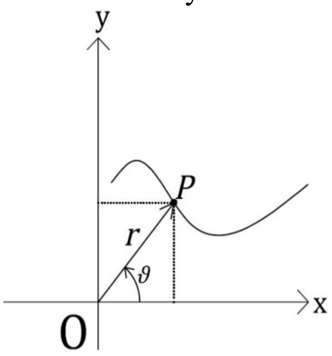

Figure 2.4. *Representation of planar rigid motion by introducing a new coordinate system*

$$1 = \hat{e}_1 \cdot \hat{e}_1 = \alpha_{11}^2 + \alpha_{12}^2$$
$$1 = \hat{e}_2 \cdot \hat{e}_2 = \alpha_{21}^2 + \alpha_{22}^2$$

and

$$\hat{e}_1 \cdot \hat{e}_2 = \alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} = 0$$

which is of course the case, since the projections on the axes of the fixed frame write:

$$\alpha_{11} = \cos\vartheta\,; \;\; \alpha_{12} = \sin\vartheta\,;$$
$$\alpha_{21} = -\sin\vartheta\,; \;\; \alpha_{22} = \cos\vartheta$$

Then, given $Q(t)$ and $\vartheta(t)$ all motion is known.

Introducing the matrix

$$R = \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have $RR^T = R^T R = I$ (the identity matrix) and

$$\hat{e}_h = R\hat{\imath}_h$$

For instance

$$\hat{e}_1 = R\hat{\imath}_1 = \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \\ 0 \end{pmatrix} = \cos\vartheta\,\hat{\imath}_1 + \sin\vartheta\,\hat{\imath}_2$$

Note that column, not row, contributes. For

$$P \in \mathcal{B}, \qquad P = (y_1(P), y_2(P), y_3(P)),$$

where $y_h(P)$ are the coordinates in the frame at rest with $\mathcal{B}$, one may write:

$$(P(t) - O) = (Q(t) - O) + \big(P(t) - Q(t)\big) = (Q(t) - O) + \sum_{h=1}^{3} y_h(P)\hat{e}_h(t) \tag{2.26}$$

where y_h are the coordinates of P in the reference frame in which $\mathcal{B}$ is at rest, hence they are constant in time because $\mathcal{B}$ is rigid, and we may write:

$$\big(P(t) - Q(t)\big) = \sum_{h=1}^{3} y_h(P)\hat{e}_h(t) = \sum_{h=1}^{3} y_h(P)R(t)\hat{\imath}_h \tag{2.27}$$

so that everything is expressed in the laboratory frame. In general, non-planar motion, we may write:

$$\begin{cases} \hat{e}_1 = \alpha_{11}\hat{\imath}_1 + \alpha_{21}\hat{\imath}_2 + \alpha_{31}\hat{\imath}_3 \\ \hat{e}_2 = \alpha_{12}\hat{\imath}_1 + \alpha_{22}\hat{\imath}_2 + \alpha_{32}\hat{\imath}_3 \\ \hat{e}_3 = \alpha_{13}\hat{\imath}_1 + \alpha_{23}\hat{\imath}_2 + \alpha_{33}\hat{\imath}_3 \end{cases} \tag{2.28}$$

where the following conditions apply:

$$\hat{e}_h \cdot \hat{\imath}_k = \alpha_{kh}, \quad \hat{e}_h \cdot \hat{e}_h = 1, \quad \hat{e}_h \cdot \hat{e}_k = 0 \;\text{ if }\; h \neq k, \quad \hat{e}_1 \wedge \hat{e}_2 \cdot \hat{e}_3 = 1 \tag{2.29}$$

This means that

$$R = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

is an orthogonal transformation, a rotation in fact, *i.e.* a linear transformation such that $\det R = 1$ and $RR^T = R^T R = I$. In addition, one may write:

$$\hat{e}_h = R\hat{\imath}_h \; ; \qquad \hat{\imath}_h = R^T \hat{e}_h$$

and, analogously to Eq.(2.27), we may write:

$$P(t) = Q(t) + R(t)\vec{p}, \qquad \vec{p} = \sum_{h=1}^{3} y_h(P)\hat{\imath}_h \tag{2.30}$$

Given the conditions (2.29) on the unit vectors $\hat{e}_h$, there remain only 3 independent entries in R. One possible way to determine them is based on the Euler angles.

EULER ANGLES: Suppose $\hat{e}_3 \nparallel \hat{\imath}_3$, *i.e.* not parallel, which means $\hat{\imath}_3 \wedge \hat{e}_3 \neq \mathbf{0}$, and introduce the vector

$\hat{n} = (\hat{\imath}_3 \wedge \hat{e}_3)/|\hat{\imath}_3 \wedge \hat{e}_3|$, the unitary vector of the line that is the intersection of the planes orthogonal to $\hat{\imath}_3$ and to $\hat{e}_3$, *i.e.* those containing the pairs $(\hat{e}_1, \hat{e}_2)$ and $(\hat{\imath}_1, \hat{\imath}_2)$. This line is called *axis of nodes*.

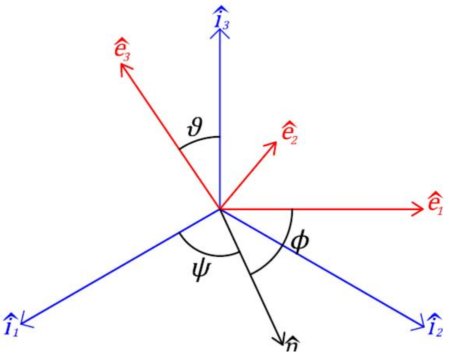

Figure 2.5. *Representation of Euler angles*

Three angles give the mutual orientation of the two reference frames:

NUTATION ϑ: the angle between $\hat{e}_3$ and $\hat{\imath}_3$ which lies in $(0, \pi)$

PRECESSION ψ : the angle between $\hat{\imath}_1$ and $\hat{n}$, which takes values in $\mathbb{R}$, but can be limited to $[0, 2\pi)$

PROPER ROTATION ϕ: the angle between $\hat{n}$ and $\hat{e}_1$, that takes values in $\mathbb{R}$, and can be limited to $[0, 2\pi)$.

Clearly, the axis of nodes is not defined if $\hat{\imath}_3 \parallel \hat{e}_3$. However, as long as $\hat{\imath}_3 \wedge \hat{e}_3 \neq \mathbf{0}$, there is a direct relation between the orientation of the body $\mathcal{B}$ and the triple (ϑ, ψ, ϕ). Unfortunately, during the motion it is not excluded that $\hat{\imath}_3$ turns parallel to $\hat{e}_3$. Therefore, more sophisticated approaches, such as those based on *quaternions* must be used *e.g.* in performing numerical simulations of the motion of satellites. In simpler situations, one may assume that $\hat{\imath}_3 \parallel \hat{e}_3$ never happens, and the description based on Euler angles is considered satisfactory.

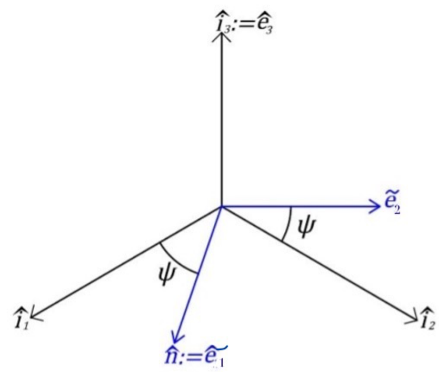

Figure 2.6. *Rotation by* ψ *about* $\hat{\imath}_3$

Given (ϑ, ψ, ϕ), three rotations overlap the frame at rest with $\mathcal{B}$ with the other (laboratory) reference frame. Therefore, let us start from the reference frame of unit vectors $(\hat{\imath}_1, \hat{\imath}_2, \hat{\imath}_3)$, and by performing the mentioned rotations, let us transform it in the $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ frame.

1) Rotation by ψ about $\hat{\imath}_3$, so that $\hat{\imath}_1 \to \hat{n}$, yields a new frame whose unit vectors are expressed by:

$$\begin{aligned} \tilde{e}_1 &= \cos\psi\, \hat{\imath}_1 + \sin\psi\, \hat{\imath}_2 = \hat{n}. \\ \tilde{e}_2 &= -\sin\psi\, \hat{\imath}_1 + \cos\psi\, \hat{\imath}_2 \\ \tilde{e}_3 &= \hat{\imath}_3 \end{aligned} \tag{2.31}$$

The vectors $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ identify a new reference frame, rotated by ψ with respect to the original one.

Another reference frame is obtained rotating $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$:

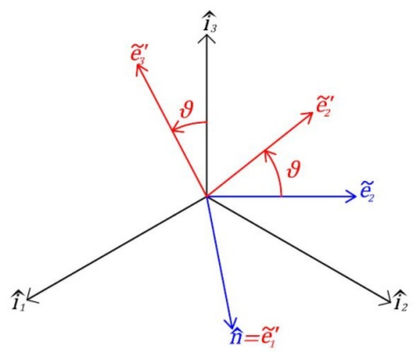

Figure 2.7. *Rotation by ϑ about $\tilde{e}_1 = \hat{n}$*

2) Rotation by ϑ about $\tilde{e}_1 = \hat{n}$ to obtain

$$
\begin{aligned}
\tilde{e}_1' &:= \tilde{e}_1 \\
\tilde{e}_2' &:= \cos\vartheta\,\tilde{e}_2 + \sin\vartheta\,\tilde{e}_3 \qquad (2.32)\\
\tilde{e}_3' &:= -\sin\vartheta\,\tilde{e}_2 + \cos\vartheta\,\tilde{e}_3 = \hat{e}_3
\end{aligned}
$$

This leads to the new frame $(\tilde{e}_1', \tilde{e}_2', \tilde{e}_3')$, that may also be rotated as follows:

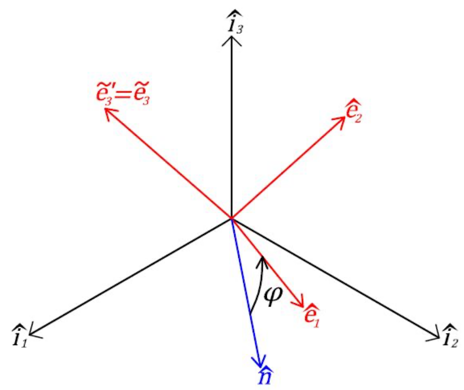

Figure 2.8. *Rotation by ϕ about $\tilde{e}_3'$*

3) Rotation by ϕ about $\tilde{e}_3'$ to obtain

$$
\begin{aligned}
\hat{e}_1 &:= \cos\phi\,\tilde{e}_1' + \sin\phi\,\tilde{e}_2' \\
\hat{e}_2 &:= -\sin\phi\,\tilde{e}_1' + \cos\phi\,\tilde{e}_2' \qquad (2.33)\\
\hat{e}_3 &:= \tilde{e}_3'
\end{aligned}
$$

This finally produces the reference frame identified by the triple of unit vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$.

One can then write:

$$
\begin{aligned}
\hat{e}_1 &= \cos\phi\,\tilde{e}_1 + \sin\phi\,(\cos\vartheta\,\tilde{e}_2 + \sin\vartheta\,\tilde{e}_3) \\
&= \cos\phi\,[\cos\psi\,\hat{\imath}_1 + \sin\psi\,\hat{\imath}_2] + \sin\phi\,[\cos\vartheta\,(-\sin\psi\,\hat{\imath}_1 + \cos\psi\,\hat{\imath}_2) + \sin\vartheta\,\hat{\imath}_3] \qquad (2.34)\\
&= (\cos\psi\cos\phi - \sin\psi\sin\phi\cos\vartheta)\hat{\imath}_1 + (\sin\psi\cos\phi + \cos\psi\cos\vartheta\sin\phi)\hat{\imath}_2 + \sin\phi\sin\vartheta\,\hat{\imath}_3
\end{aligned}
$$

$$
\hat{e}_2 = -(\cos\psi\sin\phi + \sin\psi\cos\phi\cos\vartheta)\hat{\imath}_1 + (-\sin\psi\sin\phi + \cos\psi\cos\phi\cos\vartheta)\hat{\imath}_2 + \cos\phi\sin\vartheta\,\hat{\imath}_3 \qquad (2.35)
$$

$$
\hat{e}_3 = \sin\psi\sin\vartheta\,\hat{\imath}_1 - \cos\psi\sin\vartheta\,\hat{\imath}_2 + \cos\vartheta\,\hat{\imath}_3 \qquad (2.36)
$$

which are the explicit forms of the components of the following system of equations:

$$
\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix} = R \begin{pmatrix} \hat{\imath}_1 \\ \hat{\imath}_2 \\ \hat{\imath}_3 \end{pmatrix} \qquad (2.37)
$$

The inverse rotation R^T yields:

$$
\hat{\imath}_1 = (\cos\psi\cos\phi - \sin\psi\sin\phi\cos\vartheta)\hat{e}_1 - (\cos\psi\sin\phi + \sin\psi\cos\phi\cos\vartheta)\hat{e}_2 + \sin\psi\sin\vartheta\,\hat{e}_3 \qquad (2.38)
$$

$$
\hat{\imath}_2 = (\sin\psi\cos\phi + \cos\psi\sin\phi\cos\vartheta)\hat{e}_1 + (-\sin\psi\sin\phi + \cos\psi\cos\phi\cos\vartheta)\hat{e}_2 - \cos\psi\sin\vartheta\,\hat{e}_3 \qquad (2.39)
$$

$$
\hat{\imath}_3 = \sin\phi\sin\vartheta\,\hat{e}_1 + \cos\phi\sin\vartheta\,\hat{e}_2 + \cos\vartheta\,\hat{e}_3 \qquad (2.40)
$$

Therefore, provided the condition $\hat{\imath}_3 \wedge \hat{e}_3 \neq \mathbf{0}$ is satisfied, the rigid body motion is determined by only six quantities:

$$
x_Q(t), y_Q(t), z_Q(t), \phi(t), \psi(t), \vartheta(t)
$$

On the contrary, $\hat{\imath}_3 \wedge \hat{e}_3 = \mathbf{0}$ means $\vartheta = 0, \pi$ i.e. $\cos\vartheta = \pm 1, \sin\vartheta = 0$ which implies an ambiguity concerning the orientation of the body, because one may only write:

$$\hat{e}_1 = (\cos\psi\cos\phi \mp \sin\psi\sin\phi)\hat{\imath}_1 + (\sin\psi\cos\phi \pm \cos\psi\sin\phi)\hat{\imath}_2$$
$$= \cos(\psi \pm \phi)\,\hat{\imath}_1 + \sin(\psi \pm \phi)\,\hat{\imath}_2 \tag{2.41}$$

and, analogously,

$$\hat{e}_2 = -\sin(\psi \pm \phi)\,\hat{\imath}_1 - \cos(\psi \pm \phi)\,\hat{\imath}_2 \tag{2.42}$$
$$\hat{e}_3 = \pm\hat{\imath}_3 \tag{2.43}$$

In other words, the position of $\hat{\imath}_1, \hat{\imath}_2$ is not determined, since only $\psi \pm \phi$ is known and not ψ, ϕ separately. This situation is called *polar singularity*.

Because $\hat{e}_h \cdot \hat{e}_h = 1$, one has:

$$\frac{d\hat{e}_h}{dt} \cdot \hat{e}_h = 0, \qquad h = 1,2,3 \tag{2.44}$$

and because $\hat{e}_h \cdot \hat{e}_k = 0$ for $h \neq k$, one has

$$\frac{d\hat{e}_h}{dt} \cdot \hat{e}_k = -\frac{d\hat{e}_k}{dt} \cdot \hat{e}_h \tag{2.45}$$

which leads to

THEOREM 2.1 (Poisson): *Let $\mathcal{B}$ be a rigid object at rest with respect to $\{\hat{e}_h\}_1^3$. Then,* $\exists!\,\vec{\omega}$ such that

$$\boxed{\frac{d\hat{e}_h}{dt} = \vec{\omega} \wedge \hat{e}_h, \qquad h = 1,2,3} \tag{2.46}$$

The equations (2.46) are called Poisson formulae*. The vector $\vec{\omega}$ is called angular velocity and it does not depend on the choice of the reference frame $\{\hat{e}_h\}$.*

Proof: observe that Eq.(2.46) yields:

$$\sum_{h=1}^{3} \hat{e}_h \wedge \frac{d\hat{e}_h}{dt} = \sum_{h=1}^{3} \hat{e}_h \wedge (\vec{\omega} \wedge \hat{e}_h) = \sum_{h=1}^{3} [(\hat{e}_h \cdot \hat{e}_h)\vec{\omega} - (\vec{\omega} \cdot \hat{e}_h)\hat{e}_h] = 3\vec{\omega} - \vec{\omega} = 2\vec{\omega} \tag{2.47}$$

where the second equality is due to the following property of the double vector product: for any three vectors $\vec{u}, \vec{v}, \vec{w}$ one has:

$$(\vec{u} \wedge \vec{v}) \wedge \vec{w} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{w} \cdot \vec{v})\vec{u} \tag{2.48}$$

as can be directly verified. Then, an angular velocity vector satisfying Eq.(2.46) exists:

$$\vec{\omega} = \frac{1}{2}\sum_{h=1}^{3} \hat{e}_h \wedge \frac{d\hat{e}_h}{dt} \tag{2.49}$$

To prove that it is unique, observe that a second angular velocity $\vec{\omega}^*$ satisfying Eq.(2.46) would lead to:

$$\boxed{\mathbf{0} = \frac{d\hat{e}_h}{dt} - \frac{d\hat{e}_h}{dt} = (\vec{\omega} - \vec{\omega}^*) \wedge \hat{e}_h, \qquad \text{for all} \quad h = 1,2,3}$$

But that means that the vector $(\vec{\omega} - \vec{\omega}^*)$ is the null vector, *i.e.* that $\vec{\omega}^* = \vec{\omega}$. The fact that the angular velocity does not depend the reference frame is left as an exercise, and will be proven later as Corollary 2.5. This completes the proof.

COROLLARY 2.1: *Let $\vec{u} = \vec{u}(t) \in \mathbb{R}^3$ be a vector at rest with respect to $\{\hat{e}_h\}_1^3$. Then,*

$$\frac{d\vec{u}}{dt} = \vec{\omega} \wedge \vec{u}$$

Proof: it follows directly from Poisson's theorem and from linearity of the vector product.

Recalling Eq.(2.30) and using this Corollary 2.1, we obtain:

$$P(t) = Q(t) + \sum_{h=1}^{3} y_h(P)\hat{e}_h(t) \Rightarrow \frac{d}{dt}(P-Q)(t) = \vec{\omega} \wedge (P-Q) \Rightarrow \dot{P}(t) = \dot{Q}(t) + \vec{\omega} \wedge (P-Q)$$

since $(P-Q)$ is at rest in the moving frame. This can be seen recalling Eq.(2.28) and Eq.(2.30):

$$\dot{P}(t) = \dot{Q}(t) + \sum_{h=1}^{3} y_h(P)\frac{d\hat{e}_h}{dt} = \dot{Q}(t) + \sum_{h=1}^{3} y_h(P)\,\vec{\omega}(t) \wedge \hat{e}_h(t)$$
$$= \dot{Q}(t) + \vec{\omega}(t) \wedge \textstyle\sum_{h=1}^{3} y_h(P)\hat{e}_h(t) = \dot{Q}(t) + \vec{\omega} \wedge (P-Q) \qquad (2.50)$$

where the $y_h(P)$ are the coordinates of P in the reference frame identified by the triple $\{\hat{e}_h\}_1^3$, hence they are constant in time. One can now write the fundamental formula of rigid body velocities:

$$\vec{v}_P = \vec{v}_Q + \vec{\omega} \wedge (P-Q) \qquad (2.51)$$

Whenever the motion is rigid, Eq.(2.51) holds in the reference frame identified by the triple $\{\hat{\imath}_h\}_1^3$, and the validity of Eq.(2.51) in the reference frame identified by $\{\hat{\imath}_h\}_1^3$ means that the motion is rigid. Obviously, the velocities vanish, instead, in the frame of the vectors $\{\hat{e}_h\}_1^3$.

Differentiating once more with respect to time, we obtain:

$$\ddot{P}(t) = \ddot{Q}(t) + \sum_{h=1}^{3} y_h(P)\frac{d^2\vec{e}_h}{dt^2} = \qquad (2.52)$$
$$= \ddot{Q}(t) + \textstyle\sum_{h=1}^{3} y_h(P)\frac{d}{dt}[\vec{\omega} \wedge \vec{e}_h] = \ddot{Q}(t) + \sum_{h=1}^{3} y_h(P)\left[\dot{\vec{\omega}} \wedge \vec{e}_h + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{e}_h)\right]$$

i.e.

$$\vec{a}_P = \vec{a}_Q + \dot{\vec{\omega}} \wedge (P-Q) + \vec{\omega} \wedge [\vec{\omega} \wedge (P-Q)] \qquad (2.53)$$

where we have introduced the acceleration $\vec{a}_P$ of the point *P*, and $\vec{a}_Q$, the acceleration of point Q.

Depending on $\vec{\omega}$, rigid motions are classified as:

1. *translational* if $\vec{\omega}(t) = 0\ \forall t \in \mathbb{R}$,
2. *roto-translational* if $\vec{\omega}(t)$ has constant direction,
3. *helical* if $\vec{\omega}(t)$ has constant direction and one point of the rigid body moves on a straight line parallel to $\vec{\omega}(t)$
4. *planar* if there is plane π such that $\vec{\omega} \perp \pi$,
5. *spherical* if $\exists c \in \mathcal{B}$ that does not move, which means $\vec{v}_c = 0$, hence

$$\vec{v}_P = \vec{\omega} \wedge (P-c) \qquad (2.54)$$

Scalarly multiplying $\vec{v}_P$ by $\vec{\omega}$, we have

$$\vec{v}_P \cdot \vec{\omega} = \vec{v}_Q \cdot \vec{\omega} + \vec{\omega} \wedge (P-Q) \cdot \vec{\omega} = \vec{v}_Q \cdot \vec{\omega} \qquad (2.55)$$

which means that the scalar quantity

$$I = \vec{v}_P \cdot \vec{\omega} \qquad (2.56)$$

does not depend on the reference point P. For this reason, I is called a *kinematic invariant scalar*. Multiplying $\vec{v}_P$ by $(P-Q)$, which is interesting for $P \neq Q$, Eq.(2.50) yields:

$$\vec{v}_P \cdot (P-Q) = \vec{v}_Q \cdot (P-Q) + \vec{\omega} \wedge (P-Q) \cdot (P-Q) = \vec{v}_Q \cdot (P-Q) \tag{2.57}$$

In particular, taking Q and P on a straight line parallel to $\vec{\omega}$, Eq.(2.50) yields

$$\vec{v}_P = \vec{v}_Q \tag{2.58}$$

Suppose $C \in \mathcal{B}$ is a known fixed point belonging to the body $\mathcal{B}$, *i.e.* a point of $\mathcal{B}$ whose velocity vanishes, $\vec{v}_C = \mathbf{0}$. The corresponding rigid motion is purely rotational, and taking $\vec{v}_C = \mathbf{0}$ and $Q = C$ in Eq.(2.51), one obtains:

$$\vec{v}_P = \vec{\omega} \wedge (P-C) \tag{2.59}$$

Alternatively, provided Q and $\vec{\omega}$ are known, a fixed point C can be found taking $P = C$ in Eq.(2.51), and solving the equation

$$\vec{v}_C = \vec{v}_Q + \vec{\omega} \wedge (C-Q) = \mathbf{0} \quad \text{which yields} \quad (C-Q) \wedge \vec{\omega} = \vec{v}_Q$$

from which C can be deduced. Indeed, this is a system of equations of the kind $\vec{a} \wedge \vec{v} = \vec{b}$, where $\vec{a}, \vec{b}$ are known while $\vec{v}$ is not. The solution can be obtained as follows:

1) If $\vec{a}, \vec{b} \neq 0 \Rightarrow \vec{a} \perp \vec{b}$, otherwise no solution
2) Multiplying by $\vec{a}$ on the right, we have $(\vec{a} \wedge \vec{v}) \wedge \vec{a} = \vec{b} \wedge \vec{a}$.

Equation (2.48) means that

$$(\vec{a} \wedge \vec{v}) \wedge \vec{a} = (\vec{a} \cdot \vec{a})\vec{v} - (\vec{a} \cdot \vec{v})\vec{a} = a^2\vec{v} - (\vec{a} \cdot \vec{v})\vec{a} = \vec{b} \wedge \vec{a} \tag{2.60}$$

This leads to $\vec{v}$ as a sum of one orthogonal and one parallel vector to $\vec{a}$:

$$\vec{v} = \frac{1}{a^2}\left[\vec{b} \wedge \vec{a} + (\vec{a} \cdot \vec{v})\vec{a}\right] = \frac{1}{a^2}\left[\vec{b} \wedge \vec{a} + a^2|\vec{v}| \cos\vartheta \, \hat{a}\right] \tag{2.61}$$

with $\hat{a} = \vec{a}/a$. Changing the length $|\vec{v}|$ of $\vec{v}$, one also changes the projection of the solution $\vec{v}$ on $\vec{a}$. As there is no constraint on $|\vec{v}| > 0$, $|\vec{v}| \cos\vartheta$ can be any real number, therefore there are ∞^1 solutions. Considering that $(\vec{v} \wedge \vec{a}) \wedge \vec{a} = \vec{a} \wedge \vec{b}$, we finally get:

$$(C-Q) = \frac{\vec{\omega} \wedge \vec{v}_Q}{\omega^2} + \lambda\vec{\omega}$$

Note: this says that the points of vanishing velocity of a rigid body form a line which can be expressed as above. It does not mean that whenever points are expressed like above, their velocity vanishes. This leads to the axis of motion, also known as Mozzi axis, and to the following theorem, which shows that points lying on that line have minimum magnitude velocity, but not necessarily null.

THEOREM 2.2: *Consider the roto-translational motion of a rigid body $\mathcal{B}$ and the instantaneous velocities of its points given by* $\vec{v}_P = \vec{v}_Q + \vec{\omega} \wedge (P-Q)$ *with* $\vec{\omega} \neq \mathbf{0}$. *Then the velocity* $\vec{v}_C$ *of any point* C *lying on the line defined by*

$$(C-Q) = \frac{\vec{\omega} \wedge \vec{v}_Q}{\omega^2} + \lambda\vec{\omega} = C_0 + \lambda\vec{\omega}, \quad C_0 = \frac{\vec{\omega} \wedge \vec{v}_Q}{\omega^2}, \quad \lambda \in \mathbb{R} \tag{2.62}$$

called Mozzi axis, is parallel to the angular velocity, $\vec{v}_C \parallel \vec{\omega}$, *and is the smallest among the velocities of the points of* $\mathcal{B}$.

Proof: Rigid motion means that points in $\mathcal{B}$ have constant distance from each other. For a point C lying on the line Eq.(2.62) we have:

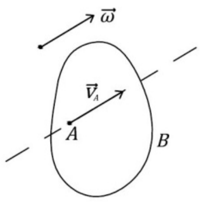

Figure 2.9. *Roto-translational motion of a rigid body B with Mozzi axis*

$$\begin{aligned}\vec{v}_C &= \vec{v}_Q + \vec{\omega} \wedge (C - Q) = \vec{v}_Q + \vec{\omega} \wedge \left[\frac{\vec{\omega} \wedge \vec{v}_Q}{\omega^2} + \lambda\vec{\omega}\right] \\ &= \vec{v}_Q + \frac{1}{\omega^2}\vec{\omega} \wedge (\vec{\omega} \wedge \vec{v}_Q) + \lambda\,\vec{\omega} \wedge \vec{\omega} = \vec{v}_Q + \frac{1}{\omega^2}(\vec{v}_Q \wedge \vec{\omega}) \wedge \vec{\omega} \\ &= \vec{v}_Q + \frac{1}{\omega^2}\left[(\vec{v}_Q \cdot \vec{\omega})\,\vec{\omega} - (\vec{\omega} \cdot \vec{\omega})\,\vec{v}_Q\right] \\ &= \vec{v}_Q + \frac{1}{\omega^2}(\vec{v}_Q \cdot \vec{\omega})\,\vec{\omega} - \vec{v}_Q = \frac{I}{\omega^2}\,\vec{\omega}\end{aligned} \tag{2.63}$$

which is indeed parallel to $\vec{\omega}$. Moreover, a point P that does not belong to the straight line (2.62) has velocity $\vec{v}_P = \vec{v}_C + \vec{\omega} \wedge (P - C)$, with one component along the direction of $\vec{\omega}$ and one component perpendicular to $\vec{\omega}$. Therefore,

$$|\vec{v}_P| = \sqrt{v_C{}^2 + |\vec{\omega} \wedge (P - C)|^2} > |\vec{v}_C| \tag{2.64}$$

If P belongs to the $Mozzi's\ axis$, $\vec{v}_P = \vec{v}_C$ *i.e.* all points in Mozzi axis have same velocity.

Note: Eq.(2.62) represents a straight line (∞^1 points) passing through the point C_0 parallel to $\vec{\omega}$ and parametrized by λ. This line may change in time, as C_0 and $\vec{\omega}$ may, but at each time there is one, necessarily unique, such a line.

Note: we distinguish the notion of motion that takes a finite time to occur, from the *instantaneous* motion that is described by $\vec{v}_P = \vec{v}_Q + \vec{\omega} \wedge (P - Q)$ at a given time *t*. For instance, the motion may not be rotational, but at any time instant *t* there may be one point C that changes in time and that allows us to express the instantaneous velocity of any other point *P* as in Eq.(2.58). In this case, C is called instantaneous center of rotation. For instance, the points of the tires of a car that touch the ground have (approximately) this property.

REMARK: the most general rigid motion is *roto-translational*.

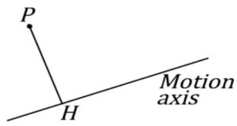

Figure 2.10. *Orthogonal projection of P on motion axis*

Recalling Eq.(2.59), the last term of Eq.(2.53) may be written as:

$$\vec{\omega} \wedge (\vec{\omega} \wedge (P - Q)) = (\vec{\omega} \cdot (P - Q))\vec{\omega} - (\vec{\omega} \cdot \vec{\omega})(P - Q) \tag{2.65}$$

Let H be the orthogonal projection of the point P on a straight line parallel to $\vec{\omega}$, and take $Q = H$. We may then rewrite Eq.(2.53) as:

$$\begin{aligned}\vec{a}_P &= \vec{a}_H + \dot{\vec{\omega}} \wedge (P - H) + \vec{\omega} \wedge (\vec{\omega} \wedge (P - H)) \\ &= \vec{a}_H + \dot{\vec{\omega}} \wedge (P - H) + \underbrace{(\vec{\omega} \cdot (P - H))}_{\vec{\omega} \perp (P-H) \Rightarrow \overset{\parallel}{0}}\vec{\omega} - \omega^2(P - H)\end{aligned}$$

i.e.

$$\vec{a}_P = \vec{a}_H + \dot{\vec{\omega}} \wedge (P - H) - \omega^2(P - H) \tag{2.66}$$

If H and $\vec{\omega}$ are fixed,

$$\vec{a}_P = -\omega^2(P - H) \tag{2.67}$$

which means that the acceleration of P is directed towards the Mozzi axis, *i.e.* it is a centripetal acceleration, and its magnitude is given by $\omega^2|P - H|$, where $|P - H|$ is the distance of P from the Mozzi axis.

In the case of planar rigid motions, one may take $\hat{e}_3 = \hat{\imath}_3$, so that the angle $\vartheta(t)$ suffices to determine the orientation of the frame at rest with respect to $\mathcal{B}$

$$\frac{d\hat{e}_1}{dt} = -\dot{\vartheta}\sin\vartheta\,\hat{\imath}_1 + \dot{\vartheta}\cos\vartheta\,\hat{\imath}_2 \tag{2.67}$$

$$\frac{d\hat{e}_2}{dt} = -\dot{\vartheta}\cos\vartheta\,\hat{\imath}_1 - \dot{\vartheta}\sin\vartheta\,\hat{\imath}_2 \tag{2.68}$$

$$\frac{d\hat{e}_3}{dt} = \mathbf{0} \tag{2.69}$$

Then, recalling Poisson formulae (2.46) we have:

$$\frac{d\hat{e}_2}{dt}\cdot\hat{e}_3 = \vec{\omega}\wedge\hat{e}_2\cdot\hat{e}_3 = \begin{vmatrix}\omega_1 & \omega_2 & \omega_3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} = \omega_1$$

$$\frac{d\hat{e}_3}{dt}\cdot\hat{e}_1 = \vec{\omega}\wedge\hat{e}_3\cdot\hat{e}_1 = \begin{vmatrix}\omega_1 & \omega_2 & \omega_3\\ 0 & 0 & 1\\ 1 & 0 & 0\end{vmatrix} = \omega_2$$

$$\frac{d\hat{e}_1}{dt}\cdot\hat{e}_2 = \vec{\omega}\wedge\hat{e}_1\cdot\hat{e}_2 = \begin{vmatrix}\omega_1 & \omega_2 & \omega_3\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix} = \omega_3$$

and we can write:

$$\vec{\omega} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3 = \left(\frac{d\hat{e}_2}{dt}\cdot\hat{e}_3\right)\hat{e}_1 + \left(\frac{d\hat{e}_3}{dt}\cdot\hat{e}_1\right)\hat{e}_2 + \left(\frac{d\hat{e}_1}{dt}\cdot\hat{e}_2\right)\hat{e}_3 \tag{2.70}$$

Because $\hat{e}_3 = \hat{\imath}_3$, we can write

$$\omega_1 = \frac{d\hat{e}_2}{dt}\cdot\hat{e}_3 = -\dot{\vartheta}(\cos\vartheta\,\hat{\imath}_1\cdot\hat{\imath}_3 + \sin\vartheta\,\hat{\imath}_2\cdot\hat{\imath}_3) = 0$$

Then, (2.69) implies $\omega_2 = 0$, and (2.70) eventually turns:

$$\vec{\omega} = \left(\frac{d\hat{e}_1}{dt}\cdot\hat{e}_2\right)\hat{\imath}_3 \tag{2.71}$$

i.e.

$$\vec{\omega} = \left(-\dot{\vartheta}\sin\vartheta\,\hat{\imath}_1 + \dot{\vartheta}\cos\vartheta\,\hat{\imath}_2\right)\cdot(-\sin\vartheta\,\hat{\imath}_1 + \cos\vartheta\,\hat{\imath}_2)\hat{\imath}_3 = \dot{\vartheta}(\sin^2\vartheta + \cos^2\vartheta)\hat{\imath}_3$$

which means

$$\vec{\omega} = \dot{\vartheta}\hat{\imath}_3 \qquad \text{and} \qquad \omega = \dot{\vartheta} \tag{2.72}$$

THEOREM 2.3: *for points $A, B \in \mathcal{B}$ and planar rigid motion, let the straight line perpendicular to the velocity $\vec{v}_A$ and that perpendicular to the velocity $\vec{v}_B$ meet in a single point H (not necessarily belonging to $\mathcal{B}$). Then, H is the center of rotation of the instantaneous motion.*

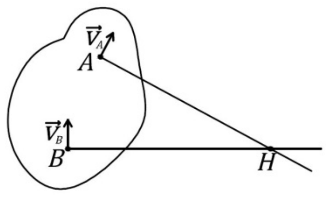

Figure 2.11. *The center of rotation of the instantaneous motion*

Equation (2.59) and following shows how to find the point that is at rest, *i.e.* verifying $\vec{v}_C = \mathbf{0}$, which means $(C - A)\wedge\vec{\omega} = \vec{v}_A$ as well as $(C - B)\wedge\vec{\omega} = \vec{v}_B$. But this means that the segments joining C to A and C to B are perpendicular to $\vec{v}_A$ and $\vec{v}_B$, respectively. Thus, the intersection of the line through A perpendicular to $\vec{v}_A$ and the one through B perpendicular to $\vec{v}_B$ is the point with zero velocity at time t.

Recall that $P(t) = Q(t) + R(t)\vec{p}$. Deriving it with respect to time, one obtains $\vec{v}_P = \vec{v}_Q + \dot{R}\vec{p}$, because $\vec{p}$ is the constant vector $y_1(P)\hat{\imath}_1 + y_2(P)\hat{\imath}_2 + y_3(P)\hat{\imath}_3$. It is then interesting to consider some property of R and of $\dot{R}$. Write $R^TP = R^TQ + R^TR\vec{p}$ and $(R^TP - R^TQ) = R^T(P - Q) = \vec{p}$. Note that:

$\frac{d}{dt}(RR^T) = \dot{R}R^T + R\dot{R}^T$ hence that $\frac{d}{dt}(RR^T) = \frac{d}{dt}I = \mathbf{0}$, which implies $\dot{R}R^T = -R\dot{R}^T$ (2.73)

where $\dot{R}^T = (\dot{R})^T$ and $R\dot{R}^T = (\dot{R}R^T)^T$. Then, $\dot{R}R^T + (\dot{R}R^T)^T = \mathbf{0}$. Thetrefore, $W = \dot{R}R^T$ is *anti-symmetric*: $W = -W^T$. One may also explicitly verify that $W\vec{a} \cdot \vec{a} = \vec{a} \cdot W^T\vec{a}$ (which is a property of algebra: $\vec{u} \cdot W\vec{v} = W^T\vec{u} \cdot \vec{v}$, check directly by yourself). Then,
$\vec{a} \cdot W^T\vec{a} = -\vec{a} \cdot W\vec{a}$, which means that $\vec{a} \cdot W\vec{a} = -\vec{a} \cdot W\vec{a}$ *i.e.* is orthogonal to $\vec{a}$. Now, there $\exists!\ \vec{\omega}$ such that:

$$W\vec{a} = \vec{\omega} \wedge \vec{a}, \quad \forall \vec{a} \in \mathbb{R}^3 \tag{2.74}$$

Therefore:

$$\vec{v}_P = \vec{v}_Q + \dot{R}\vec{p} = \vec{v}_Q + \underbrace{\dot{R}R^T}_{W}\ \underbrace{(P-Q)}_{\vec{a}} = \vec{v}_Q + \vec{\omega} \wedge (P-Q) \tag{2.75}$$

which is another way of expressing $\vec{\omega}$.

2.2 RELATIVE KINEMATICS

Classical mechanics postulates that the distance between two points, and the time between two events do not depend on the reference frame. For this reason, *"rest"* and *"mobile"* frames are purely conventional. So, we call rest frame the one with origin in O and axes $\hat{\imath}_h$, and mobile frame the one with origin in Q and axes $\hat{e}_h$. Here, none of the frames is meant to be at rest with respect to a possible rigid object $\mathcal{B}$, but the reference frames themselves, with their basis vectors, can be considered rigid.

Given a point P, let $\vec{v}_a$ be its velocity with respect to the rest frame, and $\vec{v}_r$ its velocity with respect to the mobile frame, where the index *a* stands for "absolute" and the index *r* stands for "relative". Then we have:

$$(P-O) = (Q-O) + (P-Q)$$

with

$$(P-Q) = y_1\hat{e}_1 + y_2\hat{e}_2 + y_3\hat{e}_3$$

and

$$\vec{v}_a(P) = \frac{d(P-O)}{dt} = \frac{d}{dt}\left[(Q-O) + \textstyle\sum_{h=1}^3 y_h\hat{e}_h\right] = \vec{v}_a(Q) + \textstyle\sum_{h=1}^3 \left[\dot{y}_h\hat{e}_h + y_h\frac{d\hat{e}_h}{dt}\right] \tag{2.76}$$

(using Eq. (2.46)) $\quad = \vec{v}_a(Q) + \vec{v}_r(P) + \sum_{h=1}^3 y_h\frac{d\hat{e}_h}{dt} = \vec{v}_a(Q) + \vec{v}_r(P) + \vec{\omega} \wedge (P-Q)$

because $y_h\vec{\omega} \wedge \hat{e}_h = \vec{\omega} \wedge y_h\hat{e}_h$ and summing over *h* yields $(P-Q)$. Here, it is assumed that *P* moves in both frames, hence $\dot{y}_h \neq 0$ in general, with $\vec{v}_r(P) = \sum_{h=1}^3 \dot{y}_h\hat{e}_h$. Then we have:

$$\vec{v}_a(P) = \vec{v}_r(P) + \vec{v}_\tau(P), \tag{2.77}$$

where we have introduced the *drag velocity*

$$\vec{v}_\tau(P) = \vec{v}_a(Q) + \vec{\omega} \wedge (P-Q) \tag{2.78}$$

It appears that $\vec{v}_\tau(P)$ would be the velocity of *P*, if *P* were at rest with respect to the mobile frame, as in the rigid body case, because the reference frames, *i.e.* their basis vectors, are indeed kind of rigid bodies. Note: the time derivative d/dt is taken in the rest frame, considered the frame of the observer. An observer in the mobile frame is like a passenger in a train, that observes the same phenomenon, while passing. Also, the projection y_h of $(P-Q)$ on the axis of $\hat{e}_h$ is a length, hence is as well as its variation is the same in the rest and in the mobile frames. Differentiating further we have:

$$\vec{a}_a(P) = \frac{d\vec{v}_a(P)}{dt} = \frac{d\vec{v}_r(P)}{dt} + \frac{d\vec{v}_\tau(P)}{dt}$$

where

$$\frac{d\vec{v}_r}{dt} = \frac{d}{dt}\textstyle\sum_{h=1}^3 \dot{y}_h\hat{e}_h = \sum_{h=1}^3 \ddot{y}_h\hat{e}_h + \sum_{h=1}^3 \dot{y}_h\,\vec{\omega} \wedge \sum_{h=1}^3 \hat{e}_h = \vec{a}_r + \vec{\omega} \wedge \vec{v}_r \tag{2.80}$$

and

$$\frac{d\vec{v}_\tau}{dt} = \frac{d}{dt}(\vec{v}_a(Q) + \vec{\omega} \wedge (P - Q)) = \vec{a}_a(Q) + \frac{d}{dt}\left(\vec{\omega} \wedge \sum_{h=1}^{3} y_h \hat{e}_h\right)$$

$$= \vec{a}_a(Q) + \dot{\vec{\omega}} \wedge (P - Q) + \vec{\omega} \wedge [\textstyle\sum_{h=1}^{3} \dot{y}_h \hat{e}_h + \vec{\omega} \wedge \sum_{h=1}^{3} y_h \hat{e}_h] \quad (2.81)$$

$$= \vec{a}_a(Q) + \dot{\vec{\omega}} \wedge (P - Q) + \vec{\omega} \wedge \vec{v}_r(P) + \vec{\omega} \wedge \big(\vec{\omega} \wedge (P - Q)\big)$$

$\vec{a}_a(Q)$ being the acceleration of Q in the rest frame. Moreover, we can write:

$$\vec{a}_a(P) = \vec{a}_r(P) + \vec{a}_\tau(P) + \vec{a}_c(P) \quad (2.82)$$

where

$$\vec{a}_\tau(P) = \vec{a}_a(Q) + \dot{\vec{\omega}} \wedge (P - Q) + \vec{\omega} \wedge (\vec{\omega} \wedge (P - Q)) \text{ the } \textit{drag acceleration} \quad (2.83)$$

$$\vec{a}_c(P) = 2\vec{\omega} \wedge \vec{v}_r(P) \qquad \text{the } \textit{Coriolis acceleration} \quad (2.84)$$

Interestingly, the Coriolis acceleration vanishes in both cases $\vec{\omega} = \mathbf{0}$ and $\vec{v}_r = \mathbf{0}$.

Question: given a rigid body $\mathcal{B}$ may one write

$$\vec{v}_a(P) = \vec{v}_a(H) + \vec{\omega}_a \wedge (P - H)$$

as for the fundamental formula Eq.(2.51), for some $\vec{\omega}$ which does not depend on the reference frame? It suffices to use the decomposition

$$\vec{\omega}_a = \vec{\omega}_r + \vec{\omega} \quad (2.85)$$

in terms of the angular velocity $\vec{\omega}$ of the mobile frame and of the angular velocity $\vec{\omega}_r$ of $\mathcal{B}$ within the mobile frame, which implies $\vec{\omega} = \vec{\omega}_a - \vec{\omega}_r$.

Proof: Let $P, H \in \mathcal{B}$ and assume Eq.(2.84) holds. Then, Eq.(2.51) for the rigid body writes:

$$\vec{v}_a(P) = \vec{v}_a(H) + \vec{\omega}_a \wedge (P - H)$$

The same holds in the mobile frame:

$$\vec{v}_r(P) = \vec{v}_r(H) + \vec{\omega}_r \wedge (P - H) \quad (2.86)$$

In addition, the drag velocities take the form:

$$\vec{v}_\tau(P) = \vec{v}_a(Q) + \vec{\omega} \wedge (P - Q) \quad \text{and} \quad \vec{v}_\tau(H) = \vec{v}_a(Q) + \vec{\omega} \wedge (H - Q) \quad (2.87)$$

where Q is the origin of the mobile frame and $\vec{v}_a(Q)$ is its velocity with respect to the rest frame. Then, Eq.(2.77) yields:

$$\vec{v}_a(P) = \vec{v}_r(P) + \vec{v}_a(Q) + \vec{\omega} \wedge (P - Q) \quad \text{and} \quad \vec{v}_a(H) = \vec{v}_r(H) + \vec{v}_a(Q) + \vec{\omega} \wedge (H - Q) \quad (2.88)$$

Because $\vec{v}_a(Q)$ occurs in both expressions of Eq.(2.88), it follows that:

$$\vec{v}_a(P) - \vec{v}_r(P) = \vec{v}_a(H) - \vec{v}_r(H) + (\vec{\omega}_a - \vec{\omega}_r) \wedge [(P - Q) - (H - Q)]$$

$$= \vec{v}_a(H) - \vec{v}_r(H) + (\vec{\omega}_a - \vec{\omega}_r) \wedge (P - H)$$

where P and H are arbitrary points. Now recalling the Euler angles, we can write:

$$\vec{\omega} = \dot{\psi}\hat{\imath}_3 + \dot{\vartheta}\tilde{e}_1 + \dot{\phi}\tilde{e}_3 = \dot{\psi}\hat{\imath}_3 + \dot{\vartheta}(\cos\psi\,\hat{\imath}_1 + \sin\psi\,\hat{\imath}_2) + \dot{\phi}(\cos\vartheta\,\tilde{e}_3 - \sin\vartheta\,\tilde{e}_2) \quad (2.89)$$

where $\tilde{e}_2 = -\sin\psi\,\hat{\imath}_1 + \cos\psi\,\hat{\imath}_2$, and $\tilde{e}_1 = \hat{\imath}_3$ so that

$$\vec{\omega} = \big(\dot{\vartheta}\cos\psi + \dot{\phi}\sin\vartheta\sin\psi\big)\hat{\imath}_1 + \big(\dot{\vartheta}\sin\psi - \dot{\phi}\sin\vartheta\cos\psi\big)\hat{\imath}_2 + \big(\dot{\psi} + \dot{\phi}\cos\vartheta\big)\hat{\imath}_3 \quad (2.90)$$

and

$$\vec{\omega} = \big(\dot{\vartheta}\cos\phi + \dot{\psi}\sin\vartheta\sin\phi\big)\hat{e}_1 - \big(\dot{\vartheta}\sin\phi - \dot{\psi}\sin\vartheta\cos\phi\big)\hat{e}_2 + \big(\dot{\phi} + \dot{\psi}\cos\vartheta\big)\hat{\imath}_3 \quad (2.91)$$

2.3 Precession

A rigid motion with a fixed point is called spherical. The spherical motions that can be decomposed as a *rotation* with respect to a fixed axis (*precession axis*) plus a rotation about one axis belonging to

$\mathcal{B}$ (*body axis*) in such a way that the angle between these two axes is constant and different from 0 and π , is called *precession*. Suppose $\hat{e}_3$ is the body axis and $\hat{\imath}_3$ the precession axis. Then, we have:

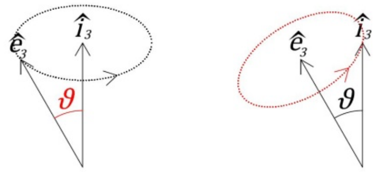

Figure 2.12. *Spherical motions: rotation about precession axis(left) and body axis(right)*

$$\boxed{\vec{\omega} = \dot{\psi}\hat{\imath}_3 + \dot{\phi}\hat{e}_3} \tag{2.92}$$

i.e. recalling Eq.(2.36):

$$\boxed{\vec{\omega} = \dot{\phi}\sin\vartheta\,(\sin\psi\,\hat{\imath}_1 - \cos\psi\,\hat{\imath}_2) + \left(\dot{\psi} + \dot{\phi}\cos\vartheta\right)\hat{\imath}_3} \tag{2.93}$$

Let $\vec{c}(t)$ be any vector. In different reference frames, its evolution appears different. Let $\dot{\vec{c}}$ be its time derivative with respect to the rest frame, $\vec{c}'$ its time derivative with respect to the mobile frame, and $\vec{\omega}$ the angular velocity of the mobile frame (seen as a rigid body) with respect to the rest frame. We can prove the following

THEOREM 2.4:
$$\dot{\vec{c}} = \vec{c}' + \vec{\omega} \wedge \vec{c} \tag{2.94}$$

Proof: in the mobile frame, we have $\vec{c} = \sum_{h=1}^{3} c_h \hat{e}_h$. The Poisson formulae (2.46) then yield:

$$\dot{\vec{c}} = \underbrace{\textstyle\sum_{h=1}^{3} \dot{c}_h \hat{e}_h}_{\vec{c}\prime} + \vec{\omega} \wedge \underbrace{\textstyle\sum_{h=1}^{3} c_h \hat{e}_h}_{\vec{c}} \tag{2.95}$$

COROLLARY 2.5:
$$\dot{\vec{\omega}} = \vec{\omega}' \tag{2.96}$$

Proof: taking $\vec{c} = \vec{\omega}$ in Eq.(2.94), yields

$$\dot{\vec{\omega}} = \vec{\omega}' + \overbrace{\vec{\omega} \wedge \vec{\omega}}^{\mathbf{0}}$$

So the time derivative angular acceleration is intrinsic to the motion and it does not depend on the reference frame. Then, $\vec{\omega}$ differs at most by a constant in the two frames, and if the constant is initially 0, it remains so and it does not depend on the chosen mobile frame with respect to which a rigid body may be at rest.