ESERCIZI SU SPAZI DI SOBOLEV A LEZIONE

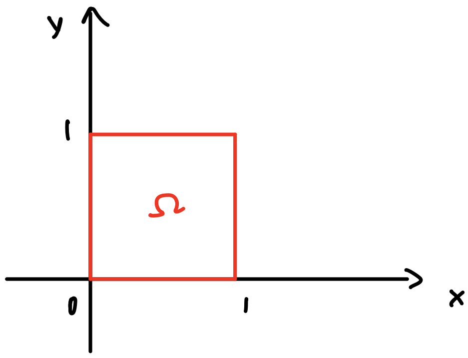
Esercizio 1 Sia $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1 \text{ e } 0 < y < 1\}$; si consideri la funzione:

$$u(x,y) = x^a + y \quad \text{per } (x,y) \in \Omega \text{ e } a \in \mathbb{R}_+ .$$

a) Determinare per quali valori di $a \in \mathbb{R}_+$ risulta $u \in L^2(\Omega)$.

b) Determinare per quali valori di $a \in \mathbb{R}_+$ u é derivabile in senso debole rispetto a x e rispetto a y.

c) Determinare per quali valori di $a \in \mathbb{R}_+$ risulta $u \in H^1(\Omega)$.



a) $u \in C^1(\bar{\Omega}) \subset L^\infty(\Omega) \subset L^2(\Omega)$

b) $u \in C^1(\Omega) \implies$ DERIVATE CLASSICHE COINCIDONO CON LE DEBOLI

$$\underbrace{u_x(x,y) = a\,x^{a-1}}_{\in L^1_{loc}(\Omega)} \quad e \quad u_y(x,y) = 1 \in L^1_{loc}(\Omega)$$

$(x,y) \in \Omega$

c) - $u \in L^2(\Omega)$ già verificato

- $u_y(x,y) = 1 \in L^2(\Omega)$

- $u_x \in L^2(\Omega)$?

$$\int_\Omega u_x^2 \, dx\, dy = \int_0^1 \int_0^1 a^2 x^{2a-2} \, dx \, dy = \int_0^1 a^2 x^{2a-2} dx < +\infty$$

$\Updownarrow$

$2 - 2a < 1$

$\iff a > \frac{1}{2}$

Per $a > \frac{1}{2} \implies u \in H^1(\Omega)$ $\square$

Esercizio 5. Si consideri la funzione

$$u_\alpha(x) = \frac{1}{\|x\|^\alpha} \quad \text{in } \mathbb{R}^n \setminus \{0\}\,, \alpha > 0\,. \qquad \underline{n \geq 2}$$

1) Dire per quali valori di $\alpha > 0$, $u \in L^1_{loc}(\mathbb{R}^n)$;
2) Provare che u è derivabile in senso debole per ogni $0 < \alpha < n-1$;
3) Dire per quali valori di $0 < \alpha < n-1$, $u \in H^1(B_1(0))$ dove $B_1(0) \subset \mathbb{R}^n$ è la palla unitaria centrata nell'origine.

1) $u_\alpha \in L^1_{loc}(\mathbb{R}^n)$?

Sia $K \subset \mathbb{R}^n$ compatto e $R > 0$: $K \subseteq B_R(0)$

$$\int_K |u_\alpha(x)|\,dx \leq \int_{B_R(0)} |u_\alpha(x)|\,dx = n\omega_n \int_0^R \frac{1}{r^\alpha} \cdot r^{n-1}\,dr < +\infty$$

$$\Updownarrow$$

$$\alpha - n + 1 < 1$$

$$\Updownarrow$$

$$\alpha < n$$

2) Per $0 < \alpha < n-1$ determino le deriv. deboli

$$u_\alpha(x) = (x_1^2 + \dots x_n^2)^{-\frac{\alpha}{2}}$$

$$(u_\alpha)_{x_i}(x) = -\alpha x_i (x_1^2 + \dots + x_n^2)^{-\frac{\alpha}{2}-1} = -\frac{\alpha x_i}{\|x\|^{\alpha+2}} \qquad \underline{x \neq 0}$$

deriv. $\underline{\text{classica}}$

$\underline{(u_\alpha)_{x_i}} \in L^1_{loc}(\mathbb{R}^n)$?

Per $\alpha + 1 < n$

$$|(u_\alpha)_{x_i}| = \frac{\alpha |x_i|}{\|x\|^{\alpha+2}} \leq \frac{\alpha \|x\|}{\|x\|^{\alpha+2}} = \frac{\alpha}{\|x\|^{\alpha+1}} \in L^1_{loc}(\mathbb{R}^n)$$

$\underline{\underline{\alpha < n-1}}$

DEFINIZIONE di DERIVATA DEBOLE :

$$\int_{\mathbb{R}^n} u_\alpha \varphi_{x_i}\,dx = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} u_\alpha \varphi_{x_i}\,dx$$

$\varphi\in\mathcal{D}(\mathbb{R}^n)$

$$= \lim_{\varepsilon\to 0^+}\Big[\underbrace{\int_{\partial B_\varepsilon(0)} u_\alpha\varphi n_i\,dS}_{o(1)\ ?} - \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} (u_\alpha)_{x_i}\varphi\,dx\Big]$$

TEO. LEBESGUE $\quad (u_\alpha)_{x_i}\in L^1_{loc}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n}(u_\alpha)_{x_i}\varphi\,dx$$

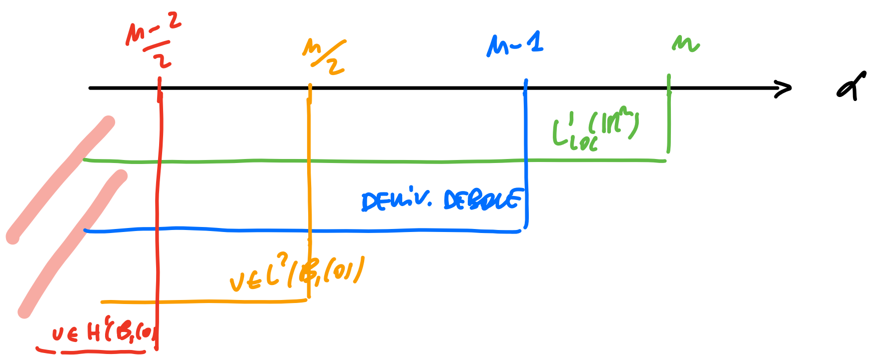
$$\left|\int_{\partial B_\varepsilon(0)} u_\alpha\varphi n_i\,dS\right| \le \frac{1}{\varepsilon^\alpha}\|\varphi\|_\infty \underbrace{\|\vec{n}\|_{\mathbb{R}^n}}_{=1} \overbrace{\int_{\partial B_\varepsilon(0)} 1\,dS}^{n\omega_n\varepsilon^{n-1}}$$

$$u_\alpha\big|_{\partial B_\varepsilon} = \frac{1}{\varepsilon^\alpha}$$

$$= \|\varphi\|_\infty\, n\,\omega_n\,\varepsilon^{n-1-\alpha} \longrightarrow 0$$

$n-1-\alpha>0$

$\iff \quad \alpha<n-1$



3) $u_\alpha \in L^2(B_1(0))$?

$$\int_{B_1(0)} |u_\alpha(x)|^2 \, dx = n\,\omega_n \int_0^1 \frac{1}{r^{2\alpha}} \, r^{n-1} \, dr < +\infty$$

$2\alpha - n + 1 < 1$

$\Leftrightarrow \quad \alpha < \frac{n}{2}$

$\frac{n}{2} \leq n-1 \quad \Leftrightarrow \quad n \leq 2n-2$

$\Leftrightarrow \quad n \geq 2$

GRADIENTE DEBOLE di UNA FUNZ. RADIALE

$$u_\alpha(x) = v_\alpha(\|x\|)$$

$$(u_\alpha)_{x_i}(x) = v_\alpha'(\|x\|) \frac{x_i}{\|x\|}$$

$$|\nabla u_\alpha(x)|^2 = \sum_{i=1}^n (v_\alpha'(\|x\|))^2 \cdot \frac{x_i^2}{\|x\|^2}$$

$$= (v_\alpha'(r))^2 \underbrace{\sum_{i=1}^n \frac{x_i^2}{\|x\|^2}}_{=1} = (v_\alpha'(r))^2$$

$$\boxed{|\nabla u_\alpha(x)|^2 = (v_\alpha'(r))^2}$$

!!!

$$u_\alpha(x) = \frac{1}{\|x\|^\alpha} \longrightarrow v_\alpha(r) = \frac{1}{r^\alpha}$$

$$v_\alpha'(r) = -\frac{\alpha}{r^{\alpha+1}}$$

$$\boxed{\int_{B_1(0)} |\nabla u_\alpha(x)|^2 \, dx = n\,\omega_n \int_0^1 (v_\alpha'(r))^2 \, r^{n-1} \, dr}$$

VALE $\forall$ FUNZ. RADIALE

$$= n\,\omega_n \int_0^1 \frac{\alpha^2}{r^{2\alpha+2}}\, r^{n-1}\, dr < +\infty$$

$$\Updownarrow$$

$$2\alpha + 2 - n + 1 < 1$$

$$\Updownarrow$$

$$\alpha < \frac{n-2}{2}$$

Esercizio 9. Sia $n \geq 1$. Date $u \in H^1(\mathbb{R}^n)$ e $\psi \in C_c^\infty(\mathbb{R}^n)$, dimostrare che

$$u\psi \in H^1(\mathbb{R}^n) \quad \text{e} \quad (u\psi)_{x_i} = u_{x_i}\psi + u\psi_{x_i} \quad \text{(derivate in senso debole) per ogni } i = 1, ..., n.$$

1. $u\psi \in L^2(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (u\psi)^2 \, dx \leq \|\psi\|_\infty^2 \int_{\mathbb{R}^n} u^2 \, dx < +\infty$$

2. $u\psi \in L^2(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n)$, cerco deriv. deboli

$$\int_{\mathbb{R}^n} (u\psi)\, \varphi_{x_i} \, dx = \boxed{\int_{\mathbb{R}^n} u\, (\psi\varphi)_{x_i} \, dx} - \int_{\mathbb{R}^n} u\, \psi_{x_i} \varphi \, dx$$

$\varphi \in C_c^\infty(\mathbb{R}^n)$

$$(\psi\varphi)_{x_i} = \psi_{x_i}\varphi + \psi\varphi_{x_i} \implies \psi\varphi_{x_i} = (\psi\varphi)_{x_i} - \psi_{x_i}\varphi$$

u deriv. in senso debole

$$= -\int_{\mathbb{R}^n} u_{x_i} \psi\varphi \, dx - \int_{\mathbb{R}^n} u\, \psi_{x_i} \varphi$$

$$\int_{\mathbb{R}^n} (u\psi)\, \varphi_{x_i} \, dx = -\int_{\mathbb{R}^n} \left(u_{x_i}\psi + u\psi_{x_i} \right) \varphi \, dx \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^n)$$

$$\implies (u\psi)_{x_i} = \underline{u_{x_i}\psi} + u\psi_{x_i} \quad \overset{?}{\in} \underline{L^2(\mathbb{R}^n)}$$

$$\| u_{x_i}\psi + u\psi_{x_i} \|_{L^2(\mathbb{R}^n)} \leq \| u_{x_i}\psi \|_{L^2(\mathbb{R}^n)} + \| u\psi_{x_i} \|_{L^2(\mathbb{R}^n)}$$

$$= \left(\int_{\mathbb{R}^n} u_{x_i}^2 \psi^2 \, dx \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^n} u^2 \psi_{x_i}^2 \, dx \right)^{\frac{1}{2}}$$

$$\leq \|\varphi\|_{L^\infty(\mathbb{R}^n)} \underbrace{\|u_{x_i}\|_{L^2(\mathbb{R}^n)}}_{\text{FINITO}} + \|\varphi_{x_i}\|_{L^\infty(\mathbb{R}^n)} \underbrace{\|u\|_{L^2(\mathbb{R}^n)}}_{\text{FINITO}}$$

$u \in H^1(\mathbb{R}^n)$

$\blacksquare$

Esercizio 8. Data $v \in L^1_{loc}(I)$, $I = (a,b) \subseteq \mathbb{R}$, sia $\quad -\infty \le a < b \le +\infty$

$$w(x) := \int_{x_0}^{x} v(s)\,ds \quad x_0 \in I \text{ fissato.} \qquad x_0 \in (a,b)$$

Provare che w è derivabile in senso debole e $w' = v$.

- $w \in L^1_{loc}(a,b)$ $\qquad$ K COMPATTO $\subset (a,b)$

$$\int_K |w(x)|\,dx \le \int_c^d \left| \int_{x_0}^{x} v(s)\,ds \right| dx \le \underbrace{\int_c^d \int_c^d |v(s)|\,ds\,dx}_{<+\infty}$$

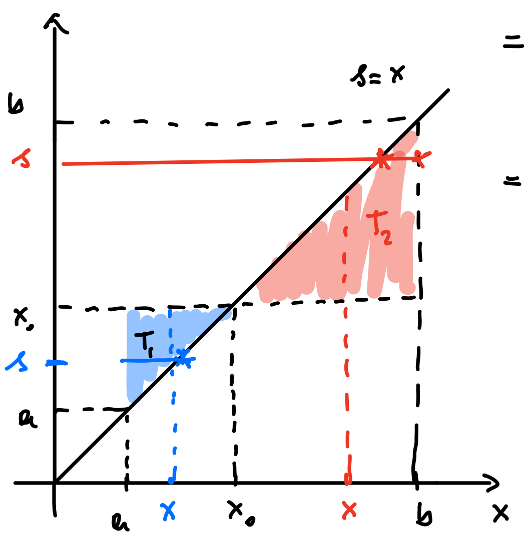
SIA (c,d) INTERVALLO CHE CONTIENE K e x_0

perchè $v \in L^1_{loc}(a,b)$

- w è DERIVABILE IN SENSO DEBOLE?

$$-\langle w', \varphi \rangle = \langle w, \varphi' \rangle = \int_a^b w(x)\,\varphi'(x)\,dx$$

$\forall \varphi \in C_c^\infty(a,b)$ $\qquad$ $w \in L^1_{loc}(a,b)$



$$= \int_a^b \left(\int_{x_0}^{x} v(s)\,ds \right) \varphi'(x)\,dx$$

$$= -\int_a^{x_0} \int_x^{x_0} v(s)\,ds\,\varphi'(x)\,dx$$

$$+ \int_{x_0}^{b} \int_{x_0}^{x} v(s)\,ds\,\varphi'(x)\,dx$$

$$= -\iint_{T_1} v\varphi'\,ds\,dx + \iint_{T_2} v\varphi'\,ds\,dx$$

$$= -\int_a^{x_0} v(s) \left(\int_a^{s} \varphi'(x)\,dx \right) ds + \int_{x_0}^{b} v(s) \left(\int_s^{b} \varphi'(x)\,dx \right) ds$$

$$= -\int_a^{x_0} v(s)\,\varphi(s)\,ds - \int_{x_0}^{b} v(s)\,\varphi(s)\,ds$$

$$\left(\int_a^s \varphi'(x)\,dx = \varphi(s) - \underbrace{\varphi(a)}_{=0 \text{ PERCHÉ } \varphi \in C_c^\infty(a,b)} \right)$$

$$= -\int_a^b v(s)\,\varphi(s)\,ds$$

DUNQUE

$$-\langle w', \varphi \rangle = -\int_a^b v(s)\,\varphi(s)\,ds = -\langle v, \varphi \rangle$$

$$\forall \varphi \in C_c^\infty(a,b)$$

$$\Rightarrow \quad w' = v$$

la deriv. distrib. di w è v, siccome $v \in L^1_{loc}(a,b)$ ne concludo che w è DERIVABILE IN SENSO DEBOLE.