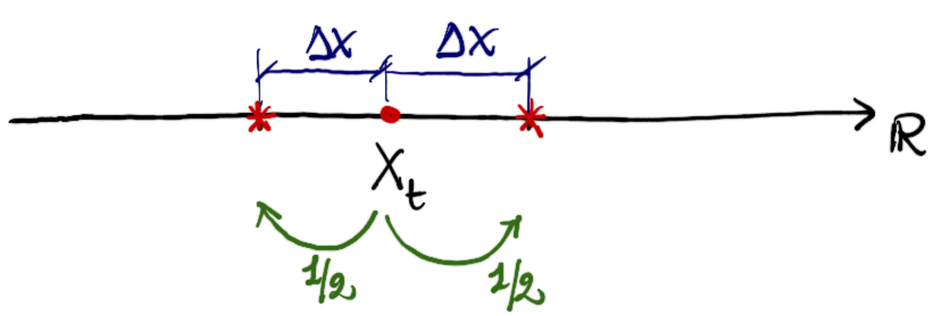
Equazione del calore

$$\partial_t u - D\Delta u = 0 \quad \text{in } Q = \Omega \times (0, +\infty)$$

dove $D > 0$ è costante e $\Omega \subseteq \mathbb{R}^n$ aperto

Derivazione dell'equazione da un processo microscopico particellare

Consideriamo $n = 1$, $\Omega = \mathbb{R}$



$$\boxed{X_{t+\Delta t} = X_t + TM\Delta x}, \qquad \Delta t, \Delta x > 0$$

- M: v.a., $M \in \{-1, 1\}$ con la seguente legge:

$$\text{Prob}(M = -1) = \text{Prob}(M = 1) = \frac{1}{2}.$$

- T: v.a., $T \in \{0, 1\}$

$$T \sim \text{Bernoulli}\left(2D \frac{\Delta t^\beta}{\Delta x^\alpha}\right) \qquad \text{con } \alpha, \beta \geq 1.$$

Descrizione del comportamento medio

$$\varphi \in C_c^\infty(\mathbb{R}) = \mathcal{D}(\mathbb{R})$$

$$\varphi(X_{t+\Delta t}) = \varphi(X_t + TM\Delta x)$$

$$\mathbb{E}[\varphi(X_{t+\Delta t})] = \mathbb{E}[\varphi(X_t + TM\Delta x)]$$

$$= \mathbb{E}[\varphi(X_t)]\,\mathrm{Prob}(T=0) + \mathbb{E}[\varphi(X_t + M\Delta x)]\,\mathrm{Prob}(T=1)$$

$$= \mathbb{E}[\varphi(X_t)]\left(1 - 2D\frac{\Delta t^\beta}{\Delta x^\alpha}\right) + \mathbb{E}[\varphi(X_t + M\Delta x)]\,2D\frac{\Delta t^\beta}{\Delta x^\alpha}$$

$$= \mathbb{E}[\varphi(X_t)]\left(1 - 2D\frac{\Delta t^\beta}{\Delta x^\alpha}\right) + \left\{\mathbb{E}[\varphi(X_t - \Delta x)]\cdot\frac{1}{\cancel{2}}\right.$$

$$\left. + \mathbb{E}[\varphi(X_t + \Delta x)]\cdot\frac{1}{\cancel{2}}\right\}\cancel{2}D\frac{\Delta t^\beta}{\Delta x^\alpha}$$

$$\varphi(X_t - \Delta x) = \varphi(X_t) - \varphi'(X_t)\Delta x + \frac{1}{2}\varphi''(X_t)\Delta x^2 + o(\Delta x^2)$$

$$\varphi(X_t + \Delta x) = \varphi(X_t) + \varphi'(X_t)\Delta x + \frac{1}{2}\varphi''(X_t)\Delta x^2 + o(\Delta x^2)$$

$$\Rightarrow \mathbb{E}[\varphi(X_t - \Delta x) + \varphi(X_t + \Delta x)] = \mathbb{E}[2\varphi(X_t) + \varphi''(X_t)\Delta x^2]$$

$$+ o(\Delta x^2)$$

Continuando il calcolo precedente otteniamo:

$$\mathbb{E}[\varphi(X_{t+\Delta t})] = \mathbb{E}[\varphi(X_t)]\left(1 - \cancel{2D\frac{\Delta t^\beta}{\Delta x^\alpha}}\right)$$

$$+ \mathbb{E}\left[\cancel{2\varphi(X_t)} + \varphi''(X_t)\Delta x^2\right] D\frac{\Delta t^\beta}{\Delta x^\alpha}$$

$$\underbrace{+ o(\Delta x^2) D\frac{\Delta t^\beta}{\Delta x^\alpha}}_{= D\Delta t^\beta o(\Delta x^{2-\alpha})}$$

$$\mathbb{E}[\varphi(X_{t+\Delta t})] - \mathbb{E}[\varphi(X_t)] = \mathbb{E}[\varphi''(X_t)] D\frac{\Delta t^\beta}{\Delta x^{\alpha-2}} + D\Delta t^\beta o(\Delta x^{2-\alpha})$$

$$\frac{\mathbb{E}[\varphi(X_{t+\Delta t})] - \mathbb{E}[\varphi(X_t)]}{\Delta t^\beta} = D\,\mathbb{E}[\varphi''(X_t)]\,\Delta x^{2-\alpha} + o(\Delta x^{2-\alpha})$$

Scegliamo: $\alpha = 2$, $\beta = 1$

$$\frac{\mathbb{E}[\varphi(X_{t+\Delta t})] - \mathbb{E}[\varphi(X_t)]}{\Delta t} = D\,\mathbb{E}[\varphi''(X_t)] + o(1).$$

Passando al limite $\Delta t, \Delta x \to 0^+$:

$$\frac{d}{dt}\mathbb{E}[\varphi(X_t)] = D\,\mathbb{E}[\varphi''(X_t)]. \qquad (*)$$

Sia $u = u(x,t)$ la distribuzione di probabilità (in x) di X_t:

$$\mathrm{Prob}\left(X_t \in A\right) = \int_A u(x,t)\,dx.$$

Riscriviamo $(*)$ usando u:

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} \varphi(x)u(x,t)\,dx &= D\int_{\mathbb{R}} \varphi''(x)u(x,t)\,dx \\
&= -D\int_{\mathbb{R}} \varphi'(x)\partial_x u(x,t)\,dx \\
&= D\int_{\mathbb{R}} \varphi(x)\,\partial_x^2 u(x,t)\,dx
\end{aligned}$$

$$\Rightarrow \int_{\mathbb{R}} \varphi(x)\left(\partial_t u(x,t) - D\partial_x^2 u(x,t)\right)dx = 0$$

Per l'arbitrarietà di $\varphi \in \mathcal{D}(\mathbb{R})$ otteniamo

$$\partial_t u - D\partial_x^2 u = 0,$$

cioè la distribuzione di probabilità della posizione X_t della particella risolve l'equazione del calore.

Soluzione fondamentale

Consideriamo $n=1$, $\Omega = \mathbb{R}$

$$\begin{cases} \partial_t u - D\partial_x^2 u = 0 & \text{in } \mathbb{R}\times(0,+\infty) \\ u = u_0 & \text{in } \mathbb{R},\ t=0 \end{cases}$$

prendiamo $u_0 = \delta_0$ (delta di Dirac centrata in $x=0$).

Cerchiamo $u(\cdot,t) \in L^1(\mathbb{R})$, $\forall t>0$. Una tale u è Fourier-trasformabile:

$$\hat{u}(\xi,t) := \int_{-\infty}^{+\infty} u(x,t)\, e^{-i\xi x}\, dx.$$

Applichiamo la trasformata di Fourier $\mathcal{F}$ ad entrambi i membri della PDE:

$$\mathcal{F}\left[\partial_t u - D\partial_x^2 u\right] = 0$$

$$\mathcal{F}[\partial_t u] - D\,\mathcal{F}[\partial_x^2 u] = 0$$

$$\partial_t \hat{u} - D\left[(i\xi)^2 \hat{u}\right] = 0$$

$$\partial_t \hat{u} + D\xi^2 \hat{u} = 0$$

$$e^{D\xi^2 t}\left(\partial_t \hat{u} + D\xi^2 \hat{u}\right) = 0$$

$$\partial_t \left(e^{D\xi^2 t} \hat{u}\right) = 0$$

$$e^{D\xi t^2} \hat{u}(\xi, t) = \hat{u}(\xi, 0) = \hat{u}_0(\xi) = 1$$

$$\mathcal{F}[\delta_0]$$

$$\Rightarrow \hat{u}(\xi, t) = e^{-D\xi^2 t}.$$