Riprendiamo esercizio 2 della scorsa volta. Domanda e)

Se $K = \frac{mg}{8\ell}$, allora $\lambda = 2$, $(\theta_1, x_1) = (\frac{\pi}{2}, 0)$ stabile.

$$\underline{\underline{H}}(\theta,x) = \begin{pmatrix} -mg\ell\sin\theta - 2k\ell x\cos\theta & -2k\ell\sin\theta \\ -2k\ell\sin\theta & -K \end{pmatrix} \Rightarrow \underline{\underline{H}}(\theta_1,x_1) = \begin{pmatrix} -mg\ell & -2k\ell \\ -2k\ell & -K \end{pmatrix}$$

Se $K = \frac{mg}{8\ell} \Rightarrow \underline{\underline{H}}(\theta_1,x_1) = \begin{pmatrix} -mg\ell & -\frac{mg}{4} \\ -\frac{mg}{4} & -\frac{mg}{8\ell} \end{pmatrix}$; $T = \frac{2}{3}m\ell^2\dot\theta^2 + \frac{3}{4}m\dot x^2$

Equazione dei piccoli moti: $\underline{\underline{A}}(\underline{q}_e)\ddot{\underline{\eta}} - \underline{\underline{H}}(\underline{q}_e)\underline{\eta} = \underline{0}$, dove $\underline{\eta} = \underline{q} - \underline{q}_e$

$\eta_1 = \theta - \frac{\pi}{2}$, $\eta_2 = x$ $\qquad \underline{\underline{A}}(\underline{q})$ matrice di massa: $T = \frac{1}{2}\underline{\underline{A}}(\underline{q})\underline{\dot{\omega}}\cdot\underline{\dot{\omega}}$

$\underline{\underline{A}}(\theta,x) = \begin{pmatrix} \frac{4}{3}m\ell^2 & 0 \\ 0 & \frac{3}{2}m \end{pmatrix} = \underline{\underline{A}}(\theta_1,x_1)$ (non dipende da θ e x) $\qquad T = \frac{1}{2}\left(\frac{4}{3}m\ell^2\dot\theta^2 + \frac{3}{2}m\dot x^2\right)$

$$\begin{cases} \frac{4}{3}m\ell^2\ddot\eta_1 + mg\ell\eta_1 + \frac{mg}{4}\eta_2 = 0 \\ \frac{3}{2}m\ddot\eta_2 + \frac{mg}{4}\eta_1 + \frac{mg}{8\ell}\eta_2 = 0 \end{cases}$$

Cerco soluzioni $\underline{\eta} = e^{\mu t}\underline{\eta}_0$: $\dot{\underline{\eta}} = \mu e^{\mu t}\underline{\eta}_0$: $\ddot{\underline{\eta}} = \mu^2 e^{\mu t}\underline{\eta}_0$

$$\underline{\underline{A}}(\underline{q}_e)\ddot{\underline{\eta}} - \underline{\underline{H}}(\underline{q}_e)\underline{\eta} = 0 \longrightarrow \mu^2\underline{\underline{A}}(\underline{q}_e)\underline{\eta}_0 - \underline{\underline{H}}(\underline{q}_e)\underline{\eta}_0 = 0$$

Ha soluzioni se $\det\left(\underline{\underline{H}}(\underline{q}_e) - \mu^2\underline{\underline{A}}(\underline{q}_e)\right) = 0$, μ^2 autovalori di $\underline{\underline{H}}(\underline{q}_e)$ rispetto ad $\underline{\underline{A}}(\underline{q}_e)$

Se $\mu^2 < 0$, $\underline{\eta} = e^{\pm i\sqrt{-\mu^2}\,t}\underline{\eta}_0$, $\omega = \sqrt{-\mu^2}$ pulsazione propria o autofrequenza.

Devo risolvere:

$$\begin{vmatrix} -mg\ell - \mu^2\frac{4}{3}m\ell^2 & -\frac{mg}{4} \\ -\frac{mg}{4} & -\frac{mg}{8\ell} - \mu^2\frac{3}{2}m \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} -g\ell - \mu^2\frac{4}{3}\ell^2 & -\frac{g}{4} \\ -\frac{g}{4} & -\frac{g}{8\ell} - \mu^2\frac{3}{2} \end{vmatrix} = 0$$

$$2\ell^2\mu^4 + \frac{3}{2}g\ell\mu^2 + \frac{1}{6}g\ell\mu^2 + \frac{1}{8}g^2 - \frac{1}{16}g^2 = 0 \Leftrightarrow \mu^4 + \frac{5}{6}\frac{g}{\ell}\mu^2 + \frac{1}{32}\frac{g^2}{\ell^2} = 0$$

Risolvendola $\rightarrow$ pulsazioni proprie

$$\mu^2 = \frac{-\frac{5}{6} \pm \sqrt{\frac{25}{36} - \frac{1}{8}}}{2}\frac{g}{\ell} = \left(-\frac{5}{12} \pm \frac{1}{2}\sqrt{\frac{50-9}{72}}\right)\frac{g}{\ell} = \left(-\frac{5}{12} \pm \frac{1}{12}\sqrt{\frac{41}{2}}\right)\frac{g}{\ell}$$

$$\omega_1 = \sqrt{\frac{5}{12} - \frac{1}{12}\sqrt{\frac{41}{2}}}\sqrt{\frac{g}{\ell}}$$

$$\omega_2 = \sqrt{\frac{5}{12} + \frac{1}{12}\sqrt{\frac{41}{2}}}\sqrt{\frac{g}{\ell}}$$

Autovettori: $\underline{\varphi}^{(1)}$ relativo a ω_1 e $\underline{\varphi}^{(2)}$ relativo a ω_2.

$$\left(\underline{\underline{H}}(\underline{q}_e) - \mu_1^2 \underline{\underline{A}}(\underline{q}_e)\right) \underline{\varphi}^{(1)} = \underline{0} \qquad \left(\underline{\underline{H}}(\underline{q}_e) - \mu_2^2 \underline{\underline{A}}(\underline{q}_e)\right) \underline{\varphi}^{(2)} = \underline{0}$$

Per la prima:

$$\begin{pmatrix} -mgl + \left(\frac{5}{12} - \frac{1}{12}\sqrt{\frac{41}{2}}\right)\frac{g}{l}\frac{4}{3}ml^2 & -\frac{mg}{4} \\ -\frac{mg}{4} & -\frac{mg}{8l} + \left(\frac{5}{12} - \frac{1}{12}\sqrt{\frac{41}{2}}\right)\frac{g}{l}\frac{3}{2}m \end{pmatrix} \begin{pmatrix} \varphi_1^{(1)} \\ \varphi_2^{(2)} \end{pmatrix} = 0$$

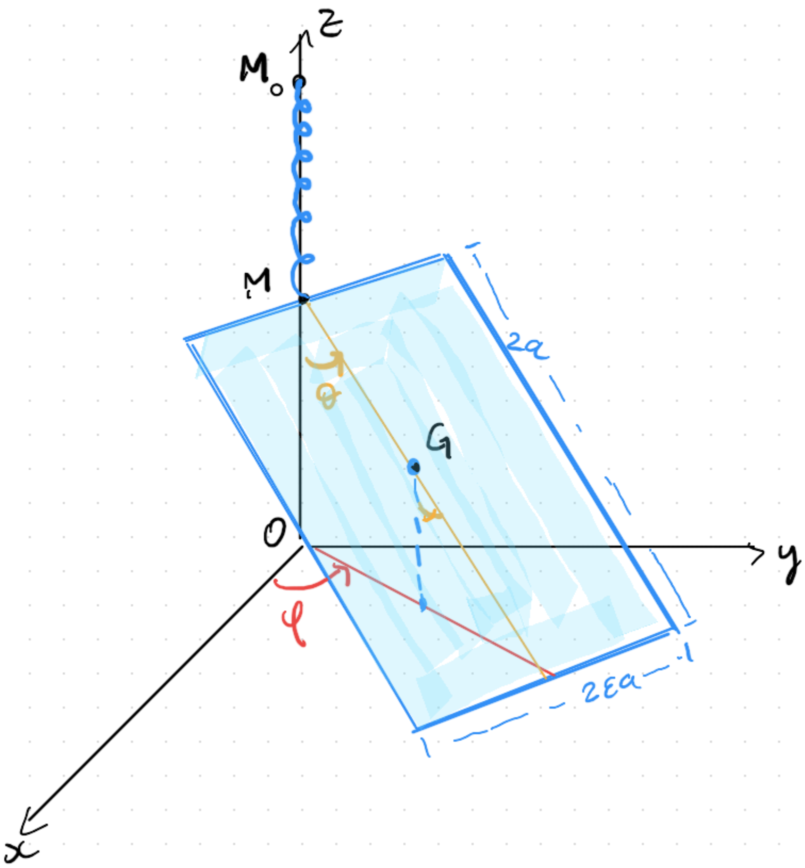
Eliminando mg (prima riga)

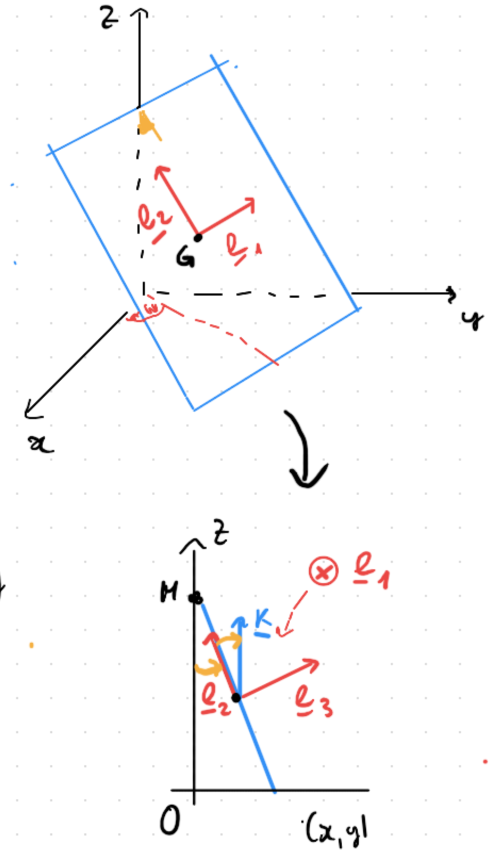
$$\left(-l + \left(\frac{5}{12} - \frac{1}{12}\sqrt{\frac{41}{2}}\right)\frac{4}{3}l\right)\varphi_1^{(1)} - \frac{1}{4}\varphi_2^{(1)} = 0 \iff \left(-\frac{4}{9} - \frac{1}{9}\sqrt{\frac{41}{2}}\right)l\,\varphi_1^{(1)} - \frac{1}{4}\varphi_2^{(1)} = 0$$

$$\left(-\frac{16}{9} - \frac{4}{9}\sqrt{\frac{41}{2}}\right)l\,\varphi_1^{(1)} - \varphi_2^{(1)} = 0 \implies \underline{\varphi}^{(1)} = \begin{pmatrix} 1 \\ -\frac{16}{9}l - \frac{4}{9}\sqrt{\frac{41}{2}}\,l \end{pmatrix}$$

Per $\underline{\varphi}^{(2)}$ cambia segno davanti alla radice:

$$\implies \underline{\varphi}^{(2)} = \begin{pmatrix} 1 \\ -\frac{16}{9}l + \frac{4}{9}\sqrt{\frac{41}{2}}\,l \end{pmatrix}$$





ESERCIZIO "LAMINA"

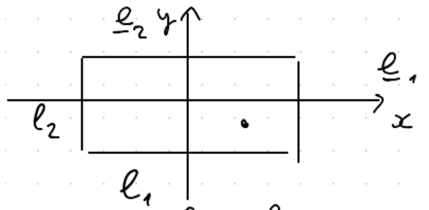
$M_0 = (0,0,4a)$; $M = (0,0,2a\cos\theta)$; $G = (a\sin\theta\cos\varphi, a\sin\theta\sin\varphi, a\cos\theta)$

$U = -mg z_G - \frac{1}{2} k |M_0M|^2 + cost =$ $\quad -8ka^2 = \text{costante}$

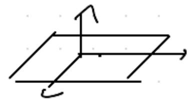
$= -mga\cos\theta - \frac{1}{2} k (4a - 2a\cos\theta)^2 + cost = -mga\cos\theta + 8ka^2\cos\theta - 2ka^2\cos^2\theta + cost$

Energia cinetica:

$T = \frac{1}{2} m v_G^2 + \frac{1}{2} \underline{\underline{I}}_G \underline{\omega} \cdot \underline{\omega}$

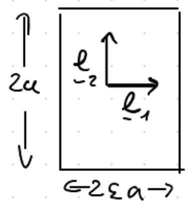


Momenti di inerzia di una lamina rettangolare, rispetto ai suoi tre assi principali di inerzia



$$I_1 = \sigma \int_{-\frac{\ell_1}{2}}^{\frac{\ell_1}{2}} dx \int_{-\frac{\ell_2}{2}}^{\frac{\ell_2}{2}} dy\, y^2 = \frac{m}{\cancel{\ell_1}\ell_2}\cancel{\ell_1}\int_{-\frac{\ell_2}{2}}^{\frac{\ell_2}{2}} y^2 dy = \frac{1}{12}\frac{m}{\ell_2}\ell_2^3 = \frac{1}{12} m \ell_2^2$$

$$I_2 = \frac{1}{12} m \ell_1^2 \quad ; \quad I_3 = \frac{1}{12} m(\ell_1^2 + \ell_2^2)$$



In un'opportuna terna solidale alla lamina:

$$\underline{\underline{I}}_G = \begin{pmatrix} \frac{1}{12}m(2a)^2 & 0 & 0 \\ 0 & \frac{1}{12}m(2\varepsilon a)^2 & 0 \\ 0 & 0 & \frac{1}{12}m[(2a)^2+(2\varepsilon a)^2] \end{pmatrix} = \begin{pmatrix} \frac{1}{3}ma^2 & 0 & 0 \\ 0 & \frac{1}{3}m\varepsilon^2 a^2 & 0 \\ 0 & 0 & \frac{1}{3}m(1+\varepsilon^2)a^2 \end{pmatrix}$$

$$\underline{e}_1 = -\sin\varphi\, \underline{i} + \cos\varphi\, \underline{j}$$

$$\underline{e}_2 = -\sin\theta\cos\varphi\, \underline{i} - \sin\theta\sin\varphi\, \underline{j} + \cos\theta\, \underline{k}$$

$$\underline{e}_3 = \underline{e}_1 \wedge \underline{e}_2 = (-\sin\varphi\, \underline{i} + \cos\varphi\, \underline{j}) \wedge (-\sin\theta\cos\varphi\, \underline{i} - \sin\theta\sin\varphi\, \underline{j} + \cos\theta\, \underline{k}) =$$
$$= \sin\theta\sin^2\varphi\, \underline{k} + \cos\theta\sin\varphi\, \underline{j} + \sin\theta\cos^2\varphi\, \underline{k} + \cos\theta\cos\varphi\, \underline{i} =$$
$$= \cos\theta\cos\varphi\, \underline{i} + \cos\theta\sin\varphi\, \underline{j} + \sin\theta\, \underline{k}$$

$$\begin{cases} \underline{e}_1 = -\sin\varphi\, \underline{i} + \cos\varphi\, \underline{j} \\ \underline{e}_2 = -\sin\theta\cos\varphi\, \underline{i} - \sin\theta\sin\varphi\, \underline{j} + \cos\theta\, \underline{k} \\ \underline{e}_3 = \cos\theta\cos\varphi\, \underline{i} + \cos\theta\sin\varphi\, \underline{j} + \sin\theta\, \underline{k} \end{cases}$$

Inoltre $\underline{k} = \underline{e}_2 \cos\theta + \underline{e}_3 \sin\theta$

Devo ancora determinare $\underline{v}_G$ e $\underline{\omega}$.

$$\underline{v}_G - \underline{v}_M = \underline{\omega} \wedge (G-M)$$

$$\begin{cases} \underline{v}_G = a(\dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi)\underline{i} + a(\dot\theta\cos\theta\sin\varphi + \dot\varphi\sin\theta\cos\varphi)\underline{j} - a\dot\theta\sin\theta\,\underline{K}\,; \\ \underline{v}_M = -2a\dot\theta\sin\theta\,\underline{K}\,; \quad G-M = a\sin\theta\cos\varphi\,\underline{i} + a\sin\theta\sin\varphi\,\underline{j} - a\cos\theta\,\underline{K} \end{cases}$$

$$\Rightarrow a(\dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi)\underline{i} + a(\dot\theta\cos\theta\sin\varphi + \dot\varphi\sin\theta\cos\varphi)\underline{j} + a\dot\theta\sin\theta\,\underline{K} =$$

$$= (\omega_x\underline{i} + \omega_y\underline{j} + \omega_z\underline{K}) \wedge (a\sin\theta\cos\varphi\,\underline{i} + a\sin\theta\sin\varphi\,\underline{j} - a\cos\theta\,\underline{K})$$ (elimino a)

$$(\dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi)\underline{i} + (\dot\theta\cos\theta\sin\varphi + \dot\varphi\sin\theta\cos\varphi)\underline{j} + \dot\theta\sin\theta\,\underline{K} =$$

$$= \omega_x\sin\theta\sin\varphi\,\underline{K} + \omega_x\cos\theta\,\underline{j} - \omega_y\sin\theta\cos\varphi\,\underline{K} - \omega_y\cos\theta\,\underline{i} + \omega_z\sin\theta\cos\varphi\,\underline{j} - \omega_z\sin\theta\sin\varphi\,\underline{i}$$

Componente per componente:

$$\begin{cases} \dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi = -\omega_y\cos\theta - \omega_z\sin\theta\sin\varphi \\ \dot\theta\cos\theta\sin\varphi + \dot\varphi\sin\theta\cos\varphi = \omega_x\cos\theta + \omega_z\sin\theta\cos\varphi \\ \dot\theta\sin\theta = \omega_x\sin\theta\sin\varphi - \omega_y\sin\theta\cos\varphi \end{cases} \qquad \begin{cases} \omega_x = \dot\theta\sin\varphi \\ \omega_y = -\dot\theta\cos\varphi \\ \omega_z = \dot\varphi \end{cases}$$

$$\underline{\omega} = \dot\theta\sin\varphi\,\underline{i} - \dot\theta\cos\varphi\,\underline{j} + \dot\varphi\,\underline{K}$$

$$\underline{\omega} = \dot\theta\sin\varphi\,\underline{i} - \dot\theta\cos\varphi\,\underline{j} + \dot\varphi\,\underline{K} \rightarrow \underline{\omega} = -\dot\theta\,\underline{e}_1 + \dot\varphi\cos\theta\,\underline{e}_2 + \dot\varphi\sin\theta\,\underline{e}_3$$

$$v_G^2 = a^2(\dot\theta^2\cos^2\theta\cos^2\varphi \cancel{- 2\dot\theta\dot\varphi\sin\theta\cos\theta\sin\varphi\cos\varphi} + \dot\varphi^2\sin^2\theta\sin^2\varphi) +$$

$$+ a^2(\dot\theta^2\cos^2\theta\sin^2\varphi \cancel{+ 2\dot\theta\dot\varphi\sin\theta\cos\theta\sin\varphi\cos\varphi} + \dot\varphi^2\sin^2\theta\cos^2\varphi) + a^2\dot\theta^2\sin^2\theta =$$

$$= a^2\dot\theta^2\cos^2\theta\cos^2\varphi + a^2\dot\theta^2\cos^2\theta\sin^2\varphi + a^2\dot\varphi^2\sin^2\theta\sin^2\varphi + a^2\dot\varphi^2\sin^2\theta\cos^2\varphi + a^2\dot\theta^2\sin^2\theta =$$

$$= a^2\dot\theta^2\cos^2\theta + a^2\dot\varphi^2\sin^2\theta + a^2\dot\theta^2\sin^2\theta = a^2\dot\theta^2 + a^2\sin^2\theta\,\dot\varphi^2$$

Calcolo $\underline{\underline{I}}_G\underline{\omega}\cdot\underline{\omega}$ nella terna mobile:

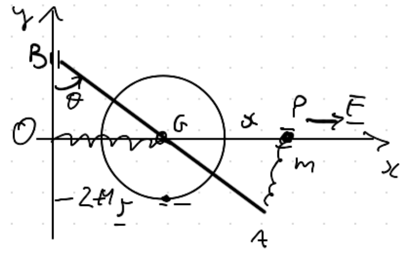
$$\underline{\underline{I}}_G\underline{\omega}\cdot\underline{\omega} = \frac{1}{3}ma^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & 1+\varepsilon^2 \end{pmatrix}\begin{pmatrix} -\dot\theta \\ \dot\varphi\cos\theta \\ \dot\varphi\sin\theta \end{pmatrix}\cdot\begin{pmatrix} -\dot\theta \\ \dot\varphi\cos\theta \\ \dot\varphi\sin\theta \end{pmatrix} = \frac{1}{3}ma^2\begin{pmatrix} -\dot\theta \\ \varepsilon^2\dot\varphi\cos\theta \\ (1+\varepsilon^2)\dot\varphi\sin\theta \end{pmatrix}\cdot\begin{pmatrix} -\dot\theta \\ \dot\varphi\cos\theta \\ \dot\varphi\sin\theta \end{pmatrix}$$

$$= \frac{1}{3}ma^2(\dot\theta^2 + \varepsilon^2\cos^2\theta\,\dot\varphi^2 + \sin^2\theta\,\dot\varphi^2 + \varepsilon^2\sin^2\theta\,\dot\varphi^2) = \frac{1}{3}ma^2(\dot\theta^2 + \varepsilon^2\dot\varphi^2 + \sin^2\theta\,\dot\varphi^2)$$

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}\underline{\underline{I}}_G\underline{\omega}\cdot\underline{\omega} = \frac{1}{2}ma^2\left(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2 + \frac{1}{3}\dot\theta^2 + \frac{1}{3}\varepsilon^2\dot\varphi^2 + \frac{1}{3}\sin^2\theta\,\dot\varphi^2\right) =$$

$$= \frac{2}{3}ma^2\dot\theta^2 + \frac{2}{3}ma^2\sin^2\theta\,\dot\varphi^2 + \frac{1}{6}m\varepsilon^2a^2\dot\varphi^2$$

ESAME 30/06/2010



$B = (0, \ell\cos\theta)$; $G = (\ell\sin\theta, 0)$; $P = (x, 0)$

$A = (2\ell\sin\theta, -\ell\cos\theta)$; $C = (\ell\sin\theta, -R)$

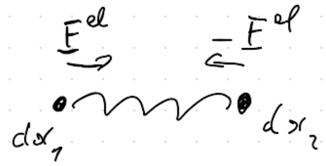
- Equilibrio e stabilità. $\underset{\sim}{F}\,\underline{i} \cdot dx\,\underline{i}$

$$U = -\frac{1}{2}k|OG|^2 - \frac{1}{2}k|AP|^2 + \int \underline{F} \cdot dP + cost = -\frac{1}{2}k|OG|^2 - \frac{1}{2}k|AP| + Fx + cost$$

$$= -\frac{1}{2}k\ell^2\sin^2\theta - \frac{1}{2}k\left[(x - 2\ell\sin\theta)^2 + \ell^2\cos^2\theta\right] + cost =$$

$$= -\frac{1}{2}k\ell^2\sin^2\theta - \frac{1}{2}kx^2 + 2k\ell x\sin\theta - 2k\ell^2\sin^2\theta - \frac{1}{2}k\ell^2\cos^2\theta + cost$$

Lo svolgeremo la settimana prossima



$$U = -\frac{1}{2}k|x_1 - x_2|^2$$

Compito di Meccanica Razionale del 30 Giugno 2010 - Tema A

1) Cinematica relativa. (Max 1 pagina)

2) In un piano verticale Oxy un disco omogeneo di massa M e raggio R è vincolato a rotolare senza strisciare sull'asse $y = -R$; un'asta omogenea AB di egual massa M e lunghezza 2ℓ ha l'estremo B mobile sull'asse delle y ed il baricentro G incernierato al centro del disco. Un punto materiale P di massa m è mobile sull'asse delle x sollecitato da una forza orizzontale costante $\vec{F}$, con $F > 0$, e collegato con una molla ideale, di costante $k > 0$, all'estremo A dell'asta. Una forza elastica di costante $h = k$ è infine applicata al centro del disco e diretta verso l'origine O. Si suppongano i vincoli perfetti e si scelgano come coordinate lagrangiane l'angolo di rotazione ϑ che l'asta forma con l'asse verticale e l'ascissa x del punto P.

- Determinare le posizioni di equilibrio del sistema e studiarne la stabilità.
- Determinare la velocità angolare del disco.
- Calcolare le equazioni differenziali del moto nella forma di Lagrange.
- Linearizzare tali equazioni nell'intorno di una posizione di equilibrio stabile e determinare le pulsazioni proprie.