CHAPTER 7

Dynamics of Systems and of Rigid Bodies

Consider a point P of mass m subjected to forces which sum up to $\vec{F}(P, \vec{v}, t)$, so that its acceleration obeys the differential equation:

$$m\vec{a} = m\frac{d^2(P-O)}{dt^2} = \vec{F}(P, \vec{v}, t), \qquad \text{with initial conditions} \quad P(t_0) = P_0, \ \ \vec{v}(t_0) = \vec{v}_0 \tag{7.1}$$

It turns out that conservative forces are associated with *constants of motion.* Indeed, letting U be the potential of $\vec{F}$, so that $\vec{F} = \nabla U(q)$, one may write:

$$0 = m\vec{a} - \vec{F} = m\dot{\vec{v}} - \vec{F} \tag{7.2}$$

and then, scalarly multiplying by $\vec{v}$:

$$0 = \vec{v} \cdot \left(m\dot{\vec{v}} - \vec{F}\right) = m\vec{v} \cdot \dot{\vec{v}} - \vec{v} \cdot \nabla U = m\vec{v} \cdot \dot{\vec{v}} - \dot{\vec{q}} \cdot \nabla U(q) = \frac{d}{dt}\left[\frac{1}{2}mv^2 - U\right] \tag{7.3}$$

The constant quantity

$$E = \frac{1}{2}mv^2 - U = const \tag{7.4}$$

is called *mechanical energy*, the quantity

$$T = \frac{1}{2}mv^2 \tag{7.5}$$

is called *kinetic energy* and the quantity

$$V = -U \tag{7.6}$$

is called *potential energy*, and one may also write $E = T + V$.

In the presence of constraints, one writes:

$$m\vec{a} = \vec{F} + \vec{\Phi} \tag{7.7}$$

as if the constraints have the properties of forces. Unfortunately, it is not known or only partially known how $\vec{\Phi}$ should be expressed, because constraints are not like forces. Again, one ought to keep in mind that a mathematical model of a given real phenomenon has the purpose of describing a relevant aspect of the phenomenon, but obviously it does not coincide with the phenomenon. Typically, in our field, the relevant aspect is something that allows predictions/forecasts. In the case of constraints, one assumes that they can exert whatever force is required to obtain a certain result, hence they are assumed to exert an infinitely large set of forces, and very precisely oriented forces, that can change at will, as the conditions of the objects of interest change. Clearly, our model of a constraint does not correspond to our model of a force. Nevertheless, we will see that Eq.(7.7) is extremely useful.

Let us begin considering holonomic constraints, for instance the point mass P constrained to move on a 1-dimensional rail of length L, $\gamma \subset \mathbb{R}^3$, like, for instance a ring on a beam or a ball in a cylinder. Then, let us introduce a curvilinear coordinate s in γ:

$$P_\gamma : [0, L] \to \mathscr{E} \subset \mathbb{R}^3 \tag{7.8}$$

Given $s = s(t)$, the expression of the position at time t, $P_\gamma(s) = P_\gamma(s(t))$, recall that:

$$\vec{v} = \frac{d(P_\gamma - O)}{ds}\frac{ds}{dt} = \dot{s}\hat{t}\,, \qquad \hat{t} = \hat{t}(s) \tag{7.9}$$

expresses the velocity, while the acceleration of the point mass is given by:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\dot{s}\hat{t})}{dt} = \ddot{s}\hat{t} + \dot{s}\frac{d\hat{t}}{ds}\frac{ds}{dt} = \ddot{s}\hat{t} + \frac{\dot{s}^2}{\rho}\hat{n} \tag{7.10}$$

where ρ is the curvature radius of γ in P. With these definitions, one may introduce the mobile frame $(\hat{t}, \hat{n}, \hat{b})$, where $\hat{b} = \hat{t} \wedge \hat{n}$ is called *binormal* to γ. Then, one may write:

$$m\vec{a} = m\ddot{s}\hat{t} + m\frac{\dot{s}^2}{\rho}\hat{n} = \vec{F} + \vec{\Phi} \tag{7.11}$$

which, projected onto the mobile axes yields:

$$m\vec{a} \cdot \hat{t} = m\ddot{s} = (\vec{F} + \vec{\Phi}) \cdot \hat{t} = F_t(s, \dot{s}, t) + \Phi_t \tag{7.12}$$

$$m\vec{a} \cdot \hat{n} = m\frac{\dot{s}^2}{\rho} = (\vec{F} + \vec{\Phi}) \cdot \hat{n} = F_n(s, \dot{s}, t) + \Phi_n \tag{7.13}$$

$$m\vec{a} \cdot \hat{b} = 0 = (\vec{F} + \vec{\Phi}) \cdot \hat{b} = F_b(s, \dot{s}, t) + \Phi_b \tag{7.14}$$

Furthermore, one may introduce the generalized component of the active forces along γ:

$$\boxed{Q_s = \vec{F} \cdot \frac{d(P-O)}{ds} = \vec{F} \cdot \hat{t} = F_t} \tag{7.15}$$

As $\vec{\Phi}$ simply prevents the motion perpendicular to γ, *i.e.* perpendicular to $\hat{t}$, the sum of forces along $\hat{n}$ and $\hat{b}$ have to vanish, hence equilibrium positions for P are identified by:

$$\boxed{Q_s = 0} \tag{7.16}$$

If there is no friction, *i.e.* the rail is smooth, we also have:

$$\Phi_t = 0 \ \text{ which implies } \quad \boxed{m\ddot{s} = F_t} \tag{7.17}$$

and we can compute Φ_n and Φ_b. In the case $F_t = 0$, we have motion due to inertia, and $\dot{s} = s_0$, is a constant. Suppose, now, the tangent force

$$F_t = F_t(s) \quad \text{hence} \qquad m\ddot{s} - F_t(s) = 0 \tag{7.18}$$

is known. Then, multiplying the last expression by $\dot{s}$ one then obtains:

$$\dot{s}\big(m\ddot{s} - F_t(s)\big) = 0 \quad i.e. \quad \frac{d}{dt}\left[\frac{1}{2}m\dot{s}^2 - U\right] = 0 \tag{7.19}$$

where U is any anti-derivative of F_t :

$$F_t = \frac{dU}{ds}; \qquad \frac{dU}{dt} = \frac{dU}{ds}\frac{ds}{dt} = \dot{s}F_t \tag{7.20}$$

$$U(s) = \int_{s_0}^{s} F_t(s')\, ds' \tag{7.21}$$

In other words, we can write:

$$Q_s(s) = F_t(s) = U'(s) \tag{7.22}$$

When this is the case, forces are not necessarily conservative, because only F_t, not the whole force, is derived from a kind of "potential", but it all looks as in the conservative case. The trick is made by the proper choice of coordinates, given the known constraints.

In cases of friction, $\vec{\Phi}$ acts also tangentially to the path: $\Phi_t \neq 0$. For instance, the Coulomb-Morin law, expressed by:

$$\vec{\Phi}_t = -f_d\sqrt{{\Phi_n}^2 + {\Phi_b}^2}\ \frac{\dot{s}}{|\dot{s}|}\hat{t} \tag{7.23}$$

may hold, where f_d is known as the dynamic friction coefficient. This coefficient is experimentally determined and results smaller than f_s, the static friction coefficient. Because

$$\Phi_n = \frac{m\dot{s}^2}{\rho} - F_n \quad \text{and} \quad \Phi_b = -F_b, \tag{7.24}$$

we eventually obtain:

$$m\ddot{s} = F_t - f_d \frac{\dot{s}}{|\dot{s}|} \sqrt{\left(\frac{m\dot{s}^2}{\rho} - F_n\right)^2 + F_b{}^2} \tag{7.25}$$

and the motion is determined by the active forces since friction merely opposes the motion. In case γ is a straight line, $\rho = \infty$, and we obtain:

$$m\ddot{s} = F_t - f_d \sqrt{F_n{}^2 + F_b{}^2} \frac{\dot{s}}{|\dot{s}|} \tag{7.26}$$

As in Chapter 2, after the point dynamics, we consider that of system of points. Therefore, we define two quantities that will be very important: *(linear) momentum* and *angular momentum.*

MOMENTUM: For a system of mass points, we define the total momentum as:

$$\vec{Q} = \sum_{i=1}^{n} m_i \vec{v}_i \tag{7.27}$$

In the case of a continuous body $\mathcal{B}$ is defined by:

$$\vec{Q} = \int_{\mathcal{B}} \rho \vec{v} d\tau \tag{7.28}$$

THEOREM 7.1: *the total momentum is given by*

$$\vec{Q} = m\vec{v}_G \tag{7.29}$$

where m *is the total mass, and* v_G *is the velocity of the center of mass.*

Proof: By definition $m(G - O) = \sum_{i=1}^{n} m_i(P_i - O)$, then, differentiating one gets $m\vec{v}_G = \sum_{i=1}^{n} m_i \vec{v}_i$. The same holds for a continuum $\mathcal{B}$. Then, differentiating we find that the total momentum $\vec{Q}^{(G)}$ vanishes in the center of mass. Let $\vec{v}_{i,r}$ be the relative velocity of the point P_i in the cener of mass frame:

$$\vec{Q}^{(G)} = \sum_{i=1}^{n} m_i \vec{v}_{i,r} = \sum_{i=1}^{n} m_i(\vec{v}_i - \vec{v}_G) = \sum_{i=1}^{n} m_i \vec{v}_i - m\vec{v}_G = \mathbf{0}$$

ANGULAR MOMENTUM: the angular momentum with respect to a point O for a system of point masses is defined by

$$\vec{K}_O = \textstyle\sum_{i=1}^{n} (P_i - O) \wedge m_i \vec{v}_i \tag{7.30a}$$

while for a for a continuous body of density ρ, it is defined by

$$\vec{K}_O = \int_{\mathcal{B}} (P - O) \wedge \rho \vec{v}\, d\tau \tag{7.30b}$$

First note that changing reference point, one obtains:

$$\vec{K}_Q = \sum_{i=1}^{n} (P_i - Q) \wedge m_i \vec{v}_i = \sum_{i=1}^{n} (O - Q) \wedge m_i \vec{v}_i + \sum_{i=1}^{n} (P_i - O) \wedge m_i \vec{v}_i = (O - Q) \wedge \vec{Q} + \vec{K}_O$$

which means that $\vec{K}_Q = \vec{K}_O$, independent of the reference point, if the total momentum $\vec{Q}$ vanishes (notation: Q is a point while $\vec{Q}$ is a vector). This makes the center of mass particularly interesting as a reference point. Therefore, we define the motion with respect to the center of mass G as the motion with respect to a reference frame whose axes have origin in G, and do not change orientation in time, with respect to a "fixed" reference frame. Then, as direct calculations of angular momentum via Eq.(7.30a) or (7.30b) is seldom easy or fast, the following theorems greatly simplify the process, suggesting a "good" choice of the reference point O.

THEOREM 7.2: *For any material system, one has*

$$\vec{K}_Q = (G-Q)\wedge m\vec{v}_G + \vec{K}_G = (G-Q)\wedge m\vec{v}_G + \vec{K}_Q^{(G)} \tag{7.31}$$

where $\vec{K}_Q^G$ is the angular momentum with respect to the point Q, computed in the center of mass frame. Furthermore, the equality $\vec{K}_G = \vec{K}_Q^{(G)}$ holds for every point Q.

Proof: Recalling Eq.(5.6), which holds for the moments of any system of vectors that sum to $\vec{R}$, and any reference point Q, $\vec{M}_Q = (P-Q)\wedge\vec{R} + \vec{M}_P$ for any other point P, the first equality of Eq.(7.31) immediately follows. Repeating the calculation, one has:

$$\vec{K}_Q = \sum_{i=1}^{n}(P_i - Q)\wedge m_i\,\vec{v}_i = \sum_{i=1}^{n} m_i[(P_i-G)+(G-Q)]\wedge\vec{v}_i =$$
$$= m(G-Q)\wedge\vec{v}_G + \sum_{i=1}^{n}(P_i-G)\wedge m_i\,\vec{v}_i = (G-Q)\wedge m\vec{v}_G + \vec{K}_G \tag{7.32}$$

Concerning the second equality, consider the motion with respect to the center of mass, so that the drag velocity $\vec{v}_\tau$ coincides with $\vec{v}_G$, and recall that $m\vec{v}_G = \sum_{i=1}^{n} m_i\vec{v}_i$. In this case, using Eq.(2.88), we can write:

$$\vec{v}_a = \vec{v}_r + \vec{v}_\tau = \vec{v}_r + \vec{v}_G + \vec{\omega}\wedge(P-G) = \vec{v}_r + \vec{v}_G$$

because the axes of the center of mass frame do not rotate, $\vec{\omega} = \mathbf{0}$. Noting that

$$\sum_{i=1}^{n} m_i(P_i - Q) = m(G-Q)$$

one can write:

$$\vec{K}_Q = \sum_{i=1}^{n}(P_i-Q)\wedge m_i\,\vec{v}_i = \sum_{i=1}^{n} m_i(P_i-Q)\wedge\left(\vec{v}_G + \vec{v}_{i,r}\right)$$
$$= m(G-Q)\wedge\vec{v}_G + \sum_{i=1}^{n} m_i(P_i-Q)\wedge\vec{v}_{i,r} = (G-Q)\wedge m\vec{v}_G + \vec{K}_Q^{(G)}$$

because $(P_i - Q)$ does not depend on the reference frame: its magnitude and orientation in space do not change, only the projections on different axes change. Comparing the second equality of Eq.(7.31) with the first implies that $\vec{K}_Q^{(G)}$ equals $\vec{K}_G$ and does not depend on Q. This completes the proof.

We conclude that $\vec{K}_Q$, which is computed with respect to the fixed reference frame, coincides with $\vec{K}_Q^{(G)}$ if $O = G$, or $(G-Q)\parallel\vec{v}_G$. Consider now a rigid body and a point O in space. Let $\vec{v}_O$ be the velocity of the body's point passing through O at a given time. Then, the following holds.

THEOREM 7.3: *the angular momentum of a rigid body $\mathcal{B}$ obeys the following expression:*

$$\vec{K}_Q = m(G-Q)\wedge\vec{v}_Q + I_Q\vec{\omega} \tag{7.33}$$

where $\vec{\omega}$ is the angular velocity, I_Q the matrix of inertia with respect to Q, and $\vec{v}_Q$ the velocity of the point of $\mathcal{B}$ which is passing through the point Q.

Proof: Recall that $\vec{v}_P = \vec{v}_Q + \vec{\omega} \wedge (P - Q)$. For a continuum, but equivalently for a discrete set of points, substituting in the definition (7.30b) with Q in place of O, we obtain:

$$\vec{K}_Q = \int_{\mathcal{B}} \rho(P - Q) \wedge \vec{v}_Q \, d\tau + \int_{\mathcal{B}} \rho(P - Q) \wedge (\vec{\omega} \wedge (P - Q)) \, d\tau$$

In the first integral, $\vec{v}_Q$ is the same for all volume elements $d\tau$, hence can be taken out of the integral, and then the remaining integrand yields the mass time the vector joining Q to the center of mass:

$$\int_{\mathcal{B}} \rho(P - Q) \wedge \vec{v}_Q \, d\tau = m(G - Q) \wedge \vec{v}_Q \quad \text{because} \quad \int_{\mathcal{B}} (P - Q)\rho \, d\tau = m(G - Q)$$

The second integral contains a double vector product, which can be rewritten using Eq. (2.60) as:

$$\int_{\mathcal{B}} \rho(P - Q) \wedge (\vec{\omega} \wedge (P - Q)) \, d\tau = \left[\int_{\mathcal{B}} \rho(P - Q) \cdot (P - Q) \, d\tau\right] \vec{\omega} - \int_{\mathcal{B}} \rho((P - Q) \cdot \vec{\omega})(P - Q) \, d\tau$$

Taking an orthonormal basis $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ and writing $\vec{\omega} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$ and $(P - Q) = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$, we have: $(P - Q) \cdot \vec{\omega} = x_1\omega_1 + x_2\omega_2 + x_3\omega_3$,

$$\left[\int_{\mathcal{B}} \rho(P - Q) \cdot (P - Q) \, d\tau\right] \vec{\omega} = \left[\int_{\mathcal{B}} \rho(x_1^2 + x_2^2 + x_3^2) \, d\tau\right] (\omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3)$$

and

$$\begin{aligned}\int_{\mathcal{B}} \rho((P - Q) \cdot \vec{\omega})(P - Q) \, d\tau = &\left[\int_{\mathcal{B}} \rho(x_1^2\omega_1 + x_1x_2\omega_2 + x_1x_3\omega_3) \, d\tau\right] \hat{e}_1 \\ &+ \left[\int_{\mathcal{B}} \rho(x_1x_2\omega_1 + x_2^2\omega_2 + x_2x_3\omega_3) \, d\tau\right] \hat{e}_2 \\ &+ \left[\int_{\mathcal{B}} \rho(x_1x_3\omega_1 + x_2x_3\omega_2 + x_3^2\omega_3) \, d\tau\right] \hat{e}_3\end{aligned}$$

Therefore, the components of the double vector product are expressed by:

$$\left[\int_{\mathcal{B}} \rho(x_2^2 + x_3^2) \, d\tau\right] \omega_1 - \left[\int_{\mathcal{B}} \rho x_1x_2 \, d\tau\right] \omega_2 - \left[\int_{\mathcal{B}} \rho x_1x_3 \, d\tau\right] \omega_3 = I_{Q1}\omega_1 + I_{Q12}\omega_2 + I_{Q13}\omega_3$$

$$\left[\int_{\mathcal{B}} \rho(x_1^2 + x_3^2) \, d\tau\right] \omega_2 - \left[\int_{\mathcal{B}} \rho x_1x_2 \, d\tau\right] \omega_1 - \left[\int_{\mathcal{B}} \rho x_2x_3 \, d\tau\right] \omega_3 = I_{Q12}\omega_1 + I_{Q2}\omega_2 + I_{Q23}\omega_3$$

$$\left[\int_{\mathcal{B}} \rho(x_1^2 + x_2^2) \, d\tau\right] \omega_3 - \left[\int_{\mathcal{B}} \rho x_1x_3 \, d\tau\right] \omega_1 - \left[\int_{\mathcal{B}} \rho x_2x_3 \, d\tau\right] \omega_2 = I_{Q13}\omega_1 + I_{Q23}\omega_2 + I_{Q3}\omega_3$$

which amounts to $I_Q\vec{\omega}$. This completes the proof.

It is most interesting to note that, taking $Q \equiv G$, one obtains:

$$\vec{K}_G = I_G\vec{\omega} \tag{7.34}$$

which, for $Q \neq G$ and recalling Eq. (7.31) leads to:

$$\vec{K}_Q = m(G - Q) \wedge \vec{v}_G + I_G\vec{\omega} \tag{7.35}$$

Another particularly simple situation, analogous to Eq.(7.34), occurs when $Q = C$ is a center of rotations, instantaneously at rest ($\vec{v}_C = 0$), or the vector $(G - C)$ is parallel to $\vec{v}_C$. In that case one analogously has:

$$\vec{K}_C = I_C\vec{\omega}$$

In the first part of this Chapter we introduced the *kinetic energy* of a material point. We now give the general definition of this crucial quantity for a system of points and for a continuous object.

DEFINITION 7.1: *the kinetic energy of a systems of* n *points, or of an extended object is respectively defined by:*

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 \qquad \text{or} \qquad T = \frac{1}{2}\int_B \rho v^2 d\tau \tag{7.36}$$

A simple but powerful theorem immediately follows.

THEOREM 7.4 (KÖNIG): *the kinetic energy satisfies the following relation:*

$$\boxed{T = \frac{1}{2}m{v_G}^2 + T^{(G)}} \tag{7.37}$$

with T_G the kinetic energy in the center of mass frame.

Proof: Consider the discrete case; the continuous case is analogous. Recall that

$$\vec{v}_{i,a} = \vec{v}_{i,r} + \vec{v}_G$$

which implies

$$\begin{aligned} T &= \frac{1}{2}\sum_{i=1}^{n} m_i\left(\vec{v}_{i,r} + \vec{v}_G\right)^2 = \frac{1}{2}\sum_{i=1}^{n} m_i\left({\vec{v}_{i,r}}^2 + 2\vec{v}_{i,r}\cdot\vec{v}_G + {\vec{v}_G}^2\right) \\ &= \frac{1}{2}m{v_G}^2 + \vec{v}_G \cdot \underbrace{\sum_{i=1}^{n} m_i\,\vec{v}_{i,r}}_{Q^{(G)}=0} + \underbrace{\frac{1}{2}\sum_{i=1}^{n} m_i {v_{i,r}}^2}_{T^{(G)}} = T^{(G)} + \frac{1}{2}m{v_G}^2 \end{aligned}$$

This completes the proof since the case of continuous objects merely replaces sums with integrals.

Although Eq.(7.37) is the most useful way of writing the kinetic energy, one may also consider the following formula: for a rigid body, one has:

$$T = \frac{1}{2}m{v_Q}^2 + m\vec{v}_Q \cdot \vec{\omega} \wedge (G - Q) + \frac{1}{2}I_Q\vec{\omega}\cdot\vec{\omega}$$

where Q is a point of the rigid body, G is its center of mass, I_Q is its moment of inertia with respect to the point Q and is the angular $\vec{\omega}$ velocity. This form arises using the general formula of rigid motion velocities $\vec{v}_P = \vec{v}_Q + \vec{\omega} \wedge (P - Q)$, taking its square and summing over all points.

FIRST EQUATION OF DYNAMICS: *Let $\vec{R}^{(e)}$be the sum of the external forces acting on a material system and let $\vec{Q}$ be its total momentum. Then*

$$\vec{R}^{(e)} = \dot{\vec{Q}} \tag{7.38}$$

Proof: denoting by $\vec{F}_i^{(i)}$ the internal forces, and by $\vec{F}_i^{(e)}$ the external ones, observe that

$$m_i\vec{a}_i = \vec{F}_i^{(e)} + \vec{F}_i^{(i)} \Rightarrow \underbrace{\sum_{i=1}^{n}\vec{F}_i^{(i)}}_{\vec{R}^{(i)}=0} + \underbrace{\sum_{i=1}^{n}\vec{F}_i^{(e)}}_{\vec{R}^{(e)}} = \sum_{i=1}^{n} m_i\vec{a}_i = \frac{d}{dt}\sum_{i=1}^{n} m_i\vec{v}_i = \dot{\vec{Q}}$$

Again, the extension to continuous objects is immediate.

SECOND EQUATION OF DYNAMICS: *Let $\vec{M}_\Omega^{(e)}$ be the total moment of the external forces with respect to a point Ω. Let $\vec{v}_\Omega$ be the velocity of Ω and $\vec{K}_\Omega$ the corresponding angular momentum.* $Then, the\ following\ holds$:

$$\boxed{\vec{M}_\Omega^{(e)} = \dot{\vec{K}}_\Omega + \vec{v}_\Omega \wedge \vec{Q}} \tag{7.39}$$

Proof: We have

$$(P_i - \Omega) \wedge \left(\vec{F}_i^{(e)} + \vec{F}_i^{(i)}\right) = (P_i - \Omega) \wedge m_i\vec{a}_i$$

and:

$$\underbrace{\sum_{i=1}^{n}(P_i - \Omega) \wedge \vec{F}_i^{(i)}}_{\vec{M}_\Omega^{(i)}=\mathbf{0}} + \underbrace{\sum_{i=1}^{n}(P_i - \Omega) \wedge \vec{F}_i^{(e)}}_{\vec{M}_\Omega^{(e)}} = \sum_{i=1}^{n}(P_i - \Omega) \wedge m_i\vec{a}_i$$

where the first term vanishes if we consider, as common, only internal forces acting along the lines joining points, hence with vanishing arm. Furthermore, one obtains:

$$\begin{aligned}
\vec{M}_\Omega^{(e)} &= \sum_{i=1}^{n}(P_i - \Omega) \wedge m_i\frac{d\vec{v}_i}{dt} = \sum_{i=1}^{n}\left[\frac{d}{dt}[(P_i - \Omega) \wedge m_i\vec{v}_i] - \frac{d(P_i - \Omega)}{dt} \wedge m_i\vec{v}_i\right] \\
&= \frac{d}{dt}\sum_{i=1}^{n}(P_i - \Omega) \wedge m_i\vec{v}_i - \sum_{i=1}^{n}\underbrace{(\vec{v}_i - \vec{v}_\Omega)}_{\substack{velocity\ of\\ P_i\ with\\ respect\ to\ \Omega}} \wedge m_i\vec{v}_i = \dot{\vec{K}}_\Omega + \vec{v}_\Omega \wedge \sum_{i=1}^{n} m_i\vec{v}_i \\
&= \dot{\vec{K}}_\Omega + \vec{v}_\Omega \wedge \vec{Q}
\end{aligned}$$

This completes the proof, since extended objects can be treated in the same way.

FIRST INTEGRALS: Given the system $\{(P_i, m_i)\}_{i=1}^{n}$, a function $f(P_1, \cdots P_n, \vec{v}_1, \cdots, \vec{v}_n; t)$ is called a *first integral* if

$$f(P_1(t), \cdots P_n(t), \vec{v}_1(t), \cdots, \vec{v}_n(t); t) = f(P_1(t_0), \cdots P_n(t_0), \vec{v}_1(t_0), \cdots, \vec{v}_n(t_0); t_0) \ , \ \forall t \tag{7.40}$$

which simply means that f is constant during the motion.

CONSERVATION OF $\vec{Q}$: *Suppose one of the components of the total external force vanishes. Then, the corresponding component of the momentum $\vec{Q}$ is preserved:*

$$\vec{R}^{(e)} \cdot \hat{\imath}_j = 0 \Rightarrow \vec{Q} \cdot \hat{\imath}_j = Q_j = mv_{G,j} = const \tag{7.41}$$

CONSERVATION OF $\vec{K}_\Omega$: *Take Ω such that $\vec{v}_\Omega \wedge \vec{Q} = 0$ and suppose $\vec{M}_\Omega^{(e)} \cdot \hat{\imath}_j = 0,$ then*

$$\vec{K}_\Omega \cdot \hat{\imath}_j = K_{\Omega,j} = const \tag{7.42}$$

THEOREM 7.5: *the center of mass moves like one point of mass equal to the total mass subjected to the external forces:*

$$\boxed{m\vec{a}_G = \vec{R}^{(e)}} \tag{7.43}$$

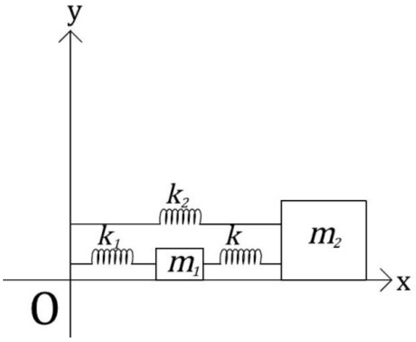

Figure 7.1. *Two masses sliding on a frictionless rail*

Ex1: Consider a frictionless rail. Absence of friction means that the constraints act only perpendicularly to the rail. Then:

$$\begin{cases} m_1\vec{a}_1 = -k_1(P_1 - O_1) + k(P_2 - P_1) + m_1\vec{g} + \vec{\Phi}_1 \\ m_2\vec{a}_2 = -k_2(P_2 - O_2) - k(P_2 - P_1) + m_2\vec{g} + \vec{\Phi}_2 \end{cases}$$

and

$$\vec{\Phi}_1 = -m_1\vec{g}; \quad \vec{\Phi}_2 = -m_2\vec{g}$$

where O_1 and O_2 hold the springs on the vertical axis. For the calculations, let us assume the masses are just points located in the x axis, with coordinates x_1 and x_2. Then, projecting along the x axis, we have:

$$m_1\ddot{x}_1 = -k_1x_1 + k(x_2 - x_1)$$
$$m_2\ddot{x}_2 = -k_2x_2 - k(x_2 - x_1)$$

The first equation of the dynamics is:

$$m\vec{a}_G = -k_1(P_1 - O_1) - k_2(P_2 - O_2) - m\vec{g} + \vec{\Phi}_1 + \vec{\Phi}_2; \qquad m = m_1 + m_2$$
$$\Rightarrow \quad m\ddot{x}_G = -k_1x_1 - k_2x_2$$

This cannot be solved by itself, because it contains three unknowns x_G, x_1, x_2.
As $mx_G = m_1x_1 + m_2x_2,\ x_1 = \frac{1}{m_1}[mx_G - m_2x_2]$, we have:

$$m\ddot{x}_G = -\frac{k_1}{m_1}[mx_G - m_2x_2] - k_2x_2 \Rightarrow m\ddot{x}_G = -\frac{k_1}{m_1}mx_G + \frac{k_1m_2 - k_2m_1}{m_1}x_2$$

which becomes an equation for x_G if and only if $k_1m_2 = k_2m_1$. However, for very large m_1 and weak k_2, the problem can be approximately solved, since $m\ddot{x}_G \approx -k_1x_G$.

THEOREM (KINETIC ENERGY) 7.6: *the time derivative of the kinetic energy, $\dot{T}$, equals the power Π of all forces acting on the system:*

$$\boxed{\dot{T} = \Pi} \tag{7.44}$$

Proof: By definition $\Pi = \sum_{i=1}^{n} \vec{F}_i \cdot \vec{v}_i$, where $\vec{F}_i$ is the sum of forces acting on P_i and $\vec{v}_i$ is the velocity of P_i. Then,

$$\vec{F}_i \cdot \vec{v}_i = m_i\vec{a}_i \cdot \vec{v}_i = \frac{d}{dt}\left(\frac{1}{2}m_i{v_i}^2\right), \qquad \forall i = 1, \cdots, n.$$

One now easily obtains a series of results, including the following.

THEOREM 7.7: *The power due to a time independent ideal bilateral (i.e. all displacements are reversible) constraint is zero.*

Proof: If we indicate with $\vec{\Phi}$ the constraint's reaction then its virtual work is non-negative since the constraint is ideal: $\delta L^{(v)} = \vec{\Phi} \cdot \delta(\mathrm{P} - \mathrm{O}) = \vec{\Phi} \cdot \vec{\mathrm{v}}dt \geq 0$ where $\vec{\mathrm{v}}$ is a virtual velocity. As the constraint is also bilateral, $-\vec{\mathrm{v}}$ is a virtual velocity if $\vec{\mathrm{v}}$ is, hence $\vec{\Phi} \cdot \vec{\mathrm{v}}dt = 0$. Lastly, the constraint is fixed, so the real velocity of the point P is one of the virtual ones and then:

$$\vec{\Phi} \cdot \frac{\mathrm{d}(\mathrm{P} - \mathrm{O})}{\mathrm{dt}}\mathrm{dt} = 0 \Rightarrow \vec{\Phi} \cdot \frac{d(P - O)}{dt} = \Pi^{(v)} = 0$$

This completes the proof.

THEOREM 7.8: *The power due to conservative forces of potential U is given by*

$$\boxed{\Pi_U = \dot{U}} \tag{7.45}$$

Proof: $\Pi_U = \vec{F} \cdot \vec{v} = \nabla U \cdot \frac{d(\mathrm{P}-\mathrm{O})}{dt} = \frac{d}{dt} U\big((\mathrm{P}-\mathrm{O})(t)\big) = \dot{U}$

THEOREM 7.9: *The variation of T equals the total work done by internal and external forces.*

THEOREM 7.10: *For a system subjected to conservative forces of potential U and ideal bilateral fixed constraints, E is preserved in time:*

$$\boxed{E = T - U = const} \tag{7.46}$$

THEOREM 7.11: *Let $Q \equiv G$ or be fixed and belonging at all times to the instantaneous rotation axis of a rigid body. We then have:*

$$m\vec{a}_G = \vec{R}^{(e)}; \qquad I_Q \dot{\vec{\omega}} + \vec{\omega} \wedge I_Q \vec{\omega} = \vec{M}_Q^{(e)} \tag{7.47}$$

Proof: the first part of Eq.(7.47) has already been proven. Then, because of Eq.(7.39) (which now reads $\vec{M}_{\mathrm{Q}}^{(e)} = \dot{\vec{K}}_{\mathrm{Q}} + \vec{v}_{\mathrm{Q}} \wedge \vec{Q}$), we have $\vec{M}_{\mathrm{Q}}^{(e)} = \dot{\vec{K}}_{\mathrm{Q}}$ if $\vec{v}_{\mathrm{Q}} = \mathbf{0}$, or $\mathrm{Q} = G$ (in which case $\vec{v}_{\mathrm{Q}} = \vec{v}_{\mathrm{G}} \parallel \vec{Q}$), as our hypotheses require. Using Eq.(7.33), which is $\vec{K}_Q = m(G-Q) \wedge \vec{v}_Q + I_Q\vec{\omega}$, and the fact that $Q = G$, or $\vec{v}_Q = \mathbf{0}$ because Q is fixed, we have $\vec{K}_Q = I_Q\vec{\omega}$. Differentiating, we obtain:

$$\dot{\vec{K}}_Q = \vec{\omega} \wedge I_Q\vec{\omega} + I_Q\dot{\vec{\omega}}$$

This equality derives from the fact that $\vec{K}_G = I_G\vec{\omega}$ and that I_Q is constant in the frame that moves rigidly with the object, and from the fact that the time derivatives ($\dot{\vec{u}}$ and $\vec{u}'$) of a vector in two different frames are related by:

$$\dot{\vec{u}} = \vec{u}' + \vec{\omega} \wedge \vec{u}$$

Indeed, in the mobile frame we can write $\vec{u} = \sum_{h=1}^{3} u_h \hat{e}_h$ and the Poisson formulae (2.46) then yield:

$$\dot{\vec{u}} = \underbrace{\textstyle\sum_{h=1}^{3} \dot{u}_h \hat{e}_h}_{\vec{u}'} + \vec{\omega} \wedge \underbrace{\textstyle\sum_{h=1}^{3} u_h \hat{e}_h}_{\vec{u}}$$

which completes the proof.

What is important to observe is the following. Let $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ be the principal axes with origin in Q. Then, the second of Eqs.(7.47) can be expanded in its components as:

$$\begin{cases} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = M_1^{(e)} \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = M_2^{(e)} \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = M_3^{(e)} \end{cases} \tag{7.48}$$

where I_i are the principal moments of inertia in Q and ω_i are the components of the angular velocity $\vec{\omega}$ in the principal reference frame.

7.1 Euler angles and equations

From relative kinematics and Euler angles we have:

$$\vec{\omega} = \overbrace{\big(\dot{\vartheta}\cos\varphi + \dot{\psi}\sin\varphi\sin\vartheta\big)}^{\omega_1}\hat{\imath}_1 + \overbrace{\big(\dot{\psi}\cos\varphi\sin\vartheta - \dot{\vartheta}\sin\varphi\big)}^{\omega_2}\hat{\imath}_2 + \overbrace{\big(\dot{\psi}\cos\vartheta + \dot{\varphi}\big)}^{\omega_3}\hat{\imath}_3 \tag{7.49}$$

which can be differentiated to yield

$$\dot{\omega}_1 = \ddot{\vartheta}\cos\varphi + \ddot{\psi}\sin\varphi\sin\vartheta - \dot{\vartheta}\dot{\varphi}\sin\varphi + \dot{\psi}\dot{\varphi}\cos\varphi\sin\vartheta + \dot{\psi}\dot{\vartheta}\sin\varphi\cos\vartheta$$
$$\dot{\omega}_2 = \ddot{\psi}\cos\varphi\sin\vartheta - \ddot{\vartheta}\sin\varphi - \dot{\psi}\dot{\varphi}\sin\varphi\sin\vartheta + \dot{\psi}\dot{\vartheta}\cos\varphi\cos\vartheta - \dot{\vartheta}\dot{\varphi}\cos\varphi$$
$$\dot{\omega}_3 = \ddot{\psi}\cos\vartheta + \ddot{\varphi} - \dot{\psi}\dot{\vartheta}\sin\varphi$$

In matrix form, one can write:

$$A\begin{pmatrix}\ddot{\vartheta}\\ \ddot{\psi}\\ \ddot{\varphi}\end{pmatrix} = b \tag{7.50}$$

with

$$A = \begin{pmatrix}\cos\varphi & \sin\varphi\sin\vartheta & 0\\ -\sin\varphi & \cos\varphi\sin\vartheta & 0\\ 0 & \cos\vartheta & 1\end{pmatrix} \tag{7.51}$$

and

$$b = \begin{pmatrix}\dfrac{M_1^{(e)}}{I_1} + \dfrac{I_2 - I_3}{I_1}\omega_2\omega_3 + \dot{\vartheta}\dot{\varphi}\sin\varphi - \dot{\psi}\big(\dot{\varphi}\cos\varphi\sin\vartheta + \dot{\vartheta}\sin\varphi\cos\vartheta\big)\\ \dfrac{M_2^{(e)}}{I_2} + \dfrac{I_3 - I_1}{I_2}\omega_1\omega_3 + \dot{\vartheta}\dot{\varphi}\cos\varphi - \dot{\psi}\big(\dot{\vartheta}\cos\varphi\cos\vartheta - \dot{\varphi}\cos\varphi\sin\vartheta\big)\\ \dfrac{M_3^{(e)}}{I_3} + \dfrac{I_1 - I_2}{I_3}\omega_1\omega_2 + \dot{\psi}\dot{\vartheta}\sin\vartheta\end{pmatrix} \tag{7.52}$$

where $\det A = \sin\vartheta$, $\vartheta = (0,\pi)$. So $\det A \neq 0$ where Euler angles can be used and, in that case A^{-1} exist, and we may write

$$\begin{pmatrix}\ddot{\vartheta}\\ \ddot{\psi}\\ \ddot{\varphi}\end{pmatrix} = A^{-1}b$$

DEFINITION 7.2: *A rigid body motion is called Poinsot, if* $\exists Q$ *fixed, or* $Q = G$, *such that*

$$\vec{M}_Q^{(e)} = \mathbf{0} \tag{7.53}$$

For such motions, one considers first the rotations about the fixed point or about the center of mass, for which Euler angles suffice, and later one may use the first cardinal equation to compute the motion of the center of mass. Clearly, the angular momentum of such motions is:

$$\vec{K}_Q = I_Q\vec{\omega} = const \tag{7.54}$$

where I_Q is the matrix of inertia. Indeed, the second equation of mechanics, referred to Q, implies $\dot{\vec{K}}_Q = \vec{M}_Q^{(e)} = \mathbf{0} \Rightarrow \vec{K}_Q = const$. Note: this does not mean $\vec{\omega} = const$, in general, for a rigid body in a Poinsot motion, because $\vec{K}_Q = I_Q\vec{\omega}$, and I_Q may vary as the object moves: its mass distribution with respect to the reference frame axes may change (we are not in the frame in which the body is at rest, hence no motion takes place). Moreover, $\vec{K}_\Omega$ with generic Ω does not need to be constant. For instance, taking $Q = G$ (then $\vec{K}_G = const$ by hypothesis), Eq.(7.31) yields:

$$\dot{\vec{K}}_\Omega = \frac{d}{dt}\big[m(G-\Omega)\wedge\vec{v}_G + \vec{K}_G\big] = \frac{d}{dt}\big(m(G-\Omega)\wedge\vec{v}_G\big) \neq \mathbf{0}$$

Proposition 7.1: *For Poinsot motion the quantity:*

$$T_0 = \frac{1}{2}I_Q\vec{\omega}\cdot\vec{\omega} = \frac{1}{2}\vec{K}_Q\cdot\vec{\omega} \tag{7.55}$$

is a constant of motion. If Q is fixed, T_0 is the kinetic energy. If $Q \equiv G$, then $T_0 = T^{(G)}$ is the kinetic energy measured in the center of mass frame (where G does not move, of course).

Proof: just apply definitions and recall the velocities involved vanish, like moments do.

Note: in agreement with the above, this proposition states that the projection of $\vec{\omega}$ along the direction of $\vec{K}_Q$, not $\vec{\omega}$ itself, is constant.

DEFINITION 7.3: *Poinsot motions with $\vec{\omega} = const$ are called permanent rotations.*

THEOREM 7.12: *A Poinsot motion is a permanent rotation if and only if the initial angular velocity is parallel to a principal axis with respect to G.*

Proof: $\dot{\vec{K}}_Q = I_Q\dot{\vec{\omega}} + \vec{\omega} \wedge I_Q\vec{\omega}$, as observed above. Because $\dot{\vec{K}}_Q = \mathbf{0}$ by assumption, we have $\dot{\vec{\omega}} = \mathbf{0}$ if and only if $\vec{\omega} \wedge I_Q\vec{\omega} = \mathbf{0}$ *i.e.* if and only if $I_Q\vec{\omega} \parallel \vec{\omega}$. Then, $I_Q\vec{\omega} = \lambda\vec{\omega}$, which means that $\vec{\omega}$ is an eigenvector of I_Q and is parallel to the principal axis.

DEFINITION 7.4: *A rigid body is called gyroscope of gyroscopic axis $\hat{e}_3$ if $I_{G1} = I_{G2}$.*

Suppose a gyroscope has a fixed point on the axis passing through G. Then we can state the following:

Proposition 7.2: *the Poinsot motion of a gyroscope has*

$$\vec{\omega} = \frac{\vec{K}_Q}{I_1} - \alpha\omega_{30}\hat{e}_3 \quad \text{with} \quad \alpha = \frac{I_3 - I_1}{I_1} \tag{7.56}$$

where ω_{30} is the initial third component of $\vec{\omega}$.

In this case ω_3 is constant along with $\vec{K}_Q$ and T_0, which are also constant. In particular, the angles between $\vec{\omega}$ and $\vec{K}_Q$ and between $\vec{\omega}$ and $\hat{e}_3$ are preserved in time.

Proposition 7.3: *All motions of a gyroscope referred to a fixed point Q or to the center of mass, $Q \equiv G$, are regular precessions, of precession axis parallel to $\vec{K}_Q$ and axis of proper rotation corresponding to the gyroscope axis.*

A motion of an object with a fixed point is called precession if its angular velocity $\vec{\omega}$ lies in the plane identified by one fixed axis $\hat{\imath}_3$ and one axis at rest with the object $\hat{e}_3$, and it is regular if the projections of $\vec{\omega}$ on such axes are constant in time. For instance, the Earth is not perfectly spherical: it is squeezed at the poles, so that

$$I_1 = I_2 = 0.9966 I_3 \Rightarrow \alpha = 3.4 \cdot 10^{-3}$$

Some calculations show that the Earth axis replaces winter with summer every 13,000 years.

Consider now a relevant special case. Let $\mathcal{B}$ be a rigid body with Q fixed in the laboratory frame and in the rest reference frame, whose configuration is identified by the Euler angles $\big(\vartheta(t), \psi(t), \varphi(t)\big)$. Then the virtual work of constraints is

$$\delta L^{(v)} = \vec{R}^{(v)} \cdot \delta Q + \vec{M}_Q^{(v)} \cdot \vec{\varepsilon}', \qquad \vec{\varepsilon}' = \vec{\omega} dt \tag{7.57}$$

As $\delta Q = \mathbf{0}$, one obtains $\delta L^{(v)} = \vec{M}_Q^{(v)} \cdot \vec{\varepsilon}'$. This means that the constraint is ideal if and only if $\vec{M}_Q^{(v)} = \mathbf{0}$ i.e. $\vec{R}^{(v)} = \vec{\Phi}_Q$ applied in Q. As we have not bounded the space of configurations, displacements are reversible and constraints bilateral. This is natural in the present case. Suppose we have only one active force, such as gravity, $m\vec{g}$, then the second equation of mechanics reads:

$$\dot{\vec{K}}_Q = \vec{Q} \wedge m\vec{g} \tag{7.58}$$

Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be at rest with respect to $\mathcal{B}$ and directed along principal directions in Q, with I_1, I_2, I_3. Let (x_1, x_2, x_3) be the coordinates of G in this rest frame $\{\hat{e}_h\}_{h=1}^3$, and let (u_1, u_2, u_3) be the unit vertical vector $\hat{u} = \hat{\imath}_3$, such that $\vec{g} = -g\hat{u}$ and $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ in $\{\hat{e}_h\}_{h=1}^3$. The Euler equations can now be written as:

$$\begin{cases} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = mg(u_2x_3 - u_3x_2) \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = mg(u_3x_1 - u_1x_3) \\ I_3\dot{\omega}_1 - (I_1 - I_2)\omega_1\omega_2 = mg(u_1x_2 - u_2x_1) \end{cases}$$

with

$$\begin{cases} u_1 = \sin\varphi\cos\vartheta \\ u_2 = \cos\varphi\sin\vartheta \\ u_3 = \cos\vartheta \end{cases}$$

Then, the Poisson formula yields:

$$0 = \frac{d\hat{u}}{dt} = \underbrace{\vec{u}'}_{\substack{mobile \\ frame}} + \vec{\omega} \wedge \underbrace{\vec{u}}_{\substack{fixed \\ frame}} \tag{7.59}$$

i.e.

$$\begin{cases} \dot{u}_1 + u_3\omega_2 - u_2\omega_3 = 0 \\ \dot{u}_2 + u_1\omega_3 - u_3\omega_1 = 0 \\ \dot{u}_3 + u_2\omega_1 - u_1\omega_2 = 0 \end{cases}$$

RIGID BODY WITH FIXED AXIS: take fixed and rest frames with coinciding third axes, $\hat{\imath}_3 = \hat{e}_3$. It suffices to consider ψ between $\hat{\imath}_1$ and $\hat{e}_1$ to determine the position of $\mathcal{B}$. The angular velocity is $\vec{\omega} = \dot{\psi}\hat{\imath}_3$. Let Ω be at rest and on the fixed axis, which means that it is fixed both in the laboratory reference frame and in the frame at rest with $\mathcal{B}$, so that $\delta\Omega = \mathbf{0}, \vec{\varepsilon}' = \varepsilon'\hat{\imath}_3$. Then,

$$\delta L^{(v)} = \vec{R}^{(v)} \cdot \delta\Omega + \vec{M}_{\Omega}^{(v)} \cdot \varepsilon'\hat{\imath}_3 = \vec{M}_{\Omega}^{(v)} \cdot \varepsilon'\hat{\imath}_3$$

In case the constraint is bilateral and ideal, $\delta L^{(v)} = 0,$ one has $\vec{M}_{\Omega}^{(v)} \cdot \hat{\imath}_3 = 0$. Then the second of the equations of mechanics yields:

$$\frac{d\vec{K}_{\Omega}}{dt} = \vec{M}_{\Omega}^{(a)} + \vec{M}_{\Omega}^{(v)}$$

and

$$\hat{\imath}_3 \cdot \frac{d\vec{K}_{\Omega}}{dt} = \vec{M}_{\Omega}^{(a)} \cdot \hat{\imath}_3$$

Also, $\hat{e}_3 = \hat{\imath}_3$, are fixed, which implies:

$$\frac{d\vec{K}_{\Omega}}{dt} \cdot \hat{\imath}_3 = \frac{d}{dt}\left(\vec{K}_{\Omega} \cdot \hat{\imath}_3\right) = \frac{d}{dt}\left(I_3\dot{\psi}\right) = I_3\ddot{\psi}$$

because $I_Q\vec{\omega} \cdot \hat{e}_1 = I_{Q1}\omega_1$ when $\hat{e}_1 \parallel \vec{\omega}$. In other words:

$$I_3\ddot{\psi} = M_{\Omega 3}^{a}\left(\psi, \dot{\psi}, t\right) \tag{7.60}$$

In the case active forces depend only on position, this kind of motion has one integral of motion like the mechanical energy. Indeed, let $M_{\Omega 3}^{a}$ depend only on ψ, its antiderivative U obeys: $I_3\dot{\psi}\ddot{\psi} = U'(\psi)\dot{\psi}$, which implies:

$$\frac{d}{dt}\left(\frac{1}{2}I_3\dot{\psi}^2 - U(\psi)\right) = 0 \;\text{ hence }\; \frac{1}{2}I_3\dot{\psi}^2 - U(\psi) = T - U = \text{constant} = E$$

As previously observed, this always possible in case of one-dimensional systems subjected to positional forces, whether conservative or not. This further shows that a potential is more general a concept than the potential energy, which only concerns conservative forces.

Ex2: consider a ladder of length l and mass m that falls, sliding on frictionless walls. There is only one degree of freedom, which we may identify with the angle ϑ. Then, we can write:

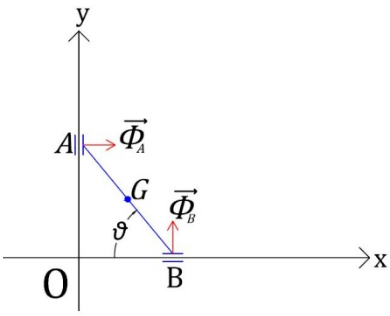

Figure 7.2. *ladder sliding on frictionless walls*

$$(G-O)=\frac{l}{2}\cos\vartheta\,\hat{\imath}_1+\frac{l}{2}\sin\vartheta\,\hat{\imath}_2;$$

$$\vec{v}_G=\frac{l}{2}\dot{\vartheta}[(-\sin\vartheta)\hat{\imath}_1+\cos\vartheta\,\hat{\imath}_2]$$

$$\vec{a}_G=\frac{l}{2}\ddot{\vartheta}[\cos\vartheta\,\hat{\imath}_2-\sin\vartheta\,\hat{\imath}_1]+\frac{l}{2}\dot{\vartheta}[-\dot{\vartheta}\sin\vartheta\,\hat{\imath}_2-\dot{\vartheta}\cos\vartheta\,\hat{\imath}_1]$$

$$=\hat{\imath}_1\left[-\frac{l}{2}\right]\left[\ddot{\vartheta}\sin\vartheta+\dot{\vartheta}^2\cos\vartheta\right]+\hat{\imath}_2\left[\frac{l}{2}\right]\left[\ddot{\vartheta}\cos\vartheta-\dot{\vartheta}^2\sin\vartheta\right]$$

In the center of mass frame, we can then write $\vec{K}_G=I_G\vec{\omega}\cdot\vec{\omega}=\frac{1}{12}ml^2\dot{\vartheta}\hat{\imath}_3$,

$T^{(G)}=\frac{1}{2}I_G\vec{\omega}\cdot\vec{\omega}=\frac{1}{2}\vec{K}_G\cdot\vec{\omega}=\frac{1}{24}ml^2\dot{\vartheta}^2$; $\quad T=\frac{1}{2}mv_G^2+T^{(G)}=\frac{1}{8}ml^2\dot{\vartheta}^2+\frac{1}{24}ml^2\dot{\vartheta}^2=\frac{1}{6}ml^2\dot{\vartheta}^2$

and $\quad U=-\frac{1}{2}mgl\sin\vartheta$.

Then, the kinetic energy theorem states:

$$\dot{T}=\frac{1}{2}\frac{d}{dt}\left(mv_G^2+\vec{K}_G\cdot\vec{\omega}\right)=m\vec{a}_G\cdot\vec{v}_G+\frac{1}{2}\left(\dot{\vec{K}}_G\cdot\vec{\omega}+\vec{K}_G\cdot\dot{\vec{\omega}}\right)$$

These relations can be combined to find the motion as a function of time. Note: this is not a Poinsot motion, so $\vec{K}_G$ is not constant.