ESERCITAZIONI

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Vettori e cinematica

1) $\underline{u} = 2\underline{i} + \underline{j}$, $\underline{v} = \underline{i} + \underline{j}$

a) $3\underline{u} + 2\underline{v} = 3(2\underline{i} + \underline{j}) + 2(\underline{i} + \underline{j}) = 8\underline{i} + 5\underline{j}$

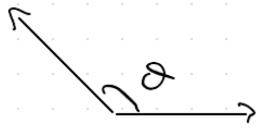
b) Angolo θ tra $\underline{u}$ e $\underline{v}$:

$\underline{u} \cdot \underline{v} = |\underline{u}||\underline{v}|\cos\theta$

$\cos\theta = \dfrac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|}$

$\underline{u} \cdot \underline{v} = (2,1)\cdot(1,1) = 2\cdot 1 + 1\cdot 1 = 3$

$|\underline{u}| = \sqrt{\underline{u}\cdot\underline{u}} = \sqrt{2^2+1^2} = \sqrt{5}$

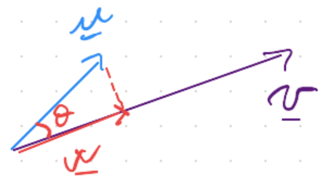


$|\underline{v}| = \sqrt{\underline{v}\cdot\underline{v}} = \sqrt{1^2+1^2} = \sqrt{2}$

$\cos\theta = \dfrac{3}{\sqrt{5}\cdot\sqrt{2}} = \dfrac{3}{\sqrt{10}}$ $\qquad \theta \simeq 0.32\ldots$

c) Proiezione ortogonale di $\underline{u}$ su $\underline{v}$

La chiamo $\underline{w}$



1° metodo

$\underline{u} \cdot \underline{v} = |\underline{u}||\underline{v}|\cos\theta$

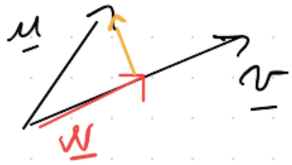
$|\underline{u}|\cos\theta$ è la proiezione (con segno) di $\underline{u}$ su $\underline{v}$

$|\underline{u}|\cos\theta = \dfrac{\underline{u}\cdot\underline{v}}{|\underline{v}|} = \dfrac{3}{\sqrt{2}}$

$|\underline{w}| = \big||\underline{u}|\cos\theta\big| = \dfrac{3}{\sqrt{2}}$

$\underline{w} = \dfrac{3}{\sqrt{2}}\hat{v} = \dfrac{3}{\sqrt{2}}\dfrac{\underline{v}}{|\underline{v}|} = \dfrac{3}{\sqrt{2}}\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = \left(\dfrac{3}{2}, \dfrac{3}{2}\right) = \dfrac{3}{2}\underline{i} + \dfrac{3}{2}\underline{j}$

2° metodo)



$\underline{u} = \underline{w} + \underline{u} - \underline{w}$

$\underline{w} = \lambda \underline{v}$

$\underline{u} = \lambda \underline{v} + \underline{u} - \lambda \underline{v}$

Per fissare λ: $\quad u - \lambda \underline{v} \perp \underline{v}$

$(\underline{u} - \lambda \underline{v}) \cdot \underline{v} = 0$

$\underline{u} \cdot \underline{v} - \lambda |\underline{v}|^2 = 0$

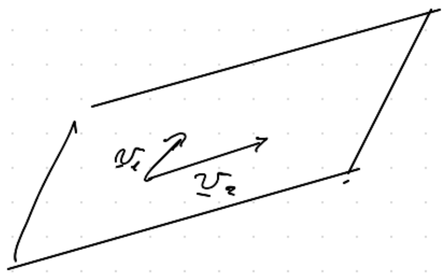
$3 - 2\lambda = 0$

$\lambda = \frac{3}{2}$

$\underline{w} = \lambda \underline{v} = \frac{3}{2}(1, 1) = \left(\frac{3}{2}, \frac{3}{2}\right)$

$|\underline{w}| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{9}{4}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

2) $\quad \underline{v}_1 = (t, t, 1-t)$; $\quad \underline{v}_2 = (1, t, 2)$; $\quad \underline{v}_3 = (0, 1, 1)$



se $\underline{v}_1 \cdot (\underline{v}_2 \wedge \underline{v}_3) = 0$ allora complanari.

$\underline{v}_1 \cdot (\underline{v}_2 \wedge \underline{v}_3) = \begin{vmatrix} t & t & 1-t \\ 1 & t & 2 \\ 0 & 1 & 1 \end{vmatrix} =$

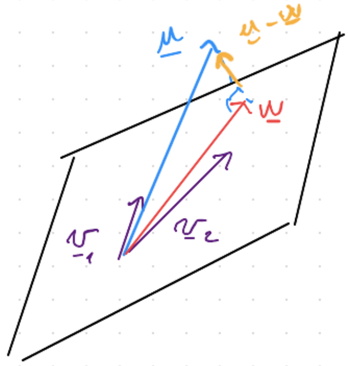
$= -\begin{vmatrix} t & 1-t \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} t & t \\ 1 & t \end{vmatrix} = -2t + 1 - t + t^2 - t = t^2 - 4t + 1$

Sono complanari se e solo se $t^2 - 4t + 1 = 0$

$t_{1,2} = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

3) $\underline{u} = (1,1,-2)$; $\underline{v}_1 = (1,0,-1)$; $\underline{v}_2 = (1,2,-1)$

$\underline{w}$ la proiezione ortogonale di $\underline{u}$ sul piano $(\underline{v}_1, \underline{v}_2)$



1° metodo)

$\underline{w} = \lambda \underline{v}_1 + \mu \underline{v}_2$

$\underline{u} = \underline{w} + \underline{u} - \underline{w}$

Impongo $(\underline{u} - \underline{w}) \cdot \underline{v}_1 = 0$

$(\underline{u} - \underline{w}) \cdot \underline{v}_2 = 0$

$$\begin{cases} (\underline{u} - \lambda \underline{v}_1 - \mu \underline{v}_2) \cdot \underline{v}_1 = 0 \\ (\underline{u} - \lambda \underline{v}_1 - \mu \underline{v}_2) \cdot \underline{v}_2 = 0 \end{cases} \qquad \begin{cases} \underline{u} \cdot \underline{v}_1 - \lambda |\underline{v}_1|^2 - \mu\, \underline{v}_2 \cdot \underline{v}_1 = 0 \\ \underline{u} \cdot \underline{v}_2 - \lambda\, \underline{v}_1 \cdot \underline{v}_2 - \mu |\underline{v}_2|^2 = 0 \end{cases}$$

$\underline{u} \cdot \underline{v}_1 = (1,1,-2) \cdot (1,0,-1) = 1+2 = 3$

$\underline{u} \cdot \underline{v}_2 = (1,1,-2) \cdot (1,2,-1) = 1+2+2 = 5$

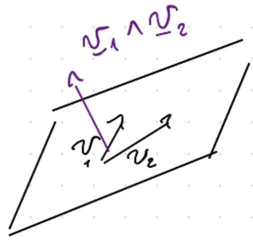
$|\underline{v}_1|^2 = 2$; $|\underline{v}_2|^2 = 6$; $\underline{v}_1 \cdot \underline{v}_2 = \underline{v}_2 \cdot \underline{v}_1 = (1,0,-1) \cdot (1,2,-1) = 2$

$$\begin{cases} 3 - 2\lambda - 2\mu = 0 \\ 5 - 2\lambda - 6\mu = 0 \end{cases} \qquad \begin{cases} 2\lambda + 2\mu = 3 \\ 2\lambda + 6\mu = 5 \end{cases} \qquad 2^a - 1^a \begin{cases} 4\mu = 2 \\ 2\lambda + 6\mu = 5 \end{cases} \qquad \begin{cases} \mu = \frac{1}{2} \\ \lambda = 1 \end{cases}$$

$\underline{w} = \lambda \underline{v}_1 + \mu \underline{v}_2 = \underline{v}_1 + \frac{1}{2} \underline{v}_2 = (1,0,-1) + \frac{1}{2}(1,2,-1) = \left(\frac{3}{2}, 1, -\frac{3}{2}\right)$

2° metodo)

$\underline{v}_1 \wedge \underline{v}_2$ è $\perp$ al piano di $\underline{v}_1$ e $\underline{v}_2$



$\underline{u} = \underbrace{\lambda \underline{v}_1 + \mu \underline{v}_2}_{\underline{w}} + \gamma\, \underline{v}_1 \wedge \underline{v}_2$

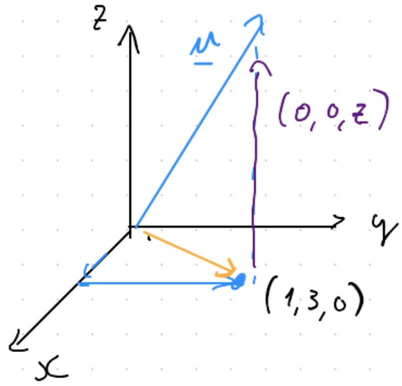
$(1,1,-2) = \lambda (1,0,-1) + \mu (1,2,-1) + \gamma (2,0,2)$

$$\underline{v}_1 \wedge \underline{v}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -1 \\ 1 & 2 & -1 \end{vmatrix} = 2\underline{i} - 0\underline{j} + 2\underline{k} = 2\underline{i} + 2\underline{k} = (2,0,2)$$

$$\begin{cases} 1 = \lambda + \mu + 2\gamma \\ 1 = 2\mu \\ -2 = -\lambda - \mu + 2\gamma \end{cases} \quad \begin{cases} \lambda + 2\gamma = \frac{1}{2} \\ \mu = \frac{1}{2} \\ \lambda - 2\gamma = \frac{3}{2} \end{cases} \quad \begin{cases} 4\gamma = -1 \\ \mu = \frac{1}{2} \\ \lambda = \frac{3}{2} + 2\gamma \end{cases} \quad \begin{cases} \gamma = -\frac{1}{4} \\ \mu = \frac{1}{2} \\ \lambda = 1 \end{cases}$$

$$\underline{w} = \lambda \underline{v}_1 + \mu \underline{v}_2 = (1,0,-1) + \frac{1}{2}(1,2,-1) = \left(\frac{3}{2}, 1, -\frac{3}{2}\right) \checkmark$$

4) $(1,3,0)$



$$\underline{u} = (1,3,0) + (0,0,z) = (1,3,z)$$

$$|\underline{u}| = \sqrt{1^2+3^2+z^2} = \sqrt{10+z^2} = a$$

$a \geq 0$

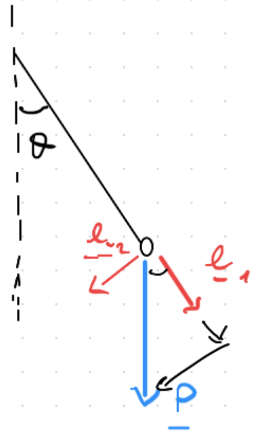
$10 + z^2 = a^2$

$z^2 = a^2 - 10$

$z = \pm\sqrt{a^2-10}$ $\qquad a \geq \sqrt{10}$ $\qquad$ (scartiamo $a \leq -\sqrt{10}$ perchè $a > 0$)

$$\underline{u} = \left(1, 3, \pm\sqrt{a^2-10}\right)$$

5) a)



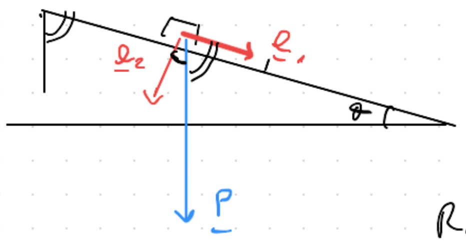
$$\underline{P} = (\underline{P}\cdot\underline{e}_1)\,\underline{e}_1 + (\underline{P}\cdot\underline{e}_2)\,\underline{e}_2 =$$

$$= P\cos\theta\,\underline{e}_1 + P\sin\theta\,\underline{e}_2$$

Risposta: $P\cos\theta$ $\parallel$

$P\sin\theta$ $\perp$

b)



$$\underline{P} = (\underline{P}\cdot\underline{e}_1)\,\underline{e}_1 + (\underline{P}\cdot\underline{e}_2)\,\underline{e}_2 =$$

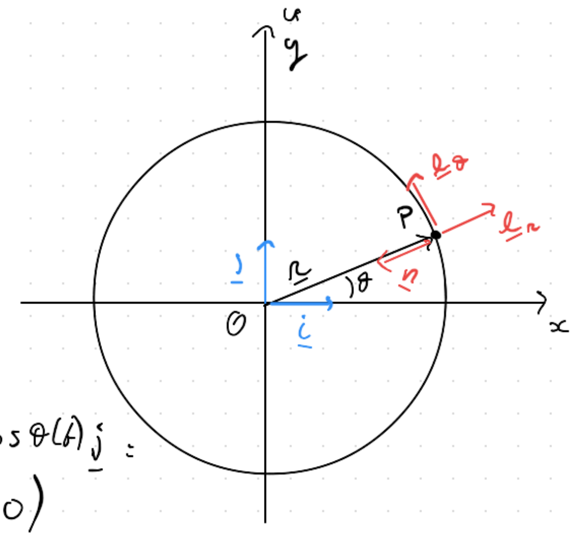
$$= P\sin\theta\,\underline{e}_1 + P\cos\theta\,\underline{e}_2$$

Risposta: $P\sin\theta$ $\parallel$

$P\cos\theta$ $\perp$

6)

a) velocità di P.



$$P(t) = (r\cos\theta(t),\ r\sin\theta(t),\ 0) =$$

$$= r\cos\theta(t)\,\underline{i} + r\sin\theta(t)\,\underline{j}$$

$$\underline{v}(t) = \dot{P}(t) = -r\dot{\theta}(t)\sin\theta(t)\,\underline{i} + r\dot{\theta}(t)\cos\theta(t)\,\underline{j} =$$

$$= r\dot{\theta}(t)\,(-\sin\theta(t),\ \cos\theta(t),\ 0)$$

Notare che $\underline{e}_r = (\cos\theta(t), \sin\theta(t), 0)$

$$\underline{e}_\theta = \left(\cos\left(\theta+\tfrac{\pi}{2}\right),\ \sin\left(\theta+\tfrac{\pi}{2}\right),\ 0\right) = (-\sin\theta(t),\ \cos\theta(t),\ 0)$$

$$\begin{cases} \underline{e}_r = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} \\ \underline{e}_\theta = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j} \end{cases}$$

Quindi $P(t) = r\,\underline{e}_r$

$\underline{v}(t) = r\dot{\theta}\,\underline{e}_\theta$

b) $\underline{v}(t) = -r\dot{\theta}(t)\sin\theta(t)\,\underline{i} + r\dot{\theta}(t)\cos\theta(t)\,\underline{j}$

$$\underline{a}(t) = -r\ddot{\theta}(t)\sin\theta(t)\,\underline{i} - r(\dot{\theta}(t))^2\cos\theta(t)\,\underline{i} +$$
$$+ r\ddot{\theta}(t)\cos\theta(t)\,\underline{j} - r(\dot{\theta}(t))^2\sin\theta(t)\,\underline{j} =$$
$$= r\ddot{\theta}(t)\left(-\sin\theta(t)\,\underline{i} + \cos\theta(t)\,\underline{j}\right) + r(\dot{\theta}(t))^2\left(-\cos\theta(t)\,\underline{i} - \sin\theta(t)\,\underline{j}\right)$$

$\underline{e}_r = \cos\theta(t)\,\underline{i} + \sin\theta(t)\,\underline{j}$

$\underline{e}_\theta = -\sin\theta(t)\,\underline{i} + \cos\theta(t)\,\underline{j}$

$\underline{n} = -\cos\theta(t)\,\underline{i} - \sin\theta(t)\,\underline{j}$

In generale:
$\underline{a}(t) = (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\underline{e}_\theta + (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r$

$\Rightarrow \underline{a}(t) = r\ddot{\theta}(t)\,\underline{e}_\theta - r\dot{\theta}^2(t)\,\underline{e}_r = r\ddot{\theta}(t)\,\underline{e}_\theta + r\dot{\theta}^2(t)\,\underline{n}$

Componente trasversa: $r\ddot{\theta}(t)(-\sin\theta(t)\,\underline{i} + \cos\theta(t)\,\underline{j}) = r\ddot{\theta}(t)\,\underline{e}_\theta$

Componente radiale: $r\dot{\theta}^2(t)(-\cos\theta(t)\,\underline{i} - \sin\theta(t)\,\underline{j}) = -r\dot{\theta}^2(t)\,\underline{e}_r$

Componente normale: " " $= r\dot{\theta}^2(t)\,\underline{n}$

c) Verifica di $\underline{v} = \underline{\omega} \wedge \underline{r}$ con $\underline{\omega} := \dot{\theta}\,\underline{k}$

$\underline{\omega} = (0, 0, \dot{\theta})$ $\qquad \underline{r} = (r\cos\theta, r\sin\theta, 0)$

$$\underline{\omega} \wedge \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \dot{\theta} \\ r\cos\theta & r\sin\theta & 0 \end{vmatrix} = -r\dot{\theta}\sin\theta\,\underline{i} + r\dot{\theta}\cos\theta\,\underline{j} + 0\underline{k} =$$
$$= -r\dot{\theta}\sin\theta\,\underline{i} + r\dot{\theta}\cos\theta\,\underline{j}$$

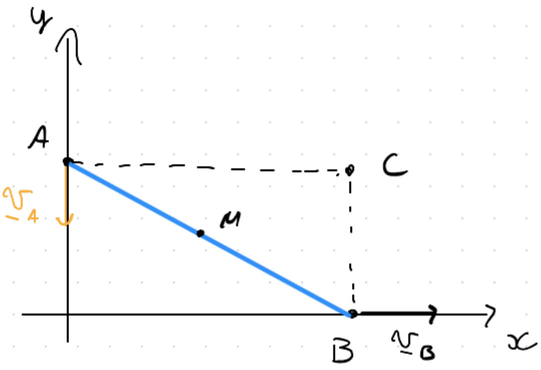
Sappiamo che $\underline{v} = -r\dot{\theta}\sin\theta\,\underline{i} + r\dot{\theta}\cos\theta\,\underline{j}$

7)

a) Teorema di Chasles:

centro istantaneo di rotazione

= intersezione delle perpendicolari alle velocità (presi due punti)



C = centro istantaneo di rotazione

$C = (x_c, y_c, 0) = (x_B, y_A, 0)$

$\underline{v}_B = v_0 \, \underline{i}$

Devo scrivere le risposte in funzione dei dati del problema.

$A = (0, y_A, 0)$; $B = (x_B, 0, 0)$

$x_B = v_0 t$

Vincolo di rigidità: $\ell^2 = x_B^2 + y_A^2 \iff y_A^2 = \ell^2 - v_0^2 t^2$
$(v_0^2 t^2 \leq \ell^2)$

$A = (0, \sqrt{\ell^2 - v_0^2 t^2}, 0)$; $B = (v_0 t, 0, 0)$; $C = (v_0 t, \sqrt{\ell^2 - v_0^2 t^2}, 0)$

b) $\underline{\omega} = ?$ Moto piano $\longrightarrow$ $\underline{\omega} = \omega \, \underline{k}$

Legge di distribuzione delle velocità:

$* \quad \underline{v}_B - \underline{v}_A = \underline{\omega} \wedge (B - A)$

$B - A = v_0 t \, \underline{i} - \sqrt{\ell^2 - v_0^2 t^2} \, \underline{j}$

Devo determinare $\underline{v}_B - \underline{v}_A$. $\underline{v}_B = (v_0, 0, 0)$

$$\underline{v}_A = \frac{d}{dt}(0, \sqrt{\ell^2 - v_0^2 t^2}, 0) = \left(0, -\frac{v_0^2 t}{\sqrt{\ell^2 - v_0^2 t^2}}, 0\right)$$

$$* \quad v_0 \, \underline{i} + \frac{v_0^2 t}{\sqrt{\ell^2 - v_0^2 t^2}} \, \underline{j} = \omega \, \underline{k} \wedge (v_0 t \, \underline{i} - \sqrt{\ell^2 - v_0^2 t^2} \, \underline{j})$$

$$v_0 \, \underline{i} + \frac{v_0^2 t}{\sqrt{\ell^2 - v_0^2 t^2}} \, \underline{j} = \omega v_0 t \, \underline{j} + \omega \sqrt{\ell^2 - v_0^2 t^2} \, \underline{i}$$

Lungo $\underline{i}$: $v_0 = \omega\sqrt{\ell^2 - v_0^2 t^2}$

Lungo $\underline{j}$: $\dfrac{v_0^2 t}{\sqrt{\ell^2 - v_0^2 t^2}} = \omega v_0 t$

$\omega = \dfrac{v_0}{\sqrt{\ell^2 - v_0^2 t^2}}$ Risposta: $\underline{\omega} = \dfrac{v_0}{\sqrt{\ell^2 - v_0^2 t^2}}\,\underline{K}$

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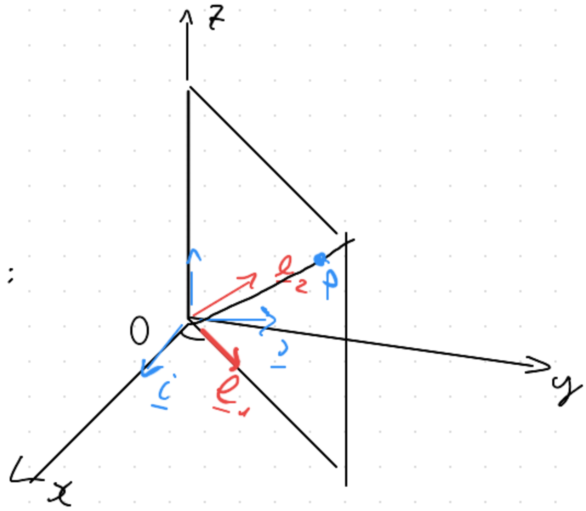
c) $M = \frac{1}{2}(A+B) = \frac{1}{2}\left(v_0 t, \sqrt{\ell^2 - v_0^2 t^2}, 0\right)$

$x_M = \frac{1}{2} v_0 t$; $y_M = \frac{1}{2}\sqrt{\ell^2 - v_0^2 t^2}$

$x_M^2 + y_M^2 = \cancel{\frac{1}{4} v_0^2 t^2} + \frac{1}{4}\ell^2 - \cancel{\frac{1}{4} v_0^2 t^2} = \frac{1}{4}\ell^2$

M disegna un arco di circonferenza di raggio $\frac{\ell}{2}$

8)



Terna del sistema assoluto:

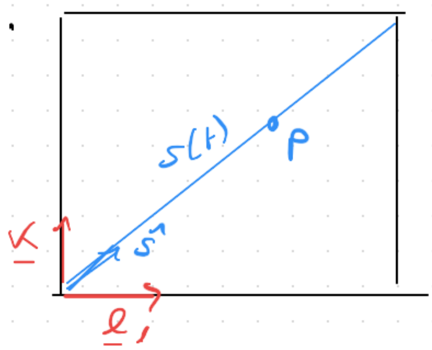
$(\underline{i}, \underline{j}, \underline{K})$

Terna solidale (relativa) al moto:

$(\underline{e}_1, \underline{e}_2, \underline{K})$

$\hat{s} = \frac{\sqrt{2}}{2}\underline{e}_1 + \frac{\sqrt{2}}{2}\underline{K}$

$P(t) = s(t)\,\hat{s}$ $\lambda = \frac{\sqrt{2}}{2}$



$$\begin{cases} \underline{e}_1 = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} \\ \underline{e}_2 = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j} \end{cases}$$

$P(t) = \lambda s(t)\left(\cos\theta\,\underline{i} + \sin\theta\,\underline{j} + \underline{K}\right)$