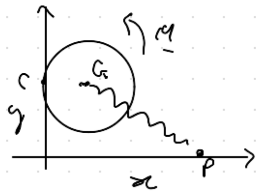
Tema 31/01/2014



$C=(0,y)$; $G=(R,y)$; $P=(x,0)$; $P-G=(x-R,-y)$

$$U=-\frac{1}{2}K|GP|^2-m_d g y_G+F x_P+\int \underline{M}\cdot d\theta \underline{k} \qquad \left(\delta L_{rot}=\underline{M}\cdot\underline{\varepsilon}'\right) \quad \delta\theta\underline{k}$$

$$\underbrace{dG-dC}_{=0}=d\theta\underline{k}\wedge(G-C) \rightarrow dy\,\underline{j}=d\theta\,\underline{k}\wedge R\underline{i} \rightarrow Rd\theta\,\underline{j}=dy\,\underline{j}$$

$$d\theta=\frac{dy}{R} \rightarrow \int\underline{M}\cdot d\theta\,\underline{k}=\int M\underline{k}\cdot\frac{dy}{R}\underline{k}=\frac{M}{R}\int dy=\frac{M}{R}y+cost$$

$$U=-\frac{1}{2}K(x^2-2Rx+R^2+y^2)-m_d g y+Fx+\frac{M}{R}y+cost=$$

$$=-\frac{1}{2}Kx^2+KRx-\frac{1}{2}Ky^2-m_d g y+Fx+\frac{M}{R}y+cost \quad \leftarrow -\frac{1}{2}KR^2$$

$$\frac{\partial U}{\partial x}=-Kx+KR+F \quad ; \quad \frac{\partial U}{\partial y}=-Ky-m_d g+\frac{M}{R}$$

1. Equilibrio $\begin{cases} x=R+\dfrac{F}{K} \\ y=\dfrac{M}{KR}-\dfrac{m_d g}{K}\end{cases}$ Unico equilibrio

Stabilità: $\dfrac{\partial^2 U}{\partial x^2}=-K$; $\dfrac{\partial^2 U}{\partial y\partial x}=\dfrac{\partial^2 U}{\partial x\partial y}=0$; $\dfrac{\partial^2 U}{\partial y^2}=-K$

$$\underline{\underline{H}}(x,y)=\begin{pmatrix}-K & 0\\ 0 & -K\end{pmatrix}$$ 2 autovalori negativi: è stabile.

2. Energia cinetica

Abbiamo già determinato $d\theta\underline{k}=\dfrac{dy}{R}\underline{k}$; $\underline{\omega}=\omega\underline{k}=\dot\theta\underline{k}=\dfrac{\dot y}{R}\underline{k}$

$$T=\frac{1}{2}m_p\dot x^2+\frac{1}{2}\frac{3}{2}m_d R^2\frac{\dot y^2}{R^2}=\frac{1}{2}m_p\dot x^2+\frac{3}{4}m_d\dot y^2=\frac{1}{2}\left(m_p\dot x^2+\frac{3}{2}m_d\dot y^2\right)$$

$P_x=\dfrac{\partial T}{\partial\dot x}=m_p\dot x$ $\quad P_y=\dfrac{\partial T}{\partial\dot y}=\dfrac{3}{2}m_d\dot y$ (momenti cinetici)

3. Equazioni del moto di Lagrange.

$\dfrac{d}{dt}\dfrac{\partial T}{\partial\dot x}=m_p\ddot x$, $\dfrac{\partial T}{\partial x}=0$; $\dfrac{d}{dt}\dfrac{\partial T}{\partial\dot y}=\dfrac{3}{2}m_d\ddot y$, $\dfrac{\partial T}{\partial y}=0$

$$\begin{cases} m_p\ddot x=-Kx+KR+F \\ \frac{3}{2}m_d\ddot y=-Ky-m_d g+\frac{M}{R}\end{cases}$$

4. Sono già lineari. $\rightarrow \omega_1 = \sqrt{\frac{K}{m_p}}$; $\omega_2 = \sqrt{\frac{K}{\frac{3}{2}m_d}} = \sqrt{\frac{2K}{3m_d}}$

$$T_1 = 2\pi\sqrt{\frac{m_p}{K}} \ ; \ T_2 = 2\pi\sqrt{\frac{3m_d}{2K}}$$

In effetti: $\underline{\underline{A}} = \begin{pmatrix} m_p & 0 \\ 0 & \frac{3}{2}m_d \end{pmatrix}$; $\underline{\underline{H}} = \begin{pmatrix} -K & 0 \\ 0 & -K \end{pmatrix}$; $\underline{\underline{A}}\,\ddot{\underline{\eta}} - \underline{\underline{H}}\,\underline{\eta} = 0$

$\eta_1 = x - x_{eq} = x - R - \frac{F}{K}$; $\eta_2 = y - y_{eq} = y - \frac{M}{KR} + \frac{m_d g}{K}$

$$\begin{cases} m_p\ddot{\eta}_1 + K\eta_1 = 0 \\ \frac{3}{2}m_d\ddot{\eta}_2 + K\eta_2 = 0 \end{cases} \longrightarrow \begin{cases} \ddot{\eta}_1 + \frac{K}{m_p}\eta_1 = 0 \\ \ddot{\eta}_2 + \frac{2K}{3m_d}\eta_2 = 0 \end{cases}$$

$\downarrow$

$$\begin{cases} m_p\ddot{x} + K\left(x - R - \frac{F}{K}\right) = 0 \\ \frac{3}{2}m_d\ddot{y} + K\left(y - \frac{M}{KR} + \frac{m_d g}{K}\right) = 0 \end{cases} \longrightarrow \text{le stesse di Lagrange}$$

5. $\underline{\Phi}_c$ reazione vincolare

1ª eq. card. disco: $\underline{\Phi}_c - m_d g\,\underline{j} + K(P-G) = \underline{0}$

$$P - G = (x - R, -y) \xrightarrow{(x_{eq},\,y_{eq})} \left(\cancel{R} + \frac{F}{K} - \cancel{R}, \frac{m_d g}{K} - \frac{M}{KR}\right)$$

$$\underline{\Phi}_c - m_d g\,\underline{j} + K\left[\frac{F}{K}\underline{i} + \left(\frac{m_d g}{K} - \frac{M}{KR}\right)\underline{j}\right] = \underline{0}$$

$$\underline{\Phi}_c - \cancel{m_d g}\,\underline{j} + F\,\underline{i} + \cancel{m_d g}\,\underline{j} - \frac{M}{R}\underline{j} = \underline{0}$$

$$\underline{\Phi}_c = -F\,\underline{i} + \frac{M}{R}\underline{j} \ ; \quad \underline{\Phi}_c = \left(-F, \frac{M}{R}\right)$$

17/09/2012 $\quad D = (0, R)$; $B = (0, R - l\cos\theta)$; $A = (l\sin\theta, R)$

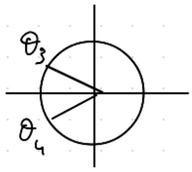
$C = (l\sin\theta, 0)$; $G = \left(\frac{l}{2}\sin\theta, R - \frac{l}{2}\cos\theta\right)$; $D - A = (-l\sin\theta, 0)$

1. $U = -\frac{1}{2}k|AD|^2 - mg\,y_G + cost = -\frac{1}{2}k\,l^2\sin^2\theta + mg\frac{l}{2}\cos\theta + cost$

$$U' = -kl^2\sin\theta\cos\theta - mg\frac{l}{2}\sin\theta = -l\sin\theta\left(kl\cos\theta + \frac{mg}{2}\right)$$

Equilibri: $1°$ caso: $\sin\theta = 0$ $\begin{cases}\theta_1 = 0 \\ \theta_2 = \pi\end{cases}$

$2°$ caso: $kl\cos\theta + \frac{mg}{2} = 0 \qquad \cos\theta = -\lambda \qquad \boxed{\lambda = \frac{mg}{2kl}}$



$\begin{cases}\sin\theta_3 = \sqrt{1-\lambda^2} \\ \cos\theta_3 = -\lambda\end{cases} \qquad \begin{cases}\sin\theta_4 = -\sqrt{1-\lambda^2} \\ \cos\theta_4 = -\lambda\end{cases}$

4 configurazioni di equilibrio se $\lambda < 1$

(se $\lambda = 1$ $\theta_2 = \theta_3 = \theta_4$)

$$U'' = -l\cos\theta\left(kl\cos\theta + \frac{mg}{2}\right) + kl^2\sin^2\theta = -kl^2\cos^2\theta + kl^2\sin^2\theta - mg\frac{l}{2}\cos\theta$$

$U''(0) = -kl^2 - mg\frac{l}{2} < 0$ stabile sempre; $U''(\pi) = -kl^2 + mg\frac{l}{2} = -kl^2(1-\lambda)$ stabile se $\lambda < 1$

$$U''(\theta_3) = U''(\theta_4) = -kl^2\lambda^2 + kl^2(1-\lambda^2) + mg\frac{l}{2}\lambda =$$
$$= kl^2(-\lambda^2 + 1 - \lambda^2 + \lambda^2) = kl^2(1-\lambda^2)$$

Instabili ($\lambda < 1$)

Per $\lambda = 1$, essendo ad una variabile non è difficile stabilire la stabilità guardando alle derivate successive di U...

2.
asta: $\underline{\omega}_a = \omega_a \underline{k} \qquad \omega_a = -\dot\theta$

disco: $\underline{v}_A - \underbrace{\underline{v}_C}_{=0} = \omega_d \underline{k} \wedge (A - C) \xrightarrow{\text{risolvendo}} \underline{\omega}_d = -\frac{l\cos\theta\,\dot\theta}{R}\underline{k}$

$\underline{v}_G = \dot\theta\frac{l}{2}(\cos\theta, \sin\theta)$

$$T = \underbrace{\frac{1}{2}m v_G^2 + \frac{1}{2}I_G^{(a)}\dot\theta^2}_{T^{(a)}} + \underbrace{\frac{1}{2}I_C^{(d)}\frac{l^2\cos^2\theta\,\dot\theta^2}{R^2}}_{T^{(d)}} = \left(\frac{1}{8}ml^2 + \frac{1}{24}ml^2 + \frac{3}{4}Ml^2\cos^2\theta\right)\dot\theta^2$$

$$T = \left(\frac{1}{6}m + \frac{3}{4}M\cos^2\theta\right)l^2\dot\theta^2 = \frac{1}{2}\left(\frac{1}{3}m + \frac{3}{2}M\cos^2\theta\right)l^2\dot\theta^2$$

$$p_\theta = \frac{\partial T}{\partial\dot\theta} = \left(\frac{1}{3}m + \frac{3}{2}M\cos^2\theta\right)l^2\dot\theta$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot\theta} = \left(\frac{1}{3}m + \frac{3}{2}M\cos^2\theta\right) l^2 \ddot\theta - 3Ml^2 \sin\theta\cos\theta\,\dot\theta^2$$

$$\frac{\partial T}{\partial \theta} = -\frac{3}{2}Ml^2\sin\theta\cos\theta\,\dot\theta^2$$

Eq. Lagrange $\quad \frac{d}{dt}\frac{\partial T}{\partial \dot\theta} - \frac{\partial T}{\partial \theta} = Q_\theta$

$$\left(\frac{1}{3}m + \frac{3}{2}M\cos^2\theta\right) l^2\ddot\theta - \frac{3}{2}Ml^2\sin\theta\cos\theta\,\dot\theta^2 = -l\sin\theta\left(Kl\cos\theta + \frac{mg}{2}\right)$$

5. Linearizzare $\quad A(\theta) = \left(\frac{1}{3}m + \frac{3}{2}M\cos^2\theta\right)l^2$

scegliamo $\theta_1 = 0 \qquad \eta = \theta \qquad A(0) = \left(\frac{1}{3}m + \frac{3}{2}M\right)l^2$

$$U''(0) = -Kl^2 - mg\frac{l}{2}$$

$$\left(\frac{1}{3}m + \frac{3}{2}M\right)l^2\ddot\theta + \left(Kl^2 + mg\frac{l}{2}\right)\theta = 0$$

$$\left(\frac{1}{3}m + \frac{3}{2}M\right)l\ddot\theta + \left(Kl + \frac{mg}{2}\right)\theta = 0$$

4. $\underline{\Phi}_C$ in equilibrio $\theta_1 = 0$

$D - A = (-l\sin\theta, 0) \xrightarrow{\theta = 0} (0, 0)$

Statica

Suggerimento: provare se conviene usare la prima equazione cardinale della statica per tutto il sistema

1ª eq. cardinale per il disco:

$$\underline{\Phi}_C + \underline{\Phi}_A + \underbrace{K(D-A)}_{=0} - Mg\underline{j} = \underline{0} \Rightarrow \underline{\Phi}_C + \underline{\Phi}_A = Mg\underline{j}$$

2ª eq. cardinale per il disco

Scelgo A come polo: $(C - A)\wedge \underline{\Phi}_C = \underline{0} \Rightarrow \underline{\Phi}_C \parallel C - A \parallel \underline{j}$

Quindi $\underline{\Phi}_C = \Phi_C\,\underline{j}$. A sua volta, poiché $\underline{\Phi}_C + \underline{\Phi}_A = Mg\underline{j}$, allora anche $\underline{\Phi}_A = \Phi_A\,\underline{j}$

1ª eq. cardinale per l'asta

$\underline{\Phi}_B - \underline{\Phi}_A - mg\underline{j} = \underline{0} \qquad$ Essendo ideale, $\underline{\Phi}_B = \Phi_B\,\underline{i}$

$\Phi_B\underline{i} - \Phi_A\underline{j} - mg\underline{j} = \underline{0} \Rightarrow \Phi_B = 0,\ \Phi_A = -mg$: $\quad \underline{\Phi}_C = Mg\underline{j} - \underline{\Phi}_A$

$$\underline{\Phi}_C = (M+m)g\,\underline{j}$$

Esame 14/09/2018

$A=(0,y)$; $B'=(l\sin\theta, 0)$; $B=(l\sin\theta, y+l\cos\theta)$

$G=\left(\frac{l}{2}\sin\theta, y+\frac{l}{2}\cos\theta\right)$; $O-A=(0,-y)$; $B'-B=(0,-y-l\cos\theta)$

$$\delta L^{(a)} = \underline{F}_A\cdot\delta A + \underline{F}_B\cdot\delta B + mg\,\underline{j}\cdot\delta G =$$

$$= -Ky\,\underline{j}\cdot\underbrace{\delta y\,\underline{j}}_{\delta A} - K(y+l\cos\theta)\underline{j}\cdot\underbrace{[l\cos\theta\,\delta\theta\,\underline{i}+(\delta y - l\sin\theta\,\delta\theta)\underline{j}]}_{\delta B} +$$

$$+ mg\,\underline{j}\cdot\left[\frac{l}{2}\cos\theta\,\delta\theta\,\underline{i}+(\delta y - \frac{l}{2}\sin\theta\,\delta\theta)\underline{j}\right] =$$

$$= -Ky\delta y - K(y+l\cos\theta)\delta y + Kl\sin\theta(y+l\cos\theta)\delta\theta +$$

$$+ mg\,\delta y - mg\frac{l}{2}\sin\theta\,\delta\theta$$

$$Q_\theta = Kl\sin\theta(y+l\cos\theta) - mg\frac{l}{2}\sin\theta = l\sin\theta\left(Ky+Kl\cos\theta-\frac{mg}{2}\right)$$

$$Q_y = -Ky - Ky - Kl\cos\theta + mg = -2Ky - Kl\cos\theta + mg$$

1) $Q_y = 0 \Rightarrow 2Ky = -Kl\cos\theta + mg$ $\qquad Ky = -\frac{Kl}{2}\cos\theta + \frac{mg}{2}$

$Q_\theta = 0$ diventa $\frac{Kl^2}{2}\sin\theta\cos\theta = 0$ (sostituendo in essa la prima)

1° caso : $\sin\theta = 0$
- $(\theta_1, y_1) = \left(0, -\frac{l}{2}+\frac{mg}{2K}\right)$
- $(\theta_2, y_2) = \left(\pi, \frac{l}{2}+\frac{mg}{2K}\right)$

2° caso : $\cos\theta = 0$
- $(\theta_3, y_3) = \left(\frac{\pi}{2}, \frac{mg}{2K}\right)$
- $(\theta_4, y_4) = \left(\frac{3}{2}\pi, \frac{mg}{2K}\right)$

2) Stabilità

$$\frac{\partial^2 U}{\partial\theta^2} = l\cos\theta\left(Ky+Kl\cos\theta-\frac{mg}{2}\right) - Kl^2\sin^2\theta$$

$$\frac{\partial^2 U}{\partial y\partial\theta} = Kl\sin\theta = \frac{\partial^2 U}{\partial\theta\partial y} \quad ; \quad \frac{\partial^2 U}{\partial y^2} = -2K$$

Su $(\theta_{eq}, y_{eq}) \rightarrow \frac{\partial^2 U}{\partial\theta^2} = \frac{1}{2}Kl^2\cos^2\theta - Kl^2\sin^2\theta$ (Usando $Ky = -\frac{Kl\cos\theta}{2}+\frac{mg}{2}$)

$$\underline{\underline{H}}(\theta_{eq}, y_{eq}) = \begin{pmatrix} \frac{1}{2}Kl^2\cos^2\theta - Kl^2\sin^2\theta & Kl\sin\theta \\ Kl\sin\theta & -2K \end{pmatrix}$$

$$\underline{\underline{H}}(\theta_1, y_1) = \underline{\underline{H}}(\theta_2, y_2) = \begin{pmatrix} \frac{1}{2}kl^2 & 0 \\ 0 & -2k \end{pmatrix}$$ entrambe instabili.

$$\underline{\underline{H}}(\theta_3, y_3) = \begin{pmatrix} -kl^2 & kl \\ kl & -2k \end{pmatrix} \quad Tr\,\underline{\underline{H}} < 0 \quad \det \underline{\underline{H}} = 2k^2l^2 - k^2l^2 = k^2l^2 > 0$$ sempre stabile

$$\underline{\underline{H}}(\theta_4, y_4) = \begin{pmatrix} -kl^2 & -kl \\ -kl & -2k \end{pmatrix} \quad Tr\,\underline{\underline{H}} < 0 \quad \det \underline{\underline{H}} = k^2l^2 > 0$$ sempre stabile

3) Statica

$$\underline{\phi}_A + mg\underline{j} + k(O-A) + k(B'-B) = \underline{0}; \quad O-A+B'-B = (0, -2y - l\cos\theta)$$

$$\underline{\phi}_A + mg\underline{j} - (2ky + kl\cos\theta)\underline{j} = 0$$

In tutti gli equilibri $2ky = -kl\cos\theta + mg \iff 2ky + kl\cos\theta = mg$

$$\underline{\phi}_A + mg\underline{j} - mg\underline{j} = \underline{0} \Rightarrow \underline{\phi}_A = \underline{0}$$

4)-5) 1ª eq. card. dinamica

$$\underline{\phi}_A + mg\underline{j} + k(O-A) + k(B'-B) = m\,\underline{a}_G$$

$$\underline{v}_G = \left(\frac{l}{2}\cos\theta\,\dot\theta,\ \dot y - \frac{l}{2}\sin\theta\,\dot\theta\right)$$

$$\underline{a}_G = \left(\frac{l}{2}\cos\theta\,\ddot\theta - \frac{l}{2}\sin\theta\,\dot\theta^2,\ \ddot y - \frac{l}{2}\sin\theta\,\ddot\theta - \frac{l}{2}\cos\theta\,\dot\theta^2\right)$$

$$\phi_A\underline{i} + mg\underline{j} - (2ky + kl\cos\theta)\underline{j} = m\,\underline{a}_G$$

$$\phi_A = m\frac{l}{2}(\cos\theta\,\ddot\theta - \sin\theta\,\dot\theta^2) \quad \longleftarrow \text{lungo } x$$

$$m\ddot y - m\frac{l}{2}\sin\theta\,\ddot\theta - m\frac{l}{2}\cos\theta\,\dot\theta^2 = \underbrace{-2ky - kl\cos\theta + mg}_{Q_y}$$

Ho trovato una delle due equazioni di Lagrange:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot y} - \frac{\partial T}{\partial y} = Q_y;$$ Verificarla che $\underline{\omega} = -\dot\theta\,\underline{k}$

$$T = \frac{1}{2}m\dot y^2 + \frac{1}{3}ml^2\dot\theta^2 - \frac{1}{2}ml\sin\theta\,\dot\theta\,\dot y$$

6) Se: $\underline{F}_a = -h\,\underline{v}_A$ in aggiunta alle altre forze:

$\underline{v}_A = \dot{y}\,\underline{j}$

$$\delta L^{(a)} = \delta L^{(a)}_{vecchia} + \underline{F}_a \cdot \delta A = \delta L^{(a)}_{vecchia} - h\dot{y}\,\underline{j} \cdot \delta y\,\underline{j} =$$

$$= \delta L^{(a)}_{vecchia} - h\,\dot{y}\,\delta y$$

La forza $\underline{F}_a$ contribuisce solo a Q_y e non a Q_θ

$$Q_y \longrightarrow Q_y = -2Ky - Kl\cos\theta + mg - h\dot{y}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}} - \frac{\partial T}{\partial y} = Q_y$$

Eq. del moto diventa:

$$m\ddot{y} - \frac{1}{2}ml\sin\theta\,\ddot{\theta} - \frac{1}{2}ml\cos\theta\,\dot{\theta}^2 = -2Ky - Kl\cos\theta + mg - h\dot{y}$$

N.B.: come discusso in aula, nulla cambia riguardo a equilibri e stabilità.