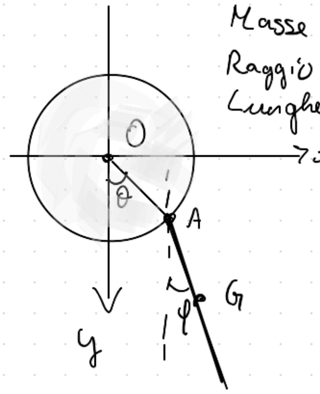
Energia cinetica del tema del 15/07/2016

$A = (R\sin\theta, R\cos\theta)$



Masse : m
Raggio : R
Lunghezza : ℓ

$G = (R\sin\theta + \frac{\ell}{2}\sin\varphi, R\cos\theta + \frac{\ell}{2}\cos\varphi)$

$\underline{\omega}_d = -\dot\theta \underline{K}$ $\qquad$ $\underline{\omega}_a = -\dot\varphi \underline{K}$

$T = T^{(d)} + T^{(a)}$

$T^{(d)} = \frac{1}{2} I_O^{(d)} \omega_d^2 = \frac{1}{2}\frac{1}{2} m R^2 \dot\theta^2 = \frac{1}{4} m R^2 \dot\theta^2$

$T^{(a)} = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G^{(a)} \omega_a^2$; $\underline{v}_G = (R\cos\theta\,\dot\theta + \frac{\ell}{2}\cos\varphi\,\dot\varphi, -R\sin\theta\,\dot\theta - \frac{\ell}{2}\sin\varphi\,\dot\varphi)$

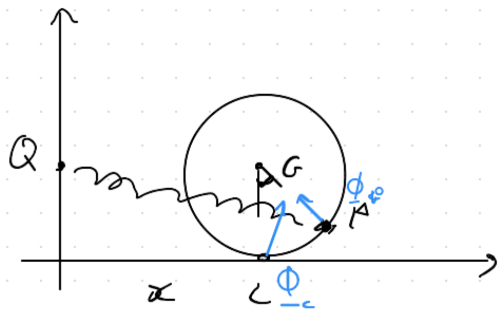
$T^{(a)} = \frac{1}{2} m \left(R^2\cos^2\theta\,\dot\theta^2 + R\ell\cos\theta\cos\varphi\,\dot\theta\dot\varphi + \frac{\ell^2}{4}\cos^2\varphi\,\dot\varphi^2 + R^2\sin^2\theta\,\dot\theta^2 + R\ell\sin\theta\sin\varphi\,\dot\theta\dot\varphi + \frac{\ell^2}{4}\sin^2\varphi\,\dot\varphi^2\right)$

$+ \frac{1}{2}\frac{1}{12} m \ell^2 \dot\varphi^2 = \frac{1}{2} m R^2 \dot\theta^2 + \frac{1}{2} m R\ell\cos(\theta-\varphi)\dot\theta\dot\varphi + \frac{1}{8} m \ell^2 \dot\varphi^2 + \frac{1}{24} m \ell^2 \dot\varphi^2$

$T = \frac{3}{4} m R^2 \dot\theta^2 + \frac{1}{2} m R\ell\cos(\theta-\varphi)\dot\theta\dot\varphi + \frac{1}{6} m \ell^2 \dot\varphi^2$

Tema del 18/02/2014

$G = (x, R)$; $Q = (0, R)$; $C = (x, 0)$



$P = (x + R\sin\theta, R - R\cos\theta)$

$Q - P = (-x - R\sin\theta, R\cos\theta)$

1. $U = -\frac{1}{2} k |PQ|^2 - m g y_P + cost =$

$= -\frac{1}{2} k (x^2 + 2Rx\sin\theta + \underbrace{R^2\sin^2\theta + R^2\cos^2\theta}_{cost}) - mg(\underbrace{R}_{cost} - \cos\theta) + cost =$

$= -\frac{1}{2} k x^2 - kRx\sin\theta + mgR\cos\theta + cost$

$Q_\theta = \frac{\partial U}{\partial \theta} = -kRx\cos\theta - mgR\sin\theta$

$Q_x = \frac{\partial U}{\partial x} = -kx - kR\sin\theta$

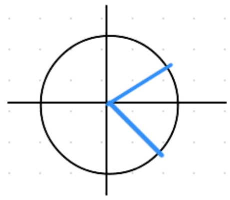
1. Da $Q_x = 0$ segue $\boxed{x = -R\sin\theta}$

Inserendo nella prima, uguagliando a zero

$$kR^2\sin\theta\cos\theta - mgR\sin\theta = 0 \iff R\sin\theta(kR\cos\theta - mg) = 0$$

1° caso: $\sin\theta = 0$ $\begin{cases} \theta_1 = 0 & x_1 = 0 & (\theta_1, x_1) = (0,0) \\ \theta_2 = \pi & x_2 = 0 & (\theta_2, x_2) = (\pi, 0) \end{cases}$ sono equilibri

2° caso: $kR\cos\theta = mg$; $\cos = \lambda$ con $\lambda = \dfrac{mg}{kR}$



$\theta_3 = \arccos\lambda \Rightarrow x_3 = -R\sqrt{1-\lambda^2}$

$\theta_4 = 2\pi - \arccos\lambda \Rightarrow x_4 = R\sqrt{1-\lambda^2}$

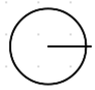
(oppure $-\arccos\lambda$)

Potevo semplicemente definirire θ_3 e θ_4 come angoli tali che:

$\begin{cases} \sin\theta_3 = \sqrt{1-\lambda^2} \\ \cos\theta_3 = \lambda \end{cases}$ $\begin{cases} \sin\theta_4 = -\sqrt{1-\lambda^2} \\ \cos\theta_4 = \lambda \end{cases}$ 4 configurazioni di equilibrio

(θ_3, x_3) e (θ_4, x_4) sono definite a patto che $\lambda \leq 1$

Se $\lambda = 1$, $(\theta_3, x_3) = (\theta_4, x_4) = (\theta_1, x_1) = (0,0)$



La 3ª e la 4ª esistono e sono distinte se $\lambda < 1$

Stabilità

$$\underline{\underline{H}}(\theta, x) = \begin{pmatrix} kRx\sin\theta - mgR\cos\theta & -kR\cos\theta \\ -kR\cos\theta & -k \end{pmatrix}$$

$$\underline{\underline{H}}(0,0) = \begin{pmatrix} -mgR & -kR \\ -kR & -k \end{pmatrix} \quad \left.\begin{array}{l} \mathrm{Tr}\,\underline{\underline{H}} < 0 \\ \det\underline{\underline{H}} = mgkR - k^2R^2 = k^2R^2(\lambda - 1) \end{array}\right\} \text{stabile se } \lambda > 1$$

$$\underline{\underline{H}}(\pi, 0) = \begin{pmatrix} mgR & -kR \\ -kR & -k \end{pmatrix} \quad \det\underline{\underline{H}} = -mgkR - k^2R^2 < 0$$

autovalori discordi sempre instabile

$$\underline{\underline{H}}(\theta_3, x_3) = \underline{\underline{H}}(\theta_4, x_4) = \begin{pmatrix} -kR^2\sin^2\theta - mgR\cos\theta & -kR\cos\theta \\ -kR\cos\theta & -k \end{pmatrix} =$$

$x = -R\sin\theta$

$$= \begin{pmatrix} -kR^2(1-\lambda^2) - mgR\lambda & -kR\lambda \\ -kR\lambda & -k \end{pmatrix}$$

$\mathrm{Tr}\,\underline{\underline{H}} = -kR^2(1-\lambda^2) - mgR\lambda - k < 0$ se $\lambda < 1$, caso di esistenza della 3ª e 4ª config. di equilibrio.

$$\det\underline{\underline{H}} = k^2R^2(1-\lambda^2) + \underbrace{mg}_{\lambda kR}kR\lambda - k^2R^2\lambda^2 = k^2R^2(1-\lambda^2+\lambda^2-\lambda^2) = k^2R^2(1-\lambda^2)$$

Dovendo essere $\lambda < 1$ perché (θ_3, x_3) e (θ_4, x_4) siano di equilibrio (distinte dalle altre), ho $\mathrm{Tr}\,\underline{\underline{H}} < 0$ e $\det\underline{\underline{H}} > 0$. Sono stabili

2. Energia e momenti cinetici.

$$\underline{v}_G - \overset{\simeq 0}{\underline{v}_C} = \omega_d \underline{k} \wedge (G-C) \xrightarrow{\text{calcolando}} \underline{\omega}_d = \omega_d \underline{k} = -\frac{\dot{x}}{R}\underline{k}$$

$$\underline{v}_P = (\dot{x} + R\cos\theta\,\dot{\theta},\ R\sin\theta\,\dot{\theta})$$

$$T = T^{(P)} + T^{(d)}$$

$$T^{(P)} = \frac{1}{2} m v_P^2 = \frac{1}{2} m \left(\dot{x}^2 + 2R\cos\theta\,\dot{\theta}\dot{x} + R^2\cos^2\theta\,\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\theta}^2\right) =$$

$$= \frac{1}{2} m R^2 \dot{\theta}^2 + m R\cos\theta\,\dot{\theta}\dot{x} + \frac{1}{2} m \dot{x}^2$$

$$T^{(d)} = \frac{1}{2} I_C^{(d)} \omega_d^2 = \frac{1}{2}\frac{3}{2} M R^2 \frac{\dot{x}^2}{R^2} = \frac{3}{4} M \dot{x}^2$$

$$T = \frac{1}{2} m R^2 \dot{\theta}^2 + m R\cos\theta\,\dot{\theta}\dot{x} + \left(\frac{1}{2}m + \frac{3}{4}M\right)\dot{x}^2 =$$

$$= \frac{1}{2}\left(m R^2 \dot{\theta}^2 + 2 m R\cos\theta\,\dot{\theta}\dot{x} + \left(m + \frac{3}{2}M\right)\dot{x}^2\right) = \frac{1}{2}\underline{\underline{A}}(\underline{q})\dot{\underline{q}}\cdot\dot{\underline{q}}$$

$$\underline{\underline{A}}(\theta, x) = \begin{pmatrix} m R^2 & m R\cos\theta \\ m R\cos\theta & m + \frac{3}{2}M \end{pmatrix}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = m R^2 \dot{\theta} + m R\cos\theta\,\dot{x} \qquad p_x = \frac{\partial T}{\partial \dot{x}} = m R\cos\theta\,\dot{\theta} + \left(m + \frac{3}{2}M\right)\dot{x}$$

3. Eq. del moto $\quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = Q_\theta \quad$ e $\quad \frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = Q_x$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = m R^2 \ddot{\theta} - m R\sin\theta\,\dot{\theta}\dot{x} + m R\cos\theta\,\ddot{x}; \quad \frac{\partial T}{\partial \theta} = -m R\sin\theta\,\dot{\theta}\dot{x}$$

$$\underbrace{m R^2 \ddot{\theta} - m R\sin\theta\,\dot{\theta}\dot{x} + m R\cos\theta\,\ddot{x} - (-m R\sin\theta\,\dot{\theta}\dot{x})}_{\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}} = \underbrace{-K R x\cos\theta - m g R\sin\theta}_{Q_\theta}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} = m R\cos\theta\,\ddot{\theta} - m R\sin\theta\,\dot{\theta}^2 + \left(m + \frac{3}{2}M\right)\ddot{x}; \quad \frac{\partial T}{\partial x} = 0$$

Equazioni di Lagrange:

$$\begin{cases} m R\ddot{\theta} + m\cos\theta\,\ddot{x} = -K x\cos\theta - m g\sin\theta \\ m R\cos\theta\,\ddot{\theta} - m R\sin\theta\,\dot{\theta}^2 + \left(m + \frac{3}{2}M\right)\ddot{x} = -K x - K R\sin\theta \end{cases}$$

(ho eliminato R)

4. Linearizzazione

Se $mg > KR$, cioè $\lambda = \frac{mg}{KR} > 1$, $(\theta_1, x_1) = (0,0)$ è stabile.
Linearizzo nel suo intorno.

$$\underline{\underline{A}}(\theta, x) = \begin{pmatrix} mR^2 & mR\cos\theta \\ mR\cos\theta & m+\frac{3}{2}M \end{pmatrix} \longrightarrow \underline{\underline{A}}(0,0) = \begin{pmatrix} mR^2 & mR \\ mR & m+\frac{3}{2}M \end{pmatrix}$$

$\underline{\underline{H}}(0,0) = \begin{pmatrix} -mgR & -KR \\ -KR & -K \end{pmatrix}$; Scrivo $\underline{\underline{A}}(0,0)\ddot{\underline{\eta}} - \underline{\underline{H}}(0,0)\underline{\eta} = \underline{0}$

con $\eta_1 = \theta$ e $\eta_2 = x$. Uso direttamente θ e x.

$$\begin{cases} mR^{\cancel{2}}\ddot{\theta} + m\cancel{R}\ddot{x} + mg\cancel{R}\theta + K\cancel{R}x = 0 \\ mR\ddot{\theta} + (m+\frac{3}{2}M)\ddot{x} + KR\theta + Kx = 0 \end{cases}$$

4b. $M = \frac{2}{3}m$; $mg = \frac{5}{3}KR$

$$|\underline{\underline{H}}(0,0) - \mu^2\underline{\underline{A}}(0,0)| = 0 \qquad \underline{\underline{A}} = \begin{pmatrix} mR^2 & mR \\ mR & m+\frac{3}{2}M \end{pmatrix} \xrightarrow{M=\frac{2}{3}m} \begin{pmatrix} mR^2 & mR \\ mR & 2m \end{pmatrix}$$

$$\underline{\underline{H}}(0,0) = \begin{pmatrix} -mgR & -KR \\ -KR & -K \end{pmatrix} \xrightarrow{mg=\frac{5}{3}KR} \begin{pmatrix} -\frac{5}{3}KR^2 & -KR \\ -KR & -K \end{pmatrix}$$

$$\begin{vmatrix} -\frac{5}{3}KR^2 - \mu^2 mR^2 & -KR-\mu^2 mR \\ -KR-\mu^2 mR & -K-\mu^2 2m \end{vmatrix} = 0$$

$$(mR^2\mu^2 + \tfrac{5}{3}KR^2)(2m\mu^2+K) - (mR\mu^2+KR)^2 = 0$$

$$2m^2R^2\mu^4 + (mKR^2 + \tfrac{10}{3}mKR^2)\mu^2 + \tfrac{5}{3}K^2R^2 - m^2R^2\mu^4 - 2mKR^2\mu^2 - K^2R^2 = 0$$

$$m^2R^2\mu^4 + \tfrac{7}{3}mKR^2\mu^2 + \tfrac{2}{3}K^2R^2 = 0 \Leftrightarrow \mu^4 + \tfrac{7}{3}\tfrac{K}{m}\mu^2 + \tfrac{2}{3}\tfrac{K^2}{m^2} = 0$$

$$\mu^4 + \frac{7}{3}\frac{K}{m}\mu^2 + \frac{2}{3}\frac{K^2}{m^2} = 0$$

$$\mu^2 = \frac{-\frac{7}{3} \pm \sqrt{\frac{49}{9} - \frac{8}{3}}}{2}\frac{K}{m} = \frac{-\frac{7}{3} \pm \sqrt{\frac{25}{9}}}{2}\frac{K}{m} = \left(-\frac{7}{6} \pm \frac{5}{6}\right)\frac{K}{m} = \begin{cases} -\frac{1}{3}\frac{K}{m} \\ -2\frac{K}{m} \end{cases}$$

$$\omega_1 = \sqrt{-\mu_1^2} = \sqrt{\frac{1}{3}}\sqrt{\frac{K}{m}} \quad ; \quad \omega_2 = \sqrt{-\mu_2^2} = \sqrt{2}\sqrt{\frac{K}{m}}$$

- Se avessi sostituito ovunque a K il valore $K = \frac{3}{5}\frac{mg}{R}$ avrei trovato da risolvere $\mu^4 + \frac{7}{5}\frac{g}{R}\mu^2 + \frac{6}{25}\frac{g^2}{R^2} = 0$

$$\mu_{1,2}^2 = \begin{cases} -\frac{1}{5}\frac{g}{R} \\ -\frac{6}{5}\frac{g}{R} \end{cases}$$

$$\theta = C_{11}\cos(\omega_1 t + \psi_{11}) + C_{12}\cos(\omega_2 t + \psi_{12})$$

$$x = C_{21}\cos(\omega_1 t + \psi_{21}) + C_{22}\cos(\omega_2 t + \psi_{22})$$

5. $\underline{\phi}_c$ per $(\theta_1, x_1) = (0,0)$

$$P = (x + R\sin\theta, R - R\cos\theta) \xrightarrow{(0,0)} (0,0); \quad Q - P \xrightarrow{(0,0)} (0,R)$$

È uno dei (rari) casi in cui mi conviene scrivere le equazioni cardinali della statica per tutto il sistema:

$$\underline{\phi}_c - Mg\underline{j} - mg\underline{j} + K(Q-P) = \underline{0}$$

$$\underline{\phi}_c = (M+m)g\underline{j} - KR\underline{j} \qquad \underline{\phi}_c = (0, (M+m)g - KR)$$

Se avessi considerato il disco ed eventualmente punto separatamente

Disco: $\underline{\phi}_c + \underline{\phi}_p - Mg\underline{j} = \underline{0}$

Punto: $-\underline{\phi}_p + K(Q-P) - mg\underline{j} = \underline{0}$

Dalla seconda $\underline{\phi}_p = K(Q-P) - mg\underline{j}$

Sostituendo nella prima: $\underline{\phi}_c + K(Q-P) - mg\underline{j} - Mg\underline{j} = \underline{0}$