CHAPTER 5

Mechanics

Mechanics investigates the causes of motion of material objects, not merely the motion itself, and it represents such causes through the notion of force. In our framework, a force $\vec{F}$ acting on a point P is a vector affecting the motion of an object of given mass localized in a point P, provided the object can move in the direction of $\vec{F}$. Constraints could indeed make that impossible, by exerting counter forces. There are numerous kinds of forces, and we consider the following:

1. CONSTANT FORCE: it is a force that does not change in time and does not depend on the position P. For instance, the weight of an object close to Earth can be seen as a constant vector pointing downward.
2. POSITION FORCE: it is a force that depends only on the position of the object: $\vec{F} = \vec{F}\,(P)$. For instance: the elastic force, the gravitational force (far from the surface of the Earth), and the Coloumb force are position forces: they depend on the distance between interacting objects.
3. VELOCITY DEPENDENT: it is a force that depends only on the velocity of the object, in symbols: $\vec{F} = \vec{F}\,(\vec{v})$. For instance, the viscous force, proportional to velocity, is like that.
4. TIME DEPENDENT: it is a force that depends only on time: $\vec{F} = \vec{F}\,(t)$. For instance, the pressure on the bottom of a swimming pool when a faucet is open and pours water in it; the amount of water grows, and it exerts a growing pressure on the bottom.
5. GENERAL FORCE: in general, a force may depend on all parameters, position, velocity and time: $\vec{F} = \vec{F}\,(P, \vec{v}, t)$.

To deal with forces, we need to treat vectors as objects applied to points. Therefore, let $\vec{u}$ be a vector and P its application point. Given the pair $(P, \vec{u})$, we call *application line* the straight line passing through P and parallel to $\vec{u}$.
Two vectors are called *concurrent* in Q if their application lines intersect in Q. A system of applied vectors

$$S = \{(P_i, \vec{u}_i), i = 1, \cdots, n\} \tag{5.1}$$

is called *planar* if all points of the given system belong to a plane π perpendicular to a vector $\hat{e}_3$: $P_i \in \pi \perp \hat{e}_3$ and $\vec{u}_i \perp \hat{e}_3$.

The *moment* of $(P, \vec{u})$ with respect to a point O is defined by

$$\vec{M}_0 = (P - O) \wedge \vec{u} \tag{5.2}$$

Given the angle ϑ between the line of $(P - O)$ and the application line of $(P, \vec{u})$, the distance between O and the application line

$$|(P - O)|_\perp = |(H - O)| = |(P - O)| \sin\vartheta \tag{5.3}$$

H being the orthogonal projection of P on the application line, is called *arm* of $(P, \vec{u})$. The moment of a force is called *torque*. Considering another point Q, we have:

$$\vec{M}_Q = (P - Q) \wedge \vec{u} = [(P - O) + (O - Q)] \wedge \vec{u} = \vec{M}_0 + (O - Q) \wedge \vec{u} \tag{5.4}$$

Equations (5.3) and (5.4) imply $\vec{M}_Q = \vec{M}_O$ if $(O - Q)$ is parallel to $\vec{u}$, and $\vec{M}_O = \mathbf{0}$ if O belongs to the application line. For a system of vectors, we often consider the resultant vector $\vec{R}$ and resultant moment $\vec{M}_O$:

$$\vec{R} = \textstyle\sum_{i=1}^{n} \vec{u}_i\,; \qquad \vec{M}_O = \textstyle\sum_{i=1}^{n} (P_i - O) \wedge \vec{u}_i \tag{5.5}$$

and again, one can write:

$$\begin{aligned}\vec{M}_Q &= \sum_{i=1}^{n} (P_i - Q) \wedge \vec{u}_i = \sum_{i=1}^{n} \big((P_i - O) + (O - Q)\big) \wedge \vec{u}_i = (O - Q) \wedge \sum_{i=1}^{n} \vec{u}_i + \vec{M}_O \\ &= (O - Q) \wedge \vec{R} + \vec{M}_O \end{aligned} \tag{5.6}$$

For a pair of applied vectors $\{(P, \vec{u}), (Q, -\vec{u})\}$, which yields $\vec{R} = \mathbf{0}$, one then obtains:

$$\vec{M}_O = (P - O) \wedge \vec{u} + (Q - O) \wedge (-\vec{u}) = \big((P - O) + (O - Q)\big) \wedge \vec{u} = (P - Q) \wedge \vec{u} \tag{5.7}$$

which does **not** depend on O. The same holds for any other system with resultant force $\vec{R} = \mathbf{0}$. The quantity

$$I \coloneqq \vec{R} \cdot \vec{M}_O \tag{5.8}$$

is called *scalar invariant*, since it does not depend on the point O:

$$\vec{R} \cdot \vec{M}_Q = \vec{R} \cdot \big((O - Q) \wedge \vec{R} + \vec{M}_0\big) = \vec{R} \cdot \vec{M}_0 \tag{5.9}$$

For planar systems, let $\hat{e}_3$ be orthogonal to the plane of the vectors π, and let $O \in \pi$. Then, $\vec{M}_O \parallel \hat{e}_3$ and $\vec{R} \perp \hat{e}_3$, which implies $\vec{R} \perp \vec{M}_O$, *i.e.* $\vec{R} \cdot \vec{M}_O = 0$. Therefore, $I = 0$ for a planar system. In the case of a system of parallel vectors, $\vec{u}_i \parallel u_j,\ \forall i, j$, let us introduce $\hat{e}_3$ such that $\vec{u}_i = u_i \hat{e}_3$; then we can write:

$$\vec{R} = \left(\textstyle\sum_{i=1}^{n} u_i\right)\hat{e}_3 \quad \text{which also implies} \quad \vec{M}_O \perp \hat{e}_3$$

and

$$\vec{M}_O = \sum_{i=1}^{n} (P_i - O) \wedge \vec{u}_i = \sum_{i=1}^{n} \big(u_i (P_i - O)\big) \wedge \hat{e}_3 \perp \vec{R}$$

which implies $I = 0$.

Two systems of forces S and S' are called *equivalent* if their resultants are equal, $\vec{R} = \vec{R}'$, and their moments with respect to a point O are also equal, $\vec{M}_0 = \vec{M}'_0$. Indeed, if they are equivalent for a given O, they are such for all points:

$$\vec{M}_Q = (O - Q) \wedge \vec{R} + \vec{M}_O = (O - Q) \wedge \vec{R}' + \vec{M}'_O = \vec{M}'_Q \tag{5.10}$$

We now observe that certain operations on a system of applied vectors produce equivalent systems. These operations include:

- Translation of a vector along its application line $(P, \vec{u}) \longmapsto (P + \lambda\vec{u}, \vec{u})$, as indeed $((P - O) + \lambda\vec{u}) \wedge \vec{u} = (P - O) \wedge \vec{u}$
- Substitution of the vectors applied in a point P with the sum of these vectors applied in P:

$$\sum_{i=1}^{m} (P - O) \wedge \vec{u}_i = (P - O) \wedge \sum_{i=1}^{m} \vec{u}_i \tag{5.11}$$

Combining these two operations, we find that also the following lead to equivalent systems:

- adding or subtracting a pair of vanishing moment;
- replacing concurrent vectors, whose application lines meet in Q with their sum applied in Q.

A system of applied vectors is called *equilibrated* if $\vec{R}$ and $\vec{M}_0$ vanish.

REDUCTION OF EQUILIBRATED SYSTEMS: given a system of applied vectors S, what is the simplest system S' equivalent to S ? To answer this question, let $\vec{R}_S$, $\vec{M}_{O_S}$ and $I_S = \vec{R}_S \cdot \vec{M}_{O_S}$ be the sum vector, moment and scalar invariant, respectively. Consider the following cases:

i. $\vec{R}_S = \mathbf{0} = \vec{M}_{0_S}$, then S is equivalent to the system made of no vectors;
ii. $\vec{R}_S = 0, \quad \vec{M}_{0_S} \neq 0$, then S is equivalent to a pair of equal and opposite vectors;
iii. $\vec{R}_S \neq 0, \quad I_S = 0$, then S is equivalent to the vector $\vec{R}_S$ applied at any point of its application line;
iv. $\vec{R}_S \neq 0, \quad I_S \neq 0$, then S is equivalent to one vector plus one pair.

Proof: (i) is trivial.
(ii) take $\{(P,\vec{u}),(Q,-\vec{u})\}$ such that $(P-Q)\wedge\vec{u} = \vec{M}_{O_S}$. These are 3 equations for 9 unknown quantities, $(P_x,P_y,P_z),(Q_x,Q_y,Q_z),(u_x,u_y,u_z)$ so typically they admit many solutions.
(iii) Take $S^* = \{(P^*,\vec{u}^*)\}$ where $\vec{u}^* = \vec{R}_S$, and impose $(P^*-O)\wedge\vec{u}^* = (P^*-O)\wedge\vec{R}_S = \vec{M}_{O_S}$. We thus have an equation of the kind $\vec{a}\wedge\vec{u} = \vec{b}, \vec{a} \neq \mathbf{0}$ which admits solutions if $\vec{a} \perp \vec{b}$ and $\vec{u} \neq \mathbf{0}$, in which case we have: $\vec{u} = \lambda\vec{a} + \frac{\vec{b}\wedge\vec{a}}{a^2}$, $\lambda \in \mathbb{R}$, that implies:

$$\overrightarrow{OP}^* = \frac{\vec{R}_S \wedge \vec{M}_{O_S}}{{R_S}^2} + \lambda\vec{R}_S, \qquad \forall\lambda \in \mathbb{R} \tag{5.12}$$

An extremely useful notion is the following.

INFINITESIMAL WORK: Let the point P move in an elementary time interval $[t, t+dt]$ under the action of a force $\vec{F}$. The corresponding infinitesimal work, given a reference frame with any origin O, is defined by:

$$dL = \vec{F}\cdot d(P-O) = \vec{F}\cdot d\vec{l} \tag{5.13}$$

where the variation $d(P-O)$ may be understood as going from P to an infinitesimally close point P', as in

$$(P'-P) = (P'-O)-(P-O) = \overrightarrow{OP}' - \overrightarrow{OP}$$

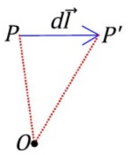

Figure 5.1. *Composition of vectors*

Writing the variation of P as (dx, dy, dz) we may express the infinitesimal work as:

$$dL = F_x dx + F_y dy + F_z dz \tag{5.14}$$

which is a *differential form*.

In general, a differential form is expressed as:

$$\psi_1(x_1,\cdots,x_n)dx_1 + \psi_2(x_1,\cdots,x_n)dx_2 + \cdots + \psi_n(x_1,\cdots,x_n)dx_n \tag{5.15}$$

where the terms ψ_i are given functions. This differential form is called *exact* or *integrable* if there exists a function $f = f(x_1,\cdots,x_n)$ such that $df = \psi_1 dx_1 + \psi_2 dx_2 + \cdots + \psi_n dx_n$. For this to be case, it is **necessary** that:

$$\psi_h = \frac{\partial f}{\partial x_h}, \qquad h = 1,\cdots,n \tag{5.16}$$

and that the ψ_h be C^1, which, by Schwartz theorem, means:

$$\frac{\partial\psi_h}{\partial x_k} = \frac{\partial\psi_k}{\partial x_h}, \qquad h \neq k \tag{5.17}$$

Equation (5.17) is then necessary for integrability to hold. In case (5.17) holds on a *simply connected domain* D, it is also **sufficient**. This is very interesting because it greatly simplifies the calculations of the form integrals:

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \psi_i \dot{x}_i(t) dt = \int_{t_1}^{t_2} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \dot{x}_i(t) dt = \int_{t_1}^{t_2} \frac{df}{dt} dt = f(t_2) - f(t_1)$$

One obtains the following theorem.

THEOREM 5.1: *for an integrable infinitesimal work, it is necessary that the force doing it depends on positions only:*

$$\vec{F} = \vec{F}(P)$$

Proof: assume dL is integrable. Let $x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = \dot{x},\ x_5 = \dot{y},\ x_6 = \dot{z},\ x_7 = t$. Comparing the expressions of dL and Eq.(5.15), one has

$$\psi_1 = F_x,\ \psi_2 = F_y,\ \psi_3 = F_z, \qquad \psi_4 = \psi_5 = \psi_6 = \psi_7 = 0$$

i.e.

$$\frac{\partial \psi_h}{\partial x_k} = 0 \quad \text{for } h \geq 4$$

Then, for Schwartz, the derivatives of ψ_1 with respect to x_4, x_5, x_6, x_7 must obey:

$$\frac{\partial \psi_1}{\partial x_h} = 0 \quad i.e. \quad \frac{\partial F_x}{\partial \dot{x}} = \frac{\partial F_x}{\partial \dot{y}} = \frac{\partial F_x}{\partial \dot{z}} = \frac{\partial F_x}{\partial t} = 0 \tag{5.18}$$

which means that F_x does not depend on velocities and time. The same holds for $\psi_2 = F_y$ and $\psi_3 = F_z$. Was this not the case, *i.e.* would the force depend on velocity or time, the infinitesimal work would not be exact.

WORK ALONG FINITE PATH: in general, let $\vec{F} = \vec{F}(P, \vec{v}, t)$ act during a time interval $[t_1, t_2]$. To compute the work done, knowledge of $\vec{F}$ is not enough. Suppose we also know the position as a function of time, $P = P(t)$. Then, introducing the *power* Π, we can write:

$$dL = \vec{F}(P(t), \vec{v}_P(t), t) \cdot \frac{d}{dt}(P - O)\, dt = \Pi(\mathrm{t}) dt \tag{5.19}$$

which implies:

$$L = \int_{t_1}^{t_2} \Pi(t) dt \tag{5.20}$$

Therefore, changing trajectory, with fixed extremes $P_1 = P(t_1),\ P_2 = P(t_2)$, the work may change. In particular, letting the initial and final point coincide, $P_1 \equiv P_2$, does not necessarily lead to $L = 0$. For instance, taking $\vec{F} = -\psi(P, \vec{v})\vec{v}$ with $\psi \geq 0$, we have:

$$\Pi(t) = -\psi v^2 \leq 0 \tag{5.21}$$

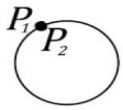

Figure 5.2. *work on a closed loop*

hence $L \leq 0$, where 0 would be a very special case.

POSITION FORCES: $\vec{F} = \vec{F}(P)$. To make dL exact we need

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}; \quad \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}; \quad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y} \tag{5.22}$$

If one of these equalities is violated, dL is not exact even though $\vec{F}$ is a position force. Moreover, the region delimited by a closed trajectory could have a hole, in which case, Schwarz equality is only a necessary condition, not sufficient, hence dL is again not exact even if $\vec{F}$ is a position force.

When $\vec{F}$ and $P = P(s)$ are known, with s a curvilinear coordinate along the trajectory, one has:

$$dL = \vec{F}(P(s)) \cdot \frac{d(P-O)}{ds} ds = \vec{F}(P(s)) \cdot \hat{t}(s) ds = F_{\parallel}(s) ds \tag{5.23}$$

where we have used Eq.(2.12), which defines the unit vector tangent to the path of P:

$$\frac{d(P-O)}{ds} = \hat{t}$$

and $F_{\parallel} = \vec{F} \cdot \hat{t}$ is the projection of $\vec{F}$ on the direction of $\hat{t}$. Then, the work is expressed by:

$$L = \int_{t_1}^{t_2} F(s) ds \quad \text{with final and initial positions } P_1 = P(t_1),\ P_2 = P(t_2) \tag{5.24}$$

In this case, different paths yield different works, but, for a fixed path shape, L does not change if the path is traversed at different speeds, *i.e.* if $s = s(t)$ differs from case to case along the given path. One way or the other, having $P_1 = P_2$ does not necessarily imply $L = 0$.

CONSERVATIVE FORCES: Suppose $\vec{F}$ is positional and all equalities like $\frac{\partial F_h}{\partial x_k} = \frac{\partial F_k}{\partial x_h}$ are verified. Then dL is exact and there exists a function U such that $dL = dU$. Such a function is called *potential* of the force $\vec{F}$, and it obeys the following relations:

$$F_x = \frac{\partial U}{\partial x}, \quad F_y = \frac{\partial U}{\partial y}, \quad F_z = \frac{\partial U}{\partial z} \tag{5.25}$$

which implies:

$$L = \int_{U_1}^{U_2} dU = U(P_2) - U(P_1)\,, \tag{5.26}$$

where P_1 and P_2 are the initial and final points. Thus, L is path independent: it only depends on initial and final points. Furthermore, in a loop, in which $P_1 = P_2$, the work vanishes, $L = 0$, if the domain is simply connected. In this case, the force $\vec{F}$ is called *conservative*.

Consider a trajectory, $\gamma = \{P = P(s) = (x(s), y(s), z(s))\}$, where s is a curvilinear coordinate parametrizing the line of motion. One may write:

$$\frac{dU}{ds} = \frac{\partial U}{\partial x}\frac{dx}{ds} + \frac{\partial U}{\partial y}\frac{dy}{ds} + \frac{\partial U}{\partial z}\frac{dz}{ds} = F_x \frac{dx}{ds} + F_y \frac{dy}{ds} + F_z \frac{dz}{ds} = \vec{F} \cdot \hat{t} = F_t(s) \tag{5.27}$$

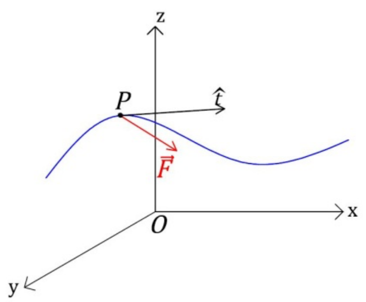

Figure 5.3. *trajectory with tangent vector and force acting on mass point*

In other words, the derivative of U along the direction of γ is the tangent component of the gradient $\vec{F}$ of U. One may then think of the motion along this 1-dimensional space as due to a kind of conservative force, even if that is not the case in the real 3-dimensional space. The force determining the motion in the line is indeed derived from a potential defined on each point $P = P(s)$ of the line.

Ex1: consider a constant force $\vec{F}$ (hence it is conservative because it is positional, it satisfies Schwarz equalities in its domain, and such domain is simply connected). The work it does on a material point P is then expressed by:

$$dL = \vec{F} \cdot d(P - O) = d[\vec{F} \cdot (P - O)] = dU$$

where

$$U = U(P) = \vec{F} \cdot (P - O) + k \tag{5.28}$$

is defined up to an additive constant. For instance, $U(y) = mgy + k$, for gravity near the Earth surface, is like that.

Ex2: consider a central Force, *i.e.* a force directed along line joining P and a fixed point O, called *force center*. One can write:

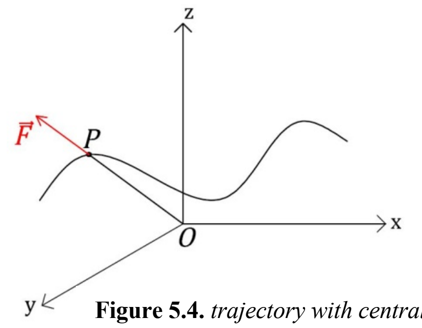

Figure 5.4. *trajectory with central force acting on mass point P.*

$$\vec{F} = \psi(r)\hat{u} \quad \text{where} \quad \hat{u} = \frac{(P-O)}{|P-O|} \quad \text{and} \quad r = |P - O| \tag{5.29}$$

Hence, a central force is conservative. Moreover, a force parallel to $(P - O)$ is given by

$$F_{\parallel} = \frac{dU}{ds}$$

where, in this case, we have $s = r$ and $\hat{t} = \hat{u}$. Consequently, we can write:

$$\frac{dU}{dr} = \psi(r)\hat{u} \cdot \hat{u} = \psi(r) \quad \text{and} \quad U(r) = \int \psi(r) dr$$

For instance, the *elastic force* is expressed by:

$$\vec{F} = k(P - O), \qquad k > 0, \qquad \psi(r) = -kr \tag{5.30}$$

and yields:
$$U(r) = -\int krdr = -\frac{1}{2}kr^2 + C \tag{5.31}$$

Another example of central force is the gravitational force, expressed by:

$$\vec{F} = -\frac{hmM}{r^2}\hat{u} \text{ , which is derived from the potential } \quad U(r) = \frac{hmM}{r} + C \tag{5.32}$$

POTENTIAL ENERGY: we call *potential energy* the opposite of the potential:

$$V = -U \tag{5.33}$$

For a set of n forces applied to n mass points, $S = \{(P_i, \vec{F}_i)\}_{i=1}^{n}$, one has $dL = \sum_{i=1}^{n} \vec{F}_i \cdot d(P_i - O)$. If this set of points, $\mathcal{B}$ say, is rigid, the relative positions of points do not change, and one can write:

$$d(P_i - O) = d(Q - O) + \vec{\varepsilon} \wedge (P_i - Q) \text{ , where } \vec{\varepsilon} = \vec{\omega} dt \text{ and } Q \in \mathcal{B} \tag{5.34}$$

Therefore, the elementary work can be expressed as:

$$dL = \left(\sum_{i=1}^{n} \vec{F}_i\right) \cdot d\,(Q - O) + \sum_{i=1}^{n} \vec{F}_i \cdot \vec{\varepsilon} \wedge (P_i - Q)$$

$$= \left(\sum_{i=1}^{n} \vec{F}_i\right) \cdot d(Q - O) + \left(\sum_{i=1}^{n}(P_i - Q) \wedge \vec{F}_i\right) \cdot \vec{\varepsilon} \tag{5.35}$$

and setting $\vec{R} = \sum_{i=1}^{n} \vec{F}_i$, $\quad \vec{M}_Q = \sum_{i=1}^{n}(P_i - Q) \wedge \vec{F}_i$, we have:

$$dL = \vec{R} \cdot d(Q - O) + \vec{M}_Q \cdot \vec{\varepsilon} \tag{5.36}$$

In turn, the work per unit time, which is the *power*, is given by:

$$\Pi = \vec{R} \cdot \vec{v}_Q + \vec{M}_Q \cdot \vec{\omega} \tag{5.37}$$

This means that work and power do not change if the system S is replaced by an equivalent system S', *e.g.* by an applied vector and a torque. As the reference point Q is arbitrary, one may take a point on a central axis, if $\vec{R} \neq \mathbf{0}$. Analogously, the virtual work is given by:

$$\delta L = \vec{R} \cdot \delta(Q - O) + \vec{M}_Q \cdot \vec{\varepsilon} \tag{5.38}$$

where $\delta(Q - O)$ is a virtual displacement.

HOLONOMIC CONSTRAINTS: Let a system S of n points P_i be described by N parameters $(q_1, \cdots, q_N)$:

$$P_i = P_i(q_1, \cdots, q_N, t), \qquad i = 1, \cdots, n \tag{5.39}$$

it follows that

$$\delta L = \textstyle\sum_{i=1}^{n} \vec{F}_i \cdot \delta(P_i - O) \quad \text{and} \quad \delta(P_i - O) = \sum_{h=1}^{N} \frac{\partial P_i}{\partial q_h} \delta q_h \tag{5.40}$$

hence that

$$\delta L = \sum_{i=1}^{n} \vec{F}_i \cdot \sum_{h=1}^{N} \frac{\partial(P_i - O)}{\partial q_h} \delta q_h = \sum_{h=1}^{N} \delta q_h \sum_{i=1}^{n} \vec{F}_i \cdot \frac{\partial(P_i - O)}{\partial q_h}$$

$$= \textstyle\sum_{h=1}^{N} Q_h \delta q_h \, ; \;\; Q_h = \sum_{i=1}^{n} \vec{F}_i \cdot \frac{\partial(P_i - O)}{\partial q_h} \tag{5.41}$$

Introducing the *generalized force* $\Lambda = (Q_1, \cdots, Q_N)$, and the set of coordinates $q = (q_1, \cdots, q_N)$, with virtual displacements $\delta q = (\delta q_1, \cdots, \delta q_N)$, the virtual work is expressed by the scalar product

$$\delta L = \Lambda \cdot \delta q \tag{5.42}$$

The generalized force Λ and its entries are also called *Lagrangian* or *generalized components* of the *active forces*. In the case in which $dL = dU$, we can write:

$$Q_h = \frac{\partial U}{\partial q_h} \tag{5.43}$$

LAWS OF MECHANICS: The purpose is to relate the notion of force to that of motion of material objects, keeping in mind that the description of motion depends on the reference frame, while forces are supposed not to depend on the reference frame. In this endeavor, one begins from the following postulate.

INERTIAL REFERENCE FRAMES: there are reference frames in which isolated mass points are stationary or have vanishing acceleration. Then, an isolated point in an inertial reference frame, obeys:

$$\text{FIRST LAW (of mechanics):} \qquad \vec{a} = \mathbf{0} \tag{5.44}$$

This postulate is commonly associated with another one, due to Mach: in an inertial reference frame, the accelerations $\vec{a}_1$ and $\vec{a}_2$ of two points P_1 and P_2, in otherwise empty space, are directed along $(P_1 - P_1)$, have opposite sign, and their magnitudes obey:

$$\frac{|\vec{a}_1|}{|\vec{a}_2|} = \sigma(P_1, P_2) \tag{5.45}$$

which does not depend on position, on velocity and on time.

Assuming that every mass point P_i can constitute an isolated pair with any other mass point P_j , one may further postulate that any three points $\{P_i, P_j, P_h\}$ obey

$$\sigma(P_i, P_j)\sigma(P_j, P_h) = \sigma(P_i, P_h) \tag{5.46}$$

This allows us to introduce an equivalence relation for the points P_i :

$$P_i \sim P_j \Leftrightarrow \sigma(P_i, P_j) = 1 \tag{5.47}$$

Equation (5.47) defines indeed an equivalence relation since it is reflexive, symmetric and transitive, as required. Let the equivalence class of P_i be characterized by a parameter m_i and suppose one may write

$$\sigma(P_i, P_j) = \frac{m_j}{m_i} \tag{5.48}$$

for two isolated points. For known m_i and m_j, this may be taken as the definition of σ. Alternatively, given σ, this is the definition of m_i (up to a constant). The parameter m_i is called *mass* of P_i and, given two isolated points, it allows us to write:

$$m_1\vec{a}_1 + m_2\vec{a}_2 = \mathbf{0} \tag{5.49}$$

$$(P_1 - O) \wedge m_1\vec{a}_1 + (P_2 - O) \wedge m_2\vec{a}_2 = \mathbf{0} \tag{5.50}$$

Points are here considered to be equivalent if they have same mass. It follows that, calling force $\vec{F}_i$ on P_i the only possible action affecting the motion of P_i, one may write:

SECOND LAW (of mechanics): $$\vec{F}_i = m_i\vec{a}_i \tag{5.51}$$

In the case of an isolated system of material points, let $\vec{F}_{ij}$ denote the force on any point P_i due to any other point P_j, and $\vec{F}_{ji}$ the one on P_j due to P_i. Introducing the *intensity of the force* on P_i due to the point P_j, φ_{ij} say, as $\vec{F}_{ij} = \varphi_{ij}(P_j - P_i)/|P_j - P_i|$, one may write:

$$\vec{F}_{ij} = \varphi_{ij}\hat{e}_{ij}; \quad \vec{F}_{ji} = \varphi_{ji}\hat{e}_{ji}; \quad \text{where} \quad \hat{e}_{ij} = \frac{(P_i - P_j)}{|P_i - P_j|} = -\hat{e}_{ji} \tag{5.52}$$

If there are *n* points, we now assume that the different interactions do not depend on each other, *i.e.* that one can write:

$$\vec{F}_i = m_i\vec{a}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \vec{F}_{ij}$$

which is known as the principle of superposition of effects. Eq.(5.49) further yields:

THIRD LAW (of mechanics): $$\vec{F}_{ij} = \varphi_{ij}\frac{\overrightarrow{P_iP_j}}{|\overrightarrow{P_iP_j}|} = \varphi\frac{\overrightarrow{P_iP_j}}{|\overrightarrow{P_iP_j}|} = -\varphi\frac{\overrightarrow{P_jP_i}}{|\overrightarrow{P_jP_i}|} = -\vec{F}_{ji} \tag{5.53}$$

which states that the force exerted by P_i on P_j has equal intensity and opposite direction to that exerted by P_j on P_i. This is also known as the principle of action and reaction. Note, it is not universal. For instance, forces acting in directions other than the line joining points are not required, here, to obey this principle.

As accelerations have been related to forces, knowledge of forces and of initial conditions lead to full knowledge of the motion, *i.e.* to a determined unique trajectory, whenever the corresponding system of ODE admits a unique solution.

THEOREM 5.2 (CAUCHY): *Given the system of ODE equations $\dot{X} = F(X)$, $X \in \mathbb{R}^n$, the solution satisfying the initial condition $X(0) = X_0$ is unique if F is Lipschitz continuous.*

DEFINITION 5.1: *The function $F: \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous* in x_1 $\exists K > 0$ *such that*

$$\|F(x_1'', x_2, \cdots, x_n) - F(x_1', x_2, \cdots, x_n)\| \le K|x_1'' - x_1'|$$

The function is Lipschitz continuous, if it is so in all its variables.

This condition is weaker than differentiability but stronger than mere continuity. Under this condition, the evolution in time can be called *deterministic*. Note, however, that determinism does not imply predictability. Determinism implies that each initial condition is followed by a unique future trajectory, fully specified by the law of motion and by the initial condition itself. Predictability means that we can infer what the trajectory will do. That, however, may be impossible even if the law of motion is simple and the initial condition is known with high accuracy. Systems known as *chaotic* are not predictable, except for short or very short times.

Forces exerted on each other by the points of a given system S are called *internal forces*; any other force is called *external*. Let $S = S_1 \oplus S_2$ be the sum of two subsystems such that each of its points belongs either to S_1 or to S_2 and not to both. If $P \in S_1$, forces due to other points of S_1 are called internal, while those due to points of S_2 are called external. Because of the action-reaction principle, the set of internal forces consists of pairs of vanishing arm, if they all obey Eq.(5.53). Hence, both their sum and their total momentum vanish. However, the internal work does not necessarily vanish:

$$dL = \sum_{i\in S}\left(\sum_{j\neq i\in S} \vec{F}_{ij}\cdot d(P_i - 0)\right) = \sum_{i\in S}\left(\sum_{j>i\in S} \varphi_{ij}\hat{e}_{ij}\cdot\left[(P_i - 0) - \left(P_j - 0\right)\right]\right) \qquad (5.54)$$

because the two forces are equal and opposite, and we eventually obtain:

$$dL = \frac{1}{2}\sum_{i\in S}\left(\sum_{j\neq i\in S} \varphi_{ij} d(P_i - P_j)\right)$$

If the system is rigid, $d(P_i - P_j) = 0 = d(P_j - P_i)$ by definition, and the internal work vanishes.

Consider the case in which $P_i^{(1)} \in S_1$ and $P_j^{(2)} \in S_2$, with $m_j^{(2)} \gg m_i^{(1)}$. In this case, the accelerations of the points of S_2 are very small compared to those of S_1, because the mutual forces have the same magnitude. This means that the points of S_2 are little affected by those of S_1. One may then consider as known the behavior of the points of S_2 (*e.g.* rest or uniform rectilinear motion) and unknown that of the points of S_1. Consequently, to a good approximation, one may write:

$$m_i^{(1)}\, \vec{a}_i^{(1)} = \sum_{\substack{j\in S_1\\ j\neq i}} \vec{F}_{ij}^{(I)}\left(\vec{P}_i^{(1)}, \vec{P}_j^{(1)}, \vec{v}_i^{(1)}, \vec{v}_j^{(1)}, t\right) + \sum_{h\in S_2} \vec{F}_{ih}^{(E)}\left(\vec{P}_i^{(1)}, \vec{v}_i^{(1)}, t\right) \qquad (5.55)$$

where the positions and velocities of particles in S_2 are automatically included. These equations are called *fundamental dynamical system of the system of mass points.* If S_1 contains only P, there are no internal forces, and

$$m\vec{a} = \vec{F}(P, \vec{v}, t) \qquad (5.56)$$

expresses the Newton equation, which is the fundamental equation of the dynamics of a mass point.

NON-INERTIAL REFERENCE FRAMES: we have postulated that inertial frames exist; we have not excluded other reference frames also exist. Therefore, in the case of non-inertial reference frames, the Newton equation must be modified. From our understanding of the properties of accelerations, we know that in rotating frames, and then presumably in other situations, forces are not directed along the lines joining points, because accelerations have components outside those lines. We moreover know how accelerations behave, therefore we try the following form, for non-inertial frames:

$$m_i\vec{a}_i = \vec{F}_i^{(I)} + \vec{F}_i^{(E)} - m_i\vec{a}_\tau(P_i) - 2m_i\vec{\omega}\wedge\vec{v}_i \qquad (5.57)$$

where: $\vec{a}_\tau$ is the drag acceleration and $\vec{\omega}$ the angular velocity associated with the non-inertial frame seen from an inertial one. Consequently, $-m_i\vec{a}_\tau(P_i)$ can be called *drag force*, and $-2m_i\vec{\omega}\wedge\vec{v}_i$ can be called *Coriolis force*. In addition, there may be internal and external forces, whose form is not precisely known, and will have to be inferred from experience, but we assume to be like the two sums, respectively in Eq.(5.55). Experience will tell whether this framework is appropriate or not. Today we know it is very successful.

PRINCIPLE OF CONSTRAINTS: Suppose P is constrained to lie on a curve or on a surface. Then, its accelerations must belong to certain subsets of all possible accelerations. In case of regular holonomic constraints acting on P, the configurations allowed by the constraints at a given time

t, C_t say, is a smooth submanifold of $\mathbb{R}^3$. The set of possible velocities, $\nu(P,t), \forall P \in C_t$, can then be written as:

$$\vec{v} = \vec{v}_t + \vec{v}_n \tag{5.58}$$

where $\vec{v}_t = \vec{v}_t(P,t)$ lies on the plane tangent to C_t in P, and $\vec{v}_n = \vec{v}_n(P,t)$ is orthogonal to $\vec{v}_t$. If the constraint does not depend on t, one has

$$\vec{v}_n(P,t) = \mathbf{0} \tag{5.59}$$

Moreover, given $P \epsilon C$ and $\vec{v} \in \nu(P,t)$, the set of accelerations of P, $\mathscr{A}(P,\vec{v},t)$, is made of vectors of the kind

$$\vec{a} = \vec{a}_t + \vec{a}_n(P,\vec{v},t) \tag{5.60}$$

where $\vec{a}_t$ is tangent to S_t (which means any possible acceleration) in P, $\vec{a}_n \perp \vec{a}_t$, and $\vec{a}_n$ is univocally determined by $P, \vec{v}, t$.

Ex3: Let the material point P be constrained to lie on a circle in the plane z=0, with varying distance from the origin of axes:

$$x^2(t) + y^2(t) = R^2(t)$$

The position of P can be expressed in terms of this distance and of the angle with respect to the x-axis:

$$\begin{cases} x(t) = R(t)\cos\vartheta(t) \\ y(t) = R(t)\sin\vartheta(t) \\ z(t) = 0 \end{cases} \tag{5.61}$$

which allows us to write:

$$\begin{cases} \dot{x} = \dot{R}\cos\vartheta - R\dot{\vartheta}\sin\vartheta \\ \dot{y} = \dot{R}\sin\vartheta + R\dot{\vartheta}\cos\vartheta \\ \dot{z} = 0 \end{cases}$$

or, equivalently, considering the part tangent to the circle at a given time, so with fixed R, and the part normal to that with varying R:

$$\begin{cases} \vec{v}_t = -R\dot{\vartheta}\sin\vartheta\,\hat{e}_1 + R\dot{\vartheta}\cos\vartheta\,\hat{e}_2 \\ \vec{v}_n = \dot{R}\cos\vartheta\,\hat{e}_1 + \dot{R}\sin\vartheta\,\hat{e}_2 \end{cases} \tag{5.62}$$

If the constraint only requires $z = 0$, the values of $R(t)$ and of $\vec{v}_t$ can be arbitrarily large or small, and arbitrarily directed, because the constraint is automatically satisfied by equations (5.61) and (5.62), while $\vec{v}_n$ depends on the characteristics of the motion. For instance, one obtains $\vec{v}_n = \mathbf{0}$ if the motion lies on a ring, in which $R(t)$ is constant. For the accelerations we have:

$$\begin{cases} \vec{a}_t = -\left(2R\ddot{\vartheta} + \dot{R}\dot{\vartheta}\right)[\sin\vartheta\,\hat{e}_1 - \cos\vartheta\,\hat{e}_2] \\ \vec{a}_n = \left(\ddot{R} - R\dot{\vartheta}^2\right)[\cos\vartheta\,\hat{e}_1 + \sin\vartheta\,\hat{e}_2] \end{cases} \tag{5.63}$$

where again the magnitude of $\vec{a}_t$ does not depend on $R(t)$, while $\vec{a}_n$ is determined by $R(t)$, $\vec{v}(t)$ and $P(t)$. Here, however, $\vec{a}_n \neq \mathbf{0}$ even if $R = const$.

This example shows why it is problematic to express the 2nd law for systems subjected to constraints: while the motion, hence the accelerations, must obey the constraint, the active forces remain largely arbitrary. Nevertheless, one may formally proceed writing:

$$m\vec{a} = \vec{F}(P,\vec{v},t) + \vec{\Phi} \tag{5.64}$$

where $\vec{F}$ is the sum of the active forces and $\vec{\Phi}$ implements the action of the constraint as if it were a force; $\vec{\Phi}$ is then called *constraint reaction*. One thus postulates that a constraint amounts to a force, although it has features that do not correspond to our notion of force.