Computational geometry is a branch of computer science devoted to the study of algorithms which can be stated in terms of geometry

We consider the <u>D-DIMENSIONAL</u> <u>EUCLIDEAN SPACE</u> E^d, i.e. the space of tuples $\underline{x} = (x_1, \dots, x_d)^T$ s.t. $x_i \in \mathbb{R}$ $\forall i \in [1, d]$ and with the ℓ^2-norm

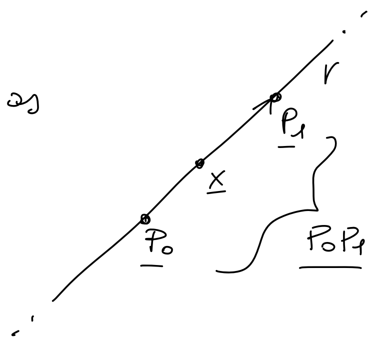
$$\|\underline{x}\|_2 = \sqrt{x_1^2 + \dots + x_d^2} = \left(\sum_{i=1}^{d} x_i^2\right)^{1/2}$$

In particular we will consider the case $d=2$ or $d=3$.

$\underline{x}$ is called <u>POINT</u> of E^d.

Given two points $\underline{P_0}, \underline{P_1}$ a <u>LINE</u> r is defined as

$$r = \left\{ \underline{x} \in E^d : \underline{x} = \alpha \underline{P_1} + (1-\alpha)\underline{P_0}, \alpha \in \mathbb{R} \right\}$$



In particular, a <u>SEGMENT</u> $\underline{P_0 P_1}$ is

$$\underline{P_0 P_1} = \left\{ \underline{x} \in E^d : \underline{x} = \alpha \underline{P_1} + (1-\alpha)\underline{P_0}, \alpha \in [0,1] \right\}$$

<u>NOTE</u>:

$$\underline{x} = \alpha \underline{P_1} + (1-\alpha)\underline{P_0} \iff \underline{x} = \underline{P_0} + \alpha \underbrace{(\underline{P_1} - \underline{P_0})}_{\underline{t}}$$

PARAMETRIC FORM of the line

α is the <u>CURVILINEAR COORDINATE</u> of r,

$\underline{P_0}$ is the <u>APPLICATION POINT</u> and

$\underline{t} = (\underline{P_1} - \underline{P_0})$ is the <u>TANGENT VECTOR</u>.

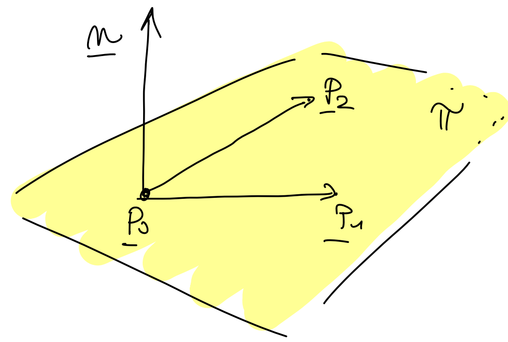
In general we define a <u>$(k-1)$-DIMENSIONAL VARIETY</u> v, with $k \le d$, given k points $\{\underline{P_i}\}_{i=(0,k-1)}$

$$v = \left\{ \underline{x} \in E^d : \underline{x} = \alpha_0 \underline{P_{k-1}} + \alpha_1 \underline{P_{k-2}} + \dots + \alpha_{k-2} \underline{P_1} + \left(1 - \sum_{i=0}^{k-2} \alpha_i\right) \underline{P_0}, \underline{\alpha} = (\alpha_0, \dots, \alpha_{k-1}) \in \mathbb{R}^k \right\}$$

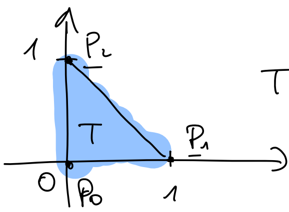
EXAMPLE 1

- A PLANE is a 2-DIMENSIONAL VARIETY

$$\pi = \left\{ \underline{X} \in E^d : \underline{X} = \alpha_0 \underline{P}_2 + \alpha_1 \underline{P}_1 + (1-\alpha_0-\alpha_1)\underline{P}_0,\ \underline{\alpha} = (\alpha_0, \alpha_1) \in \mathbb{R}^2 \right\}$$



- A TRIANGLE is a 2-Dimensional variety with some conditions on $\underline{\alpha}$

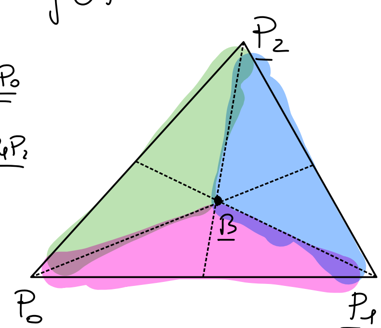


$$T = \underbrace{\alpha_0}_{\lambda_1} \underline{P}_2 + \underbrace{\alpha_1}_{\lambda_2} \underline{P}_1 + \underbrace{(1-\alpha_0-\alpha_1)}_{\lambda_3} \underline{P}_0$$

with (α_0, α_1) s.t. $0 \leq \alpha_0 + \alpha_1 \leq 1$

$\lambda_1, \lambda_2, \lambda_3$ are called BARYCENTRIC COORDINATES and work in general with all Triangles

$$\lambda_1 = \frac{\text{Area}_{BP_1P_2}}{\text{Area}_{P_0P_1P_2}} \qquad \lambda_2 = \frac{\text{Area}_{BP_2P_0}}{\text{Area}_{P_0P_1P_2}}$$



where $\underline{B}$ is the BARYCENTER of the Triangle

$$\underline{B} = \frac{1}{3}\left(\underline{P}_0 + \underline{P}_1 + \underline{P}_2\right)$$

Two useful operations are the DOT-PRODUCT (or INNER-PRODUCT) and the CROSS-PRODUCT, defined taking $\underline{P}_1, \underline{P}_2 \in E^d$

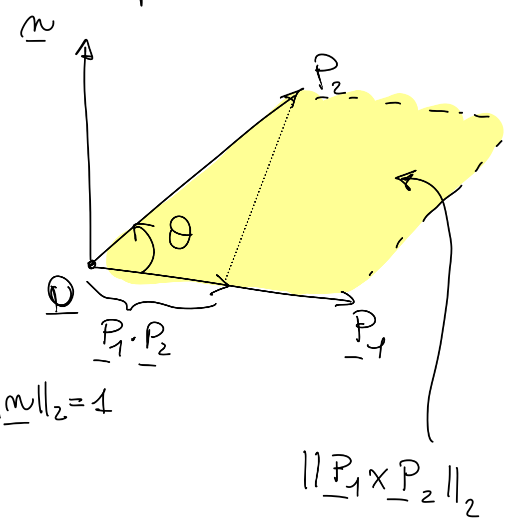
(DOT) $$\underline{P}_1 \cdot \underline{P}_2 = \underline{P}_1^T \underline{P}_2 = (x_1^1, \dots, x_d^1) \begin{pmatrix} x_1^2 \\ \vdots \\ x_d^2 \end{pmatrix} = \sum_{i=1}^{d} x_i^1 x_i^2$$

(CROSS) $d=3$ $$\underline{P}_1 \times \underline{P}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1^1 & x_2^1 & x_3^1 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = \underline{i}\left(x_2^1 x_3^2 - x_2^2 x_3^1\right) - \underline{j}\left(x_1^1 x_3^2 - x_1^2 x_3^1\right) + \underline{k}\left(x_1^1 x_2^2 - x_1^2 x_2^1\right)$$

where $\{\underline{i}, \underline{j}, \underline{k}\}$ is the ORTHONORMAL BASE of E^3,

$$\underline{i} = (1,0,0), \quad \underline{j} = (0,1,0), \quad \underline{k} = (0,0,1)$$

<u>NOTE</u>: Some useful properties of the two operations



1) $\underline{P}_1 \cdot \underline{P}_2 = \|\underline{P}_1\|_2 \, \|\underline{P}_2\|_2 \cos\theta$

2) $\underline{P}_1 \times \underline{P}_2 = \|\underline{P}_1\|_2 \, \|\underline{P}_2\|_2 \sin\theta \, \underline{n}$

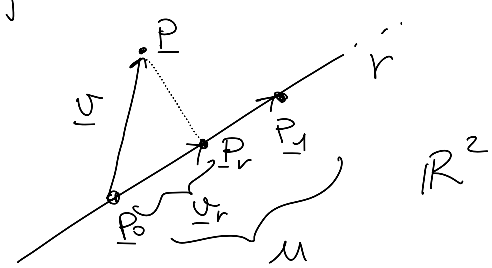
where θ is the angle formed by the two points with the origin of the space, and $\underline{n}$ is the array orthogonal to $\underline{OP_1}$, $\underline{OP_2}$, with $\|\underline{n}\|_2 = 1$

3) $\|\underline{P}_1\|_2^2 = \underline{P}_1 \cdot \underline{P}_1$

With this two operations we can solve different computational geometry problems:

<u>PROBLEM 1</u>: "Given a point $\underline{P}$ and a line r defined by points $\underline{P}_0, \underline{P}_1$, find the position of the point respect the line. We fix $d=2$"

<u>Solution</u> We use the DOT and CROSS product and define



$$\underline{u} = (\underline{P}_1 - \underline{P}_0), \quad \underline{v} = (\underline{P} - \underline{P}_0)$$

Thus

- $\underline{u} \times \underline{v} > 0 \Rightarrow \underline{P}$ on the LEFT
- $\underline{u} \times \underline{v} < 0 \Rightarrow \underline{P}$ on the RIGHT
- $\underline{u} \times \underline{v} = 0 \Rightarrow \underline{P}$ on the line r
  - $\underline{u} \cdot \underline{v} < 0 \Rightarrow \underline{P}$ before $\underline{P}_0$
  - $\underline{u} \cdot \underline{v} > 0 \Rightarrow \underline{P}$ after $\underline{P}_0$

$\cdot\ \underline{u} \cdot \underline{v} = 0 \Rightarrow \underline{P} \equiv \underline{P_0}$

Moreover, if we want to project $\underline{P}$ on the line r we use the property of dot product

$$r: \underline{x} = \underline{P_0} + \alpha(\underline{P_1} - \underline{P_0}) \Rightarrow \underline{P_r} = \underline{P_0} + \alpha_{P_r}(\underline{P_1} - \underline{P_0})$$

$$\Rightarrow \underbrace{\underline{P_r} - \underline{P_0}}_{\underline{v_r}} = \alpha_{P_r} \underbrace{(\underline{P_1} - \underline{P_0})}_{\underline{u}} \Rightarrow \alpha_{P_r} = \frac{\|\underline{v_r}\|_2}{\|\underline{u}\|_2} = \frac{1}{\|\underline{u}\|_2} \left\| \frac{\underline{u} \cdot \underline{v}}{\|\underline{u}\|_2} \right\|_2$$

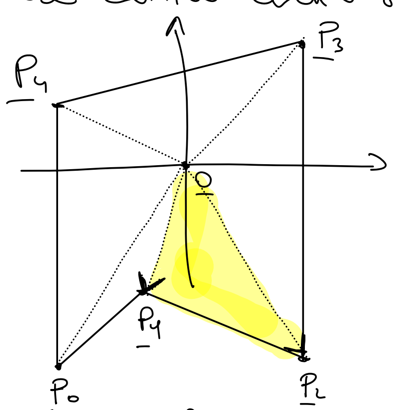
$$= \frac{\underline{u} \cdot \underline{v}}{\|\underline{u}\|_2^2}$$

$$\Rightarrow \underline{P_r} = \underline{P_0} + \frac{\underline{u} \cdot \underline{v}}{\|\underline{u}\|_2^2}(\underline{P_1} - \underline{P_0}) \qquad \square$$

PROBLEM 2 : [Given a polygon P with n points $\{\underline{P_i}\}_{i \in [0, n-1]}$ compute the Area, we fix $d = 2$]

Solution : lets make the hypothesis that the origin of the is contained inside the polygon. Moreover the vertices are counter-clockwise.

With the vector $\underline{OP_i}$ and $\underline{OP_{i+1}}$ the polygon becomes divided in n triangles $\underline{T_{OP_iP_{i+1}}}$.



$\mathbb{R}^2$

Thus

$$Area_P = \sum_{i=0}^{n-1} Area_{\underline{OP_iP_{i+1}}}$$

and, using the property of cross product we have

$$Area_{\underline{OP_iP_{i+1}}} = \frac{1}{2} \| \underline{OP_i} \times \underline{OP_{i+1}} \|_2$$

To conclude

$$Area_P = \sum_{i=0}^{m-1} \frac{1}{2} \| \underline{OP_i} \times \underline{OP_{i+1}} \|_2 =$$

$$= \frac{1}{2} \sum_{i=0}^{m-1} (x_i y_{i+1} - x_{i+1} y_i)$$

It is possible to generalize the same formula when the origin is not contained in the polygon and the vertices are not ordered counter-clockwise as

$$Area_P = \frac{1}{2} \left| \sum_{i=0}^{m-1} (x_i y_{i+1} - x_{i+1} y_i) \right|$$

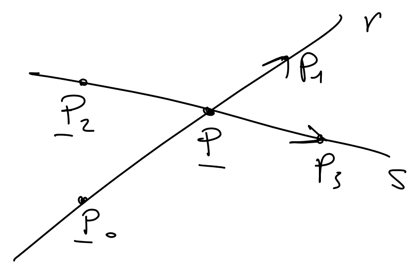
$\boxed{1}$

Sometimes the products are not enough and we have to other instruments, such as the linear system resolution.

PROBLEM 3: "Given two lines r, s find the intersection point, if exists"

SOLUTION: We want to find the intersection $\underline{P}$ two lines r, s

$$\begin{cases} r: \underline{P} = \underline{P_0} + \alpha(\underline{P_1} - \underline{P_0}) \\ s: \underline{P} = \underline{P_2} + \beta(\underline{P_3} - \underline{P_2}) \end{cases} \Rightarrow$$



$$\alpha(\underline{P_1} - \underline{P_0}) - \beta(\underline{P_3} - \underline{P_2}) = \underline{P_2} - \underline{P_0}$$

$$\Rightarrow \underbrace{(\underline{P_1} - \underline{P_0}, \ \underline{P_3} - \underline{P_2})}_{\underline{\underline{A}}} \underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{\underline{u}} = \underbrace{(\underline{P_2} - \underline{P_0})}_{\underline{b}} \quad \text{LINEAR SYSTEM}$$

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

$$\text{if } d=2 \Rightarrow \underbrace{\begin{bmatrix} x_1^1 - x_1^0 & x_1^3 - x_1^2 \\ x_2^1 - x_2^0 & x_2^3 - x_2^2 \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{\underline{u}} = \underbrace{\begin{pmatrix} x_1^2 - x_1^0 \\ x_2^2 - x_2^0 \end{pmatrix}}_{\underline{b}}$$

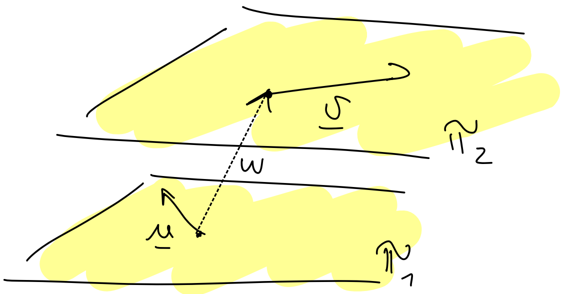
the solution exists if $\det(\underline{\underline{A}}) \neq 0$, then

$$\det(\underline{\underline{A}}) = (x_1^1 - x_1^0)(x_2^3 - x_2^2) - (x_2^1 - x_2^0)(x_1^3 - x_1^2) \neq 0$$

$$\Rightarrow \|(\underline{P}_1 - \underline{P}_0) \times (\underline{P}_3 - \underline{P}_2)\|_2 \neq 0$$ lines are not PARALLEL.

if $d=3 \Rightarrow \det(\underline{\underline{A}}) \neq 0$ if lines are not parallel and <u>COPLANAR</u>

$$\underbrace{(\underline{u} \times \underline{v}) \cdot \underline{w}}_{\text{TRIPLE PRODUCT}} = 0$$



$$\Rightarrow \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$$

$\square$