Fisica II

Esercitazione 3

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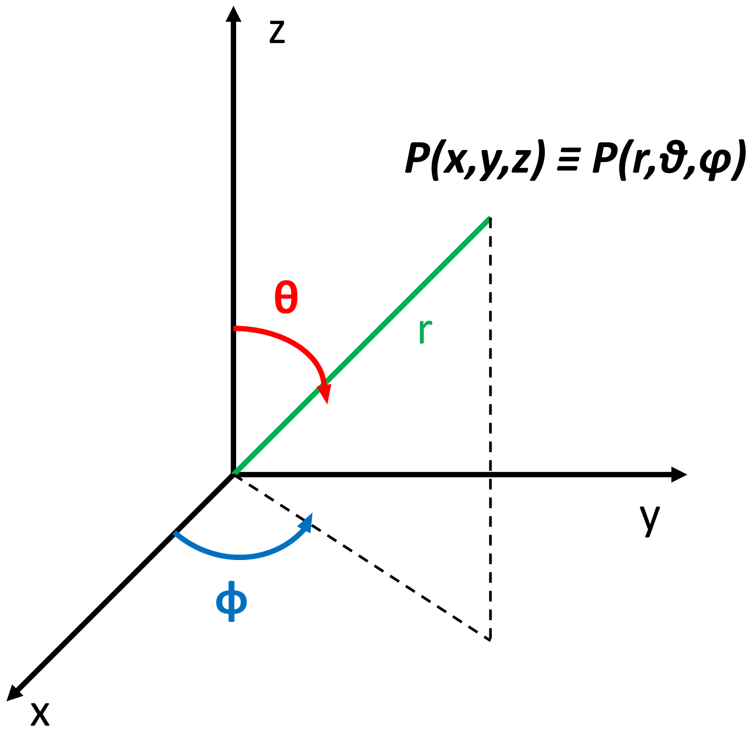
Coordinate sferiche



r: $[0,\infty)$ θ: $[0,\pi]$ ϕ: $[0,2\pi)$

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$
$$\phi = \arctan \frac{y}{x}$$

Coordinate sferiche

$$x = \mathrm{r}\sin\theta\cos\phi$$
$$y = \mathrm{r}\sin\theta\sin\phi$$
$$z = \mathrm{r}\cos\theta$$

Matrice jacobiana

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{bmatrix}$$

$$\boxed{\mathrm{dx\,dy\,dz} = |\det J|\,\mathrm{dr\,d\theta\,d\phi}}$$

$$\det \boldsymbol{J} = \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix}$$

Coordinate sferiche

$$\det \boldsymbol{J} = \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} =$$

Coordinate sferiche

$$\det \boldsymbol{J} = \begin{vmatrix} \boxed{\sin\theta\cos\phi} & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} =$$

$$= \sin\theta\cos\phi\,\big(r\sin\theta\cos\phi \cdot (-r\sin\theta) - (0 \cdot r\cos\theta\sin\phi)\big)$$

Coordinate sferiche

$$\det \boldsymbol{J} = \begin{vmatrix} \sin\theta\cos\phi & \boxed{-r\sin\theta\sin\phi} & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} =$$

$$= \sin\theta\cos\phi\left(r\sin\theta\cos\phi \cdot (-r\sin\theta) - (0 \cdot r\cos\theta\sin\phi)\right) +$$

$$+ r\sin\theta\sin\phi\left(\sin\theta\sin\phi \cdot (-r\sin\theta) - (\cos\theta \cdot r\cos\theta\sin\phi)\right) +$$

Coordinate sferiche

$$\det \boldsymbol{J} = \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & \boxed{r\cos\theta\cos\phi} \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} =$$

$$= \sin\theta\cos\phi\,\big(r\sin\theta\cos\phi \cdot (-r\sin\theta) - (0 \cdot r\cos\theta\sin\phi)\big) +$$

$$+ r\sin\theta\sin\phi\,\big(\sin\theta\sin\phi \cdot (-r\sin\theta) - (\cos\theta \cdot r\cos\theta\sin\phi)\big) +$$

$$+ r\cos\theta\cos\phi\,\big(\sin\theta\sin\phi \cdot 0 - (\cos\theta \cdot r\sin\theta\cos\phi)\big) =$$

Coordinate sferiche

$$\det \boldsymbol{J} = \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} =$$

$$= \sin\theta\cos\phi\left(r\sin\theta\cos\phi \cdot (-r\sin\theta) - \cancel{(0 \cdot r\cos\theta\sin\phi)}\right) +$$

$$+ r\sin\theta\sin\phi\left(\sin\theta\sin\phi \cdot (-r\sin\theta) - (\cos\theta \cdot r\cos\theta\sin\phi)\right) +$$

$$+ r\cos\theta\cos\phi\left(\cancel{\sin\theta\sin\phi \cdot 0} - (\cos\theta \cdot r\sin\theta\cos\phi)\right) =$$

$$= -r^2\sin^3\theta\cos^2\phi +$$

$$-r^2\sin^3\theta\sin^2\phi - r^2\sin\theta\cos^2\theta\sin^2\phi +$$

$$-r^2\cos^2\theta\sin\theta\cos^2\phi =$$

Coordinate sferiche

$$\det \boldsymbol{J} = \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} =$$

$$= -r^2\sin^3\theta\cos^2\phi\ +$$
$$-r^2\sin^3\theta\sin^2\phi - r^2\sin\theta\cos^2\theta\sin^2\phi +$$
$$-r^2\cos^2\theta\sin\theta\cos^2\phi\ =$$

$$= -r^2\sin\theta\,(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi) =$$

$$= -r^2\sin\theta[\sin^2\theta\,(\cos^2\phi + \sin^2\phi) + \cos^2\theta(\sin^2\phi + \cos^2\phi)] =$$

Coordinate sferiche

$$\det \boldsymbol{J} = \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} =$$

$$= -r^2\sin^3\theta\cos^2\phi\ +$$
$$-r^2\sin^3\theta\sin^2\phi - r^2\sin\theta\cos^2\theta\sin^2\phi +$$
$$-r^2\cos^2\theta\sin\theta\cos^2\phi\ =$$

$$= -r^2\sin\theta\,(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi) =$$

$$= -r^2\sin\theta[\sin^2\theta\,\underbrace{(\cos^2\phi + \sin^2\phi)}_{1} + \cos^2\theta\underbrace{(\sin^2\phi + \cos^2\phi)}_{1}] =$$

Coordinate sferiche

$$\det \boldsymbol{J} = \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} =$$

$$= -r^2\sin^3\theta\cos^2\phi +$$
$$-r^2\sin^3\theta\sin^2\phi - r^2\sin\theta\cos^2\theta\sin^2\phi +$$
$$-r^2\cos^2\theta\sin\theta\cos^2\phi =$$

$$= -r^2\sin\theta\,(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi) =$$

$$= -r^2\sin\theta[\sin^2\theta\,(\cos^2\phi + \sin^2\phi) + \cos^2\theta(\sin^2\phi + \cos^2\phi)] =$$

$$= -r^2\sin\theta\underbrace{[\sin^2\theta + \cos^2\theta]}_{1} =$$

Coordinate sferiche

$$\det \boldsymbol{J} = \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} =$$

$$= -r^2\sin^3\theta\cos^2\phi \ +$$

$$-r^2\sin^3\theta\sin^2\phi - r^2\sin\theta\cos^2\theta\sin^2\phi +$$

$$-r^2\cos^2\theta\sin\theta\cos^2\phi \ =$$

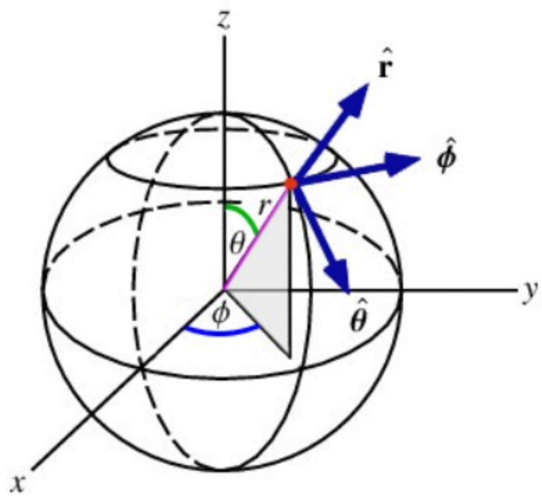
$$= -r^2\sin\theta\,(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi) =$$

$$= -r^2\sin\theta[\sin^2\theta\,(\cos^2\phi + \sin^2\phi) + \cos^2\theta(\sin^2\phi + \cos^2\phi)] =$$

$$= -r^2\sin\theta[\sin^2\theta \ + \cos^2\theta] =$$

$$\boxed{= -r^2\sin\theta}$$

Approfondimento coordinate sferiche



Gradiente in coordinate sferiche

$$\nabla\left(f(r,\theta,\varphi)\right) = \boxed{\frac{\partial f}{\partial x}}\hat{u}_x + \frac{\partial f}{\partial y}\hat{u}_y + \frac{\partial f}{\partial z}\hat{u}_z$$

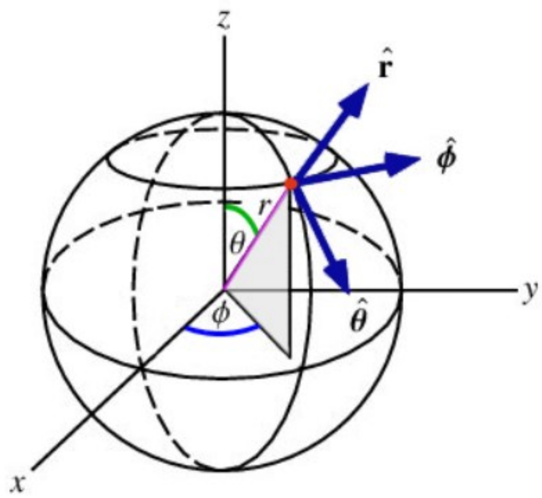
$$\frac{\partial f(r,\theta,\varphi)}{\partial x}$$

$$\boxed{\begin{array}{l} r(x,y,z) \\ \theta(x,y,z) \\ \varphi(x,y,z) \end{array}}$$

$$\frac{\partial f\left(r(x,y,z),\theta(x,y,z),\varphi(x,y,z)\right)}{\partial x}$$

$$\boxed{\begin{array}{l} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos \dfrac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \phi = \arctan \dfrac{y}{x} \end{array}}$$

Approfondimento coordinate sferiche



Gradiente in coordinate sferiche

$$\nabla\left(f(r,\theta,\varphi)\right) = \frac{\partial f}{\partial x}\hat{u}_x + \frac{\partial f}{\partial y}\hat{u}_y + \frac{\partial f}{\partial z}\hat{u}_z$$

$$\nabla\left(f(r,\theta,\varphi)\right) = \left(\frac{\partial r}{\partial x}\frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial f}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial f}{\partial \varphi}\right)\hat{u}_x + \\ + \left(\frac{\partial r}{\partial y}\frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial f}{\partial \theta} + \frac{\partial \varphi}{\partial y}\frac{\partial f}{\partial \varphi}\right)\hat{u}_y + \\ + \left(\frac{\partial r}{\partial z}\frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial f}{\partial \theta} + \frac{\partial \varphi}{\partial z}\frac{\partial f}{\partial \varphi}\right)\hat{u}_z$$

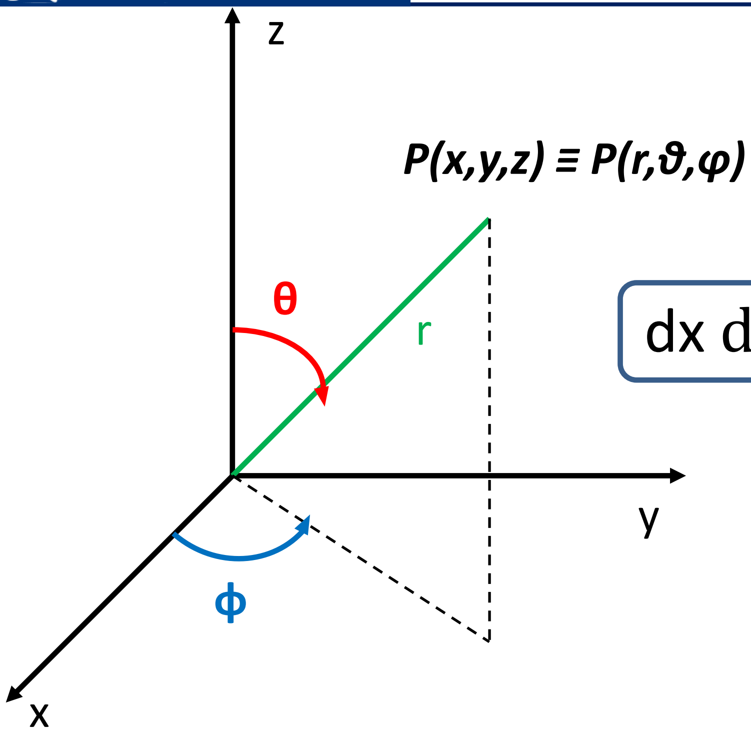
$$\hat{u}_x = \sin\theta\,\cos\varphi\,\hat{u}_r + \cos\theta\,\cos\varphi\,\hat{u}_\theta - \sin\varphi\,\hat{u}_\varphi$$

$$\hat{u}_y = \sin\theta\,\sin\varphi\,\hat{u}_r + \cos\theta\,\sin\varphi\,\hat{u}_\theta + \cos\varphi\,\hat{u}_\varphi$$

$$\hat{u}_z = \cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta$$

$$\nabla\left(f(r,\theta,\varphi)\right) = \frac{\partial f}{\partial r}\,\hat{u}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{u}_\theta + \frac{1}{r\sin\theta}\,\frac{\partial f}{\partial \varphi}\,\hat{u}_\varphi$$

Coordinate sferiche



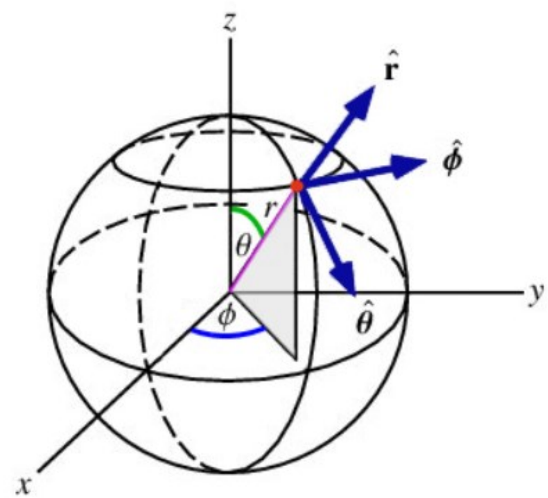
Integrazione su tutto lo spazio

$$\boxed{dV} = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\boxed{dx \, dy \, dz}$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} r^2 \, dr$$

ϕ: $[0,2\pi)$ θ: $[0,\pi]$ r: $[0,\infty)$

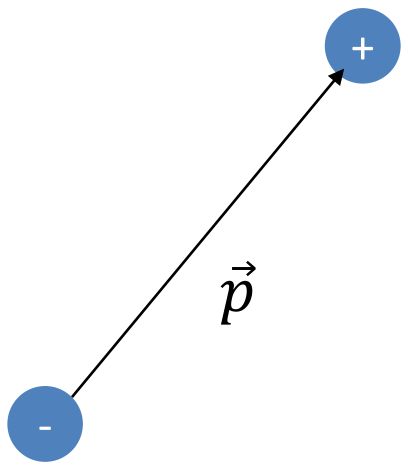


Gradiente in coordinate sferiche

$$\boldsymbol{\nabla}\left(f(r,\theta,\varphi)\right) = \frac{\partial f}{\partial r}\,\hat{u}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\,\hat{u}_\varphi$$

Dipolo elettrico

Consideriamo un sistema costituito da due cariche elettriche puntiformi *+q* e *–q*, poste ad una certa distanza *a* l'una dall'altra. Tale sistema si chiama ***dipolo elettrico***.



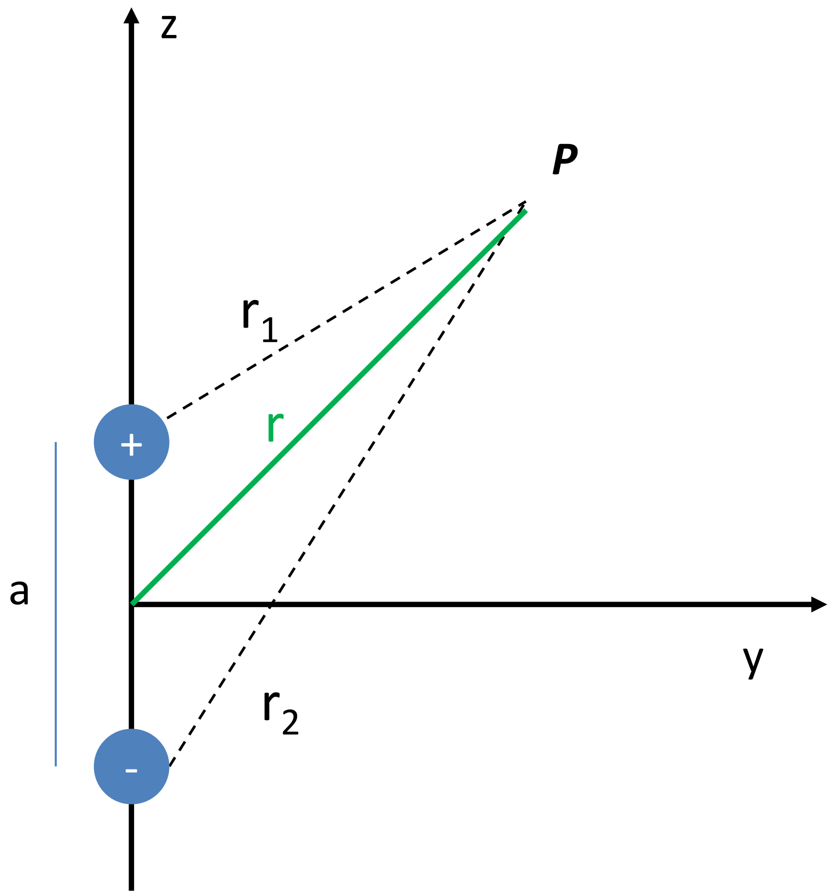
Possiamo definire il vettore ***momento di dipolo elettrico***:

$$\vec{p} = q\,a\,\hat{u}_{-,+}$$

orientato dalla carica negativa a quella positiva.

Lo scopo della lezione è studiare il **campo elettrico generato dal dipolo**. Per effettuare il conto, è conveniente utilizzare le *coordinate sferiche.*

Dipolo elettrico



Potenziale elettrostatico

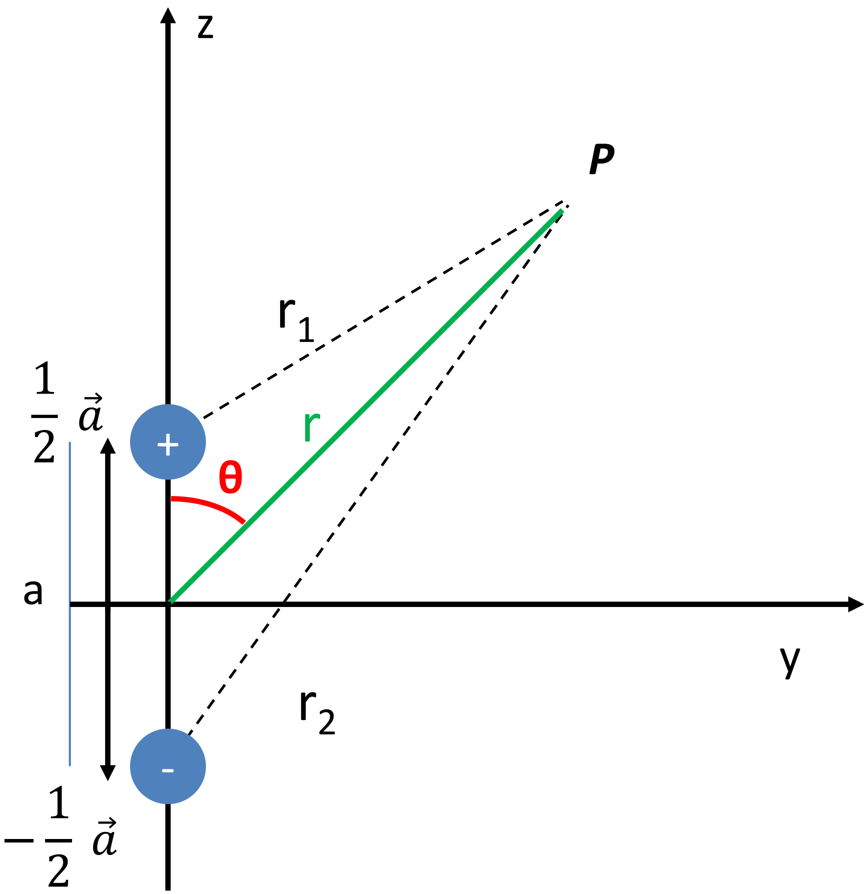
$$V(P) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$
$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{r_2 - r_1}{r_1 r_2}\right)$$

Vogliamo utilizzare

$$\vec{E} = -\vec{\nabla} V$$

per calcolare il campo elettrostatico.

Dipolo elettrico



$$r_1^2 = \left|\vec{r} - \frac{1}{2}\vec{a}\right|^2 \qquad r_2^2 = \left|\vec{r} + \frac{1}{2}\vec{a}\right|^2$$

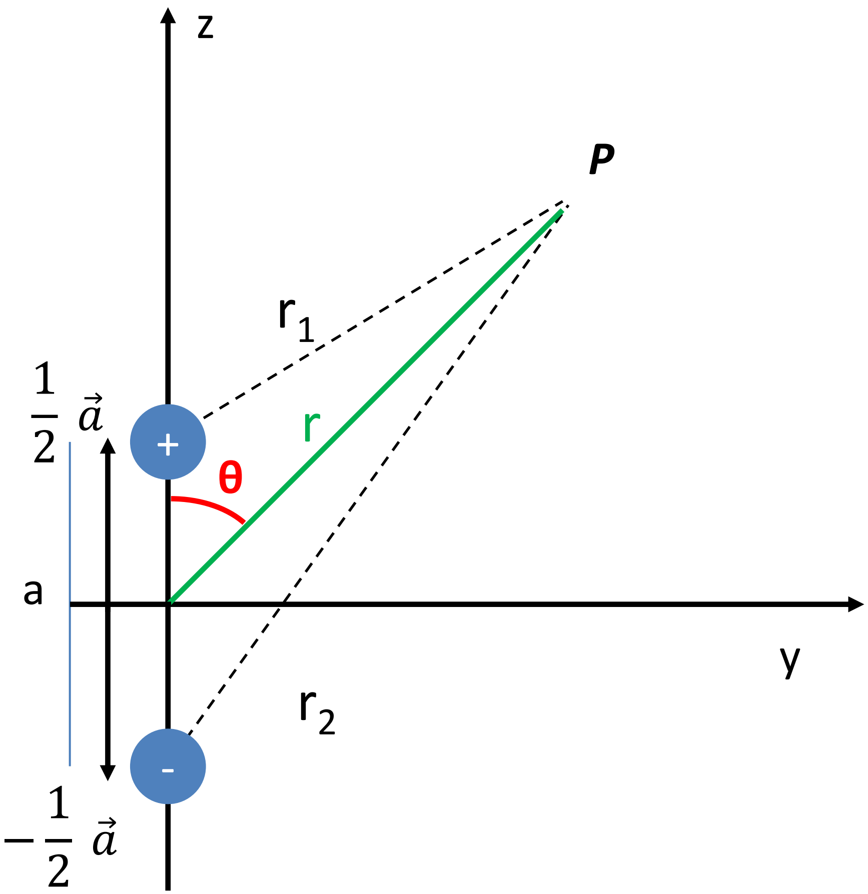
$$\left|\vec{r} + \frac{1}{2}\vec{a}\right| = \sqrt{r^2 + \frac{1}{4}a^2 + 2r\,\frac{1}{2}a\,\cos\theta}$$

Teorema del coseno

$$r_1^2 = r^2 - r\,a\,\cos\theta + \frac{a^2}{4}$$
$$= r^2\left[1 - \frac{a}{r}\cos\theta + \frac{a^2}{4r^2}\right]$$

$$r_2^2 = r^2 + r\,a\,\cos\theta + \frac{a^2}{4}$$
$$= r^2\left[1 + \frac{a}{r}\cos\theta + \frac{a^2}{4r^2}\right]$$

Dipolo elettrico



Consideriamo adesso il caso **r >> a**

$$r_1^2 = r^2\left[1 - \frac{a}{r}\cos\theta + \cancel{\frac{a^2}{4r^2}}\right]$$

$$r_2^2 = r^2\left[1 + \frac{a}{r}\cos\theta + \cancel{\frac{a^2}{4r^2}}\right]$$

$$\boxed{\begin{aligned} r_1 &\approx r\sqrt{1 - \frac{a}{r}\cos\theta} \\ r_2 &\approx r\sqrt{1 + \frac{a}{r}\cos\theta} \end{aligned}}$$

$$\mathbf{V(P) = \frac{q}{4\,\pi\,\varepsilon_0}\left(\frac{r_2 - r_1}{r_2 r_1}\right)}$$

Calcoliamo l'espressione per il potenziale in **approssimazione di grandi distanze**

Dipolo elettrico

$$r_1 \approx r\sqrt{1 - \frac{a}{r}\cos\theta}$$

$$r_2 \approx r\sqrt{1 + \frac{a}{r}\cos\theta}$$

r >> a

$$\mathbf{V(P)} = \frac{\mathbf{q}}{\mathbf{4\,\pi\,\varepsilon_0}}\left(\frac{\mathbf{r_2 - r_1}}{\mathbf{r_2 r_1}}\right)$$

$$r_2 - r_1 = r\left[\sqrt{1 + \frac{a}{r}\cos\theta} - \sqrt{1 - \frac{a}{r}\cos\theta}\right] \approx r\left[1 + \frac{1}{2}\,\frac{a}{r}\cos\theta\, - 1 + \frac{1}{2}\frac{a}{r}\cos\theta\right]$$

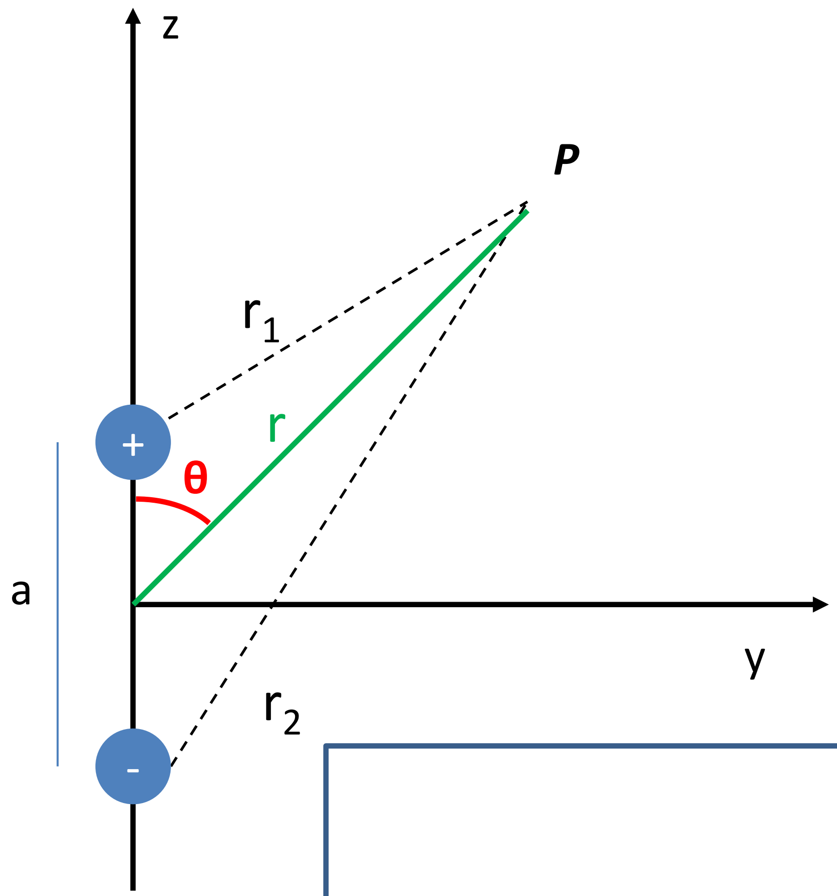
$$= a\ \cos\theta$$

$$r_1 r_2 = r^2\sqrt{\left(1 - \frac{a}{r}\ \cos\theta\right)\left(1 + \frac{a}{r}\ \cos\theta\right)} = r^2\sqrt{1 - \cancel{\frac{a^2}{r^2}\cos^2\theta}}$$

$$\approx r^2$$

Dipolo elettrico

Punto P molto lontano dal dipolo: ***r >> a***.



Approssimazioni

$$r_2 - r_1 = a\cos\theta$$

$$r_2\, r_1 = r^2$$

$$V(P) = \frac{q}{4\,\pi\,\varepsilon_0}\left(\frac{r_2 - r_1}{r_1\, r_2}\right)$$

$$V(P) = \frac{q\, a\cos\theta}{4\,\pi\,\varepsilon_0\, r^2} = \frac{\vec{p}\cdot\hat{u}_r}{4\,\pi\,\varepsilon_0\, r^2}$$

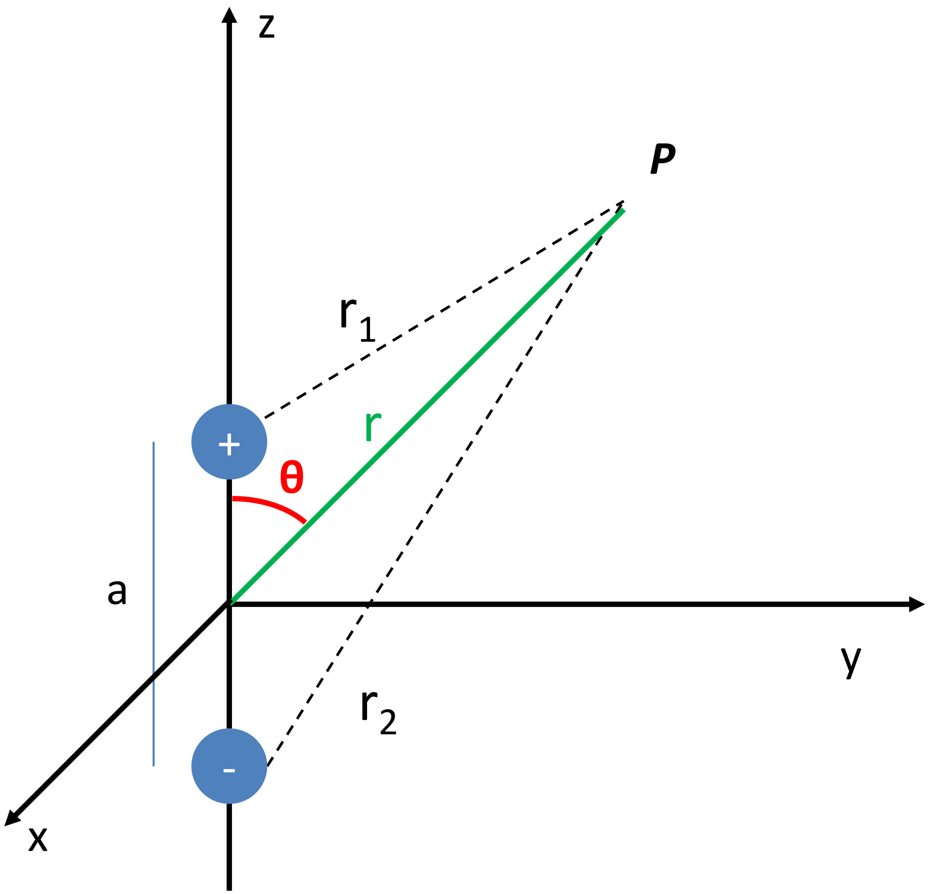
Potenziale elettrostatico in approssimazione di grandi distanze

Dipolo elettrico

Calcoliamo il campo elettrostatico a partire dal potenziale:

$$\boxed{\vec{E} = -\vec{\nabla}\, V}$$

$$\boldsymbol{\nabla}\left(V(r,\theta)\right) = \frac{\partial V}{\partial r}\,\hat{u}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\hat{u}_\theta + \frac{1}{r\sin\theta}\,\frac{\partial V}{\partial \varphi}\,\hat{u}_\varphi$$

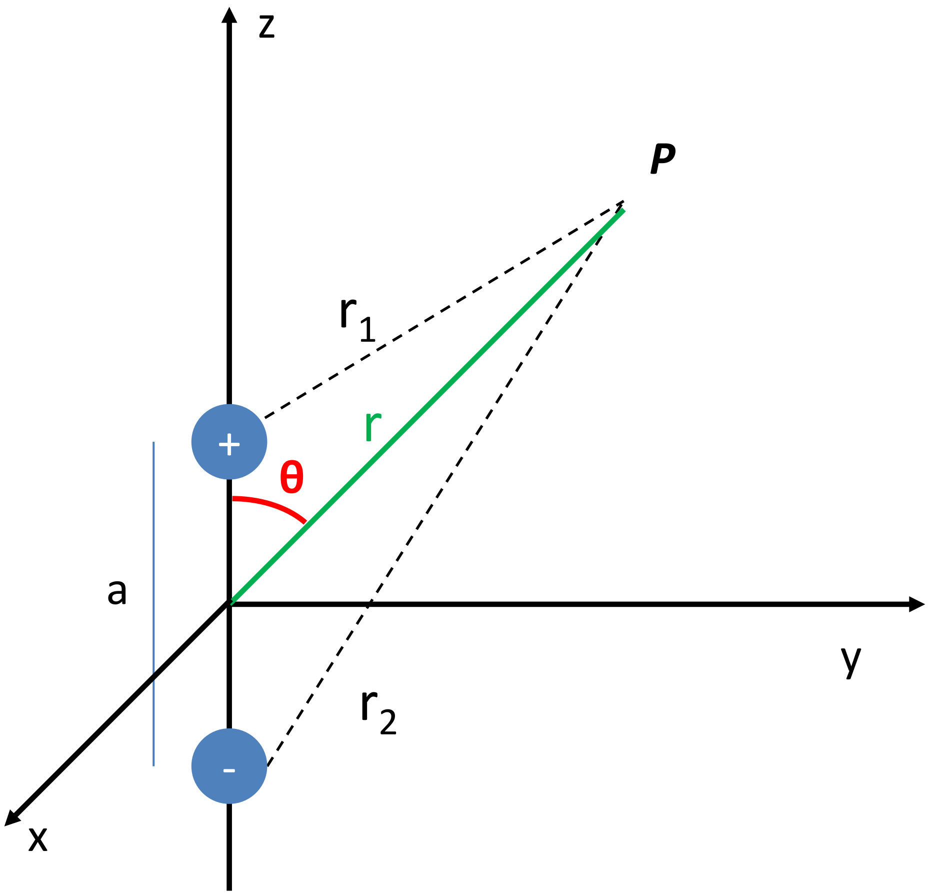


Dipolo elettrico

Calcoliamo il campo elettrostatico a partire dal potenziale:

$$\boxed{\vec{E} = -\vec{\nabla} V}$$

$$\boldsymbol{\nabla}\left(V(r,\theta)\right) = \frac{\partial V}{\partial r}\,\hat{u}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\hat{u}_\theta + \frac{1}{r\sin\theta}\cancel{\frac{\partial V}{\partial \varphi}}\,\hat{u}_\varphi$$

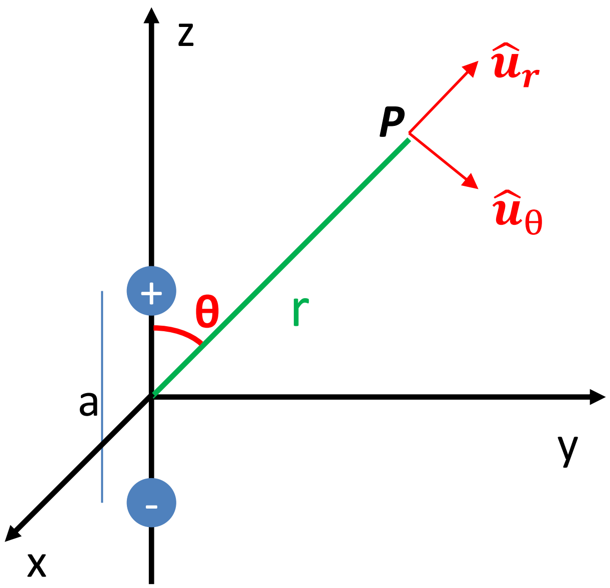


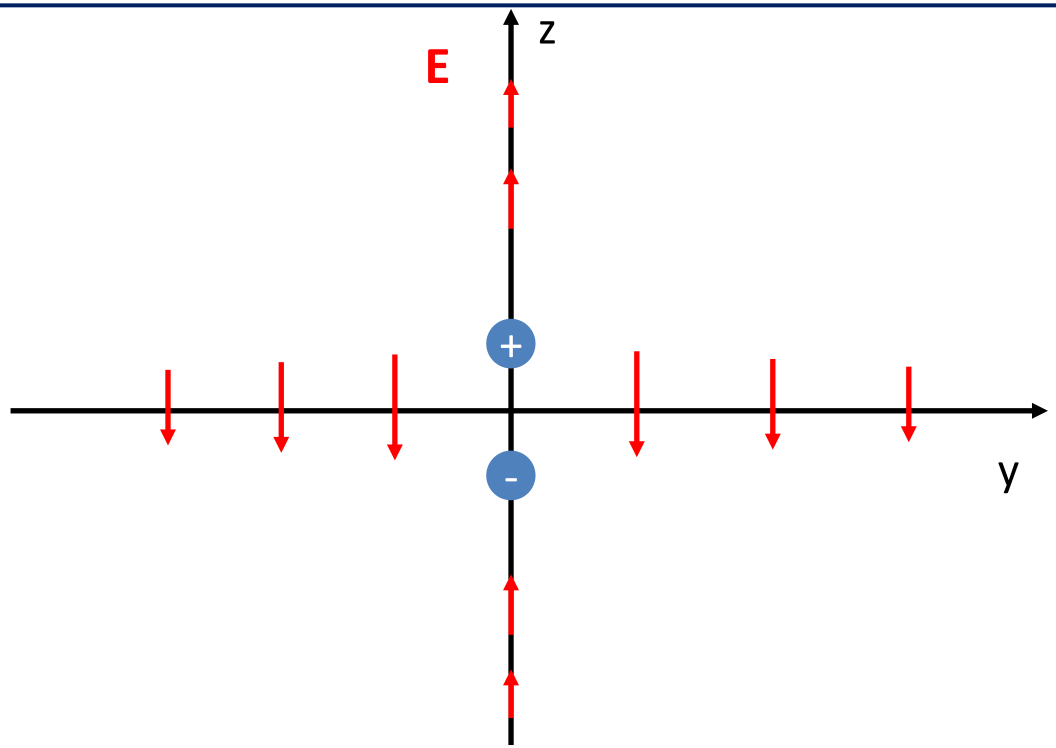
$$E_r = -\frac{\partial V}{\partial r} = \frac{2\,p\cos\theta}{4\,\pi\,\varepsilon_0\,r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\,\pi\,\varepsilon_0\,r^3}$$

$$E_\varphi = 0$$

Dipolo elettrico





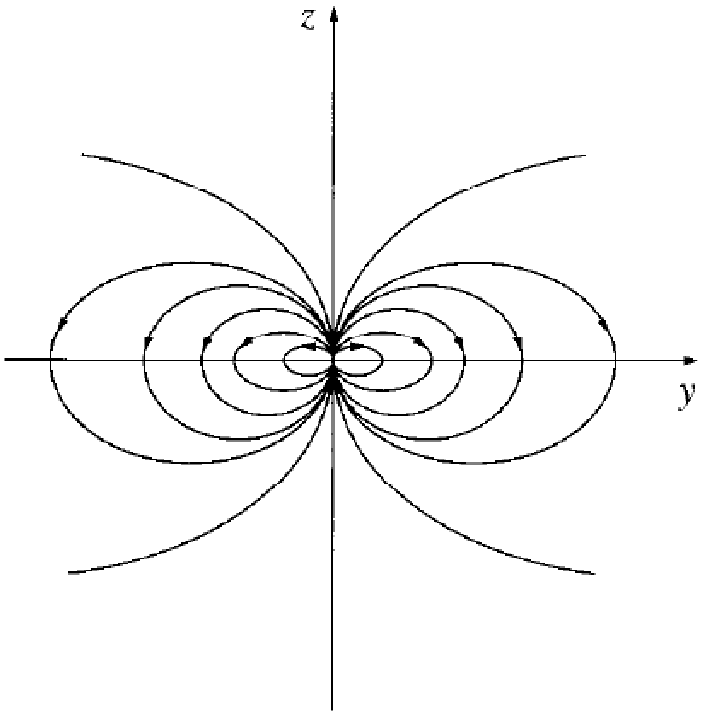
$$E_r = -\frac{\partial V}{\partial r} = \frac{2\,p\cos\theta}{4\,\pi\,\varepsilon_0\,r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\,\pi\,\varepsilon_0\,r^3}$$

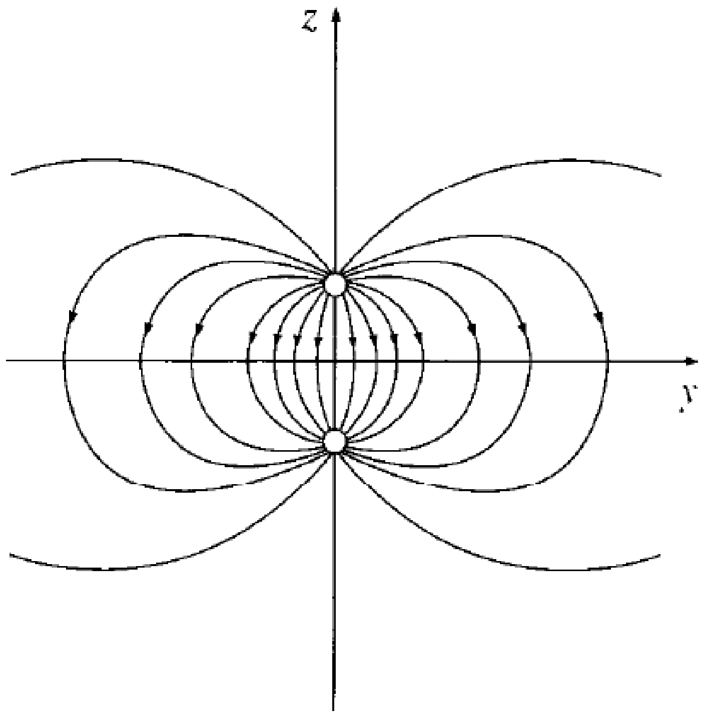
$$E_\varphi = 0$$

Nota bene: il risultato trovato è valido a grandi distanze dal dipolo, ovvero quando ***r >> a.***

Dipolo elettrico

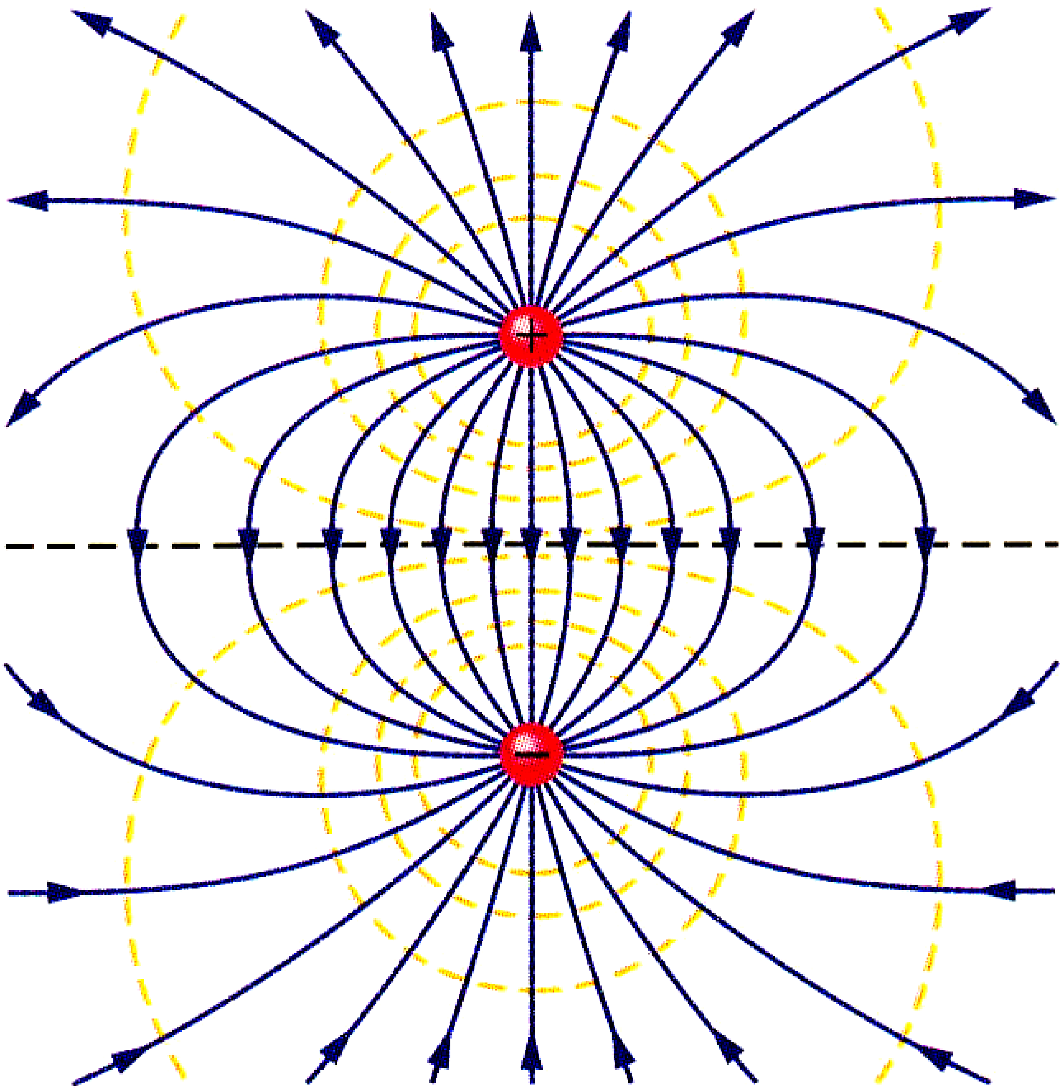


(a) Field of a "pure" dipole



(a) Field of a "physical" dipole

Dipolo elettrico



Linee di campo E

V = cost.

$$V(P) = \frac{q\, a \cos\theta}{4\,\pi\,\varepsilon_0\, r^2} = \frac{\vec{p}\cdot\hat{u}_r}{4\,\pi\,\varepsilon_0\, r^2}$$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2\,p\cos\theta}{4\,\pi\,\varepsilon_0\, r^3}$$

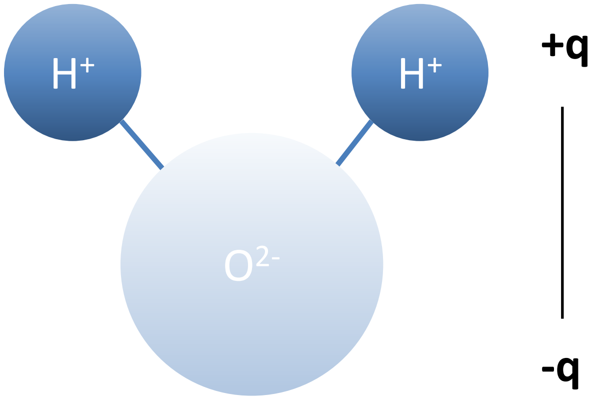
$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\,\pi\,\varepsilon_0\, r^3}$$

$$E_\varphi = 0$$

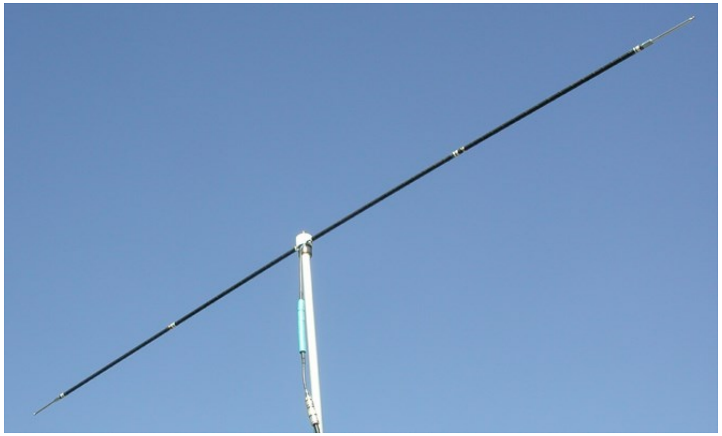
Approssimazione di dipolo

Le espressioni per il campo elettrostatico e per il potenziale sono valide per sistemi più complessi rispetto a quello presentato

La definizione di momento di dipolo può essere generalizzata a sistemi di cariche neutri costituiti da più di due cariche, su scale dimensionali molto diverse



Molecola H_2O

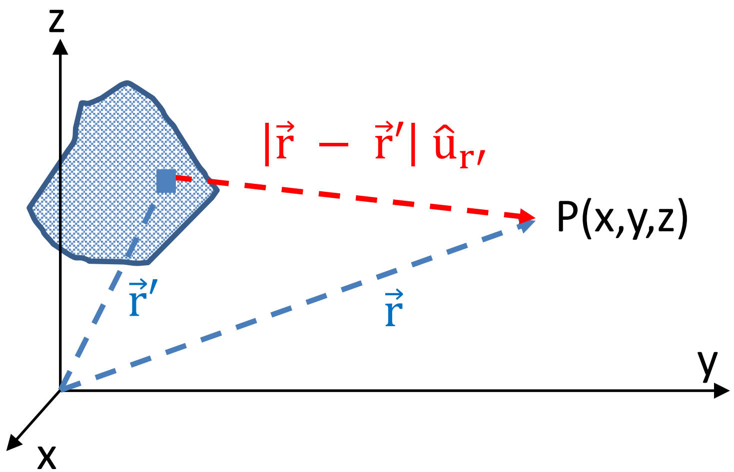


Antenna

Approssimazione di dipolo

Consideriamo una generica distribuzione di carica localizzata:

$$V(\vec{r}) = \frac{1}{4\,\pi\,\varepsilon_0}\int_\tau \frac{\rho(\vec{r}')}{|\vec{r}\,-\,\vec{r}'|}\,dV' = \frac{1}{4\,\pi\,\varepsilon_0}\int_\tau \frac{\rho(\vec{r}')}{\sqrt{r^2 - 2\vec{r}\cdot\vec{r}' + r'^2}}\,dV'$$



$$= \frac{1}{4\,\pi\,\varepsilon_0}\int_\tau \frac{\rho(\vec{r}')}{r\left[1 - 2\frac{\vec{r}\cdot\vec{r}'}{r^2} + \frac{r'^2}{r^2}\right]^{1/2}}\,dV'$$

$$= \frac{1}{4\,\pi\,\varepsilon_0}\int_\tau \frac{\rho(\vec{r}')}{r}\left[1 - 2\frac{\vec{r}\cdot\vec{r}'}{r^2} + \frac{r'^2}{r^2}\right]^{-1/2} dV'$$

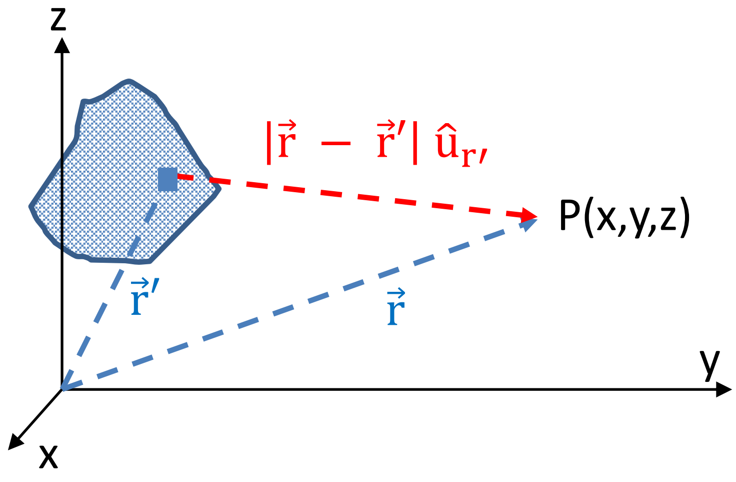
Consideriamo l'approssimazione di grandi distanze rispetto alle dimensioni della distribuzione di carica:

$$r \gg r' \qquad \forall\, r' \text{in}\ \tau$$

Approssimazione di dipolo

$$(1+x)^{-1/2} \approx 1 - \frac{1}{2}x$$

$$x = \frac{\mathrm{r}'^2}{\mathrm{r}^2} - 2\frac{\vec{\mathrm{r}}\cdot\vec{\mathrm{r}}'}{\mathrm{r}^2}$$

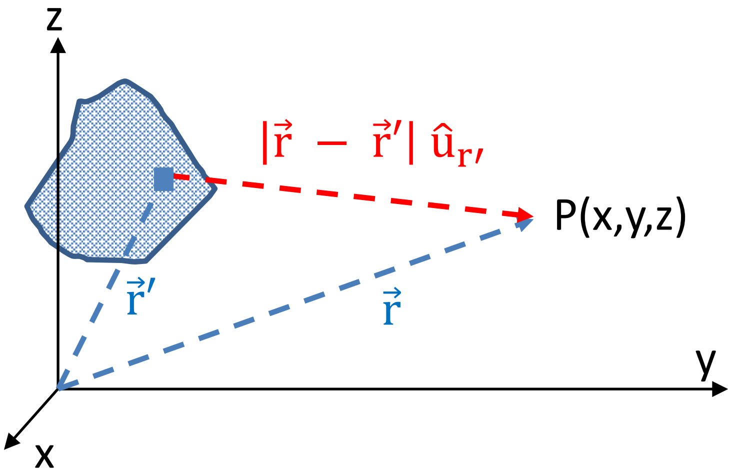


Approssimazione di dipolo

$$(1+x)^{-1/2} \approx 1 - \frac{1}{2}x$$

$$\approx \frac{1}{4\,\pi\,\varepsilon_0}\int_\tau \frac{\rho(\vec{r}')}{r}\left[1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} - \frac{1}{2}\,\frac{r'^2}{r^2}\right]\,dV'$$

$$x = \frac{r'^2}{r^2} - 2\,\frac{\vec{r}\cdot\vec{r}'}{r^2}$$

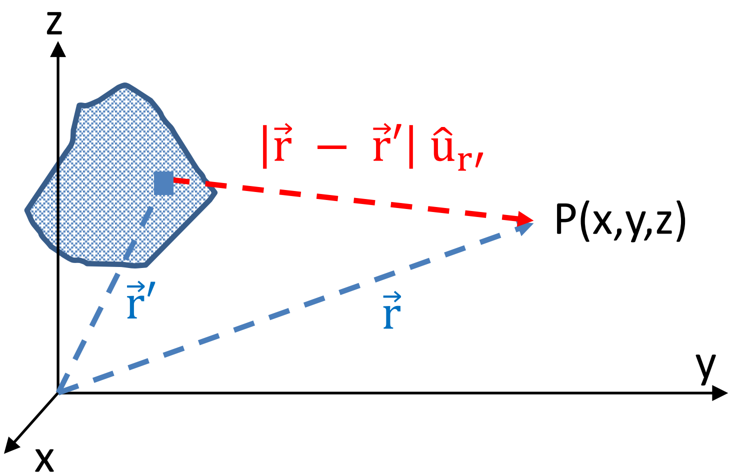


Approssimazione di dipolo

$$(1+x)^{-1/2} \approx 1 - \frac{1}{2}x$$

$$x = \frac{r'^2}{r^2} - 2\frac{\vec{r}\cdot\vec{r}'}{r^2}$$

$$\approx \frac{1}{4\,\pi\,\varepsilon_0}\int_\tau \frac{\rho(\vec{r}')}{r}\left[1 + \underbrace{\frac{\vec{r}\cdot\vec{r}'}{r^2}}_{\sim\frac{1}{r}} - \cancel{\underbrace{\frac{1}{2}\frac{r'^2}{r^2}}_{\sim\frac{1}{r^2}}}\right] dV'$$



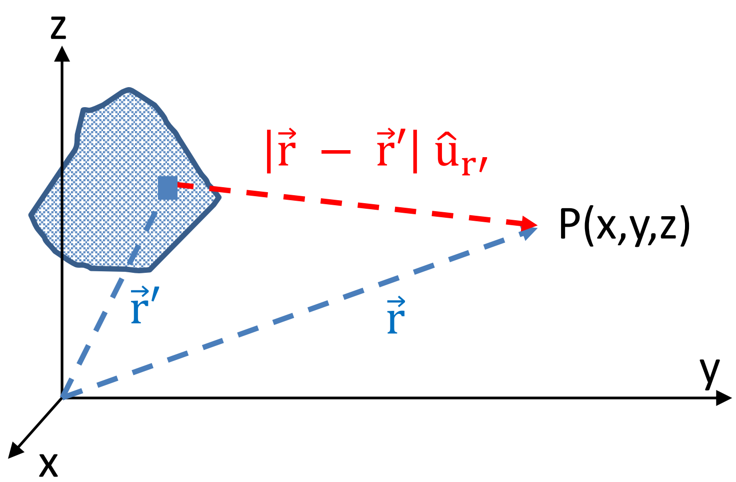
Approssimazione di dipolo

$$(1+x)^{-1/2} \approx 1 - \frac{1}{2}x$$

$$x = \frac{r'^2}{r^2} - 2\frac{\vec{r}\cdot\vec{r}'}{r^2}$$

$$\approx \frac{1}{4\,\pi\,\varepsilon_0}\int_\tau \frac{\rho(\vec{r}')}{r}\left[1 + \underbrace{\frac{\vec{r}\cdot\vec{r}'}{r^2}}_{\sim\frac{1}{r}} - \cancel{\frac{1}{2}\frac{r'^2}{r^2}}_{\sim\frac{1}{r^2}}\right] dV'$$

$$\approx \frac{1}{4\,\pi\,\varepsilon_0}\int_\tau \frac{\rho(\vec{r}')}{r}dV' + \frac{1}{4\,\pi\,\varepsilon_0}\int_\tau \frac{\rho(\vec{r}')\vec{r}\cdot\vec{r}'}{r^3}\,dV'$$



Approssimazione di dipolo

$$(1+x)^{-1/2} \approx 1 - \frac{1}{2}x$$

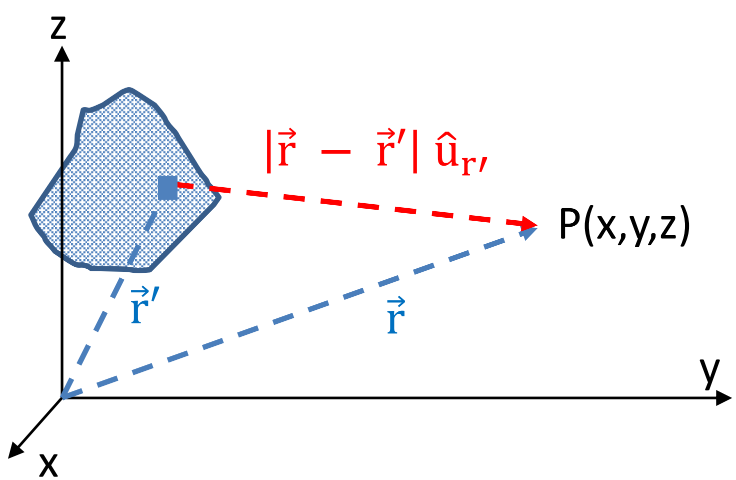
$$x = \frac{r'^2}{r^2} - 2\,\frac{\vec{r}\cdot\vec{r}'}{r^2}$$

$$\approx \frac{1}{4\,\pi\,\varepsilon_0}\int_\tau \frac{\rho(\vec{r}')}{r}\left[1 + \underbrace{\frac{\vec{r}\cdot\vec{r}'}{r^2}}_{\sim\frac{1}{r}} - \underbrace{\cancel{\frac{1}{2}\frac{r'^2}{r^2}}}_{\sim\frac{1}{r^2}}\right] dV'$$

$$\approx \frac{1}{4\,\pi\,\varepsilon_0}\int_\tau \frac{\rho(\vec{r}')}{r} dV' + \frac{1}{4\,\pi\,\varepsilon_0}\int_\tau \frac{\rho(\vec{r}')\vec{r}\cdot\vec{r}'}{r^3} dV'$$

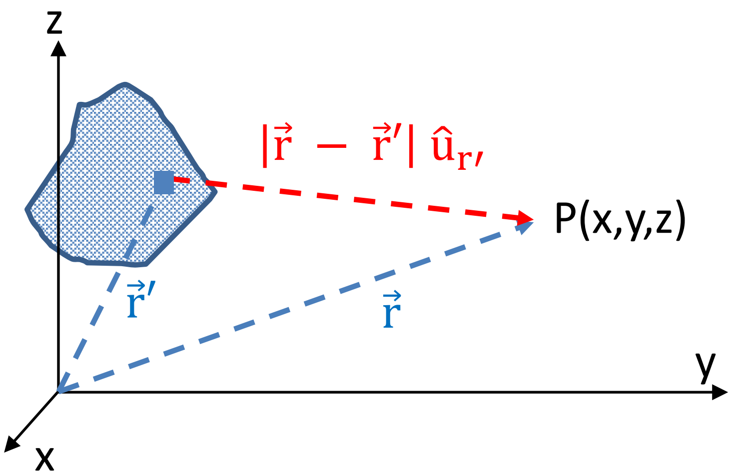
$$\vec{r}\cdot\vec{r}' = r\,r'\cos\theta$$

$$= \frac{1}{4\,\pi\,\varepsilon_0}\frac{Q}{r} + \frac{1}{4\,\pi\,\varepsilon_0}\frac{\vec{p}\cdot\hat{u}_r}{r^2}$$



Approssimazione di dipolo

$$V(\vec{r}) = \frac{1}{4\,\pi\,\varepsilon_0}\frac{Q}{r} + \frac{1}{4\,\pi\,\varepsilon_0}\frac{\vec{p}\cdot\hat{u}_r}{r^2}$$



per **sistemi neutri** la carica totale Q = 0

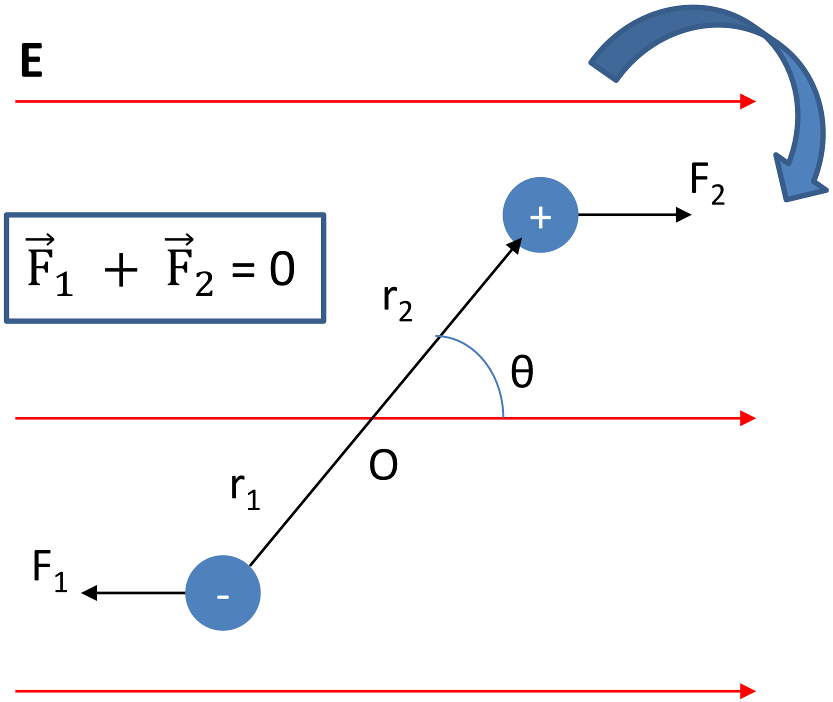
$$V(\vec{r}) = \frac{1}{4\,\pi\,\varepsilon_0}\frac{\vec{p}\cdot\hat{u}_r}{r^2}$$

A grandi distanze possiamo approssimare la distribuzione di carica con un dipolo di momento:

$$\vec{p} = \int_\tau \rho(\vec{r}')\,\vec{r}'\ d\tau'$$

Forza elettrica su dipolo

Passiamo allo studio dell'interazione tra un dipolo e un ***campo elettrico esterno uniforme***.



$$\boxed{\vec{F}_1 + \vec{F}_2 = 0}$$

Momento meccanico

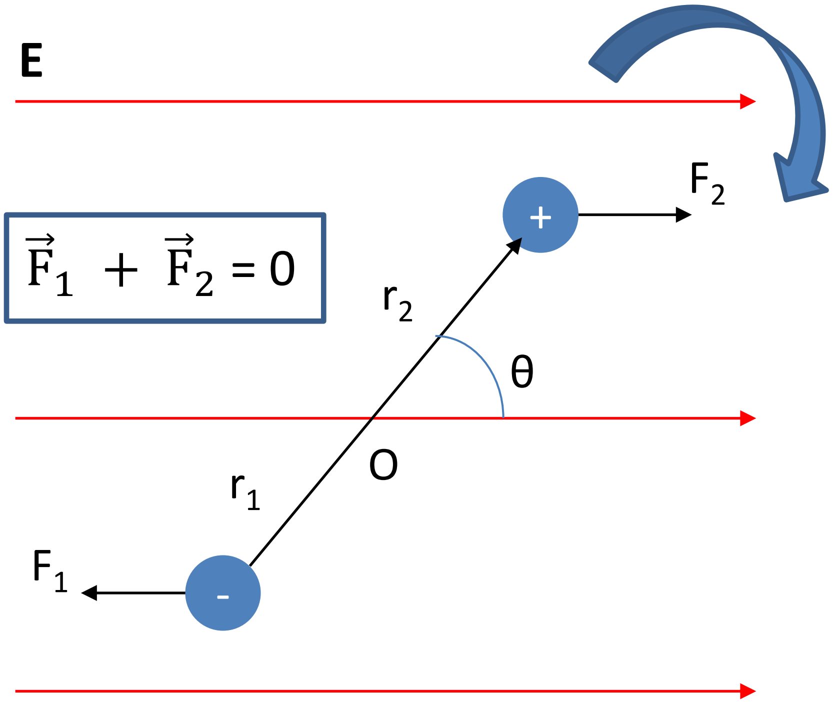
$$\overrightarrow{M} = (\vec{r}_2 - \vec{r}_1) \times \vec{F}_2 = \vec{p} \times \vec{E}$$

Il momento di dipolo ruota fino ad essere parallelo e concorde al campo elettrostatico.

$$W = \int_{\theta_0}^{\theta} \underbrace{M}_{p\,E\,\sin\theta}\, d\theta$$

Forza elettrica su dipolo

Passiamo allo studio dell’interazione tra un dipolo e un ***campo elettrico esterno uniforme***.



Momento meccanico

$$\vec{M} = (\vec{r}_2 - \vec{r}_1) \times \vec{F}_2 = \vec{p} \times \vec{E}$$

Il momento di dipolo ruota fino ad essere parallelo e concorde al campo elettrostatico.

$$W = \int_{\theta_0}^{\theta} M \, d\theta$$

$$-p\,E\cos\theta$$

Energia potenziale elettrostatica di dipolo

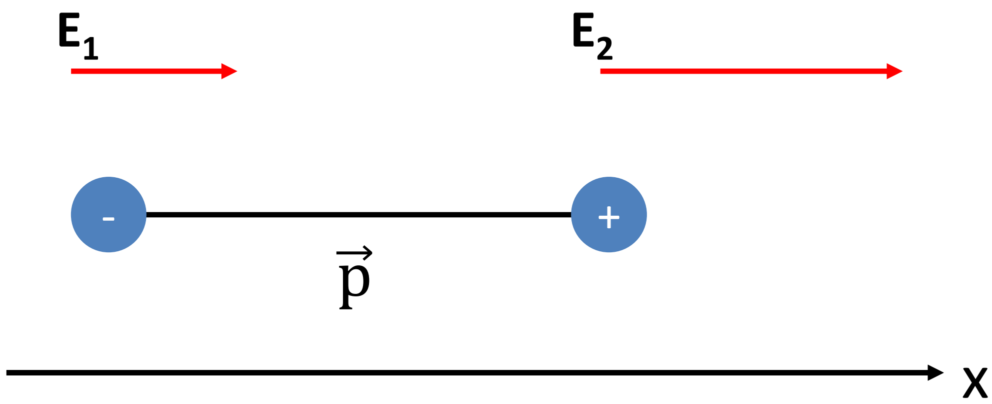
$$U_e(\theta) = -\vec{p} \cdot \vec{E}$$

Politecnico di Torino 1859

Forza elettrica su dipolo

Cosa succede quando il *campo elettrico esterno* ***non è uniforme***?

Supponiamo che E cresca al crescere di x e che il dipolo sia orientato come il campo elettrico.



Approssimazione distanza tra le cariche *a* piccola

$$E_2 = E_1 + \frac{\partial E}{\partial x} a$$

$$\boxed{F = q\,(E_2 - E_1) = p\,\frac{\partial E}{\partial x}}$$

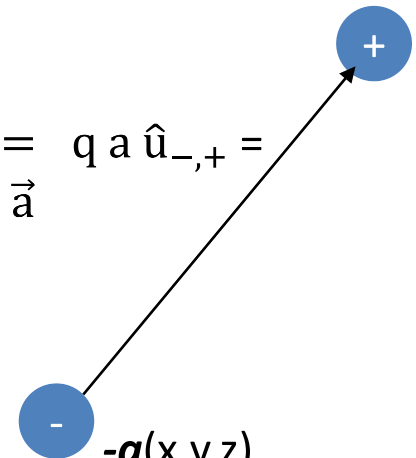
- Se il dipolo è **concorde** al campo E tende a spostarsi dove esso è **più intenso**
- Se il dipolo è **discorde**, si sposta dove il campo è **meno intenso**

Forza elettrica su dipolo

Caso più generale: **campo elettrico qualsiasi** e dipolo **non allineato**.

+q(x + a_x, y + a_y, z+ a_z)

$\vec{p} = q\, a\, \hat{u}_{-,+} =$
$= q\, \vec{a}$



-q(x,y,z)

Energia potenziale sistema

$$U_e = qV\left(x + a_x, y + a_y, z + d_z\right) - q\,V(x, y, z)$$

Approssimazione a piccolo

$$\boxed{U_e = q\, a_x \frac{\partial V}{\partial x} + q\, a_y \frac{\partial V}{\partial y} + q\, a_z \frac{\partial V}{\partial z} = -\,\vec{p} \cdot \vec{E}}$$

$$\boxed{\vec{F} = -\vec{\nabla}\, U_e}$$

- Il dipolo subisce sia l'azione del momento meccanico che tende ad allinearlo al campo elettrico, sia una forza risultante calcolabile tramite gradiente dell'energia potenziale (dato che il campo elettrico è conservativo.)