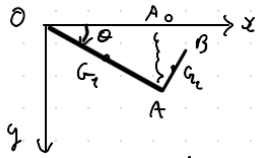
ESERCIZIO 1 (iniziato la scorsa volta)



$G_1 = (a\cos\theta, b\sin\theta)$; $A = (2a\cos\theta, 2a\sin\theta)$; $A_0 = (a\cos\theta, 0)$

$B = (2a\cos\theta + 2b\sin\theta, 2a\sin\theta - 2b\cos\theta)$; $G_2 = (2a\cos\theta + b\sin\theta, 2a\sin\theta - b\cos\theta)$

$G = \frac{1}{2}(3a\cos\theta + b\sin\theta, 3a\sin\theta - b\cos\theta)$

Dalla 1ª eq. cardinale della dinamica ($\underline{R}^{(e)} = \dot{\underline{Q}} = M\underline{a}_G$)

$\phi_{0x} = m\ddot{\theta}(-3a\sin\theta + b\cos\theta) + m\dot{\theta}^2(-3a\cos\theta - b\sin\theta)$

$\phi_{0y} = 2ka\sin\theta - mg + m\ddot{\theta}(3a\cos\theta + b\sin\theta) + m\ddot{\theta}(-3a\sin\theta + b\cos\theta)$

2ª Equazione cardinale della dinamica

$\underline{M}_\Omega^{(e)} = \dot{\underline{K}}_\Omega \leftarrow$ se Ω fisso oppure $\Omega \equiv G$

$\dot{\underline{K}}_\Omega = \underline{\omega} \wedge \underline{\underline{I}}_\Omega \underline{\omega} + \underline{\underline{I}}_\Omega \dot{\underline{\omega}}$ $\xrightarrow{\text{moto piano}}$ $\dot{\underline{K}}_\Omega = \underline{\underline{I}}_\Omega \dot{\underline{\omega}}$

Scelgo $\Omega \equiv O$ $\quad \dot{\underline{K}}_\Omega = I_{Oz}\dot{\underline{\omega}}$

$\underline{\omega} = \omega\underline{k}$ $\quad \underline{v}_A - \underset{\overset{\shortparallel}{0}}{\underline{v}_O} = \underline{\omega} \wedge (A - O) \Rightarrow \underline{\omega} = \dot{\theta}\underline{k}$

a) $\underline{M}_O^{(e)} = I_{Oz}\dot{\omega}\underline{k}$ $\qquad \underline{M}_O^{(e)} = I_{Oz}\ddot{\theta}\underline{k}$

$\underline{F} = -2ka\sin\theta\,\underline{j}$
$\underline{P} = 2mg\,\underline{j}$
$\underline{M} = -h\sin\theta\,\underline{k}$

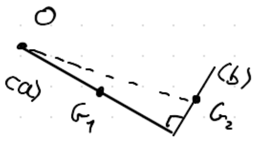
$\underline{M}_O^{(e)} = (G - O) \wedge \underline{P} + (A - O) \wedge \underline{F} + \underline{M}$

$\underline{M}_O^{(e)} = \left[\frac{1}{2}(3a\cos\theta + b\sin\theta)\underline{i} + \frac{1}{2}(3a\sin\theta - b\cos\theta)\underline{j}\right] \wedge 2mg\underline{j} +$

$+ (2a\cos\theta\,\underline{i} + 2a\sin\theta\,\underline{j}) \wedge (-2ka\sin\theta\,\underline{j}) - h\sin\theta\,\underline{k} =$

$= mg(3a\cos\theta + b\sin\theta)\underline{k} - 4ka^2\sin\theta\cos\theta\,\underline{k} - h\sin\theta\,\underline{k}$

Passiamo ad $I_{Oz}\ddot{\theta}\underline{k}$



vincolo di rigidità

$I_{Oz} = I_{Oz}^{(a)} + I_{Oz}^{(b)}$

$I_{Oz}^{(a)} = \frac{1}{3}m(2a)^2 = \frac{4}{3}ma^2$

$I_{Oz}^{(b)} = I_{G_2 z}^{(b)} + m|OG_2|^2 = \frac{1}{12}m(2b)^2 + m(4a^2 + b^2) = \frac{1}{3}mb^2 + 4ma^2 + mb^2$

$I_{Oz} = \frac{4}{3}ma^2 + 4ma^2 + \frac{4}{3}mb^2 = \frac{4}{3}m(4a^2 + b^2)$

$\frac{4}{3}m(4a^2 + b^2)\ddot{\theta} = mg(3a\cos\theta + b\sin\theta) - 4ka^2\sin\theta\cos\theta - h\sin\theta$

b) Energia cinetica:

$$T = \frac{1}{2} M v_\Omega^2 + M \underline{v}_\Omega \cdot (\underline{\omega} \wedge \Omega G) + \frac{1}{2} \underline{\underline{I}}_\Omega \underline{\omega} \cdot \underline{\omega} \qquad \text{(corpo rigido)}$$

Se Ω fisso: $T = \frac{1}{2} \underline{\underline{I}}_\Omega \underline{\omega} \cdot \underline{\omega}$

Se $\Omega \equiv G$ $\quad T = \frac{1}{2} M v_G^2 + \frac{1}{2} \underline{\underline{I}}_G \underline{\omega} \cdot \underline{\omega}$

Moto piano (scelgo un asse $/\!/ \underline{k}$) ; $\underline{\underline{I}}_\Omega \underline{\omega} = I_{\Omega z} \underline{\omega}$

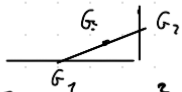
Se Ω fisso: $T = \frac{1}{2} I_{\Omega z} \omega^2$. Se $\Omega \equiv G$: $T = \frac{1}{2} M v_G^2 + \frac{1}{2} I_{Gz} \omega^2$

Scelgo $\Omega \equiv O$ (via breve) $\quad T = \frac{1}{2} I_{Oz} \omega^2 = \frac{1}{2} \frac{4}{3} m (4a^2 + b^2) \dot\theta^2 = \frac{2}{3} m (4a^2 + b^2) \dot\theta^2$

Applicando $T = \frac{1}{2} M v_G^2 + \frac{1}{2} I_{Gz} \omega^2$; $\quad \underline{v}_G = \frac{1}{2} \dot\theta (-3a\sin\theta + b\cos\theta, 3a\cos\theta + b\sin\theta)$

I_{Gz}: due modi; $I_{Gz} = I^{(a)}_{Gz} + I^{(b)}_{Gz} = I^{(a)}_{G_1 z} + m|G_1 G|^2 + I^{(b)}_{G_2 z} + m|G_2 G|^2$

Oppure $\quad I_{Gz} = I_{Oz} - 2m|OG|^2 = \frac{16}{3} m a^2 + \frac{4}{3} m b^2 - 2m|OG|^2$



$|OG|^2 = \frac{9}{4} a^2 + \frac{1}{4} b^2 \Rightarrow I_{Gz} = \frac{16}{3} m a^2 + \frac{4}{3} m b^2 - \frac{9}{2} m a^2 - \frac{1}{2} m b^2 = \frac{5}{6} m a^2 + \frac{5}{6} m b^2$

$$v_G^2 = \frac{1}{4} \dot\theta^2 (9a^2 \sin^2\theta + b^2\cos^2\theta - \cancel{6ab\sin\theta\cos\theta} + 9a^2\cos^2\theta + b^2\sin^2\theta + \cancel{6ab\sin\theta\cos\theta}) =$$

$$= \frac{9}{4} a^2 \dot\theta^2 + \frac{1}{4} b^2 \dot\theta^2$$

$$T = \underbrace{\frac{1}{2} 2m \left(\frac{9}{4} a^2 + \frac{1}{4} b^2\right) \dot\theta^2}_{\frac{1}{2} M v_G^2} + \underbrace{\frac{1}{2} \frac{5}{6} m (a^2 + b^2) \dot\theta^2}_{\frac{1}{2} I_{Gz} \omega^2} =$$

$$= m \left(\frac{9}{4} + \frac{5}{12}\right) a^2 \dot\theta^2 + m \left(\frac{1}{4} + \frac{5}{12}\right) b^2 \dot\theta^2 = \frac{8}{3} m a^2 \dot\theta^2 + \frac{2}{3} m b^2 \dot\theta^2$$

$$T = \frac{2}{3} m (4a^2 + b^2) \dot\theta^2$$

Potenziale delle forze attive.

Premessa: per la componente rotatoria il lavoro $\underline{M} \cdot \varepsilon'$

$\rightarrow dL = \underline{M} \cdot d\theta \underline{K}$

nel nostro caso $\underline{M} = -h \sin\theta \, \underline{K} \Rightarrow dL = dU = -h\sin\theta \underline{K} \cdot d\theta \underline{K} =$

$= -h \sin\theta \, d\theta$

Il contributo della coppia $\underline{M}$ ad U è:

$$U = \int (-h \sin\theta) d\theta = h\cos\theta + cost$$

$$U = 2mg\, y_G - \frac{1}{2} K |A_0 A|^2 + h\cos\theta + cost$$

$$U = mg(3a\sin\theta - b\cos\theta) - \frac{1}{2} K(-2a\sin\theta)^2 + h\cos\theta + cost$$

$$U = mg(3a\sin\theta - b\cos\theta) - 2Ka^2\sin^2\theta + h\cos\theta + cost$$

c) Equazione del moto di Lagrange

(1^a forma) $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_h} - \frac{\partial \mathcal{L}}{\partial q_h} = \begin{cases} 0 & \text{(solo forze conservative)} \\ Q_h^{(n.c.)} & \text{(con forze non cons. in gioco)} \end{cases}$

(2^a forma) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_h}\right) - \frac{\partial T}{\partial q_h} = Q_h$

$\dot{p}_h - \frac{\partial T}{\partial q_h} = Q_h$

$p_h = \frac{\partial T}{\partial \dot{q}_h}$ sono anche detti momenti cinetici.

$$\frac{d}{dt}\frac{\partial T}{\partial \dot\theta} - \frac{\partial T}{\partial \theta} = Q_\theta \qquad\qquad Q_\theta = \frac{\partial U}{\partial \theta} = \frac{d}{d\theta} U$$

$$U = mg(3a\sin\theta - b\cos\theta) - 2ka^2\sin^2\theta + h\cos\theta\,; \quad T = \frac{2}{3}m(4a^2+b^2)\dot\theta^2$$

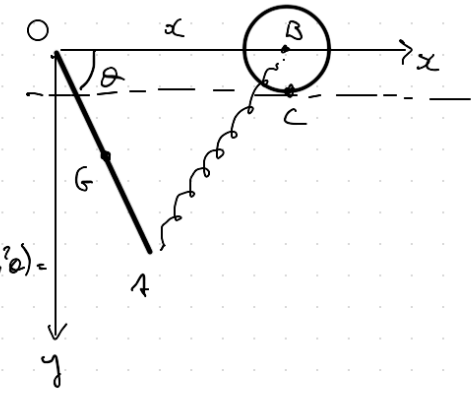
$$p_\theta = \frac{\partial T}{\partial \dot\theta} = \frac{4}{3}m(4a^2+b^2)\dot\theta\,; \quad \frac{d}{dt}\frac{\partial T}{\partial \dot\theta} = \frac{4}{3}m(4a^2+b^2)\ddot\theta\,; \quad \frac{\partial T}{\partial \theta} = 0$$

$$Q_\theta = U' = mg(3a\cos\theta + b\sin\theta) - 4ka^2\sin\theta\cos\theta - h\sin\theta.$$

Eq. di Lagrange:

$$\underbrace{\frac{4}{3}m(4a^2+b^2)\ddot\theta}_{\frac{d}{dt}\frac{\partial T}{\partial \dot\theta} - \frac{\partial T}{\partial \theta}} = \underbrace{mg(3a\cos\theta + b\sin\theta) - 4ka^2\sin\theta\cos\theta - h\sin\theta}_{Q_\theta}$$

... come avevamo ricavato dalla 2ª equazione cardinale della dinamica



$B = (x, 0)$; $C = (x, R)$; $G = (l\cos\theta, l\sin\theta)$

$A = (2l\cos\theta, 2l\sin\theta)$; $B - A = (x - 2l\cos\theta, -2l\sin\theta)$

a) $U = mg\,y_G + mg\,y_B - \frac{1}{2}k|AB|^2 =$

$= mgl\sin\theta + 0 - \frac{1}{2}k(x^2 - 4lx\cos\theta + \underbrace{4l^2\cos^2\theta + 4l^2\sin^2\theta}_{=4l^2=cost}) =$

$= mgl\sin\theta - \frac{1}{2}kx^2 + 2klx\cos\theta + cost$

$Q_\theta = \frac{\partial U}{\partial \theta} = mgl\cos\theta - 2klx\sin\theta$

$Q_x = \frac{\partial U}{\partial x} = -kx + 2kl\cos\theta$

Equilibrio: $\begin{cases} mgl\cos\theta - 2klx\sin\theta = 0 \\ -kx + 2kl\cos\theta = 0 \end{cases}$

2ª eq.:) $x = 2l\cos\theta$ $\rightarrow$ 1ª $\rightarrow$ $mgl\cos\theta - 4kl^2\sin\theta\cos\theta = 0$

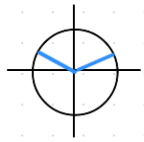
$\begin{cases} x = 2l\cos\theta \\ l\cos\theta(mg - 4kl\sin\theta) = 0 \end{cases}$ $\nearrow$ 2 sottocasi:
- $\cos\theta = 0$
- $mg - 4kl\sin\theta = 0$

• ~~Dalla~~ condizione $\cos\theta = 0$ $\begin{cases} \theta_1 = \frac{\pi}{2} & \rightarrow x_1 = 2l\cos\theta_1 = 0 \\ \theta_2 = \frac{3}{2}\pi & \rightarrow x_2 = 2l\cos\theta_2 = 0 \end{cases}$

$(\theta_1, x_1) = (\frac{\pi}{2}, 0)$; $(\theta_2, x_2) = (\frac{3}{2}\pi, 0)$ configurazioni di equilibrio

• Dalla condizione $mg - 4kl\sin\theta = 0 \Rightarrow \sin\theta = \frac{mg}{4kl} = \lambda$

Se $\lambda \leq 1$: $\theta_3 = \arcsin\lambda$
$\theta_4 = \pi - \arcsin\lambda$



$x_3 = 2l\cos\theta_3 = 2l\sqrt{1-\lambda^2}$
$x_4 = 2l\cos\theta_3 = -2l\sqrt{1-\lambda^2}$

oppure dire : $\sin\theta_3 = \lambda$, $\cos\theta_3 = \sqrt{1-\lambda^2}$ e $\sin\theta_4 = \lambda$, $\cos\theta_4 = -\sqrt{1-\lambda^2}$

$(\theta_3, x_3) = (\arcsin\lambda, 2l\sqrt{1-\lambda^2})$; $(\theta_4, x_4) = (\pi - \arcsin\lambda, -2l\sqrt{1-\lambda^2})$

Se $\lambda = 1$ $(\theta_3, x_3) = (\theta_4, x_4)$ e coincidono con $(\theta_1, x_1) = (\frac{\pi}{2}, 0)$

In realtà le config. 3 e 4 sono distinte se $\lambda < 1$

5) Stabilità. $\frac{\partial^2 U}{\partial\theta^2} = -mgl\sin\theta - 2kl x\cos\theta$; $\frac{\partial^2 U}{\partial x^2} = -k$

$$\frac{\partial^2 U}{\partial x\partial\theta} = \frac{\partial^2 U}{\partial\theta\partial x} = -2kl\sin\theta$$

$$\underline{\underline{H}}(\theta, x) = \begin{pmatrix} -mgl\sin\theta - 2klx\cos\theta & -2kl\sin\theta \\ -2kl\sin\theta & -k \end{pmatrix}$$

$$\underline{\underline{H}}(\theta_1, x_1) = \begin{pmatrix} -mgl & -2kl \\ -2kl & -k \end{pmatrix}$$

$\mathrm{Tr}\,\underline{\underline{H}}_1 < 0$ $\det\underline{\underline{H}}_1 = mgkl - 4k^2l^2$

$\det\underline{\underline{H}}_1 = 4k^2l^2(\lambda - 1)$ Stabile se $\lambda > 1$ (autovalori concordi e negativi)

$$\underline{\underline{H}}(\theta_2, x_2) = \begin{pmatrix} mgl & 2kl \\ 2kl & -k \end{pmatrix}$$

$\det\underline{\underline{H}}_2 = -mgkl - 4k^2l^2 < 0$ (autovalori discordi) Instabile sempre

Su tutte le configurazioni di equilibrio $x = 2l\cos\theta$. Lo uso per semplificare $\underline{\underline{H}}_3$ e $\underline{\underline{H}}_4$

$$\underline{\underline{H}}(\theta_{eq}, x_{eq}) = \begin{pmatrix} -mgl\sin\theta - 4Kl^2\cos^2\theta & -2Kl\sin\theta \\ -2Kl\sin\theta & -K \end{pmatrix}$$

Esistono distinte se $\lambda < 1$

$$\underline{\underline{H}}(\theta_3, x_3) = H(\theta_4, x_4) = \begin{pmatrix} -mgl\lambda - 4Kl^2(1-\lambda^2) & -2Kl\lambda \\ -2Kl\lambda & -K \end{pmatrix}$$

$$\mathrm{Tr}\,\underline{\underline{H}}_3 = \mathrm{Tr}\,\underline{\underline{H}}_4 = -mgl\lambda - 4Kl^2\underbrace{(1-\lambda^2)}_{\geq 0 \text{ quando mi interessa}} - K < 0$$

$$\det \underline{\underline{H}}_3 = \det \underline{\underline{H}}_4 = mgKl\lambda + 4K^2l^2(1-\lambda^2) - 4K^2l^2\lambda^2 =$$

$$= 4K^2l^2 \underbrace{\frac{mg}{4Kl}}_{\lambda}\lambda + 4K^2l^2 - 4K^2l^2\lambda^2 - 4K^2l^2\lambda^2 =$$

$$= 4K^2l^2(\lambda^2 + 1 - \lambda^2 - \lambda^2) = 4K^2l^2(1-\lambda^2) > 0 \quad \text{se } \lambda < 1$$

Quando le configurazioni esistono distinte sono stabili.

c) Eq. di Lagrange.

Energia cinetica : $T = T^{(a)} + T^{(d)}$ $\qquad$ (a) = asta, (d) = disco

$\underline{\omega}_a = \omega_a \underline{K} \qquad \underline{v}_G - \overset{=0}{\underline{v}_0} = \omega_a \underline{K} \wedge (G - O)$

$\dot{\theta} l(-\sin\theta\, \underline{i} + \cos\theta\, \underline{j}) = \omega_a \underline{K} \wedge l(\cos\theta\, \underline{i} + \sin\theta\, \underline{j}) \Rightarrow \omega_a = \dot{\theta}$

$\underline{\omega}_d = \omega_d \underline{K} \qquad \underline{v}_B - \overset{=0}{\underline{v}_c} = \omega_d \underline{K} \wedge (B - C)$

$\dot{x}\, \underline{i} = \omega_d \underline{K} \wedge (-R\underline{j}) \iff \dot{x}\, \underline{i} = \omega_d R\, \underline{i} \qquad \omega_d = \dfrac{\dot{x}}{R}$

$T^{(a)} = \dfrac{1}{2} I_0^{(a)} \omega_a^2 = \dfrac{1}{2}\dfrac{1}{3} m 4l^2 \dot{\theta}^2 = \dfrac{2}{3} m l^2 \dot{\theta}^2$

$T^{(d)} = \dfrac{1}{2} I_c^{(d)} \omega_d^2 = \dfrac{1}{2}\dfrac{3}{2} m R^2 \dfrac{\dot{x}^2}{R^2} = \dfrac{3}{4} m \dot{x}^2 ; \qquad T = \dfrac{2}{3} m l^2 \dot{\theta}^2 + \dfrac{3}{4} m \dot{x}^2$

$\dfrac{\partial T}{\partial \dot{\theta}} = \dfrac{4}{3} m l^2 \dot{\theta} ; \ \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\theta}} = \dfrac{4}{3} m l^2 \ddot{\theta} ; \ \dfrac{\partial T}{\partial \theta} = 0 = \dfrac{\partial T}{\partial x} ; \ \dfrac{\partial T}{\partial \dot{x}} = \dfrac{3}{2} m \dot{x} ; \ \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{x}} = \dfrac{3}{2} m \ddot{x}$

$$\begin{cases} \dfrac{4}{3} m l^2 \ddot{\theta} = mgl\cos\theta - 2Klx\sin\theta \\ \dfrac{3}{2} m \ddot{x} = -Kx + 2Kl\cos\theta \end{cases}$$

Equazioni di Lagrange