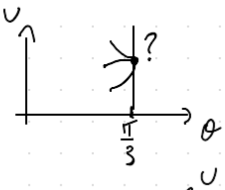
Riprendiamo l'esercizio della scorsa volta, ora diventato 1)

$$Q(\theta) = mgl\sin\theta\left(1 - \frac{\lambda}{\cos^3\theta}\right) = U'(\theta)$$

$$Q'(\theta) = mgl\cos\theta\left(1 - \frac{\lambda}{\cos^3\theta}\right) - 3kl^2\,\frac{\sin^2\theta}{\cos^4\theta} = U''(\theta)$$

Avevamo visto che se $\theta = \frac{\pi}{3}$:

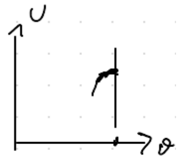
$Q(\frac{\pi}{3}) = mgl\,\frac{\sqrt{2}}{2}(1-8\lambda)$ Equilibrio se $1-8\lambda \geq 0 \Leftrightarrow \lambda \leq \frac{1}{8}$



Stabilità: se $1-8\lambda > 0$ $(\lambda < \frac{1}{8})$, $U' > 0$
è stabile;

se $1-8\lambda = 0$, guardiamo a $U''(\theta)$

$U''(\frac{\pi}{3}) = \frac{mgl}{2}(1-80\lambda) \xrightarrow[\lambda=\frac{1}{8}]{} U''(\frac{\pi}{3}) = -\frac{9}{2}mgl < 0$



stabile

Se $\theta = 0$? $Q_\theta = U'(\theta) = 0$ sempre di equilibrio

Dobbiamo guardare a $U''(\theta)$

$U''(0) = mgl(1-\lambda)$ $1-\lambda > 0$ instabile $(\lambda < 1)$

$1-\lambda < 0$ stabile $(\lambda > 1)$

Se $\lambda = 1 \Rightarrow U''(0) = 0$

fissiamo $\lambda = 1$

$$U'(\theta) = mgl\sin\theta\left(1 - \frac{1}{\cos^3\theta}\right)$$

Se $\theta = \varepsilon$ $\sin\theta \simeq \varepsilon$, $\cos\theta \simeq 1 - \frac{\varepsilon^2}{2}$

$$U'(\varepsilon) \simeq mgl\varepsilon\left(1 - \frac{1}{(1-\frac{\varepsilon^2}{2})^3}\right) \simeq mgl\varepsilon\left(1 - 1\left(1 + \frac{3}{2}\varepsilon^2\right)\right) = -\frac{3}{2}mgl\varepsilon^3$$

Stabile

b) Equazioni cardinali della statica : 1) $\underline{R}^{(e)} = \underline{0}$

2) $\underline{M}_{\Omega}^{(e)} = \underline{0}$

1) $\underline{F} + \underline{P} + \underline{\phi}_Q + \underline{\phi}_A = \underline{0}$

$\underline{\phi}_Q = \phi_x \underline{i} + \phi_y \underline{j}$; $\underline{\phi}_A = \phi_A \underline{j}$

$$K(O-A) - mg\underline{j} + \phi_{Qx}\underline{i} + \phi_{Qy}\underline{j} + \phi_A \underline{j} = \underline{0}$$

$$-Kl\tan\theta\,\underline{i} - mg\underline{j} + \phi_{Qx}\underline{i} + \phi_{Qy}\underline{j} + \phi_A\underline{j} = \underline{0}$$

$$\begin{cases} \phi_{Qx} = Kl\tan\theta \\ \phi_{Qy} + \phi_A = mg \end{cases}$$

2ª) Scegliamo Q come polo: $\underline{M}_Q = \underline{0}$

$$\underline{M}_Q = (G-Q)\wedge\underline{P} + (A-Q)\wedge\underline{F} + (A-Q)\wedge\underline{\phi}_A$$

$$\underline{M}_Q = [l(\tan\theta - \sin\theta)\underline{i} - l\underline{j}]\wedge(-mg\underline{j}) +$$

$$+(l\tan\theta\,\underline{i} - l\underline{j})\wedge(-Kl\tan\theta\,\underline{i}) + (l\tan\theta\,\underline{i} - l\underline{j})\wedge\phi_A\underline{j} =$$

$$= mgl(\sin\theta - \tan\theta)\underline{K} - Kl^2\tan\theta\,\underline{K} + \phi_A l\tan\theta\,\underline{K} = \underline{0}$$

Terza equazione scalare : $\phi_A l\tan\theta = mgl(\tan\theta - \sin\theta) + Kl^2\tan\theta$

$$\begin{cases} \phi_{Qx} = Kl\tan\theta \\ Q_{Qy} + \phi_A = mg \\ \phi_A\tan\theta = mg(\tan\theta - \sin\theta) + Kl\tan\theta \end{cases}$$

c) $\underline{\Phi}_Q = \Phi_Q \cos\theta\, \underline{i} + \Phi_Q \sin\theta\, \underline{j}$

$$\begin{cases} \Phi_Q \cos\theta = k\ell \tan\theta \\ \Phi_Q \sin\theta + \Phi_A = mg \\ mg(\tan\theta - \sin\theta) + k\ell \tan\theta = \Phi_A \tan\theta \end{cases}$$

$$\Phi_Q = k\ell \frac{\sin\theta}{\cos^2\theta}$$

$$k\ell \frac{\sin^2\theta}{\cos^2\theta} + \Phi_A = mg \longrightarrow \Phi_A = mg - k\ell \frac{\sin^2\theta}{\cos^2\theta}$$

$$mg(\tan\theta - \sin\theta) + k\ell \tan\theta = mg\tan\theta - k\ell \tan\theta \frac{\sin^2\theta}{\cos^2\theta}$$

$$mg\sin\theta - k\ell\tan\theta\left(1 + \frac{\sin^2\theta}{\cos^2\theta}\right) = 0 \qquad \theta \neq 0$$

$$mg - k\ell \frac{1}{\cos^3\theta} = 0 \iff mg\left(1 - \frac{k\ell}{mg}\frac{1}{\cos^3\theta}\right) = 0 \iff mg\left(1 - \frac{\lambda}{\cos^3\theta}\right) = 0$$

2) $G_1 = (\ell\cos\varphi, \ell\sin\varphi)$; $A = (2\ell\cos\varphi, 2\ell\sin\varphi)$

$$P = A + P - A = (2\ell\cos\varphi - \ell\sin\theta, 2\ell\sin\varphi + \ell\cos\theta) \equiv G_2$$

$$Q = A + Q - A = (2\ell\cos\varphi + \ell\sin\theta, 2\ell\sin\varphi - \ell\cos\theta)$$

$$Q_0 = (2\ell\cos\varphi + \ell\sin\theta, 0)$$

$$G = \frac{2m G_1 + m G_2}{3m} = \frac{1}{3}(2G_1 + G_2) = \frac{1}{3}(4\ell\cos\varphi - \ell\sin\theta, 4\ell\sin\varphi + \ell\cos\theta)$$

I^a eq. card.) $\underline{P} + \underline{F} + \underline{\Phi}_0 = \underline{0}$; $\underline{P} = 3mg\underline{j}$, $\underline{F} = k(Q_0 - Q)$

$$3mg\underline{j} + k\ell(\cos\theta - 2\sin\varphi)\underline{j} + \Phi_{0x}\underline{i} + \Phi_{0y}\underline{j} = \underline{0}$$

$$\begin{cases} \Phi_{0x} = 0 \\ \Phi_{0y} = -3mg + k\ell(2\sin\varphi - \cos\theta) \end{cases}$$

II^a) Scelgo O come polo: $\underline{M}_O^{(e)} = \underline{0}$

$$\underline{M}_O^{(e)} = (G-O)\wedge \underline{P} + (Q-O)\wedge \underline{F}$$

$$\underline{P} = 3mg\,\underline{j} \quad ; \quad \underline{F} = K\ell(\cos\theta - 2\sin\varphi)\,\underline{j}$$

$$\frac{1}{3}[\underbrace{(4\ell\cos\varphi - \ell\sin\theta)\underline{i} + (4\ell\sin\varphi + \ell\cos\theta)\underline{j}}_{G-O}]\wedge \underbrace{3mg\,\underline{j}}_{P} +$$

$$+ [\underbrace{(2\ell\cos\varphi + \ell\sin\theta)\underline{i} + (2\ell\sin\varphi - \ell\cos\theta)\underline{j}}_{Q-O}]\wedge \underbrace{K\ell(\cos\theta - 2\sin\varphi)\underline{j}}_{F} = \underline{0}$$

$$mg\ell(4\cos\varphi - \sin\theta)\,\underline{K} + K\ell^2(2\cos\varphi + \sin\theta)(\cos\theta - 2\sin\varphi)\,\underline{K} = \underline{0}$$

$$mg(4\cos\varphi - \sin\theta) + K\ell(2\cos\varphi\cos\theta - 4\cos\varphi\sin\varphi + \sin\theta\cos\theta - 2\sin\theta\sin\varphi) = 0$$

$$mg(4\cos\varphi - \sin\theta) + K\ell\left(2\cos(\theta+\varphi) - 2\sin(2\varphi) + \frac{1}{2}\sin(2\theta)\right) = 0$$

$$\begin{cases} \Phi_{Ox} = 0 \\ \Phi_{Oy} = -3mg + K\ell(2\sin\varphi - \cos\theta) \\ mg(4\cos\varphi - \sin\theta) + K\ell\left(2\cos(\theta+\varphi) - 2\sin(2\varphi) + \frac{1}{2}\sin(2\theta)\right) = 0 \end{cases}$$

b) I^a strada) $\delta L^{(a)}$ e poi integriamo. $\quad \delta L^{(a)} = \sum_i \underline{F}_i \cdot \delta P_i = \sum_h Q_h \delta q_h$

$$\delta L^{(a)} = \underline{P}\cdot\delta G + \underline{F}\cdot\delta Q \qquad \underline{P} = 3mg\,\underline{j} \;;\; \underline{F} = K\ell(\cos\theta - 2\sin\varphi)\,\underline{j}$$

$$\delta G = \frac{\ell}{3}[(-4\sin\varphi\,\delta\varphi - \cos\theta\,\delta\theta)\underline{i} + (4\cos\varphi\,\delta\varphi - \sin\theta\,\delta\theta)\underline{j}]$$

$$\delta Q = \ell[(-2\sin\varphi\,\delta\varphi + \cos\theta\,\delta\theta)\underline{i} + (2\cos\varphi\,\delta\varphi + \sin\theta\,\delta\theta)\underline{j}]$$

$$\underline{P}\cdot\delta G = 3mg\,\underline{j}\cdot\frac{\ell}{3}(4\cos\varphi\,\delta\varphi - \sin\theta\,\delta\theta)\,\underline{j} \qquad (\text{perché } \underline{j}\cdot\underline{i} = 0)$$

$$\underline{F}\cdot\delta Q = K\ell(\cos\theta - 2\sin\varphi)\,\underline{j}\cdot\ell(2\cos\varphi\,\delta\varphi + \sin\theta\,\delta\theta)\,\underline{j}$$

$$\delta L^{(a)} = mg\ell(4\cos\varphi\,\delta\varphi - \sin\theta\,\delta\theta) + K\ell^2(\cos\theta - 2\sin\varphi)(2\cos\varphi\,\delta\varphi + \sin\theta\,\delta\theta) =$$

$$= [\underbrace{4mg\ell + 2K\ell^2(\cos\theta - 2\sin\varphi)}_{Q_\varphi}]\cos\varphi\,\delta\varphi + [\underbrace{-mg\ell + K\ell^2(\cos\theta - 2\sin\varphi)}_{Q_\theta}]\sin\theta\,\delta\theta$$

Osservazione: equilibrio $Q_\varphi = 0$ e $Q_\theta = 0$

Ma! $Q_\varphi + Q_\theta = mg\ell(4\cos\theta - \sin\theta) + K\ell^2[2\cos(\theta+\varphi) - 2\sin(2\varphi) + \frac{1}{2}\sin(2\theta)]$
che è la 3^a eq. scalare di prima

$$U = \int Q_\varphi \, d\varphi + f(\theta) \quad ; \quad U = \int Q_\theta \, d\theta + g(\varphi)$$

$U = 4mgl\sin\varphi + 2Kl^2\cos\theta\sin\varphi - 2Kl^2\sin^2\varphi + f(\theta)$ (integrando in $d\varphi$)

$U = mgl\cos\theta + \frac{1}{2}Kl^2\sin^2\theta + 2Kl^2\sin\varphi\cos\theta + g(\varphi)$

Dal confronto: $f(\theta) = mgl\cos\theta + \frac{1}{2}Kl^2\sin^2\theta$

$g(\varphi) = 4mgl\sin\varphi - 2kl^2\sin^2\varphi$

$$\underline{U = 4mgl\sin\varphi + mgl\cos\theta + 2kl^2\cos\theta\sin\varphi - 2Kl^2\sin^2\varphi + \frac{1}{2}Kl^2\sin^2\theta} + c$$

II° metodo) $U = 3mg\,y_G - \frac{1}{2}k|Q - Q_0|^2 + c$

$U = 3mg\,\frac{l}{3}(4\sin\varphi + \cos\theta) - \frac{1}{2}kl^2(2\sin\varphi - \cos\theta)^2 + c =$

$$= \underline{4mgl\sin\varphi + mgl\cos\theta - 2Kl^2\sin^2\varphi + 2Kl^2\cos\theta\sin\varphi - \frac{1}{2}Kl^2\cos^2\theta} + c$$

$\frac{1}{2}Kl^2\sin^2\theta = \underbrace{\frac{1}{2}Kl^2}_{\text{Cost}} - \frac{1}{2}Kl^2\cos^2\theta$

3) a)

I^a eq. cardinale della dinamica: $\underline{R}^{(e)} = \dot{\underline{Q}}$

$\underline{Q} = M\,\underline{v}_G$

$\dot{\underline{Q}} = M\,\underline{a}_G$

Qui $M = 2m$

$\underline{R}^{(e)} = \underline{P} + \underline{F} + \underline{\phi}_0 =$

$= 2mg\,\underline{j} - 2ka\sin\theta\,\underline{j} + \phi_{0x}\,\underline{i} + \phi_{0y}\,\underline{j}$

$\underline{v}_G = \frac{1}{2}\dot{\theta}\,(-3a\sin\theta + b\cos\theta,\ 3a\cos\theta + b\sin\theta)$

$\underline{a}_G = \frac{1}{2}\ddot{\theta}(-3a\sin\theta + b\cos\theta,\ 3a\cos\theta + b\sin\theta) + \frac{1}{2}\dot{\theta}^2(-3a\cos\theta - b\sin\theta,\ -3a\sin\theta + b\cos\theta)$

$2mg\,\underline{j} - 2ka\sin\theta\,\underline{j} + \phi_{0x}\,\underline{i} + \phi_{0y}\,\underline{j} =$

$= m\ddot{\theta}\left[(-3a\sin\theta + b\cos\theta)\,\underline{i} + (3a\cos\theta + b\sin\theta)\,\underline{j}\right] +$

$+ m\dot{\theta}^2\left[(-3a\cos\theta - b\sin\theta)\,\underline{i} + (-3a\sin\theta + b\cos\theta)\,\underline{j}\right]$

$$\begin{cases} \phi_{0x} = m\ddot{\theta}(-3a\sin\theta + b\cos\theta) + m\dot{\theta}^2(-3a\cos\theta - b\sin\theta) \\ \phi_{0y} = m\ddot{\theta}(3a\cos\theta + b\sin\theta) + m\dot{\theta}^2(-3a\sin\theta + b\cos\theta) - 2mg + 2ka\sin\theta \end{cases}$$

II^a eq. card. $\underline{M}_\Omega^{(e)} = \dot{\underline{K}}_\Omega + \dot{\Omega} \wedge \underline{Q}$

Se $\Omega \equiv G$ oppure fisso $\underline{M}_\Omega^{(e)} = \dot{\underline{K}}_\Omega$

$\underline{K}_\Omega = m\, \Omega G \wedge \underline{v}_\Omega + \underline{\underline{I}}_\Omega \underline{\omega}$

di nuovo se Ω fisso o $\Omega \equiv G$

$\underline{K}_\Omega = \underline{\underline{I}}_\Omega \underline{\omega}$

$\underline{M}_\Omega^{(e)} = \dot{\underline{K}}_\Omega = \underline{\omega} \wedge \underline{\underline{I}}_\Omega \underline{\omega} + \underline{\underline{I}}_\Omega \dot{\underline{\omega}}$ se Ω fisso oppure $\Omega \equiv G$

Se il sistema è piano, z è asse principale di inerzia

$$\underline{\underline{I}}_\Omega \underline{\omega} = \begin{pmatrix} I_x & I_{xy} & 0 \\ I_y & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = I_z \underline{\omega} \qquad \underline{\omega} = \omega \underline{k}$$

Quindi $\underline{\underline{I}}_\Omega \underline{\omega} \parallel \underline{\omega} \Rightarrow \underline{M}_\Omega^{(e)} = \underline{\underline{I}}_\Omega \dot{\underline{\omega}}$

Scelgo O :

$$(G-O) \wedge \underline{P} + (A-O) \wedge \underline{F} + \underline{M} = \underline{\underline{I}}_{Oz} \dot{\underline{\omega}}$$

$\uparrow \quad \dot{\underline{\omega}} = \ddot{\theta}\, \underline{k}$