CHAPTER 6

Statics

Consider a force that depends both on the position P and on the velocity $\vec{v}$ of the mass point to which it is applied, but that does not depend on time; denote it by $\vec{F} = \vec{F}(P, \vec{v})$. The solution of the Newton equations $m\vec{a} = \vec{F}(P, \vec{v})$ may include a rest solution, *i.e.* a solution with a position that does not depend on time:

$$P(t) = P^*, \qquad \forall t \geq t_0 \tag{6.1}$$

if the initial condition is given by $(P(t_0) = P^*;\ \vec{v}(t_0) = \mathbf{0})$. Differently, a configuration P^* is called *equilibrium* if

$$\vec{F}(P^*, \mathbf{0}) = \mathbf{0} \tag{6.2}$$

A rest solution is also an equilibrium because P^* would move if $\vec{F}(P^*, \mathbf{0}) \neq \mathbf{0}$. However, given a force that vanishes in P^* (P^* is then an equilibrium), and the initial condition $(P(t_0) = P^*;\ \vec{v}(t_0) = \mathbf{0})$ the solution may not be unique: Eq. (6.1) is not guaranteed to hold. That depends on $\vec{F}$ and on its properties, not just the fact that it vanishes in P^*. For instance, the evolution determined by:

$$m\ddot{x} = a\sqrt[3]{x}, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$$

has the constant $x(t) = 0$ as a rest solution, but also the time evolving solution

$$x(t) = \left(\frac{a}{6m}\right)^{3/2} t^3 \,,$$

as it can be verified by inspection. Nevertheless, a physically relevant mechanical model cannot be affected by such ambiguities, in general, it can only have a single solution that exists globally in time.

DEFINITION 6.1: *A system of mass points is in equilibrium when all its points are.*

This is a situation that requires $\vec{F}_i = \mathbf{0}$ for all unconstrained points i and $\vec{F}_j + \vec{\Phi}_j = \mathbf{0}$ for all constrained points j, where $\vec{F}_i$ are active forces, and $\vec{\Phi}_j$ are constraining reactions.

Let us consider one example. Let P belong to the surface Σ or let it lie in one of the two sides (half spaces), Ω^+ say, of Σ. In absence of friction, the constraining force is orthogonal in P to the surface, $\vec{\Phi} \perp \Sigma$, in order to cancel possible orthogonal components of the active forces. Its projection along the outward normal $\hat{n}$ to Σ is non-negative, $\vec{\Phi} \cdot \hat{n} \geq 0$, because $\vec{\Phi}$ in some sense "repels" objects from the surface. For a virtual displacement δP, the virtual work is:

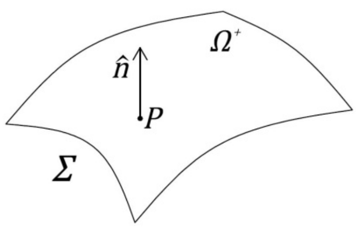

Figure 6.1. *constraining surface*

$$\delta L^{(v)} = \vec{\Phi} \cdot \delta P \geq 0 \tag{6.3}$$

because δP is either tangent to Σ, zero, or it points away from Σ, and $P + \delta P$ it belongs to Ω^+. Differently, friction implies a negative contribution, hence $\delta L^{(v)}$ is not guaranteed to be nonnegative.

Consider a disk rolling without slipping on a straight line. Here friction is needed for point C to be at rest; if C is not at rest, the disk slips on its rail. In the presence of friction and no slip, the friction does no work:

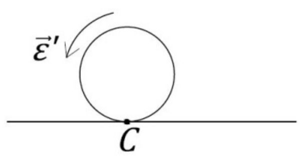

Figure 6.2. *disk rolling on a horizontal rail*

$$\delta L^{(v)} = \vec{\Phi} \cdot \delta C = 0 \tag{6.4}$$

DEFINITION 6.2: *A virtual displacement δP is called reversible if $-\delta P$ is a virtual displacement as well.*

DEFINITION 6.3: *The constraints for which $\delta L^{(v)} \geq 0$ for every virtual δP starting in every allowed configuration are called ideal.*

THEOREM 6.1 (or PRINCIPLE OF VIRTUAL WORK): *the configuration C of a system subjected to ideal constraints is an equilibrium if and only if* $\boxed{\delta L^{(a)} \leq 0}$ *$\forall \delta P$ starting in C, where "=" holds for any reversible δP.* [Note: ≤ 0, all δP, and starting in C]

Proof (sort of): Let C be an equilibrium, J_0 the set of unconstrained points and J_ϕ the set of constrained ones.

$$\delta L^{(TOT)} = \sum_{i=1}^{n} \vec{F}_i \cdot \delta P_i = \sum_{P_i \in J_0} \vec{F}_i \cdot \delta P_i + \sum_{P_j \in J_\phi} \vec{F}_j \cdot \delta P_j \ ; \qquad \vec{F}_j = \vec{F}_j^{(a)} + \vec{\Phi}_j$$

Equilibrium for $P_j \in J_\phi$ implies $\vec{F}_j^{(a)} = -\vec{\Phi}_j$, while $\vec{F}_i = \vec{F}_i^{(a)} = \mathbf{0}$ for $P_i \in J_0$. Then, $\delta L^{(TOT)} = 0$ and we may write:

$$\delta L^{(a)} = \sum_{P_i \in J_0} \mathbf{0} \cdot \delta P_i + \sum_{P_j \in J_\phi} \vec{F}_j^{(a)} \cdot \delta P_j = -\sum_{P_i \in J_\phi} \vec{\Phi}_j \cdot \delta P_j = -\delta L^{(v)} \leq 0$$

For the opposite implication, we take $\delta L^{(a)} \leq 0$ ("=" for reversible δP_i). Let us then begin from unconstrained points, whose virtual displacements are reversible, because no mechanism prevents any displacement. Then, we obtain

$$\delta L_0^{(a)} = \sum_{P_i \in J_0} \vec{F}_i^{(a)} \cdot \delta P_i = 0$$

because $-\delta P_i$ is allowed whenever δP_i is allowed, and the only way to realize $\delta L^{(a)} \leq 0$ is that $\delta L^{(a)} = 0$. Now, consider a single $P_k \in J_0$. If we move it keeping all other points fixed, we obtain: $\vec{F}_k \cdot \delta P_k = 0$, and this holds $\forall \delta P_k$, which implies $\vec{F}_k = \mathbf{0}$ for all $P_k \in J_0$. Therefore, the work of the active forces, if any, is limited to the work done by them on the constrained points only. Consequently, according to the assumed inequality for $\delta L^{(a)}$, we may write:

$$\textstyle\sum_{P_i \in J_\phi} \vec{F}_i \cdot \delta P_i = \delta L^{(a)} \leq 0 \ ; \quad \text{where} \ \ \vec{F}_i = \vec{F}_i^{(a)}$$

Now assume that an ideal constraint exerts any force necessary to obtain the desired limitation of the motion, in particular, that it can take the value $\vec{\Phi}_i^* = -\vec{F}_i$ if needed. Is that really the case? Only rule constraints must satisfy is: $\delta L^{(v)} \geq 0$, because they are ideal. Then, $\vec{\Phi}_i^* = -\vec{F}_i$ works, since it yields:

$$\delta L^{(v*)} \equiv \sum_{P_i \in J_\phi} \vec{\Phi}_i^* \cdot \delta P_i = -\sum_{P_i \in J_\phi} \vec{F}_i \cdot \delta P_i = -\delta L^{(a)} \geq 0$$

The question remains whether this is the actually the case, because the uniqueness of solutions is not guaranteed. So, this is not exactly a theorem but a principle… within a mathematical theory. On the other hand, we are speaking of a mathematical model. Real constraints, in reality, do what they do.

STATICS OF HOLONOMIC SYSTEMS: recall that the virtual works of the active forces for a holonomic system are expressed as:

$$\delta L^{(a)} = \sum_{h=1}^{N} Q_h \delta q_h = Q \cdot \delta q \tag{6.5}$$

Moreover, bilateral constraints imply reversible displacements, hence statics requires

$$\delta L^{(a)} = Q \cdot \delta q = 0 \quad \forall \delta q \, .$$

As the q_h are free coordinates, we may vary only one of them: $\delta q_k \neq 0$, keeping the others fixed: $\delta q_h = 0$ for $h \neq k$ which implies $Q_h = 0$. Repeating $\forall h$, we obtain:

$$\begin{cases} Q_1(q_1, \cdots, q_N) = 0 \\ \qquad \vdots \\ Q_N(q_1, \cdots, q_N) = 0 \end{cases} \tag{6.6}$$

Differently, for unilateral constraints, we must distinguish *ordinary* from *boundary* configurations. Let Ω be the set of all allowed configurations, Ω^0 the set of the ordinary (interior) ones, and $\partial\Omega$ the boundary ones, so that $\Omega = \Omega^0 \cup \partial\Omega$.

For a configuration $q \in \Omega^0$, all displacements are allowed, and one has reversible displacements. Therefore, equilibrium is obtained if and only if $Q(q) = 0, \forall q \in \Omega^0$. For boundary configurations, $q \in \partial\Omega$, let us suppose first that Ω be of the following kind:

$$\Omega = \{q_h \in [a_h, b_h], h = 1, \cdots, N\}; \text{ with } a_h, b_h \in \mathbb{R}$$

Then, $q \in \partial\Omega$ means that $q_h = a_h$ or $q_h = b_h$ for some h. For instance, let each of the first $j \leq N$ coordinates equal one value a_h , and let the remaining coordinates belong to (a_h, b_h):

$$q_h = a_h \text{ for } h = 1, \cdots, j \leq N, \text{ and } q_h \in (a_h, b_h) \text{ for } h = j+1, \cdots, N$$

In this case, the first j virtual displacements cannot be negative, $\delta q_h \geq 0$ for $h = 1, \cdots, j$, while the remaining ones, δq_h for $h = j+1, \cdots, N$, are arbitrary and reversible. This means that:

$$\sum_{h=j+1}^{N} Q_h \delta q_h = 0, \qquad \forall \delta q_h$$

which implies:

$$\begin{cases} Q_{j+1}(a_1, \cdots, a_j; q_{j+1}, \cdots, q_N) = 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ Q_N(a_1, \cdots, a_j; q_{j+1}, \cdots, q_N) = 0 \end{cases} \tag{6.7}$$

This is a system of $N - j$ homogeneous equations for $N - j$ unknowns. Now, irreversible displacements allow $q_h > 0$ for $h \in \{1, \cdots, j\}$. Let us take $h = 1$ and $\delta q_k = 0$ for $k > 1$, which is a possible set of virtual displacement. Then, we have

$$\delta L^{(a)} = Q_1 \delta q_1 \leq 0, \qquad\qquad \forall \delta q_1 > 0,$$

which means $Q_1 \leq 0$. Repeating this argument for all $h \leq j$, we obtain the following set of inequalities:

$$\begin{cases} Q_1(a_1, \cdots, a_j; q_{j+1}, \cdots, q_N) \leq 0 \\ \qquad \vdots \\ Q_j(a_1, \cdots, a_j; q_{j+1}, \cdots, q_N) \leq 0 \end{cases} \tag{6.8}$$

In the end, given the solutions $(q^*_{j+1}, \cdots, q^*_N)$ of Eq.(6.7), we find that the configuration

$$q = (a_1, \cdots, a_j; q^*_{j+1}, \cdots, q^*_N)$$

is an equilibrium if also Eq.(6.8) verified. We thus conclude that:

Proposition 6.1: *boundary configurations are equilibria if:*

$$Q_h = 0 \text{ *when* } \delta q_h \text{ *is reversible*}$$
$$Q_h \leq 0 \text{ *when* } \delta q_h \geq 0$$
$$Q_h \geq 0 \text{ *when* } \delta q_h \leq 0$$

In the case of conservative forces, that derive from a potential U, one has

$$Q_h = \frac{\partial U}{\partial q_h}$$

THEOREM 6.2: *Bulk configurations C of a holonomic system subjected to conservative forces are equilibria if and only if*

$$\boxed{\left.\frac{\partial U}{\partial q_h}\right|_C = 0} \tag{6.9}$$

For instance, given the potential profile represented by the blue line in Figure 6.3, let the red dots lying on it identify bulk equilibria. In turn, $q = b$ is also an equilibrium because

$$\left.\frac{\partial U}{\partial q_h}\right|_b > 0 \text{ and } \delta q \leq 0$$

which implies

$$\delta L^{(a)} = Q\delta q \ \leq 0$$

On the contrary, $q = a$ has

$$\left.\frac{\partial U}{\partial q_h}\right|_a > 0 \text{ but } \delta q \geq 0$$

the virtual work of the active forces is positive, $\delta L^{(a)} = Q\delta q \geq 0$, which means that $q = a$ is not an equilibrium: the "wall" at b "repels" the system.

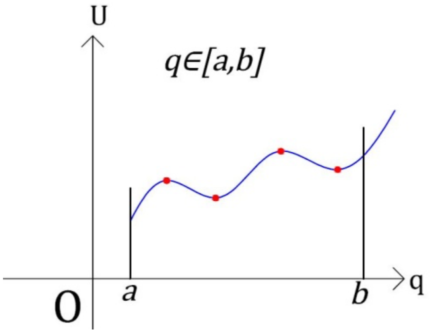

Figure 6.3. *equilibria for potential with boundaries*

A new question arises: is a given equilibrium stable? To make sense of this question, we need some definition.

DEFINITION 6.4: *let $I(q^*, \varepsilon)$ be a neighborhood of the equilibrium configuration q^* of radius ε. This equilibrium is called stable in the static sense if $\exists \varepsilon > 0$ s.t. $\forall \tilde{q} \in I(q^*, \varepsilon) \backslash \{q^*\}$ (I without q^*) the effective work needed to take the configuration from q^* to $\tilde{q}$ is negative:*

$$\boxed{\Delta L^{(a)}_{q^* \to \tilde{q}} < 0} \tag{6.10}$$

Note: here we do not speak of virtual but of real work, which is finite and not infinitesimal. If Eq.(6.10) holds, q is driven back to q^*, if moved away from q^*. Differently, the virtual works principle identifies equilibrium looking at q^* only and testing the ≤ 0 inequality. If the force derives from a potential, we can then state:

Proposition 6.2: *the equilibrium configuration q^* is statically stable for a holonomic conservative system if and only if $U(q^*)$ is a relative isolated maximum or, equivalenty, the potential energy $V(q^*) = -U(q^*)$ is an isolated relative minimum.*

Note: this holds also for boundary configurations. Figure 6.4 illustrates what happens in the case of a ball under the action of the gravitational force, when it is inside an idealized cup and when it is on an idealized hill, whose shape resembles that of the potential energy.

Figure 6.4. *stable and unstable equilibria*

Ex1: Consider a beam of length $2l$, mass m, subjected to elastic force $\vec{F}_e = -kx\hat{\imath}_1$ and to unilateral constraints such that $\vartheta \in \left[0, \frac{\pi}{2}\right]$. The beam further slips with no friction along both vertical and horizontal rails. The active forces are conservative and include the spring and the gravitational force:

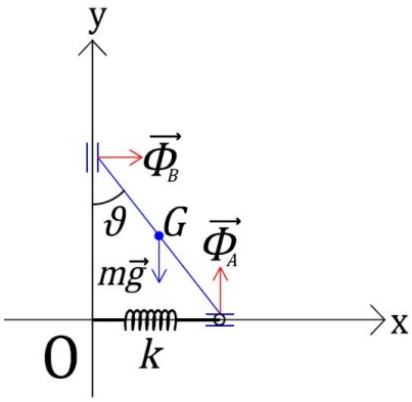

Figure 6.5. *beam sliding on horizontal and vertical rails*

$$U(\vartheta) = -mgl\cos\vartheta - \frac{k}{2}(4l^2\sin^2\vartheta) = -mgy_G - \frac{k}{2}x_A^2$$

Note, the sign of potential terms must be such that the derivatives with respect to the cartesian coordinates, which are the real forces, point in their correct direction. The derivatives with respect to the Langrangian coordinates are not forces, but generalized forces. Let us introduce the parameter

$$\lambda = \frac{mg}{4kl} \text{ (which is >0) hence } mg = 4kl\lambda$$

to compare weight and elastic force. We can write:

$$U = -4kl^2\lambda\cos\vartheta - \frac{k}{2}4l^2\sin^2\vartheta = 4kl^2\left[-\lambda\cos\vartheta - \frac{1}{2}\sin^2\vartheta\right]$$

and

$$Q = \frac{\partial U}{\partial \vartheta} = 4kl^2\left[\lambda\sin\vartheta - \frac{1}{2}2\sin\vartheta\cos\vartheta\right] = 4kl^2\sin\vartheta\,[\lambda - \cos\vartheta]$$

For ordinary configurations, *i.e.* away from the boundaries, $\vartheta \in \left(0, \frac{\pi}{2}\right)$, equilibrium requires $Q = 0$, which means:

$$\lambda + \cos\vartheta = 0, \text{ whose unique solution is given by } \vartheta_1^* = \cos^{-1}(-\lambda)\,.$$

This is an ordinary equilibrium configuration, but such a solution exists if and only if $0 < \lambda < 1$. For $\lambda \geq 1$, this *ordinary* solution does not exist, and only boundary equilibrium configurations are allowed. In particular,

$$\vartheta_2^* = 0 \quad \text{(which means vertical configuration)}$$

is always a solution. The configuration

$$\vartheta_3^* = \frac{\pi}{2}$$

must also be considered. In this case, the virtual displacement can only be negative, because ϑ cannot exit the interval $\left[0, \frac{\pi}{2}\right]$. As we have

$$Q\left(\frac{\pi}{2}\right) = 4kl^2\lambda > 0$$

this leads to the conclusion that ϑ_3^* is a stable equilibrium, since it corresponds to negative work for all allowed displacements, which are negative: $\delta\vartheta < 0$. In other words, moving a bit away from this boundary, the active forces tend to drive the system back to it. If almost horizontal, gravity wins over the spring. Note: in this case we cannot use the test of the second derivative U'', because $U(\vartheta_3^*)$ is neither a local maximum nor a local minimum. Let us examine the stability of the other equilibrium configurations. First, observe that:

$$U'' = 4kl^2\cos\vartheta\,(\lambda - \cos\vartheta) + 4kl^2\sin\vartheta\sin\vartheta$$

For $\lambda < 1$, let us take $\lambda = \cos\vartheta_1^*$. This implies:

$$U''(\vartheta_1^*) = 4kl^2\sin^2\vartheta_1^* = 4kl^2(1 - \lambda^2) > 0$$

Which means that the equilibrium is unstable. For $\vartheta_2^* = 0$, one has $Q = 0$ so the effective work is not immediately clear. However, one also has $U''(0) = 4kl^2(\lambda - 1)$, which implies that

$\lambda < 1$ corresponds to stable equilibrium
$\lambda > 1$ corresponds to unstable equilibrium

For $\lambda = 1$, we have $U'(0) = 0$, but $U'(\vartheta) > 0$ for $\vartheta > 0$ and $U'(\vartheta) < 0$ for $\vartheta < 0$, which means that $U'(0)$ is a minimum of the potential, hence the equilibrium is unstable. These results may be understood considering that λ is the ratio of the gravitational to elastic forces. Large λ means that the elastic force cannot balance the weight of the beam, hence the vertical equilibrium is unstable. Small λ means the opposite: the weight is not too big for the spring to keep it standing. For $\lambda = 1$ the horizontal equilibrium is stable: if moved up a bit, the beam falls back to the ground, as it is already too heavy for the spring. This stability is due to the unilateral constraint, that stops the beam when it reaches the horizontal position.

Why even the light bar (λ <1) is stable at $\vartheta_3^* = \pi/2$? Can't the spring lift it up if the bar is infinitesimally displaced up in B? It cannot. An infinitesimal displacement of the angle ϑ does not alter the fact that the active forces are balanced by the constraints. Therefore, it may grow only if the moment generated by the spring exceeds the moment generated by gravity. Now, the arm of the pair generated by gravity remains l: the vertical distance of G from A (the center of rotations) is first order, while the horizontal distance is second order in $\delta\vartheta$:

$$x_A - x_G = l\sin\left(\frac{\pi}{2} - \delta\vartheta\right) = l - \frac{1}{2}\delta\vartheta^2\,; \qquad y_A - y_G = l\,\delta\vartheta$$

Hence the gravitational moment tends to a positive value $mg \times l$, while the elastic moment $k2l \times l\delta\vartheta$ vanishes for $\delta\vartheta \to 0$. For sufficiently small $\delta\vartheta$, the moment of gravity is bound to exceed that of the spring; an infinitesimal displacement will not be enough for whatever spring to win over gravity!

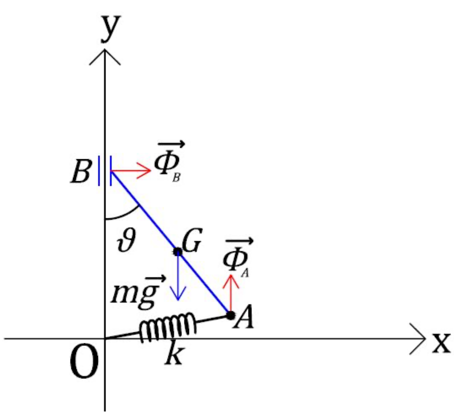

Figure 6.6. *beam sliding on vertical rail*

For the constraints, let us consider the configuration ϑ_1^* and do as if the constraint in A does not exist, but there is one extra active force $\vec{\Phi}_A$, say. In this situation, there are two degrees of freedom $(y = y_B, \vartheta)$. Then, we can write:

$$\begin{aligned}\delta L^{(a)} &= \delta(G - O)\cdot m\vec{g} + \left(k(O - A) + \vec{\Phi}_A\right)\cdot \delta A\\ &= Q_y\delta y + Q_\vartheta\delta\vartheta\end{aligned}$$

Here, the first contribution to the virtual work is:

$$\begin{aligned}\delta L_y &= Q_y\delta y\\ &= m\vec{g}\cdot(\delta y\hat{\imath}_2) + k(O - A)\cdot(\delta y\hat{\imath}_2) + \Phi_A\cdot(\delta y\hat{\imath}_2)\end{aligned}$$

where we move y keeping ϑ fixed, so to obtain:

$$\delta L_y = Q_y\delta y = \left[\Phi_{Ay} - mg - k(y - 2l\cos\vartheta)\right]\delta y$$

Analogously, the second contribution, obtained keeping y fixed and varying ϑ, is given by:

$$\begin{aligned}\delta L_\vartheta = Q_\vartheta\delta\vartheta = m\vec{g}\cdot &\overbrace{[y - l\cos(\vartheta + \delta\vartheta) - (y - l\cos\vartheta)]}^{take\ difference\ with\ fixed\ y\ and\ moving\ \vartheta}\ \hat{\imath}_2\\ &+\vec{\Phi}_A\cdot\hat{\imath}_2\,[y - 2l\cos(\vartheta + \delta\vartheta) - (y - 2l\cos\vartheta)] - k\,\overbrace{2l\sin\vartheta}^{>0}\,\hat{\imath}_1\cdot\overbrace{[2l\sin(\vartheta + \delta\vartheta) - 2l\sin\vartheta]}^{>0\ in\ (0,\pi/2)\ for\ \delta\vartheta>0}\,\hat{\imath}_1\\ &-k\,\overbrace{(y - 2l\cos\vartheta)}^{>0}\,\hat{\imath}_2\cdot\overbrace{[y - 2l\cos(\vartheta + \delta\vartheta) - (y - 2l\cos\vartheta)]}^{<0\ for\ \delta\vartheta>0}\,\hat{\imath}_2\\ &= -mgl[-(\cos\vartheta - \sin\vartheta\cdot\delta\vartheta) + \cos\vartheta] + \Phi_{Ay}2l[-(\cos\vartheta - \sin\vartheta\cdot\delta\vartheta) + \cos\vartheta]\\ &\quad -4kl^2\sin\vartheta\,[\sin\vartheta + \cos\vartheta\,\delta\vartheta - \sin\vartheta] - 2lk(y - 2l\cos\vartheta)[-(\cos\vartheta - \sin\vartheta\,\delta\vartheta) + \cos\vartheta]\\ &= -mgl\sin\vartheta\,\delta\vartheta + 2l\Phi_{Ay}\sin\vartheta\,\delta\vartheta - 4kl^2\sin\vartheta\cos\vartheta\,\delta\vartheta - 2lky\sin\vartheta\,\delta\vartheta + 4kl^2\cos\vartheta\sin\vartheta\,\delta\vartheta\\ &= \left[-mgl\sin\vartheta + 2l\sin\vartheta\,\Phi_{Ay} - 2kly\sin\vartheta\right]\delta\vartheta\end{aligned}$$

where a Taylor expansion to first order has been done for small $\delta\vartheta$. Note: although A moves both vertically and horizontally when ϑ is perturbed at constant y, the scalar products of $\delta(A-O)$ with the gravity force and with the constraint reaction in A only need the vertical component of the displacement. Also $m\vec{g}\cdot\hat{\imath}_2=-mg$ and $\vec{\Phi}_A\cdot\hat{\imath}_2=\Phi_{Ay}$. Now, the point A lies on the horizontal rail if $y=2l\cos\vartheta$, which implies $Q_y=\Phi_{Ay}-mg$. At equilibrium, one also has $Q_y=0$, hence one obtains:

$$\Phi_{Ay}=mg$$

Then, we also have:

$$\begin{aligned}Q_\vartheta&=-l[mg\sin\vartheta-2mg\sin\vartheta+2ky\sin\vartheta]\\&=-l\sin\vartheta\,[-mg+2k2l\cos\vartheta]=-l\sin\vartheta\,[4lk\cos\vartheta-mg]\end{aligned}$$

which automatically verifies $Q_\vartheta=0$ for $\vartheta=0$ and for $\cos\vartheta=mg/4lk$. The reaction in B can be computed analogously, replacing the constraint $\vec{\Phi}_B$ with an active force.

6.1 Cardinal equations of statics

The first equation of statics requires the sum of external forces to vanish, and the second requires the total moment of the external forces to vanish:

$$\boxed{\vec{R}^{(e)}=\mathbf{0}}\qquad\text{first cardinal equation of statics}\qquad(6.11a)$$

$$\boxed{\vec{M}_\Omega^{(e)}=\mathbf{0}}\qquad\text{second cardinal equation of statics}\qquad(6.11b)$$

where Ω is a given reference point.

THEOREM 6.3*: a rigid system is in equilibrium if and only if its cardinal equations of statics hold.*

Proof: NECESSITY: equilibrium needs the total vectors $\vec{R}$ and $\vec{M}_\Omega$ to vanish at all points. But internal forces exert pairs of opposite forces, generating torques of vanishing arm, therefore, one does not need to worry about them. But non-equilibrated forces imply motion, according to the second law.

SUFFICIENCY: the worst case for statics is that of ideal constraints. Indeed, experience tells that friction does not alter equilibria; if anything, it contributes to the stability of the equilibria obtained without friction, and a consistent mathematical model of friction must produce that effect. Therefore, let us assume ideal constraints and suppose the cardinal equations hold:

$$\vec{R}^{(e,a)}+\vec{R}^{(e,v)}=\mathbf{0}\quad\text{and}\quad\vec{M}_\Omega^{(e,a)}+\vec{M}_\Omega^{(e,v)}=\mathbf{0}$$

having distinguished active and constraining forces. The virtual work on a rigid body is given by

$$\delta L=\vec{R}\cdot\overrightarrow{\delta\Omega}+\vec{M}_\Omega\cdot\vec{\varepsilon}'$$

implying:

$$\mathbf{0}\cdot\overrightarrow{\delta\Omega}=\left(\vec{R}^{(e,a)}+\vec{R}^{(e,v)}\right)\cdot\overrightarrow{\delta\Omega}=0\,;\quad\mathbf{0}\cdot\vec{\varepsilon}'=\vec{M}_\Omega^{(e,a)}\cdot\vec{\varepsilon}'+\vec{M}_\Omega^{(e,v)}\cdot\vec{\varepsilon}'=0$$

In this case, the virtual work of (external) active and constraining forces vanishes:

$$\delta L=\delta L^{(e,a)}+\delta L^{(e,v)}=0$$

Internal forces make no work, because in a rigid body there is no relative motion of different points, and this implies:

$$\delta L^{(i,a)}=0=\delta L^{(i,v)}$$

The total virtual work eventually yields:

$$\delta L^{(a)} + \delta L^{(v)} = \delta L^{(e,a)} + \delta L^{(i,a)} + \delta L^{(e,v)} + \delta L^{(i,v)} = 0$$

By definition, $\delta L^{(v)} \geq 0$ for ideal constraints, hence $\delta L^{(a)} \leq 0$, which means we have an equilibrium.

Ex2: two bars of length $2l$ and mass m, joined by a hinge and by a spring. One end of the first bar is only allowed to rotate about the origin of the axis. Search for equilibrium positions in $\vartheta \in \left(0, \frac{\pi}{2}\right)$.

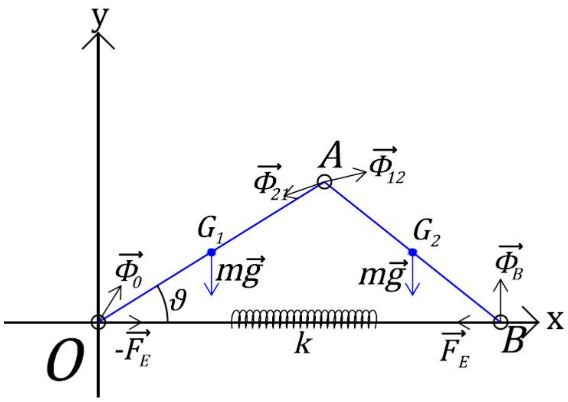

Figure 6.7. *mechanism with two bars*

There are various unknowns, including:

$$\vartheta, \Phi_{By}, \Phi_{0x}, \Phi_{0y}$$

In A, the principle of action and reaction yields

$$\vec{\Phi}_{12} = -\vec{\Phi}_{21}$$

which are also unknown. Statics requires:

$$\begin{cases} m\vec{g} + k\overrightarrow{OB} + \vec{\Phi}_0 + \vec{\Phi}_{21} = \mathbf{0} \\ \overrightarrow{OG_1} \wedge m\vec{g} + \overrightarrow{OA} \wedge \vec{\Phi}_{21} = \mathbf{0} \end{cases} \quad \text{for bar 1}$$

and

$$\begin{cases} m\vec{g} - k(B - O) + \vec{\Phi}_B - \vec{\Phi}_{21} = \mathbf{0} \\ (G_2 - B) \wedge m\vec{g} - (A - B) \wedge \vec{\Phi}_{21} = \mathbf{0} \end{cases} \quad \text{for bar 2}$$

where

$$(A - O) = (2l\cos\vartheta, 2l\sin\vartheta); \quad (G_1 - O) = (l\cos\vartheta, l\sin\vartheta)$$
$$(B - O) = (4l\cos\vartheta, 0); \quad (G_2 - O) = (3l\cos\vartheta, l\sin\vartheta)$$

Project these equations on the two axes. We obtain:

$$\left(m\vec{g} + k(B - O) + \vec{\Phi}_0 + \vec{\Phi}_{21}\right) \cdot \hat{\imath}_1 = 0 \text{ implies } \boxed{4lk\cos\vartheta + \Phi_{0x} + \Phi_{21,x} = 0} \tag{6.12}$$

$$\left(m\vec{g} + k(B - O) + \vec{\Phi}_0 + \vec{\Phi}_{21}\right) \cdot \hat{\imath}_2 = 0 \text{ implies } \boxed{-mg + \Phi_{0y} + \Phi_{21,y} = 0} \tag{6.13}$$

$$(G_1 - O) \wedge m\vec{g} = \begin{vmatrix} \hat{\imath}_1 & \hat{\imath}_2 & \hat{\imath}_3 \\ l\cos\vartheta & l\sin\vartheta & 0 \\ 0 & -mg & 0 \end{vmatrix} = \hat{\imath}_3(-mgl\cos\vartheta)$$

hence

$$(G_1 - O) \wedge m\vec{g}) \cdot \hat{\imath}_3 = -mgl\cos\vartheta$$

$$(A - O) \wedge \vec{\Phi}_{21} = \begin{vmatrix} \hat{\imath}_1 & \hat{\imath}_2 & \hat{\imath}_3 \\ 2l\cos\vartheta & 2l\sin\vartheta & 0 \\ \Phi_{21,x} & \Phi_{21,y} & 0 \end{vmatrix} = \hat{\imath}_3(2l)\left(\Phi_{21,y}\cos\vartheta - \Phi_{21,x}\sin\vartheta\right) \tag{6.14}$$

$$\left(m\vec{g} + k\overrightarrow{OB} + \vec{\Phi}_0 + \vec{\Phi}_{21}\right) \cdot \hat{\imath}_2 = 0 \;\Rightarrow\; \boxed{-mg + \Phi_{0y} + \Phi_{21,y} = 0}$$

The second equation becomes

$$\boxed{-mgl\cos\vartheta + 2l\left(\Phi_{21,y}\cos\vartheta - \Phi_{21,x}\sin\vartheta\right) = 0}$$

Analogously for bar 2 we have

$$\boxed{\begin{array}{l} 4kl\cos\vartheta + \Phi_{21,x} = 0, \qquad -mg + \Phi_{By} - \Phi_{21,y} = 0 \\ mg\cos\vartheta + 2\left(\Phi_{21,y}\cos\vartheta + \Phi_{21,x}\sin\vartheta\right) = 0 \end{array}} \tag{6.15}$$

Let us introduce the parameter

$$\lambda = \frac{mg}{8kl}$$

which measures the strength of weight compared to that of the spring. For $\lambda < 1$ (light weight, compared to the spring), the first of Eq.(6.15) yields:

$$\boxed{\Phi_{21,x} = -4kl\cos\vartheta} \tag{6.16}$$

and the second line of Eq.(16.5) becomes

$$mg\cos\vartheta + 2(\Phi_{21,y}\cos\vartheta - 4kl\cos\vartheta\sin\vartheta) = 0$$

which immediately leads to

$$\boxed{\Phi_{21,y} = \frac{1}{2}(8kl\sin\vartheta - mg)} \tag{6.17}$$

In turn, the third of Eqs.(6.15) becomes:

$$\begin{aligned}
&-mg\cos\vartheta + 2\left(\frac{1}{2}(8kl\sin\vartheta - mg)\cos\vartheta + 4kl\cos\vartheta\sin\vartheta\right) = 0\\
&-mg + 8kl\sin\vartheta - mg + 8kl\sin\vartheta = 0\\
&16kl\sin\vartheta = 2mg
\end{aligned}$$

and, eventually:

$$\boxed{\sin\vartheta = \frac{mg}{8kl} = \lambda} \tag{6.18}$$

The second of Eq.(6.15)

$$\Phi_{By} = mg + 4kl\frac{mg}{8kl} - \frac{mg}{2} \quad \text{gives:} \quad \boxed{\Phi_{By} = mg} \tag{6.19}$$

In turn, Eq.(6.16) yields

$$\boxed{\Phi_{21,x} = -4kl\sqrt{1-\lambda^2}} \tag{6.20}$$

Eq.(6.17) yields

$$\boxed{\Phi_{21,y} = 4kl\frac{mg}{8kl} - \frac{1}{2}mg = 0} \tag{6.21}$$

Eq.(6.12) yields:

$$\boxed{\Phi_{0x} = 4kl\sqrt{1-\lambda^2} - 4kl\sqrt{1-\lambda^2} = 0} \tag{6.22}$$

and Eq.(6.13) yields:

$$\boxed{\Phi_{0y} = mg - 0 = mg} \tag{6.23}$$

Note also that

$$\begin{aligned}
U &= -\frac{k}{2}|B-O|^2 - mg(y_{G_1} + y_{G_2}) = -k8l^2\cos^2\vartheta - mg(2l\sin\vartheta)\\
&= -8kl^2\cos^2\vartheta - 8kl\lambda 2l\sin\vartheta = -8kl^2(\cos^2\vartheta + 2\lambda\sin\vartheta)
\end{aligned}$$

so that

$$Q = \frac{\partial U}{\partial\vartheta} = -8kl^2(-2\cos\vartheta\sin\vartheta + 2\lambda\cos\vartheta) = 16kl^2(\sin\vartheta - \lambda)\cos\vartheta$$

which in $\vartheta \in (0, \pi/2)$ vanishes only when $\sin\vartheta = \lambda$, in case $\lambda < 1$. Otherwise, only the boundaries can be equilibria. To be equilibrium, one boundary configuration requires $\delta L^{(a)} \leq 0$. For $\vartheta = 0$, $\delta\vartheta \geq 0$ and $Q < 0$, so $\delta L^{(a)} \leq 0$ independently of λ. Therefore, $\vartheta = 0$ is an equilibrium configuration. Moreover, moving away from this point with $\delta\vartheta > 0$, the virtual work is negative, so the equilibrium is also stable. It is correct: bars lie on the floor, and gravity wins over spring, as in previous example Ex1. For $\vartheta = \pi/2$, $\delta\vartheta \leq 0$, while $Q = 0$. Then, $\delta L^{(a)} \leq 0$, and we have an equilibrium. For its stability, we need to move a bit away from the configuration itself and check the virtual work of the active forces. For $\vartheta < \pi/2$, the sign of Q depends on λ. For $\lambda > 1$, Q is negative and $\delta\vartheta < 0$ implies $\delta L^{(a)}_{q^*\to\tilde{q}} < 0$, *i.e.* an unstable equilibrium; correct: gravity cannot be balanced by the spring. For $\lambda < 1$, Q is positive, so $\delta\vartheta < 0$ implies $\delta L^{(a)}_{q^*\to\tilde{q}} < 0$, *i.e.* the equilibrium is stable. The virtual work can also be computed as follows:

$$\delta L^{(a)} = mg \cdot \delta G_1 + mg \cdot \delta G_2 + k(O - B) \cdot \delta B$$

Proposition 6.3: *Given a rigid body subjected to a system of forces, one may replace those forces with an equivalent system preserving the equilibrium configurations.*

DEFINITION 6.5: *Let $\mathcal{B}$ be a rigid body touching a horizontal frictionless plane π. Let $\{P_i\}_{i=1}^n$ be the points of $\mathcal{B}$ touching π. We call **basis polygon** the polygon satisfying:*

- *i.* *The vertices of the polygon belong to $\{P_i\}_{i=1}^n$*
- *ii.* *The polygon is convex*
- *iii.* *The points P_i that are not vertices do not lie outside the polygon*

THEOREM 6.4: *C is an equilibrium configuration for a rigid body $\mathcal{B}$ lying on a plane π if and only if the projection G^* of the center of mass G on π does not lie outside the basis polygon.*

Proof: weight forces are equivalent to their sum applied in G^* because they are all vertical. The constraints due to the plane are also vertical and equivalent to their sum applied in their center $G^{(v)}$ which is not outside the polygon. If G^* were outside polygon, there would be torques and no equilibria.

For instance, equilibrium of the ladder in Fig.6.7, friction at A is needed, so that $\vec{\Phi}_A$ is not vertical, because the total momentum does not vanish, if the forces do not concur in a single point. However, friction at B is not required.

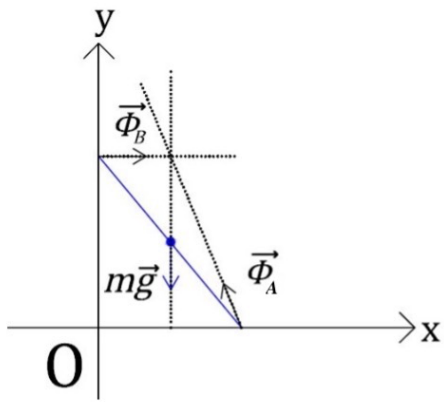

Figure 6.7. *graphical determination of constraint forces*