CHAPTER 4

Geometry of masses

Real objects look *continuous*, and cover a given region of space, with a given mass distribution, or *density*, which may vary from point to point. If $\mathcal{B}$ represents one such domain in space, one may write:

$$m = \int_{\mathcal{B}} \rho \, d\tau \quad \text{where} \quad \rho(x) = \begin{cases} \rho > 0 & if\ x \in \mathcal{B} \\ \rho = 0 & if\ x \notin \mathcal{B} \end{cases} \tag{4.1}$$

The quantity m is the mass of $\mathcal{B}$, ρ is its density and $d\tau$ is a volume element. More generally, one may consider just a part $C \subset \mathcal{B}$, in which case $m_C = \int_C \rho \, d\tau$ is the mass within the portion C of $\mathcal{B}$, and one may consider space and time dependent mass density, $\rho = \rho\ (P,\ t)$. The object is called *homogeneous* if $\rho = const$ in space. Various definitions are in order.

CENTRE OF MASS: *for a system made of* n *point masses* m_i *located in positions* P_i*, the centre of mass* G *is defined by*

$$(G - O) = \frac{1}{m} \sum_{i=1}^{n} m_i (P_i - O); \quad \text{where} \quad m = \sum_{i=1}^{n} m_i \tag{4.2}$$

with coordinates

$$x_i(G) = \frac{1}{m} \sum_{l=1}^{n} m_l x_i(P_l)\ , \quad i = 1,2,3 \tag{4.3}$$

For a continuous body, one analogously defines the center of mass as

$$(G - O) = \frac{1}{m} \int_B \rho (P - O) d\tau\, ; \quad m = \int_B \rho d\tau \tag{4.4}$$

In homogeneous bodies, G is the geometric centre, or "average" point. In inhomogeneous objects, it is a weighted average. In general, the centre of mass has the following properties:

1) The centre of mass of two point masses lies in the line joining them, and is located at inversely proportional distances from their masses; indeed, taking $O \equiv G$ one has:

$$m_1(P_1 - G) + m_2(P_2 - G) = 0$$

2) The centre of mass of a system lying on a plane lies in that plane
3) For a body made of k parts:

$$\mathcal{B} = \bigcup_{i=1}^{k} B_i, \qquad B_i \cap B_j = \emptyset, \qquad i \neq j \tag{4.5}$$

one has

$$\int_{\mathcal{B}} \rho (P - O) d\tau = \sum_{i=1}^{k} \int_{B_i} \rho (P - O) d\tau = \sum_{i=1}^{k} (G_i - O) \int_{B_i} \rho d\tau \tag{4.6}$$

$$\int_{\mathcal{B}} \rho d\tau = \sum_{i=1}^{k} \int_{B_i} \rho d\tau = \sum_{i=1}^{k} m_i = m \tag{4.7}$$

which, denoting by G_i the center of mass of part B_i, yields

$$\boxed{(G - O) = \frac{1}{\sum_{i=1}^{k} m_i} \sum_{i=1}^{k} m_i (G_i - O)} \tag{4.8}$$

4) G is within the boundary $\partial\mathcal{B}$ of $\mathcal{B}$ if this boundary is convex.
5) Every conjugate plane contains G.

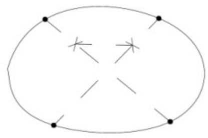


Figure 4.1. *Center of mass lies inside boundary of object if object is convex*

DEFINITION 1: A plane π is called conjugated to direction $\hat{u}$ if the points of $\mathcal{B}$ not belonging to π come in pairs *(P, P')* of equal masses or density, so that $(P'-P) \parallel \hat{u}$ and π cuts in half the segment of length $|(P'-P)|$. π is called symmetry plane if it is conjugated to its normal direction.

Ex1: Triangle. The hypotenuse lies on the straight line through (a, 0), (0, b) *i.e.* the line of equation

$$y = -\frac{b}{a}x + b \tag{4.9}$$

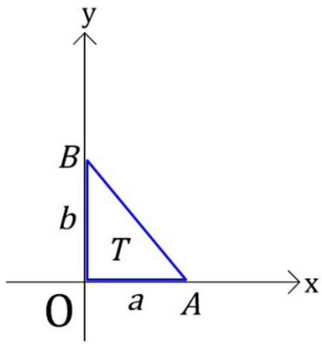


Figure 4.2. *Right triangle with homogeneous density*

Let $\rho = const$ then

$$m = \int_T \rho \overbrace{dxdy}^{d\tau} = \frac{1}{2}\rho ab \tag{4.10}$$

and

$$\begin{aligned} x_G &= \frac{1}{m}\int_0^a \int_0^{-\frac{b}{a}x+b} \rho x dxdy = \frac{2\rho}{\rho ab}\int_0^a x(-\frac{b}{a}x+b)dx \\ &= \frac{2}{a}\left[-\frac{1}{a}\frac{x^3}{3}+\frac{x^2}{2}\right]_0^a = \frac{2}{a}\left[-\frac{1}{a}\frac{a^3}{3}+\frac{a^2}{2}\right] \\ &= -\frac{2}{3}a + a \\ &= \frac{a}{3} \end{aligned} \tag{4.11}$$

$$\begin{aligned} y_G &= \frac{2}{\rho ab}\int_0^a \int_0^{-\frac{b}{a}x+b} \rho y dxdy = \frac{2}{ab}\int_0^a \frac{1}{2}\,(-\frac{b}{a}x+b)^2\,dx = \frac{1}{ab}\int_b^0 t^2\left(-\frac{a}{b}\right)dt = \frac{1}{b^2}\int_0^b t^2\,dt \\ &= \frac{b}{3} \end{aligned} \tag{4.12}$$

Ex2: Homogeneous circular plate of radius R with a hole of radius $R/2$. Because homogenous, the centers of mass of the two disks are their centers:

$$G_{full} = (0,0),\ m_{full} = \pi R^2 \rho \tag{4.13}$$

$$G_{hole} = \left(\frac{R}{2}, 0\right),\ m_{hole} = \pi\frac{R^2}{2}\rho = \frac{\pi R^2}{4}\rho \tag{4.14}$$

Therefore, one obtains:

$$\begin{aligned} (G_{full} - O) = \frac{1}{m_{full}}\big[&m_{figure}(G_{figure} - O) \\ &+ m_{hole}(G_{hole} - O)\big] \end{aligned} \tag{4.15}$$

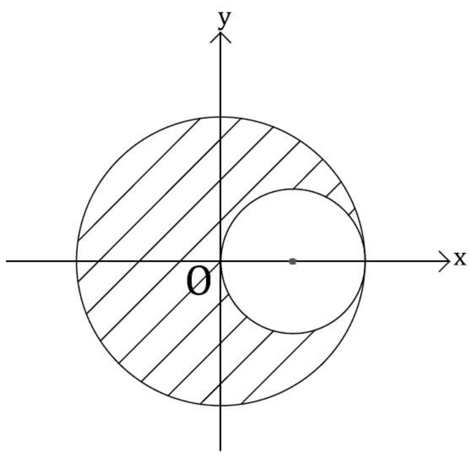


Figure 4.3. *Homogeneous disk with a circular hole*

$$
\begin{aligned}
(G_{figure} - O) &= \frac{1}{m_{figure}}\left[m_{full}(G_{full} - O) - m_{hole}(G_{hole} - O)\right] \\
&= \frac{1}{\pi\left(R^2\rho - \frac{R^2}{4}\rho\right)}\left[\pi R^2\rho(0,0) - \pi\frac{R^2}{4}\rho\left(\frac{R}{2},0\right)\right] = \frac{4}{3}\left[(0,0) - \left(\frac{R}{8},0\right)\right] \\
&= \left(-\frac{R}{6},0\right) \qquad (4.16)
\end{aligned}
$$

MOMENTS OF INERTIA: *the moment of inertia of a point mass P, with respect to an axis a, is defined by:*

$$\boxed{I_a = mr^2} \qquad (4.17)$$

where m is the mass of P, and r *is the distance of P from the straight line a. For a system of n point masses, one takes the sum of the corresponding moments of inertia over all points, and for a continuous object* $\mathcal{B}$*, one takes*

$$\boxed{I_a = \int_B \rho\, r^2 d\tau} \qquad (4.18)$$

The quantity

$$\delta_a = \sqrt{\frac{I_a}{m}} \qquad (4.19)$$

is called g*yration radius*. Then Eq.(4.18) takes the form:

$$I_a = m{\delta_a}^2 \qquad (4.20)$$

For a homogeneous $\mathcal{B}$, *i.e.* an object with uniform mass density ρ, one has

$$I_a = \rho\int_{\mathcal{B}} r^2\, d\tau = \rho i_a \qquad (4.21)$$

Where i_a is called geometric moment of inertia.

Ex3: Consider a rigid beam of length l. If it is homogeneous, one can write:

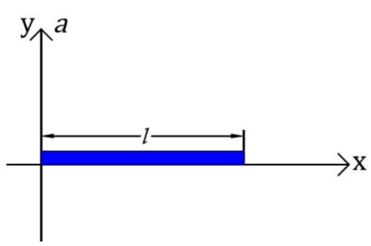
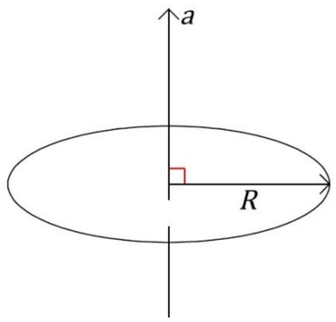


Figure 4.4. *Homogeneous rigid bar*

$$I_a = I_y = \rho\int_0^l x^2 \overbrace{dx}^{d\tau} = \rho\frac{l^3}{3} = \frac{ml^2}{3} \qquad (4.22)$$

where $d\tau$ is the “volume” element and ρ is a *linear* density of mass which, multiplied by l, gives the mass m of the beam.

Ex4: For a homogeneous disk of radius *R*, one has:



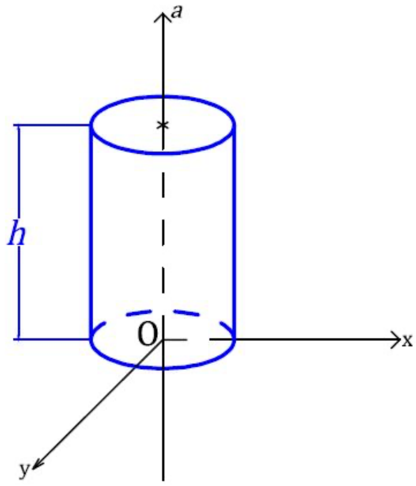
$$I_a = \rho \int_0^{2\pi} \int_0^R r^2 \overbrace{rdrd\varphi}^{d\tau} = 2\pi\rho\frac{R^4}{4} = \frac{\pi}{2}\rho R^4$$

where $d\tau$ is the "volume" element and ρ is the *surface* mass density, so that $\pi\rho R^2$ is the mass m of the disk:

$$m = \pi R^2 \rho \Rightarrow \quad I_a = \frac{mR^2}{2} \tag{4.23}$$

Figure 4.5. *Homogeneous disk perpendicular to axis of reference*

Ex5: For a homogeneous cylinder, of height h and radius R, one has:



$$I_a = \rho \int_0^h \int_0^{2\pi} \int_0^R r^2 \overbrace{rdrd\varphi dz}^{d\tau} = \rho 2\pi h \frac{R^4}{4}$$
$$= \frac{\pi R^2 h\rho}{2} R^2 = \frac{mR^2}{2} \tag{4.24}$$

where $d\tau$ is the volume element, and the total mass is given by density times volume:

$$m = \pi R^2 h\rho. \tag{4.25}$$

Figure 4.6. *Homogeneous cylinder*

HUYGENS-STEINER THEOREM: *The moment of inertia of a body $\mathcal{B}$ with respect to any axis "a" is given by*

$$\boxed{I_a = I_{a_G} + md^2} \tag{4.26}$$

where I_{a_G} is the moment of inertia with respect to a_G, the axis parallel to a containing the center of mass G of $\mathcal{B}$, and d is the distance between the lines a and a_G.

Proof: take the origin of a reference frame in G, and let its z axis be parallel to a. Let $(x_a, y_a, 0)$ be the intersection of the line a with the (x, y) plane. Then, one can write:

$$I_a = \int_{\mathcal{B}} \rho r^2 d\tau = \int_{\mathcal{B}} \rho[(x_a - x)^2 + (y_a - y)^2] \overbrace{d\tau}^{dxdydz}$$
$$= \int_{\mathcal{B}} [x_a{}^2 - 2x_a x + x^2 + y_a{}^2 - 2y_a y + y^2]\rho d\tau$$
$$= \underbrace{(x_a{}^2 + y_a{}^2)}_{d^2} \underbrace{\int_{\mathcal{B}} \rho d\tau}_{m} + \underbrace{\int_{\mathcal{B}} \overbrace{(x^2 + y^2)}^{r^2} \rho d\tau}_{I_{a_G}} - 2x_a \int_{\mathcal{B}} \rho x d\tau - 2y_a \int_{\mathcal{B}} \rho y d\tau$$
$$= md^2 + I_{a_G} - 2mx_a x_G - 2my_a y_G = I_{a_G} + md^2$$

because $x_G = y_G = 0$. This means that the minimum moment of inertia is the one with respect to the axis passing through G, with the chosen direction. Moreover, given two parallel axes, a_1 and a_2, one has

$$I_{a_1} = I_{a_G} + m{d_1}^2 \quad \text{and} \quad I_{a_2} = I_{a_G} + m{d_2}^2$$

which leads to

$$\boxed{I_{a_2} = I_{a_1} + m({d_2}^2 - {d_1}^2)} \tag{4.27}$$

Ex6: consider a homogeneous square of side $3l$, and cut out a square of side l from its lower right corner. Let r be an axis passing through A making 45^o with the x axis. Take $\rho = m/l^2$. The mass of the full square is $m_{full} = 9m$; the removed mass is $m_{hole} = m$; and the object of interest has mass $m_{figure} = 8m$. Moreover, the center of mass of the full square has coordinates

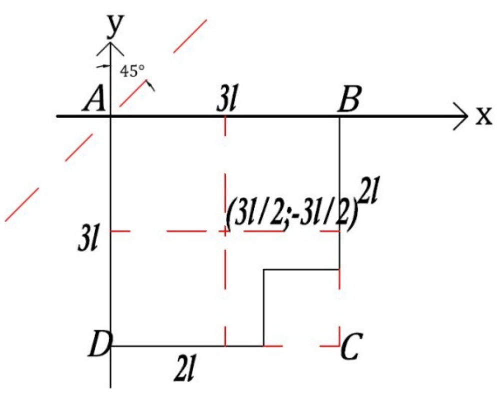


Figure 4.7. *Homogeneous square with a missing corner*

$$x_{G_{full}} = \frac{3}{2}l; \qquad y_{G_{full}} = -\frac{3}{2}l$$

while the center of mass of the removed part has:

$$x_{G_{hole}} = 2l + \frac{l}{2}; \; y_{G_{hole}} = -\frac{5}{2}l$$

Then one can write: $\frac{3}{2}l = \frac{1}{9m}\left[x_{G_{figure}}8m + x_{G_{hole}}m\right]$ which means that $\frac{27}{2}l = 8x_{G_{figure}} + x_{G_{hole}}$ and leads to

$$x_{G_{figure}} = \frac{1}{8}\left(\frac{27}{2}l - \frac{5}{2}l\right) = \frac{11}{8}l$$

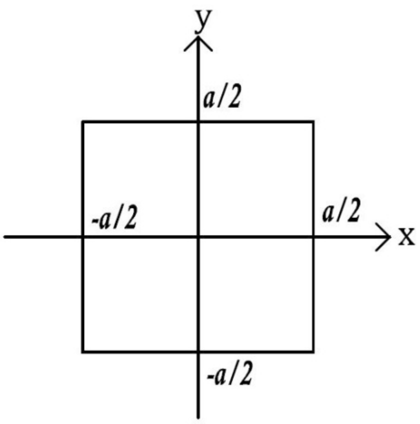


Figure 4.8. *Homogeneous square*

For the moments of inertia, begin from the full square and take the axis r' passing through the center of mass, parallel to the chosen one. Move the orgin of the reference frame to G, as in Figure 4.8, so that r' has equation $y = x$. The distance squared of a point (x_0, y_0) from r' is $(x_0 + y_0)^2/2$ and then

$$I_{r'} = \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} \rho \frac{(x-y)^2}{2} dxdy$$

$$= \rho \int_{-a/2}^{a/2} \left[\frac{x^3}{3} - x^2y + xy^2\right]_{-a/2}^{a/2} dy$$

$$= \rho \int_{-a/2}^{a/2} \left(\frac{a^3}{12} + ay^2\right) dy = \rho\frac{a^4}{12} = \frac{Ma^2}{12}$$

where $a = 3l$, $M = 9m$. Doing the same for the hole, moving the origin of the reference frame to its center of mass, G', introducing r'' through G' parallel to r, and taking $a = l$, $M = m$, we can write:

$$I^{(G)}_{r'full} = \frac{9m9l^2}{12} = \frac{27}{4}ml^2; \quad I^{(G')}_{r''hole} = \frac{1}{12}ml^2 \tag{4.28}$$

Finally, moving from r' and r'' to r, *i.e.* from the centers of mass to point A, the Huygens-Steiner theorem yields

$$I^{(A)}_{full} = \frac{27}{4}ml^2 + 9m\left[\frac{9}{4}l^2 + \frac{9}{4}l^2\right] = \frac{27}{4}ml^2 + \frac{81}{2}ml^2 = \frac{189}{4}ml^2 \tag{4.29}$$

$$I^{(A)}_{hole} = \frac{ml^2}{12} + m\left[\frac{25}{4}l^2 + \frac{25}{4}l^2\right] = \frac{ml^2}{12} + \frac{25}{2}ml^2 = \frac{151}{12}ml^2 \tag{4.30}$$

(d^2 of Eq.(4.26) inside $[\,\cdot\,]$). Now, consider that $I^{(A)}_{full} = I^{(A)}_{figure} + I^{(A)}_{hole}$, hence we can write:

$$I^{(A)}_{figure} = \frac{189}{4}ml^2 - \frac{151}{12}ml^2 = \frac{104}{3}ml^2 \tag{4.31}$$

What if a makes an angle ϑ with the x or y axis?

To rotate by any angle ϑ the axis, let $\hat{u}$ be the direction of a and I_u the corresponding moment of inertia. Observe that the distance of any point P from a is given by:

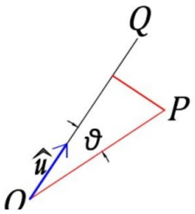


Figure 4.9. *Rotation by an angle*

$$r = |(P-O)\wedge\vec{u}| = |(P-O)|\sin\vartheta \tag{4.32}$$

Let us take a reference frame with origin in O so that we can write:

$$(P-O) = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3; \quad \hat{u} = \alpha\hat{e}_1 + \beta\hat{e}_2 + \gamma\hat{e}_3 \tag{4.33}$$

where $\alpha^2 + \beta^2 + \gamma^2 = 1$. Then, we have:

$$\begin{aligned}(P-O)\wedge\vec{u} &= \begin{vmatrix}\hat{e}_1 & \hat{e}_2 & \hat{e}_3\\ x & y & z\\ \alpha & \beta & \gamma\end{vmatrix}\\ &= (\gamma y - \beta z)\hat{e}_1 - (\gamma x - \alpha z)\hat{e}_2 + (\beta x - \alpha y)\hat{e}_3\end{aligned} \tag{4.34}$$

which yields

$$I_{u=}\int_{\mathcal{B}} \rho\,[(\gamma y - \beta z)^2 + (\gamma x - \alpha z)^2 + (\beta x - \alpha y)^2]d\tau$$

i.e.

$$\begin{aligned}I_u = \alpha^2\int_{\mathcal{B}}\rho(y^2+z^2)\,d\tau + \beta^2\int_{\mathcal{B}}\rho(x^2+z^2)\,d\tau + \gamma^2\int_{\mathcal{B}}\rho(x^2+y^2)\,d\tau - 2\alpha\gamma\int_{\mathcal{B}}\rho xz d\tau\\ -2\alpha\beta\int_{\mathcal{B}}\rho xy d\tau - 2\beta\gamma\int_{\mathcal{B}}\rho yz d\tau\end{aligned} \tag{4.35}$$

Here, the first 3 integrals are the moments of inertia I_x, I_y, I_z with respect to coordinate axes, and the remaining integrals are called *products of inertia*. Denoting by I_{xy}, I_{yz}, I_{xz}, these products of inertia, one writes:

$$\boxed{I_u = \alpha^2 I_x + \beta^2 I_y + \gamma^2 I_z + 2\alpha\gamma I_{xz} + 2\alpha\beta I_{xy} + 2\beta\gamma I_{yz}} \tag{4.36}$$

where

$$\boxed{I_{xz} = -\int_{\mathcal{B}}\rho xz d\tau\,; \quad I_{xy} = -\int_{\mathcal{B}}\rho xy d\tau\,; \quad I_{yz} = -\int_{\mathcal{B}}\rho yz d\tau} \tag{4.37}$$

The products of inertia enjoy the symmetry

$$\boxed{I_{kl} = I_{lk}} \tag{4.38}$$

It is useful to introduce the symmetric matrix

$$\boxed{I_0 = \begin{pmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{pmatrix}} \tag{4.39}$$

so that we can write

$$I_u = \hat{u} \cdot (I_0 \hat{u}) = \hat{u}^T I_0 \hat{u} \tag{4.40}$$

where $\hat{u}$ is here intended as a column vector, and $\hat{u}^T$ its transpose.

Apart cases in which $\mathcal{B}$ is localized at the point O, or lies in a line passing through O, one integrates over positive distances, which implies that I_0 is *positive definite*: being symmetric, it can be diagonalized via rotations, and then the eigenvalues are positive. The reference frame in which I_0 is diagonal is called *principal frame*. Its axes are called *principal axes* and correspond to the *eigendirections* of I_0.

The eigenvalues of I_0 are called *principal moments of inertia.* If $O = G$, one speaks of *central* axes and of *central* moments of inertia, instead of principal axes and moments. Introduce

$$\begin{cases} A = I_x, \qquad B = I_y, \qquad C = I_z & (4.41) \\ C' = -I_{xy}, \quad B' = -I_{xz}, \quad A' = -I_{yz} & (4.42) \end{cases}$$

and the vector

$$(Q - O) = \frac{k}{\sqrt{I_u}} \hat{u} = (x, y, z) = \left(\frac{\alpha}{\sqrt{I_u}}, \frac{\beta}{\sqrt{I_u}}, \frac{\gamma}{\sqrt{I_u}} \right) \tag{4.43}$$

where k is a dimensional constant, of value 1 and dimensions: $[k] = L^2\sqrt{M}$, meant to make $k/\sqrt{I_u}$ dimensionless. Without loss of generality, k can be taken to equal to 1. This means we can write:

$$\alpha = \sqrt{I_u}\, x, \qquad \beta = \sqrt{I_u}\, y, \qquad \gamma = \sqrt{I_u}\, z$$

and Eq.(4.36) as

$$I_u = x^2 I_u I_x + y^2 I_u I_y + z^2 I_u I_z + 2xz I_u I_{xz} + 2xy I_u I_{xy} + 2yz I_u I_{yz} \tag{4.44}$$

i.e.

$$\boxed{x^2 I_x + y^2 I_y + z^2 I_z + 2xz I_{xz} + 2xy I_{xy} + 2yz I_{yz} = 1} \tag{4.45}$$

which is a quadratic surface centered in O. Apart from degenerate cases, this is an ellipsoid called *ellipsoid of inertia*. If $O = G$, it is called *central ellipsoid of inertia.* This geometrical figure is very useful because any axis a_G passing through the center of mass G of $\mathcal{B}$ intersects the central ellipsoid in two pints Q and Q' which obey:

$$\boxed{|(Q - G)| = |(Q' - G)| = 1/\sqrt{I_{a_G}}} \tag{4.46}$$

When I_{a_G} is known, the Huygens-Steiner theorem easily yields I_a, with a not passing through G. Rotating the axes, Eq.(4.45) turns into:

$$I_\xi \xi^2 + I_\eta \eta^2 + I_\varsigma \varsigma^2 = 1 \tag{4.47}$$

where $\xi,\ \eta,\ \varsigma$ are the principal axes of inertia and $I_\xi,\ I_\eta,\ I_\varsigma$ the principal moments.

When the moments with respect to the axes parallel to $\hat{e}_1$ and $\hat{e}_2$ are equal, the axis perpendicular to them (parallel to $\hat{e}_3$) is called *gyroscopic axis,* and the body is called a *gyroscope.*

Symmetries simplify calculations:

1. If π is a symmetry plane for $\mathcal{B}$, the normal to π through any point $O \in \pi$ is a principal axis.
2. The intersection a of two symmetry planes is a principal axis with respect to any $O \in a$.
3. a is a principal axis if and only if its products of inertia with the other axes vanish.
4. Central axes are principal with respect to all points in them.

The calculation of the principal axes can be made computing eigenvalues and eigenvectors of I_0, which are used to diagonalize the matrix of inertia I_0. Once in diagonal form, I_u is easily seen to be continuous in a compact set, hence it takes maximum and minimum value in it, together with all intermediate values. Writing

$$I_u = I_\xi \alpha'^2 + I_\eta \beta'^2 + I_\varsigma \gamma'^2 \tag{4.48}$$

and assuming, without loss of generality, $I_\xi \le I_\eta \le I_\varsigma$, the search for principal axes may also proceed looking for extrema of I_u on a surface of radius 1, because $\alpha'^2 + \beta'^2 + \gamma'^2 = |\hat{u}| = 1$. Then, let us use Lagrange multipliers to find these extremes:

$$\begin{cases} \Phi := I_u - \lambda(\alpha^2 + \beta^2 + \gamma^2 - 1) = 0 & (4.49) \\ I_u = \alpha^2 I_x + \beta^2 I_y + \gamma^2 I_z + 2\alpha\gamma I_{xz} + 2\alpha\beta I_{xy} + 2\beta\gamma I_{yz} & (4.50) \end{cases}$$

and let us compute

$$\frac{\partial \Phi}{\partial \alpha} = 2\left(\alpha I_x + \beta I_{xy} + \gamma I_{xz} - \lambda\alpha\right) = 0 \tag{4.51}$$

$$\frac{\partial \Phi}{\partial \beta} = 2\left(\beta I_y + \alpha I_{xy} + \gamma I_{yz} - \lambda\beta\right) = 0 \tag{4.52}$$

$$\frac{\partial \Phi}{\partial \gamma} = 2\left(\gamma I_z + \alpha I_{xz} + \beta I_{yz} - \lambda\gamma\right) = 0 \tag{4.53}$$

$$\frac{\partial \Phi}{\partial \lambda} = \alpha^2 + \beta^2 + \gamma^2 - 1 = 0 \tag{4.54}$$

The first three equations constitute a homogeneous system, which has non vanishing solutions (α, β, γ) if and only if

$$\begin{vmatrix} I_x - \lambda & I_{xy} & I_{xz} \\ I_{xy} & I_y - \lambda & I_{yz} \\ I_{xz} & I_{yz} & I_z - \lambda \end{vmatrix} = 0 \tag{4.55}$$

which shows the equivalence with the previous method based on eigenvalues and eigenvectors.

Planar system: here, $z = 0$ for all points of the system. Then, taking any 2 axes on the plane π, the third axis is a principal axis with respect to the origin of the reference frame. This allows us to write:

$$I_x = \int_{\mathcal{B}} \rho\, y^2 d\tau, \qquad I_y = \int_{\mathcal{B}} \rho\, x^2 d\tau, \qquad I_z = \int_{\mathcal{B}} \rho\,(x^2 + y^2) d\tau = I_x + I_y \tag{4.56}$$

To determine the other 2 principal axes, let

$$\begin{cases} x = \xi \cos\vartheta + \eta \sin\vartheta \\ y = \xi \sin\vartheta - \eta \cos\vartheta \end{cases}$$

which, substituting in Eq.(4.45), yields:

$$I_x(\xi^2 \cos^2\vartheta + 2\xi\eta \sin\vartheta\cos\vartheta + \eta^2 \sin^2\vartheta) \\ + I_y(\xi^2 \sin^2\vartheta - 2\xi\eta \sin\vartheta\cos\vartheta \\ + \eta^2 \cos^2\vartheta) + 0$$

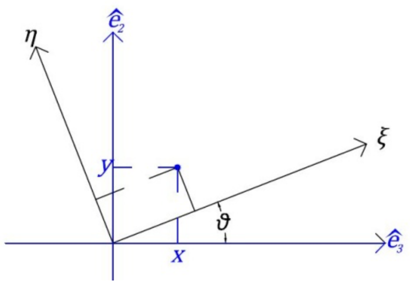


Figure 4.10. *Rotated plane reference frames*

$$+2I_{xy}(\xi^2 \sin\vartheta\cos\vartheta - \xi\eta\cos^2\vartheta + \xi\eta\sin^2\vartheta - \eta^2\sin\vartheta\cos\vartheta) + 0 + 0 = 1 \tag{4.57}$$

i.e.

$$\xi^2\left(I_x\cos^2\vartheta + I_y\sin^2\vartheta + 2I_{xy}\sin\vartheta\cos\vartheta\right) + \eta^2\left(I_x\sin^2\vartheta + I_y\cos^2\vartheta - 2I_{xy}\sin\vartheta\cos\vartheta\right)$$
$$+\xi\eta\left(2I_x\sin\vartheta\cos\vartheta - 2I_y\sin\vartheta\cos\vartheta - 2I_{xy}\cos^2\vartheta + 2I_{xy}\sin^2\vartheta\right) = 1 \tag{4.58}$$

Then some algebra leads to

$$\frac{1}{2}\{I_x + I_y + (I_x - I_y)\cos 2\vartheta + 2I_{xy}\sin 2\vartheta\}\xi^2 + \frac{1}{2}\{I_x + I_y + (I_y - I_x)\cos 2\vartheta - 2I_{xy}\sin 2\vartheta\}\eta^2$$
$$+\{(I_x - I_y)\sin 2\vartheta - 2I_{xy}\cos 2\vartheta\}\xi\eta = 1 \tag{4.59}$$

Apart from the trivial case $I_x = I_y$, one may take

$$\tan 2\vartheta = \frac{2I_{xy}}{I_x - I_y} \quad \text{which means} \quad 2\vartheta = \tan^{-1}\frac{2I_{xy}}{I_x - I_y}$$

and one obtains

$$I_\xi = \frac{1}{2}\left\{I_x + I_y - \sqrt{(I_x - I_y)^2 + 4I_{xy}{}^2}\right\} \tag{4.60}$$

$$I_\eta = \frac{1}{2}\left\{I_x + I_y + \sqrt{(I_x - I_y)^2 + 4I_{xy}{}^2}\right\} \tag{4.61}$$

Ex7: consider a homogeneous right-angled triangle of sides a and b and density ρ. We can write:

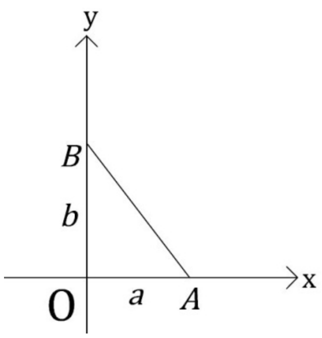


Figure 4.11. *homogeneous right triangle*

$$I_x = \int_0^a \int_0^{-\frac{b}{a}x+b} \rho y^2 dxdy = \int_0^a \rho \frac{y^3}{3}\Big|_0^{-\frac{b}{a}x+b} = \frac{\rho}{3}\int_0^a \left(-\frac{b}{a}x + b\right)^3 dx$$

$$t = -\frac{b}{a}x + b \;\Rightarrow dt = -\frac{b}{a}dx \Rightarrow$$

$$I_x = \frac{\rho}{3}\int_b^0 t^3\left(-\frac{b}{a}\right)dt = \frac{\rho}{3}\frac{a}{b}\int_0^b t^3 dt = \frac{\rho a}{3b}\frac{b^4}{4} = \frac{\rho ab^3}{12} = \frac{mb^2}{6} \tag{4.62}$$

Similarly,

$$I_y = \frac{ma^2}{6} \tag{4.63}$$

Then

$$I_{xy} = -\int_0^a \int_0^{-\frac{b}{a}x+b} \rho xydxdy = -\rho\int_0^a x \int_0^{-\frac{b}{a}x+b} \rho ydydx = -\rho\int_0^a \frac{1}{2}\left(-\frac{b}{a} + b\right)^2 xdx$$

$$= -\rho\int_0^a \frac{x}{2}\left(\frac{b^2}{a^2}x^2 - 2\frac{b^2}{a}x + b^2\right)dx = -\rho\int_0^a \left(\frac{b^2}{2a^2}x^3 - \frac{b^2}{a}x^2 + \frac{b^2}{2}x\right)dx$$

$$= -\rho\left[\frac{a^2b^2}{8} - \frac{a^2b^2}{3} + \frac{a^2b^2}{4}\right] = -m\left(\frac{ab}{4} - \frac{4ab}{6} + \frac{ab}{2}\right) = -\frac{1}{12}mab \tag{4.64}$$

$I_z = I_x + I_y$, because $\mathcal{B}$ lies on a plane, so we obtain

$$I_z = \frac{m}{6}(a^2 + b^2); \quad I_{xz} = I_{yz} = 0 \tag{4.65}$$

and

$$
\begin{aligned}
I_\xi &= \frac{1}{2}\left[\frac{m}{6}(a^2+b^2) - \sqrt{\frac{m^2}{36}(a^2-b^2)^2 + 4\left(\frac{1}{12}\right)^2 m^2a^2b^2}\right] \\
&= \frac{m}{12}\left[(a^2+b^2) - \sqrt{a^4 - 2a^2b^2 + b^4 + a^2b^2}\right] \\
&= \frac{m}{12}\left[(a^2+b^2) - \sqrt{a^4 - a^2b^2 + b^4}\right] \qquad (4.66)
\end{aligned}
$$

Analogously, one finally has:

$$I_\xi = \frac{m}{12}\left[a^2 + b^2 + \sqrt{a^4 - a^2b^2 + b^4}\right] \qquad (4.67)$$

Then $I_\varsigma = I_z$ and the angle between principal axes and (x, y) is

$$2\vartheta = \tan^{-1}\frac{ab}{a^2-b^2} \qquad (4.68)$$