Affine transformations

giovedì 27 aprile 2023 16:12

AFFINE TRANSFORMATIONS

We define AFFINE TRANSFORMATION the geometric function $\mathcal{T}$

$$\mathcal{T}: \quad \underline{x} = \underline{\underline{A}}\,\hat{\underline{x}} + \underline{c} \quad , \quad \forall\, \hat{\underline{x}}, \underline{x} \in \mathbb{E}^d$$

with $\det(\underline{\underline{A}}) \neq 0$, $\underline{\underline{A}} \in \mathbb{R}^{d\times d}$, $\underline{c} \in \mathbb{R}^d$.

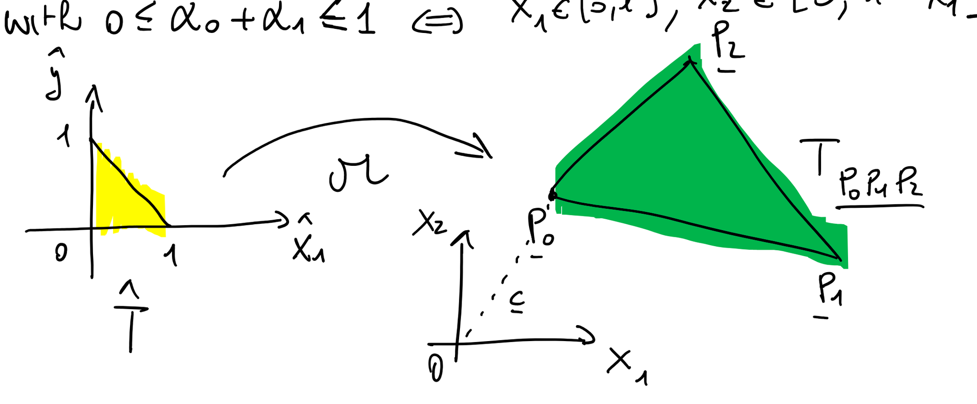
EXAMPLE we recall the definition of the Triangle $\underline{T_{P_0P_1P_2}} \in \mathbb{E}^2$

$$\mathcal{T}: \quad \underline{x} = \alpha_0 \underline{P_2} + \alpha_1 \underline{P_1} + (1-\alpha_0-\alpha_1)\underline{P_0} =$$

$$= \underline{P_0} + \alpha_0(\underline{P_2}-\underline{P_0}) + \alpha_1(\underline{P_1}-\underline{P_0}) =$$

$$= \underbrace{\underline{P_0}}_{\underline{c}} + \underbrace{\left[\underline{P_1}-\underline{P_0}\,,\ \underline{P_2}-\underline{P_0}\right]}_{\underline{\underline{A}}} \underbrace{\begin{pmatrix}\alpha_1\\ \alpha_0\end{pmatrix}}_{\hat{\underline{x}} = (\hat{x}_1, \hat{x}_2)^T}$$

with $0 \le \alpha_0 + \alpha_1 \le 1 \iff \hat{x}_1 \in [0,1],\ \hat{x}_2 \in [0, 1-\hat{x}_1]$

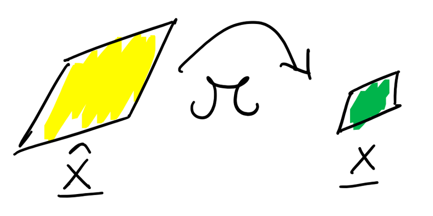


NOTE the condition $\det(\underline{\underline{A}}) \neq 0$ is required to guarantee the existence of $\mathcal{T}^{-1}$

$$\mathcal{T}^{-1}: \quad \underline{x} = \underline{\underline{A}}^{-1}(\hat{\underline{x}} - \underline{c})$$

Particular cases of affine transformations are:

1) SIMILARITY : $\underline{\underline{A}}^T\underline{\underline{A}} = \lambda^2 \underline{\underline{I}}$, maintains the ratio of distances between points conserved. Indeed, if $\underline{c} = \underline{0}$



$$\|\underline{x}\|_2^2 = \underline{x}^T\underline{x} = (\underline{\underline{A}}\hat{\underline{x}})^T(\underline{\underline{A}}\hat{\underline{x}}) = \hat{\underline{x}}^T\underline{\underline{A}}^T\underline{\underline{A}}\hat{\underline{x}} = \lambda^2 \hat{\underline{x}}^T\hat{\underline{x}} = \lambda^2\|\hat{\underline{x}}\|_2^2$$

Moreover, given $\underline{x}, \underline{y} \in \mathbb{E}^d$

$$\begin{cases} \underline{x}\cdot\underline{y} = \|\underline{x}\|_2\|\underline{y}\|_2\cos(\theta) = \lambda^2\|\hat{\underline{x}}\|_2\|\hat{\underline{y}}\|_2\cos(\theta) \\ \underline{x}\cdot\underline{y} = (\underline{\underline{A}}\hat{\underline{x}})^T(\underline{\underline{A}}\hat{\underline{y}}) = \lambda^2\hat{\underline{x}}^T\hat{\underline{y}} = \lambda^2\hat{\underline{x}}\cdot\hat{\underline{y}} = \lambda^2\|\hat{\underline{x}}\|_2\|\hat{\underline{y}}\|_2\cos(\hat{\theta}) \end{cases} \quad \cos(\theta) = \cos(\hat{\theta})$$

thus the angles are preserved

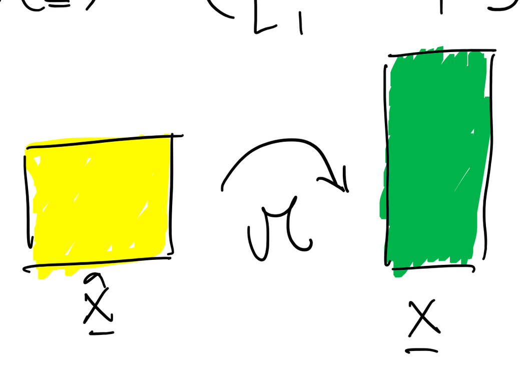
2) EQUIAFFINE : $\det(\underline{\underline{A}}) = \pm 1$

the volume is preserved, indeed given a set of points $\underline{x}_1, \dots, \underline{x}_n$ the volume is defined as

$$V = \det\left(\begin{bmatrix} \underline{x}_1 & \cdots & \underline{x}_n \\ 1 & & 1\end{bmatrix}\right) = \det(\underline{\underline{A}})\det\left(\begin{bmatrix} \hat{\underline{x}}_1 & \cdots & \hat{\underline{x}}_n \\ 1 & & 1\end{bmatrix}\right)$$

$$= \pm\hat{V}$$

thus the VOLUME is preserved

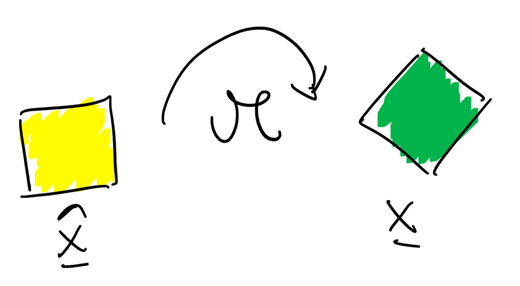


3) RIGID MOTIONS : both SIMILARITY and EQUIAFFINE, thus

$$|\underline{\underline{A}}^T\underline{\underline{A}}| = |\underline{\underline{A}}^T|\,|\underline{\underline{A}}| = |\underline{\underline{A}}|^2 = 1$$

$$\Rightarrow \underline{\underline{A}}^T\underline{\underline{A}} = \underline{\underline{I}} \text{ , rotations}$$

here the DISTANCES are preserved



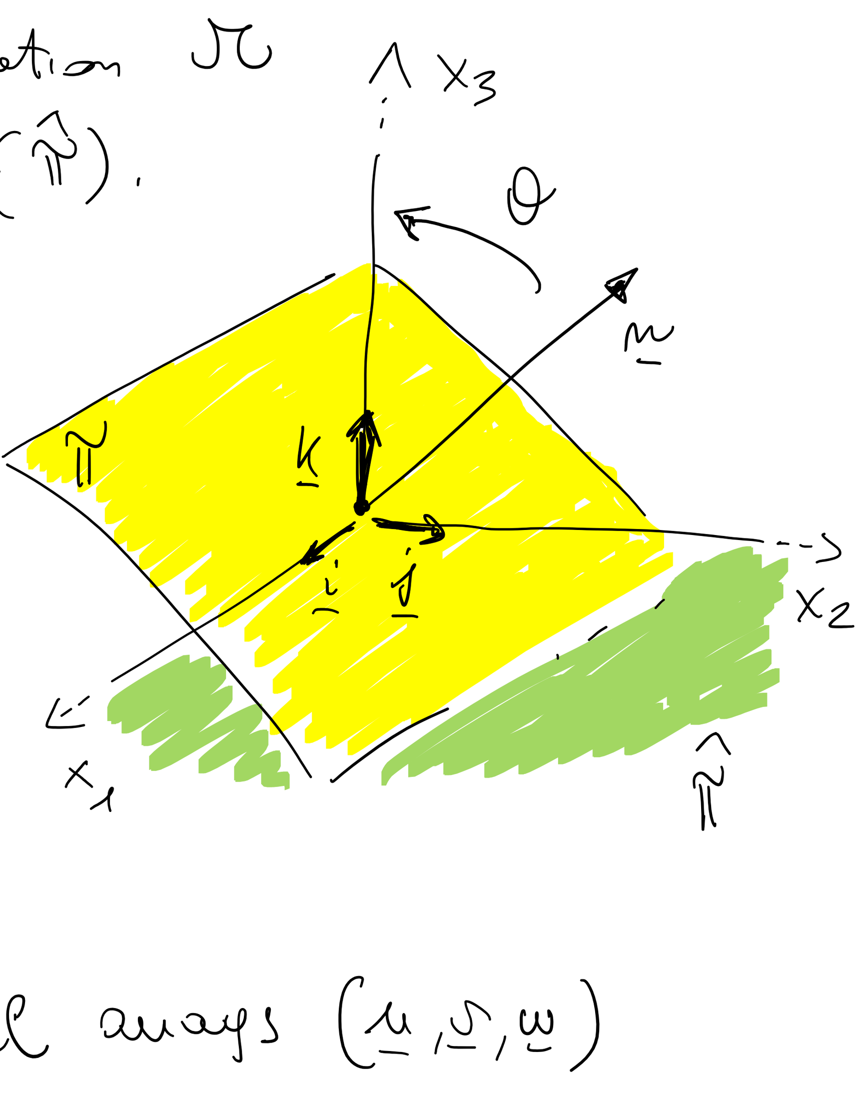
$$d(\underline{x},\underline{y})^2 = (\underline{x}-\underline{y})\cdot(\underline{x}-\underline{y}) =$$

$$= \underline{x}\cdot\underline{x} + \underline{y}\cdot\underline{y} - \underline{x}\cdot\underline{y} - \underline{y}\cdot\underline{x} =$$

$$= \hat{\underline{x}}\cdot\hat{\underline{x}} + \hat{\underline{y}}\cdot\hat{\underline{y}} - \hat{\underline{x}}\cdot\hat{\underline{y}} - \hat{\underline{y}}\cdot\hat{\underline{x}} = d(\hat{\underline{x}},\hat{\underline{y}})^2$$

Problem 5 : Given a plane Π with normal $\underline{n}$ find the RIGID MOTION (ROTATION matrix) which map the points on Π to plane xy

Solution : We need to find the transformation $\mathcal{T}$ which maps Π to xy-plane ($\hat{\Pi}$).



$$\mathcal{T}: \quad \underline{x} = \underline{\underline{Q}}\,\hat{\underline{x}} + \underline{c}$$

Let's require the plane passing from the origin $\underline{O}$, thus

$$\Pi: \underline{n}^T\underline{x} = 0 \Rightarrow \underline{c} = \underline{0}$$

Now we need to find $\underline{\underline{Q}}$ s.t.

$$\underline{x} = \underline{\underline{Q}}\,\hat{\underline{x}}$$

The idea is to find three orthogonal arrays $(\underline{u}, \underline{v}, \underline{w})$ s.t.

$$\underline{\underline{Q}}\,\underline{i} = \underline{u}, \quad \underline{\underline{Q}}\,\underline{j} = \underline{v}, \quad \underline{\underline{Q}}\,\underline{k} = \underline{w} \Rightarrow$$

$$\Rightarrow \underline{\underline{Q}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix} | & | & | \\ \underline{u} & \underline{v} & \underline{w} \\ | & | & | \end{pmatrix}$$

where $(\underline{i}, \underline{j}, \underline{k})$ are the orthonormal basis of $\mathbb{R}^3$.

We choose $\underline{w} = \underline{n} \Rightarrow \underline{\underline{Q}}\,\underline{k} = \underline{n}$

To find $\underline{u}$ we compute the tangent vector of the line intersection between Π and $\hat{\Pi}$

$$\begin{cases} \underline{n}^T\underline{x} = 0 \\ \underline{k}^T\underline{x} = 0 \end{cases} \Rightarrow \underline{u} = \frac{\underline{n}\times\underline{k}}{\|\underline{n}\times\underline{k}\|_2} = \begin{vmatrix} i & j & k \\ n_x & n_y & n_z \\ 0 & 0 & 1\end{vmatrix}\frac{1}{\|\underline{n}\times\underline{k}\|} =$$

$$= \frac{1}{\sqrt{n_x^2+n_y^2}}\left(n_y, -n_x, 0\right)$$

Finally, to compute $\underline{v}$ we require that $\underline{u}$, $\underline{v}$ and $\underline{w}$ are orthogonal $\Rightarrow$

$$\underline{v} = \frac{\underline{u}\times\underline{n}}{\|\underline{u}\times\underline{n}\|_2} = \frac{1}{\|\underline{u}\times\underline{n}\|_2}\begin{vmatrix} i & j & k \\ \frac{n_y}{\sqrt{n_x^2+n_y^2}} & \frac{-n_x}{\sqrt{n_x^2+n_y^2}} & 0 \\ n_x & n_y & n_z \end{vmatrix} =$$

$$= \underbrace{\frac{1}{\|\underline{u}\times\underline{n}\|_2}}_{=1}\left(-\frac{n_x n_z}{\sqrt{n_x^2+n_y^2}}, -\frac{n_y n_z}{\sqrt{n_x^2+n_y^2}}, \sqrt{n_x^2+n_y^2}\right)$$

Thus

$$\underline{\underline{Q}} = \begin{pmatrix} n_y & -\frac{n_x n_z}{\sqrt{n_x^2+n_y^2}} & n_x \\ -n_x & -\frac{n_y n_z}{\sqrt{n_x^2+n_y^2}} & n_y \\ 0 & \sqrt{n_x^2+n_y^2} & n_z \end{pmatrix}$$

$\square$