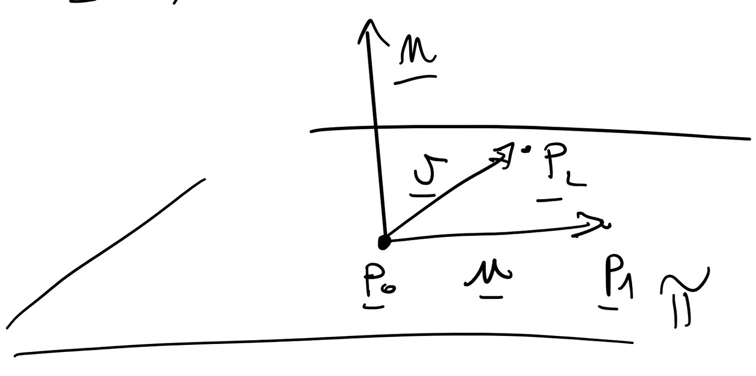
Tolerances

PROBLEM 4: given two planes π_1, π_2, find the intersection line r

Solution: we recall the definition of the plane in $d=3$

$$\pi = \{ \underline{x} \in E^3 \mid \underline{x} = \alpha_0 \underline{P}_2 + \alpha_1 \underline{P}_1 + (1-\alpha_0-\alpha_1)\underline{P}_0 \ , \ \underline{\alpha} = (\alpha_0, \alpha_1) \in \mathbb{R}^2 \}$$



now, we rewrite as

$$\underline{x} = \underline{P}_0 + \alpha_0 \underbrace{(\underline{P}_2 - \underline{P}_0)}_{\underline{u}} + \alpha_1 \underbrace{(\underline{P}_1 - \underline{P}_0)}_{\underline{v}}$$

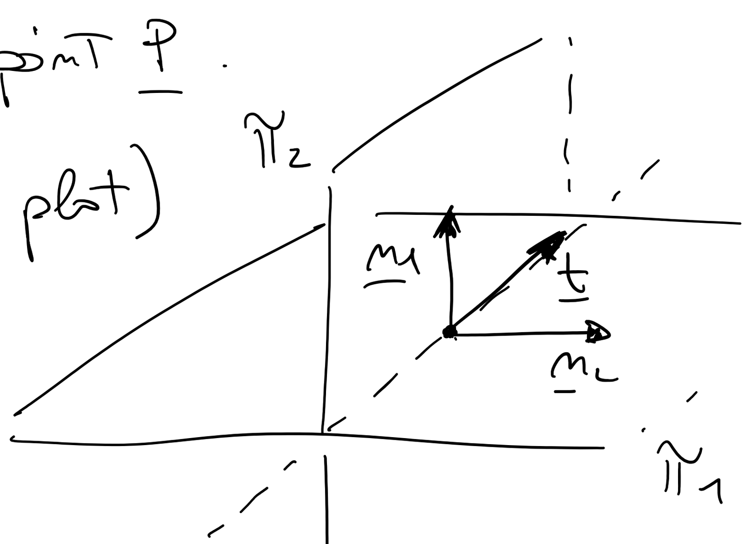
if we define $\underline{n} = \dfrac{\underline{u} \times \underline{v}}{\|\underline{u}\|_2 \|\underline{v}\|_2}$, we can rewrite the equation as

$$\underline{n}^T \underline{x} = \underline{n}^T \underline{P}_0 + \alpha_0 \underbrace{\cancel{\underline{n}^T \underline{u}}}_{\underline{n} \perp \underline{u}} + \alpha_1 \underbrace{\cancel{\underline{n}^T \underline{v}}}_{\underline{n} \perp \underline{v}}$$

obtaining the PLANE IMPLICIT FORM

$$\underline{n}^T \underline{x} = \underline{n}^T \underline{P}_0 \Rightarrow n_1 x_1 + n_2 x_2 + n_3 x_3 = d = n_1 P_1^0 + n_2 P_2^0 + n_3 P_3^0$$

To find the line r intersection of π_1, π_2 we have to compute the TANGENT $\underline{t}$ and a point $\underline{P}$.



1) To compute $\underline{t}$ is sufficient (from the plot)

$$\underline{t} = \underline{n}_1 \times \underline{n}_2$$

2) To compute $\underline{P}$ we find the intersection of π_1, π_2 and a plane π_3 orthogonal to $\underline{t}$ and passing from $\underline{O}$

$$\underline{P} : \begin{cases} \underline{n}_1^T \underline{P} = d_1 \\ \underline{n}_2^T \underline{P} = d_2 \\ \underline{t}^T \underline{P} = 0 \end{cases} \Rightarrow \underbrace{\begin{bmatrix} - \underline{n}_1 - \\ - \underline{n}_2 - \\ - \underline{t} - \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}}_{\underline{u}} = \underbrace{\begin{pmatrix} d_1 \\ d_2 \\ 0 \end{pmatrix}}_{\underline{b}}$$

Again there exists a solution if $\det(\underline{\underline{A}}) \neq 0$

$\Rightarrow$ NO COPLANARITY $\quad t \cdot (\underline{n}_1 \times \underline{n}_2) \neq 0$

$\square$

To implement all the algorithms we have to remember that we have a FINITE ALGEBRA, thus the numbers are in MACHINE NUMBERS

$$a \in \mathbb{R} \Rightarrow \bar{a} \in \mathcal{F} \ , \ \frac{|a - \bar{a}|}{|a|} < \varepsilon \ , \ |\varepsilon| \leq \varepsilon_m$$

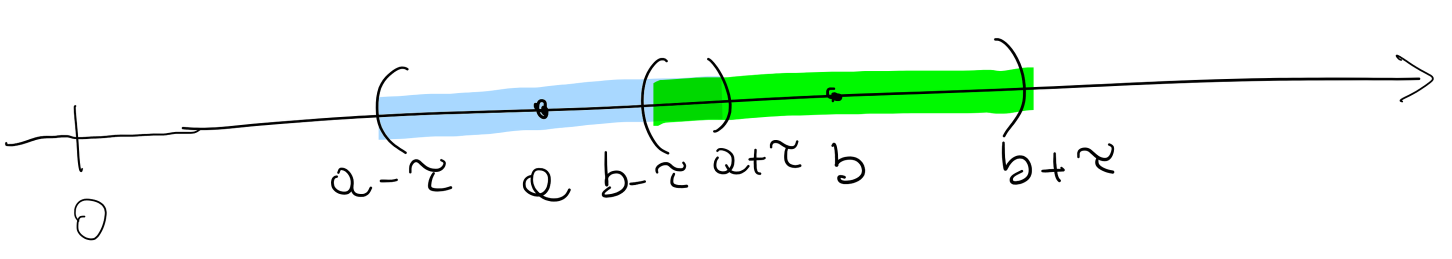
where $\mathcal{F}$ is the MACHINE NUMBERS SET and ε_m is the MACHINE PRECISION

$$\mathcal{F} = \{ a \in \mathbb{R} : a = (-1)^s 0.\underbrace{a_1 a_2 \cdots a_t}_{t} \cdot N^q \ , \ 0 \leq a_i < N, \ a_1 \neq 0 \ , \ L \leq q \leq U \ , \ s \in \{0,1\} \} \cup \{0\}$$

$$\varepsilon_m = \frac{1}{2} N^{1-t}$$, with t the length of the MANTISSA.

For this reason, when we have to perform the comparisons on the algorithms we need to take in account a TOLERANCE τ

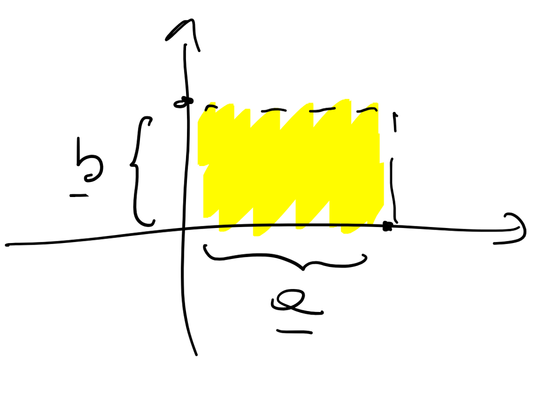
EXAMPLE: $a == b \ , \ a, b \in \mathbb{R} \Rightarrow \dfrac{|a-b|}{\max(|a|,|b|,1)} < \tau$



$a > b \ , \ a, b \in \mathbb{R} \Rightarrow \dfrac{a-b}{\max(|a|,|b|,1)} > \tau$
$a, b \neq 0$

NOTE: It is a good practice in a code to fix a SINGLE TOLERANCE for all the operations, in order to have consistency in all the operations.

EXAMPLE: τ is the tolerance for the 1D operations. For the 2D operations, such as the AREA computation we take $\approx \tau^2$ as tolerance. Indeed, we admit only polygons with edges with length $> \tau$. Thus, choosing τ^2 we say that the smallest polygon admissible is the rectangle with edges $|a| = \tau$ and $|b| = \tau$.



Another important thing to consider is the CATASTROPHIC CANCELLATION (cancellazione numerica), i.e. the approximation of $x_1 - x_2$ leads to a loss of the original numbers

Example $L_1 = 253.5 \text{ cm}$ $L_2 = 252.4 \text{ cm}$
$\bar{L}_1 = 254 \text{ cm}$ $\bar{L}_2 = 252 \text{ cm}$

with $\varepsilon_m = \frac{1}{2} 10^{1-3} = \frac{1}{2} 10^{-2} = 0.005$

$\overline{0.2535} = 0.254$
$\overline{0.2524} = 0.252$
$t = 3$

$$\frac{|L_1 - \bar{L}_1|}{|L_1|} = 0.002 < \varepsilon_m$$

$$\frac{|L_2 - \bar{L}_2|}{|L_2|} = 0.005 < \varepsilon_m$$

However $L_1 - L_2 \sim 1 \text{ cm}$ but $\bar{L}_1 - \bar{L}_2 \sim 2 \text{ cm}$, thus

$$\left| \frac{(L_1 - L_2) - (\bar{L}_1 - \bar{L}_2)}{(L_1 - L_2)} \right| = 1 \gg \varepsilon_m$$

$\square$

For example, in $E^2 \equiv \mathbb{R}^2$, for the cross product $(\underline{w})_3 = (\underline{u} \times \underline{v})_3 = x_1 y_2 - x_2 y_1$

if $x_1 y_2 \sim x_2 y_1$ then catastrophic cancellation may happen.

In the area of a polygon an error can be made if we do not take in account this phenomenon.