Martingales

Martingales

Definition (Martingale)

A stochastic process $(X_i)_{i\in I}$ is a martingale with respect to a filtration $(\mathcal{F}_i)_{i\in I}$ if

- X_i is $\mathcal{F}_i$-measurable for each $i \in I$;
- $\mathbb{E}[|X_i|] < \infty$ for each $i \in I$;
- $\mathbb{E}[X_j|\mathcal{F}_i] = X_i$ a.s. for each $i < j \in I$.

We simply say that a stochastic process is a martingale if it is so with respect to its natural filtration.

Martingales are *fair games*: if we are playing in a fair game, with $\mathcal{F}_i$ being the information on what happened in the game up to time i, and X_i is our wealth at time i, then the expected value of our wealth in the future, given what has happened so far, is our current wealth.

By the tower property, $\mathbb{E}[X_j] = \mathbb{E}[\mathbb{E}[X_j|\mathcal{F}_i]] = \mathbb{E}[X_i]$ for all $i < j \in I$.

Submartingales

Definition (Submartingale)

A stochastic process $(X_i)_{i \in I}$ is a submartingale with respect to a filtration $(\mathcal{F}_i)_{i \in I}$ if

- X_i is $\mathcal{F}_i$-measurable for each $i \in I$;
- $\mathbb{E}[|X_i|] < \infty$ for each $i \in I$;
- $\mathbb{E}[X_j | \mathcal{F}_i] \geq X_i$ a.s. for each $i < j \in I$.

We simply say that a stochastic process is a submartingale if it is so with respect to its natural filtration.

Submartingales are *favourable games*.

By the tower property, $\mathbb{E}[X_j] = \mathbb{E}[\mathbb{E}[X_j | \mathcal{F}_i]] \geq \mathbb{E}[X_i]$ for all $i < j \in I$.

Supermartingales

Definition (Supermartingale)

A stochastic process $(X_i)_{i \in I}$ is a supermartingale with respect to a filtration $(\mathcal{F}_i)_{i \in I}$ if

- X_i is $\mathcal{F}_i$-measurable for each $i \in I$;
- $\mathbb{E}[|X_i|] < \infty$ for each $i \in I$;
- $\mathbb{E}[X_j | \mathcal{F}_i] \leq X_i$ a.s. for each $i < j \in I$.

We simply say that a stochastic process is a supermartingale if it is so with respect to its natural filtration.

Supermartingales are *unfavourable games*.

By the tower property, $\mathbb{E}[X_j] = \mathbb{E}[\mathbb{E}[X_j | \mathcal{F}_i]] \leq \mathbb{E}[X_i]$ for all $i < j \in I$.

Closed martingales

Closed martingales

Theorem

Consider a filtration $(\mathcal{F}_i)_{i\in I}$ of σ-algebras contained in $\mathcal{F}$, and let X be an $\mathcal{F}$-measurable random variable with $E[|X|] < \infty$. Then, the stochastic process $(X_i)_{i\in I}$ defined by

$$X_i = \mathbb{E}[X|\mathcal{F}_i]$$ V.A. MISURABILE RISPETTO A $\mathcal{F}_i$

is a martingale w.r.t. $(\mathcal{F}_i)_{i\in I}$.

Definition

Martingales of the type described in the theorem are called closed martingales.

Closed martingales

Proof.

By definition of conditional expectation, we have that X_i id $\mathcal{F}_i$-measurable for all $i \in I$. We have that for all $i \in I$

$$\mathbb{E}[|X_i|] = \mathbb{E}\Big[\big|\mathbb{E}[X|\mathcal{F}_i]\big|\Big] \leq \mathbb{E}\Big[\mathbb{E}[|X||\mathcal{F}_i]\Big] \overset{\text{T.P.}}{=} \mathbb{E}[|X|] < \infty.$$

Moreover, for all $\underline{i < j} \in I$,

$\mathcal{F}_i \subseteq \mathcal{F}_j$

$$\mathbb{E}[X_j|\mathcal{F}_i] = \mathbb{E}\Big[\underbrace{\mathbb{E}[X|\mathcal{F}_j]}|\mathcal{F}_i\Big] \underset{\text{T.P.}}{=} \mathbb{E}[X|\mathcal{F}_i] = X_i.$$

□

Example 1

Let $(X_t)_{t\in[0,\infty)}$ be a finite CTMC with a transient state z. Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be the natural filtration of $(X_t)_{t\in[0,\infty)}$. The random variable

$$N(z) = \int_0^\infty \mathbb{1}_{\{X_t = z\}}\,dt$$

is measurable w.r.t. $\mathcal{F} = \sigma\{X(s) : s \in [0,\infty)\}$. The process $(M_t)_{t\in[0,\infty)}$ defined by

$$M_t = \mathbb{E}[N(z)|\mathcal{F}_t] \overset{\text{MARKOV PROPERTY}}{=} \underbrace{\mathbb{E}_{X_t}[N(z)]}_{} + \overbrace{\int_0^t \mathbb{1}_{\{X_s = z\}}\,ds}^{\text{TEMPO DI VISITA A Z IN } [0,t]}$$

is a closed martingale with respect to $(\mathcal{F}_t)_{t\in[0,\infty)}$, and almost surely $\lim_{t\to\infty} M_t = N(z)$.

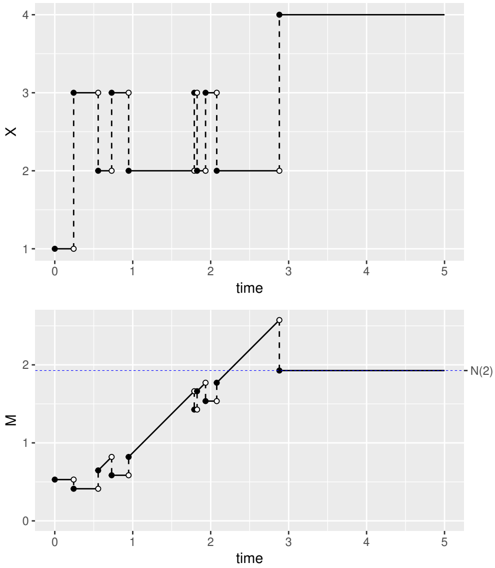
Example 1

Example with $S = \{1, 2, 3, 4\}$,

$$Q = \begin{pmatrix} -4 & 2 & 2 & 0 \\ 1 & -3 & 1 & 1 \\ 1 & 3 & -6 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$M_t = \mathbb{E}[N(2) | \mathcal{F}_t]$$



Example 2

Let $(X_t)_{t\in[0,\infty)}$ be a recurrent, irreducible CTMC. Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be the natural filtration of $(X_t)_{t\in[0,\infty)}$. Then, fixed two recurrent states x and y, the random variable

$$V(x,y) = \begin{cases} 1 & \text{if } x \text{ is visited before } y \\ 0 & \text{otherwise} \end{cases}$$

is measurable w.r.t. $\mathcal{F} = \sigma\{X(s) : s \in [0,\infty)\}$. The process $(M_t)_{t\in[0,\infty)}$ defined by

$$M_t = \mathbb{E}[V(x,y)|\mathcal{F}_t] \overset{\text{M.P.}}{=} \begin{cases} P_{X_t}(V(x,y)=1) & \text{IF } x,y \text{ NOT VISITED YET} \\ V(x,y) & \text{IF EITHER } x,y \text{ VISITED IN } [0,t] \end{cases}$$

is a closed martingale, and almost surely $\lim_{t\to\infty} M_t = V(x,y)$.

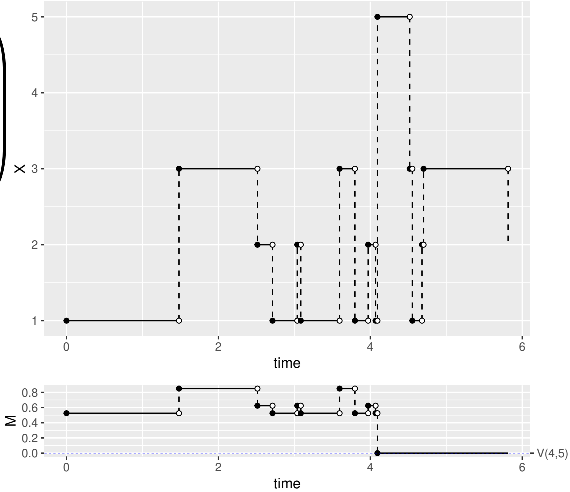
Example 2

Example with $S = \{1,2,3,4,5\}$,

$$Q = \begin{pmatrix} -4 & 2 & 1 & 0 & 1 \\ 2 & -6 & 2 & 1 & 1 \\ 1 & 3 & -4 & 1 & 0 \\ 0 & 1 & 2 & -3 & 0 \\ 1 & 0 & 3 & 1 & -5 \end{pmatrix}$$

and

$$M_t = \mathbb{E}[V(4,5)|\mathcal{F}_t]$$



Example 2

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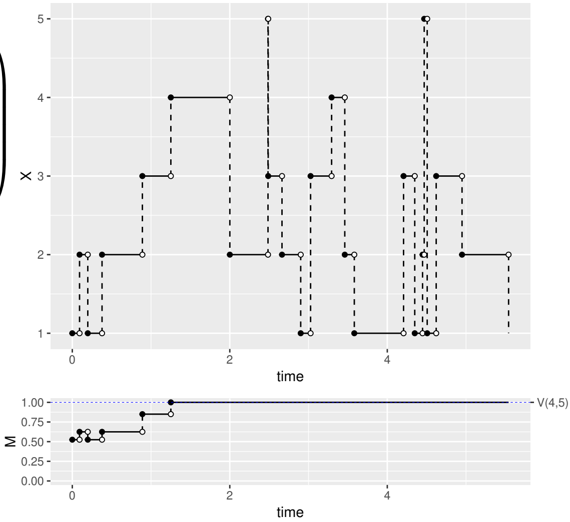
and

$$M_t = \mathbb{E}[V(4,5)|\mathcal{F}_t]$$



Closed submartingales and supermartingales

Definition

A submartingale $(X_i)_{i\in I}$ w.r.t. $(\mathcal{F}_i)_{i\in I}$ is closed if there exists an $\mathcal{F}$-measurable random variable X such that $\mathbb{E}[|X|] < \infty$ and for all $i \in I$

$$X_i \leq \mathbb{E}[X|\mathcal{F}_i] \quad \text{a.s.}$$

Definition

A supermartingale $(X_i)_{i\in I}$ w.r.t. $(\mathcal{F}_i)_{i\in I}$ is closed if there exists an $\mathcal{F}$-measurable random variable X such that $\mathbb{E}[|X|] < \infty$ and for all $i \in I$

$$X_i \geq \mathbb{E}[X|\mathcal{F}_i] \quad \text{a.s.}$$

Basic properties of the increments

Uncorrelated increments

Theorem

Let $(X_i)_{i \in I}$ be a martingale w.r.t. $(\mathcal{F}_i)_{i \in I}$, such that $\mathbb{E}[X_i^2] < \infty$ for all $i \in I$. Then, for any $i_1 < i_2 \leq i_3 < i_4$ we have

$$Cov(X_{i_2} - X_{i_1}, X_{i_4} - X_{i_3}) = 0.$$

Proof.

We have

TOW. PROP. $\quad \mathcal{F}_{i_3}$-MEAS. $\quad i_1 \le i_2 \le i_3 \le i_4$

$$\mathbb{E}[(X_{i_2} - X_{i_1})(X_{i_4} - X_{i_3})] \overset{\downarrow}{=} \mathbb{E}\left[\mathbb{E}[(X_{i_2} - X_{i_1})(X_{i_4} - X_{i_3}) \,|\, \boxed{\mathcal{F}_{i_3}}]\right]$$
$$= \mathbb{E}\left[(X_{i_2} - X_{i_1})\,\underbrace{\mathbb{E}[X_{i_4} - X_{i_3} \,|\, \mathcal{F}_{i_3}]}\right] = 0.$$

Similarly, by the martingale property

$E[X_{i_4} | \mathcal{F}_{i_3}] - X_{i_3} = 0$

DEF. DI MART.

$$\mathbb{E}[X_{i_2} - X_{i_1}] = \mathbb{E}[X_{i_4} - X_{i_3}] = 0.$$

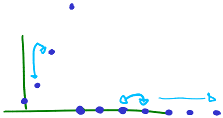
$\overset{\text{T.P.}}{=} E[E[X_{i_2} - X_{i_1} | \mathcal{F}_{i_1}]] \overset{\text{DEF. MART.}}{=} 0$

Hence,

$$\mathrm{Cov}(X_{i_2} - X_{i_1}, X_{i_4} - X_{i_3}) = \mathbb{E}[(X_{i_2} - X_{i_1})(X_{i_4} - X_{i_3})] - \mathbb{E}[X_{i_2} - X_{i_1}]\mathbb{E}[X_{i_4} - X_{i_3}]$$
$$= 0.$$

□

Uncorrelated does not mean independent!



Let $(B_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $P(B_i = -1) = P(B_i = 1) = 1/2$. Consider

$$X_n = \prod_{i=1}^{n}(1 + B_i) = \begin{cases} 2^n & \text{if } B_1 = B_2 = \cdots = B_n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then $(X_n)_{n=1}^{\infty}$ is a martingale, and as such its increments are uncorrelated. But they are not independent!

$$P(X_5 - X_4 = 0, X_9 - X_8 > 0) = 0$$

but $P(X_5 - X_4 = 0) > 0$ and $P(X_9 - X_8 > 0) > 0$.

Quadratic increments

Theorem (Pitagora's Theorem for Martingales)

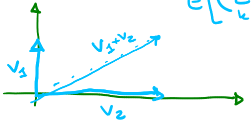
Let $(X_i)_{i\in I}$ be a martingale w.r.t. $(\mathcal{F}_i)_{i\in I}$, such that $\mathbb{E}[X_i^2] < \infty$ for all $i \in I$. Then, for any $i < j \in I$ we have

$$\mathbb{E}[(X_j - X_i)^2 | \mathcal{F}_i] = \mathbb{E}[X_j^2 - X_i^2 | \mathcal{F}_i]$$

and as a consequence (by TOWER PROP.)

$$\mathbb{E}[(X_j - X_i)^2] = \mathbb{E}[X_j^2 - X_i^2]$$

$$\| \quad E\left[\left(\sum_{k=i}^{j-1}(X_{k+1}-X_k)\right)^2\right] = E\left[\sum_{k=i}^{j-1}(X_{k+1}-X_k)^2\right]$$



$$\|v_1+v_2\|^2 = \|v_1\|^2 + \|v_2\|^2$$

Quadratic increments

Proof.

We have

$$
\begin{aligned}
\mathbb{E}[(X_j - X_i)^2 | \mathcal{F}_i] &= \mathbb{E}[X_j^2 + X_i^2 - 2X_iX_j | \mathcal{F}_i] \\
&= \mathbb{E}[X_j^2 | \mathcal{F}_i] + X_i^2 - 2X_i\underbrace{\mathbb{E}[X_j | \mathcal{F}_i]}_{X_i} \\
&= \mathbb{E}[X_j^2 | \mathcal{F}_i] + X_i^2 - 2X_i^2 \\
&= \mathbb{E}[X_j^2 - X_i^2 | \mathcal{F}_i].
\end{aligned}
$$

As a consequence,

$$
\mathbb{E}[(X_j - X_i)^2] \overset{T.P.}{=} \mathbb{E}\Big[\mathbb{E}[(X_j - X_i)^2 | \mathcal{F}_i]\Big] = \mathbb{E}\Big[\mathbb{E}[X_j^2 - X_i^2 | \mathcal{F}_i]\Big] \overset{T.P.}{=} \mathbb{E}[X_j^2 - X_i^2].
$$

□

Predictable processes and the first hard lesson

Predictable processes

For simplicity, we only give the following definition for discrete-time processes (a continuous-time analougus exists but we will not cover it - you will need to know about continuous predictable processes for stochastic analysis or mathematical finance)

Definition

A stochastic process $(H_n)_{n=1}^{\infty}$ is predictable with respect to a filtration $(\mathcal{F}_n)_{n=0}^{\infty}$ if H_n is $\mathcal{F}_{n-1}$ measurable for all $n \geq 1$.

Thinking about games, a predictable process at time n is a function of all previous rounds of the game. It is something we can construct a strategy with to try and win the game!

Constructing a strategy

Our goal is to transform a martingale, or a supermartingale, into a submartingale. The typical control gamblers are given is how much to play at the next round given what they have observed so far. This way, the wealth at time n is given by

$$W_n = W_{n-1} + \underbrace{H_n}_{\substack{\text{how much to bet,}\\ \text{depending on previous}\\ \text{observations}}} \cdot \Big(\underbrace{X_n - X_{n-1}}_{\substack{\text{increment of}\\ \text{the (super)martingale}}} \Big)$$

By recursion, we can write

$$W_n = X_0 + \sum_{i=1}^{n} H_i \cdot (X_i - X_{i-1}).$$

Example: doubling strategy

Consider a fair game, where with probability $1/2$ we win the value we bet, and with probability $1/2$ we lose it. If we always bet 1 euro and we allow debts, we get the random walk, which is a martingale.

A famous strategy is this: start with gambling 1 euro and if losing at the previous round, double the amount to bet!

When the gambler eventually wins at the jth round, the net gain is

$$2^j - \sum_{i=0}^{j-1} 2^i = 2^j - \frac{2^j - 1}{2 - 1} = 1 \text{ euro.}$$

Example: doubling strategy

In this case, our wealth after the nth round is

$$W_n = W_{n-1} + \underbrace{H_n}_{\substack{\text{how much we bet} \\ \text{at the } n\text{th round}}} \cdot (\underbrace{X_n - X_{n-1}}_{\text{either 1 or -1}})$$

with

$$H_n = 2^m, \quad m = \#\text{number of the last consecutive losses.}$$

Bad news

Theorem

Let $(X_n)_{n=0}^{\infty}$ a martingale (or supermartingale, or submartingale) w.r.t. a filtration $(\mathcal{F}_n)_{n=0}^{\infty}$, and let $(H_n)_{n=1}^{\infty}$ be a positive, predictable process with respect to the same filtration with $H_n < c_n < \infty$ for all $n \geq 1$. Then, the process $(W_n)_{n=0}^{\infty}$ defined by

$$W_n = X_0 + \sum_{i=1}^{n} H_i \cdot (X_i - X_{i-1})$$

is a martingale (or supermartingale, or submartingale) w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$.

Proof.

$E[|W_n|] < \infty$ because $H_j < c_j < \infty$ and $\mathbb{E}[|X_j|] < \infty$ for all $1 \leq j \leq n$. Moreover,

$$\mathbb{E}[W_{n+1}|\mathcal{F}_n] = \mathbb{E}[W_n + \underbrace{H_{n+1}}_{\mathcal{F}_n\text{-MEAS.}}(X_{n+1} - X_n)|\mathcal{F}_n] = W_n + \underbrace{H_{n+1}}_{>0}\underbrace{(\mathbb{E}[X_{n+1}|\mathcal{F}_n] - X_n)}$$

$=0$ (X_n) MART.

≤ 0 (X_n) SUPER

≥ 0 (X_n) SUB. $\square$

Example: doubling strategy

With the doubling strategy, we get

$$W_n = \#\text{wins by the } n\text{th round} - 2^g$$

where g is the length of the last stretch of consecutive losses. So, even if g is almost surely finite, it occasionally gets big and make the gambler lose a lot of money!

A predictable martingale is constant

Theorem

Let $(X_n)_{n=0}^{\infty}$ a martingale w.r.t. a filtration $(\mathcal{F}_n)_{n=0}^{\infty}$. If $\boxed{(X_n)_{n=1}^{\infty} \text{ is predictable}}$ w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$, then $X_n = X_0$ for all $n \in \mathbb{N}$ almost surely.

Proof.

If X_{n+1} is $\mathcal{F}_n$-measurable, then $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \underline{X_{n+1}}$. However, by the martingale property we have $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \underline{X_n}$. Hence, for any $n \in \mathbb{N}$ we have $X_n = X_{n+1}$. $\square$

Doob's decomposition theorem

Theorem (Doob's decomposition theorem, part 1)

Let $(X_n)_{n=0}^{\infty}$ be a discrete-time stochastic process with $\mathbb{E}[|X_n|] < \infty$ for all $n \in \mathbb{N}$, and let $(\mathcal{F}_n)_{n=0}^{\infty}$ be its natural filtration. Then, there exists a unique decomposition

$$X_n = M_n + A_n$$

$M_n = X_n - A_n$

where

- $(M_n)_{n=0}^{\infty}$ *is a martingale w.r.t.* $(\mathcal{F}_n)_{n=0}^{\infty}$;
- $A_0 = 0$ *and* $(A_n)_{n=1}^{\infty}$ *is predictable w.r.t.* $(\mathcal{F}_n)_{n=0}^{\infty}$.

In particular, for all $n \geq 1$

$$A_n = \sum_{i=1}^{n} \overbrace{\mathbb{E}[X_i - X_{i-1} | \mathcal{F}_{i-1}]}^{\text{V.A. } \mathcal{F}_{n-1}\text{-MEAS.}}. \leftarrow \quad A_n \ \mathcal{F}_{n-1}\text{-MEAS.}$$

Doob's decomposition theorem

Theorem (Doob's decomposition theorem, part 2)

Moreover,

- *If $(X_n)_{n=0}^{\infty}$ is a supermartingale then $A_{n+1} \leq A_n$ for all $n \in \mathbb{N}$ almost surely;*
- *If $(X_n)_{n=0}^{\infty}$ is a submartingale then $A_{n+1} \geq A_n$ for all $n \in \mathbb{N}$ almost surely.*

Doob's decomposition theorem

The version we state and prove is slightly more general of what typically stated, where only submartingales or supermartingales are considered.

A similar decomposition for continuous time processes exists, but it holds under more technical assumptions (especially the uniqueness). The continuous time version is known as Doob-Meyer decomposition.

The process $(A_n)_{n=0}^{\infty}$ is called the compensator of the process $(X_n)_{n=0}^{\infty}$.