Linear averaging and flow dynamics

Giacomo Como, DISMA, Politecnico di Torino

Fabio Fagnani, DISMA, Politecnico di Torino

French-De Groot learning model

Community of individuals

Interpersonal influence $\Rightarrow$ Learning process $\Rightarrow$ *Social power*
Consensus

- J.R.P. French, *A Formal Theory of Social Power*, Psychological Review 63, 1956.
- M.H. DeGroot, *Reaching a Consensus*, Journal of the American Statistical Association 69, 1974.

French-De Groot learning model

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ weighted directed graph representing a social network

- $(i, j) \in \mathcal{E}$ if i is influenced by j
- W_{ij} *strength* of this influence
- $P_{ij} = w_i^{-1} W_{ij}$ $(w_i = \sum_k W_{ik})$ normalized weight matrix

French-De Groot learning model

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ weighted directed graph representing a social network

- $(i, j) \in \mathcal{E}$ if i is influenced by j
- W_{ij} *strength* of this influence
- $P_{ij} = w_i^{-1} W_{ij}$ $(w_i = \sum_k W_{ik})$ normalized weight matrix

- Each agent $i \in \mathcal{V}$ has an initial opinion on some fact or event $x_i(0) \in \mathbb{R}$.
- Learning model: agents modify their opinion in time averaging the opinions of their neighbors:

$x_i(t+1) = \sum_j P_{ij} x_j(t)$

Linear averaging dynamics

French-De Groot learning model

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ weighted directed graph representing a social network

- $(i, j) \in \mathcal{E}$ if i is influenced by j
- W_{ij} *strength* of this influence
- $P_{ij} = w_i^{-1} W_{ij}$ ($w_i = \sum_k W_{ik}$) normalized weight matrix

- Each agent $i \in \mathcal{V}$ has an initial opinion on some fact or event $x_i(0) \in \mathbb{R}$.
- Learning model: agents modify their opinion in time averaging the opinions of their neighbors:

  $$x_i(t+1) = \sum_j P_{ij} x_j(t)$$

  Linear averaging dynamics
- Compact notation $x(t) \in \mathbb{R}^n$:

  $$x(t+1) = Px(t)$$
  $$x(t) = P^t x(0)$$

French-De Groot learning model

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ weighted directed graph representing a social network

- $(i, j) \in \mathcal{E}$ if i is influenced by j
- W_{ij} *strength* of this influence
- $P_{ij} = w_i^{-1} W_{ij}$ $(w_i = \sum_k W_{ik})$ normalized weight matrix

- Each agent $i \in \mathcal{V}$ has an initial opinion on some fact or event $x_i(0) \in \mathbb{R}$.
- Learning model: agents modify their opinion in time averaging the opinions of their neighbors:

  $$x_i(t+1) = \sum_j P_{ij} x_j(t)$$

  Linear averaging dynamics
- Compact notation $x(t) \in \mathbb{R}^n$:

  $$x(t+1) = Px(t)$$
  $$x(t) = P^t x(0)$$
- Behavior when $t \to +\infty$?

A little digression: dynamical systems

A little digression: dynamical systems

- A *dynamical system*: Ω set, $f : \Omega \to \Omega$ map,
- $x(0) \in \Omega$ *initial condition*
- *Evolution*:

$$x(0) \mapsto x(1) = f(x(0)) \mapsto x(2) = f(x(1)) \ \ldots$$

More formally, $x(t)$ is the sequence defined recursively by

$$x(t+1) = f(x(t))$$

A little digression: dynamical systems

- A *dynamical system*: Ω set, $f : \Omega \to \Omega$ map,
- $x(0) \in \Omega$ *initial condition*
- *Evolution*:

$$x(0) \mapsto x(1) = f(x(0)) \mapsto x(2) = f(x(1)) \ \ldots$$

More formally, $x(t)$ is the sequence defined recursively by

$$x(t+1) = f(x(t))$$

- x_0 *equilibrium* if $f(x_0) = x_0$;

$$x(0) = x_0 \ \Rightarrow \ x(t) = x_0 \ \forall t$$

A little digression: dynamical systems

- A *dynamical system*: Ω set, $f:\Omega\to\Omega$ map,
- $x(0)\in\Omega$ *initial condition*
- *Evolution*: $x(t)$ is the sequence defined recursively by
$$x(t+1)=f(x(t))$$
- x_0 *equilibrium* if $f(x_0)=x_0$;
$$x(0)=x_0 \;\Rightarrow\; x(t)=x_0\ \forall t$$
- Ω metric space (e.g. $\Omega=\mathbb{R}^n$), f continuous:
$$\text{If } \lim_{t\to+\infty} x(t)=\bar{x} \;\Rightarrow\; f(\bar{x})=\bar{x} \text{ (equilibrium)}$$

A little digression: dynamical systems

- A *dynamical system*: Ω set, $f : \Omega \to \Omega$ map,
- $x(0) \in \Omega$ *initial condition*
- *Evolution*: $x(t)$ is the sequence defined recursively by
$$x(t+1) = f(x(t))$$
- x_0 *equilibrium* if $f(x_0) = x_0$;
$$x(0) = x_0 \;\Rightarrow\; x(t) = x_0 \;\forall t$$
- Ω metric space (e.g. $\Omega = \mathbb{R}^n$), f continuous:
$$\text{If } \lim_{t \to +\infty} x(t) = \bar{x} \;\Rightarrow\; f(\bar{x}) = \bar{x} \text{ (equilibrium)}$$

Proof:

A little digression: dynamical systems

- A *dynamical system*: Ω set, $f : \Omega \to \Omega$ map,
- $x(0) \in \Omega$ *initial condition*
- *Evolution*: $x(t)$ is the sequence defined recursively by
$$x(t+1) = f(x(t))$$
- x_0 *equilibrium* if $f(x_0) = x_0$;
$$x(0) = x_0 \;\Rightarrow\; x(t) = x_0 \;\forall t$$
- Ω metric space (e.g. $\Omega = \mathbb{R}^n$), f continuous:
$$\text{If } \lim_{t \to +\infty} x(t) = \bar{x} \;\Rightarrow\; f(\bar{x}) = \bar{x} \text{ (equilibrium)}$$

Proof: $x(t) \to \bar{x} \Rightarrow x(t+1) = f(x(t)) \to f(\bar{x})$
But $x(t+1) \to \bar{x}$. Hence, $f(\bar{x}) = \bar{x}$ ∎

Recap of PF theory for stochastic matrices

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, $P = D^{-1}W$ normalized weight matrix

Theorem (**Spectral properties of stochastic matrices)**

- *Dominant eigenvalue* $\lambda_P = 1$, $P\mathbb{1} = \mathbb{1}$,
- *Invariant distributions* $\{\pi \in \mathbb{R}^n_+, \ \mathbb{1}'\pi = 1, \ P'\pi = \pi\}$ *form a simplex in* $\mathbb{R}^{\mathcal{V}}$ *with* $s_{\mathcal{G}}$ *vertices.*
- *For every sink component with nodes* $\mathcal{W}$, *there exists an invariant distribution* π *such that* $\pi_i > 0$ *if and only if* $i \in \mathcal{W}$.
- *The invariant distribution is unique if and only if* $s_{\mathcal{G}} = 1$.
- *If* $\mathcal{G}$ *is strongly connected, then* 1 *is simple and* $\pi_i > 0$ *for all* i;
- *If* $\mathcal{G}$ *is strongly connected and aperiodic, then every eigenvalue* $\mu \neq 1$ *is s.t.* $|\mu| < 1$.

Analysis of the French-De Groot model

- Dynamical system with $\Omega = \mathbb{R}^n$, $f(x) = Px$
- Evolution from the initial condition $x(0)$: $x(t) = P^t x(0)$
- Equilibria: $Px_0 = x_0$, right eigenvectors of P relative to the dominant eigenvalue 1
- *Consensus vectors* $\alpha\mathbb{1}$ are always equilibria
- If $s_{\mathcal{G}} = 1$, the only equilibria are the consensus vectors.

Analysis of the French-De Groot model

- Dynamical system with $\Omega = \mathbb{R}^n$, $f(x) = Px$
- Evolution from the initial condition $x(0)$: $x(t) = P^t x(0)$
- Equilibria: $Px_0 = x_0$, right eigenvectors of P relative to the dominant eigenvalue 1
- *Consensus vectors* $\alpha\mathbb{1}$ are always equilibria
- If $s_{\mathcal{G}} = 1$, the only equilibria are the consensus vectors.
- IMPORTANT: for any invariant distribution π

$$\pi' x(t+1) = \pi' P x(t) = \pi' x(t) \; \forall t \;\Rightarrow\; \pi' x(t) = \pi' x(0) \; \forall t$$

The quantity $\pi' x(t)$ is a *motion invariant*.

Analysis of the French-De Groot model

$x(t) = P^t x(0)$

Suppose that for a given $x(0)$, $x(t) \to \bar{x}$ for $t \to +\infty$

Then,

Analysis of the French-De Groot model

$x(t) = P^t x(0)$

Suppose that for a given $x(0)$, $x(t) \to \bar{x}$ for $t \to +\infty$

Then,

- $\bar{x}$ is an equilibrium.
- $\pi' \bar{x} = \pi' x(0)$ for every invariant distribution π

Analysis of the French-De Groot model

$x(t) = P^t x(0)$

Suppose that for a given $x(0)$, $x(t) \to \bar{x}$ for $t \to +\infty$

Then,

- $\bar{x}$ is an equilibrium.
- $\pi' \bar{x} = \pi' x(0)$ for every invariant distribution π
  Proof: $\pi' x(t) \to \pi' \bar{x}$, $\pi' x(t) = \pi' x(0)\ \forall t$. Hence, $\pi' \bar{x} = \pi' x(0)$.

Analysis of the French-De Groot model

$x(t) = P^t x(0)$

Suppose that for a given $x(0)$, $x(t) \to \bar{x}$ for $t \to +\infty$

Then,

- $\bar{x}$ is an equilibrium.
- $\pi'\bar{x} = \pi' x(0)$ for every invariant distribution π
  Proof: $\pi' x(t) \to \pi'\bar{x}$, $\pi' x(t) = \pi' x(0)\ \forall t$. Hence, $\pi'\bar{x} = \pi' x(0)$.
- When $s_{\mathcal{G}} = 1$, $x(t) \to \bar{x} = \alpha\mathbb{1}$ and

$$\alpha = \pi'\bar{x} = \pi' x(0) = \sum_{k \in \mathcal{V}} \pi_k x_k(0)$$

  where π is the only invariant distribution of P

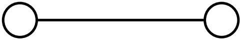
Analysis of the French-De Groot model

$$x(t) = P^t x(0)$$

When does it converge?

A counterexample

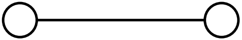
$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ 2-line simple graph:



$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

A counterexample

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ 2-line simple graph:

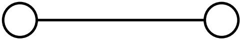


$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$x(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \; x(1) = Px(0) = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

A counterexample

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ 2-line simple graph:



$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

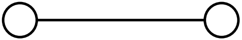
$$x(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \; x(1) = Px(0) = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

$$t \text{ even } x(t) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad t \text{ odd } x(t) = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

If $\alpha \neq \beta$, no convergence!

A counterexample

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ 2-line simple graph:



$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$x(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \; x(1) = Px(0) = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

$$t \text{ even } x(t) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad t \text{ odd } x(t) = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

If $\alpha \neq \beta$, no convergence!

The graph above is not aperiodic....

Asymptotics of French-De Groot learning model

Theorem

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ *strongly connected, aperiodic.* $x(t) = P^t x(0)$

$$\lim_{t \to +\infty} x(t) = \alpha \mathbb{1}$$

with
$$\alpha = \pi' x(0) = \sum_k \pi_k x_k(0) \quad \forall i$$

where $\pi = P'\pi$ *is the unique invariant distribution of* P.

Asymptotics of French-De Groot learning model

Theorem

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

$$\lim_{t \to +\infty} x(t) = \alpha \mathbb{1}$$

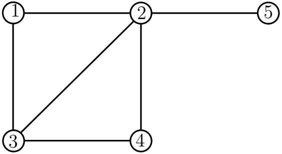
with
$$\alpha = \pi' x(0) = \sum_k \pi_k x_k(0) \quad \forall i$$

where $\pi = P'\pi$ is the unique invariant distribution of P.

- All opinions converge to a common value: CONSENSUS
- The consensus value $\pi' x(0)$ is a convex combination of the original opinions weighted by the *invariant distribution centralities* of the various agents.
- The invariant distribution centrality is a measure of the social power of an agent in the French-De Groot learning process.

Example

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$



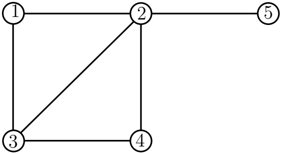
$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$x(t) = P^t x(0)$, $x(0) = (0, 1, 2, 5, 3)$

$$\lim_{t \to +\infty} x(t)?$$

Example

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$



$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

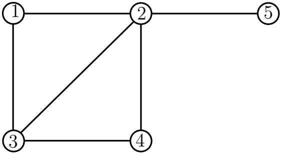
$x(t) = P^t x(0)$, $x(0) = (0, 1, 2, 5, 3)$

$$\lim_{t \to +\infty} x(t)?$$

- $\mathcal{G}$ is strongly connected and aperiodic
- $\lim_{t \to +\infty} x_i(t) = \pi' x(0)$ for every i

Example

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$



$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

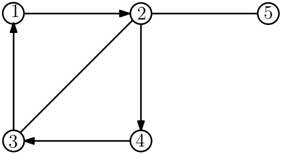
$x(t) = P^t x(0)$, $x(0) = (0, 1, 2, 5, 3)$

$$\lim_{t \to +\infty} x(t)?$$

- $\mathcal{G}$ is strongly connected and aperiodic
- $\lim_{t \to +\infty} x_i(t) = \pi' x(0)$ for every i
- $\mathcal{G}$ is undirected, unweighted $\Rightarrow$ $\pi = (1/6, 1/3, 1/4, 1/6, 1/12)$
- $\pi' x(0) = 1/3 + 1/2 + 5/6 + 1/4 = 23/12$

Example

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$



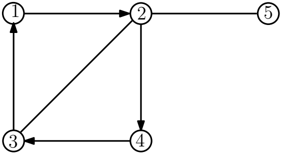
$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$x(t) = P^t x(0)$, $x(0) = (0, 1, 2, 5, 3)$

$$\lim_{t \to +\infty} x(t)?$$

Example

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$



$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

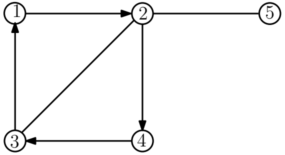
$x(t) = P^t x(0)$, $x(0) = (0, 1, 2, 5, 3)$

$$\lim_{t \to +\infty} x(t)?$$

- $\mathcal{G}$ str. conn. and aperiodic
- $\lim_{t \to +\infty} x_i(t) = \pi' x(0)$ for every i
- Computation of π:

Example

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$



$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$x(t) = P^t x(0)$, $x(0) = (0, 1, 2, 5, 3)$

$$\lim_{t\to+\infty} x(t)?$$

- $\mathcal{G}$ str. conn. and aperiodic
- $\lim_{t\to+\infty} x_i(t) = \pi' x(0)$ for every i
- Computation of π:

$$\begin{cases} \pi_1 &= \pi_3/2 \\ \pi_2 &= \pi_1 + \pi_3/2 + \pi_5 \\ \pi_3 &= \pi_2/3 + \pi_4 \\ \pi_5 &= \pi_4 = \pi_2/3 \end{cases}$$

$\pi = (1/8, 3/8, 1/4, 1/8, 1/8)$

- $\pi' x(0) =$
  $1/2 + 1/2 + 5/8 + 3/8 = 2$

Wisdom of crowds and wise societies

- Social network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ str. connected aperiodic
- The initial opinion is a noisy measurement of a true variable μ:

$$x_i(0) = \mu + N_i, \quad N_i \text{ independent r.v. } \mathbb{E}[N_i] = 0 \ \mathrm{Var}(N_i) = \sigma^2$$

- French-De Groot learning model $x(t+1) = Px(t)$

Wisdom of crowds and wise societies

- Social network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ str. connected aperiodic
- The initial opinion is a noisy measurement of a true variable μ:

$$x_i(0) = \mu + N_i, \quad N_i \text{ independent r.v. } \mathbb{E}[N_i] = 0 \ \mathrm{Var}(N_i) = \sigma^2$$

- French-De Groot learning model $x(t+1) = Px(t)$
- $x_i(t) \to \pi' x(0) = \sum_k \pi_k(\mu + N_k) = \mu + \sum_k \pi_k N_k$

Wisdom of crowds and wise societies

- Social network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ str. connected aperiodic
- The initial opinion is a noisy measurement of a true variable μ:

  $$x_i(0) = \mu + N_i, \quad N_i \text{ independent r.v. } \mathbb{E}[N_i] = 0 \ \mathrm{Var}(N_i) = \sigma^2$$

- French-De Groot learning model $x(t+1) = Px(t)$
- $x_i(t) \to \pi' x(0) = \sum_k \pi_k(\mu + N_k) = \mu + \sum_k \pi_k N_k$
- $\mathrm{Var}(\sum \pi_k N_k) = \sigma^2 \sum \pi_k^2$
  $\mathcal{G}$ strongly connected, $\pi_i > 0$ for all i, then, $\sum \pi_k^2 < \sum \pi_k = 1$.

Wisdom of crowds and wise societies

- Social network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ str. connected aperiodic
- The initial opinion is a noisy measurement of a true variable μ:

$$x_i(0) = \mu + N_i, \quad N_i \text{ independent r.v. } \mathbb{E}[N_i] = 0 \ \mathrm{Var}(N_i) = \sigma^2$$

- French-De Groot learning model $x(t+1) = Px(t)$
- $x_i(t) \to \pi' x(0) = \sum_k \pi_k(\mu + N_k) = \mu + \sum_k \pi_k N_k$
- $\mathrm{Var}(\sum \pi_k N_k) = \sigma^2 \sum \pi_k^2$
  $\mathcal{G}$ strongly connected, $\pi_i > 0$ for all i, then, $\sum \pi_k^2 < \sum \pi_k = 1$.

  $\mathrm{Var}(\sum \pi_k N_k) < \sigma^2$ Crowd is wiser than a single!

Wisdom of crowds and wise societies

- Social network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ str. connected aperiodic
- The initial opinion is a noisy measurement of a true variable μ:

$$x_i(0) = \mu + N_i, \quad N_i \text{ independent r.v. } \mathbb{E}[N_i] = 0 \ \mathrm{Var}(N_i) = \sigma^2$$

- French-De Groot learning model $x(t+1) = Px(t)$
- $x_i(t) \to \pi' x(0) = \sum_k \pi_k(\mu + N_k) = \mu + \sum_k \pi_k N_k$
- $\mathrm{Var}(\sum \pi_k N_k) = \sigma^2 \sum \pi_k^2$

  $\mathcal{G}$ strongly connected, $\pi_i > 0$ for all i, then, $\sum \pi_k^2 < \sum \pi_k = 1$.

  $\mathrm{Var}(\sum \pi_k N_k) < \sigma^2$ Crowd is wiser than a single!
- *Asymptotic wisdom:* $\lim_{n \to +\infty} \pi' x(0) = \mu \Leftrightarrow \lim_{n \to +\infty} \max_k \pi_k = 0$

  (Golub and Jackson, 2010)

The many applications of averaging dynamics

- A simple model for opinion fusion, social power, and consensus formation
- The basis of many distributed algorithms
  - Decentralized computation in sensor networks
  - Load balancing in computer networks
  - Clock syncronization
  - Relative localization
  - Coordination dynamics of robot networks.

Asymptotics of French-De Groot learning model

Theorem

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

$$\lim_{t \to +\infty} x(t) = \alpha \mathbb{1}$$

with

$$\alpha = \pi' x(0) = \sum_k \pi_k x_k(0) \quad \forall i$$

where $\pi = P'\pi$ is the unique invariant distribution of P.

We are left with proving convergence.

All remaining facts follow from previous arguments.

Asymptotics of French-De Groot learning model

Theorem

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

$$\lim_{t \to +\infty} x(t) = \alpha \mathbb{1}$$

with

$$\alpha = \pi' x(0) = \sum_k \pi_k x_k(0) \quad \forall i$$

where $\pi = P'\pi$ is the unique invariant distribution of P.

We are left with proving convergence.

All remaining facts follow from previous arguments.

There are extensions for more general graphs.

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

- $M = P - \mathbb{1}\pi'$

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

- $M = P - \mathbb{1}\pi'$
- $$\begin{aligned} M^2 &= (P - \mathbb{1}\pi')(P - 1\pi') \\ &= P^2 - \mathbb{1}\pi' P - P\mathbb{1}\pi' + \mathbb{1}\pi'\mathbb{1}\pi' \\ &= P^2 - \mathbb{1}\pi' - \mathbb{1}\pi' + \mathbb{1}\pi' \\ &= P^2 - \mathbb{1}\pi' \end{aligned}$$

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

- $M = P - \mathbb{1}\pi'$
- $M^t = P^t - \mathbb{1}\pi'$

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

- $M = P - \mathbb{1}\pi'$
- $M^t = P^t - \mathbb{1}\pi'$
- Eigenvalues of M:

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

- $M = P - \mathbb{1}\pi'$
- $M^t = P^t - \mathbb{1}\pi'$
- Eigenvalues of M:
  - $M\mathbb{1} = P\mathbb{1} - \mathbb{1}\pi'\mathbb{1} = \mathbb{1} - \mathbb{1} = 0$, $\pi' M = 0$

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

- $M = P - \mathbb{1}\pi'$
- $M^t = P^t - \mathbb{1}\pi'$
- Eigenvalues of M:
  - $M\mathbb{1} = P\mathbb{1} - \mathbb{1}\pi'\mathbb{1} = \mathbb{1} - \mathbb{1} = 0$, $\pi' M = 0$
  - $Mx = \lambda x$ with $\lambda \neq 0$ and $x \neq 0$
    $0 = \pi' Mx = \lambda \pi' x$, hence $\pi' x = 0$ and $Mx = Px$,

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

- $M = P - \mathbb{1}\pi'$
- $M^t = P^t - \mathbb{1}\pi'$
- Eigenvalues of M:
  - $M\mathbb{1} = P\mathbb{1} - \mathbb{1}\pi'\mathbb{1} = \mathbb{1} - \mathbb{1} = 0$, $\pi' M = 0$
  - $Mx = \lambda x$ with $\lambda \neq 0$ and $x \neq 0$
    $0 = \pi' M x = \lambda \pi' x$, hence $\pi' x = 0$ and $Mx = Px$,
  - $Px = \lambda x$

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

- $M = P - \mathbb{1}\pi'$
- $M^t = P^t - \mathbb{1}\pi'$
- Eigenvalues of M:
  - $M\mathbb{1} = P\mathbb{1} - \mathbb{1}\pi'\mathbb{1} = \mathbb{1} - \mathbb{1} = 0$, $\pi' M = 0$
  - $Mx = \lambda x$ with $\lambda \neq 0$ and $x \neq 0$
    $0 = \pi' Mx = \lambda \pi' x$, hence $\pi' x = 0$ and $Mx = Px$,
  - $Px = \lambda x$
  - $\pi' x = 0$ implies $x \neq \alpha \mathbb{1}$. Hence $\lambda \neq 1$

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

- $M = P - \mathbb{1}\pi'$
- $M^t = P^t - \mathbb{1}\pi'$
- Eigenvalues of M:
  - $M\mathbb{1} = P\mathbb{1} - \mathbb{1}\pi'\mathbb{1} = \mathbb{1} - \mathbb{1} = 0$, $\pi' M = 0$
  - $Mx = \lambda x$ with $\lambda \neq 0$ and $x \neq 0$
    $0 = \pi' Mx = \lambda \pi' x$, hence $\pi' x = 0$ and $Mx = Px$,
  - $Px = \lambda x$
  - $\pi' x = 0$ implies $x \neq \alpha\mathbb{1}$. Hence $\lambda \neq 1$
  - $\mathcal{G}$ str. connected, aperiodic yields $|\lambda| < 1$

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

- $M = P - \mathbb{1}\pi'$
- $M^t = P^t - \mathbb{1}\pi'$
- Eigenvalues of M:
  - $M\mathbb{1} = P\mathbb{1} - \mathbb{1}\pi'\mathbb{1} = \mathbb{1} - \mathbb{1} = 0$, $\pi' M = 0$
  - $Mx = \lambda x$ with $\lambda \neq 0$ and $x \neq 0$
    $0 = \pi' M x = \lambda \pi' x$, hence $\pi' x = 0$ and $Mx = Px$,
  - $Px = \lambda x$
  - $\pi' x = 0$ implies $x \neq \alpha\mathbb{1}$. Hence $\lambda \neq 1$
  - $\mathcal{G}$ str. connected, aperiodic yields $|\lambda| < 1$
  - $\lambda \in \sigma(M) \Rightarrow |\lambda| < 1$

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

- $M = P - \mathbb{1}\pi'$
- $M^t = P^t - \mathbb{1}\pi'$
- $\lambda \in \sigma(M) \;\Rightarrow\; |\lambda| < 1$

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

- $M = P - \mathbb{1}\pi'$
- $M^t = P^t - \mathbb{1}\pi'$
- $\lambda \in \sigma(M) \;\Rightarrow\; |\lambda| < 1$
- General fact: if M is a squared matrix with all eigenvalues λ such that $|\lambda| < 1$, then $M^t \to 0$ (proof using Jordan form)

The quickest proof of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic. $x(t) = P^t x(0)$

π the unique invariant distribution of P ($\pi = P'\pi$ or $\pi' = \pi' P$)

- $M = P - \mathbb{1}\pi'$
- $M^t = P^t - \mathbb{1}\pi'$
- $\lambda \in \sigma(M) \;\Rightarrow\; |\lambda| < 1$
- General fact: if M is a squared matrix with all eigenvalues λ such that $|\lambda| < 1$, then $M^t \to 0$ (proof using Jordan form)
- $M^t = P^t - \mathbb{1}\pi' \to 0 \;\Rightarrow\; P^t x(0) - \mathbb{1}\pi' x(0) \to 0$
  Hence, $P^t x(0) \to (\pi' x(0))\mathbb{1}$

An alternative road

Contractive properties of stochastic matrices

An alternative road

Contractive properties of stochastic matrices

Q stochastic matrix: $Q_{ij} \geq 0$, $\sum_j Q_{ij} = 1$ for every i.

Consider a vector x and $y = Qx$.
Put $x_{\max} = \max x_j$, $x_{\min} = \min x_j$.

An alternative road

Contractive properties of stochastic matrices

Q stochastic matrix: $Q_{ij} \geq 0$, $\sum_j Q_{ij} = 1$ for every i.

Consider a vector x and $y = Qx$.
Put $x_{\max} = \max x_j$, $x_{\min} = \min x_j$.

$$y_i = \sum_j Q_{ij} x_j \begin{cases} \leq \sum_j Q_{ij} x_{\max} = x_{\max} \\ \geq \sum_j Q_{ij} x_{\min} = x_{\min} \end{cases}$$

An alternative road

Contractive properties of stochastic matrices

Q stochastic matrix: $Q_{ij} \geq 0$, $\sum_j Q_{ij} = 1$ for every i.

Consider a vector x and $y = Qx$.
Put $x_{\max} = \max x_j$, $x_{\min} = \min x_j$.

$$y_i = \sum_j Q_{ij} x_j \begin{cases} \leq \sum_j Q_{ij} x_{\max} = x_{\max} \\ \geq \sum_j Q_{ij} x_{\min} = x_{\min} \end{cases}$$

$$\Rightarrow x_{\min} \leq y_{\min} \leq y_{\max} \leq x_{\max}$$

An alternative road

Contractive properties of stochastic matrices

Q stochastic matrix: $Q_{ij} \geq 0$, $\sum_j Q_{ij} = 1$ for every i.

Consider a vector x and $y = Qx$.
Put $x_{\max} = \max x_j$, $x_{\min} = \min x_j$.

$$y_i = \sum_j Q_{ij} x_j \begin{cases} \leq \sum_j Q_{ij} x_{\max} = x_{\max} \\ \geq \sum_j Q_{ij} x_{\min} = x_{\min} \end{cases}$$

$$\Rightarrow x_{\min} \leq y_{\min} \leq y_{\max} \leq x_{\max}$$

$$\Rightarrow y_{\max} - y_{\min} \leq x_{\max} - x_{\min}$$

An alternative road

Contractive properties of stochastic matrices

Q stochastic matrix: $Q_{ij} \geq 0$, $\sum_j Q_{ij} = 1$ for every i.

Consider a vector x and $y = Qx$.
Put $x_{\max} = \max x_j$, $x_{\min} = \min x_j$.

$$y_i = \sum_j Q_{ij} x_j \begin{cases} \leq \sum_j Q_{ij} x_{\max} = x_{\max} \\ \geq \sum_j Q_{ij} x_{\min} = x_{\min} \end{cases}$$

$$\Rightarrow x_{\min} \leq y_{\min} \leq y_{\max} \leq x_{\max}$$

$$\Rightarrow y_{\max} - y_{\min} \leq x_{\max} - x_{\min}$$

Not yet a real contraction... we have not used connectivity and aperiodicity so far!

An alternative road

Lemma

Let Q be a stochastic matrix for which there exist $\alpha > 0$ and an index k such that $Q_{ik} \geq \alpha$ for all i. Then, $y = Qx$ satisfies

$$y_{\max} - y_{\min} \leq (1-\alpha)\big(x_{\max} - x_{\min}\big)$$

$$\begin{aligned} y_i &= \sum_j Q_{ij} x_j = \sum_j Q_{ij}(x_j - x_{\min}) + \sum_j Q_{ij} x_{\min} \\ &\geq \alpha(x_k - x_{\min}) + x_{\min} = \alpha x_k + (1-\alpha) x_{\min} \end{aligned}$$

$$\begin{aligned} y_i &= \sum_j Q_{ij} x_j = \sum_j Q_{ij}(x_j - x_{\max}) + \sum_j Q_{ij} x_{\max} \\ &\leq \alpha(x_k - x_{\max}) + x_{\max} = \alpha x_k + (1-\alpha) x_{\max} \end{aligned}$$

An alternative road

Lemma

Let Q be a stochastic matrix for which there exist $\alpha > 0$ and an index k such that $Q_{ik} \geq \alpha$ for all i. Then, $y = Qx$ satisfies

$$y_{\max} - y_{\min} \leq (1-\alpha)\big(x_{\max} - x_{\min}\big)$$

For every node i:

$$\begin{aligned} y_i \geq & \alpha x_k + (1-\alpha) x_{\min} \\ y_i \leq & \alpha x_k + (1-\alpha) x_{\max} \end{aligned}$$

An alternative road

Lemma

Let Q be a stochastic matrix for which there exist $\alpha > 0$ and an index k such that $Q_{ik} \geq \alpha$ for all i. Then, $y = Qx$ satisfies

$$y_{\max} - y_{\min} \leq (1-\alpha)\big(x_{\max} - x_{\min}\big)$$

For every node i:

$$\begin{aligned} y_i \geq& \alpha x_k + (1-\alpha)x_{\min} \\ y_i \leq& \alpha x_k + (1-\alpha)x_{\max} \end{aligned}$$

Putting these two inequalities together gives:

$$\begin{aligned} y_{\max} - y_{\min} &\leq \alpha x_k + (1-\alpha)x_{\max} - \alpha x_k - (1-\alpha)x_{\min} \\ &= (1-\alpha)(x_{\max} - x_{\min}), \end{aligned}$$

An alternative road

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, $P_{ij} = w_i^{-1} W_{ij}$, $x(t) = P^t x(0)$

Suppose there exist $\alpha > 0$ and k such that $P_{ik} \geq \alpha$ for all i

An alternative road

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, $P_{ij} = w_i^{-1} W_{ij}$, $x(t) = P^t x(0)$

Suppose there exist $\alpha > 0$ and k such that $P_{ik} \geq \alpha$ for all i

$$[x_{\max}(t+1) - x_{\min}(t+1)] \leq (1-\alpha)[x_{\max}(t) - x_{\min}(t)]$$

$$[x_{\max}(t) - x_{\min}(t)] \leq (1-\alpha)^t [x_{\max}(0) - x_{\min}(0)]$$

- $\lim_{t \to +\infty} [x_{\max}(t) - x_{\min}(t)] = 0$

An alternative road

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, $P_{ij} = w_i^{-1} W_{ij}$, $x(t) = P^t x(0)$

Suppose there exist $\alpha > 0$ and k such that $P_{ik} \geq \alpha$ for all i

$$[x_{\max}(t+1) - x_{\min}(t+1)] \leq (1-\alpha)[x_{\max}(t) - x_{\min}(t)]$$

$$[x_{\max}(t) - x_{\min}(t)] \leq (1-\alpha)^t [x_{\max}(0) - x_{\min}(0)]$$

- $\lim_{t \to +\infty} [x_{\max}(t) - x_{\min}(t)] = 0$
- $x_{\min}(t) \leq x_{\min}(t+1) \leq x_{\max}(t+1) \leq x_{\max}(t)$ monotonicity

$$\lim_{t \to +\infty} x_{\max}(t) = \bar{x} = \lim_{t \to +\infty} x_{\min}(t)$$

An alternative road

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, $P_{ij} = w_i^{-1} W_{ij}$, $x(t) = P^t x(0)$

Suppose there exist $\alpha > 0$ and k such that $P_{ik} \geq \alpha$ for all i

$$[x_{\max}(t+1) - x_{\min}(t+1)] \leq (1-\alpha)[x_{\max}(t) - x_{\min}(t)]$$

$$[x_{\max}(t) - x_{\min}(t)] \leq (1-\alpha)^t [x_{\max}(0) - x_{\min}(0)]$$

- $\lim_{t \to +\infty} [x_{\max}(t) - x_{\min}(t)] = 0$
- $x_{\min}(t) \leq x_{\min}(t+1) \leq x_{\max}(t+1) \leq x_{\max}(t)$ monotonicity

$$\lim_{t \to +\infty} x_{\max}(t) = \bar{x} = \lim_{t \to +\infty} x_{\min}(t) \Rightarrow \lim_{t \to +\infty} x_i(t) = \bar{x} \ \forall i \text{ consensus}$$

An alternative road

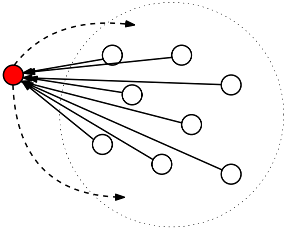
$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic.

When P satisfies the assumption of the Lemma: $P_{ik} \geq \alpha \ \forall i$?

An alternative road

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic.

When P satisfies the assumption of the Lemma: $P_{ik} \geq \alpha \; \forall i$?



There must exist a *global influencer*, a node to which all others are directly connected!

This is a quite strong assumption!

An alternative road

However,

Lemma

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, P)$ strongly connected, aperiodic. There exists $m \in \mathbb{N}$ such that $P^m_{ij} > 0$ for every i, j.

The alternative proof

- There exists $m \in \mathbb{N}$ such that $P_{ij}^m > 0$ for every i, j.
- Put $\alpha = \min_{ij} P_{ij}^m$.

The alternative proof

- There exists $m \in \mathbb{N}$ such that $P_{ij}^m > 0$ for every i, j.
- Put $\alpha = \min_{ij} P_{ij}^m$.
- Apply the Contraction lemma to P^m:

The alternative proof

- There exists $m \in \mathbb{N}$ such that $P^m_{ij} > 0$ for every i, j.
- Put $\alpha = \min_{ij} P^m_{ij}$.
- Apply the Contraction lemma to P^m:
  - $[x_{\max}(tm) - x_{\min}(tm)] \leq (1-\alpha)^t [x_{\max}(0) - x_{\min}(0)]$

The alternative proof

- There exists $m \in \mathbb{N}$ such that $P^m_{ij} > 0$ for every i, j.
- Put $\alpha = \min_{ij} P^m_{ij}$.
- Apply the Contraction lemma to P^m:
  - $[x_{\max}(tm) - x_{\min}(tm)] \leq (1-\alpha)^t [x_{\max}(0) - x_{\min}(0)]$
  - $\lim_{t \to +\infty} [x_{\max}(tm) - x_{\min}(tm)] = 0$

The alternative proof

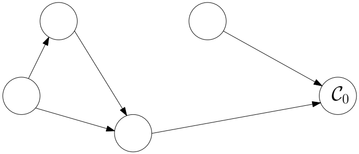
- There exists $m \in \mathbb{N}$ such that $P_{ij}^m > 0$ for every i, j.
- Put $\alpha = \min_{ij} P_{ij}^m$.
- Apply the Contraction lemma to P^m:
  - $[x_{\max}(tm) - x_{\min}(tm)] \leq (1-\alpha)^t [x_{\max}(0) - x_{\min}(0)]$
  - $\lim_{t \to +\infty} [x_{\max}(tm) - x_{\min}(tm)] = 0$
- $x_{\min}(t) \leq x_{\min}(t+1) \leq x_{\max}(t+1) \leq x_{\max}(t)$ monotonicity

The alternative proof

- There exists $m \in \mathbb{N}$ such that $P^m_{ij} > 0$ for every i, j.
- Put $\alpha = \min_{ij} P^m_{ij}$.
- Apply the Contraction lemma to P^m:
  - $[x_{\max}(tm) - x_{\min}(tm)] \leq (1-\alpha)^t [x_{\max}(0) - x_{\min}(0)]$
  - $\lim_{t \to +\infty} [x_{\max}(tm) - x_{\min}(tm)] = 0$
- $x_{\min}(t) \leq x_{\min}(t+1) \leq x_{\max}(t+1) \leq x_{\max}(t)$ monotonicity
- $\lim_{t \to +\infty} x_{\max}(t) = \bar{x} = \lim_{t \to +\infty} x_{\min}(t)$
- $\Rightarrow \lim_{t \to +\infty} x_i(t) = \bar{x}$

Extensions of the main theorem

$\mathcal{G}$ possesses a globally reachable connected component $\mathcal{C}_0$ (equivalently, the condensation graph has just one sink $s_{\mathcal{G}} = 1$) that is aperiodic.



- Global reachability: Fix $k \in \mathcal{C}_0$. For every i, $P_{ik}^{m_i} > 0$ for some m_i.
- Aperiodicity lemma $P_{kk}^{q} > 0$ for every $q \geq m$
- $P_{ik}^{s} > 0$ for every i, where $s = \max m_i + m$.
- Apply the contractive lemma again!

The most general result

Theorem

If $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ possesses a globally reachable aperiodic component $\mathcal{C}_0$, then

$$\lim_{t \to +\infty} x_i(t) = \pi' x(0) \quad \forall i$$

where $\pi = P'\pi$ is the invariant measure centrality of the graph

π has support only on the globally reachable component.

The opinions of agents not belonging to the globally reachable component have no influence on the final consensus value.

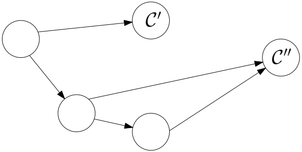
A remark

$P^t \to \mathbb{1}\pi'$

Namely, P^t converges to a matrix where the rows are all the same, equal to the invariant distribution centrality π.

The most general result

No further generalization is possible:



No influence between nodes in $\mathcal{C}'$ and $\mathcal{C}''$.

Nodes in $\mathcal{C}'$ and $\mathcal{C}''$ (if the components are aperiodic) will reach separated consensus depending on their own initial opinions.

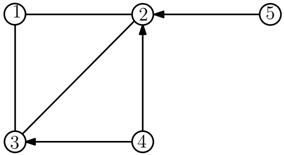
No global consensus

The most general result

- Lack of periodicity $\Rightarrow$ no convergence.
- More than one sink in the condensation graph $\Rightarrow$ no consensus.

Example

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$



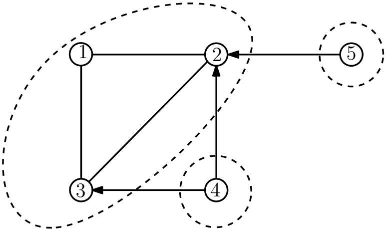
$P =???$

$x(t) = P^t x(0)$, $x(0) = (0, 1, 2, 5, 3)$

$$\lim_{t \to +\infty} x(t)?$$

Example

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$



$P =???$

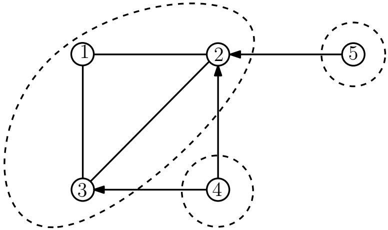
$x(t) = P^t x(0)$, $x(0) = (0, 1, 2, 5, 3)$

$$\lim_{t \to +\infty} x(t)?$$

- $\mathcal{G}$ has an aperiodic globally reachable component

Example

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$



$P = ???$

$x(t) = P^t x(0)$, $x(0) = (0, 1, 2, 5, 3)$

$\lim_{t \to +\infty} x(t)$?

- $\mathcal{G}$ has an aperiodic globally reachable component
- $\lim_{t \to +\infty} x_i(t) = \pi' x(0)$ for every i
- $\pi = (1/3, 1/3, 1/3, 0, 0)$
- $\pi' x(0) = 1$

Other applications of the linear averaging dynamics

The basis of many distributed algorithms

- Decentralized computation in sensor networks
- Load balancing in computer networks
- Clock syncronization
- Relative localization
- Coordination dynamics of robot networks.

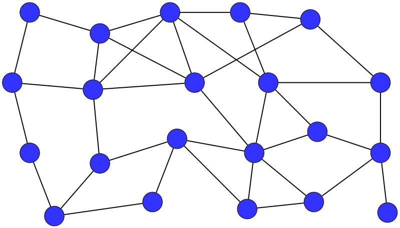
Other applications of the linear averaging dynamics

The basis of many distributed algorithms

- Decentralized computation in sensor networks
- Load balancing in computer networks
- Clock syncronization
- Relative localization
- Coordination dynamics of robot networks.

Decentralized computation of global functions

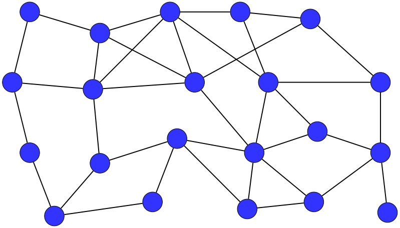
$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ set of units (sensors) connected through a network



$i \in \mathcal{V} \to x_i$ state

Decentralized computation of global functions

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ set of units (sensors) connected through a network

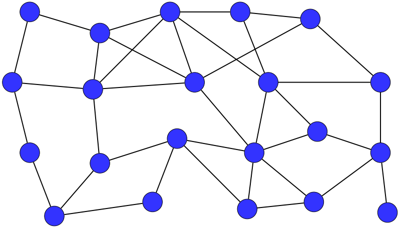


$i \in \mathcal{V} \to x_i$ state

Goal: compute some global function $f(x_i \,|\, i \in \mathcal{V})$

Decentralized computation of global functions

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ set of units (sensors) connected through a network



$i \in \mathcal{V} \to x_i$ state

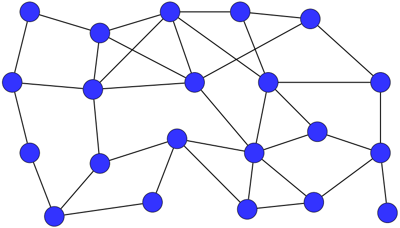
Goal: compute some global function $f(x_i \,|\, i \in \mathcal{V})$

Constraints:

- no supervision, decentralized design;
- use only the available communication links;
- time and computation complexity scaling well w.r. to size $n = |\mathcal{V}|$;

Decentralized computation of global functions

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ set of units (sensors) connected through a network



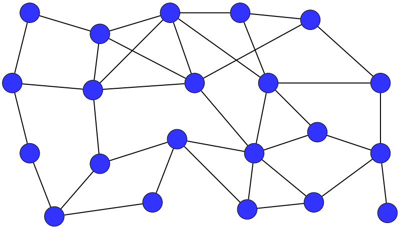
$i \in \mathcal{V} \to x_i$ state

Goal: compute some global function $f(x_i \,|\, i \in \mathcal{V})$

Examples:
x_i internal state (e.g. level of energy) or the result of a measurement (temperature, presence detection, fire detection)

Decentralized computation of global functions

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ set of units (sensors) connected through a network



$i \in \mathcal{V} \to x_i$ state

Goal: compute some global function $f(x_i \,|\, i \in \mathcal{V})$

Examples:
x_i internal state (e.g. level of energy) or the result of a measurement (temperature, presence detection, fire detection)

f computing some statistics of the data:

- average value $\bar{x} = n^{-1} \sum x_i$
- variance $n^{-1} \sum (x_i - \bar{x})^2$
- $\max x_i$, $\min x_i$
- fraction of nodes s.t. $x_i \geq \alpha$

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected str. connected, $i \in \mathcal{V} \to x_i$ state

Goal: compute $\bar{x} = n^{-1} \sum x_i$

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected str. connected, $i \in \mathcal{V} \to x_i$ state

Goal: compute $\bar{x} = n^{-1} \sum x_i$ Idea: use averaging dynamics.

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected str. connected, $\quad i \in \mathcal{V} \to x_i$ state

Goal: compute $\bar{x} = n^{-1} \sum x_i$ $\quad$ Idea: use averaging dynamics.

W adjacency matrix of $\mathcal{G}$, $P = (D^{-1}W + I)/2$

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected str. connected, $\quad i \in \mathcal{V} \to x_i$ state

Goal: compute $\bar{x} = n^{-1} \sum x_i \quad$ Idea: use averaging dynamics.

W adjacency matrix of $\mathcal{G}$, $P = (D^{-1}W + I)/2$

Important remark: in this case, there is not an a-priori choice for the matrix P: this just becomes a *design* choice.

Given an $n \times n$ matrix P , we can consider the associated graph $\mathcal{G}_P = (\mathcal{V}, \mathcal{E})$ where

$$\mathcal{V} = \{1, \ldots, n\},\ \mathcal{E} = \{(i,j) \,|\, P_{ij} > 0\}$$

Notice that, with the choice above, $\mathcal{G}_P = \mathcal{G} \cup \{\mathit{selfloops}\}$: to implement P we only need to communicate along the edges of the graph $\mathcal{G}$; use of self-loops is not an issue.

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected str. connected, $i \in \mathcal{V} \to x_i$ state

Goal: compute $\bar{x} = n^{-1} \sum x_i$ Idea: use averaging dynamics.

W adjacency matrix of $\mathcal{G}$, $P = (D^{-1}W + I)/2$

$$x(t+1) = Px(t), \; x(0)_i = x_i$$

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected str. connected, $\quad i \in \mathcal{V} \to x_i$ state

Goal: compute $\bar{x} = n^{-1} \sum x_i$ $\quad$ Idea: use averaging dynamics.

W adjacency matrix of $\mathcal{G}$, $P = (D^{-1}W + I)/2$

$$x(t+1) = Px(t), \; x(0)_i = x_i$$

$x(t+1)_i = \frac{1}{2}x(t)_i + \frac{1}{2w_i} \sum_{j \in N_i} x(t)_j$

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected str. connected, $\quad i \in \mathcal{V} \to x_i$ state

Goal: compute $\bar{x} = n^{-1} \sum x_i$ Idea: use averaging dynamics.

W adjacency matrix of $\mathcal{G}$, $P = (D^{-1}W + I)/2$

$$x(t+1) = Px(t), \; x(0)_i = x_i$$

$\mathcal{G}_P = \mathcal{G} \cup \{selfloops\}$ str. connected, aperiodic

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected str. connected, $\quad i \in \mathcal{V} \to x_i$ state

Goal: compute $\bar{x} = n^{-1} \sum x_i$ $\quad$ Idea: use averaging dynamics.

W adjacency matrix of $\mathcal{G}$, $P = (D^{-1}W + I)/2$

$$x(t+1) = Px(t), \; x(0)_i = x_i$$

$\mathcal{G}_P = \mathcal{G} \cup \{selfloops\}$ str. connected, aperiodic

$$\Rightarrow x(t)_i \to x^* = \sum \pi_j x_j \qquad \pi_j = w_j/|\mathcal{E}| \text{ centrality}$$

$((D^{-1}W)'\pi = \pi \Leftrightarrow P'\pi = \pi)$

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected str. connected, $\quad i \in \mathcal{V} \to x_i$ state

Goal: compute $\bar{x} = n^{-1} \sum x_i$ $\quad$ Idea: use averaging dynamics.

W adjacency matrix of $\mathcal{G}$, $P = (D^{-1}W + I)/2$

$$x(t+1) = Px(t), \; x(0)_i = x_i$$

$\mathcal{G}_P = \mathcal{G} \cup \{selfloops\}$ str. connected, aperiodic

$$\Rightarrow x(t)_i \to x^* = \sum \pi_j x_j \qquad \pi_j = w_j/|\mathcal{E}| \text{ centrality}$$

$((D^{-1}W)'\pi = \pi \Leftrightarrow P'\pi = \pi)$

$$x^* = \bar{x} \Leftrightarrow \mathcal{G} \text{ is regular}$$

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected str. connected, $\quad i \in \mathcal{V} \to x_i$ state

Goal: compute $\bar{x} = n^{-1} \sum x_i$ $\quad$ Idea: use averaging dynamics.

W adjacency matrix of $\mathcal{G}$, $P = (D^{-1}W + I)/2$

$$x(t+1) = Px(t),\ x(0)_i = x_i$$

$\mathcal{G}_P = \mathcal{G} \cup \{selfloops\}$ str. connected, aperiodic

$$\Rightarrow x(t)_i \to x^* = \sum \pi_j x_j \qquad \pi_j = w_j/|\mathcal{E}| \text{ centrality}$$

$((D^{-1}W)'\pi = \pi \Leftrightarrow P'\pi = \pi)$

$x^* = \bar{x} \Leftrightarrow \mathcal{G}$ is regular $\quad$ What if $\mathcal{G}$ is not regular?

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected str. connected, $\quad i \in \mathcal{V} \to x_i$ state

Goal: compute $\bar{x} = n^{-1} \sum x_i$

Possible solutions:

- Use P as follows

$$\begin{cases} x(t+1) = Px(t), & x(0)_i = x_i / w_i \\ y(t+1) = Py(t), & y(0)_i = 1/w_i \end{cases}$$

Check (exercise): $\frac{x(t)_i}{y(t)_i} \to \bar{x}$

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ undirected str. connected, $\quad i \in \mathcal{V} \to x_i$ state

Goal: compute $\bar{x} = n^{-1} \sum x_i$

Possible solutions:

- Use P as follows

$$\begin{cases} x(t+1) = Px(t), & x(0)_i = x_i / w_i \\ y(t+1) = Py(t), & y(0)_i = 1/w_i \end{cases}$$

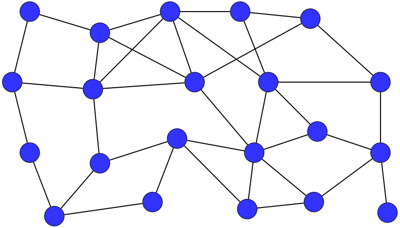
  Check (exercise): $\frac{x(t)_i}{y(t)_i} \to \bar{x}$

- Find Q stochastic symmetric s.t. $\mathcal{G}_Q \subseteq \mathcal{G} \cup \{\text{selfloops}\}$, $\mathcal{G}_Q$ aperiodic (exercise)

$$x(t+1) = Qx(t),\ x(0)_i = x_i, \quad x(t)_i \to \bar{x}$$

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ set of units (sensors) connected through a network



$i \in \mathcal{V} \to x_i$ state

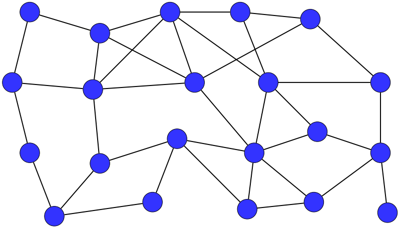
Goal: compute $\bar{x} = n^{-1} \sum x_i$

Using average consensus algorithms:

- no supervision, decentralized design;
- use only the available communication links;

Average computation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ set of units (sensors) connected through a network



$i \in \mathcal{V} \to x_i$ state

Goal: compute $\bar{x} = n^{-1} \sum x_i$

Using average consensus algorithms:

- no supervision, decentralized design;
- use only the available communication links;
- time and computation complexity?;

Speed of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ str. connected, $P = (D^{-1}W + I)/2$

$x(t+1) = Px(t), \quad x(t) \to \mathbb{1}(\pi' x(0))$

How fast $x(t)$ converges to consensus?

Speed of convergence

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ str. connected, $P = (D^{-1}W + I)/2$

$x(t+1) = Px(t), \quad x(t) \to \mathbb{1}(\pi' x(0))$

How fast $x(t)$ converges to consensus?

Theorem

$\mathcal{G}$ undirected str. connected. $P = (D^{-1}W + I)/2$.

- *Eigenvalues of P: $1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq 0$*
- $$||x(t) - \mathbb{1}\pi' x(0)||_2 \leq \sqrt{\frac{\max \pi_i}{\min \pi_i}} \lambda_2^t ||x(0)||_2$$

Speed of convergence and computational complexity

$$||x(t) - \mathbb{1}\pi' x(0)||_2 \leq \frac{\max \sqrt{\pi}_i}{\min \sqrt{\pi}_i} \lambda_2^t ||x(0)||_2$$

Speed of convergence and computational complexity

$$||x(t) - \mathbb{1}\pi' x(0)||_2 \leq \frac{\max \sqrt{\pi}_i}{\min \sqrt{\pi}_i} \lambda_2^t ||x(0)||_2$$

$$\sqrt{\frac{\max \pi_i}{\min \pi_i}} \lambda_2^t \leq \epsilon \Leftrightarrow t \geq \frac{\log(\epsilon^{-1} \max \pi_i / \min \pi_i)}{2 \log \lambda_2^{-1}}$$

Speed of convergence and computational complexity

$$||x(t) - \mathbb{1}\pi' x(0)||_2 \leq \frac{\max \sqrt{\pi}_i}{\min \sqrt{\pi}_i} \lambda_2^t ||x(0)||_2$$

$$\sqrt{\frac{\max \pi_i}{\min \pi_i}} \lambda_2^t \leq \epsilon \Leftrightarrow t \geq \frac{\log(\epsilon^{-1} \max \pi_i / \min \pi_i)}{2 \log \lambda_2^{-1}}$$

- *Convergence time:* $\tau_{conv}(\epsilon) := \frac{\log(\epsilon^{-1} \max \pi_i / \min \pi_i)}{2 \log \lambda_2^{-1}}$

$\log \lambda_2^{-1} \sim 1 - \lambda_2$ *spectral gap* (for $\lambda_2 \to 1$)

Speed of convergence and computational complexity

$$||x(t) - \mathbb{1}\pi' x(0)||_2 \leq \frac{\max \sqrt{\pi}_i}{\min \sqrt{\pi}_i} \lambda_2^t ||x(0)||_2$$

$$\sqrt{\frac{\max \pi_i}{\min \pi_i}} \lambda_2^t \leq \epsilon \Leftrightarrow t \geq \frac{\log(\epsilon^{-1} \max \pi_i / \min \pi_i)}{2 \log \lambda_2^{-1}}$$

- *Convergence time:* $\tau_{conv}(\epsilon) := \dfrac{\log(\epsilon^{-1} \max \pi_i / \min \pi_i)}{2 \log \lambda_2^{-1}}$

  $\log \lambda_2^{-1} \sim 1 - \lambda_2$ *spectral gap* (for $\lambda_2 \to 1$)

- *Computation complexity per node:* $\gamma(\epsilon) = \dfrac{\tau_{conv}(\epsilon)|\mathcal{E}|}{|\mathcal{V}|}$

Performance comparison

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ family of graphs with increasing size $n = |\mathcal{V}|$

	$1 - \lambda_2$	τ_{conv}	diam	γ
Line, cycle	C/n^2	Cn^2	Cn	Cn^2
***d*-dimensional Grids**	$Cn^{2/d}$	$n^{2/d}$	$Cn^{1/d}$	$Cdn^{2/d}$
Complete	$1/2$	1	1	Cn
Expanders	C	$C \log n$	$C \log n$	$C \log n$

Performance comparison

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ family of graphs with increasing size $n = |\mathcal{V}|$

	$1-\lambda_2$	τ_{conv}	diam	γ
Line, cycle	C/n^2	Cn^2	Cn	Cn^2
***d*-dimensional Grids**	$Cn^{2/d}$	$n^{2/d}$	$Cn^{1/d}$	$Cdn^{2/d}$
Complete	$1/2$	1	1	Cn
Expanders	C	$C\log n$	$C\log n$	$C\log n$

Expanders $\supseteq$ Random graphs, Barabasi, Small world

Network flow dynamics

- Models of transport phenomena
- They find applications in several fields: infrastructure networks, epidemiology, ecology, pharmacokinetics
- In some of the literature they are referred to as *compartmental systems*

Network flow dynamics

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ physical network

- $\mathcal{V}$ cells containing a homogeneous mass of some matter,
- $\mathcal{E}$ physical constraints: the matter can flow directly from cell i to cell j if $(i, j) \in \mathcal{E}$,
- $y_i(t)$ mass present in node i at time t.
- $f_{ij}(t)$ mass flowing from i to j at time t ($f_{ij}(t) = 0$ if $(i, j) \notin \mathcal{E}$)

Mass conservation law prescribes that

$$y_i(t+1) = y_i(t) + \sum_j f_{ji}(t) - \sum_j f_{ij}(t)$$

Linear network flow dynamics

$$y_i(t+1) = y_i(t) + \sum_j f_{ji}(t) - \sum_j f_{ij}(t)$$

We study the *linear* case: $f_{ij}(t) = y_i(t) P_{ij}$ where $P = D^{-1} W$.

Linear network flow dynamics

$$y_i(t+1) = y_i(t) + \sum_j f_{ji}(t) - \sum_j f_{ij}(t)$$

We study the *linear* case: $f_{ij}(t) = y_i(t)P_{ij}$ where $P = D^{-1}W$.
The amount of matter flowing from i to j is proportional to the amount present in i.

$$y_i(t+1) = y_i(t) + \sum_j y_j(t)P_{ji} - \sum_j y_i(t)P_{ij} = \sum_j y_j(t)P_{ji}$$

Linear network flow dynamics

$$y_i(t+1) = y_i(t) + \sum_j f_{ji}(t) - \sum_j f_{ij}(t)$$

We study the *linear* case: $f_{ij}(t) = y_i(t)P_{ij}$ where $P = D^{-1}W$.
The amount of matter flowing from i to j is proportional to the amount present in i.

$$y_i(t+1) = y_i(t) + \sum_j y_j(t)P_{ji} - \sum_j y_i(t)P_{ij} = \sum_j y_j(t)P_{ji}$$

Compact notation:

$$y(t+1) = P'y(t)$$

- Equilibria: $P'y = y$ eigenvectors of P' of eigenvalue 1
- Invariant distributions are equilibria (this is why they are called invariant!)
- $\mathbb{1}'y(t+1) = \mathbb{1}'P'y(t) = \mathbb{1}'y(t)$ Total mass is a motion invariant

Linear network flow dynamics

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic.

$P = D^{-1}W$, $\pi = P'\pi$ invariant distribution centrality

Linear network flow dynamics

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic.

$P = D^{-1}W$, $\pi = P'\pi$ invariant distribution centrality

The only equilibria are scalar multiples of π: $y = a\pi$.

Linear network flow dynamics

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic.

$P = D^{-1}W$, $\pi = P'\pi$ invariant distribution centrality

The only equilibria are scalar multiples of π: $y = a\pi$.

$$\lim_{t \to +\infty} P^t = \mathbb{1}\pi'$$

Linear network flow dynamics

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, aperiodic.

$P = D^{-1}W$, $\pi = P'\pi$ invariant distribution centrality

The only equilibria are scalar multiples of π: $y = a\pi$.

$$\lim_{t \to +\infty} P^t = \mathbb{1}\pi'$$

$$\lim_{t \to +\infty} y(t) = \lim_{t \to +\infty} P'^t y(0) = \pi \mathbb{1}' y(0)$$

Asymptotically, mass distributes according to π

Continuous time dynamics models

The linear averaging and the linear network flow dynamics on a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ have an analogue in continuous time:

$$\dot{x} = -Lx, \qquad \dot{y} = -L'y$$

where $L = D - W$ is the Laplacian of $\mathcal{G}$.

Theorem

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph and let L be its Laplacian. If $s_{\mathcal{G}} = 1$, then

$$\lim_{t \to +\infty} x(t) = \mathbb{1}\overline{\pi}' x(0) \tag{1}$$

$$\lim_{t \to +\infty} y(t) = \overline{\pi}\mathbb{1}' y(0) \tag{2}$$

where $\overline{\pi}$ is the unique Laplace invariant probability distribution of $\mathcal{G}$ ($L'\overline{\pi} = 0$, $\mathbb{1}'\overline{\pi} = 1$).