Algebraic graph theory and centrality measures

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Weight of walks and powers of the weight matrix

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$

$\gamma = (i_0, i_1, \ldots, i_l)$ walk, $W_\gamma = \prod_{1 \le h \le l} W_{i_{h-1} i_h}$

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Proof By induction on $l \geq 1$. $l = 1$ trivial.

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It is proven for $l+1$. By induction result is proven.

Properties of the products of the weight matrix

Theorem

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a graph. Then,

1. $(W^l)_{ij} > 0$ *if and only if there exists a walk of length* l *from* i *to* j;
2. $\mathcal{G}$ *is strongly connected iff for every* $i, j \in \mathcal{V}$*, there exists* $l > 0$ *such that* $(W^l)_{ij} > 0$.
3. $\mathcal{G}$ *is strongly connected and aperiodic iff there exists* $N > 0$ *such that* $(W^N)_{ij} > 0$ *for every* $i, j \in \mathcal{V}$.

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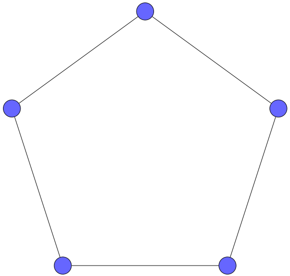
Comments on the proof:

- 1. and 2. are consequences of previous theorem.
- 3. is more involved.

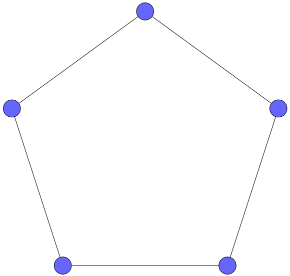
Examples

C_5



Examples

C_5



$$W = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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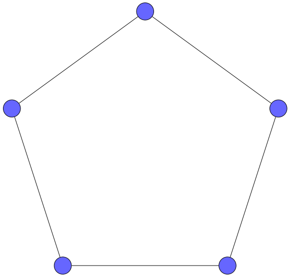

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$$W^2 = \begin{pmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{pmatrix}$$

Examples

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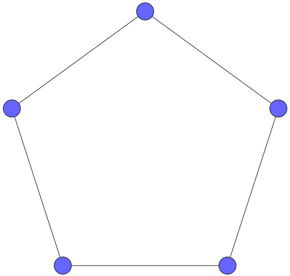

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Examples

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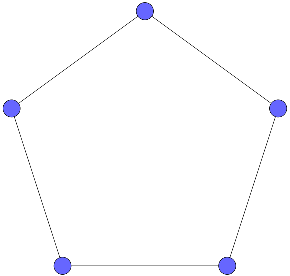

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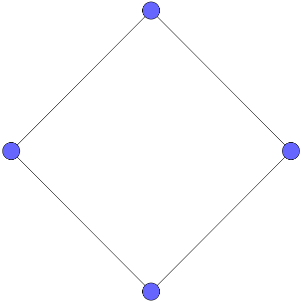
$$W^4 = \begin{pmatrix} 6 & 1 & 4 & 4 & 1 \\ 1 & 6 & 1 & 4 & 4 \\ 4 & 1 & 6 & 1 & 4 \\ 4 & 4 & 1 & 6 & 1 \\ 1 & 4 & 4 & 1 & 6 \end{pmatrix}$$

Examples

C_4



Examples

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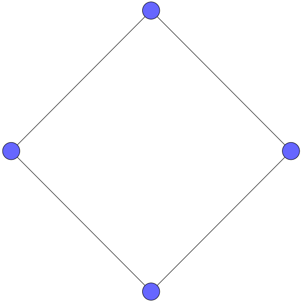

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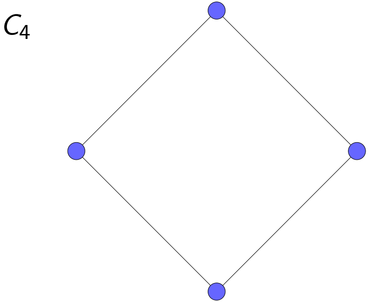
Examples



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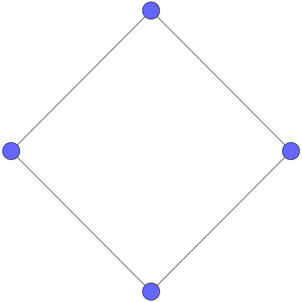

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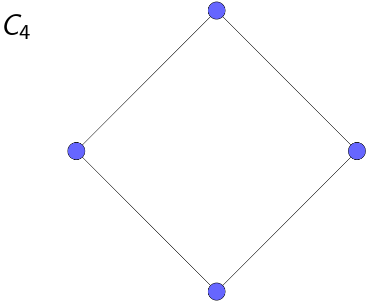
Examples



$$W = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$W^4 = \begin{pmatrix} 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \\ 8 & 0 & 8 & 0 \\ 0 & 8 & 0 & 8 \end{pmatrix}$$

Counting objects in simple graphs

Corollary

For a simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, we have that:

(i) $(W^2)_{ii} = w_i$ *for every* $i \in \mathcal{V}$*;*

(ii) $Tr(W^2) = |\mathcal{E}|$*;*

(iii) $Tr(W^3) = 6 \cdot$ *number of triangles.*

The normalized weight matrix

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, $w_i = \sum_j W_{ij}$ out-degrees

$P_{ij} = w_i^{-1} W_{ij}$ *normalized weight matrix* of $\mathcal{G}$.

More compactly, $P = D^{-1} W$ where D is diagonal with $D_i = w_i$

P is a *stochastic* matrix: $P_{ij} \geq 0$, $P\mathbb{1} = \mathbb{1}$

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- Topology of $\mathcal{G} \leftrightarrow$ Spectral properties of P;
- Through P we can describe interesting dynamical systems over $\mathcal{G}$;
- P can be interpreted as the transition matrix of a Markov chain, a *random walk* over $\mathcal{G}$.

The Laplacian matrix

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, D diagonal matrix with $D_i = w_i$ out-degrees

$$L = D - W$$

$$L_{ij} = \begin{cases} -W_{ij} & \text{if } i \neq j \\ w_i - W_{ii} & \text{if } i = j \end{cases}$$

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- Topology of $\mathcal{G} \leftrightarrow$ Spectral properties of L;
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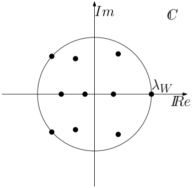
Perron-Frobenius theory



$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$$

Theorem (**Perron-Frobenius**)

There exists $\lambda_W \geq 0$ and non-negative vectors $x \neq 0$, $y \neq 0$ s.t.

- $Wx = \lambda_W x, \quad W'y = \lambda_W y;$
- *every eigenvalue μ of W is such that $|\mu| \leq \lambda_W$;*

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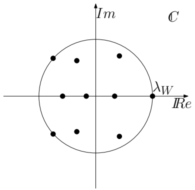

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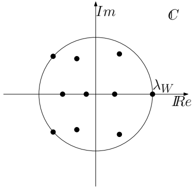

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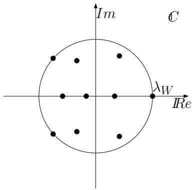

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λ_W *dominant eigenvalue* of W.

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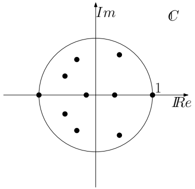

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Theorem (Spectral properties of stochastic matrices)

- $\lambda_P = 1$;
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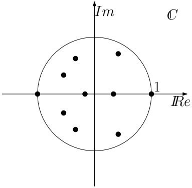

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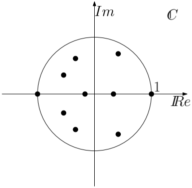

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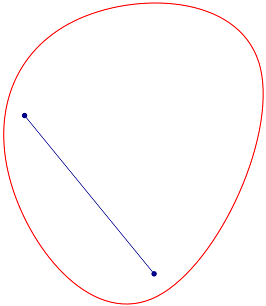
π invariant distribution

A digression: convex sets...

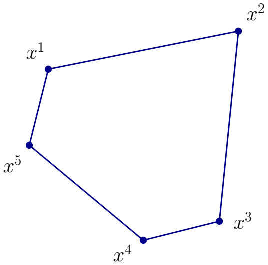
A subset $\mathcal{K} \subseteq \mathbb{R}^n$ is called *convex* if given any two points in $\mathcal{K}$, the segment joining them is all inside $\mathcal{K}$



polytopes...

A *polytope* is a convex subset that can be obtained by taking convex combinations of $k+1$ vectors $x^1, \dots, x^{k+1} \in \mathbb{R}^n$,

$$\mathcal{K} = \{x = \sum_i \lambda_i x^i \mid \lambda_i \geq 0, \ \sum_i \lambda_i = 1\}$$



x^i extremal points of the polytope.

Simplexes

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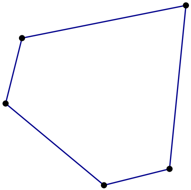
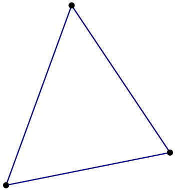


polytope, not a symplex



symplex

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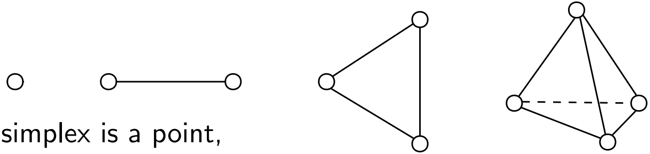

- 0-simplex is a point,
- 1-simplex is a segment,
- 2-simplex is a triangle,
- ...

Simplexes

Example: $x^i = e^i \in \mathbb{R}^n$ canonical basis

$$\mathcal{K} = \{x = (\lambda_1, \ldots, \lambda_n); \mid \lambda_i \geq 0,\ \sum_i \lambda_i = 1\}$$

$(n-1)$-simplex of probability vectors in $\mathbb{R}^n$

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Perron-Frobenius theory $\Rightarrow \exists$ a fixed point $\pi \in \mathcal{K}$: $\quad P'\pi = \pi$

Perron-Frobenius theory applied to graphs

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A more refined result.

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A more refined result.

$\mathcal{H}_{\mathcal{G}}$ condensation graph, $s_{\mathcal{G}}$ number of sinks in $\mathcal{H}_{\mathcal{G}}$.

Theorem

- *Invariant distributions* π

$$\pi \geq 0,\ \mathbb{1}'\pi = 1,\ P'\pi = \pi$$

  form a simplex in $\mathbb{R}^{\mathcal{V}}$ *with* $s_{\mathcal{G}}$ *vertices.*
- *For every sink component with nodes* $\mathcal{W}$*, there exists an invariant distribution* π *such that* $\pi_i > 0$ *if and only if* $i \in \mathcal{W}$*.*
- *The invariant distribution is unique if and only if* $s_{\mathcal{G}} = 1$*.*

The special case of balanced graphs

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, $P = D^{-1}W$.

A remarkable fact.

Suppose $\mathcal{G}$ is balanced ($w_i = w_i^-$ for every $i \in \mathcal{V}$). Then,

The special case of balanced graphs

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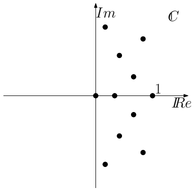
$$P' w = w$$

- the out-degree vector w is an eigenvector of eigenvalue 1
- $\pi = \frac{1}{|\mathcal{E}|} w$ is an invariant distribution of $\mathcal{G}$!
- If the graph is strongly connected, this is the unique invariant distribiution.

Perron-Frobenius theory applied to graphs



$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$

$L = D - W$ Laplacian matrix

Theorem (Spectral properties of the Laplacian)

- *$L\mathbb{1} = 0$, 0 is an eigenvalue of L*
- *$\exists \bar{\pi} \geq 0$ s.t. $\mathbb{1}'\bar{\pi} = 1$ and $L'\bar{\pi} = 0$;*
- *All other eigenvalues λ have $\mathcal{R}e(\lambda) > 0$;*
- *If $\mathcal{G}$ is strongly connected, then 0 is simple and $\bar{\pi}_i > 0$ for all i;*
- *$L'\bar{y} = 0 \Leftrightarrow P'(D\bar{y}) = (D\bar{y})$;*

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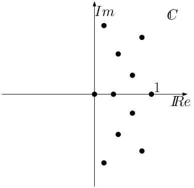

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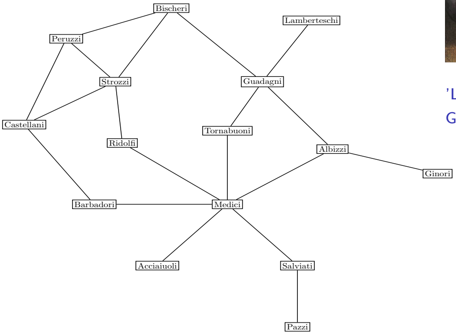
$\bar{\pi}$ Laplace-invariant distribution

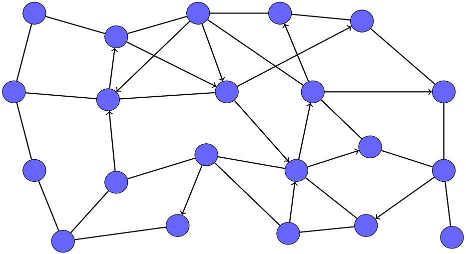
Who is the most central node?



'Lorenzo de' Medici'
G. Vasari, 1534

Network centralities

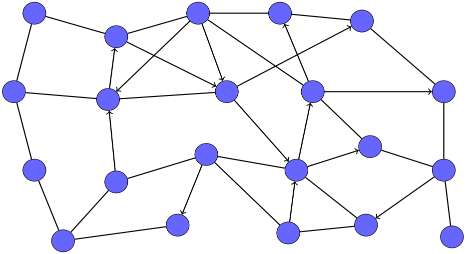
$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$



$z_i \geq 0$ centrality of node i

Degree centrality

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$

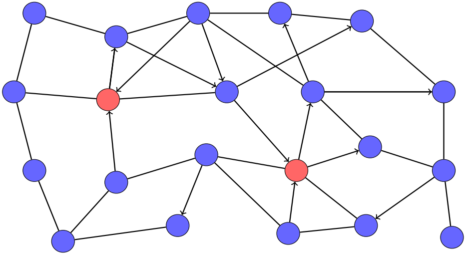


$z_i = w_i^-$ in-degree of node i *degree centrality*

Example: number of citations of an article, of followers on Twitter

Degree centrality

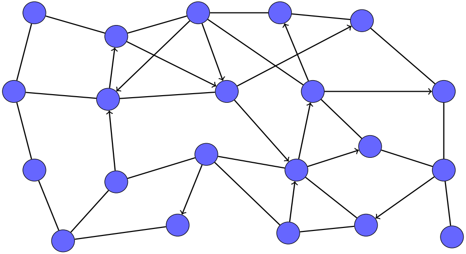
$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$$



The two nodes with the highest degree centrality.

Eigenvector centrality

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$



Drawback of degree-centrality: all incoming links are considered the same.

A different approach: $z_i \propto \sum_{j \in N_i^-} z_j$

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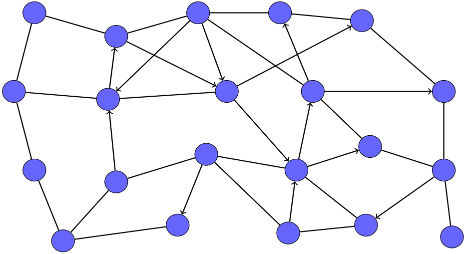

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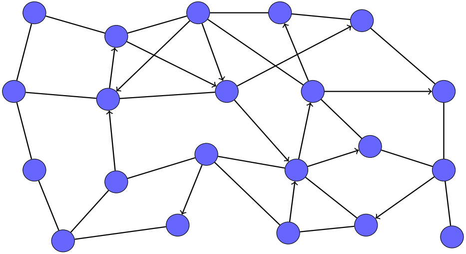

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$\lambda z = W' z$ with $\lambda = \lambda_W$ *eigenvector centrality*

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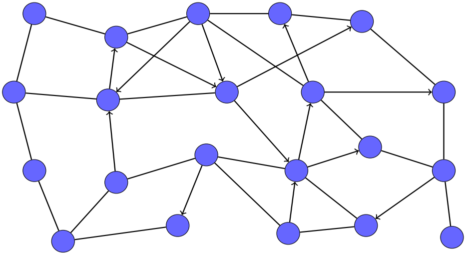


$\lambda z = W'z$ with $\lambda = \lambda_W$

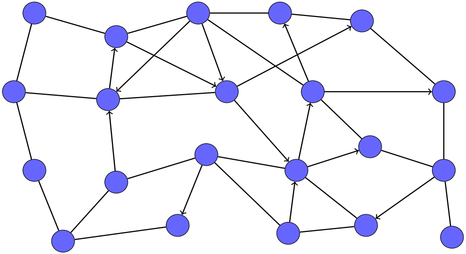
Remark: If all nodes have the same in-degree: $w_i^- = \delta$ for all i

$W'\mathbb{1} = \delta\mathbb{1} \;\Rightarrow\; z = \mathbb{1}$

Drawback: node j contributes proportional to its out-degree w_j

Invariant distribution centrality

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$



Another centrality measure: $z_i \propto \sum_{j \in N_i^-} \frac{1}{w_j} z_j$

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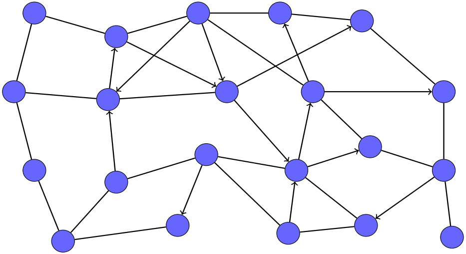

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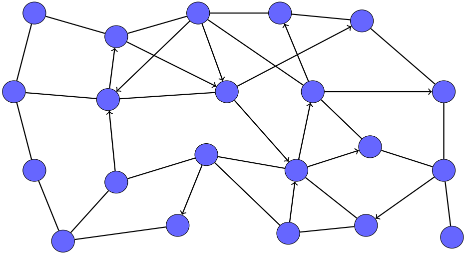

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$z = P'z \Rightarrow$ *$z = \pi$ invariant distribution centrality*

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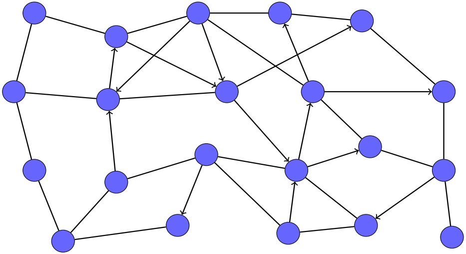

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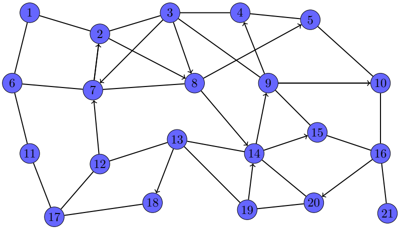
$z = P'z \Rightarrow$ $z = \pi$ *invariant distribution centrality*

If $\mathcal{G}$ is balanced, $\pi = w = w^- \Rightarrow$ inv. dist. centr. = deg centr.

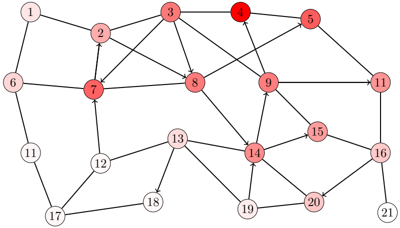
A comparison of the various centralities



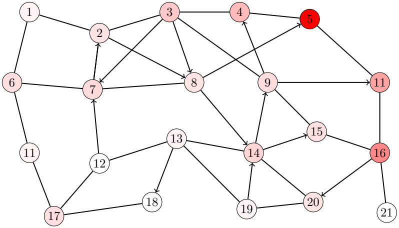
	Deg	Eig	Inv. dist.
1	0.0345	0.0348	0.0313
2	0.0517	0.0581	0.0451
3	0.0517	0.0664	0.0613
4	0.0517	0.0689	0.0680
5	0.0517	0.0680	0.0869
6	0.0517	0.0430	0.0490
7	0.0690	0.0678	0.0514
8	0.0517	0.0661	0.0444
9	0.0517	0.0659	0.0491
10	0.0517	0.0627	0.0761
11	0.0345	0.0226	0.0324
12	0.0345	0.0215	0.0240
13	0.0517	0.0399	0.0317
14	0.0690	0.0640	0.0548
15	0.0517	0.0613	0.0464
16	0.0517	0.0484	0.0817
17	0.0517	0.0225	0.0481
18	0.0345	0.0215	0.0240
19	0.0345	0.0307	0.0300
20	0.0517	0.0492	0.0441
21	0.0172	0.0166	0.0204

A comparison of the various centralities

Eigenvalue centrality



Invariant distribution centrality



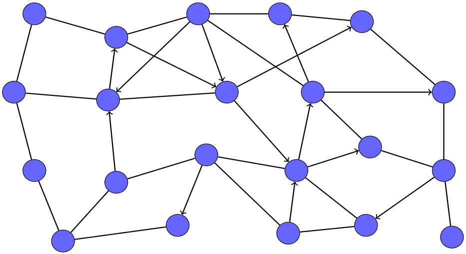
Drawbacks of eigenvalue and invariant distribution centr.

In general, they are not uniquely defined if the graph has more than one sink component

if the graph has just one sink component, only nodes in that component have non zero centrality

Katz centrality

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$



A convex comb. of *network centrality* and *intrinsic centrality*

$$z = \frac{1-\beta}{\lambda_W} W' z + \beta \mu$$

$z = (I - (1-\beta)\lambda_W^{-1} W')^{-1} \beta \mu$ *Katz centrality*

Bonacich centrality

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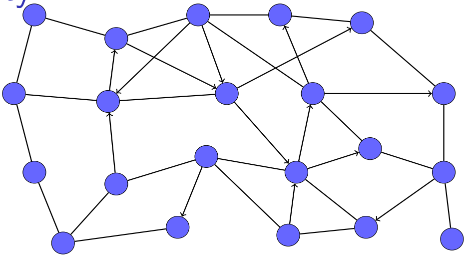

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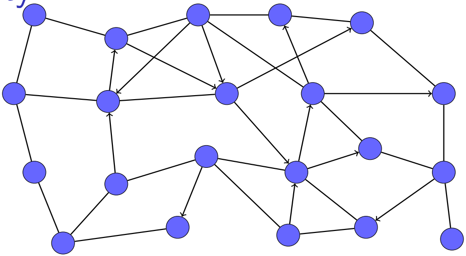

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Page-rank (Google): $\mu = n^{-1}\mathbb{1}$, $\beta \sim 0.15$

The structure of Bonacich centrality

$$z = (I - (1-\beta)P')^{-1}\beta\mu$$

$$z_i = \beta \sum_{k=0}^{\infty} (1-\beta)^k \sum_j \mu_j (P^k)_{ji}$$

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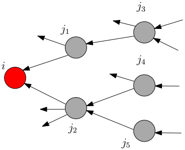

$$= \beta \left[\mu_i + (1-\beta)(\mu_{j_1} P_{j_1 i} + \mu_{j_2} P_{j_2 i}) + (1-\beta)^2 (\mu_{j_3} P_{j_3 j_1} P_{j_1 i} + \mu_{j_4} P_{j_4 j_2} P_{j_2 i} + \mu_{j_5} P_{j_5 j_2} P_{j_2 i})\right] + \cdots$$

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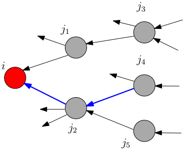

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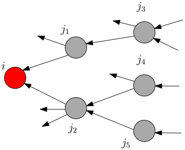

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- The Bonacich centrality is uniquely defined for every graph $\mathcal{G}$
- $\beta = 0 \;\Rightarrow\;$ Intrinsic centrality (no network)
- $\beta \to 1 \;\Rightarrow\;$ Invariant distribution centrality (only network)

The various centralities

Eigenvector centrality	Invariant distribution
$z = \frac{1}{\lambda_W} W' z$	$z = P' z$
Katz centrality	**Bonacich centrality**
$z = \frac{1-\beta}{\lambda_W} W' z + \beta \mu$	$z = (1-\beta) P' z + \beta \mu$

Some references

- L. Katz; A new status index derived from sociometric analysis, *Psychometrica*, 18, 3–43, 1953.
- P. Bonacich; Power and Centrality: A Family of Measures, *American Journal of Sociology*, 1987.