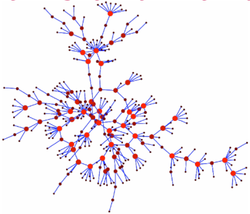
01RMHNG–03RMHPF–01RMING
Network Dynamics
Week 7
Markov Chains and Random Walks



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Torino, November 7-12, 2024

This week

- Discrete-Time Markov Chains and Random Walks
- Convergence in Probability and Ergodic Theorem
- Hitting Times and Absorbing Probabilities
- Reversible Markov Chains
- Birth-and-Death Chains
- Continuous-Time Markov Chains

Random walks

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ (weighted, directed) graph, $P = D^{-1}W$

Random walk on $\mathcal{G}$:

- start in an initial node $i \in \mathcal{V}$ with probability p_i;
- from every node i jump to any other node j with probability P_{ij};

$X(t) =$ position at time t. It is a random process:

1. Markov property:

$$\mathbb{P}(X(t+1) = i_{t+1}, | X(0) = i_0, \dots X(t) = i_t) = P_{i_t i_{t+1}}$$

   future is conditionally independent from the past given the present

2. Initial condition: $\mathbb{P}(X(0) = i) = \pi_i(0)$

Random walks and Markov chains

(1) Markov property:

$$\mathbb{P}(X(t+1) = i_{t+1}, \,|\, X(0) = i_0, \ldots X(t) = i_t) = P_{i_t i_{t+1}}$$

(2) Initial condition: $\mathbb{P}(X(0) = i) = \pi_i(0)$

Random process $X(t)$ on finite set $\mathcal{V}$ satisfying (1) and (2) is a (homogeneous) Markov chain with transition probability matrix P and initial distribution $\pi(0)$.

If we consider $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ where $\mathcal{V} = \{1, \ldots, n\}$, $\mathcal{E} = \{(i,j) \,|\, P_{ij} > 0\}$, $W_{ij} = P_{ij}$, we have that $X(t)$ coincides with a random walk on $\mathcal{G}$.

Random walks and Markov chains

- Markov chain $X(t)$, transition probability matrix P, initial distrib. $\pi(0)$
- Finite time distributions:

$$\mathbb{P}(X(0) = i_0, X(1) = i_1, \dots, X(t) = i_t)$$
$$= \mathbb{P}(X(0) = i_0)\mathbb{P}(X(1) = i_1, \mid X(0) = i_0) \cdots \mathbb{P}(X(t) = i_t, \mid X(t-1) = i_{t-1})$$
$$= \pi_{i_0}(0) P_{i_0 i_1} \cdots P_{i_{t-1} i_t}$$

Random walks and Markov chains

- Markov chain $X(t)$, transition probability matrix P, initial distrib. $\pi(0)$
- Finite time distributions:

$$\mathbb{P}(X(0) = i_0, X(1) = i_1, \dots, X(t) = i_t)$$
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$$= \pi_{i_0}(0) P_{i_0 i_1} \cdots P_{i_{t-1} i_t}$$

- Marginal distributions $\pi_i(t) = \mathbb{P}(X(t) = i)$ satisfy following recursion:

$$\begin{array}{rcl} \pi_i(t+1) & = & \mathbb{P}(X(t+1) = i) \\ & = & \sum_j \mathbb{P}(X(t) = j, X(t+1) = i) \\ & = & \sum_j \mathbb{P}(X(t) = j)\mathbb{P}(X(t+1) = i | X(t) = j) \\ & = & \sum_j \pi_j(t) P_{ji} \end{array}$$

The same evolution of linear flow dynamics

$$\pi(t+1) = P'\pi(t)$$

Limit behavior of Markov chains

- $X(t)$ discrete-time random walk on $\mathcal{G}$
- given initial probability distribution $\pi(0)$

$$\pi(t+1) = P'\pi(t)$$

- Definition: π stationary probability distribution for $X(t)$ if

$$\pi(0) = \pi \qquad \Longrightarrow \qquad \pi(t) = \pi \quad \forall t \geq 0$$

Theorem: $\mathcal{G}$ connected graph. Then,

(i) $\exists$ unique stationary probability distribution $\pi = P'\pi$ and

$$\pi_i > 0 \qquad \forall i \in \mathcal{V}$$

If $\mathcal{G}$ is also aperiodic, then:

(ii) $\pi(t) \stackrel{t \to +\infty}{\longrightarrow} \pi$ for every $\pi(0)$. Equivalently, $X(t)$ converges in distribution to a random variable X with distribution π.

The ergodic theorem

Theorem (Ergodic theorem)

If $\mathcal{G}$ connected, then, for every $f : \mathcal{V} \to \mathbb{R}$,

$$\lim_{t \to +\infty} \frac{1}{t} \sum_{s=0}^{t-1} f(X(s)) = \sum_{i \in \mathcal{V}} \pi_i f(i)$$

with probability 1*.*

Time Average $\quad = \quad$ Spatial average

- Note: $X(t)$ ergodic if $\mathcal{G}$ is connected.

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- Note: $X(t)$ ergodic if $\mathcal{G}$ is connected.

- Example: $f(j) = \delta_j^i = \begin{cases} 1 & \text{if} \quad j = i \\ 0 & \text{if} \quad j \neq i \end{cases}$

$$\lim_{t\to+\infty} \frac{\text{number of visits in } i \text{ before time } t}{t} = \pi_i \qquad \text{w.p. } 1$$

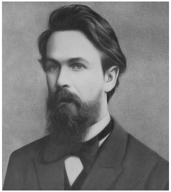
A generalization of the law of large numbers

- Markov Chain Monte Carlo (MCMC): simulate $X(t)$ to estimate π

A historical remark

Andrei Andreevich Markov

1906: Extension of the law of large numbers to dependent quantities, Izvestiia Fiz.-Matem. Obsch. Kazan Univ



Beginning 20th century: a debate in Russia regarding the interpretation of certain regularity observed in social behaviors.

Quetelet: laws governing social phenomena exactly as in physics.

Nekrasov: theological arguments (free will) against social physics. Law of large numbers only holds for independent variables.

Markov invented the chains just to disprove this claim!

The many roles of the invariant distribution centrality

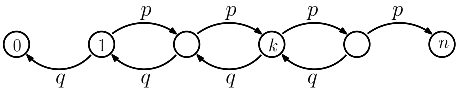
$$\pi = P'\pi$$

- it measures centrality in networks;
- it determines the consensus value in linear averaging dynamics;
- it describes the fraction of time spent in the various nodes by a random walk on the graph;
- it describes the asymptotics of other network dynamics (gossip dynamics, voter model) (details next week)

Gambler's ruin

- At each time step a gambler bets one euro.
- He wins with probability p and loses with probability q.
- The gambler's initial capital is $k > 0$ euros, and the gambler quits when he either gets broke (corresponding to a capital of 0 euros) or reaches a pre-defined capital target of $n > k$ euro.

Random walk on a weighted line graph:



The chain is not ergodic! $\mathcal{G}$ has two sinks.

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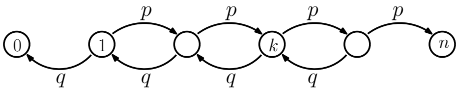

The chain is not ergodic! $\mathcal{G}$ has two sinks.

Interesting questions:

- What is the expected time the game will last?
- What is the probability that the better reaches her n euro target?

Hitting times

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, $P = D^{-1}W$,

$X(t)$ random walk on $\mathcal{G}$ with some initial distribution $\pi(0)$

Notation: $\mathbb{P}_i$, $\mathbb{E}_i$ probability and expected value for the chain $X(t)$ assuming $\pi(0) = \delta^{(i)}$ (i.e. random walk starts in i)

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- Definition: hitting time on $\mathcal{S} \subseteq \mathcal{V}$:

$$T_{\mathcal{S}} = \min\{t \geq 0 \,|\, X(t) \in \mathcal{S}\} \in \{0, 1, 2, \ldots, +\infty\}$$

- Note 1: $T_{\mathcal{S}} = 0 \Leftrightarrow X(0) \in \mathcal{S}$
- Note 2: we can have $T_{\mathcal{S}} = +\infty$ with positive probability

Finiteness of expected hitting times

- Theorem: $X(t)$ discrete-time random walk on graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. If $\mathcal{S} \subseteq \mathcal{V}$ globally reachable, then

$$\mathbb{P}_i(T_{\mathcal{S}} < +\infty) = 1, \qquad \mathbb{E}_i[T_{\mathcal{S}}] < +\infty$$

for all $i \in \mathcal{V}$.

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- Corollary: $X(t)$ discrete-time random walk on graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. If $\mathcal{S} \subseteq \mathcal{V}$ globally reachable and trapping, then, with probability 1, $\exists T_{\mathcal{S}} < +\infty$ s.t.

$$X(t) \in \mathcal{S}\,, \qquad \forall t \geq T_{\mathcal{S}}$$

- When $\mathcal{S}$ trapping: hitting time = absorbing time

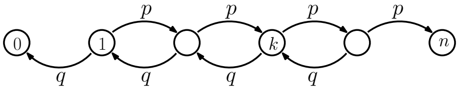
Expected hitting times

Theorem (Recursive relations for expected hitting times)

$X(t)$ random walk on $\mathcal{G}$, $\mathcal{S} \subseteq \mathcal{V}$ globally reachable. Then, the expected hitting times related to $\mathcal{S}$ are the only family of finite values satisfying the relations:

$$\begin{aligned} \mathbb{E}_i[T_{\mathcal{S}}] &= 0 && i \in \mathcal{S} \\ \mathbb{E}_i[T_{\mathcal{S}}] &= 1 + \sum_{j \in \mathcal{V}} P_{ij} \mathbb{E}_j[T_{\mathcal{S}}] && i \notin \mathcal{S}\,. \end{aligned}$$

Expected hitting time for the gambler's ruin

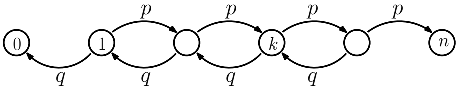


$q = 1 - p$

$\mathcal{V} = \{0, 1, \ldots, n\}$, $P_{i,i+1} = p \qquad P_{i,i-1} = q$

$$\mathbb{E}_0[T_{\mathcal{S}}] = \mathbb{E}_n[T_{\mathcal{S}}] = 0\,, \quad \mathbb{E}_i[T_{\mathcal{S}}] = 1 + q\mathbb{E}_{i-1}[T_{\mathcal{S}}] + p\mathbb{E}_{i+1}[T_{\mathcal{S}}]\,, \quad 1 \le i < n\,,$$

Expected hitting time for the gambler's ruin



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Solution for $p = q = 1/2$:

$$\mathbb{E}_i[T_{\mathcal{S}}] = i(n-i)$$

A useful technical result

Lemma

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, $P = D^{-1}W$, $\mathcal{S} \subseteq \mathcal{V}$ globally reachable, $\mathcal{R} = \mathcal{V} \setminus \mathcal{S}$
$Q = P_{|\mathcal{R}\times\mathcal{R}}$ is such that

- $||Q^t \mathbb{1}||_\infty \leq \beta^t$ *for some* $\beta < 1$
- $\lim\limits_{t\to+\infty} Q^t = 0$ $\quad Q$ *asymptotically stable*
- $(I - Q)$ *invertible as* 1 *is not an eigenvalue*
- $(I - Q)^{-1} = \sum_{t=0}^{+\infty} Q^t$ *geometric series*

Proof Proven in the lecture on stubborn nodes models.

Expected hitting times

Theorem (Recursive relations for expected hitting times)

$X(t)$ random walk on $\mathcal{G}$, $\mathcal{S} \subseteq \mathcal{V}$ globally reachable. Then, the expected hitting times related to $\mathcal{S}$ are the only family of finite values satisfying the relations:

$$\begin{aligned} &\mathbb{E}_i[T_{\mathcal{S}}] = 0 && i \in \mathcal{S} \\ &\mathbb{E}_i[T_{\mathcal{S}}] = 1 + \sum_{j\in\mathcal{V}} P_{ij}\mathbb{E}_j[T_{\mathcal{S}}] && i \notin \mathcal{S}\,. \end{aligned}$$

Proof

- Expected hitting times are finite (previous result)
- Conditioning with respect to the first jump:

$$\mathbb{E}_i[T_{\mathcal{S}}] = \sum_{j\in\mathcal{V}} P_{ij}\mathbb{E}_i[T_{\mathcal{S}} \,|\, X(1) = j] = \sum_{j\in\mathcal{V}} P_{ij}(1 + \mathbb{E}_j[T_{\mathcal{S}}])$$

Expected hitting times

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Proof

- Uniqueness: put $\mathcal{R} = \mathcal{V} \setminus \mathcal{S}$ and $Q = P_{|\mathcal{R} \times \mathcal{R}}$, $\tau \in \mathbb{R}^{\mathcal{R}}$, $\tau_i = \mathbb{E}_i[T_{\mathcal{S}}]$

$$\tau = \mathbb{1} + Q\tau \iff (I - Q)\tau = \mathbb{1} \iff \tau = (I - Q)^{-1}\mathbb{1}$$

■

Return times

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, $P = D^{-1}W$,

$X(t)$ random walk on $\mathcal{G}$ with some initial distribution p

$h \in \mathcal{V}, \quad T_h^+ = \min\{t \geq 1 \,|\, X(t) = h\}$ return time

$\mathbb{E}_i[T_h^+] = \mathbb{E}_i[T_h]$ if $i \neq h$

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Theorem

Expected return times satisfy the relations

$$\mathbb{E}_i[T_i^+] = 1 + \sum_{j \in \mathcal{V}} P_{ij}\mathbb{E}_j[T_i]$$

Proof Exercise (conditioning argument...)

Return times

Theorem (Kac's formula)

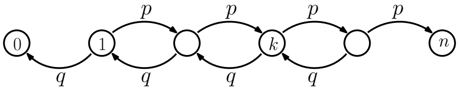
$X(t)$ random walk on $\mathcal{G}$ strongly connected, π unique invariant distribution. For every $i \in \mathcal{V}$, it holds

$$\pi_i = \frac{1}{\mathbb{E}_i[T_i^+]}$$

In a graph $\mathcal{G}$, the invariant distribution centrality of a node i is the reciprocal of the expected return time of the random walk on $\mathcal{G}$ that starts in node i.

Back to the gambler's ruin

- At each time step a gambler bets one euro.
- He wins with probability p and loses with probability q.
- The gambler's initial capital is $k > 0$ euros, and the gambler quits when he either gets broke (corresponding to a capital of 0 euros) or reaches a pre-defined capital target of $n > k$ euros.



Interesting questions:

- What is the expected time the game will last?
- What is the probability that he will achieve the capital of n euros?

Hitting probabilities

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, $P = D^{-1}W$, $X(t)$ random walk on $\mathcal{G}$

$\mathcal{S} \subseteq \mathcal{V}$, $T_{\mathcal{S}} = \min\{t \geq 0 \,|\, X(t) \in \mathcal{S}\}$

$$H_{i,s} = \mathbb{P}_i(X(T_{\mathcal{S}}) = s) \quad \textit{hitting probabilities}$$

$H_{i,s}$ is the probability that $X(t)$ first hits $\mathcal{S}$ in $s \in \mathcal{S}$

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Theorem (Hitting probabilities)

$X(t)$ random walk on $\mathcal{G}$ and $\mathcal{S} \subseteq \mathcal{V}$ globally reachable. Then, for all $s \in \mathcal{S}$, the hitting probabilities $\{H_{i,s}\}_{i \in \mathcal{V}}$ is the only family of values satisfying the relations:

$$H_{i,s} = \sum_{j \in \mathcal{V}} P_{ij} H_{j,s} \;\; i \notin \mathcal{S}$$
$$H_{s,s} = 1, \;\; H_{i,s} = 0 \; i \in \mathcal{S} \setminus \{s\}$$

Hitting probabilities

H_s vector of hitting probabilities in s starting from the various $i \in \mathcal{V}$

$\Rightarrow \ ((I - P)H_s)_i = 0 \ \ i \in \mathcal{R} = \mathcal{V} \setminus \mathcal{S}$

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- $H_{i,s}$ can be interpreted as the asymptotic opinion of agent i when nodes in $\mathcal{S}$ are stubborn with $u_s = 1$ and $u_{s'} = 0$ for every $s' \in S \setminus \{s\}$

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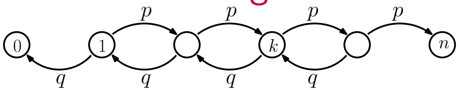
- $H_{i,s}$ can be interpreted as the asymptotic opinion of agent i when nodes in $\mathcal{S}$ are stubborn with $u_s = 1$ and $u_{s'} = 0$ for every $s' \in S \setminus \{s\}$
- If $\mathcal{G}$ is undirected, $H_{i,s}$ can be interpreted as the voltage of node i when the voltage for the nodes in $\mathcal{S}$ is $u_s = 1$ and $u_{s'} = 0$ for every $s' \in S \setminus \{s\}$

When $\mathcal{S}$ trapping: hitting probabilities $=$ absorbing probabilities

Absorbing probabilities for the gambler's ruin



$0 < q = 1 - p < 1$, $\qquad H_{i,0} + H_{i,n} = 1$

$$H_{0,n} = 0\,, \qquad H_{n,n} = 1$$

$$H_{i,n} = qH_{i-1,n} + pH_{i+1,n}\,, \qquad 1 \le i < n$$

Let $y_i = H_{i,n} - H_{i-1,n}$. Then,

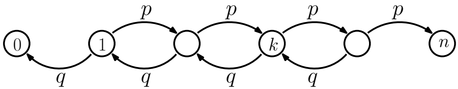
$$qy_i = py_{i+1}\,, \qquad 1 \le i < n$$

so that

$$y_i = (q/p)^{i-1}\, y_1\,, \qquad 1 \le i \le n$$

$$\sum_{1 \le k \le n} y_k = H_{nn} - H_{0n} = 1 \qquad \Longrightarrow \qquad y_i = \frac{(q/p)^{i-1}}{\sum_{1 \le k \le n} (q/p)^{k-1}}$$

Absorbing probabilities for the gambler's ruin



Let $\rho = p/q$. Then, we have found that

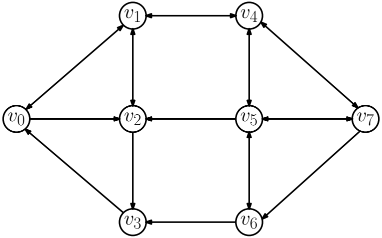
$$y_i = H_{i,n} - H_{i-1,n} = \frac{\rho^{-(i-1)}}{\sum_{1 \le k \le n} \rho^{-(k-1)}}$$

Hence, $H_{k,n} = \dfrac{\sum_{1 \le i \le k} \rho^{-(i-1)}}{\sum_{1 \le i \le n} \rho^{-(i-1)}} = \begin{cases} k/n & \text{if} \quad \rho = 1 \\ \dfrac{1-\rho^{-k}}{1-\rho^{-n}} & \text{if} \quad \rho \neq 1 \end{cases}$

- Note: if $\rho < 1$ then

$n - k \to +\infty \;\Rightarrow\; H_{k,n} \to 0 \qquad\qquad j = \text{const.} \;\Rightarrow\; H_{n-j,n} \to \rho^j$

Example



$$W = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad w = (2, 3, 2, 1, 3, 4, 2, 3)$$

Example continued

$$W = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

- Stationary distribution $\pi = P'\pi$

$$\pi = (0.1887, 0.2264, 0.1887, 0.1132, 0.1132, 0.0755, 0.0377, 0.0566)$$

Example continued

$$W = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

- Expected hitting times on $\mathcal{S} = \{v_0, v_7\}$, $\tau_i = \mathbb{E}_i[T_\mathcal{S}]$:

$$\mathbb{E}_i[T_\mathcal{S}] = 0 \quad \text{if} \quad i \in \mathcal{S}$$

$$\mathbb{E}_i[T_\mathcal{S}] = 1 + \textstyle\sum_j P_{ij}\mathbb{E}_j[T_\mathcal{S}] \quad \text{if} \quad i \notin \mathcal{S}$$

$$\tau = (0, 3.0506, 3.0253, 1, 3.1266, 3.3291, 3.1646, 0)$$

Example continued

$$W = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

- Probability to hit v_7 before v_0:

$$H_{i7} = \mathbb{P}(X(T_{\mathcal{S}}) = 7 | X(0) = i)$$

$$H_{i7} = \sum_j P_{ij} H_{j7} \quad \text{if} \quad i \notin \mathcal{S}$$

$$H_{77} = 1 \qquad\qquad H_{07} = 0$$

$$H_{\cdot,7} = (0, 0.2278, 0.1139, 0, 0.5696, 0.4810, 0.2405, 1)$$

Reversible Markov chains

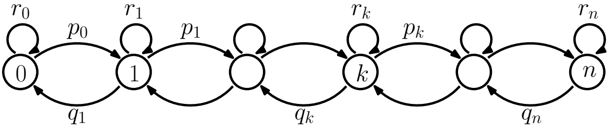
Stochastic matrix P reversible with respect to prob. vector π if

$$\pi_i P_{ij} = \pi_j P_{ji} \qquad \forall i, j \in \mathcal{X} \qquad \text{[detailed balance]}$$

- P reversible w.r.t. $\pi \quad \implies \quad \pi = P'\pi$ invariant probability
- P reversible w.r.t. π, if $X(0)$ has distribution π, then

  $$(X(0), X(1), \ldots, X(t-1), X(t)) \stackrel{\text{dist}}{=} (X(t), X(t-1), \ldots, X(1), X(0))$$
- $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ undirected $\implies P = D^{-1}W$ reversible w.r.t. $\pi \propto w$
- P reversible w.r.t. $\pi \implies \mathcal{G} = (\mathcal{V}, \mathcal{E}, W) \quad W = \text{diag}(\pi)P$ undir.

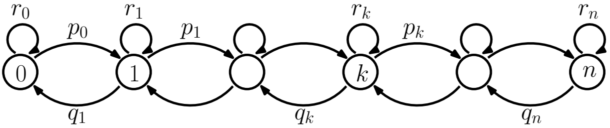
Birth-and-death chains



- Birth-and-death chains: Random walks on weighted line graphs: at every t, $X(t)$ can only increase or decrease by 1 or remain still:

$$\mathcal{V} = \{0, 1, \ldots, n\}, \qquad P_{i,i+1} = p_i, \qquad P_{i,i-1} = q_i, \qquad P_{i,i} = r_i$$

Birth-and-death chains



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- P is irreducible $\Leftrightarrow$ $p_i > 0$ for $i < n$ and $q_i > 0$ for $i > 0$

- Proposition: birth-and-death chains are always reversible w.r.t.

$$\pi_k = \frac{\prod_{j=0}^{k-1} p_j / q_{j+1}}{\sum_{h=0}^{n} \prod_{j=0}^{h-1} p_j / q_{j+1}} \qquad k = 0, 1, \ldots, n$$

Invariant distributions of birth-and-death chains

$$\pi_k = \frac{\prod_{j=1}^{k} \frac{p_{j-1}}{q_j}}{\sum_{h=0}^{n} \prod_{j=1}^{h} \frac{p_{j-1}}{q_j}}, \qquad 0 \le k \le n$$

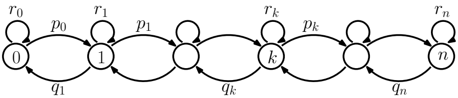
Special case: $p_k = p$, $q_k = q$ for every k, $\rho = p/q$

$$\pi_k = \frac{\rho^k}{1 + \rho + \rho^2 + \dots \rho^n}, \qquad \rho = \frac{p}{q}, \qquad k = 0, 1, \dots, n.$$

- $\rho = 1$: uniform distribution on the set $\mathcal{V} = \{0, 1, \dots, n\}$.
- $\rho < 1$: the invariant distribution is the highest at node 0 and is exponentially decreasing with the distance from it.
- $\rho > 1$: as the previous point inverting the role of 0 and n

Hitting probabilities of birth-and-death chains

$\mathcal{S} = \{0, n\}$



$$H_{0,n} = 0\,, \qquad H_{1,n} = 1\,, \qquad H_{k,n} = q_k H_{k-1,n} + r_k H_{k,n} + p_k H_{k+1,n},\,, \quad 1 \le k < n$$

$$H_{k,n} = 1 - H_{k,0} = \frac{\displaystyle\sum_{1 \le h \le k} \prod_{1 \le j < h} \frac{q_j}{p_j}}{\displaystyle\sum_{1 \le h \le n} \prod_{1 \le j < h} \frac{q_j}{p_j}}\,, \qquad k = 0, 1, \ldots, n$$

Continuous-time Markov chains

Exponential random variables and Poisson processes

S rate-r exponential random variable:

- $\mathbb{P}(S \leq t) = \begin{cases} 1 - e^{-rt} & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases} \qquad \mathbb{E}[S] = \frac{1}{r}$
- memoryless property: $\mathbb{P}(S \geq t + u | S \geq t) = \mathbb{P}(S \geq u) \qquad \forall t, u \geq 0$

- S_k, $k = 1, 2, \ldots$ independent rate-r exponential random variables
- Poisson clock:

$$T_0 = 0 \qquad T_k = \sum_{0 < h \leq k} S_h \qquad k \geq 1$$

- Poisson process: $N_t = \sup\{k : T_k \leq t\} =$ number of events in $[0, t]$

$$\mathbb{P}(N_t = k) = \frac{e^{-rt}(rt)^k}{k!} \qquad k = 0, 1, \ldots$$

Continuous-time Markov chains

Transition rate matrix Λ nonnegative, $\Lambda_{ii} = 0, \omega = \Lambda\mathbb{1}, \ \omega_* = \max_i \omega_i$

Three equivalent definitions of continuous-time Markov chain $X(t)$:

- given current state i, wait rate ω_i-exponential time, then jump to new state j chosen with probability $P_{ij} = \Lambda_{ij}/\omega_i$
- links are equipped with independent rate-Λ_{ij} Poisson clocks; if (i,j)'s clock ticks at time t, and $X(t^-) = i$, then jumps to $X(t) = j$
- rate-ω_* Poisson clock $T_0 \leq T_1 \leq \dots$, with Poisson process N_t independent jump chain $U(k)$, $k = 0, 1, \dots$, transition probabilities

$$\overline{P}_{ij} = \frac{\Lambda_{ij}}{\omega_*}\,, \qquad i \neq j\,, \qquad \overline{P}_{ii} = 1 - \sum_{j \neq i} \overline{P}_{ij}\,.$$

$$X(t) = U(N_t)$$

Continuous-time Markov chains

Continuous-time Markov chain $X(t)$ with transition rate matrix Λ

- marginal distributions $\overline{\pi}(t) = \mathbb{P}(X(t) = i)$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\pi}(t) = -L'\overline{\pi}(t)$$

  where Laplacian $L = D - \Lambda$, $D = diag(\omega)$.

- stationary probability vector $\overline{\pi}$:

$$L'\overline{\pi} = 0\,, \qquad \mathbb{1}'\overline{\pi} = 1\,.$$

- $\overline{\pi} = \overline{P}'\overline{\pi}$ is stationary also for jump chain $U(k)$

- typically, $\overline{\pi} \neq \pi$, where $\pi = P'\pi$ invariant distribution for P

$$\overline{\pi}_i = \frac{\pi_i/\omega_i}{\sum_j \pi_j/\omega_j}\,, \qquad i \in \mathcal{V}\,.$$

Continuous-time Markov chains

Theorem

$X(t)$ continuous-time Markov chain with transition rate matrix Λ. If $\mathcal{G}_\Lambda = (\mathcal{V}, \mathcal{E}, \Lambda)$ is strongly connected, then

- *$X(t)$ has a unique invariant probability vector $\overline{\pi}$*
- *for every initial probability distribution $\mathbb{P}(X(0) = i) = \overline{\pi}_i(0)$*

$$\lim_{t \to +\infty} \overline{\pi}(t) = \overline{\pi}\,.$$

- *for every given function $f : \mathcal{X} \to \mathbb{R}$, with probability 1*

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t f(X(s)) \mathrm{d}s = \sum_i \overline{\pi}_i f(i)\,,$$

Important: no aperiodicity assumption needed

Hitting times and hitting probabilities

$X(t)$ continuous-time Markov chain with transition rate matrix Λ,

$D = diag(\omega)$, $P = D^{-1}\Lambda$, $\mathcal{S} \subseteq \mathcal{V}$ globally reachable in $\mathcal{G}_\Lambda$

- Hitting time: $T_\mathcal{S} := \inf\{t \geq 0 : X(t) \in \mathcal{S}\}$

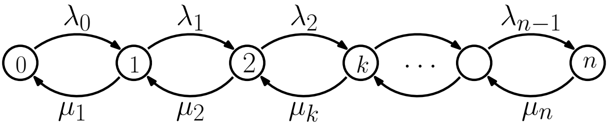
$$\begin{aligned} \mathbb{E}_i[T_\mathcal{S}] &= 0 \text{ if} & i \in \mathcal{S} \\ \mathbb{E}_i[T_\mathcal{S}] &= \tfrac{1}{\omega_i} + \textstyle\sum_j P_{ij}\mathbb{E}_j[T_\mathcal{S}] \text{ if} & i \notin \mathcal{S} \end{aligned}$$

- Hitting probabilities: $H_{is} = \mathbb{P}(X(T_\mathcal{S}) = s | X(0) = i)$

$$\begin{aligned} H_{is} &= \textstyle\sum_j P_{ij} H_{js} \quad i \notin \mathcal{S} \\ H_{ss} &= 1, \quad H_{is} = 0 \text{ for } i \in \mathcal{S} \setminus \{s\} \end{aligned}$$

Exactly the same expression as in discrete time

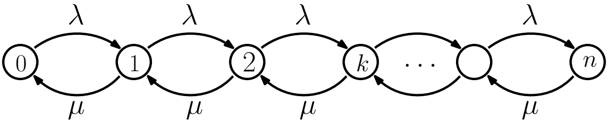
Continuous-time birth-and-death chains



$$\mathcal{V} = \{0, 1, \ldots, n\} \qquad \Lambda_{k,k+1} = \lambda_k \qquad \Lambda_{k,k-1} = \mu_k$$

$$\overline{\pi}_i = \frac{1}{\zeta} \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}}\,, \qquad \zeta = \sum_{k=0}^{n-1} \prod_{j=0}^{k-1} \frac{\lambda_j}{\mu_{j+1}}\,, \qquad i = 0, 1, \ldots, n$$

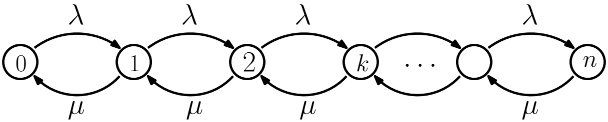
A special case



$$\mathcal{V} = \{0, 1, \ldots, n\} \qquad \Lambda_{k,k+1} = \lambda \qquad \Lambda_{k,k-1} = \mu, \qquad \rho = \lambda/\mu$$

$$\overline{\pi}_i = \frac{\rho^i}{1 + \rho + \ldots + \rho^n}, \qquad i = 0, 1, \ldots, n$$

A special case

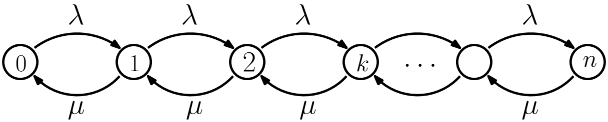


$$\mathcal{V} = \{0, 1, \dots, n\} \qquad \Lambda_{k,k+1} = \lambda \qquad \Lambda_{k,k-1} = \mu, \qquad \rho = \lambda/\mu$$

$$\overline{\pi}_i = \frac{\rho^i}{1 + \rho + \ldots + \rho^n}\,, \qquad i = 0, 1, \dots, n$$

$$\rho < 1 \;\Rightarrow\; \overline{\pi}_i \overset{n\to\infty}{\longrightarrow} (1-\rho)\rho^i$$

A special case



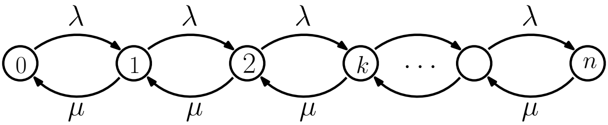
$$\mathcal{V} = \{0, 1, \ldots, n\} \qquad \Lambda_{k,k+1} = \lambda \qquad \Lambda_{k,k-1} = \mu, \qquad \rho = \lambda/\mu$$

$$\overline{\pi}_i = \frac{\rho^i}{1 + \rho + \ldots + \rho^n}, \qquad i = 0, 1, \ldots, n$$

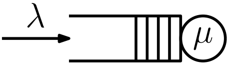
$$\rho < 1 \;\Rightarrow\; \overline{\pi}_i \xrightarrow{n\to\infty} (1-\rho)\rho^i$$

- expected stationary occupancy: $\mathbb{E}[X] = \sum_{i=0}^{n} i \cdot \overline{\pi}_i \xrightarrow{n\to\infty} \frac{\rho}{1-\rho}$
- expected sojourn time: $\mathbb{E}[T] = \frac{1+\mathbb{E}[X]}{\mu} \xrightarrow{n\to\infty} \frac{1}{\mu(1-\rho)} = \frac{1}{\mu-\lambda}$

Queueing theoretic interpretation



A M/M/1-queue (Markovian arrivals, Markovian service rates, 1 Server)

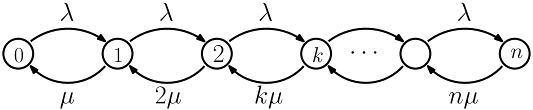


The expression for expected sojourn time

$$\mathbb{E}[T] = \frac{1}{\mu - \lambda} = \frac{\mathbb{E}[X]}{\lambda}$$

is known as Little's law.

Queueing theoretic interpretation



A M/M/n-queue (Markovian arrivals, Markovian service rates, n Servers)

$\mathcal{V} = \{0, 1, \ldots, n\} \qquad \Lambda_{k,k+1} = \lambda \qquad \Lambda_{k,k-1} = \mu = k\lambda$

$$\overline{\pi}_i = \frac{\rho^i / i!}{\sum_{j=0}^{n} \rho^j / j!}\,, \qquad i = 0, 1, \ldots, n\,.$$

In stationarity, the probability of finding the queue full is then

$$\overline{\pi}_n = \frac{\rho^n / n!}{\sum_{j=0}^{n} \rho^j / j!}$$

This is known as Erlang's formula

Discrete vs Continuous Time Markov Chains

Markov chain with state-space $\mathcal{X}$

Discrete-time

- Defined through P
- Stationary distribution $\pi = P'\pi$
- P irreducible (i.e., $\mathcal{G}$ connected) and aperiodic $\Rightarrow$ convergence to π
- Hitting time:

$$\mathbb{E}_i[T_{\mathcal{S}}] = 0,\ i \in \mathcal{S}$$
$$\mathbb{E}_i[T_{\mathcal{S}}] = 1 + \sum_j P_{ij}\mathbb{E}_j[T_{\mathcal{S}}],\ i \notin \mathcal{S}$$

- Return time: $\mathbb{E}_i[T_i^+] = \frac{1}{\pi_i}$

Continuous-time

- Defined through Λ, $\omega = \Lambda\mathbb{1},\ \omega_* = \max_i \omega_i$

$$\overline{P}_{ij} = \frac{\Lambda_{ij}}{\omega_*}\,,\ i \neq j\,,\ \overline{P}_{ii} = 1 - \sum_{j\neq i}\overline{P}_{ij}\,.$$

- Stationary distribution $\overline{\pi} = \overline{P}'\overline{\pi}$
- Λ irreducible $\Rightarrow$ convergence to $\overline{\pi}$
- Hitting time ($P = \mathrm{diag}(\omega)^{-1}\Lambda$):

$$\mathbb{E}_i[T_{\mathcal{S}}] = 0,\ i \in \mathcal{S}$$
$$\mathbb{E}_i[T_{\mathcal{S}}] = \frac{1}{\omega_i} + \sum_j P_{ij}\mathbb{E}_j[T_{\mathcal{S}}],\ i \notin \mathcal{S}$$

- Return time: $\mathbb{E}_i[T_i^+] = \frac{1}{\omega_i\overline{\pi}_i}$

- Absorbing probabilities computed in the same way