Outline

- Existence results for Nash equilibria
- Mixed strategies and Nash's theorem
- Potential Games

Strategic Form Games

- $\mathcal{V}$ finite set of players
- $\mathcal{A}_i$ set of actions (a.k.a. strategies) for player i
- $\mathcal{X} = \prod_{i \in \mathcal{V}} \mathcal{A}_i$ set of configurations (a.k.a. strategy profiles)
- $u_i : \mathcal{X} \to \mathbb{R}$ utility function
- $x \in \mathcal{X}$ configuration (a.k.a. action/strategy profile, or outcome)
- x_i action played by player i
- x_{-i} vector of actions played by everyone but i
- utility of player i when each player j plays action x_j:

$$u_i(x_i, x_{-i}) = u_i(x)$$

$\left(\mathcal{V}, \{\mathcal{A}_i\}_{i \in \mathcal{V}}, \{u_i\}_{i \in \mathcal{V}}\right)$ is called a strategic (a.k.a. normal form) game

Best Response and Nash Equilibria

- Rational choice for player i given x_{-i}: best response

$$\mathcal{B}_i(x_{-i}) = \underset{x_i \in \mathcal{A}_i}{\operatorname{argmax}}\, u_i(x_i, x_{-i})$$

- A pure strategy (P) Nash equilibrium (NE) is $x^* \in \mathcal{X}$ s.t.

$$x_i^* \in \mathcal{B}_i(x_{-i}^*)\,, \qquad \forall i \in \mathcal{V}\,.$$

- PNE $x^* \Leftrightarrow$ no player has strict incentive to unilaterally deviate
- PNE x^* is said strict if $|\mathcal{B}_i(x_{-i}^*)| = 1$ for every player i
- $\mathcal{N} = \{\text{PNE}\}$
- $\mathcal{N}$ might be empty (e.g., discoordination, Rock-Scissor-Paper)
- PNE might not be unique (e.g., coordination, anti-coordination)

Best Response and Nash Equilibria

- Rational choice for player i given x_{-i}: best response

$$\mathcal{B}_i(x_{-i}) = \operatorname*{argmax}_{x_i \in \mathcal{A}_i} u_i(x_i, x_{-i})$$

- A pure strategy (P) Nash equilibrium (NE) is $x^* \in \mathcal{X}$ s.t.

$$x_i^* \in \mathcal{B}_i(x_{-i}^*), \qquad \forall i \in \mathcal{V}.$$

- PNE $x^* \Leftrightarrow$ no player has strict incentive to unilaterally deviate
- PNE x^* is said strict if $|\mathcal{B}_i(x_{-i}^*)| = 1$ for every player i
- $\mathcal{N} = \{\text{PNE}\}$
- $\mathcal{N}$ might be empty (e.g., discoordination, Rock-Scissor-Paper)
- PNE might not be unique (e.g., coordination, anti-coordination)

When does a PNE exist? (And when is it unique?)

Strategic Equivalence

- Definition: Two games $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ and $(\mathcal{V}, \{\mathcal{A}_i\}, \{\overline{u}_i\})$ are strategically equivalent if for every player $i \in \mathcal{V}$ there exists a non-strategic term $n_i : \mathcal{X}_{-i} \to \mathbb{R}$ such that

$$u_i(x) = \overline{u}_i(x) + n_i(x_{-i})$$

- Note 1: strategic equivalence is an equivalence relation in the set of games with fixed player set $\mathcal{V}$ and action sets $\{\mathcal{A}_i\}$

- Note 2: strategically equivalent games have same best response correspondences, hence the same set of PNE (but not vice versa)

Pure strategy Nash equilibria of continuous games

Continuous strategies: general results for existence/uniqueness

Theorem (Debreu, Glicksberg, Fan, '52)
Consider a game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ such that for each $i \in \mathcal{V}$:

- $\mathcal{A}_i \subseteq \mathbb{R}^{q_i}$ is nonempty, compact, and convex;
- $u_i(x)$ is continuous in x and concave in x_i for all x_{-i}.

Then there exists at least one PNE x^*.

Example: Cournot oligopoly

- $p(q) =$ continuous concave non-increasing inverse demand function with $p(0) > 0$ and $p(\overline{q}) = 0$ for some $\overline{q} > 0$
- $c_i(x_i) =$ continuous convex production costs

$$u_i(x) = x_i \cdot p\left(\sum_j x_j\right) - c_i(x_i)$$

- while $\mathbb{R}_+$ is not compact, we can restrict to compact action space $\mathcal{A}_i = [0, \overline{q}]$ (for every action $x_i > \overline{q}$ of player i is strictly dominated by $x_i^* = 0$)

Pure strategy Nash equilibria of continuous games

Continuous strategies: general results for existence/uniqueness

Theorem (Debreu, Glicksberg, Fan, '52)

Consider a game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ such that for each $i \in \mathcal{V}$:

- $\mathcal{A}_i \subseteq \mathbb{R}^{q_i}$ is nonempty, compact, and convex;
- $u_i(x)$ is continuous in x and concave in x_i for all x_{-i}.

Then a pure strategy Nash equilibrium $x^* \in \mathcal{N}$ exists.

Proof is based on Kakutami's Fixed Point Theorem:

Let $\mathcal{B} : \mathcal{X} \rightrightarrows \mathcal{X}$ be a correspondence (i.e., set-valued function) s.t.

- $\mathcal{X}$ is nonempty, compact, and convex;
- $\mathcal{B}(x) \subseteq \mathcal{X}$ is nonempty and convex for all $x \in \mathcal{X}$;
- $\mathcal{B}$ has closed graph, i.e., $x^n \to x$, $y^n \to y$ and $y^n \in \mathcal{B}(x^n)$ $\forall n$ imply that $y \in \mathcal{B}(x)$

Then, there exists at least one fixed point $x^* \in \mathcal{B}(x^*)$.

Proof of Debreu, Glicksberg, Fan Theorem

Apply Kakutami's Fixed Point with $\mathcal{X} = \prod_i \mathcal{A}_i$, $\mathcal{B} : \mathcal{X} \rightrightarrows \mathcal{X}$ with

$$\mathcal{B}(x) = \prod_{i \in \mathcal{V}} \mathcal{B}_i(x_{-i})$$

- clearly $\mathcal{X}$ is nonempty, compact, and convex;
- for all $i \in \mathcal{V}$ and $x_{-i} \in \mathcal{X}_{-i}$, the set $\mathcal{B}_i(x_{-i}) \subseteq \mathcal{A}_i$ is nonempty (since u_i continuous in x_i) and convex (as u_i concave in x_i), hence $\mathcal{B}(x) \subseteq \mathcal{X}$ is nonempty and convex;
- by contradiction, $\exists\ x^n \to x$, $y^n \to y$, $y^n \in \mathcal{B}(x^n)$, $y \notin \mathcal{B}(x)$ then $\exists\ i \in \mathcal{V}$ and $\overline{y}_i \in \mathcal{A}_i$ s.t.

  $$u_i(\overline{y}_i, x_{-i}) > u_i(y_i, x_{-i})$$

  But $y^n \in \mathcal{B}(x^n)$ implies that for every n

  $$u_i(\overline{y}_i, x^n_{-i}) \leq u_i(y^n_i, x^n_{-i})$$

  and taking the limit on both sides gives the contradiction

  $$u_i(\overline{y}_i, x_{-i}) = \lim_n u_i(\overline{y}_i, x^n_{-i}) \leq \lim_n u_i(y^n_i, x^n_{-i}) = u_i(y_i, x_{-i})$$

Pure strategy Nash equilibria of continuous games

Continuous strategies: general results for existence/uniqueness

Theorem (Debreu, Glicksberg, Fan, '52)
Consider a game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ such that for each $i \in \mathcal{V}$:

- $\mathcal{A}_i \subseteq \mathbb{R}^{q_i}$ is nonempty, compact, and convex;
- $u_i(x)$ is continuous in x and concave in x_i for all x_{-i}.

Then there exists at least one PNE.

- quasi-concavity of $u_i(x_i, x_{-i})$ in x_i is sufficient;
- [Rosen'65]: sufficient conditions for uniqueness of PNE: strictly diagonally concave game;
- does not apply to finite games: e.g., Matching Penny, Rock-Scissor-Paper, ...

Outline

- Existence results for Nash equilibria
- Mixed strategies and Nash's theorem
- Potential Games

Mixed strategies

Finite game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$

- Mixed strategy for a player i is a probability distribution on $\mathcal{A}_i$:

$$z_i \in \mathcal{P}(\mathcal{A}_i)$$

- $z_{i,a}$ = probability of choosing action $a \in \mathcal{A}_i$
- action $a \in \mathcal{A}_i \leftrightarrow$ pure strategy $z = \delta^a$ concentrated on a
- mixed strategy profile space: $\mathcal{Z} = \prod_{i \in \mathcal{V}} \mathcal{P}(\mathcal{A}_i)$
- multilinear utilities: expected values assuming independent plays:

$$\overline{u}_i(z) = \sum_x \left(\prod_j z_{j,x_j} \right) u_i(x) \qquad \forall z \in \mathcal{Z}, \, \forall i \in \mathcal{V}$$

- Definition: a mixed strategy Nash equilibrium is $z^* \in \mathcal{Z}$ s.t.

$$\overline{u}_i(z^*) \geq \overline{u}_i(z) \qquad \forall i \in \mathcal{V}, \; \forall z \in \mathcal{Z} \text{ s.t. } z_{-i} = z^*_{-i}$$

Example 1: Matching Penny

	-1	+1
-1	+1,-1	-1,+1
+1	-1,+1	+1,-1

- Recall that there is no PNE
- Mixed strategy profile z^* with $z_1^* = z_2^* = (1/2, 1/2)$ is a MNE.

Example 1: Matching Penny

	-1	+1
-1	+1,-1	-1,+1
+1	-1,+1	+1,-1

- Recall that there is no PNE
- Mixed strategy profile z^* with $z_1^* = z_2^* = (1/2, 1/2)$ is a MNE.

Indeed, for every mixed strategy $z_1 = (1-p, p)$ of player 1,

$$\begin{aligned}\overline{u}_1(z_1, z_2^*) &= \frac{1-p}{2} u_1(-1,-1) + \frac{1-p}{2} u_1(-1,+1) + \frac{p}{2} u_1(+1,-1) + \frac{p}{2} u_1(+1,+1) \\ &= (1-p)\frac{u_1(-1,-1) + u_1(-1,+1)}{2} + p\frac{u_1(+1,-1) + u_1(+1,+1)}{2} = 0\end{aligned}$$

and for every mixed strategy $z_2 = (1-q, q)$ of player 2,

$$\overline{u}_2(z_1^*, z_2) = (1-q)\frac{u_2(-1,-1) + u_2(+1,-1)}{2} + q\frac{u_2(-1,+1) + u_2(+1,+1)}{2} = 0$$

- There are no other MNE.

Example 2: Rock-Scissor-Paper

	R	S	P
R	0,0	1,-1	-1,1
S	-1,1	0,0	1,-1
P	1,-1	-1,1	0,0

- Recall that there is no PNE
- Mixed strategy profile z^* with $z_1^* = z_2^* = (1/3, 1/3, 1/3)$ is a MNE. Indeed, for every mixed strategy z_1 of player 1,

$$\overline{u}_1(z_1, z_2^*) = 0$$

and for every mixed strategy z_2 of player 2,

$$\overline{u}_2(z_1^*, z_1) = 0$$

- There are no other MNE.

Existence of mixed strategy Nash equilibria for finite games

- Theorem[Nash, 1950]:
  Every finite game admits a mixed strategy Nash equilibrium.

Existence of mixed strategy Nash equilibria for finite games

- Theorem[Nash, 1950]:
Every finite game admits a mixed strategy Nash equilibrium.

- Proof: We apply the Debreu, Glicksberg, Fan Theorem to the mixed strategy extension of the finite game, i.e., the game with player set $\mathcal{V}$, configuration space $\mathcal{Z} = \prod_i \mathcal{P}(\mathcal{A}_i)$ and multilinear utilities

$$u_i(z) = \sum_x \left(\prod_j z_{j,x_j}\right) u_i(x) \qquad \forall z \in \mathcal{Z}, \forall i \in \mathcal{V}$$

Indeed, for every player i:

the simplex $\mathcal{P}(\mathcal{A}_i) \subseteq \mathbb{R}^{q_i}$, $q_i = |\mathcal{A}_i|$ is nonempty convex compact;

$u_i(z)$ continuous in z; $u_i(z_i, z_{-i})$ linear (hence concave) in z_i $\forall z_{-i}$.

Hence, mixed strategy extension of the game admits PNE, i.e., original finite game admits MNE.

- in his thesis, Nash proposed another proof, based on Brouwer's fixed point theorem

Mixed strategies for continuous games

- the idea of mixed strategy can be extended beyond finite games
- e.g., if the action set of player i is a measurable set $\mathcal{A}_i \subseteq \mathbb{R}^{q_i}$, then a a mixed strategy z_i for player i is a probability measure z_i over $\mathcal{A}_i$
- one can then associate to every mixed strategy profile z a utility

$$\overline{u}_i(z) = \int_{\mathcal{X}} u_i(x_1, \dots, x_n) \mathrm{d}z_1(x_1) \dots \mathrm{d}z_n(x_n)$$

and define a MNE as a mixed strategy profile z^* such that

$$\overline{u}_i(z_i^*, z_{-i}^*) \geq \overline{u}_i(z_i, z_{-i}^*) \qquad \forall i$$

- Theorem [Glicksberg,1952] If $\mathcal{A}_i$ nonempty and compact and u_i continuous for all i in $\mathcal{V}$, then a MNE exists.

Outline

- Existence results for Nash equilibria
- Mixed strategies and Nash's theorem
- Potential Games

Exact Potential Games

- Definition: A game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ is an (exact) potential game if there exists $\Phi : \mathcal{X} \to \mathbb{R}$ (called potential function) such that

$$u_i(y_i, x_{-i}) - u_i(x_i, x_{-i}) = \Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}),$$

for every $x \in \mathcal{X}$, $i \in \mathcal{V}$, and $y_i \in \mathcal{A}$, equivalently if

$$x_{-i} = y_{-i} \quad \Longrightarrow \quad u_i(y) - u_i(x) = \Phi(y) - \Phi(x)$$

- In an exact potential game, for any configuration x, the utility variation incurred by player i when changing action unilaterally is the same as the corresponding variation in the potential function

Example: symmetric 2×2 games

Every symmetric 2×2 game is an exact potential game

	-	+
-	a,a	d,c
+	c,d	b,b

Φ	-	+
-	a-c	0
+	0	b-d

- Coordination and anti-coordination games are potential games
- Prisoner's dilemma is a potential game

	C	S
C	-3,-3	0,-5
S	-5,0	-1,-1

Φ	C	S
C	2	0
S	0	-1

Equivalent Definition of Exact Potential Games

- Lemma: A strategic game is an exact potential game $\iff$ it is strategically equivalent to game with identical utilities, i.e., $\exists \Phi(x)$ s.t.

$$u_i(x) = \Phi(x) + n_i(x_{-i}) \tag{1}$$

for every player i in $\mathcal{V}$ and configuration x in $\mathcal{X}$.

- Proof: If (1) holds true, then, for every y in $\mathcal{X}$ s.t. $y_{-i} = x_{-i}$,

$$u_i(y) - u_i(x) = \Phi(y) + n_i(x_{-i}) - \Phi(x) - n_i(x_{-i}) = \Phi(y) - \Phi(x)$$

Conversely, if $u_i(y) - u_i(x) = \Phi(y) - \Phi(x)$ whenever $y_{-i} = x_{-i}$, then, for an arbitrary choice of $y_i = a$ in $\mathcal{A}_i$, we get

$$u_i(x) = \Phi(x) + u_i(y) - \Phi(y) = \Phi(x) + \underbrace{u_i(a, x_{-i}) - \Phi(a, x_{-i})}_{n_i(x_{-i})}$$

Special case: finite symmetric 2-player games

- Proposition: Finite symmetric two-player game with utilities

$$u_1(a, b) = U_{ab} \qquad u_2(a, b) = U_{ba}$$

is exact potential if and only if

$$U = S + C$$

where S symmetric and C constant on columns

- Proof: Follows from previous lemma with $S \leftrightarrow$ equal interest part; $C \leftrightarrow$ non-strategic part

Properties of Exact Potential Games

- Lemma: If $\Phi(x)$ and $\overline{\Phi}(x)$ exact potential functions for same game, then $\exists$ constant C

$$\Phi(x) = \overline{\Phi}(x) + C \qquad \forall x \in \mathcal{X}$$

- Theorem: Game is exact potential if and only if

$$\sum_{i=1}^{4} u_{i_k}(x^{(k)}) - u_{i_k}(x^{(k-1)}) = 0$$

$\forall (x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)} = x^{(0)})$ s.t. $x_{-i_k}^{(k-1)} = x_{-i_k}^{(k)}$, $1 \leq k \leq 4$

null "cirquitation" on 4-cycles

Smooth potential games

- Proposition: Game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$, $\mathcal{A}_i \subseteq \mathbb{R}$ interval, $u_i \in \mathcal{C}^2$.

Then, the game is an exact potential if and only if

$$\frac{\partial^2}{\partial x_i x_j} u_i(x) = \frac{\partial^2}{\partial x_j x_i} u_j(x)$$

for every $i, j \in \mathcal{V}$ and $x \in \mathcal{X}$. Moreover, in this case a potential function is

$$\Phi(x) = \int_{\Gamma_{\overline{x} \to x}} f(s) \cdot \mathrm{d}s$$

where $\Gamma_{\overline{x} \to x}$ is any simple curve from $\overline{x}$ to x, and

$$f(x) = \left(\frac{\partial u_1}{\partial x_1}(x), \dots, \frac{\partial u_n}{\partial x_n}(x) \right)$$

Ordinal potential games

- Definition: A game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ is an ordinal potential game if there exists $\Phi : \mathcal{X} \to \mathbb{R}$ (called ordinal potential function) s.t.

$$u_i(y_i, x_{-i}) - u_i(x_i, x_{-i}) > 0 \iff \Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}) > 0$$

for every $x \in \mathcal{X}$, $i \in \mathcal{V}$, and $y_i \in \mathcal{A}$.

- In an ordinal potential game, the sign of the utility variation incurred by player i when changing action unilaterally is the same as the sign of corresponding variation in the potential function:

$$\operatorname{sgn}\left(u_i(y_i, x_{-i}) - u_i(x_i, x_{-i})\right) = \operatorname{sgn}\left(\Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i})\right)$$

Ordinal potential games

- Definition: A game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ is an ordinal potential game if there exists $\Phi : \mathcal{X} \to \mathbb{R}$ (called ordinal potential function) s.t.

$$u_i(y_i, x_{-i}) - u_i(x_i, x_{-i}) > 0 \iff \Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}) > 0$$

for every $x \in \mathcal{X}$, $i \in \mathcal{V}$, and $y_i \in \mathcal{A}$.

- In an ordinal potential game, the sign of the utility variation incurred by player i when changing action unilaterally is the same as the sign of corresponding variation in the potential function:

$$\operatorname{sgn}\left(u_i(y_i, x_{-i}) - u_i(x_i, x_{-i})\right) = \operatorname{sgn}\left(\Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i})\right)$$

exact potential $\overset{\Longrightarrow}{\underset{\not\Longleftarrow}{}}$ weighted potential $\overset{\Longrightarrow}{\underset{\not\Longleftarrow}{}}$ ordinal potential

Example: Symmetric Cournot Oligopoly

- arbitrary inverse demand (price) function $p(q)$
- identical linear cost functions $c_i(x_i) = cx_i$ for firms $i = 1, \ldots, n$
- profit (utility) of firm i producing $x_i > 0$ is

$$u_i(x) = x_i p \left(\sum\nolimits_j x_j \right) - cx_i$$

- this is ordinal potential game with potential function

$$\Phi(x) = \left(\prod\nolimits_i x_i \right) \left(p \left(\sum\nolimits_i x_i \right) - c \right)$$

Indeed

$$\Phi(x) = \left(\prod\nolimits_{j \neq i} x_j \right) u_i(x)$$

Example: Cournot Oligopoly with Affine Inverse Demand

- inverse demand (price) function $F(q) = \alpha - \beta q$
- arbitrary differentiable cost functions $c_i(x_i)$ for firms $i = 1, \ldots, n$
- profit (utility) of firm i producing $x_i > 0$ is

$$u_i(x) = \alpha x_i - \beta x_i \sum_j x_j - c_i(x_i)$$

- this is exact potential game with potential function

$$\Phi(x) = \alpha \sum_i x_i - \beta \sum_i x_i^2 - \frac{\beta}{2} \sum_i \sum_{j \neq i} x_i x_j - \sum_i c_i(x_i)$$

Potential games have Pure Strategy Nash Equilibria

Proposition: For an ordinal potential game, every global max point of the ord. potential function $\Phi(x)$ is a pure Nash equilibrium, i.e.,

$$\mathcal{N} \supseteq \mathcal{N}_{max} := \operatorname*{argmax}_{x \in \mathcal{X}} \Phi(x)$$

Potential games have Pure Strategy Nash Equilibria

Proposition: For an ordinal potential game, every global max point of the ord. potential function $\Phi(x)$ is a pure Nash equilibrium, i.e.,

$$\mathcal{N} \supseteq \mathcal{N}_{max} := \operatorname*{argmax}_{x \in \mathcal{X}} \Phi(x)$$

Proof: Since

$$\mathrm{sgn}(u_i(y_i, x_{-i}) - u_i(x_i, x_{-i})) = \mathrm{sgn}(\Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}))$$

we have that $x^* \in \mathcal{X}$ is PNE if and only if

$$\Phi(y_i, x^*_{-i}) \leq \Phi(x^*_i, x^*_{-i}) \qquad \forall i \in \mathcal{V},\ \forall y_i \in \mathcal{A}_i \tag{2}$$

Potential games have Pure Strategy Nash Equilibria

Proposition: For an ordinal potential game, every global max point of the ord. potential function $\Phi(x)$ is a pure Nash equilibrium, i.e.,

$$\mathcal{N} \supseteq \mathcal{N}_{max} := \operatorname*{argmax}_{x \in \mathcal{X}} \Phi(x)$$

Proof: Since

$$\operatorname{sgn}(u_i(y_i, x_{-i}) - u_i(x_i, x_{-i})) = \operatorname{sgn}(\Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}))$$

we have that $x^* \in \mathcal{X}$ is PNE if and only if

$$\Phi(y_i, x^*_{-i}) \leq \Phi(x^*_i, x^*_{-i}) \qquad \forall i \in \mathcal{V},\ \forall y_i \in \mathcal{A}_i \tag{2}$$

- Note: There might be pure Nash equilibria outside $\operatorname*{argmax}_{x \in \mathcal{X}} \Phi(x)$

$\mathcal{N} =$ “local maximum points”

Potential games have Pure Strategy Nash Equilibria

Proposition: For an ordinal potential game, every global max point of the ord. potential function $\Phi(x)$ is a pure Nash equilibrium, i.e.,

$$\mathcal{N} \supseteq \mathcal{N}_{max} := \operatorname*{argmax}_{x \in \mathcal{X}} \Phi(x)$$

- Corollary 1: Every finite ordinal potential game admits a PNE
- Corollary 2: Every continuous ordinal potential game with compact strategy space admits a PNE