Averaging with inputs, Reversibility, Electrical Networks

Giacomo Como, DISMA, Politecnico di Torino

Fabio Fagnani, DISMA, Politecnico di Torino

Averaging models with exogenous inputs

We consider the averaging dynamics model where certain nodes, instead of following the usual averaging update rule, have their state evolving according to an exogeneous signal.

- Model the presence of uninfluenced nodes playing the role of leaders
- Model the presence of primitive beliefs in a community
- Coordination of mobile agents
- Connection with the Bonacich centrality

Averaging models with exogenous inputs

- $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, $P = D^{-1}W$,
- A partition of the node set: $\mathcal{V} = \mathcal{R} \cup \mathcal{S}$
- Corresponding block structure:

$$P = \begin{array}{cc} \begin{array}{cc} \mathcal{R} & \mathcal{S} \end{array} & \\ \left[\begin{array}{cc} Q & E \\ F & G \end{array}\right] & \begin{array}{c} \mathcal{R} \\ \mathcal{S}\,. \end{array} \end{array} \qquad x(t) = \left[\begin{array}{c} \underline{x}(t) \\ u(t) \end{array}\right] \begin{array}{c} \mathcal{R} \\ \mathcal{S}\,, \end{array}$$

- The linear averaging model on $\mathcal{G}$ with input set $\mathcal{S}$:

$$\underline{x}(t+1) = Q\underline{x}(t) + Eu(t)\,,$$

Averaging models with exogenous inputs

- $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, $P = D^{-1}W$,
- A partition of the node set: $\mathcal{V} = \mathcal{R} \cup \mathcal{S}$
- Corresponding block structure:

$$P = \begin{array}{c c} & \begin{array}{cc} \mathcal{R} & \mathcal{S} \end{array} \\ & \left[\begin{array}{cc} Q & E \\ F & G \end{array}\right] \begin{array}{c} \mathcal{R} \\ \mathcal{S} \, . \end{array} \end{array} \qquad x(t) = \left[\begin{array}{c} \underline{x}(t) \\ u(t) \end{array}\right] \begin{array}{c} \mathcal{R} \\ \mathcal{S} \, , \end{array}$$

- The linear averaging model on $\mathcal{G}$ with input set $\mathcal{S}$:

$$\underline{x}(t+1) = Q\underline{x}(t) + Eu(t) \, ,$$

- A special case: $u(t) = u$ const. $\mathcal{S}$ *Stubborn nodes*, $\mathcal{R}$ *Regular nodes*

Averaging models with stubborn nodes

$$\underline{x}(t+1) = Q\underline{x}(t) + Eu\,,$$

- A new dynamical system:

$$\underline{x} \in \mathbb{R}^{\mathcal{R}} \mapsto Q\underline{x} + Eu \in \mathbb{R}^{\mathcal{R}}$$

- Equilibria:

$$\underline{x} = Q\underline{x} + Eu \;\Leftrightarrow\; (I-Q)\underline{x} = Eu$$

- Is $(I-Q)$ invertible?

Sub-stochastic matrices

$$P = \begin{array}{c} \\ \end{array}\begin{array}{cc} \mathcal{R} & \mathcal{S} \end{array}$$

$$P = \left[\begin{array}{cc} Q & E \\ F & G \end{array}\right]$$

- Q is a *sub-stochastic* matrix: $\sum_j Q_{ij} \leq 1$ for all $i \in \mathcal{R}$

Sub-stochastic matrices

$$P = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \mathcal{R} & \mathcal{S} \\ \left[\begin{matrix} Q \\ F \end{matrix}\right. & \left.\begin{matrix} E \\ G \end{matrix}\right] \end{array}$$

- Q is a *sub-stochastic* matrix: $\sum_j Q_{ij} \leq 1$ for all $i \in \mathcal{R}$
- If $\rho(Q) < 1$, then $(I - Q)$ would be invertible

Sub-stochastic matrices

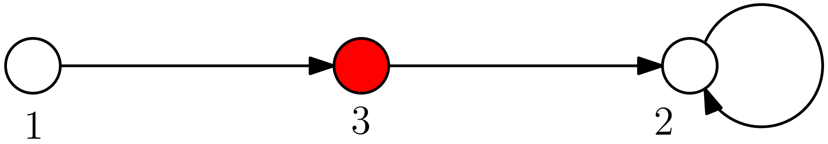
$$P = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \mathcal{R} & \mathcal{S} \\ \end{array}$$

$$P = \begin{array}{c c} & \begin{array}{cc} \mathcal{R} & \mathcal{S} \end{array} \\ & \left[\begin{array}{cc} Q & E \\ F & G \end{array}\right] \end{array}$$

- Q is a *sub-stochastic* matrix: $\sum_j Q_{ij} \leq 1$ for all $i \in \mathcal{R}$
- If $\rho(Q) < 1$, then $(I - Q)$ would be invertible
- However, if $E = 0$, Q is stochastic and $\rho(Q) = 1$
- This would correspond to the situation that $\mathcal{R}$ is trapping: no link from regular nodes to the stubborn nodes.
- What if $E \neq 0$?

Sub-stochastic matrices

Example:



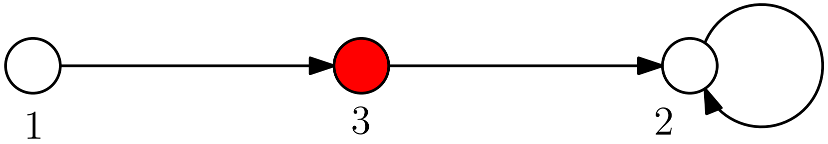
$$\mathcal{R} = \{1,2\}, \quad \mathcal{S} = \{3\}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \Rightarrow \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

1 is an eigenvalue of Q $\Rightarrow$ $(I - Q)$ not invertible

Sub-stochastic matrices

Example:



$\mathcal{R} = \{1, 2\}, \quad \mathcal{S} = \{3\}$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \Rightarrow \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

1 is an eigenvalue of Q $\quad \Rightarrow \quad$ $(I - Q)$ not invertible

Problem: The stubborn node set is not reachable from 2.

Sub-stochastic matrices

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, $P = D^{-1}W$, $\mathcal{V} = \mathcal{R} \cup \mathcal{S}$

$$P = \begin{array}{c} \\ \\ \end{array} \begin{array}{cc} \mathcal{R} & \mathcal{S} \\ \left[\begin{array}{c} Q \\ F \end{array}\right. & \left.\begin{array}{c} E \\ G \end{array}\right] \end{array}$$

Theorem

If $\mathcal{S}$ is globally reachable in $\mathcal{G}$, the following facts hold true:

1. *There exists $m \in \mathbb{N}$ and $\alpha < 1$ such that $\sum_j Q_{ij}^m \leq \alpha$ for all $i \in \mathcal{R}$.*
2. *$\rho(Q) < 1$ and $(I - Q)^{-1} = \sum_{t=0}^{+\infty} Q^t$*

Proof of the theorem

- A matrix, $||A||_\infty = \max\limits_{||x||_\infty \leq 1} ||Ax||_\infty = \max_i \sum_j |A_{ij}|$ induced norm

  $||AB||_\infty \leq ||A||_\infty \cdot ||B||_\infty, \quad |A_{ij}| \leq ||A||_\infty$ for every i, j

Proof of the theorem

- A matrix, $||A||_\infty = \max\limits_{||x||_\infty \leq 1} ||Ax||_\infty = \max_i \sum_j |A_{ij}|$ induced norm

  $||AB||_\infty \leq ||A||_\infty \cdot ||B||_\infty, \quad |A_{ij}| \leq ||A||_\infty$ for every i, j
- Q stochastic $||Q||_\infty = 1$, sub-stochastic $||Q||_\infty \leq 1$

Proof of the theorem

- A matrix, $||A||_\infty = \max\limits_{||x||_\infty \leq 1} ||Ax||_\infty = \max_i \sum_j |A_{ij}|$ induced norm

  $||AB||_\infty \leq ||A||_\infty \cdot ||B||_\infty, \quad |A_{ij}| \leq ||A||_\infty$ for every i, j
- Q stochastic $||Q||_\infty = 1$, sub-stochastic $||Q||_\infty \leq 1$
- There exists m such that $\sum_j Q^m_{ij} < 1$ for every $i \in \mathcal{R}$

Proof of the theorem

- A matrix, $||A||_\infty = \max\limits_{||x||_\infty \leq 1} ||Ax||_\infty = \max_i \sum_j |A_{ij}|$ induced norm

  $||AB||_\infty \leq ||A||_\infty \cdot ||B||_\infty, \quad |A_{ij}| \leq ||A||_\infty$ for every i, j
- Q stochastic $||Q||_\infty = 1$, sub-stochastic $||Q||_\infty \leq 1$
- There exists m such that $\sum_j Q^m_{ij} < 1$ for every $i \in \mathcal{R}$

  $||Q^m||_\infty = \max_i \sum_j Q^m_{ij} = \alpha < 1$

Proof of the theorem

- A matrix, $||A||_\infty = \max_{||x||_\infty \leq 1} ||Ax||_\infty = \max_i \sum_j |A_{ij}|$ induced norm

  $||AB||_\infty \leq ||A||_\infty \cdot ||B||_\infty, \quad |A_{ij}| \leq ||A||_\infty$ for every i, j
- Q stochastic $||Q||_\infty = 1$, sub-stochastic $||Q||_\infty \leq 1$
- There exists m such that $\sum_j Q^m_{ij} < 1$ for every $i \in \mathcal{R}$

  $||Q^m||_\infty = \max_i \sum_j Q^m_{ij} = \alpha < 1$
- $t = qm + r \;\; q = \lfloor t/m \rfloor$

$$||Q^t||_\infty = ||(Q^m)^q Q^r||_\infty \leq ||Q^m||^q_\infty ||Q||^r_\infty \leq \alpha^q = \alpha^{\lfloor t/m \rfloor}$$

Proof of the theorem

- A matrix, $||A||_\infty = \max_{||x||_\infty \leq 1} ||Ax||_\infty = \max_i \sum_j |A_{ij}|$ induced norm
  $||AB||_\infty \leq ||A||_\infty \cdot ||B||_\infty, \quad |A_{ij}| \leq ||A||_\infty$ for every i, j
- Q stochastic $||Q||_\infty = 1$, sub-stochastic $||Q||_\infty \leq 1$
- There exists m such that $\sum_j Q^m_{ij} < 1$ for every $i \in \mathcal{R}$
  $||Q^m||_\infty = \max_i \sum_j Q^m_{ij} = \alpha < 1$
- $t = qm + r \;\; q = \lfloor t/m \rfloor$

$$||Q^t||_\infty = ||(Q^m)^q Q^r||_\infty \leq ||Q^m||^q_\infty ||Q||^r_\infty \leq \alpha^q = \alpha^{\lfloor t/m \rfloor}$$

- $Q^t_{ij} \leq \alpha^{\lfloor t/m \rfloor} \to 0 \;\Rightarrow\; Q^t \to 0 \;\Rightarrow\; \rho(Q) < 1$

Proof of the theorem

- A matrix, $||A||_\infty = \max\limits_{||x||_\infty \le 1} ||Ax||_\infty = \max_i \sum_j |A_{ij}|$ induced norm

  $||AB||_\infty \le ||A||_\infty \cdot ||B||_\infty, \quad |A_{ij}| \le ||A||_\infty$ for every i, j
- Q stochastic $||Q||_\infty = 1$, sub-stochastic $||Q||_\infty \le 1$
- There exists m such that $\sum_j Q^m_{ij} < 1$ for every $i \in \mathcal{R}$

  $||Q^m||_\infty = \max_i \sum_j Q^m_{ij} = \alpha < 1$
- $t = qm + r \;\; q = \lfloor t/m \rfloor$

$$||Q^t||_\infty = ||(Q^m)^q Q^r||_\infty \le ||Q^m||^q_\infty ||Q||^r_\infty \le \alpha^q = \alpha^{\lfloor t/m \rfloor}$$

- $Q^t_{ij} \le \alpha^{\lfloor t/m \rfloor} \to 0 \;\Rightarrow\; Q^t \to 0 \;\Rightarrow\; \rho(Q) < 1$
- $\sum\limits_{s=0}^{+\infty} Q^s$ converges and $\sum\limits_{s=0}^{+\infty} Q^s = (I - Q)^{-1}$

Averaging models with stubborn nodes

$$\underline{x}(t+1) = Q\underline{x}(t) + Eu\,,$$

- A new dynamical system:

$$\underline{x} \in \mathbb{R}^{\mathcal{R}} \mapsto Q\underline{x} + Eu \in \mathbb{R}^{\mathcal{R}}$$

- Equilibria:

$$\underline{x} = Q\underline{x} + Eu \;\Leftrightarrow\; (I - Q)\underline{x} = Eu$$

- If $\mathcal{S}$ is globally reachable, $(I - Q)$ is invertible and the only equilibrium is

$$\underline{x} = (I - Q)^{-1}Eu$$

Asymptotics of averaging models with stubborn nodes

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, $\mathcal{V} = \mathcal{R} \cup \mathcal{S}$ partition

$$P = D^{-1}W = \begin{array}{c} \begin{array}{cc} \mathcal{R} & \mathcal{S} \end{array} \\ \left[\begin{array}{cc} Q & E \\ F & G \end{array}\right] \end{array}, \qquad \underline{x}(t+1) = Q\underline{x}(t) + Eu\,,$$

Theorem

1. $\underline{x}(t) = Q^t\underline{x}(0) + \sum_{s=0}^{t-1} Q^s Eu$ *for every* $t \geq 0$
2. *If* $\mathcal{S}$ *is globally reachable,*

$$\lim_{t \to +\infty} \underline{x}(t) = (I - Q)^{-1}Eu = \sum_{s=0}^{\infty} Q^s Eu$$

Asymptotics of averaging models with stubborn nodes

$\mathcal{G}=(\mathcal{V},\mathcal{E},W)$ graph, $\mathcal{V}=\mathcal{R}\cup\mathcal{S}$ partition

$$P=D^{-1}W=\begin{array}{cc} \mathcal{R} & \mathcal{S} \\ \left[\begin{array}{c} Q \\ F \end{array}\right. & \left.\begin{array}{c} E \\ G \end{array}\right] \end{array}, \qquad \underline{x}(t+1)=Q\underline{x}(t)+Eu\,,$$

Theorem

1. $\underline{x}(t)=Q^t\underline{x}(0)+\sum_{s=0}^{t-1}Q^sEu$ *for every* $t\geq 0$
2. *If $\mathcal{S}$ is globally reachable,*

$$\lim_{t\to+\infty}\underline{x}(t)=(I-Q)^{-1}Eu=\sum_{s=0}^{\infty}Q^sEu$$

Remark: If $\mathcal{G}$ is strongly connected, than any $\mathcal{S}\neq\emptyset$ is globally reachable.

Averaging models with stubborn nodes

$$\underline{x} = \lim_{t \to +\infty} \underline{x}(t) = (I - Q)^{-1} E u$$

Remark 1:

Averaging models with stubborn nodes

$$\underline{x} = \lim_{t \to +\infty} \underline{x}(t) = (I - Q)^{-1} E u$$

Remark 1:

$$\mathbb{1} = Q\mathbb{1} + E\mathbb{1} \;\Rightarrow\; \mathbb{1} = (I - Q)^{-1} E\mathbb{1}$$

Averaging models with stubborn nodes

$$\underline{x} = \lim_{t \to +\infty} \underline{x}(t) = (I - Q)^{-1} E u$$

Remark 1:

$$\mathbb{1} = Q\mathbb{1} + E\mathbb{1} \;\Rightarrow\; \mathbb{1} = (I - Q)^{-1} E\mathbb{1}$$

- each row of $(I - Q)^{-1}E$ sums to 1.
- The asymptotic state $\underline{x}_i$ of every agent $i \in \mathcal{R}$ is a convex combination of the states of the stubborn nodes.
- The initial state of the nodes in $\mathcal{R}$ play no role.

Averaging models with stubborn nodes

$$\underline{x} = \lim_{t \to +\infty} \underline{x}(t) = (I - Q)^{-1} E u$$

Remark 2:

Averaging models with stubborn nodes

$$\underline{x} = \lim_{t \to +\infty} \underline{x}(t) = (I - Q)^{-1} E u$$

Remark 2:

$(I - Q)\underline{x} = Eu$

Averaging models with stubborn nodes

$$\underline{x} = \lim_{t \to +\infty} \underline{x}(t) = (I - Q)^{-1} E u$$

Remark 2:

$$(I - Q)\underline{x} = Eu$$

$$P = \begin{pmatrix} Q & E \\ F & G \end{pmatrix} \qquad I - P = \begin{pmatrix} I - Q & -E \\ -F & I - G \end{pmatrix}, \quad x = \begin{pmatrix} \underline{x} \\ u \end{pmatrix}$$

Averaging models with stubborn nodes

$$\underline{x} = \lim_{t\to+\infty} \underline{x}(t) = (I - Q)^{-1}Eu$$

Remark 2:

$$(I - Q)\underline{x} = Eu$$

$$P = \begin{pmatrix} Q & E \\ F & G \end{pmatrix} \qquad I - P = \begin{pmatrix} I - Q & -E \\ -F & I - G \end{pmatrix}, \quad x = \begin{pmatrix} \underline{x} \\ u \end{pmatrix}$$

$$((I - P)x)_i = 0 \ \forall i \in \mathcal{R}$$

$$(I - P)x = (I - D^{-1}W)x = D^{-1}Lx \quad \Rightarrow \quad (Lx)_i = 0 \ \forall i \in \mathcal{R}$$

Averaging models with stubborn nodes

$$\underline{x} = \lim_{t \to +\infty} \underline{x}(t) = (I - Q)^{-1} E u$$

Remark 2:

$$(I - Q)\underline{x} = Eu$$

$$P = \begin{pmatrix} Q & E \\ F & G \end{pmatrix} \qquad I - P = \begin{pmatrix} I - Q & -E \\ -F & I - G \end{pmatrix}, \quad x = \begin{pmatrix} \underline{x} \\ u \end{pmatrix}$$

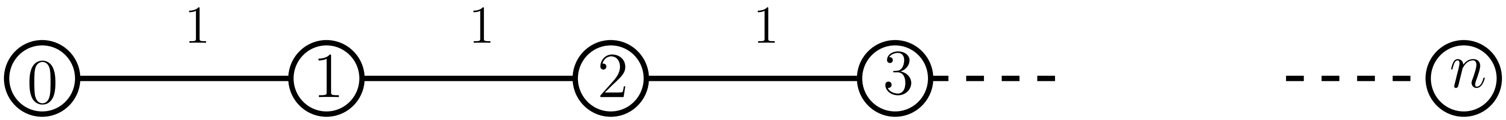
$$((I - P)x)_i = 0 \ \forall i \in \mathcal{R}$$

$$(I - P)x = (I - D^{-1}W)x = D^{-1}Lx \quad \Rightarrow \quad (Lx)_i = 0 \ \forall i \in \mathcal{R}$$

Equivalent characterization of the vector x of asymptotic opinions:

$$\begin{cases} (Lx)_{|\mathcal{R}} = 0 \\ x_{|\mathcal{S}} = u \end{cases}$$

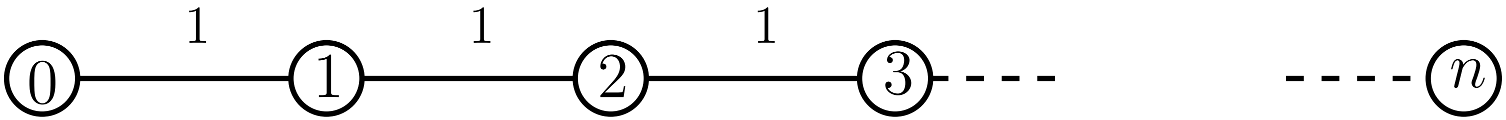
Example



Two stubborn agents 0 and n: $u_0 = 0$, $u_n = 1$.

x asymptotic vector of opinions.

Example

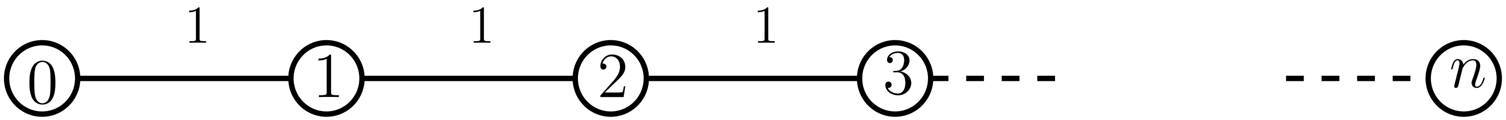


Two stubborn agents 0 and n: $u_0 = 0$, $u_n = 1$.

x asymptotic vector of opinions.

- For every $i = 1, \dots, n-1$, Laplace equation gives: $2x_i - x_{i-1} - x_{i+1} = 0 \ \Rightarrow \ x_{i+1} - x_i = x_i - x_{i-1}$

Example



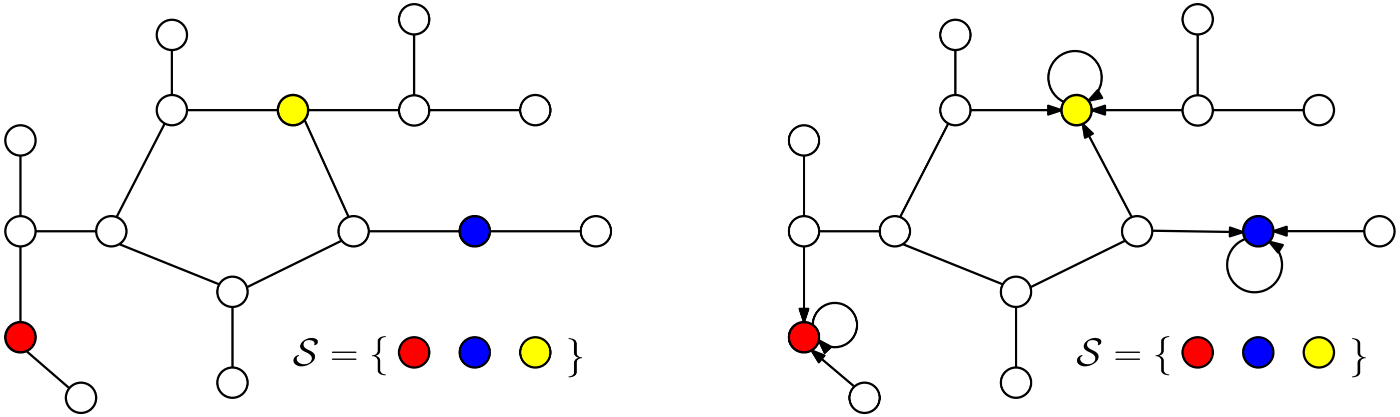
Two stubborn agents 0 and n: $u_0 = 0$, $u_n = 1$.

x asymptotic vector of opinions.

- For every $i = 1, \dots, n-1$, Laplace equation gives: $2x_i - x_{i-1} - x_{i+1} = 0 \;\Rightarrow\; x_{i+1} - x_i = x_i - x_{i-1}$
- $\sum_{i=1}^{n} (x_i - x_{i-1}) = x_n - x_0 = u_n - u_0 = 1 \;\Rightarrow\; x_i - x_{i-1} = \frac{1}{n}$ for all i
- $x_k = \sum_{i=1}^{k} (x_i - x_{i-1}) = \frac{k}{n}$

Stubborn nodes as sink components

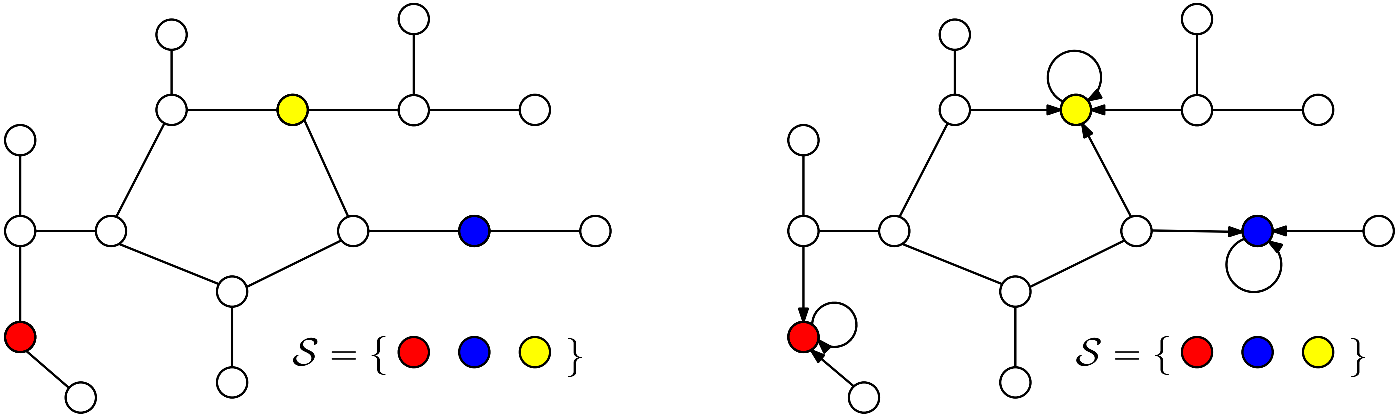
Consider the following graph transformation:



$$P = \begin{pmatrix} Q & E \\ F & G \end{pmatrix} \qquad \tilde{P} = \begin{pmatrix} Q & E \\ 0 & I \end{pmatrix}$$

Stubborn nodes as sink components

Consider the following graph transformation:



$$P = \begin{pmatrix} Q & E \\ F & G \end{pmatrix} \qquad \tilde{P} = \begin{pmatrix} Q & E \\ 0 & I \end{pmatrix}$$

- Same dynamics for the regular agents!
- $\mathcal{S}$ glob. reachable in $\mathcal{G}$ $\Rightarrow$ $\mathcal{S}$ glob. reachable and trapping in $\mathcal{G}_{\tilde{P}}$
- $\mathcal{G}$ str. connected $\Rightarrow$ $\mathcal{S}$ sinks in the condensation graph of $\mathcal{G}_{\tilde{P}}$

Flow models with exogenous inputs

- $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, $P = D^{-1}W$,
- A partition of the node set: $\mathcal{V} = \mathcal{R} \cup \mathcal{S}$
- Corresponding block structure:

$$P = \begin{array}{cc} \mathcal{R} & \mathcal{S} \\ \left[\begin{array}{c} Q \\ F \end{array}\right. & \left.\begin{array}{c} E \\ G \end{array}\right] \end{array} \begin{array}{l} \\ \mathcal{R} \\ \mathcal{S}\,. \end{array} \qquad y(t) = \left[\begin{array}{c} \underline{y}(t) \\ v(t) \end{array}\right] \begin{array}{l} \mathcal{R} \\ \mathcal{S}\,, \end{array}$$

- The linear flow model on $\mathcal{G}$ with input set $\mathcal{S}$:

$$\underline{y}(t+1) = Q'\underline{y}(t) + F'v(t)\,,$$

Flow models with exogenous inputs

- $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, $P = D^{-1}W$,
- A partition of the node set: $\mathcal{V} = \mathcal{R} \cup \mathcal{S}$
- Corresponding block structure:

$$P = \begin{array}{cc} & \begin{array}{cc} \mathcal{R} & \mathcal{S} \end{array} \\ & \left[\begin{array}{cc} Q & E \\ F & G \end{array}\right] \begin{array}{c} \mathcal{R} \\ \mathcal{S}\,. \end{array} \end{array} \qquad y(t) = \left[\begin{array}{c} \underline{y}(t) \\ v(t) \end{array}\right] \begin{array}{c} \mathcal{R} \\ \mathcal{S}\,, \end{array}$$

- The linear flow model on $\mathcal{G}$ with input set $\mathcal{S}$:

$$\underline{y}(t+1) = Q'\underline{y}(t) + F'v(t)\,,$$

- $F'v(t)$ exogenous inflow vector, new balance equation

$$\mathbb{1}'\underline{y}(t+1) - \mathbb{1}'\underline{y}(t) = \mathbb{1}'F'v(t) - \mathbb{1}'(I - Q')\underline{y}(t)$$

Asymptotics of flow models with exogenous inputs

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, $\mathcal{V} = \mathcal{R} \cup \mathcal{S}$ partition

$$P = D^{-1}W = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \mathcal{R} & \mathcal{S} \\ \left[\begin{matrix} Q & E \\ F & G \end{matrix}\right. & \left.\right] \end{array}, \qquad \underline{y}(t+1) = Q'\underline{y}(t) + F'v\,,$$

Theorem

1. $\underline{y}(t) = (Q')^t \underline{y}(0) + \sum_{s=0}^{t-1} (Q')^s F'v$ *for every* $t \geq 0$
2. *If $\mathcal{S}$ is globally reachable,*

$$\lim_{t\to+\infty} \underline{y}(t) = (I - Q')^{-1} F'v = \sum_{s=0}^{\infty} (Q')^s F'v$$

Computation of the Bonacich centrality

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, $P = D^{-1}W$

$$z = (1 - \beta)P'z + \beta\nu$$

$\beta \in (0, 1)$, ν_i intrinsic centrality of node i

Computation of the Bonacich centrality

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, $P = D^{-1}W$

$$z = (1 - \beta)P'z + \beta\nu$$

$\beta \in (0, 1)$, ν_i intrinsic centrality of node i

$$\underline{y}(t + 1) = (1 - \beta)P'\underline{y}(t) + \beta\nu$$

flow dynamics model with exogenous input

Computation of the Bonacich centrality

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ graph, $P = D^{-1}W$

$$z = (1-\beta)P'z + \beta\nu$$

$\beta \in (0,1)$, ν_i intrinsic centrality of node i

$$\underline{y}(t+1) = (1-\beta)P'\underline{y}(t) + \beta\nu$$

flow dynamics model with exogenous input

$$z = \lim_{t \to +\infty} \underline{y}(t)$$

Undirected networks and reversible matrices

Undirected networks and reversible matrices

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ undirected graph, $w = W\mathbb{1}$, $D = diag(w)$, $P = D^{-1}W$

$P'w = w \;\Rightarrow\; \pi = \frac{w}{\mathbb{1}'w}$ normalized left dominant eigenvector of P

$\mathcal{G}$ strongly connected $\Rightarrow \pi$ is the invariant distribution centrality of $\mathcal{G}$

Undirected networks and reversible matrices

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ undirected graph, $w = W\mathbb{1}$, $D = diag(w)$, $P = D^{-1}W$

$P'w = w \;\Rightarrow\; \pi = \frac{w}{\mathbb{1}'w}$ normalized left dominant eigenvector of P

$\mathcal{G}$ strongly connected $\Rightarrow \pi$ is the invariant distribution centrality of $\mathcal{G}$

Actually, more is true:

$$w_iP_{ij} = w_iw_i^{-1}W_{ij} = w_jw_j^{-1}W_{ji} = w_jP_{ji} \quad \text{balanced equation}$$

Undirected networks and reversible matrices

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ undirected graph, $w = W\mathbb{1}$, $D = diag(w)$, $P = D^{-1}W$

$P'w = w \;\Rightarrow\; \pi = \frac{w}{\mathbb{1}'w}$ normalized left dominant eigenvector of P

$\mathcal{G}$ strongly connected $\Rightarrow \pi$ is the invariant distribution centrality of $\mathcal{G}$

Actually, more is true:

$$w_iP_{ij} = w_iw_i^{-1}W_{ij} = w_jw_j^{-1}W_{ji} = w_jP_{ji} \quad \text{balanced equation}$$

The flow interpretation: if the matter is distributed among the nodes according to w, the amount of matter flowing from node i to node j is equal to the amount of matter flowing from node j to node i.

Not only the distribution is invariant (out flow equal to in flow for each node); the equilibrium is in this case maintained at the level of every single edge!

Undirected networks and reversible matrices

A stochastic matrix P is said to be *reversible* if there exists a positive vector y such that

$$y_i P_{ij} = y_j P_{ji} \qquad \forall i, j \quad \text{balance equation}$$

$$\sum_{j \in \mathcal{V}} P_{ji} y_j = \sum_{j \in \mathcal{V}} y_i P_{ij} = y_i \Rightarrow \; y = P' y$$

- y is a dominant left eigenvector of P
- If $\mathcal{G}_P$ is strongly connected, $\pi = y \frac{1}{\mathbb{1}' y}$ is the unique invariant distribution centrality.

Undirected networks and reversible matrices

A stochastic matrix P is said to be *reversible* if there exists a positive vector y such that

$$y_i P_{ij} = y_j P_{ji} \qquad \forall i, j \quad \text{balance equation}$$

$$\sum_{j \in \mathcal{V}} P_{ji} y_j = \sum_{j \in \mathcal{V}} y_i P_{ij} = y_i \Rightarrow \;\; y = P' y$$

- y is a dominant left eigenvector of P
- If $\mathcal{G}_P$ is strongly connected, $\pi = y \frac{1}{\mathbb{1}' y}$ is the unique invariant distribution centrality.

If we define $W_{ij} = y_i P_{ij}$ and consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ where $\mathcal{E} = \{(i,j) \,|\, W_{ij} > 0\}$ we have that $w_i = y_i$ and $P = D^{-1} W$

Reversible stochastic matricess

An important class of stochastic matrices with very special properties:

- The left dominant eigenvector π is easy to compute.
- They are diagonalizable.
- The rate of convergence to consensus $||(P^t - \mathbb{1}\pi')x(0)||$ can be computed in an explicit way.

They have important applications in network flows, Markov chain theory.

Reversible stochastic matrices

Lemma

Let P be a reversible irreducible stochastic matrix with invariant distribution centrality π. Put $\Pi := diag(\pi)$. Then,

- *$M := \Pi^{1/2}P\Pi^{-1/2}$ is symmetric.*
- *In particular P is diagonalizable and its eigenvalues satisfy:*

$$1 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n = -1$$

Reversible stochastic matrices

Theorem

Let P be a reversible stochastic matrix with invariant distribution centrality π. *Then,*

$$||(P^t - \mathbb{1}\pi')x(0)||_2 \leq \frac{\max\sqrt{\pi_i}}{\min\sqrt{\pi_i}}\lambda^t||x(0)||_2 \qquad \lambda := \max\{\lambda_2, |\lambda_n|\}$$

Spectral gap and Cheeger inequality

- $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected undirected, $P = D^{-1}W$
- For $\mathcal{U} \subseteq \mathcal{V}$, put
$$\Phi(\mathcal{U}) := \frac{\sum_{i\in\mathcal{U}} \sum_{j\in\mathcal{V}\setminus\mathcal{U}} W_{ij}}{w_{\mathcal{U}}} \quad \textit{bottleneck ratio of } U$$
where $w_{\mathcal{U}} := \sum_{i\in\mathcal{U}} w_i$.
- $$\Phi_{\mathcal{G}} = \min_{\substack{\mathcal{U}\subseteq\mathcal{V}: \\ 0 < w_{\mathcal{U}} \leq \frac{1}{2}\mathbb{1}'w}} \Phi(\mathcal{U}) \quad \textit{bottleneck ratio of } \mathcal{G}$$

Spectral gap and Cheeger inequality

- $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected undirected, $P = D^{-1}W$
- For $\mathcal{U} \subseteq \mathcal{V}$, put
$$\Phi(\mathcal{U}) := \frac{\sum_{i\in\mathcal{U}} \sum_{j\in\mathcal{V}\setminus\mathcal{U}} W_{ij}}{w_{\mathcal{U}}} \quad \textit{bottleneck ratio of } U$$
where $w_{\mathcal{U}} := \sum_{i\in\mathcal{U}} w_i$.
-
$$\Phi_{\mathcal{G}} = \min_{\substack{\mathcal{U}\subseteq\mathcal{V}:\\ 0<w_{\mathcal{U}}\leq\frac{1}{2}\mathbb{1}'w}} \Phi(\mathcal{U}) \quad \textit{bottleneck ratio of } \mathcal{G}$$

Theorem (Cheeger's inequality)

Let λ_2 be the second eigenvalue of P. Then,

$$\frac{1}{2}\Phi_{\mathcal{G}}^2 \leq 1 - \lambda_2 \leq 2\Phi_{\mathcal{G}}\,.$$

Spectral gap and Cheeger inequality

For a simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$:

$$\Phi_{\mathcal{G}} = \min_{\substack{\mathcal{U} \subseteq \mathcal{V}: \\ 0 < w_{\mathcal{U}} \leq \frac{1}{2} \mathbb{1}' w}} \frac{|\partial_{\mathcal{U}}|}{w_{\mathcal{U}}},$$

where $\partial_{\mathcal{U}} = \{(i,j) \in \mathcal{E} : i \in \mathcal{U}, j \in \mathcal{V} \setminus \mathcal{U}\}$ is called the boundary of $\mathcal{U}$.

Example

- Complete K_n:

$$|\mathcal{U}| = k \Rightarrow \Phi(\mathcal{U}) = k(n-k)/(k(n-1)) = (n-k)/(n-1)$$

$$\Phi_{K_n} = \begin{cases} \frac{1}{2}(1 + \frac{1}{n-1}) & \text{if } n \text{ is even} \\ \frac{1}{2}(1 + \frac{2}{n-1}) & \text{if } n \text{ is odd} \end{cases}$$

- Barbell graph with n nodes B_n

$$\Phi_{B_n} = (n^2/4 - n/2 + 1)^{-1}$$

Electrical networks

Electrical networks

- $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ strongly connected, undirected, no self-loops
- $W_{ij} = W_{ji}$ *electrical conductance* of the undirected link $\{i, j\}$.
- $1/W_{ij}$ *electrical resistance* of $\{i, j\}$.
- an exogenous net flow of currents ν on $\mathcal{G}$ $(\sum_i \nu_i = 0)$

A network flow optimization problem:

$$M(\nu) := \inf_{\substack{f \in \mathbb{R}_+^{\mathcal{E}} \\ Bf = \nu}} \sum_{(i,j) \in \mathcal{E}} \frac{1}{2W_{ij}} f_{ij}^2 .$$

The cost function $f_{ij}^2 / 2W_{ij}$ is a power dissipation.

Electrical networks

Duality approach:

- $L(f,\lambda,\nu) = \sum_{(i,j)\in\mathcal{E}} \frac{1}{2W_{ij}} f_{ij}^2 + \sum_{i\in\mathcal{V}} \lambda_i(\nu_i - (Bf)_i)$
- $D(\lambda,\nu) := \inf_{f\in\mathbb{R}_+^{\mathcal{E}}} L(f,\lambda,\nu) = -\frac{1}{4} \sum_{(i,j)\in\mathcal{E}} W_{ij}[\lambda_i - \lambda_j]^2 + \sum_{i\in\mathcal{V}} \lambda_i \cdot \nu_i \,,$
- (f^*,λ^*) solutions of, respectively, the primal and the dual problem:

$$\sum_j f_{ij}^* - \sum_j f_{ji}^* = \nu_i \quad \text{Kirchoff's law}$$

$$f_{ij}^* = W_{ij}\left[\lambda_i^* - \lambda_j^*\right]_+ \quad \text{Ohm's law}$$

Interpretation: f_{ij}^* electrical current flow, λ_i^* voltage.

Ohm's law states that current flows from an higher voltage node to a lower one and is proportional to the difference.

Notation $x = \lambda^*$ in the next slides.

Electrical networks

$$\sum_j f^*_{ij} - \sum_j f^*_{ji} = \nu_i \quad \text{Kirchoff's law}$$

$$f^*_{ij} = W_{ij}\left[x_i - x_j\right]_+ \qquad \text{Ohm's law}$$

Substituting

$$\begin{aligned} \nu_i &= \textstyle\sum_j W_{ij}[x_i - x_j]_+ - \sum_j W_{ji}[x_j - x_i]_+ \\ &= \textstyle\sum_j W_{ij}([x_i - x_j]_+ - [x_j - x_i]_+) \\ &= \textstyle\sum_j W_{ij}(x_i - x_j) = w_i x_i - \sum_j W_{ij} x_j \end{aligned}$$

Consider the Laplacian matrix $L = D - W$. The voltage x has to satisfy:

$$Lx = \nu \quad \text{Laplace equation}$$

Laplace $\Leftrightarrow$ Kirchoff $+$ Ohm

Electrical networks

L in this case is symmetric and can thus be decomposed as

$$L = \sum_{k \geq 2} \overline{\lambda}_k \overline{z}_{(k)} \overline{z}'_{(k)}$$

where $0 = \overline{\lambda}_1 < \overline{\lambda}_2 \leq \ldots \leq \overline{\lambda}_n$ are the eigenvalues of L and $\overline{z}_{(k)}$ the corresponding normalized eigenvectors with $\overline{z}_{(1)} = n^{-1/2}\mathbb{1}$.

$$Z = \sum_{k \geq 2} \frac{1}{\overline{\lambda}_k} \overline{z}_{(k)} \overline{z}'_{(k)} \quad \text{Green matrix} \tag{1}$$

$$ZL = LZ = I - n^{-1}\mathbb{1}\mathbb{1}', \quad Z\mathbb{1} = 0 \tag{2}$$

Electrical networks

Theorem

Consider a strongly connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ *with no self-loops and an exogenous net flow* ν *on* $\mathcal{G}$*. Then,* x *is a solution of the Laplace equation* $Lx = \nu$ *if and only if*

$$x = Z\nu + c\mathbb{1}$$

where c *is any constant.*

Proof

Electrical networks

Theorem

Consider a strongly connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with no self-loops and an exogenous net flow ν on $\mathcal{G}$. Then, x is a solution of the Laplace equation $Lx = \nu$ if and only if

$$x = Z\nu + c\mathbb{1}$$

where c is any constant.

Proof

$$\begin{aligned} Lx = \nu \quad &\Rightarrow \quad Z\nu = ZLx = x - n^{-1}\mathbb{1}\mathbb{1}'x \\ &\Rightarrow \quad x = Z\nu + c\mathbb{1} \quad c = n^{-1}\mathbb{1}'x \end{aligned}$$

Electrical networks

Theorem

Consider a strongly connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ *with no self-loops and an exogenous net flow* ν *on* $\mathcal{G}$*. Then,* x *is a solution of the Laplace equation* $Lx = \nu$ *if and only if*

$$x = Z\nu + c\mathbb{1}$$

where c *is any constant.*

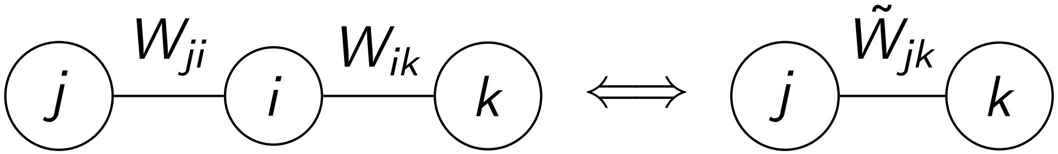
Proof

$$\begin{aligned} Lx = \nu \quad &\Rightarrow \quad Z\nu = ZLx = x - n^{-1}\mathbb{1}\mathbb{1}'x \\ &\Rightarrow \quad x = Z\nu + c\mathbb{1} \quad c = n^{-1}\mathbb{1}'x \end{aligned}$$

$$x = Z\nu + c\mathbb{1} \Rightarrow Lx = LZ\nu + cL\mathbb{1} = \nu$$

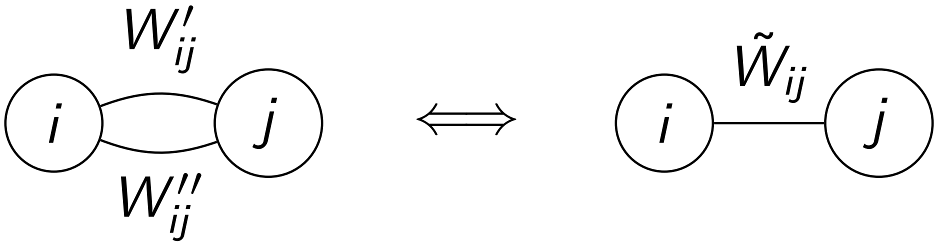
Computational tricks

1. *Series law:*



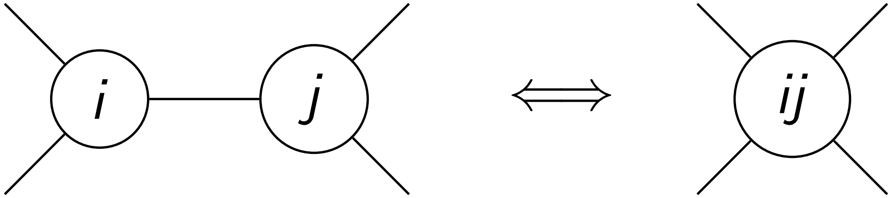
$$\tilde{W}_{jk}^{-1} = W_{ji}^{-1} + W_{ik}^{-1}$$

2. *Parallel law:*



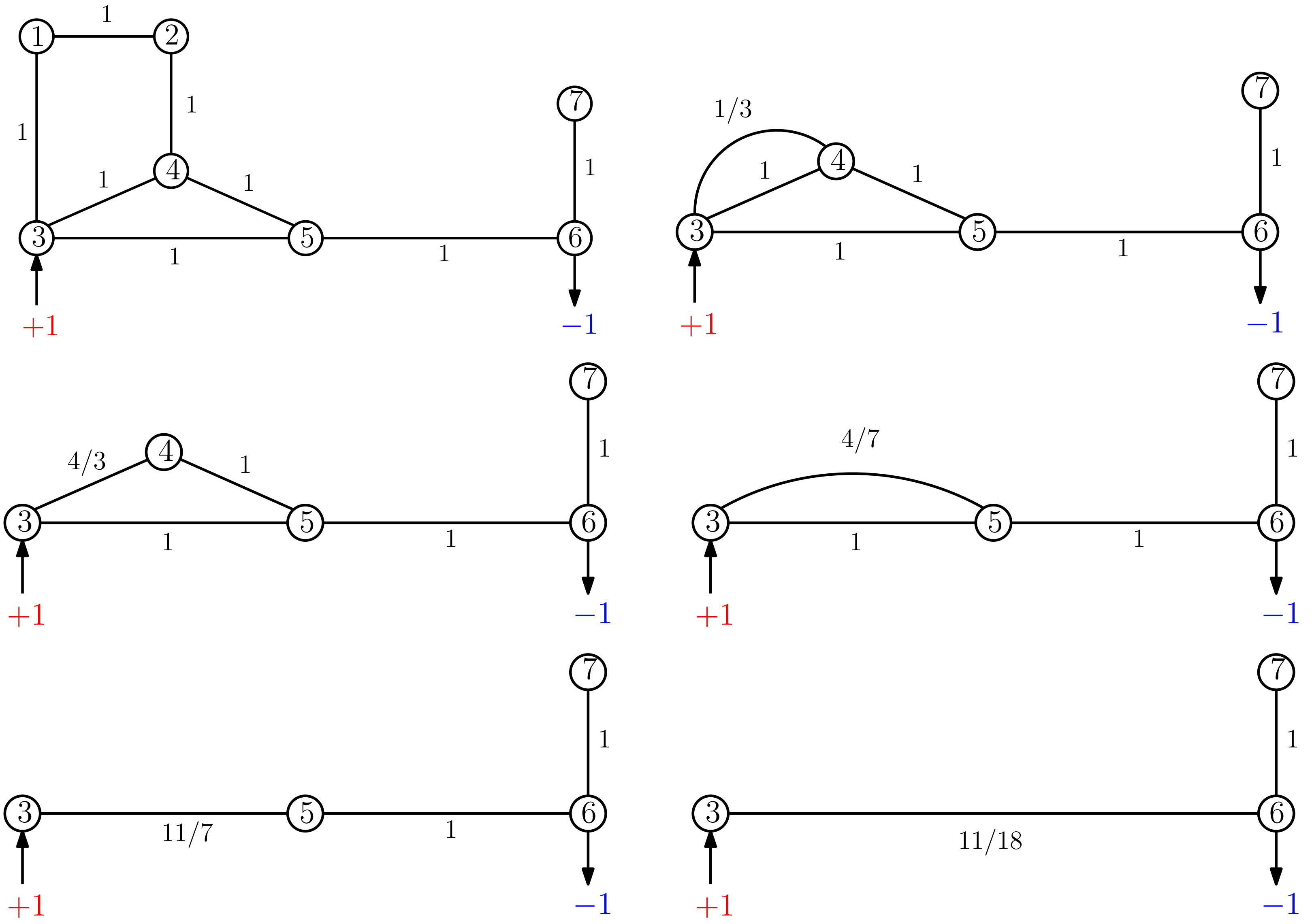
$$\tilde{W}_{ij} = W'_{ij} + W''_{ij}$$

3. *Glueing:*

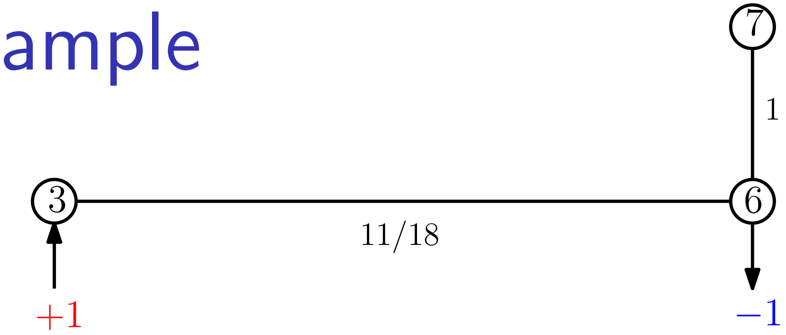


if $x_i = x_j$

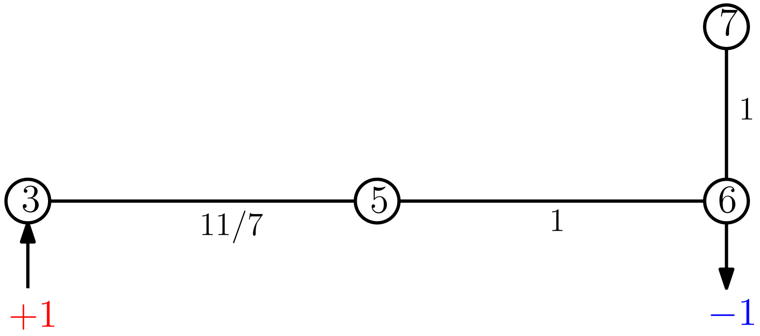
Example


1
2
3
4
5
6
7
1
1
1
1
1
1
1
1
1
+1
−1
1/3
1
1
1
1
1
+1
−1
4/3
1
1
1
1
+1
−1
4/7
1
1
1
+1
−1
11/7
1
1
+1
−1
11/18
1
+1
−1

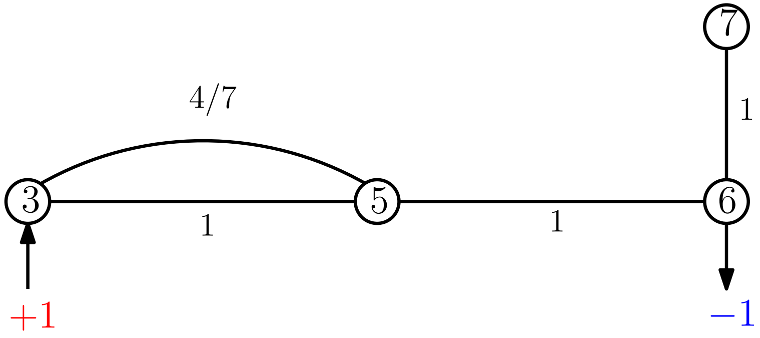
Example



$$\phi_{36} = 1,\ x_3 - x_6 = \frac{\phi_{36}}{W_{36}} = \frac{18}{11}$$
$$\phi_{67} = \phi_{76} = 0,\ x_6 = x_7$$

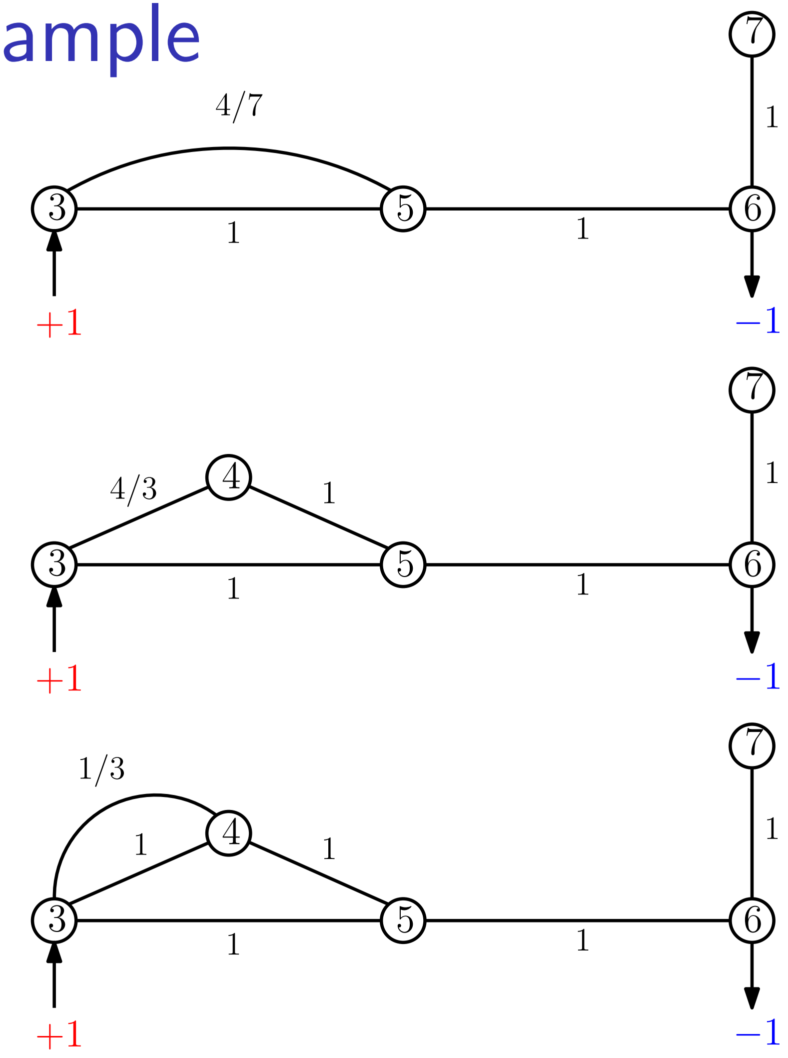


$$\phi_{35} = \phi_{56} = 1,\ x_3 - x_5 = \frac{7}{11}$$



$$\phi_{35}^{up} = \frac{4}{7} \cdot \frac{7}{11} = \frac{4}{11},\ \phi_{35}^{down} = \frac{7}{11}$$

Example

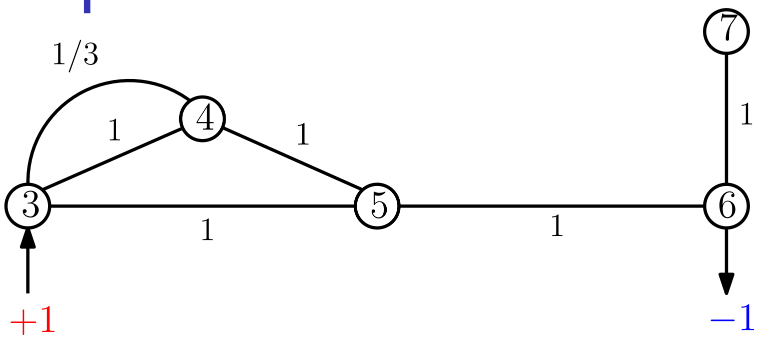


$$\phi_{35}^{up} = \frac{4}{7} \cdot \frac{7}{11} = \frac{4}{11}, \quad \phi_{35}^{down} = \frac{7}{11}$$

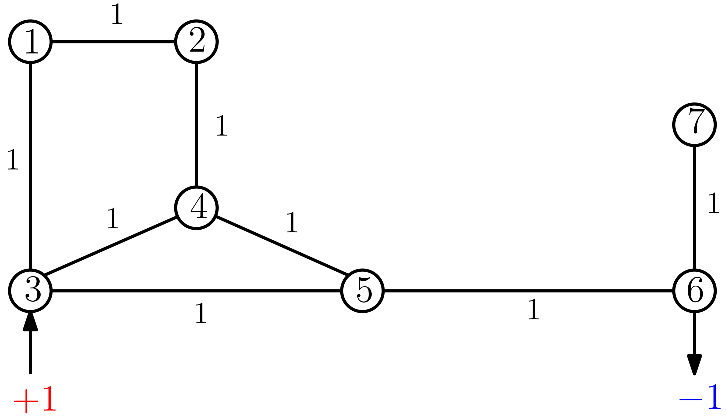
$$\phi_{34} = \phi_{45} = \frac{4}{11}, \quad x_3 - x_4 = \frac{3}{11}$$

$$\phi_{34}^{up} = \frac{1}{11}, \quad \phi_{34}^{down} = \frac{3}{11}$$

Example



$$\phi_{34}^{up} = \frac{1}{11}, \ \phi_{34}^{down} = \frac{3}{11}$$



$$\phi_{31} = \phi_{12} = \phi_{24} = \frac{1}{11}$$

$$x_3 - x_1 = x_1 - x_2 = x_2 - x_4 = \frac{1}{11}$$

$$x_6 = x_7 = 0, \ x_5 = 1, \ x_3 = \frac{18}{11}, \ x_4 = \frac{15}{11}, \ x_2 = \frac{16}{11}, \ x_2 = \frac{17}{11}$$

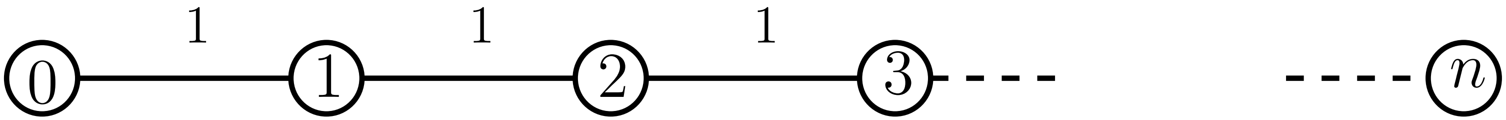
The Dirichlet problem

We assign voltages in a family of nodes $\mathcal{S} \subseteq \mathcal{V}$: u_i, $i \in \mathcal{S}$

Problem: does there exist a vector of exogenous currents ν such that the corresponding voltage x is such that $x_{|\mathcal{S}} = u$?

- Recall that $Lx = \nu$
- Dirichlet problem: $\begin{cases} (Lx)_{|\mathcal{R}} = 0 \\ x_{|\mathcal{S}} = u \end{cases}$
- If we can solve it and x is a solution, it is sufficient to put $\nu = Lx$: ν is an exogenous flow concentrated on $\mathcal{S}$ that generates x
- However, we have already solved the Dirichlet problem: x is the asymptotic opinion vector when nodes in $\mathcal{S}$ are stubborn with opinion u!
- Voltages $\leftrightarrow$ Asymptotic opinions

Example



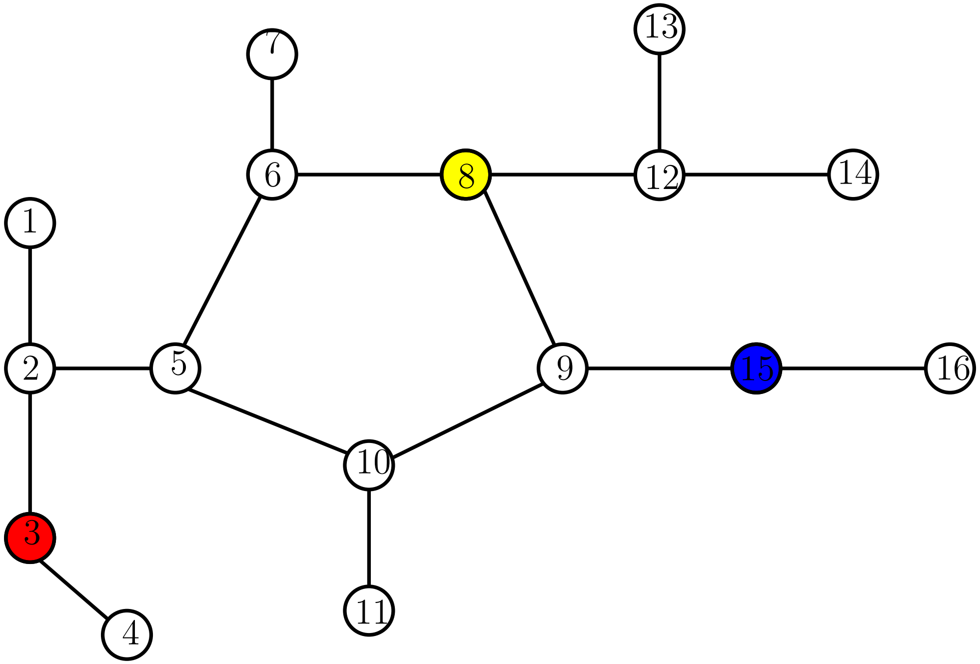
Assigned voltage on the leaf nodes: $u_0 = 0$, $u_n = 1$.

$x_k = \frac{k}{n}$ voltage in the remaining nodes.

Check using 'electrical tools'.

Example

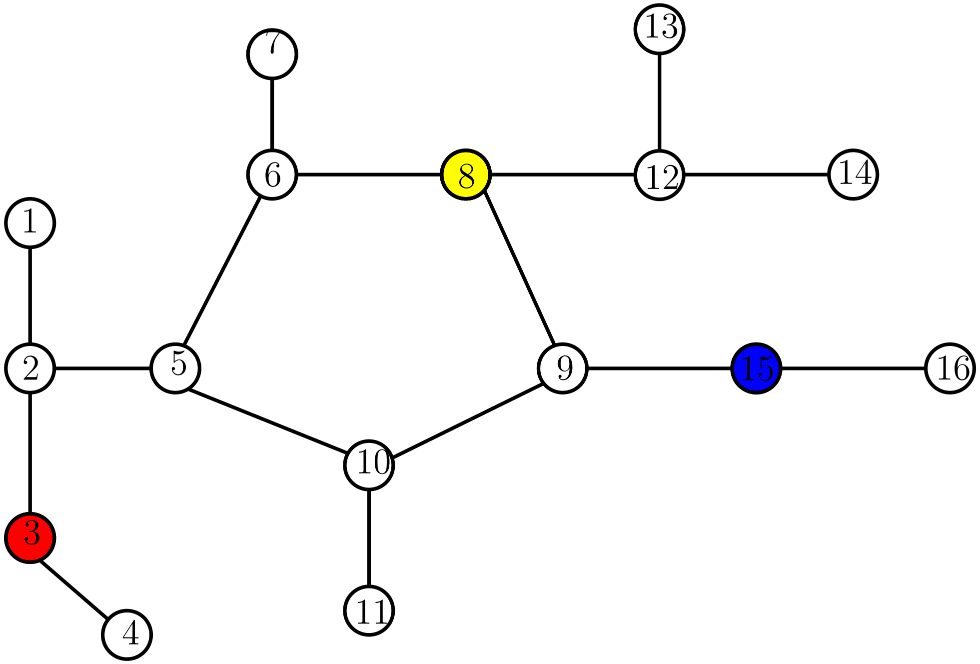
$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ simple graph below



Assume that $x_3 = +1$, $x_8 = -1$, $x_{15} = 2$. Compute x_i for $i \in \mathcal{R}$.

Example

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ simple graph below

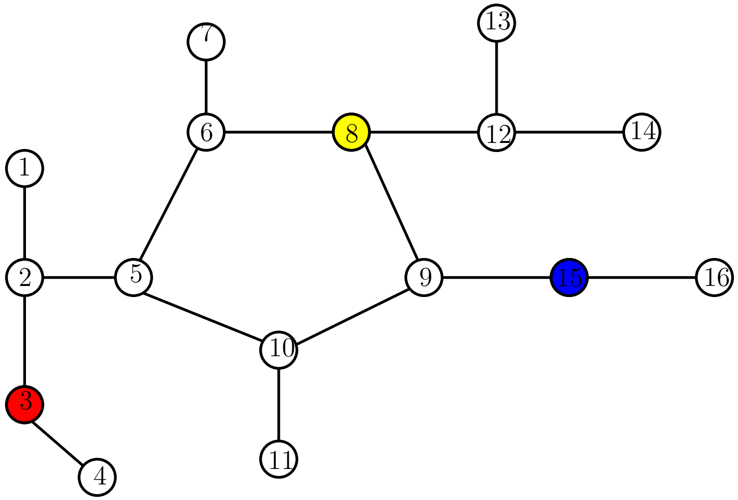


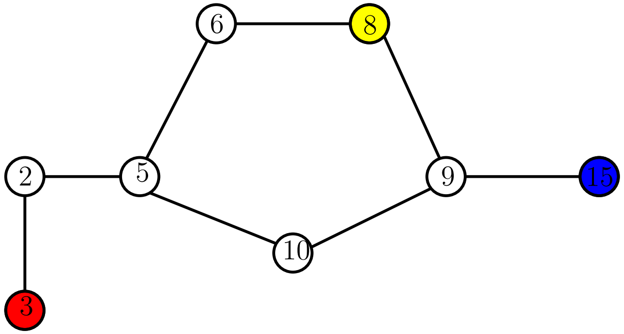
Assume that $x_3 = +1$, $x_8 = -1$, $x_{15} = 2$. Compute x_i for $i \in \mathcal{R}$.

Easy: $x_4 = x_3 = +1$, $x_{16} = x_{15} = 2$, $x_{12} = x_{13} = x_{14} = x_8 = -1$

Example

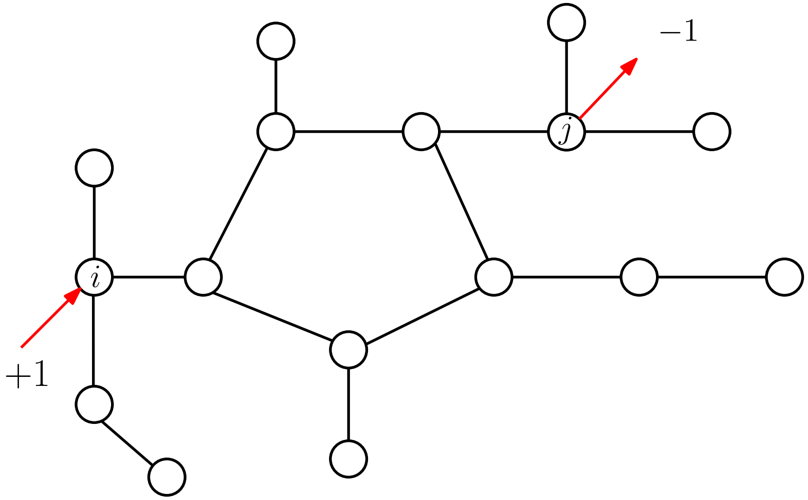
Glueing:





Effective resistance

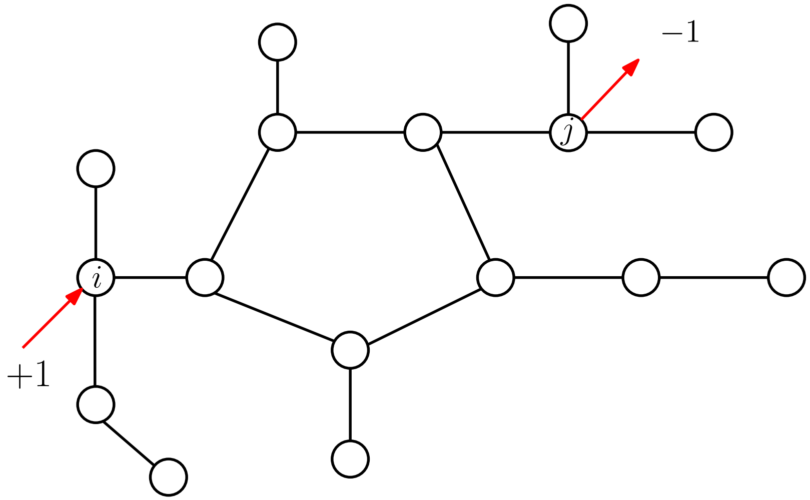
$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ undirected network with no self-loops



- Fix two nodes $i, j \in \mathcal{V}$
- Consider the exogenous net flow $\nu = \delta^{(i)} - \delta^{(j)}$ and compute the voltage $x = Z\nu$
- $R_{ij}^{\mathcal{G}} = x_i - x_j$ *effective resistance* between i and j

Effective resistance

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ undirected network with no self-loops



- Fix two nodes $i, j \in \mathcal{V}$
- Consider the exogenous net flow $\nu = \delta^{(i)} - \delta^{(j)}$ and compute the voltage $x = Z\nu$
- $R_{ij}^{\mathcal{G}} = x_i - x_j$ *effective resistance* between i and j

$$R_{ij}^{\mathcal{G}} = Z_{ii} - Z_{ij} - Z_{ji} + Z_{jj} = Z_{ii} + Z_{jj} - 2Z_{ij}$$

Effective resistance

Theorem (Thompson's principle)

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ *undirected network with no self-loops*

$$R_{ij}^{\mathcal{G}} = \min_{\substack{f \in \mathbb{R}_+^{\mathcal{V}}: \\ Bf = \delta^{(i)} - \delta^{(j)}}} \sum_{(h,k) \in \mathcal{E}} \frac{1}{W_{hk}} f_{hk}^2$$

$$\begin{aligned} M(\delta^{(i)} - \delta^{(j)}) &= M^*(\delta^{(i)} - \delta^{(j)}) \quad \text{optimum of the dual problem} \\ &= -\frac{1}{4} \sum_{(h,k)\in\mathcal{E}} W_{hk}[\lambda_h^* - \lambda_k^*]^2 + \lambda_i^* - \lambda_j^* \\ &= -\frac{1}{2} \sum_{(h,k)\in\mathcal{E}} \frac{1}{W_{hk}} (f_{hk}^*)^2 + R_{ij}^{\mathcal{G}} = -M(\delta^{(i)} - \delta^{(j)}) + R_{ij}^{\mathcal{G}} \end{aligned}$$

$$\Rightarrow \quad R_{ij}^{\mathcal{G}} = 2M(\delta^{(i)} - \delta^{(j)}) = \min_{\substack{f \in \mathbb{R}_+^{\mathcal{V}}: \\ Bf = \delta^{(i)} - \delta^{(j)}}} \sum_{(i,j)\in\mathcal{E}} \frac{1}{W_{ij}} f_{ij}^2 \,,$$

Effective resistance

Corollary (Raleigh principle)

For every $i, j \in \mathcal{V}$, $R_{ij}^{\mathcal{G}}$ *is monotonically decreasing with respect to the conductances* $\{W_{hk}\}$

Optimal targeting problems

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ undirected network with no self-loops. $v_0 \neq v_1$ stubborn agents, $u_{v_0} = 0$, $u_{v_1} = 1$, x vector of asymptotic opinions.

$$H(v_0, v_1) = \frac{1}{n} \sum_{i \in \mathcal{V}} x_i \in [0, 1] \quad \text{Harmonic influence centrality}$$

Theorem

1. $H(v_0, v_1) = \frac{1}{2} + \frac{\frac{1}{n} \sum_{i \in \mathcal{V}} R^{\mathcal{G}}_{iv_0} - \frac{1}{n} \sum_{i \in \mathcal{V}} R^{\mathcal{G}}_{iv_1}}{2 R^{\mathcal{G}}_{v_0 v_1}}$
2. *For the problem*
$$\min_{v_0 \in \mathcal{V}} \max_{v_1 \in \mathcal{V} \setminus \{v_0\}} H(v_0, v_1)$$
the optimal choice for v_0 *is to minimize* $\sum_{i \in \mathcal{V}} R^{\mathcal{G}}_{iv_0}$.
3. *This optimal choice leads to* $H(v_0, v_1) \leq 1/2$. *The first stubborn to set has an advantage!*