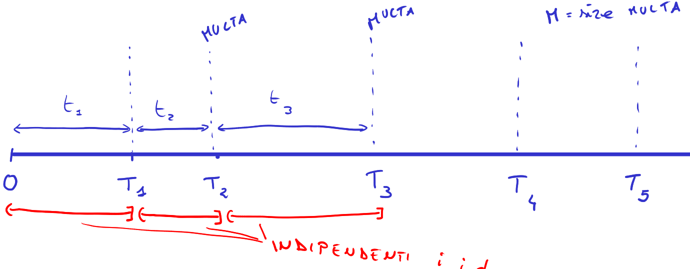
Renewal-reward processes

Suppose the manager of a restaurant should pay $\$c_1$ per day for health and safety maintenance, enough to guarantee that there will be no health violations. However, the manager is evil and wants to budget less, say $c_2 < c_1$. We want to figure out when this will make him money. Here is what we know:

- The restaurant is inspected, on average, every 45 days, and the number of days between two consecutive inspections are independent and identically distributed random variables (maybe not realistic).
- There is a probability $p = p(c_2)$ that a violation will be found on a given visit. It is a monotonically decreasing function of c_2, with $p(c_1) = 0$.
- The fines have an expected value of $c_3 > 0$ and the fine sizes are all independent and identically distributed, and independent on the inspection times.

Renewal-reward processes



t_i i.i.d.

$E[t_i] = 45$

$P(M > 0) = p(c_2)$

$E[M \mid M > 0] = c_3$

i.i.d.

RENEWAL REWARD PROCESS

MASSIMIZZARE $\lim_{t \to +\infty} \frac{R(t)}{t}$

$R(t)$ = GUADAGNO FINO A t.

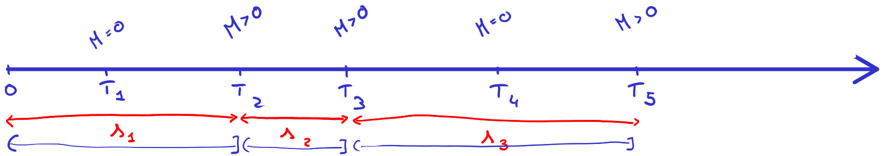
$r_i = -c_2 \cdot t_i - M_i$

$$\lim_{t \to +\infty} \frac{R(t)}{t} = \frac{E[r_1]}{E[t_1]} =$$

$$= \frac{-c_2 E[t_1] - p(c_2) \cdot E[M_i \mid M_i > 0]}{E[t_1]} = \frac{-c_2 \cdot 45 - p(c_2) \cdot c_3}{45}$$

Renewal-reward processes

s_i i.i.d.



ANCORA UN RENEWAL REWARD PROCESS CONTANDO MULTE!

$\tilde{r}_i = -C_2 \cdot s_i - \tilde{M}_i$

$E[\tilde{r}_i] = -C_2 E[s_i] - E[M_i] = -C_2 E[s_i] - C_3$

$E[s_1] = E\left[\sum_{i=1}^{G} t_i\right]$

G = # ISPEZIONI PRIMA PRIMA MULTA

$\sim \mathrm{Geom}(p(c_2))$

$= E[G]\,E[t_1]$

$= \dfrac{1}{p(c_2)} \cdot 45 \quad \rightsquigarrow \quad \dfrac{R(t)}{t} \xrightarrow[t\to+\infty]{} \dfrac{E[\tilde{r}_1]}{E[s_1]} = \dfrac{-C_2 \cdot \frac{45}{p(c_2)} - C_3}{\frac{45}{p(c_2)}}$

UGUALE A PRIMA!

PERCHE' NON CONTARE I GIORNI?

μ_i i.i.d.

$E[\mu_i] = 1$ $\quad E[\hat{r}_i] = -c_2 - \hat{M}_i$ $\quad \hat{r}_i$ NON SONO i.i.d.

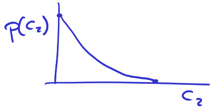
NON FUNZIONA

Renewal-reward processes

$P(0) = 1$ $P(c_1) = 0$

Assume

$$p(c_2) = \frac{1+c_1}{c_1} \cdot \frac{1}{1+c_2} - \frac{1}{c_1}$$



We want to minimize

$$g(c_2) = c_2 + c_3 \frac{p(c_2)}{45} = c_2 + \frac{c_3}{45}\left(\frac{1+c_1}{c_1} \cdot \frac{1}{1+c_2} - \frac{1}{c_1}\right).$$

We consider

$$g'(c_2) = 1 - \frac{c_3}{45} \cdot \frac{1+c_1}{c_1} \cdot \frac{1}{(1+c_2)^2}$$

and note that

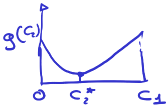
$$g''(c_2) = \frac{c_3}{45} \cdot \frac{1+c_1}{c_1} \cdot \frac{2}{(1+c_2)^3}$$

is positive for any c_2 in $[0, c_1]$.

Renewal-reward processes

Solving $g'(c_2^*) = 0$ yields

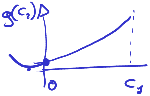
$$c_2^* = \sqrt{\frac{c_3}{45} \cdot \frac{1+c_1}{c_1}} - 1.$$



We have three cases:

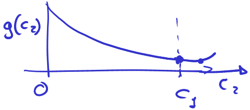
1. if c_2^* is between 0 and c_1, then it corresponds to the optimal policy the evil manager is looking for;
2. if

$$\frac{c_3}{45} \cdot \frac{1+c_1}{c_1} \le 1$$



   then the fee to pay for the violation is so low that the optimal strategy is to pay nothing for maintenance;
3. if

$$\boxed{\frac{c_3}{45} \cdot \frac{1+c_1}{c_1} \ge (1+c_1)^2}$$

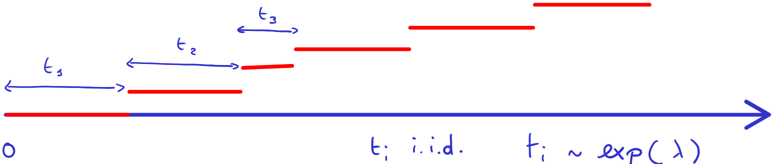


   then the fee is adequate, and the optimal policy is to pay c_1 dollars for maintenance.

Poisson processes

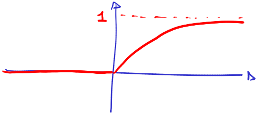
Definition

A *Poisson process with rate* λ is a counting process $(N(s))_{s\in[0,\infty)}$ with $N(0)=0$, whose inter-arrival times are i.i.d. exponential random variables with rate λ.



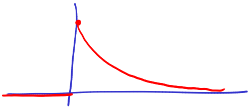
t_i i.i.d. $\quad t_i \sim exp(\lambda)$

A review of the Exponential distribution



We say that $\tau \sim \exp(\lambda)$ if any of the following holds:

$$f_\tau(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \qquad P(\tau \leq t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

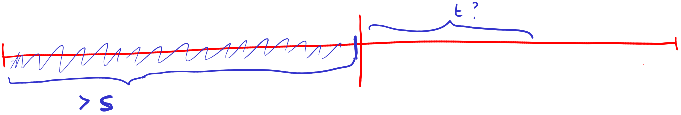


$$P(\tau > t) = \begin{cases} e^{-\lambda t} & \text{if } t \geq 0 \\ 1 & \text{if } t < 0 \end{cases}$$

$= 1 - P(\tau \leq t)$

$$E[\tau] = 1/\lambda, \quad \mathrm{Var}(\tau) = 1/\lambda^2$$

Memoryless property



For any $\underline{t > s \geq 0}$ we have $P(\tau > t + s | \tau > s) = \boxed{P(\tau > t)}$

$$P(\tau > t+s \mid \tau > s) = \frac{P(\tau > t+s,\ \tau > s)}{P(\tau > s)} = \frac{P(\tau > t+s)}{P(\tau > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = P(\tau > t)$$

Exponential random variables are the **only** continuous random variables with the memoryless property (geometric random variables are the only discrete random variables with the memoryless property).

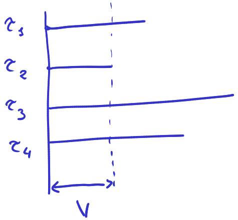
Exponential races

Exponential races: Let $\tau_1, \tau_2, \ldots, \tau_n$ be independent random variables, with $\tau_i \sim \exp(\lambda_i)$. Let

$$V = \min\{\tau_1, \tau_2, \ldots, \tau_n\} \quad \text{and} \quad I \text{ s.t. } \tau_I = V.$$

Then:

1. $V \sim \exp\left(\sum_{i=1}^n \lambda_i\right)$
2. $P(I = j) = \frac{\lambda_j}{\sum_{i=1}^n \lambda_i}$
3. I and V are independent.



$I = 2$

Exponential races

$t > 0$

$V \sim \exp\left(\sum_{i=1}^{n} \lambda_i\right)$

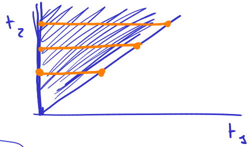
$$P(V > t) = P(\min_i \tau_i > t) = P(\tau_1 > t, \tau_2 > t, \dots, \tau_n > t)$$

$$= P(\tau_1 > t) P(\tau_2 > t) \dots P(\tau_n > t)$$

$$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdot e^{-\lambda_3 t} \cdot \dots \cdot e^{-\lambda_n t}$$

$$= e^{-(\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n) t} \quad \leadsto \quad V \sim \exp\left(\sum_{i=1}^{n} \lambda_i\right)$$

Exponential races



$P(I=j)=\frac{\lambda_j}{\sum_{i=1}^n \lambda_i}$ with two exponentials.

τ_1, τ_2

$$
\begin{aligned}
P(I=1)=P(\tau_1<\tau_2)&=\iint_{t_1<t_2} \boxed{\lambda_1 e^{-\lambda_1 t_1}}\,\boxed{\lambda_2 e^{-\lambda_2 t_2}}\,dt_1\,dt_2 \\
&=\int_0^\infty \left(\int_0^{t_2} \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2}\,dt_1\right) dt_2 \\
&=\int_0^\infty \lambda_2 e^{-\lambda_2 t_2} \left(\int_0^{t_2} \lambda_1 e^{-\lambda_1 t_1}\,dt_1\right) dt_2 \\
&=\cdots=\frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}
$$

Exponential races

$$P(I=j) = P(\underbrace{\tau_j}_{\sim \exp(\lambda_j)} < \underbrace{\boxed{\min_{i \neq j} \tau_i}}_{\sim \exp(\sum_{i\neq j}\lambda_i)}) \overset{\text{VISTO PRIMA}}{=} \frac{\lambda_j}{\lambda_j + \sum_{i \neq j} \lambda_i} = \frac{\lambda_j}{\sum_i \lambda_i}$$

$\exp(\lambda_j) \perp \exp(\sum_{i \neq j} \lambda_i)$

$$P(I=j) = \frac{\lambda_j}{\sum_{i=1}^{n} \lambda_i}.$$

$$\{I=j\} = \{\tau_j < \tau_1, \tau_j < \tau_2, \dots, \tau_j < \tau_{j-1}\}$$

Exponential races

I and V are independent.

GOAL: $P(V > t, I = j) = P(V > t)P(I = j)$ for all t, j.

$\min\limits_{i \neq j} \tau_i \sim \exp(\hat{\lambda}_j) \perp \tau_j$

$\hat{\lambda}_j = \sum\limits_{i \neq j} \lambda_i$

$$P(V > t, I = j) = P\Bigg(\overbrace{\underbrace{\min_{\substack{i=1,\dots,n\\ i\neq j}} \tau_i}_{h} > \tau_j}^{I=j}, \underbrace{\tau_j}_{t_j} > t\Bigg)$$

$$= \iint_{h > t_j > t} \hat{\lambda}_j e^{-h\hat{\lambda}_j} \lambda_j e^{-\lambda_j t_j} \, dh dt_j$$

$$= \int_0^\infty \left(\int_{t_j}^\infty \hat{\lambda}_j e^{-h\hat{\lambda}_j} \lambda_j e^{-\lambda_j t_j} \, dh \right) dt_j$$

$$= \dots = e^{-(\hat{\lambda}_j + \lambda_j)t} \frac{\lambda_j}{\hat{\lambda}_j + \lambda_j} = e^{-(\sum_i \lambda_i)t} \cdot \frac{\lambda_j}{\sum_i \lambda_i}$$

$$= P(V > t) P(I = j)$$

Alice and Bob are doing homework. Alice is done after a time $\tau_A \sim \exp(1)$ and Bob is done after a time $\tau_B \sim \exp(1/4)$. τ_A, τ_B INDIPENDENTI

What is the probability Alice is done before Bob?

$$P(\tau_A < \tau_B) = \frac{\lambda_A}{\lambda_A + \lambda_B} = \frac{1}{1 + 1/4} = \frac{4}{5}$$

Knowing the the first who is done does so after 4 hours, what is the probability Alice is done before Bob?

$$P(\tau_A < \tau_B \mid \min\{\tau_A, \tau_B\} > 4) = P(I = A \mid V > 4) = P(I = A)$$

$$= P(\tau_A < \tau_B)$$

Knowing that Alice is done before Bob, what is the expected time she finishes homework?

$$E[\min\{\tau_A, \tau_B\} \mid \tau_A = \min\{\tau_A, \tau_B\}] = E[\min\{\tau_A, \tau_B\}] = \frac{1}{\lambda_A + \lambda_B} = \frac{1}{1 + 1/4} = \frac{4}{5}$$

$$\cancel{E[\tau_A] = \frac{1}{\lambda_A} = 1}$$

$$E[\tau_A \mid \tau_A = \min\{\tau_A, \tau_B\}]$$

$$E[\tau_A \mid \tau_A < \tau_B]$$

$\tau^{(1)} = \min\{\tau_1, \tau_2, \tau_3\}$

$\tau^{(3)} = \max\{\tau_1, \tau_2, \tau_3\}$

$\tau^{(1)} < \tau^{(2)} < \tau^{(3)}$

$P(\tau_2 = \tau^{(1)}) = \dfrac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}$

INDEPENDENT

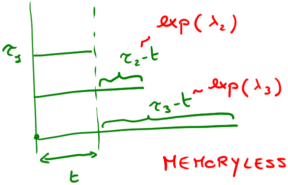
Let τ_i, $i \in \{1,2,3\}$, be exponentially distributed with parameters λ_i. For every realization, sort the three exponentials. Denote by $\tau^{(1)}$ the minimum, $\tau^{(3)}$ the maximun and $\tau^{(2)}$ the intermediate one. Calculate the probability that the second exponential also ranks second ($\tau^{(2)} = \tau_2$).

$$
\begin{aligned}
P(\tau_2 = \tau^{(2)}) = {} & P(\tau_2 = \tau^{(2)} \mid \tau_1 = \tau^{(1)}) \cdot P(\tau_1 = \tau^{(1)}) \\
& + \underbrace{P(\tau_2 = \tau^{(2)} \mid \tau_2 = \tau^{(1)})}_{0} \cdot P(\tau_2 = \tau^{(1)}) \\
& + P(\tau_2 = \tau^{(2)} \mid \tau_3 = \tau^{(1)}) \cdot P(\tau_3 = \tau^{(1)})
\end{aligned}
$$

Exponential races and forgetfulness. Exercise

$$P(\tau_2=\tau^{(2)} \mid \tau_1=\tau^{(1)}) = \int_0^{+\infty} P(\tau_2=\tau^{(2)} \mid \tau_1=\tau^{(1)}, \tau_1=t)\, f_{\tau_1}(t)\, dt$$

$$= \int_0^{+\infty} P(\tau_2=\tau^{(2)} \mid \tau_1=t, \underbrace{\tau_2>t, \tau_3>t}_{\tau_1=\tau^{(1)}})\, f_{\tau_1}(t)\, dt$$



$$= \int_0^{+\infty} P(\tau_2<\tau_3 \mid \cancel{\tau_1=t}, \tau_2>t, \tau_3>t)\, f_{\tau_1}(t)\, dt$$

$$= \int_0^{+\infty} P(\tau_2-t<\tau_3-t \mid \tau_2>t, \tau_3>t) \cdot f_{\tau_1}(t)\, dt$$

$$= \int_0^{+\infty} P(\tau_2<\tau_3) \cdot f_{\tau_1}(t)\, dt = P(\tau_2<\tau_3) \int_0^{\infty} f_{\tau_1}(t)\, dt$$

$$= P(\tau_2<\tau_3)$$

Exponential races and forgetfulness. Exercise

$$P(\tau_2 = \tau^{(2)}) = \boxed{P(\tau_2 = \tau^{(2)} \mid \tau_1 = \tau^{(1)})} \cdot P(\tau_1 = \tau^{(1)})$$
$$+ \boxed{P(\tau_2 = \tau^{(2)} \mid \tau_3 = \tau^{(1)})} \cdot P(\tau_3 = \tau^{(1)})$$

MEMORYLESS

$$= \boxed{P(\tau_2 < \tau_3)} \cdot P(\tau_1 = \tau^{(1)}) + \boxed{P(\tau_2 < \tau_1)} \cdot P(\tau_3 = \tau^{(1)})$$

$$= \frac{\lambda_2}{\lambda_2 + \lambda_3} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}$$

$$= \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \left(\frac{\lambda_1}{\lambda_2 + \lambda_3} + \frac{\lambda_3}{\lambda_1 + \lambda_2} \right)$$

Sum of exponential random variables

Let $\tau_1, \tau_2, \dots, \tau_n$ be independent **and identically distributed** exponential random variables, with common rate λ. Then, $T_n = \tau_1 + \tau_2 + \cdots + \tau_n$ has a gamma distribution with parameters n and λ, that is has density

$$f_{T_n}(t) = \begin{cases} \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Independent increments

Definition

A stochastic process $\{X_i : i \in I\}$ has *independent increments* if for any sequence $i_0 < i_1 < i_2 < \cdots < i_n$ we have that

$$(X_{i_1} - X_{i_0}), (X_{i_2} - X_{i_1}), \ldots, (X_{i_n} - X_{i_{n-1}})$$

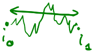
are independent.

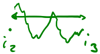
Alternative definition:

Definition

A stochastic process $\{X_i : i \in I\}$ has *independent increments* if for any $i_0 < i_1$ we have that $(X_{i_1} - X_{i_0})$ is independent of $\mathcal{F}_{i_0}$, where $\{\mathcal{F}_i : i \in I\}$ is the natural filtration.

Stationary increments





Definition

A stochastic process $\{X_i : i \in I\}$ has *stationary increments* if for any sequence $i_0 < i_1$ and $i_2 < i_3$ such that $i_1 - i_0 = i_3 - i_2$ we have that

$$(X_{i_1} - X_{i_0}) \quad \text{and} \quad (X_{i_3} - X_{i_2})$$

have the same distribution.

A review of the Poisson distribution

We say that $N \sim \text{Pois}(\lambda)$ if

$$P\{N = n\} = e^{-\lambda} \frac{(\lambda)^n}{n!}. \qquad \forall \quad n \in \{0, 1, 2, 3, \dots\}$$

Properties:

- $E[X] = \lambda$ and $\text{Var}(X) = \lambda$.
- Let $X_1, X_2, \dots, X_n$ be independent random variables, with $X_i \sim \text{Pois}(\lambda_i)$. Then,

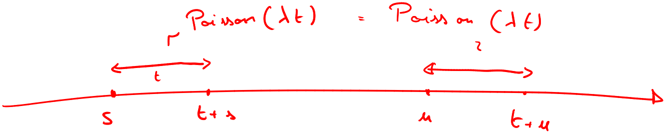
$$X_1 + X_2 + \cdots + X_n \sim \text{Pois}\left(\sum_{i=1}^{n} \lambda_i\right)$$

Theorem

$(N(s))_{s\in[0,\infty)}$ is a Poisson process with rate λ if and only if it is a counting process such that

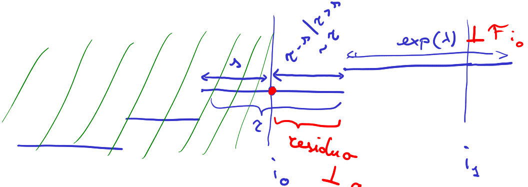
- (i) $N(0) = 0$,
- (ii) *it has independent increments;*
- (iii) $N(t+s) - N(s) \sim Poisson(\lambda t)$. ⇝ STATIONARY INCREMENTS

we split the proof into several parts. The first bullet is obvious



$N(t)$ has independent increments.

SKETCH



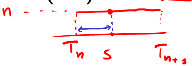
⊥ quanto già aspettato

MEMORYLESS

Poisson Process

$t_{n+1} > s - T_n$

If $(N(s))_{s\in[0,\infty)}$ is a Poisson process, $N(s) \sim Poisson(\lambda s)$. Indeed $N(s) = n$ if and only if $T_n \le s < T_{n+1}$. That is, for $n \ge 0$,



$N(s) - N(0)$

$$P\{N(s) = n\} = P\{T_n \le s, t_{n+1} > s - T_n\} = \int_0^s \int_{s-t}^{\infty} f_{T_n}(t) f_{t_{n+1}}(r)\, dr\, dt$$

GAMMA(n,λ)

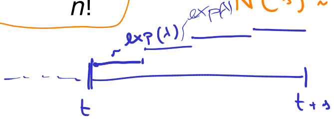
CONGIUNTA = PRODOTTO DENSITA'

$$= \int_0^s \int_{s-t}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \cdot \lambda e^{-\lambda r} dr\, dt$$

$$= \frac{\lambda^n}{(n-1)!} \int_0^s e^{-\lambda t} t^{n-1} \cdot e^{-\lambda(s-t)}\, dt = \frac{\lambda^n}{(n-1)!} \int_0^s t^{n-1} e^{-\lambda s}\, dt$$

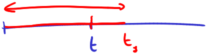
$$= \boxed{e^{-\lambda s} \frac{(\lambda s)^n}{n!}}.$$

$\Longrightarrow N(s) \sim \text{Poiss}(\lambda s)$



$\boxed{N(t+s) - N(t)} \sim N(s) - N(0)$

Reverse: GOAL : $t_i \sim exp(\lambda)$ INDEP.

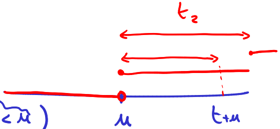


① $t_1 \sim exp(\lambda)$

$$P(t_1 > t) = P(N(t)=0)$$

$$= e^{-\lambda t} \cdot \frac{(\lambda t)^0}{0!} = e^{-\lambda t} \cdot \frac{1}{1} = e^{-\lambda t} \Rightarrow t_1 \sim exp(\lambda)$$

② $P(t_2 > t \mid t_1 = u)$



$$= P(N(t+u) - N(u) = 0 \mid \overbrace{N(u)=1,\ N(w)=0\ \forall\, w<u}^{\mathcal{F}_u})$$

$$= P(N(t+u) - N(u) = 0) = e^{-\lambda t} \cdot \frac{(\lambda t)^0}{0!} = e^{-\lambda t} \Rightarrow t_2 \sim exp(\lambda) \perp t_1$$

Poisson Process

Theorem

$(N(s))_{s\in[0,\infty)}$ is a Poisson process with rate λ if and only if it is a counting process such that:

1. $N(0) = 0$;
2. *it has independent increment;*
3. *it has stationary increments;*
4. $\displaystyle\lim_{h\to 0} \frac{P(N(h)=1)}{h} = \lambda$ *and* $\displaystyle\lim_{h\to 0} \frac{P(N(h)\geq 2)}{h} = 0$.