Homework 2

hand-in by Wednesday January 15th at 23:59

1. Consider the following continuous-time birth and death models, and for all of them decide if they are explosive or not. If they are not explosive, determine whether they are transient, null recurrent, or positive recurrent.

   (a) $S = \{0, 1, 2, \dots\}$ and $q(x, x+1) = 2^x$ for all $x \geq 0$, $q(x, x-1) = 2^{x+1}$ for all $x \geq 1$;

   (b) $S = \{0, 1, 2, \dots\}$ and $q(x, x+1) = 2^x$ for all $x \geq 0$, $q(x, x-1) = 2^x$ for all $x \geq 1$;

   (c) $S = \{0, 1, 2, \dots\}$ and $q(x, x+1) = 2^{2x+1}$ for all $x \geq 0$, $q(x, x-1) = 2^{2x}$ for all $x \geq 1$;

2. A computer lab has 3 laser printers and 5 toner cartridges. Each machine requires one toner cartridge which lasts for an exponentially distributed amount of time (independent of anything else) with mean 6 days. When a toner cartridge is empty it takes an exponential amount of time with mean 1 day to refill it. Up to 2 toner cartridges can be refilled at the same time, and we assume that cartridge replacements is instantaneous.

   (a) On the long run, how often are all three printers working?

   (b) Assuming that each printer prints 1000 pages per day (when working), how many pages per day will be approximately printed at the end of one year?

3. Consider a production system consisting of a machine center followed by an inspection station. Arrivals from outside the system occur only at the machine center and follow a Poisson process with rate λ per hour. The machine center and inspection station are each single-server operations with rates μ_1 per hour and μ_2 per hour. Suppose that each item independently passes inspection with probability p. When an object fails inspection it is sent to the machine center for reworking.

   (a) Find the conditions on the parameters that are necessary for the system to be positive recurrent, and under these conditions write a formula for the stationary distribution.

   (b) Assume the system is at stationary regime. What is the probability that less than 3 pieces pass inspection in two hours?

   (c) Assume the system is at stationary regime. Given that 10 pieces have left the machine center so far and currently 5 pieces are under inspection, what is the average number of pieces at the machine center?

4. In a urn there is a total of $n \geq 2$ balls, some are black and some are white. We play the following game: we draw two balls uniformly at random and

   - If the balls have the same color than we put them back in the urn;
   - If the balls have different colors then we select one of the two colors uniformly at random and put back in the urn two balls of the same color.

   (a) Will the balls have the same colour eventually? Prove it.

   (b) What is the probability that all balls will be white at the end, given that we start with 10 white balls and 20 black balls in the urn at the beginning?

   (c) Let σ be the first time that the number of white balls is a multiple of 3. As in the previous point, we start with 10 white balls and 20 black balls in the urn at the beginning. What is the probability that all balls will be white at the end, given that at time σ we will have 12 white balls?

5. (From personal experience) Let $(X_n)_{n=0}^{\infty}$ be a recurrent discrete-time Markov chain with state space S, with fixed initial condition $X_0 = x_0$, and such that for every state $x \in S$ we have $p(x, y) \neq 0$ only for finitely many states y. Let $f, g \colon S \to \mathbb{R}$ be such that

$$\mathbb{E}[f(X_{n+1}) - f(X_n)|X_n] = \mathbb{E}[g(X_{n+1}) - g(X_n)|X_n].$$

Show that, if $f(x) \geq g(x)$ for all $x \in S$, then necessarily $f(x) = g(x) + C$ for some non-negative constant $C > 0$.