Doob's optional sampling theorem in discrete time

Theorem (Doob's optional sampling theorem in discrete time)

Let $(X_n)_{n=0}^{\infty}$ be a martingale, submartingale, or supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$. Let τ and σ be two stopping times w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$, such that almost surely,

$$\sigma \leq \tau \leq C$$

for a constant $C \in \mathbb{R}$. Then, $\mathbb{E}[|X_\tau|], \mathbb{E}[|X_\sigma|] < \infty$ and

- *if $(X_n)_{n=0}^{\infty}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ then $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] = X_\sigma$ a.s.;*
- *if $(X_n)_{n=0}^{\infty}$ is a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ then $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] \geq X_\sigma$ a.s.;*
- *if $(X_n)_{n=0}^{\infty}$ is a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ then $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] \leq X_\sigma$ a.s..*

Doob's optional stopping theorem in discrete time

Theorem (Doob's optional stopping theorem in discrete time)

Let $(X_n)_{n=0}^{\infty}$ *be a martingale, submartingale, or supermartingale w.r.t.* $(\mathcal{F}_n)_{n=0}^{\infty}$*. Let* τ *and* σ *be two stopping times w.r.t.* $(\mathcal{F}_n)_{n=0}^{\infty}$*, such that*

- *almost surely,* $\sigma \leq \tau < \infty$*;*
- $\mathbb{E}[|X_\sigma|], \mathbb{E}[|X_\tau|] < \infty$*;*
- $\lim_{n\to\infty} \mathbb{E}[|X_n| \mathbb{1}_{\{\tau > n\}}] = 0$*.*

Then,

- *if* $(X_n)_{n=0}^{\infty}$ *is a martingale w.r.t.* $(\mathcal{F}_n)_{n=0}^{\infty}$ *then* $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] = X_\sigma$ *a.s.;*
- *if* $(X_n)_{n=0}^{\infty}$ *is a submartingale w.r.t.* $(\mathcal{F}_n)_{n=0}^{\infty}$ *then* $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] \geq X_\sigma$ *a.s.;*
- *if* $(X_n)_{n=0}^{\infty}$ *is a supermartingale w.r.t.* $(\mathcal{F}_n)_{n=0}^{\infty}$ *then* $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] \leq X_\sigma$ *a.s..*

We used it for

- Calculate the expectation of hitting probabilities and hitting time of a symmetric random walk: if $A, B \geq 0$, $\tau = \inf\{n : X_n \leq -A \text{ or } X_n \geq B\}$ and $x_0 \in [-A, B]$, then

$$P_{x_0}(X_\tau = -A) = \frac{B - x_0}{A + B} \quad \text{and} \quad P_{x_0}(X_\tau = B) = \frac{A + x_0}{A + B}$$

  and

$$\mathbb{E}_{x_0}[\tau] = \frac{A^2(B - x_0) + B^2(A + x_0)}{A + B} - x_0^2 = (A + x_0)(B - x_0).$$

- Finding out that the hitting probabilities for a realistic doubling strategy are the same as for the symmetric random walk.
- Calculate the expectation of hitting probabilities of an asymmetric random walk ($p \neq 1/2$):

$$P_{x_0}(X_\tau = -A) = \frac{(q/p)^{x_0} - (q/p)^B}{(q/p)^{-A} - (q/p)^B}, \quad P_{x_0}(X_\tau = B) = \frac{(q/p)^{-A} - (q/p)^{x_0}}{(q/p)^{-A} - (q/p)^B}.$$

Stopped asymmetric random walk

What about the expected hitting time $\mathbb{E}[\tau]$? We consider the martingale

$$M_n = (X_n - n(p-q))_{n=0}^{\infty}.$$

The assumptions of Doob's optional stopping theorem are satisfied (check it!), so

$$\begin{aligned} x_0 &= \mathbb{E}_{x_0}[M_0] = \mathbb{E}_{x_0}[M_\tau] = \mathbb{E}_{x_0}[X_\tau - \tau(p-q)] \\ &= -A\frac{(q/p)^{x_0} - (q/p)^B}{(q/p)^{-A} - (q/p)^B} + B\frac{(q/p)^{-A} - (q/p)^{x_0}}{(q/p)^{-A} - (q/p)^B} - \mathbb{E}[\tau](p-q). \end{aligned}$$

Solving for $\mathbb{E}[\tau]$ yields

$$\mathbb{E}[\tau] = \frac{1}{p-q}\left[\frac{A(q/p)^B + B(q/p)^{-A} - (A+B)(q/p)^{x_0}}{(q/p)^{-A} - (q/p)^B} - x_0\right].$$

Easier proofs of recurrence and transience

Consider a discrete time, partially reflected random walk:

$$p(x, x+1) = p \quad \text{for } x \geq 0, \quad p(x, x-1) = q \quad \text{for } x \geq 1, \quad \text{and } p(0,0) = q.$$

$$\begin{tikzcd}
0 \arrow[loop left, "q"] \arrow[r, bend left, "p"] & 1 \arrow[l, bend left, "q"] \arrow[r, bend left, "p"] & 2 \arrow[l, bend left, "q"] \arrow[r, bend left, "p"] & 3 \arrow[l, bend left, "q"] \arrow[r, bend left, "p"] & \cdots \arrow[l, bend left, "q"]
\end{tikzcd}$$

The model is irreducible, so to study the recurrence it suffices to study $P_0(T_0 < \infty)$. By conditioning on the first step and by the continuity of measure,

$$\begin{aligned}
P_0(T_0 < \infty) = q + pP_1(T_0 < \infty) &= q + pP_1\left(\bigcup_{n=2}^{\infty}\{T_0 < T_n\}\right) \\
&= q + p \lim_{n\to\infty} P_1(T_0 < T_n).
\end{aligned}$$

So, we have recurrence if and only if $\lim_{n\to\infty} P_1(T_0 < T_n) = 1$.

Easier proofs of recurrence and transience

- If $p = q = 1/2$ then

$$\lim_{n\to\infty} P_1(T_0 < T_n) = \lim_{n\to\infty} \frac{n-1}{0+n} = 1.$$

The model is recurrent!

- If $p < q$ then $q/p > 1$ and

$$\lim_{n\to\infty} P_1(T_0 < T_n) = \lim_{n\to\infty} \frac{q/p - (q/p)^n}{1-(q/p)^n} = 1.$$

The model is recurrent!

- If $p > q$ then $q/p < 1$ and

$$\lim_{n\to\infty} P_1(T_0 < T_n) = \lim_{n\to\infty} \frac{q/p - (q/p)^n}{1-(q/p)^n} = \frac{q}{p} < 1.$$

The model is transient! Moreover, we can explicitly calculate

$$P_0(T_0 < \infty) = q + p \lim_{n\to\infty} P_1(T_0 < T_n) = 2q.$$

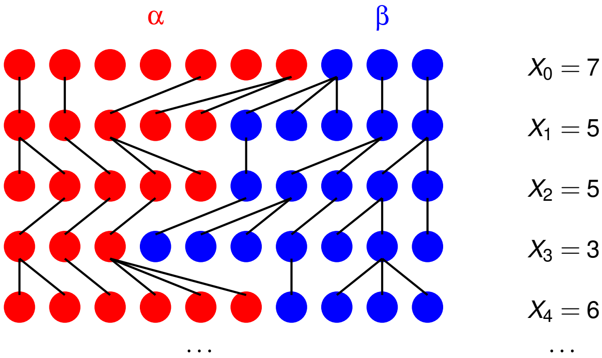
The Wright-Fisher model for genetic drift

In a population two genetic treats α and β are present. The genetic difference is neutral, i.e. it does not increase the survival probability or the number of children of the individual possessing it. It is assumed that the population has fixed size N and, consistently with the neutrality of the genetic treats, the types of the offspring are a sequence of independent uniform samples (with replacement) from the treats available in the previous generation.

If the first generation had a individuals of type α and $N - a$ individuals of type β, what is the fixation probability of treat α, i.e. , what is the probability that eventually all the individuals in the population will have the treat α?

The Wright-Fisher model for genetic drift

Let X_n be the number of individuals with treat α in the nth generation



It's a Markov chain with $S = \{0, 1, 2, \ldots, N\}$ and $X_{n+1} | X_n = i \sim \mathrm{Bin}(N, i/N)$, so

$$p(i,j) = \binom{N}{j} \left(\frac{i}{N}\right)^j \left(1 - \frac{i}{N}\right)^{N-j}$$

The Wright-Fisher model for genetic drift

$(X_n)_{n=0}^{\infty}$ is a martingale:

$$X_{n+1}|X_n \sim \mathrm{Bin}\left(N, \frac{X_n}{N}\right).$$

Hence, $\mathbb{E}[|X_n|] < N$ for all $n \geq 0$ and

$$\mathbb{E}[X_{n+1}|X_n] = N \cdot \frac{X_n}{N} = X_n.$$

By Doob's optional stopping theorem, if $\tau = \inf\{n \in \mathbb{N} : X_n = 0 \text{ or } X_n = N\}$ and $X_0 = x$,

$$x = \mathbb{E}[X_\tau] = 0 \cdot P_x(X_\tau = 0) + NP_x(X_\tau = N).$$

Hence, the fixation probability of α is

$$P_x(X_\tau = N) = \frac{x}{N}$$

where x is the initial counts of individuals with genetic treat α.

More wrong ideas about making money with unfavourable games

So far we have seen two wrong ideas to make easy money

If we have an unfavourable game (that is, a supermartingale), we cannot transform it into a favourable game by

- applying predictable strategies to decide how much to bet (and of course we can't look into the future!);
- stopping the game at favourable stopping times, especially if we have a lower bound on how much we can invest (stopped processes mantain the nature of the original process).

Another idea to make easy money

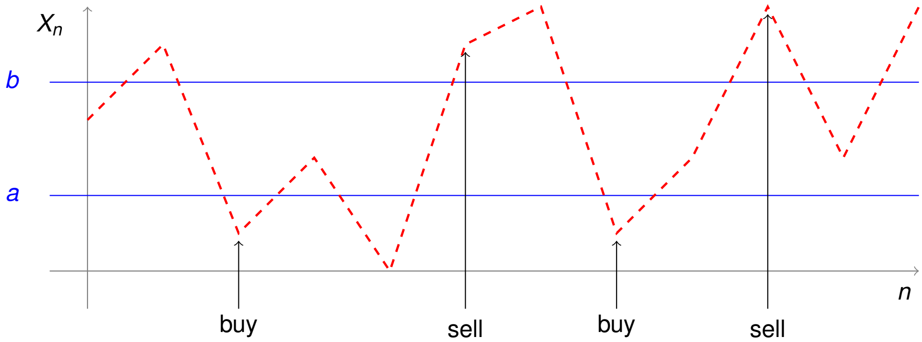
Imagine the price of a financial stock day by day follows a supermartingale. We still want to make money out of it, so we come up with the following brilliant plan: we fix two numbers $a < b$ and

- buy the stock **as soon as** its price is less than or equal to a;
- sell the stock **as soon as** its price is greater then or equal to b;
- repeat!

This will make us infinite money. What can go wrong?

Upcrossing

Let $(X_n)_{n=0}^{\infty}$ be a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ and fix $a < b$.



In the picture above we see two upcrossings of the interval $[a, b]$, i.e. two sequences of states starting below a and finishing above b.

Upcrossing

We are interested in quantifying the number of upcrossings in our supermartingale. We define the following quantities:

$$\sigma_1^{a,b} = \inf\{n \in \mathbb{N} : X_n \leq a\}$$

and recursively, for $i \geq 1$

$$\tau_i^{a,b} = \inf\{n \geq \sigma_i : X_n \geq b\} \quad \text{and} \quad \sigma_{i+1}^{a,b} = \inf\{n \geq \tau_i : X_n \leq a\}.$$

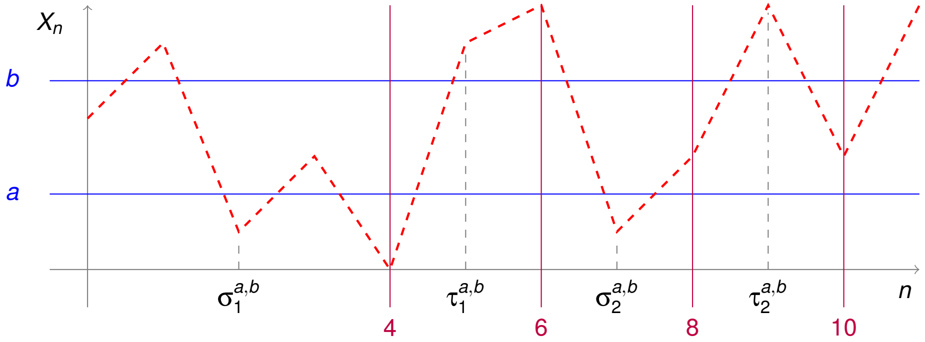
Moreover, let

$$U^{a,b}(N) = \sup\{k \in \mathbb{N} : \tau_k \leq N\}.$$

In words, $\sigma_i^{a,b}$ and $\tau_i^{a,b}$ mark the beginning and the end of the ith upcrossing of $[a,b]$, and $U^{a,b}(N)$ is the number of upcrossings of $[a,b]$ up to time N.

Upcrossing

Let $(X_n)_{n=0}^{\infty}$ be a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ and fix $a < b$.



We have

$$U^{a,b}(4) = 0, \quad U^{a,b}(6) = 1, \quad U^{a,b}(8) = 1, \quad U^{a,b}(10) = 2.$$

Upcrossing

The total number of upcrossing of $[a, b]$ is

$$U^{a,b} = \sup\{k \in \mathbb{N} : \tau_k < \infty\}.$$

Since $U^{a,b}(N)$ is monotone in N, we have

$$\lim_{N\to\infty} U^{a,b}(N) = \sup_N U^{a,b}(N) = U^{a,b}.$$

Upcrossing

Theorem (Doob's upcrossing inequality in discrete time)

Let $(X_n)_{n=0}^{\infty}$ be a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ and fix $a < b$. Then,

$$(b-a)\mathbb{E}[U^{a,b}(N)] \leq \mathbb{E}[(X_N - a)^-] \leq \mathbb{E}[|X_N - a|]$$

where as usual $k^- = |\min\{k, 0\}|$ is the negative part of k and as such $0 < k^- < |k|$.

Corollary

Let $(X_n)_{n=0}^{\infty}$ be a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ and fix $a < b$. If

$$\sup_n \mathbb{E}[X_n^-] < \infty,$$

then $\mathbb{E}[U^{a,b}] < \infty$ and as a consequence $U^{a,b}$ is almost surely finite.

Again, no easy money!

Proof of the corollary.

It is a simple application of the monotone convergence theorem: since $U^{a,b}(N)$ is monotone in N and $\lim_{N\to\infty} U^{a,b}(N) = U^{a,b}$ almost surely, then

$$\mathbb{E}[U^{a,b}] = \lim_{N\to\infty} \mathbb{E}[U^{a,b}(N)] \leq \limsup_{N\to\infty} \frac{\mathbb{E}[(X_N - a)^-]}{b-a}$$
$$\leq \frac{a^+}{b-a} + \limsup_{N\to\infty} \frac{\mathbb{E}[X_N^-]}{b-a} < \infty,$$

where we used $(k_1 - k_2)^- \leq k_1^- + k_2^+$. $\square$

Upcrossing

Proof of Doob's upcrossing inequality in disccrete time .

Consider the process $(Y_n)_{n=0}^{\infty}$ defined by

$$Y_n = X_0 + \underbrace{\sum_{i=1}^{U^{a,b}(n)} (X_{\tau_i} - X_{\sigma_i})}_{\text{increment in upcrossings}} + \underbrace{(X_n - X_{\sigma_{U^{a,b}(n)+1}})\mathbb{1}_{\{\sigma_{U^{a,b}(n)+1} \leq n\}}}_{\text{increment in the last potentially uncompleted upcrossing}} .$$

Clearly,

$$Y_N \geq X_0 + (b-a)U^{a,b}(N) - (X_n - a)^- .$$

Hence,

$$\mathbb{E}[Y_N] \geq \mathbb{E}[X_0] + (b-a)\mathbb{E}[U^{a,b}(N)] - \mathbb{E}[(X_n - a)^-].$$

To prove the theorem , we just need to show $\mathbb{E}[Y_N] \leq \mathbb{E}[X_0]$.

$\square$

Upcrossing

Proof of Doob's upcrossing inequality in disccrete time .

To show $\mathbb{E}[Y_N] \le \mathbb{E}[X_0]$ we use a nice trick: by using telescoping sums we have

$$
\begin{aligned}
Y_n &= X_0 + \sum_{i=1}^{U^{a,b}(n)} (X_{\tau_i} - X_{\sigma_i}) + (X_n - X_{\sigma_{U^{a,b}(n)+1}}) \mathbb{1}_{\{\sigma_{U^{a,b}(n)+1} \le n\}} \\
&= X_0 + \sum_{m=1}^{n} C_m (X_m - X_{m-1})
\end{aligned}
$$

where

$$
C_m = \begin{cases} 1 & \text{if } \exists i : \sigma_i \le m-1 < \tau_i \\ 0 & \text{otherwise} \end{cases}
$$

$(C_m)_{m=1}^{\infty}$ is predictable w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$! Hence, Y_n is a supermartingale and

$$
\mathbb{E}[Y_N] \le \mathbb{E}[Y_0] = \mathbb{E}[X_0].
$$

□

Convergence theorems in discrete time

First Doob's Convergence Theorem in discrete time

Theorem (First Doob's Convergence Theorem in discrete time)

Let $(X_n)_{n=0}^{\infty}$ be a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ (a filtration on $(\Omega, \mathcal{F}, P)$) and assume that $\sup_n \mathbb{E}[X_n^-] < \infty$. Then there exists a $\mathcal{F}$-measurable random variable X with $\mathbb{E}[|X|] < \infty$ and

$$\lim_{n\to\infty} X_n = X \quad \textit{almost surely.}$$

First Doob's Convergence Theorem in discrete time

Proof.

The non-existence of $\lim_{n\to\infty} X_n$ corresponds to

$$\liminf_{n\to\infty} X_n \neq \limsup_{n\to\infty} X_n,$$

which implies the existence of two numbers a, b with

$$\liminf_{n\to\infty} X_n < a < b < \limsup_{n\to\infty} X_n$$

However, if this were the case then the sequence $(X_n)_{n=0}^{\infty}$ would be infinitely many times lower than a, and infinitely many times higher than b, thus the number of upcrossing of $[a, b]$ would be infinite. This is a contradiction with the corollary to Doob's upcossing inequality.

$\square$

First Doob's Convergence Theorem in discrete time

Proof.

We are left to show that the random variable defined by $X = \lim_{n\to\infty} X_n$ is such that $\mathbb{E}[|X|] < \infty$. First, we prove that $\sup_n \mathbb{E}[|X_n|] < \infty$:

- since $(X_n)_{n=0}^{\infty}$ is a supermartingale, $\mathbb{E}[X_n] = \mathbb{E}[X_n^+] - \mathbb{E}[X_n^-] \leq \mathbb{E}[X_0]$, hence

$$\sup_n \mathbb{E}[X_n^+] \leq \mathbb{E}[X_0] + \sup_n \mathbb{E}[X_n^-] < \infty.$$

- we have

$$\sup_n \mathbb{E}[|X_n|] = \sup_n (\mathbb{E}[X_n^+] + \mathbb{E}[X_n^-]) \leq \sup_n \mathbb{E}[X_n^+] + \sup_n \mathbb{E}[X_n^-] < \infty.$$

Now note that $|X| = \lim_{n\to\infty} |X_n|$ and by Fatou's lemma

$$\mathbb{E}[|X|] \leq \liminf_{n\to\infty} \mathbb{E}[|X_n|] < \infty.$$

$\square$

First Doob's Convergence Theorem in discrete time

Theorem (First Doob's Convergence Theorem in discrete time)

Let $(X_n)_{n=0}^{\infty}$ *be a submartingale w.r.t.* $(\mathcal{F}_n)_{n=0}^{\infty}$ *(a filtration on* $(\Omega, \mathcal{F}, P)$*) and assume that* $\sup_n \mathbb{E}[X_n^+] < \infty$*. Then there exists a* $\mathcal{F}$*-measurable random variable* X *with* $\mathbb{E}[|X|] < \infty$ *and*

$$\lim_{n\to\infty} X_n = X \quad \textit{almost surely.}$$

Proof.

if $(X_n)_{n=0}^{\infty}$ is a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$, then $(-X_n)_{n=0}^{\infty}$ is a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$, and we can apply the first Doob's Convergence Theorem for supermartingales. □

First Doob's Convergence Theorem in discrete time

Corollary

Let $(X_n)_{n=0}^{\infty}$ be a closed submartingale, martingale, or supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ (a filtration on $(\Omega, \mathcal{F}, P)$). Then there exists a $\mathcal{F}$-measurable random variable X with $\mathbb{E}[|X|] < \infty$ and

$$\lim_{n\to\infty} X_n = X \quad \textit{almost surely.}$$

Proof.

If $(X_n)_{n=0}^{\infty}$ is a closed submartingale, then there exists a random variable Y that is measurable with respect to $\mathcal{F}$ such that $\mathbb{E}[|Y|] < \infty$ and almost surely $X_n \leq \mathbb{E}[Y|\mathcal{F}_n]$. Then,

$$\sup_n \mathbb{E}[X_n^+] \leq \sup_n \mathbb{E}[|X_n|] \leq \sup_n \mathbb{E}[|\mathbb{E}[Y|\mathcal{F}_n]|] \leq \sup_n \mathbb{E}[\mathbb{E}[|Y|\,|\mathcal{F}_n]] = \mathbb{E}[|Y|] < \infty$$

and we conclude by Doob's Convergence Theorem. □

First Doob's Convergence Theorem in discrete time

Corollary

Let $(X_n)_{n=0}^{\infty}$ be a closed submartingale, martingale, or supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ (a filtration on $(\Omega, \mathcal{F}, P)$). Then there exists a $\mathcal{F}$-measurable random variable X with $\mathbb{E}[|X|] < \infty$ and

$$\lim_{n\to\infty} X_n = X \quad \textit{almost surely.}$$

Proof.

If $(X_n)_{n=0}^{\infty}$ is a closed supermartingale, then $(-X_n)_{n=0}^{\infty}$ is a closed submartingale and the result holds.
If $(X_n)_{n=0}^{\infty}$ is a closed martingale, then it is both a closed submartingale and a closed supermartingale, and the result holds.

$\square$

Different convergence types

Remember the following types of convergence of random variables defined on a same probability space $(\Omega, \mathcal{F}, P)$:

almost sure convergence : when $\lim_{n\to\infty} X_n = X$ almost surely;

$\mathcal{L}_1$ convergence : when $\lim_{n\to\infty} \mathbb{E}[|X_n - X|] = 0$;

convergence in probability : when for all $\varepsilon > 0$, $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$.

Remember:

$$X_n \xrightarrow[n\to\infty]{\mathcal{L}_1} X \quad \Longrightarrow \quad X_n \xrightarrow[n\to\infty]{P} X$$
$$X_n \xrightarrow[n\to\infty]{\text{a.s.}} X \quad \Longrightarrow \quad X_n \xrightarrow[n\to\infty]{P} X$$

The converse statements do not hold. Also, almost sure convergence and $\mathcal{L}_1$ convergence do not imply each other. Finally, limits in probability are unique (if they exist). Hence, if we have both almost sure convergence and $\mathcal{L}_1$ convergence, they need to be to the same limit X.

Second Doob's Convergence Theorem in discrete time

The second Doob's Convergence theorem is concerned with $\mathcal{L}_1$ convergence:

$$\lim_{n\to\infty} \mathbb{E}[|X - X_n|] = 0.$$

This is not implied by the first Doob's Convergence Theorem. We see an example.

Example of failure of $\mathcal{L}_1$ convergence

Let $(X_n)_{n=0}^{\infty}$ be the random walk with $p = 1/2$ and $X_0 = 5$. Let

$$T = \inf\{n \geq 0 \, : \, X_n = 0\}.$$

Then, $(X_{n\wedge T})_{n=0}^{\infty}$ is a non-negative martingale (hence, a supermartingale). Specifically, $\sup_n \mathbb{E}[X_{n\wedge T}^-] = 0 < \infty$. Then, there exists $X = \lim_{n\to\infty} X_n$ which in this case is $X = 0$ because T is almost surely finite. However, by using $X = 0$ and the martingale property,

$$\mathbb{E}[|X_{n\wedge T} - X|] = \mathbb{E}[X_{n\wedge T}] = \mathbb{E}[X_0] = 5 \nrightarrow 0.$$

Uniform integrability and $\mathcal{L}_1$ convergence

Definition

Let $(X_n)_{n=0}^{\infty}$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$. We say that they are uniformly integrable if for every $\varepsilon > 0$ there exists $K > 0$ such that

$$\mathbb{E}[|X_n| \mathbb{1}_{\{|X_n| > K\}}] < \varepsilon \quad \text{for all } n.$$

Theorem

- *If $(X_n)_{n=0}^{\infty}$ is uniformly integrable, then $\sup_n \mathbb{E}[|X_n|] < \infty$. The converse is not necessarily true.*
- *If there exists $p > 1$ with $\sup_n \mathbb{E}[|X_n|^p] < \infty$, then $(X_n)_{n=0}^{\infty}$ is uniformly integrable. The converse is not necessarily true.*

We will not prove it.

Uniform integrability and $\mathcal{L}_1$ convergence

Theorem

Let $(X_n)_{n=0}^{\infty}$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}[|X_n|] < \infty$ for all n, and let X be a random variable on $(\Omega, \mathcal{F}, P)$ such that $(X_n)_{n=0}^{\infty}$ converges in probability to X. Then, the following statements are equivalent:

1. $(X_n)_{n=0}^{\infty}$ *converges in* $\mathcal{L}_1$ *to* X*;*
2. $(X_n)_{n=0}^{\infty}$ *is uniformly integrable;*
3. $\lim_{n\to\infty} \mathbb{E}[|X_n|] = \mathbb{E}[|X|] < \infty$.

We will not prove it.

As a consequence,

$$\begin{array}{ccc} & & X_n \xrightarrow[n\to\infty]{a.s.} X + \text{unif. integrability} \\ & & \Downarrow \\ X_n \xrightarrow[n\to\infty]{\mathcal{L}_1} X & \iff & X_n \xrightarrow[n\to\infty]{P} X + \text{unif. integrability} \end{array}$$

Second Doob's Convergence Theorem in discrete time

Theorem (Second Doob's Convergence Theorem in discrete time)

Let $(X_n)_{n=0}^\infty$ be a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$ (a filtration on $(\Omega, \mathcal{F}, P)$). The following properties are equivalent:

1. *there exists a $\mathcal{F}$-measurable random variable X with $X_n \xrightarrow[n\to\infty]{\mathcal{L}_1} X$;*
2. *there exists a $\mathcal{F}$-measurable random variable X such that $\mathbb{E}[|X|] < \infty$, $\lim_{n\to\infty} X_n = X$ almost surely,*
$$X_n \leq \mathbb{E}[X|\mathcal{F}_n] \quad \text{almost surely for all } n,$$
and $\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X]$;
3. *$(X_n)_{n=0}^\infty$ is uniformly integrable.*

Second Doob's Convergence Theorem in discrete time

Proof (skipped in class).

We first prove (1) $\implies$ (2).

By triangular inequality

$$\mathbb{E}[|X|] - \lim_{n\to\infty} \mathbb{E}[|X_n - X|] \leq \lim_{n\to\infty} \mathbb{E}[|X_n|] \leq \mathbb{E}[|X|] + \lim_{n\to\infty} \mathbb{E}[|X_n - X|]$$

hence $\lim_{n\to\infty} \mathbb{E}[|X_n|] = \mathbb{E}[|X|]$ and $\sup_n \mathbb{E}[X_n^+] \leq \sup_n \mathbb{E}[|X_n|] < \infty$. By the first Doob's Convergence Theorem we have almost sure convergence to X. For all $m > n$ we have

$$\mathbb{E}[X_m | \mathcal{F}_n] \geq X_n$$

and by dominated convergence theorem we let $m \to \infty$ and obtain

$$\mathbb{E}[X | \mathcal{F}_n] \geq X_n.$$

$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X]$ is implied by $\mathcal{L}_1$ convergence. $\square$

Proof (skipped in class).

We now prove (2) $\implies$ (3).

We have that (X_n^+) is a submartingale by Jensen's inequality (taking the positive part is a convex function). In particular, $\mathbb{E}[X^+|\mathcal{F}_n] \geq X_n^+$ which implies

$$\mathbb{E}[X^+ \mathbb{1}_{\{X_n^+ > K\}}] = \int_{\{X_n^+ > K\}} \mathbb{E}[X^+|\mathcal{F}_n](\omega) dP(\omega) \geq \mathbb{E}[X_n^+ \mathbb{1}_{\{X_n^+ > K\}}]$$

hence $(X_n^+)_{n=0}^\infty$ is uniformly integrable. This, together with $\lim_{n\to\infty} X_n^+ = X^+$ almost surely, imply that $\lim_{n\to\infty} \mathbb{E}[X_n^+] = \mathbb{E}[X^+]$. Since by hypothesis $\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X]$, it follows that $\lim_{n\to\infty} \mathbb{E}[X_n^-] = \mathbb{E}[X^-]$. This, togehter with $\lim_{n\to\infty} X_n^- = X^-$ almost surely, imply that $(X_n^-)_{n=0}^\infty$ is uniformly integrable. Since both $(X_n^+)_{n=0}^\infty$ and $(X_n^-)_{n=0}^\infty$ are uniformly integrable, so is $(X_n)_{n=0}^\infty$.

$\square$

Second Doob's Convergence Theorem in discrete time

Proof (skipped in class).

We finally prove (3) $\implies$ (1).

By uniform integrability, $\sup_n \mathbb{E}[X_n^+] \leq \sup_n \mathbb{E}[|X_n|] < \infty$. By the first Doob's Convergence Theorem we have almost sure convergence to X. Almost sure convergence and uniform integrability imply convergence in $\mathcal{L}_1$.

$\square$

Second Doob's Convergence Theorem in discrete time

Theorem (Second Doob's Convergence Theorem in discrete time)

Let $(X_n)_{n=0}^{\infty}$ be a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ (a filtration on $(\Omega, \mathcal{F}, P)$). The following properties are equivalent:

1. *there exists a $\mathcal{F}$-measurable random variable X with $X_n \xrightarrow[n\to\infty]{\mathcal{L}_1} X$;*
2. *there exists a $\mathcal{F}$-measurable random variable X such that $\mathbb{E}[|X|] < \infty$, $\lim_{n\to\infty} X_n = X$ almost surely,*
$$X_n \geq \mathbb{E}[X|\mathcal{F}_n] \quad \textit{almost surely for all } n,$$
and $\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X]$;
3. *$(X_n)_{n=0}^{\infty}$ is uniformly integrable.*

Proof.

If $(X_n)_{n=0}^{\infty}$ is a supermartingale, then $(-X_n)_{n=0}^{\infty}$ is a submartingale. □

Second Doob's Convergence Theorem in discrete time

Theorem (Second Doob's Convergence Theorem in discrete time)

Let $(X_n)_{n=0}^{\infty}$ be a martingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ (a filtration on $(\Omega, \mathcal{F}, P)$). The following properties are equivalent:

1. *there exists a $\mathcal{F}$-measurable random variable X with $X_n \xrightarrow[n\to\infty]{\mathcal{L}_1} X$;*
2. *there exists a $\mathcal{F}$-measurable random variable X such that $\mathbb{E}[|X|] < \infty$, $\lim_{n\to\infty} X_n = X$ almost surely, and*
$$X_n = \mathbb{E}[X|\mathcal{F}_n] \quad \textit{almost surely for all } n.$$
3. *$(X_n)_{n=0}^{\infty}$ is uniformly integrable.*

Proof.

If $(X_n)_{n=0}^{\infty}$ is a martingale, then it is both a supermartingale and a submartingale. □

Second Doob's Convergence Theorem in discrete time

Things worth noticing:

If $(X_n)_{n=0}^{\infty}$ is a submartingale, a supermartingale, or a martingale with respect to some filtration, then

$$X_n \xrightarrow[n\to\infty]{\mathcal{L}_1} X \Longrightarrow X_n \xrightarrow[n\to\infty]{a.s.} X$$

If $(X_n)_{n=0}^{\infty}$ is a submartingale, a supermartingale, or a martingale with respect to some filtration, and

$$X_n \xrightarrow[n\to\infty]{\mathcal{L}_1} X,$$

then $(X_n)_{n=0}^{\infty}$ is closed.

Feel the Martingale power! Made to prove stuff since the 1940s.

Theorem (More general strong law of large numbers)

Let $(X_n)_{n=1}^{\infty}$ *be a sequence of independent random variables, with null mean* $\mathbb{E}[X_n] = 0$ *and finite second moments* $\mathbb{E}[X_n^2] < \infty$*. Assume there exists a sequence* $(a_n)_{n=1}^{\infty}$ *of real numbers with* $0 < a_n \uparrow \infty$ *and such that the limit*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\mathbb{E}[X_i^2]}{a_i^2} = \sum_{i=0}^{\infty} \frac{\mathbb{E}[X_i^2]}{a_i^2}$$

exists and is finite. Then,

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} X_i = 0 \quad \textit{almost surely.}$$

and

$$\lim_{n\to\infty} \frac{1}{a_n} \mathbb{E}\left[\left|\sum_{i=1}^{n} X_i\right|\right] = 0.$$

Strong law of large numbers

Corollary

Let $(X_n)_{n=1}^{\infty}$ *be a sequence of independent random variables, with the same mean* $\mathbb{E}[X_n] = \mu$ *and bounded variance* $\sup_n Var(X_n) < \infty$. *Then,*

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} X_i = \mu \quad \textit{almost surely.}$$

and

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right|\right] = 0.$$

Strong law of large numbers

Proof.

We simply apply the previous result to the centered random variables $Y_n = X_n - \mu$, with the choice $a_n = n$, and note that

$$\sum_{i=1}^{n} \frac{\mathbb{E}[Y_i^2]}{i^2} = \sum_{i=1}^{\infty} \frac{\mathrm{Var}(Y_i)}{i^2}$$

is a Cauchy sequence (hence converging). Then,

$$\frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\sum_{i=1}^{n} X_i - \mu$$

converges to 0 both almost surely and in $\mathcal{L}_1$. □

Strong law of large numbers

For the proof of the stronger version of the strong law of large numbers, we need the following result, which we will not prove.

Lemma (Kronecker)

Let $(x_n)_{n=1}^{\infty}$ *and* $(a_n)_{n=1}^{\infty}$ *be two sequences of real numbers, with* $0 < a_n \uparrow \infty$ *and such that the limit*

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{x_i}{a_i}=\sum_{i=1}^{\infty}\frac{x_i}{a_i}$$

exists and is finite. Then,

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{i=1}^{n}x_i=0.$$

Strong law of large numbers

Proof of the more general result.

Simply note that $(M_n)_{n=0}^{\infty}$ defined by $M_0 = 0$ and $M_n = \sum_{i=1}^{n}(X_n/a_n)$ is a martingale:

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n + \mathbb{E}\left[\frac{X_{n+1}}{a_{n+1}}\right] = M_n.$$

By Pitagora's theorem for martingales,

$$\sup_n \mathbb{E}[M_n^2] = \sup_n \sum_{i=1}^{n} \mathbb{E}\left[\frac{X_i^2}{a_i^2}\right] = \sum_{i=1}^{\infty} \frac{\mathbb{E}[X_i^2]}{a_i^2} < \infty.$$

As mentioned some slides ago, if $\sup_n \mathbb{E}[|M_n|^p] < \infty$ for some $p > 1$, then $(M_n)_{n=0}^{\infty}$ is uniformly integrable and the second Doob's convergence theorem applies. This means that $(M_n)_{n=0}^{\infty}$ converges almost surely to a random variable X with finite mean.

$\square$

Strong law of large numbers

Proof of the more general result.

Specifically, almost surely

$$\lim_{n\to\infty} M_n(\omega) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{X_n(\omega)}{a_n} = X(\omega) < \infty,$$

and we conclude by Kronecker's lemma that almost surely

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} X_i = 0.$$

$\square$

Proof of the more general result.

To prove $\mathcal{L}_1$ convergence, note that

$$S_n = \sum_{i=1}^{n} X_i$$

is a martingale, hence by Pitagora's theorem for martingales,

$$\sup_n \mathbb{E}\left[\left(\frac{1}{a_n}\sum_{i=1}^{n} X_i\right)^2\right] = \sup_n \frac{1}{a_n^2}\sum_{i=1}^{n}\mathbb{E}[X_i^2] \leq \sup_n \sum_{i=1}^{n}\frac{\mathbb{E}[X_i^2]}{a_i^2} < \infty.$$

As before, uniform integrability of S_n/a_n follows. Since the sequence convergences almost surely (hence in probability), it converges in $\mathcal{L}_1$ as well.

$\square$

Kolmogorov's 0-1 law

The Kolmogorov 0-1 law is an extremely useful tool to see that certain events necessarily have probability zero or one.

Theorem (Kolmogorov's 0-1 law)

Let $(X_n)_{n=1}^{\infty}$ be a sequence of i.i.d. random variables. Define

$$\mathcal{T}_n = \sigma(X_{n+1}, X_{n+2}, \dots), \quad \textit{and} \quad \mathcal{T} = \bigcap_{n=1}^{\infty} \mathcal{T}_n.$$

$\mathcal{T}$ is called the tail σ-algebra. Every event $A \in \mathcal{T}$ $P(A) = 0$ or $P(A) = 1$.

Examples of tail events are

- $\{X_n \in A_n \text{infinitely many times}\}$;
- $\limsup_n X_n \geq 8$;
- $\lim_{n\to\infty} 1/n \sum_{i=1}^{n} X_i = \mu$.

Proof.

Let, $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ and $\mathcal{F} = \sigma(X_1, X_2, \dots)$. Let $A \in \mathcal{T} \subset \mathcal{F}$. Since for all n we have $A \in \mathcal{T}_n$ and $\mathcal{F}_n$ and $\mathcal{T}_n$ are independent, we get

$$\mathbb{E}[\mathbb{1}_A | \mathcal{F}_n] = \mathbb{E}[\mathbb{1}_A] = P(A).$$

Now, $(\mathbb{E}[\mathbb{1}_A | \mathcal{F}_n])_{n=1}^{\infty}$ defines a closed martingale, clearly uniformly integrable. By the second Doob's convergence theorem we have the existence of an almost sure $\mathcal{F}$-measurable limit X with

$$\mathbb{E}[\mathbb{1}_A | \mathcal{F}_n] = \mathbb{E}[X | \mathcal{F}_n] \quad \text{for all } n.$$

This implies that $\mathbb{E}[\mathbb{1}_A \mathbb{1}_C] = \mathbb{E}[X \mathbb{1}_C]$ for all $C \in \mathcal{F}$ (some details missing), and necessarily $X = \mathbb{1}_A$ almost surely. Then, almost surely

$$P(A) = \lim_{n \to \infty} \mathbb{E}[\mathbb{1}_A | \mathcal{F}_n] = \mathbb{1}_A \quad \text{and} \quad P(A) = 0 \text{ or } P(A) = 1.$$

□

A recurrence criterion

Theorem (Foster-Lyapunov recurrence criterion)

Let $(X_n)_{n=0}^{\infty}$ be an irreducible DTMC. Assume there exists a non-negative function V such that

- *for all $0 < K < \infty$ the sub-level set*

$$B_K = \{x \in S : V(x) \leq K\}$$

  is finite;
- $\mathbb{E}[V(X_{n+1})|X_n = x] \leq V(x)$ *for all but finitely many states x.*

Then, $(X_n)_{n=0}^{\infty}$ is recurrent.

A recurrence criterion

Proof.

It is enough to show that a finite set is recurrent: if a finite set is visited infinitely often, then at least one of its states is visited infinitely often and is therefore recurrent, then all states are recurrent by irreducibility.
Let $\Gamma = \{x : \mathbb{E}[V(X_{n+1})|X_n = x] > V(x)\}$, which is finite by assumption. Choose K such that B_K is not empty and $\Gamma \subseteq B_K$. Assume $X_0 \notin B_K$ and define

$$\tau = \inf\{n : X_n \in B_K\}.$$

Our goal is to show that τ is almost surely finite. But $(V(X_{n\wedge\tau}))_{n=0}^{\infty}$ is a supermartingale bounded from below by 0, hence by the first Doob's convergence theorem there exists an almost sure limit X, which is almost surely finite. This means that $\sup_n V(X_{n\wedge\tau})$ is almost surely bounded, which by the finitedness of the sub-level sets of V and the irreducibility of the DTMC can only happen if $\tau < \infty$. $\square$

A recurrence criterion

For example, consider the DTMC on $\mathbb{Z}$ defined by

$$p(i,j) = \begin{cases} 1/3 & \text{if } j = i+5; \\ 1/6 & \text{if } j = i+2; \\ 1/2 & \text{if } j = i-4; \end{cases}$$

And prove it is recurrent!