# Homework 1

hand-in by Wednesday November 20th at 23:59

1. A Mars rover takes pictures of its surronding according to a Poisson process with rate 6/h, when the camera is functioning. The probability that each picture is of good quality is 2/5, independently of the others. The battery of the camera lasts for 100h, at which point it needs to be recharged via a solar panel. The recharging times depend on the current atmospheric conditions, but for simplicity we assume they are independent and identically distributed lognormals with mean 20h and standard deviation 10h.

   After one year of operation, how many good-quality photos per hour did the rover take?

2. Alice, Bob, and Carl work in a factory. The time it takes them to create a piece are independent exponential random variables with mean 5, 10 and 20 minutes respectively.

   (a) What is the probability they will produce more than 10 pieces in total in one hour?

   (b) What is the probability that it will take more than 15 minutes to produce the first piece? What is the probability that Bob will be the first one producing a piece?

   (c) What is the probability that Bob will be the first one producing a piece, given that it will take more than 15 minutes to produce it?

   (d) What is the probability that it will take more than 15 minutes to produce the first piece, given that Bob will be the one producing it?

3. Let $\{M(t)\}_{t\in[0,+\infty)}$ be a non-homogeneous Poisson process with rate function $\lambda(t) = 100/(t+1)$ per minute. Write a pseudo-code that simulates this process until 500 jumps have taken place. Next, write a pseudo-code that simulates this process up to a time of 150 minutes.

4. Consider the discrete time Markov chain with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 0.2 & 0.1 & 0 & 0.7 \\ 0.3 & 0 & 0.1 & 0.6 & 0 \\ 0.3 & 0.5 & 0.1 & 0 & 0.1 \\ 0 & 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

   (a) Identify the transient and the recurrent states.

   (b) Let $T = \min\{n \geq 1 : X_n = 1\}$. Is $T$ a stopping time? Argue why or why not.

   (c) Find $P(X_{T+1} = 5|X_T = 1)$.

   (d) Let $U = \max\{n \geq 0 : X_n \in \{1, 2, 3\}\}$ and find $P_1(U < \infty)$.

   (e) Is $U$ a stopping time? Argue why or why not.

   (f) Find $P(X_{U+1} = 5|X_U = 1)$.

   (g) Find $P_1(U \leq 6)$.

5. Consider the so-called "renewal chain" with state space $\mathbb{N} = \{0, 1, 2, \dots\}$ in which $p(x, x-1) = 1$ when $x > 0$ and $p(0, x) = p_x > 0$ for all $x \in \mathbb{N}$.

   (a) Show that all the states in $\mathbb{N}$ are always recurrent but are positive recurrent if and only if $\sum_{x\in\mathbb{N}} xp_x < \infty$.

   (b) Prove that if $\sum_{x\in\mathbb{N}} xp_x < \infty$, then the long-term frequency of visits to a state $x$ is

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} \mathbb{1}_{\{X_m=y\}}}{n} = \frac{\sum_{x=y}^{\infty} p_x}{\sum_{x\in\mathbb{N}} xp_x}.$$

(c) Calculate

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} \mathbb{1}_{\{X_m=y\}}}{n}$$

for the case $\sum_{x\in\mathbb{N}} xp_x = \infty$ and explain your reasoning.

(d) Prove by using the ergodic theorem that if $\sum_{x\in\mathbb{N}} xp_x < \infty$, then the long-term frequency of visits to states that are greater than or equal to 100 is

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} \mathbb{1}_{\{X_m\geq 100\}}}{n} = \frac{\sum_{x=100}^{\infty}(x-99)p_x}{\sum_{x\in\mathbb{N}} xp_x}.$$

(e) Calculate

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} \mathbb{1}_{\{X_m\geq 100\}}}{n}$$

for the case $\sum_{x\in\mathbb{N}} xp_x = \infty$.