Homework II

- La soluzione degli esercizi *(ovvero un PDF contenente la risoluzione analitica degli esercizi e eventuali codici numerici* deve essere caricata sul portale del corso entro le ore 23:59 di domenica 15 Dicembre 2024, sotto il nome di Homework2. Il voto massimo in caso di consegna in ritardo è diminuito di 1 punto. In ogni caso, il PDF deve essere consegnato entro 5 giorni dalla data dell'esame orale.
- Le qualità dell'esposizione, la capacità di sintesi e la chiarezza del documento finale rientrano nella valutazione dell'homework. La scrittura del documento finale in Latex o in qualsiasi altro formato elettronico è fortemente incoraggiata. Se il documento finale è scritto a mano deve essere facilmente leggibile.
- La collaborazione e lo scambio di idee sono incoraggiati. In ogni caso, ogni studente deve sottomettere una copia del documento finale (in formato PDF) e del codice numerico (se presente), e specificare con chi ha collaborato e per quale specifica parte del lavoro.

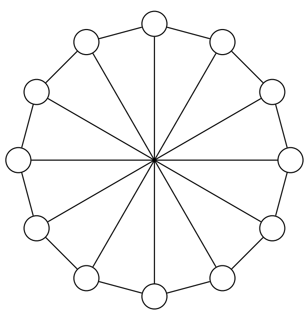
Esercizio 1. Given an even integer n, consider the simple graph R_n, in which the node i is connected by undirected links to nodes $i+1$, $i-1$ and $i+n/2$ (mod n).

(a) Consider the stochastic matrix P with entries $P_{ij} = w_i^{-1} W_{ij}$, where $w_i = (W\mathbb{1})_i$, and let $P_{lazy} = (I+P)/2$. Analyze the convergence of dynamics $x(t+1) = Px(t)$ and $x(t+1) = P_{lazy}x(t)$.

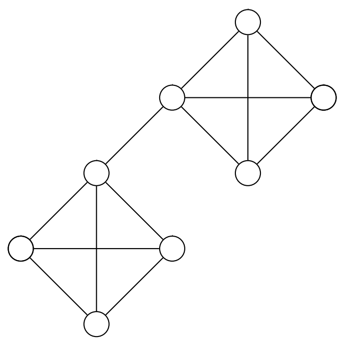
(b) Compute explicitly λ_2, i.e., the second largest eigenvalue of P_{lazy}, and determine the relaxation time $1/(1-\lambda_2)$ as a function of n, as $n \to +\infty$ (**Hint:** observe that R_n is a circulant graph).

(c) Compute an upper and a lower bound for the relaxation time on the barbell graph B_n as a function of n, as $n \to +\infty$.

(d) (**optional**) Observe that the relaxation time provides an upper bound for the convergence time of dynamics $x(t+1) = P_{lazy}x(t)$ to consensus. In light of this observation, find two initial conditions $x(0)$ and $y(0)$ on B_n (consider n multiple of 4) where half of the nodes have initial opinion 0 and half of the nodes have initial opinion 1, such that i) the convergence to consensus of $x(t)$ is *fast* (i.e., it does not scale with the size of the graph) ii) the convergence of $y(t)$ to consensus is *slow* (i.e., it scales quadratically with the size of the graph). Simulate numerically the dynamics with n sufficiently and plot the convergence time as a function of n for both the initial conditions $x(0)$ and $y(0)$. It is not asked to compute analytically the speed of convergence of $x(0)$ and $y(0)$.



Il grafo R_{12}



Barbell graph B_8

Esercizio 2. Consider the simple tree in Figure 1 and assume there are two players: player P_A, who places on the tree a stubborn node s_1 with value 1, and player P_B, who places on the tree a stubborn node s_0 with value 0. Each player aims at maximizing the infuence of the stubborn node that he places, that is, he aims at making the average asymptotic opinion of nodes in the average dynamics as close as possible to the opinion of its own stubborn node.

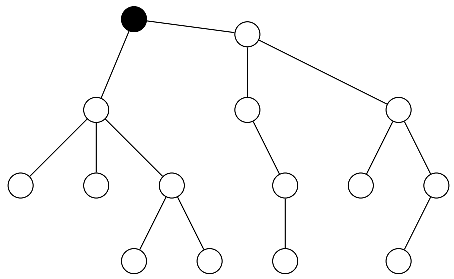


Figure 1: Tree for Exercise 2

(a) Suppose that player P_A has moved first and he has placed the stubborn node s_1 on the black node of the tree. Then, it is time for player P_B to move: where should he place the stubborn node s_0 in order to make the average opinion of nodes as close to 0 as possible?

(b) Consider now a different scenario. Player P_B is the first to move and he places the stubborn node s_0. Then it is P_A's turn to move, and he places (on a different position) the stubborn node s_1 in such a way that the average opinion of nodes is the highest possible. Where should player P_B place s_0 in order to make the average opinion of nodes as close to 0 as possible?

Esercizio 3. Consider a community of agents whose interactions are described by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ with

$$W = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{1}$$

Each agent has an initial opinion $x_i(0) = \xi_i$, where $\{\xi_i\}_{i \in \mathcal{V}}$ are independent random variables with variance

$$\sigma_1^2 = 0.1, \quad \sigma_2^2 = \sigma_4^2 = 0.2, \quad \sigma_3^2 = 0.3$$

and $E[\xi_i] = 0$ for each node i.

(a) Compute the variance of the asymptotic consensus value of the French-DeGroot dynamics.

(b) How should the invariant probability distribution π of a strongly connected aperiodic graph be in order to minimize the variance of the consensus value?

(c) Consider the graph defined by the adjacency matrix (1). Assume now that a selfloop (i, i) with arbitrary weight α_i is added to each node i. Find a non-negative vector α such that on the new graph the variance of the asymptotic consensus value is minimized. If multiple α exist, select the one that minimizes $\alpha'\mathbf{1}$.

(d) Consider again the graph (1), and assume that the initial opinions $\{\xi_i\}_{i=1}^n$ are random variables with covariance matrix given by

$$A = \begin{pmatrix} 0.2 & 0.1 & 0.1 & 0 \\ 0.1 & 0.3 & 0 & 0 \\ 0.1 & 0 & 0.2 & 0.1 \\ 0 & 0 & 0.1 & 0.3 \end{pmatrix},$$

and such that $E[\xi_i] = 0$ for each node i. Compute the variance of the asymptotic consensus value of the French-DeGroot dynamics.

(e) Consider the same setting of point (d), and find a non-negative vector α such that on the new graph the variance of the asymptotic consensus value is minimized. If multiple α exist, select the one that minimizes $\alpha'\mathbf{1}$.

Esercizio 4. Consider the epidemic model SI on an undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ of size $|\mathcal{V}| = n$. Let β be the transmission rate of each link, and let $X(t) \in \{0,1\}^{\mathcal{V}}$ denote the configuration of the epidemic model at time $t \geq 0$, where $X_i(t) = 1$ if the node i is infected and $X_i(t) = 0$ if the node is susceptible. Let

$$N(t) = \mathbb{1}'X(t) = \sum_i X_i(t), \qquad B(t) = (\mathbb{1} - X(t))'WX(t) = \sum_{i,j} W_{ij}(1 - X_i(t))X_j(t),$$

which represent respectively the number of infected nodes and the number of active links at time t. Assume that the initial configuration is such that $\mathbb{P}(N(0) \geq 1) = 1$, namely there is at least an infected node at time 0. The goal of this exercise is to compute an upper bound of the expected absorbing time

$$\tau = \mathbb{E}[T_{\mathbb{1}}]$$

of the configuration where all nodes are infected. The bound must hold true for each graph $\mathcal{G}$. We then apply the result to some specific graphs. Let the *minimum conductance profile* of the graph $\mathcal{G}$ be the function

$$\gamma : \{1, \ldots, n-1\} \to \mathbb{R}$$

such that

$$\gamma(k) = \min_{\mathcal{S} \subset \mathcal{V} : |\mathcal{S}| = k} \sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{V} \setminus \mathcal{S}} W_{ij}.$$

(a) Prove that

$$\tau \leq \frac{1}{\beta} \sum_{k=1}^{n-1} \frac{1}{\gamma(k)} \leq \frac{2}{\beta} \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{\gamma(k)}.$$

(**Hint**: use the fact that, conditioned to the state $X(t)$, the waiting time before the next infection has exponential distribution with rate $\beta B(t)$.)

(b) Consider the case where the graph $\mathcal{G}$ is a bidimensional grid with $n = m \times m$ nodes. Use the inequality $\gamma(k) \geq \sqrt{2k}$ (valid for $k \in \{1, \ldots, \lfloor n/2 \rfloor\}$) and (a) to estimate the expected absorbing time τ for the SI model on the grid. Estimate the asymptotic behavior of τ as a function of n as $n \to +\infty$.

(c) Compute $\gamma(k)$ for the barbell graph with $n + n$ nodes, and use the result to estimate by (a) the expected absorbing time of the model SI on this graph. Estimate the asymptotic behavior of τ as a function of n as $n \to +\infty$.