Martingale useful inequalities

Doob's maximal inequality in discrete time

Theorem (Doob's maximal inequality in discrete time)

Let $(X_n)_{n=0}^{\infty}$ be a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$. Then,

$$P\left(\max_{0\le m\le n} X_m \ge \varepsilon\right) \le \frac{\mathbb{E}[X_n^+]}{\varepsilon}$$

where $a^+ = \max\{0, a\}$ is the positive part of a.

The theorem resembles the Markov inequality, but it gives us probability bounds on the **whole** path of the process up to a certain time n.

This is useful to prove convergence of processes!

Example 1: process convergence

As an example of process convergence, consider the following compact-time interval convergence in probability of rescaled random walks:

Let $(X_n)_{n=0}^{\infty}$ be a random walk on $\mathbb{Z}$ with $X_0 = 0$. We "zoom out" the trajectory by rescaling time and space by the same factor. Formally, for $N \in \mathbb{N}$, let

$$I_N = \{m/N \,:\, m \in \mathbb{N}\}$$

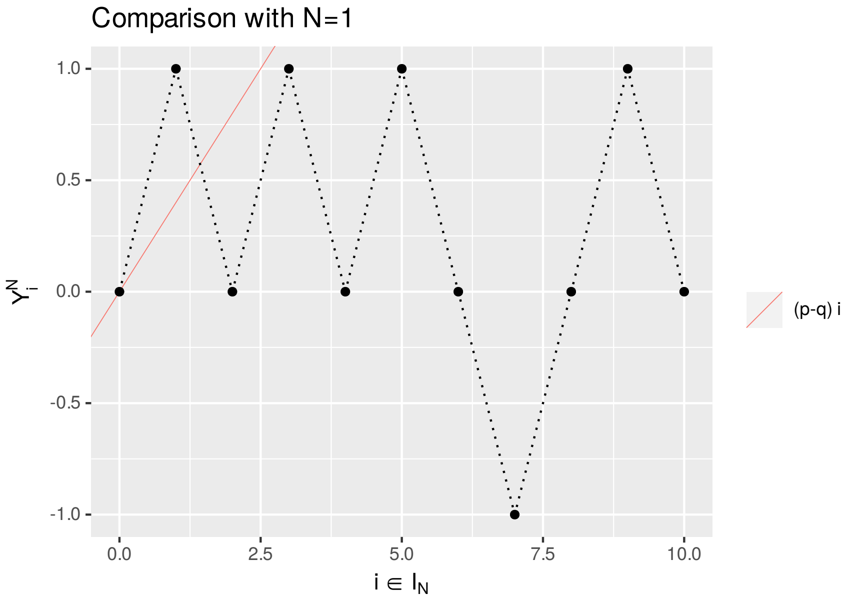
and define the process $(Y_i^N)_{i \in I_N}$ by

$$Y_i^N = \frac{1}{N} X_{Ni}.$$

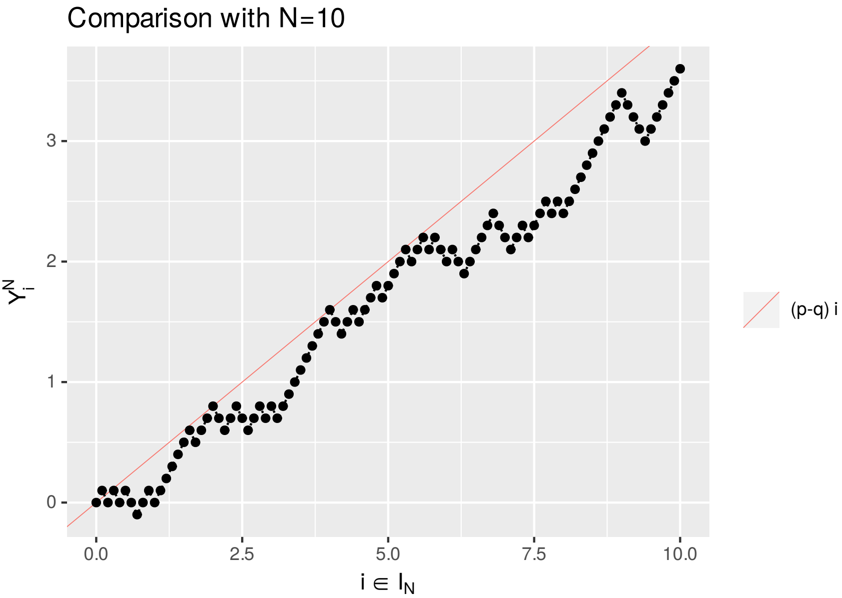
Then, for any $T > 0$ and any $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\sup_{i \in I_N \,:\, i \leq T} |Y_i^N - (p-q)i| > \varepsilon \right) = 0.$$

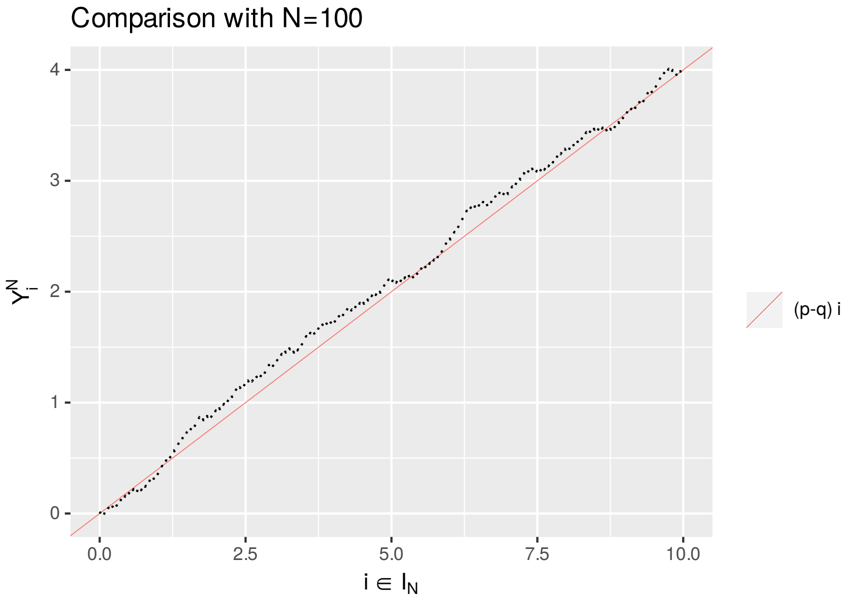
Example 1: process convergence



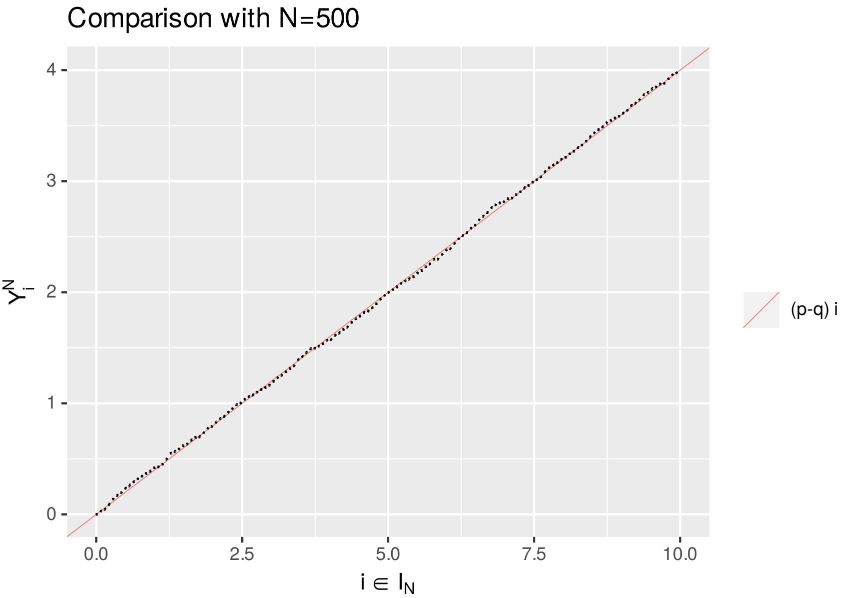
Example 1: process convergence



Example 1: process convergence



Example 1: process convergence



Example 1: process convergence

To prove the convergence we use Doob's maximal inequality. First note that

$$X_n = \sum_{i=1}^{n} B_i$$

with $(B_i)_{i=1}^{\infty}$ sequence of i.i.d. random variables with $P(B_i = 1) = p$ and $P(B_i = -1) = q$. We have $\mathbb{E}[B_i] = p - q$. Hence, by the $\mathcal{L}_1$ convergence in the law of large numbers,

$$\lim_{n \to \infty} \mathbb{E}\left[\left|\frac{1}{n} X_n - (p - q)\right|\right] = 0.$$

Example 1: process convergence

The process

$$\left(\frac{1}{N}X_{Ni}-(p-q)i\right)_{i\in I_N}$$

is a martingale:

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{N}X_{N(i+1/N)}-(p-q)\left(i+\frac{1}{N}\right)\middle|\mathcal{F}_i\right] &= \frac{1}{N}X_{Ni}-(p-q)i+\mathbb{E}\left[\frac{B_1}{N}\right]-\frac{p-q}{N}\\
&= \frac{1}{N}X_{Ni}-(p-q)i.
\end{aligned}$$

Hence, by Jensen's inequality and the convexity of $|\cdot|$,

$$\left(\left|\frac{1}{N}X_{Ni}-(p-q)i\right|\right)_{i\in I_N}$$

is a submartingale

Example 1: process convergence

By Doob's maximal inequality

$$P\left(\sup_{i\in I_N\,:\,i\leq T}\left|\frac{1}{N}X_{Ni}-(p-q)i\right|>\varepsilon\right)\leq\frac{1}{\varepsilon}\mathbb{E}\left[\left|\frac{1}{N}X_{N\frac{\lfloor NT\rfloor}{N}}-(p-q)\frac{\lfloor NT\rfloor}{N}\right|\right]$$

$$=\frac{1}{\varepsilon}\frac{\lfloor NT\rfloor}{N}\mathbb{E}\left[\left|\frac{1}{\lfloor NT\rfloor}X_{\lfloor NT\rfloor}-(p-q)\right|\right]\xrightarrow[N\to\infty]{}0.$$

Example 2: bounds on hitting times

Let $(X_n)_{n=0}^\infty$ be a random walk with $p = 1/2$, and consider the stopping times

$$\tau_k = \inf\{n : X_n \geq k\}, \quad \tau'_k = \inf\{n : |X_n| \geq k\}.$$

By Doob's maximal inequality we have bounds on the cumulative distribution functions of τ_k and τ'_k:

$$P(\tau_k \leq n) = P\left(\max_{0 \leq m \leq n} X_m \geq k\right) \leq \frac{\mathbb{E}[X_n^+]}{k}$$
$$P(\tau'_k \leq n) = P\left(\max_{0 \leq m \leq n} |X_m| \geq k\right) \leq \frac{\mathbb{E}[|X_n|]}{k}$$

because both $(X_n)_{n=0}^\infty$ and $(|X_n|)_{n=0}^\infty$ are submartingales.

Doob's maximal inequality in discrete time

Proof of Doob's maximal inequality in discrete time.

Define the stopping time $\tau = \inf\{n : X_n \geq \varepsilon\}$. Note that $\tau \wedge n$ and n are bounded stopping times with $\tau \wedge n \leq n$, so by Doob's optional sampling theorem

$$\mathbb{E}[X_n] \geq \mathbb{E}[X_{\tau\wedge n}] = \mathbb{E}[X_\tau \mathbb{1}_{\{\tau \leq n\}}] + \mathbb{E}[X_n \mathbb{1}_{\{\tau > n\}}] \geq \varepsilon P(\tau \leq n) + \mathbb{E}[X_n \mathbb{1}_{\{\tau > n\}}].$$

Hence, since

$$P(\tau \leq n) = P\left(\max_{0\leq m\leq n} X_m \geq \varepsilon\right) \quad \text{and} \quad \mathbb{E}[X_n] - \mathbb{E}[X_n \mathbb{1}_{\{\tau > n\}}] = \mathbb{E}[X_n \mathbb{1}_{\{\tau \leq n\}}]$$

we obtain

$$P\left(\max_{0\leq m\leq n} X_m \geq \varepsilon\right) \leq \frac{\mathbb{E}[X_n \mathbb{1}_{\{\tau \leq n\}}]}{\varepsilon} = \frac{\mathbb{E}[X_n^+ \mathbb{1}_{\{\tau \leq n\}}]}{\varepsilon} - \frac{\mathbb{E}[X_n^- \mathbb{1}_{\{\tau \leq n\}}]}{\varepsilon} \leq \frac{\mathbb{E}[X_n^+]}{\varepsilon}$$

$\square$

Maximal Azuma-Hoeffding inequality

Lemma (Hoeffding's lemma)

Let X be a random variable with $\mathbb{E}[X] = 0$ such that $X \in [a, b]$ almost surely, for two real numbers $a < b$. Then, for any $s > 0$

$$\mathbb{E}[e^{sX}] \leq \exp\left(\frac{s^2(b-a)^2}{8}\right).$$

The proof is based on the convexity of the function $x \mapsto e^{sx}$ and on Taylor's expansions. We skip it.

Maximal Azuma-Hoeffding inequality

Theorem (Maximal Azuma-Hoeffding inequality)

Let $(X_n)_{n=0}^{\infty}$ be a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$. Assume that are predictable processes $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ and a sequence of finite real numbers $(c_n)_{n=1}^{\infty}$ such that for all $n \geq 1$

$$A_n \leq X_n - X_{n-1} \leq B_n, \quad B_n - A_n \leq c_n \quad \textit{almost surely.}$$

Then, for all $\beta > 0$ we have

$$P\left(\max_{0 \leq m \leq n} (X_m - X_0) \geq \beta\right) \leq \exp\left(-\frac{2\beta^2}{\sum_{i=1}^{n} c_i^2}\right)$$

Clearly, under the assumptions of the theorem above,

$$P(X_n - X_0 \geq \beta) \leq P\left(\max_{0 \leq m \leq n} (X_m - X_0) \geq \beta\right) \leq \exp\left(-\frac{2\beta^2}{\sum_{i=1}^{n} c_i^2}\right)$$

Maximal Azuma-Hoeffding inequality

Proof.

By Doob's decomposition theorem, there exists a predictable process $(H_n)_{n=1}^{\infty}$ (w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$) which is almost surely non-increasing, and a martingale (w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$) $(M_n)_{n=0}^{\infty}$ such that

$$X_n = M_n + H_n,$$

where we set $H_0 = 0$. Then,

$$\underbrace{A_n - (H_n - H_{n-1})}_{\text{predictable}} \leq M_n - M_{n-1} \leq \underbrace{B_n - (H_n - H_{n-1})}_{\text{predictable}}$$

and

$$B_n - (H_n - H_{n-1}) - [A_n - (H_n - H_{n-1})] = B_n - A_n \leq c_n \quad \text{almost surely.}$$

□

Maximal Azuma-Hoeffding inequality

Proof.

Since $x \mapsto e^{sx}$ is convex, for any $s > 0$ we have that $(e^{sM_n})_{n=0}^{\infty}$ is a submartingale. Hence, by Doob's maximal inequality

$$P\left(\max_{0\le m\le n}(M_m - M_0) \geq \beta\right) = P\left(\max_{0\le m\le n} e^{s(M_m - M_0)} \geq e^{s\beta}\right) \leq \frac{\mathbb{E}\left[e^{s(M_n-M_0)}\right]}{e^{s\beta}}$$

We focus on the numerator:

$$\begin{aligned}
\mathbb{E}\left[e^{s(M_n-M_0)}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{s(M_n-M_0)} \middle| \mathcal{F}_{n-1}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{s(M_{n-1}-M_0)} e^{s(M_n-M_{n-1})} \middle| \mathcal{F}_{n-1}\right]\right] \\
&= \mathbb{E}\left[e^{s(M_{n-1}-M_0)} \mathbb{E}\left[e^{s(M_n-M_{n-1})} \middle| \mathcal{F}_{n-1}\right]\right].
\end{aligned}$$

□

Maximal Azuma-Hoeffding inequality

Proof.

By Hoeffding's lemma,

$$\mathbb{E}\left[e^{s(M_n - M_0)}\right] \leq \exp\left(\frac{s^2 c_n^2}{8}\right) \mathbb{E}\left[e^{s(M_{n-1} - M_0)}\right]$$

Hence by induction,

$$\mathbb{E}\left[e^{s(M_n - M_0)}\right] \leq \exp\left(\frac{s^2 \sum_{i=1}^n c_i^2}{8}\right).$$

Therefore, by putting things together, for any $s > 0$

$$P\left(\max_{0 \leq m \leq n} (M_m - M_0) \geq \beta\right) \leq \exp\left(\frac{s^2 \sum_{i=1}^n c_i^2}{8} - s\beta\right)$$

$\square$

Maximal Azuma-Hoeffding inequality

Proof.

The expression above attains its minimum

$$P\left(\max_{0\le m\le n}(M_m - M_0) \ge \beta\right) \le \exp\left(-\frac{2\beta^2}{\sum_{i=1}^n c_i^2}\right)$$

for

$$s = \frac{4\beta}{\sum_{i=1}^n c_i^2}.$$

We conclude by noting that $H_n \le H_{n-1}$ almost surely for all $n \ge 1$, hence

$$\begin{aligned} P\left(\max_{0\le m\le n}(X_m - X_0) \ge \beta\right) &= P\left(\max_{0\le m\le n}[(M_m - M_0) + (H_m - H_0)] \ge \beta\right) \\ &\le P\left(\max_{0\le m\le n}(M_m - M_0) \ge \beta\right) \le \exp\left(-\frac{2\beta^2}{\sum_{i=1}^n c_i^2}\right) \end{aligned}$$

□

Maximal Azuma-Hoeffding inequality

Under the hypothesis of the Azuma-Hoeffding inequality,

- If $(X_n)_{n=0}^{\infty}$ is a supermartingale,

$$P\left(\max_{0\leq m\leq n}(X_m - X_0) \geq \beta\right) \leq \exp\left(-\frac{2\beta^2}{\sum_{i=1}^n c_i^2}\right).$$

- If $(X_n)_{n=0}^{\infty}$ is a submartingale, then $(-X_n)_{n=0}^{\infty}$ is a supermartingale hence

$$P\left(\max_{0\leq m\leq n}(X_m - X_0) \leq -\beta\right) \leq \exp\left(-\frac{2\beta^2}{\sum_{i=1}^n c_i^2}\right).$$

- If $(X_n)_{n=0}^{\infty}$ is a martingale, then both inequalities above hold. Moreover, by combining them,

$$P\left(\max_{0\leq m\leq n}|X_m - X_0| \geq \beta\right) \leq 2\exp\left(-\frac{2\beta^2}{\sum_{i=1}^n c_i^2}\right).$$

Example 1: chromatic number of Erdös Renyi graphs

Definition (Erdös Renyi graph)

Consider N nodes $V = \{1, 2, \ldots, N\}$ and assume for each pair $i \neq j \in V$ the edge $\{i, j\}$ is present with probability p, independently on the others. The random graph $\mathcal{G} = (V, E)$ obtained in this way is a Erdös Renyi graph with N nodes and probability parameter p. We write $\mathcal{G} \sim G(N, p)$

Example 1: chromatic number of Erdös Renyi graphs

Definition (chromatic number)

The chromatic number of a graph $\mathcal{G} = (V, E)$ is the minimum number of colors we can associate each vertex with, in such a way that for any edge $\{i, j\} \in V$ the two vertices are of different colors. It is denoted by $\chi(\mathcal{G})$.

It may be complicated to calculate the chromatic number of a given graph (it's NP hard).

Example 1: chromatic number of Erdös Renyi graphs

Theorem (Shamir and Spencer, 87)

Let $\mathcal{G} \sim G(N,p)$*. Then, for any* $\beta > 0$

$$P(\chi(\mathcal{G}) - \mathbb{E}[\chi(\mathcal{G})] \geq \beta) \leq \exp\left(-\frac{\beta^2}{2N}\right)$$

and

$$P(\chi(\mathcal{G}) - \mathbb{E}[\chi(\mathcal{G})] \leq -\beta) \leq \exp\left(-\frac{\beta^2}{2N}\right)$$

It follows that we do not expect deviations of order higher than $\sqrt{N}$:

$$P(|\chi(\mathcal{G}) - \mathbb{E}[\chi(\mathcal{G})]| \geq \beta\sqrt{N}) \leq \exp\left(-\frac{\beta^2}{2}\right) \xrightarrow[\beta\to\infty]{} 0.$$

The inequality does not tell us anything about $\mathbb{E}[\chi(\mathcal{G})]$, but it tells us how much the random variable $\chi(\mathcal{G})$ is *concentrated* around its mean: it is a concentration inequality.

Example 1: chromatic number of Erdös Renyi graphs

Proof.

We can define a filtration $(\mathcal{F}_n)_{n=0}^{\infty}$ by $\mathcal{F}_0 = \{\emptyset\}$,

$$\mathcal{F}_n = \sigma\{B_{in} : 1 \leq i \leq n-1\} \quad \text{for } 1 \leq n \leq N,$$

and $\mathcal{F}_n = \mathcal{F}_N$ for $n \geq N$, where

$$B_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \text{ is an edge} \\ 0 & \text{otherwise.} \end{cases}$$

We know that

$$X_n = \mathbb{E}[\chi(\mathcal{G})|\mathcal{F}_n]$$

defines a martingale. It is called the vertex exposure martingale as each time the connections of a vertex with the previous ones is disclosed, and the expectation of a function of the graph (in this case the chromatic number) is recalculated accordingly. □

Example 1: chromatic number of Erdös Renyi graphs

Proof.

Altering the status of edges incident to the vertex n increases the chromatic number by at most 1, since in the worst case one can simply use an extra color for n.

On the other hand, the chromatic number cannot decrease by more than 1 after altering the status of edges incident to n: if that were the case, then upon removing the vertex n the chromatic number would increase by more than 1, which is a contradiction.

Hence, for all $1 \leq n \leq N$,

$$-1 \leq \mathbb{E}[\chi(\mathcal{G})|\mathcal{F}_n] - \mathbb{E}[\chi(\mathcal{G})|\mathcal{F}_{n-1}] \leq 1.$$

Noting that $X_0 = \mathbb{E}[\chi(\mathcal{G})|\mathcal{F}_0] = \mathbb{E}[\chi(\mathcal{G})]$ and $X_N = \mathbb{E}[\chi(\mathcal{G})|\mathcal{F}_N] = \chi(\mathcal{G})$, we conclude by using Azuma-Hoeffding inequality with $c_i = 2$.

□

Example 2: balls and bins

Theorem

Assume m balls are thrown uniformly at random into n bins, independently on each other. Let $Z_{n,m}$ be the number of empty bins. We have

$$\mathbb{E}[Z_{n,m}] = n\left(1 - \frac{1}{n}\right)^m$$

and

$$P(Z_{n,m} - \mathbb{E}[Z_{n,m}] \geq \beta) \leq \exp\left(-\frac{\beta^2}{2m}\right)$$
$$P(Z_{n,m} - \mathbb{E}[Z_{n,m}] \leq -\beta) \leq \exp\left(-\frac{\beta^2}{2m}\right).$$

Again, we expect deviations from the mean of order of $\sqrt{m}$ or lower.

Example 2: balls and bins

Proof.

Note that

$$Z_{n,m} = \sum_{i=1}^{n} B_i$$

where

$$B_i = \begin{cases} 1 & \text{if the } i\text{th bin is empty} \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\mathbb{E}[Z_{n,m}] = \sum_{i=1}^{n} \mathbb{E}[B_i] = n\left(1 - \frac{1}{n}\right)^m.$$

Note that the B_i's are not independent, hence finding the variance of $Z_{n,m}$ to apply, for example, the Chebychev inequality is complicated.

$\square$

Example 2: balls and bins

Proof.

We can define a filtration $(\mathcal{F}_i)_{i=0}^{\infty}$ by $\mathcal{F}_0 = \{\emptyset\}$,

$$\mathcal{F}_i = \sigma\{L_j : 1 \leq j \leq i\} \quad \text{for } 1 \leq i \leq m,$$

and $\mathcal{F}_i = \mathcal{F}_m$ for $i \geq m$, where $L_j \in \{1, 2, \dots, n\}$ is the bin the jth ball falls into. Define the closed martingale

$$X_i = \mathbb{E}[Z_{n,m} | \mathcal{F}_i].$$

Since changing the position of a single ball can increase or decrease $Z_{n,m}$ by at most one, for all $1 \leq i \leq m$,

$$-1 \leq \mathbb{E}[Z_{n,m} | \mathcal{F}_i] - \mathbb{E}[Z_{n,m} | \mathcal{F}_{i-1}] \leq 1.$$

Noting that $X_0 = \mathbb{E}[Z_{n,m} | \mathcal{F}_0] = \mathbb{E}[Z_{n,m}]$ and $X_m = \mathbb{E}[Z_{n,m} | \mathcal{F}_m] = Z_{n,m}$, we conclude by using Azuma-Hoeffding inequality with $c_i = 2$. $\square$

Continuous time martingales

Càdlàg versions

Definition ("Augmented filtration" or "Filtration satisfying the usual conditions")

A filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ of a probability space $(\Omega, \mathcal{F}, P)$ is augmented if

It is complete : $\mathcal{N} \in \mathcal{F}_t$ for all $t \in [0,\infty)$ where

$$\mathcal{N} = \{A \subset \Omega : \exists G \in \mathcal{F} \text{ with } A \subseteq G \text{ and } P(G) = 0\}.$$

It is right-continuous : for all $t \in [0,\infty)$,

$$\mathcal{F}_t = \bigcap_{t<s} \mathcal{F}_s.$$

It is always possible to slightly enlarge the elements of a filtration to make it *augmented*. Since we only enlarge the sigma-algebras, stochastic processes that were previously adapted are still adapted after modification.

Càdlàg versions

Definition (Versions of a stochastic process)

Let $(X_i)_{i\in I}$ be a stochastic process adapted to $(\mathcal{F}_i)_{i\in I}$. A version of $(X_i)_{i\in I}$ is a stochastic process $(Y_i)_{i\in I}$ adapted to $(\mathcal{F}_i)_{i\in I}$ such that $P(X_i = Y_i) = 1$ for all $i \in I$.

Theorem (Càdlàg versions of continuous-time martingales, submartingales, supermartingales)

Let $(X_t)_{t\in[0,\infty)}$ be a martingale (or submartingale, or supermartingale) w.r.t. an augmented filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$. Then, there exists a version $(Y_t)_{t\in[0,\infty)}$ which is still a martingale (or submartingale, or supermartingale) w.r.t. $(\mathcal{F}_t)_{t\in[0,\infty)}$ and has right-continuous, left-limited (càdlàg) paths almost surely.

We will not prove the result. But it highlights why it makes sense to assume we are working with càdlàg processes in the following slides.

Doob's optional sampling theorem in continuous time

Theorem (Doob's optional sampling theorem in continuous time)

Let $(X_t)_{t\in[0,\infty)}$ *be a martingale, submartingale, or supermartingale w.r.t.* $(\mathcal{F}_t)_{t\in[0,\infty)}$*, with càdlàg paths almost surely. Let* τ *and* σ *be two stopping times w.r.t.* $(\mathcal{F}_t)_{t\in[0,\infty)}$*, such that almost surely,*

$$\sigma \leq \tau \leq C$$

for a constant $C \in \mathbb{R}$*. Then,* $\mathbb{E}[|X_\tau|], \mathbb{E}[|X_\sigma|] < \infty$ *and*

- *if* $(X_t)_{t\in[0,\infty)}$ *is a martingale w.r.t.* $(\mathcal{F}_t)_{t\in[0,\infty)}$ *then* $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] = X_\sigma$ *a.s.;*
- *if* $(X_t)_{t\in[0,\infty)}$ *is a submartingale w.r.t.* $(\mathcal{F}_t)_{t\in[0,\infty)}$ *then* $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \geq X_\sigma$ *a.s.;*
- *if* $(X_t)_{t\in[0,\infty)}$ *is a supermartingale w.r.t.* $(\mathcal{F}_t)_{t\in[0,\infty)}$ *then* $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \leq X_\sigma$ *a.s..*

Doob's optional sampling theorem in continuous time

Proof.

Consider

$$I_n = \left\{ \frac{k}{n} : k \in \mathbb{N} \right\}.$$

We define the following sequences of approximating, bounded stopping times:

$$\tau_n = \inf\{r \in I_n : r \geq \tau\}$$
$$\sigma_n = \inf\{r \in I_n : r \geq \sigma\}.$$

We have $\tau_n \downarrow \tau$ and $\sigma_n \downarrow \sigma$ as $n \to \infty$. For any $n \geq 1$, consider the discrete time process $(X_r)_{r \in I_n}$. It is straightforward to show that $(X_r)_{r \in I_n}$ is either a martingale, a submartingale, or a supermartingale depending on the nature of $(X_t)_{t \in [0,\infty)}$.

$\square$

Doob's optional sampling theorem in continuous time

Proof.

By right-continuity of $(X_t)_{t\in[0,\infty)}$ we have that almost surely

$$\lim_{n\to\infty} X_{\sigma_n} = X_\sigma \quad \text{and} \quad \lim_{n\to\infty} X_{\tau_n} = X_\tau.$$

Hence, by Doob's optional sampling theorem in discrete time applied to $(X_r)_{r\in I_n}$,

$$\mathbb{E}[X_{\tau_n}|\mathcal{F}_{\sigma_n}] = X_{\sigma_n} \xrightarrow{a.s.} X_\sigma.$$

To conclude the proof, we only need to show that

$$\lim_{n\to\infty} \mathbb{E}[X_{\tau_n}|\mathcal{F}_{\sigma_n}] = \mathbb{E}[X_\tau|\mathcal{F}_\sigma].$$

We skip it here. You can try to fill the gap as an exercise, this will not be asked at the oral exam.

□

Doob's optional stopping theorem in continuous time

Theorem (Doob's optional stopping theorem in continuous time)

Let $(X_t)_{t\in[0,\infty)}$ be a martingale, submartingale, or supermartingale w.r.t. $(\mathcal{F}_t)_{t\in[0,\infty)}$, with càdlàg paths almost surely. Let τ and σ be two stopping times w.r.t. $(\mathcal{F}_t)_{t\in[0,\infty)}$, such that

- *almost surely, $\sigma \leq \tau < \infty$;*
- $\mathbb{E}[|X_\sigma|], \mathbb{E}[|X_\tau|] < \infty$;
- $\lim_{n\to\infty} \mathbb{E}[|X_n| \mathbb{1}_{\{\tau > n\}}] = 0$.

Then,

- *if $(X_t)_{t\in[0,\infty)}$ is a martingale w.r.t. $(\mathcal{F}_t)_{t\in[0,\infty)}$ then $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] = X_\sigma$ a.s.;*
- *if $(X_t)_{t\in[0,\infty)}$ is a submartingale w.r.t. $(\mathcal{F}_t)_{t\in[0,\infty)}$ then $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] \geq X_\sigma$ a.s.;*
- *if $(X_t)_{t\in[0,\infty)}$ is a supermartingale w.r.t. $(\mathcal{F}_t)_{t\in[0,\infty)}$ then $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] \leq X_\sigma$ a.s..*

Proof.

By approximation with discrete time processes, as before. □

Theorem (First Doob's Convergence Theorem in continuous time)

Let $(X_t)_{t\in[0,\infty)}$ be a supermartingale w.r.t. $(\mathcal{F}_t)_{t\in[0,\infty)}$ (a filtration on $(\Omega, \mathcal{F}, P)$) with càdlàg paths almost surely. Assume that $\sup_t \mathbb{E}[X_t^-] < \infty$. Then there exists a $\mathcal{F}$-measurable random variable X with $\mathbb{E}[|X|] < \infty$ and

$$\lim_{t\to\infty} X_t = X \quad \textit{almost surely.}$$

First Doob's Convergence Theorem in continuous time

Proof.

As before, consider

$$I_n = \left\{ \frac{k}{n} : k \in \mathbb{N} \right\}.$$

The discrete time stochastic process $(X_r)_{r \in I_n}$ is a supermartingale, and the first Doob's Convergence Theorem in discrete time applies. Since $I_n \subseteq I_{n+1}$ there exists a common X with $\mathbb{E}[|X|] < \infty$ such that for all $n \in \mathbb{N}$

$$\lim_{k \to \infty} X_{\frac{k}{n}} = X \quad \text{almost surely.}$$

Since $\mathbb{Q} = \cup_{n=1}^{\infty} I_n$ is dense in $\mathbb{R}$ and by right-continuity of $(X_t)_{t \in [0,\infty)}$, necessarily

$$\lim_{t \to \infty} X_t = X \quad \text{almost surely.}$$

□

Second Doob's Convergence Theorem in continuous time

Theorem (Second Doob's Convergence Theorem in continuous time)

Let $(X_t)_{t\in[0,\infty)}$ be a submartingale w.r.t. $(\mathcal{F}_t)_{t\in[0,\infty)}$ (a filtration on $(\Omega, \mathcal{F}, P)$). Let X be a $\mathcal{F}$-measurable random variable. The following properties are equivalent:

1. *there exists a $\mathcal{F}$-measurable random variable X with $X_t \xrightarrow[t\to\infty]{\mathcal{L}_1} X$;*
2. *there exists a $\mathcal{F}$-measurable random variable X such that $\mathbb{E}[|X|] < \infty$, $\lim_{t\to\infty} X_t = X$ almost surely,*

   $$X_t \leq \mathbb{E}[X|\mathcal{F}_t] \quad \text{almost surely for all } t,$$

   and $\lim_{t\to\infty} \mathbb{E}[X_t] = \mathbb{E}[X]$;
3. *$(X_t)_{t=0}^{\infty}$ is uniformly integrable.*

Proof.

The same proof for the result in discrete time holds, with few modifications. □

Doob's maximal inequality in continuous time

Theorem (Doob's maximal inequality in continuous time)

Let $(X_t)_{t\in[0,\infty)}$ be a submartingale w.r.t. $(\mathcal{F}_t)_{t\in[0,\infty)}$, with càdlàg paths almost surely. Then,

$$P\left(\sup_{0\leq t\leq T} X_t \geq \varepsilon\right) \leq \frac{\mathbb{E}[X_T^+]}{\varepsilon}.$$

Doob's maximal inequality in continuous time

Proof.

We consider

$$I_{n,T} = \{T\} \cup \left\{ \frac{k}{n} : k \in \mathbb{N} \right\}.$$

By applying Doob's maximal inequality to $(X_r)_{r \in I_{n,T}}$ we obtain that for all $n \in \mathbb{N}$

$$P\left(\sup_{0 \le k/n \le T} X_{\frac{k}{n}} \ge \varepsilon \right) \le \frac{\mathbb{E}[X_T^+]}{\varepsilon}.$$

We conclude by the fact that $(X_t)_{t \in [0,\infty)}$ has càdlàg paths. $\square$