Continuous time Markov chains

The Markov property

Definition (Alternative definition of CTMCs)

A continuous-time stochastic process $(X_t)_{t\in[0,\infty)}$ with discrete state space S is a Continuous time Markov chain if for all $t, s \in [0,\infty)$ and all $x \in S$ we have

$$P(X_{t+s} = x \mid \mathcal{F}_t^X) = P(X_{t+s} = x \mid X_t)$$

Moreover, as in the discrete time case, we will only consider time homogeneous cases, that is we always assume that

$$P(X_{t+s} = y \mid X_t = x) = P(X_s = y \mid X_0 = x).$$

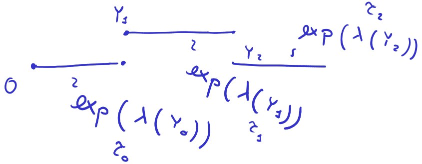
Construction 1

1. We let $(Y_n)_{n\geq 0}$ be a discrete time Markov chain on a state space S. We denote its transition probabilities by

$$r(i,j) = P(Y_{n+1} = j \mid Y_n = i),$$

with the condition that $r(i,i) = 0$ for all $i \in S$ (unless i is absorbing).

2. Let $m = 0$ and set $X_0 = Y_0$.
3. We spend a time $\tau_m \sim \text{Exp}(\lambda(Y_m))$ in state Y_m, the holding time. We assume that τ_m is independent of $(Y_n)_{n\geq 0}$ and all previous holding times.
4. After the holding time τ_m we transition to state Y_{m+1}.
5. Repeat from step 3.



Construction 2

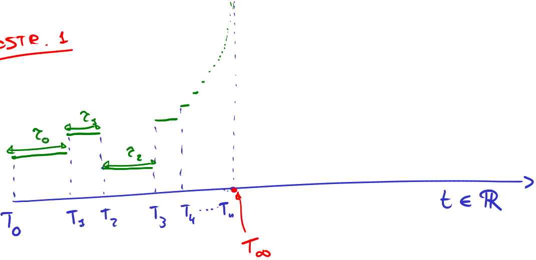
Y_0 j_1 j_2 j_3 j_4 $\sim \mathrm{Exp}(q(Y_0, j_1))$

1. Let $m = 0$ and set $Y_0 = X_0$.
2. For each $j \in S$ with $r(Y_m, j) > 0$ we consider $\tau_j \sim \mathrm{Exp}(q(i,j))$ where

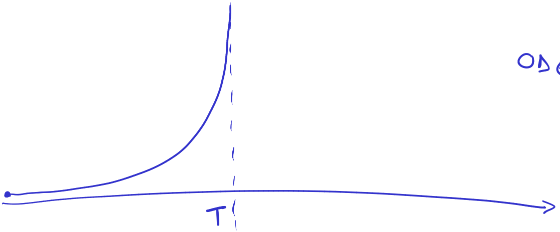
$$q(i,j) = \underline{\underline{\lambda(i)}} \cdot \underline{\underline{r(i,j)}}$$

   All exponential random variables are independent of each other and of all previous random variables.
3. Let $\tau_m = \min_j\{\tau_j\}$ and let Y_{m+1} be the index of the minimum.
4. Move to state Y_{m+1} after a holding time equal to τ_m.
5. Repeat from step 2.

COSTR. 1



Are Constructions 1 and 2 complete?



ODE

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

$x(t)$ DEFINITA SOLO IN $[0,T)$

Explosions

Costruction 1 has a potential problem: we denote by τ_m, $m = 0, 1, 2, \ldots$, the holding time in state Y_m, and define the mth jump time $T_m = \sum_{i=0}^{m-1} \tau_i$. Let

$$T_\infty = \lim_{m\to\infty} T_m = \sum_{i=0}^{\infty} \tau_i.$$

- If $T_\infty = \infty$ then then the process is defined for all $t \geq 0$.
- If $T_\infty < \infty$ then we have an explosion and the process is defined (according to Construction 1) only up to T_∞.

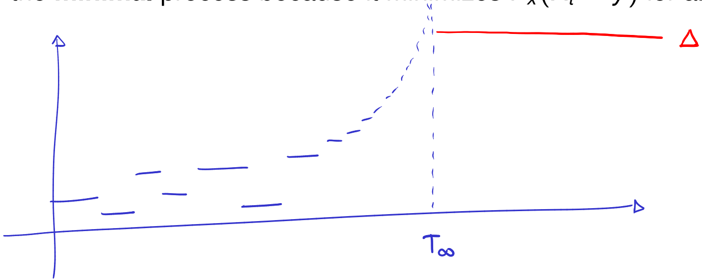
Explosions

Definition

If $P_x(T_\infty < \infty) > 0$ for some $x \in S$ then the CTMC is explosive.

If we want to define the process for all times we have different options:

- we can add a "cemetery state" Δ to S and say $X_t = \Delta$ for all $t \geq T_\infty$. This is the **minimal** process because it minimizes $P_x(X_t = y)$ for all $x, y \in S$;



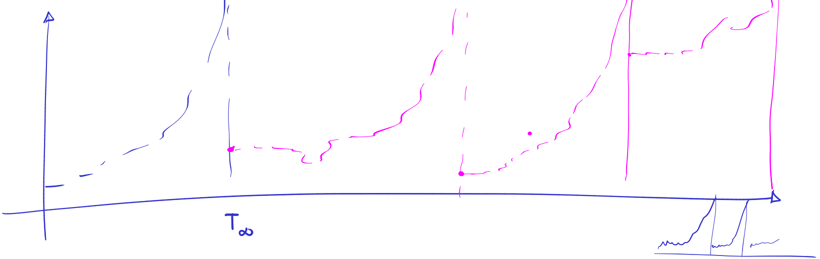
Explosions

Definition

If $P_x(T_\infty < \infty) > 0$ for some $x \in S$ then the CTMC is explosive.

If we want to define the process for all times we have different options:

- we can consider a distribution ν on S and restart the process according to ν at each explosion;



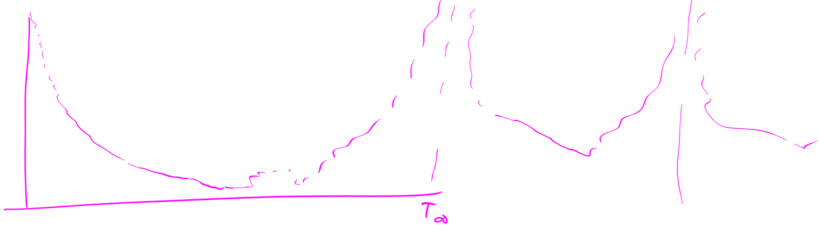
Explosions

Definition

If $P_x(T_\infty < \infty) > 0$ for some $x \in S$ then the CTMC is explosive.

If we want to define the process for all times we have different options:

- if the process is also “explosive in reverse,” that is it has the capability of coming back from infinity in a finite time, then we use this property;



Definition

If $P_x(T_\infty < \infty) > 0$ for some $x \in S$ then the CTMC is explosive.

If we want to define the process for all times we have different options:

- We can set a sequence of starting states $x_1, x_2, x_3, \dots$ for consecutive explosions. This choice is no longer Markovian.

Explosions

When do we have explosions? Preliminary:

Theorem

Let $(u_n)_{n=0}^{\infty}$ be a sequence of independent r.v. with $u_n \sim Exp(\lambda_n)$. Then

$$P\left(\sum_{n=0}^{\infty} u_n < \infty\right) = 1 \quad \iff \quad \sum_{n=0}^{\infty} \frac{1}{\lambda_n} < \infty \qquad \textit{and}$$

$$P\left(\sum_{n=0}^{\infty} u_n < \infty\right) = 0 \quad \iff \quad \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty.$$

By the monotone convergence theorem,

$$\mathbb{E}\left[\sum_{n=0}^{\infty} u_n\right] = \sum_{n=0}^{\infty} \mathbb{E}[u_n] = \sum_{n=0}^{\infty} \frac{1}{\lambda_n}.$$

$$P\left(\sum_{n=0}^{\infty} u_n = \infty\right) > 0 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty$$

Explosions

Given the embedded Markov chain $Y_0, Y_1, \cdots$, each holding time τ_m is exponentially distributed with mean $\frac{1}{\lambda(Y_m)}$, independently on the duration of all other holding times.

Due to the previous preliminary result, given the embedded Markov chain $Y_0, Y_1, \cdots$, with probability 1 an explosion occurs if and only if

$$\sum_{m=0}^{\infty} \frac{1}{\lambda(Y_m)} < \infty.$$

$$\Updownarrow$$

$$P\left(\sum_{m=0}^{+\infty} \tau_m < \infty \right) = 1$$

$$1 \xrightarrow{k} 2 \xrightarrow{k\cdot 2^2} 3 \xrightarrow{k\,3^2} 4 \xrightarrow{k\,4^2} 5 \longrightarrow \cdots$$

Consider a pure birth process, that is a CTMC $(X_t)_{t\in[0,\infty)}$ whose embedded DTMC $(Y_n)_{n=0}^{\infty}$ is a pure birth chain. Suppose that $\lambda(n) = \kappa n^2$ for some constant $\kappa > 0$, and let $X_0 = Y_0 = 1$. **Since the embedded chain is deterministic**, $Y_m = m+1$ and $\tau_m \sim \mathrm{Exp}(\kappa(m+1)^2)$.

$$= \frac{1}{k}\sum_{m=1}^{+\infty} \frac{1}{m^2} < \infty$$

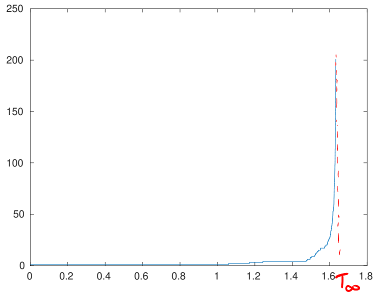
$$\sum_{m=0}^{\infty} \frac{1}{\lambda(Y_m)} = \sum_{m=0}^{\infty} \frac{1}{\kappa(m+1)^2} < \infty,$$

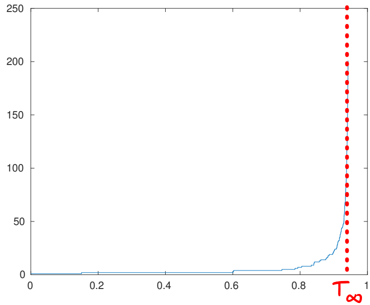
$$P\left(\sum_{m=0}^{\infty} \tau_m < \infty\right) = 1$$

we may conclude that the process is explosive.

Explosions: example 1

Here is a couple of simulations of the process (with $\kappa = 1$), up to its 200*th* jump. You can see that the process tends to increase very quickly, and it looks like there is a vertical asymptote.





Note that the explosion time is random, therefore different realization by realization.

Explosions

Theorem

If the holding rates are bounded, (it exists a constant $K > 0$ with $\lambda(x) \leq K$ for any x), then the CTMC is non-explosive.

Indeed, independently on the embedded DTMC,

$\lambda(x) \leq K$

$\frac{1}{\lambda(x)} \geq \frac{1}{K}$

$$\sum_{m=0}^{\infty} \frac{1}{\lambda(Y_m)} \geq \sum_{m=0}^{\infty} \frac{1}{K} = \infty.$$

Corollary

If the state space is finite, then the CTMC is non-explosive.

If the state space is finite, $K = \max_{x \in S} \lambda(x) < \infty$ The proof is then concluded by the previous proposition.

Explosions

Theorem

If a CTMC $(X_t)_{t\in[0,\infty)}$ is explosive, then the embedded DTMC is transient.

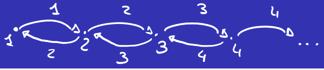
Proof.

We prove here that if the embedded DTMC $(Y_n)_{n=0}^{\infty}$ is recurrent, then necessarily the CTMC is non explosive. Denote by $Y_0 = X_0 = x_0$ the initial state. If the embedded DTMC is recurrent, then with probability 1 x_0 is visited infinitely many times. Say that for each $k \geq 0$ we have $Y_{n_k} = x_0$. Hence,

$$\sum_{m=0}^{\infty} \frac{1}{\lambda(Y_m)} \geq \sum_{k=0}^{\infty} \frac{1}{\lambda(Y_{n_k})} = \sum_{k=0}^{\infty} \frac{1}{\lambda(x_0)} = \infty.$$

Hence, the chain cannot be explosive. □

Explosions: example 2



Consider the CTMC on $S = \{1, 2, 3, \dots\}$ with the only positive rates

$$q(x, x+1) = x \quad \text{if } x \geq 1 \qquad \text{and} \qquad q(x, x-1) = x \quad \text{if } x \geq 2.$$

We have $\lambda(x) = 2x$ if $x \geq 2$, and $\lambda(1) = 1$. Is this CTMC explosive? The rates are not bounded, however if $Y_0 = x_0$ we have

$$Y_n \leq x_0 + n.$$

Since the $\lambda(x)$ are non-decreasing in x, it follows that

$$\lambda(Y_n) \leq \lambda(x_0 + n) \leq 2(x_0 + n).$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{\lambda(Y_n)} \geq \sum_{n=0}^{\infty} \frac{1}{2(x_0 + n)} = \infty,$$

which implies that the chain does not explode.

Transition probabilities

In the context of discrete time Markov chains, we defined the *n*-step transition probabilities $p^{(n)}(x,y)$. Similarly, we define the *transition probability* for a continuous time Markov chain as

$$p_t(x,y) = P(X_t = y | X_0 = x).$$

$t \in \mathbb{R}$

$t = 1.34\bar{2}$

$t = 5 \cdot \pi$

We still have the analogue of the DT Chapman-Kolmogorov equation

Theorem (Chapman-Kolmogorov)

Consider a continuous time Markov chain $(X_t)_{t \geq 0}$. Then, for any real $s, t > 0$

$$p_{s+t}(x,y) = \sum_{k \in S} p_s(x,k) p_t(k,y).$$

DTMC: $\quad p^{(n+m)}(x,y) = \sum_{k \in S} p^{(n)}(x,k)\, p^{(m)}(k,y)$

Transition probabilities and transition rates

The structure of the proof is exactly the same as in the discrete time case. We have

$$\underbrace{P(X_{s+t} = y | X_0 = x)}^{p_{s+t}(x,y)} \overset{TPT}{=} \sum_{k \in S} P(X_{s+t} = y | \underline{X_s = k}, \cancel{X_0 = x}_{MP}) P(X_s = k | X_0 = x)$$

$$= \sum_{k \in S} \boxed{P(X_{s+t} = y | X_s = k)} P(X_s = k | X_0 = x) \quad \text{(by Markov pr}$$

$$= \sum_{k \in S} \underline{\boxed{P(X_t = y | X_0 = k)}}\, \underline{P(X_s = k | X_0 = x)} \quad \text{(by homogeneit}$$

$$= \sum_{k \in S} \underline{p_t(k, y)}\, \underline{p_s(x, k)}.$$

$$P_{s+t} = P_s \cdot P_t$$

$$P_{s+t}(x,y) = p_{s+t}(x,y) = \sum_{k \in S} p_s(x,k) \cdot p_t(k,y) = (P_s \cdot P_t)(x,y)$$

Let $\tau(x)$ be a holding time in state x. We know that $\tau(x) \sim \text{Exp}(\lambda(x))$. Consider a small interval of time $[0, h]$. Given that $X_0 = x$, the probability that a jump occurs in $[0, h]$ is

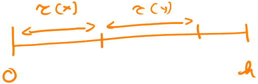
$$P(\tau(x) < h) = 1 - e^{-\lambda(x)h} \underset{\text{TAYLOR}}{=} \cancel{1} - \left(\cancel{1} - \lambda(x)h + \frac{(\lambda(x)h)^2}{2} + \dots\right) = \lambda(x)h + O(h^2).$$

From the above calculation, in particular we have

$$\lim_{h \to 0} \frac{P(\tau(x) < h)}{h} = \lambda(x) + \underbrace{\lim_{h \to 0} O(h)}_{=0} = \lambda(x).$$

The probability of two or more jumps in $[0,h]$ is $O(h^2)$. Here is why: let Y_0 and Y_1 denote the initial and the second visited states of the chain, respectively:

$$
\begin{aligned}
P(2 \text{ or more jumps in}[0,h] | Y_0 = x, Y_1 = y) &= P(\tau(x) + \tau(y) < h) \\
&\leq P(\tau(x) < h, \tau(y) < h) \\
&= P(\tau(x) < h) P(\tau(y) < h) \quad \text{(by ind.)} \\
&= \lambda(x)\lambda(y)h^2 + O(h^3) = O(h^2)
\end{aligned}
$$



$$
\begin{aligned}
&P(2 \text{ or more jumps in}[0,h] | Y_0 = x) = \\
&\qquad = \sum_{y \in S} P(2 \text{ or more jumps in}[0,h] | Y_0 = x, Y_1 = y) \cdot P(Y_1 = y | Y_0 = x) \\
&\qquad = \sum_{y \in S} O(h^2) P(Y_1 = y | Y_0 = x) = O(h^2)
\end{aligned}
$$

The last equality holds due to the dominated convergence theorem.

How many jumps in $[0, h]$ with h small

Take home message: the probability of one jump in one interval $[0, h]$ goes to zero *as* h. The probability of two or more jumps in $[0, h]$ goes to zero *as* h^2.

From transition probabilities to transition rates

Theorem

For any $i, j \in S$ (potentially $i = j$) we have

$$p_0'(i,j) = \frac{dp_t(i,j)}{dt}(0) = \lim_{h \to 0} \frac{p_h(i,j) - p_0(i,j)}{h} = q(i,j).$$

$$q(i,i) = -\lambda(i) = -\sum_{j \neq i} q(i,j)$$

From transition probabilities to transition rates

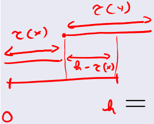
Proof.

If $x \neq y$,

$$\frac{dp_t(x,y)}{dt}(0) = \lim_{h\to 0}\frac{p_h(x,y) - \overbrace{p_0(x,y)}^{=0}}{h} = \lim_{h\to 0}\frac{p_h(x,y)}{h} = \lim_{h\to 0}\frac{P(X_h = y \mid Y_0 = X_0 = x)}{h}$$

$$= \lim_{h\to 0}\Bigg(\frac{P(X_h = y, \text{only one jump in } [0,h] \mid Y_0 = X_0 = x)}{h}$$

$$+ \frac{P(X_h = y, \text{two or more jumps in } [0,h] \mid Y_0 = X_0 = x)}{h}\Bigg)$$



$$= \lim_{h\to 0}\left(\frac{P(\tau(x) < h, Y_1 = y, \tau(y) > h - \tau(x) \mid Y_0 = X_0 = x)}{h} + \overbrace{\cancel{\frac{O(h^2)}{h}}}^{0}\right)$$

independence$\rightarrow$ $$= \lim_{h\to 0}\left(\boxed{\frac{P(\tau(x) < h)}{h}}^{\to \lambda(x)}\ \underbrace{\boxed{P(Y_1 = y \mid Y_0 = x)}}_{= r(x,y)}\ \underbrace{\boxed{P(\tau(y) > h - \tau(x) \mid \tau(x) < h)}}_{\geq P(\tau(y) > h) \xrightarrow{h\to 0} 1}\right)$$

$$= \lambda(x) \cdot r(x,y) \cdot 1 = q(x,y).$$

□

From transition probabilities to transition rates

Proof.

If $x = y$,

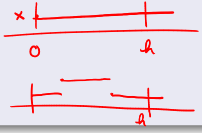
$$\lim_{h\to 0}\frac{p_h(x,x)-\overbrace{p_0(x,x)}^{=1}}{h} = \lim_{h\to 0}\frac{p_h(x,x)-1}{h} = \lim_{h\to 0}\frac{P(X_h = x \mid Y_0 = X_0 = x)-1}{h}$$

$$= \lim_{h\to 0}\Bigg(\frac{P(X_h = x, \underline{\text{no jumps in } [0,h]} \mid Y_0 = X_0 = x)-1}{h} + \frac{P(X_h = x, \underline{\text{two or more jumps in } [0,h]} \mid Y_0 = X_0 = x)}{h}\Bigg)$$

$$= \lim_{h\to 0}\left(\frac{P(\tau(x) > h \mid \cancel{Y_0 = X_0 =} x)-1}{h} + \cancelto{0}{\frac{O(h^2)}{h}}\right)$$

$$= \boxed{-\lambda(x)} = q(x,x).$$

2 CASI :



$$\frac{P(\tau(x) > h) - 1}{h} = \frac{\cancel{1} - P(\tau(x) < h) - \cancel{1}}{h} \to -\lambda(x)$$

□

Kolmogorov's equations (from trans. rates to trans. probs.)

$P_0'(i,j) = q(i,j)$ $\qquad$ $P_0' = Q$

If S is finite, the transition probabilities $p_t(x,y)$ form a finite matrix that we denote P_t. Remember that $Q = P_0'$ (component-wise) and that by the Ch-K eqns. $P_{t+h} = P_t P_h = \underline{P_h P_t}$. Therefore

P_0 $\quad$ P_0'

$$P_t' = \lim_{h\to 0} \frac{P_t P_h - P_t}{h} = P_t \cdot \boxed{\lim_{h\to 0} \frac{P_h - \tilde{I}}{h}} = P_t Q \qquad \Longrightarrow \quad \underline{P_t' = P_t Q}$$

$$P_t' = \lim_{h\to 0} \frac{P_h P_t - P_t}{h} = \boxed{\lim_{h\to 0} \frac{P_h - I}{h}} P_t = Q P_t \qquad \Longrightarrow \quad P_t' = Q P_t$$

that are called *forward* and *backword* Kolmogorov eqns. In components they read

$$p_t'(x,y) = \sum_{k\neq y} p_t(x,k)\underline{q(k,y)} - p_t(x,y)\underline{\lambda(y)}$$

$$p_t'(x,y) = \sum_{k\neq x} q(x,k)p_t(k,y) - \lambda(x)p_t(x,y)$$

Kolmogorov's equations (from trans. rates to trans. probs.)

- If S is infinite, but the chain is non explosive the equations in components are still valid and the series there appearing are convergent.
- In the explosive case, however, the backward equation still follows from the Ch-K eqns, but the solution is no longer unique and care has to be taken
- In the explosive case, moreover, the forward equation are not valid to describe $p_t(x, y)$ any more and the series there appearing might not converge.

Explosive chains are treated in details in Chung, *Markov Chains with Stationary Transition Probabilities*, Springer 1967. Whenever we apply the Kolmogorov eqns. (specially the forward one) we assume the non-explosivity of our chain.

Kolmogorov's equations (from trans. rates to trans. probs.)

We focus on Kolmogorov's backward eqn: $P_t' = QP_t$. In dimension 1 it reads $p_t' = qp_t$ with q a constant. We have the solution $p_t = p_0 e^{qt}$. Something similar happens in higher dimension: we can define the *exponential matrix*

$$e^{Qt} = \sum_{k=0}^{\infty} t^k \frac{Q^k}{k!},$$

where $Q^0 = I$. We have that $e^{Q0} = I = P_0$.

Theorem

Let $(X_t)_{t\in[0,\infty)}$ be a non-explosive CTMC. Then, for any $t \geq 0$

$$P_t = e^{Qt}.$$

Techniques are available to calculate e^{Qt} as a function of t, most symbolic computation software can do that for finite matrices.

Connectivity of CTMCs

Irreducibility

Definition

A CTMC is irreducible if for all $i, j \in S$ there exists $t > 0$ such that $p_t(i, j) > 0$.

Theorem

A non-explosive CTMC is irreducible if and only if so is the embedded DTMC.

Proof.

By definition of embedded DTMC. □

The explosive case

Consider the explosive pure birth CTMC with state space $S = \{1, 2, 3, \dots\}$ discussed ~~in the past lecture~~: BEFORE:

CTMC $\quad 1 \xrightarrow{1} 2 \xrightarrow{4} 3 \xrightarrow{9} 4 \xrightarrow{16} 5 \to \cdots$

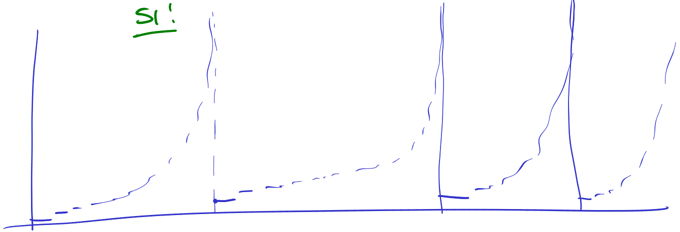
$$q(i, i+1) = i^2 \text{ for all } i \in S \text{ and } q(i,j) = 0 \text{ otherwise.}$$

eDTMC $\quad 1 \xrightarrow{1} 2 \xrightarrow{1} 3 \xrightarrow{1} 4 \xrightarrow{1} \cdots$

As a rule, set $X_{T_\infty} = 1$ at all explosion times.

Is the CTMC irreducible? Is the embedded DTMc irreducible?

SI!



$Y_0, Y_1, Y_2, \dots = 1, 2, 3, 4, 5, \dots$ DET.

NOT IRREDUCIBLE!

Aperiodicity

For CTMCs the notion of periodicity does not exist! We have

Lemma

If X_t is irreducible, then $p_t(i,j) > 0$ for any states i,j (potentially $i = j$) and ***for any*** *$t > 0$.*

Proof.

Assume the CTMC is non-explosive. Let $i = i_0, i_1, i_2, \ldots, i_{k-1}, j = i_k$ be a sequence of states with $r(i_n, i_{n+1}) > 0$, $0 \leq n \leq k-1$. Let $\tau_n \sim \exp(\lambda(i_n))$. Then

$$p_t(i,j) \geq \left(\prod_{n=0}^{k-1} P\left(\tau_n < \frac{t}{k}\right) r(i_n, i_{n+1}) \right) P(\tau_k > t) > 0$$

Similarly for the general case: exponentials can be as short as we need (formal details in the course material). □

Stationary distributions

Stationary distributions and measures

Let $(X_t)_{t=0}^{\infty}$ be a CTMC. We say that π is a stationary distribution if it is a probability distribution, and if for all $j \in S$ and all $t \geq 0$.

DTMC: $X_0 \sim \pi$
$X_n \sim \pi$
BASTAVA $X_0 \sim \pi \Rightarrow X_1 \sim \pi$
$\pi P = \pi$

$$P(X_t = j \mid X_0 \sim \pi) = \boxed{\overset{\pi P_t(j)}{\overset{\shortparallel}{\sum_{i \in S} \pi(i) p_t(i,j)}} = \pi(j)} \quad \forall t$$

$X_0 \sim \pi$

$\forall t \quad X_t \sim \pi$

Theorem

*Let $(X_t)_{t=0}^{\infty}$ be a **non-explosive** CTMC. Then, π is a stationary distribution if and only if $\pi Q = 0$ and π sums up to 1.*

Indeed, if π is stationary, $\pi = \pi P_t$. Taking derivative we have $\pi P_t' = 0$ and applying the forward Kolmogorov eqn. we conclude

SCAMBIO DI LIMITI...

$$\pi P_t' = \underbrace{\pi P_t}_{\pi} Q = \pi Q = 0.$$

We skipped the necessary technical details requires if S is infinite, but the result still holds.

On the other hand we have $\pi P_t' = \pi Q P_t$ by the backward eqn. If $\pi Q = 0$, then $\pi P_t' = 0$. Therefore πP_t is constant and $\pi P_t = \pi P_0 \overset{I}{=} \pi$

Finite irreducible chains

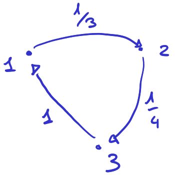
For a finite irreducible chain we can find a stationary distribution if we can solve the linear system

$$\begin{cases} \pi Q = 0 \\ \pi \cdot e = 1 \end{cases} \qquad e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Where e is a vector whose length is the same as that of π (the cardinality of S) and all entries are 1.

Example

Then, assume that the transition rate matrix is



$$Q = \begin{bmatrix} -1/3 & 1/3 & 0 \\ 0 & -1/4 & 1/4 \\ 1 & 0 & -1 \end{bmatrix}$$

Then, we can find the stationary distribution by solving $\pi Q = 0$, with the additional constraint that π sums to 1. No matter the strategy we use, in this case we obtain

$$\pi = \left(\frac{3}{8}, \frac{1}{2}, \frac{1}{8} \right).$$

Stationary measures CTMS vs. embedded DTMC

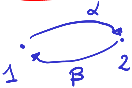
Cosider a two state CTMC with transition rate matrix

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

EXERCISE!

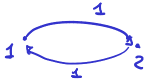
with $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$. Then, $\pi = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta} \right)$ is a stationary distribution of the CTMC. The corresponding embedded DTMC has transition matrix

(IF $\alpha, \beta > 0$)



$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

eDTMC



and its stationary distribution is $\tilde{\pi} = \left(\frac{1}{2}, \frac{1}{2} \right)$. Certainly the two differ. Is there any connections in general?

Stationary measures CTMS vs. embedded DTMC

Theorem

Consider a CTMC with no absorbing states. Then μ is a stationary measure of the embedded DTMC if and only if γ, defined entry-wise by

$$\gamma(j) = \frac{\mu(j)}{\lambda(j)},$$

HOLDING RATE

(where $\lambda(j)$ is as usual the rate of the holding time in j), satisfies $\gamma Q = 0$.

Stationary measures CTMS vs. embedded DTMC

Proof.

Since there are no absorbing states, $r(j,j) = 0$ for all j. μ is a stationary measure of the embedded chain iff $\mu P = \mu$, iff (since $\mu(j) = \gamma(j)\lambda(j)$),

$$(\mu P)(j) = \sum_{i \neq j} \underbrace{\gamma(i)\overbrace{\lambda(i)}_{\mu(i)} r(i,j)}^{q(i,j)} = \gamma(j)\lambda(j) = \mu(j)$$

for all $j \in S$. Using that $r(i,j) = q(i,j)/\lambda(i)$ we have that stationarity of μ is equivalent to

$$\sum_{i \neq j} \gamma(i) q(i,j) = \gamma(j)\underbrace{\lambda(j)}_{-Q(j,j)}$$

for all $j \in S$ and, the latter can be rewritten as

$$(\gamma \cdot Q)(j) = \left(\sum_{i \neq j} \gamma(i) Q(i,j) \right) + \gamma(j) Q(j,j) = 0$$

and it holds for any $j \in S$ if and only if $\gamma Q = 0$, concluding the proof. □

Stationary measures CTMS vs. embedded DTMC

Why is the previous theorem important?

If the CTMC is non-explosive and $\sum \gamma(i)$ is finite, then we can normalize γ to get a stationary distribution! We can also go in the opposite direction and calculate the stationary distributions (if any!) of the embedded DTMC from those of the CTMC.

Example

Consider again the 2-state CTMC with transition rate matrix

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

The stationary distribution of the embedded DTMC is $\tilde{\pi} = \left(\frac{1}{2}, \frac{1}{2}\right)$. How can we find the stationary distr. of the CTMC, without solving $\pi Q = 0$? Define

$$\gamma = \left(\frac{\tilde{\pi}(1)}{\lambda(1)}, \frac{\tilde{\pi}(2)}{\lambda(2)}\right) = \left(\frac{1}{2\alpha}, \frac{1}{2\beta}\right).$$

We know that $\gamma Q = 0$. However, γ is not a distribution. Define $M = \gamma(1) + \gamma(2) = \frac{\alpha+\beta}{2\alpha\beta}$ and let $\pi = \frac{1}{M}\gamma = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$. Then, π is a stationary distribution for the CTMC since $\pi Q = 0$ and $\pi(1) + \pi(2) = 1$ and the chain is not explosive.

Stationary measures CTMS vs. embedded DTMC

Corollary

An irreducible CTMC with finitely many states has a ***unique*** *stationary distribution* π. *Moreover, for every state* j *we have* $\pi(j) > 0$.

The embedded DTMC is irreducible and has finitely many states. Hence, it has a *unique* stationary distribution $\tilde{\pi}$. Moreover, $\tilde{\pi}(j) > 0$ for any j. We first show that π exists and then that is it unique:

Stationary measures CTMS vs. embedded DTMC

Existence: consider the vector γ defined by $\gamma(j) = \frac{\tilde{\pi}(j)}{\lambda(j)}$ for all j. Then we know that $\gamma Q = 0$. Moreover, $\sum_j \gamma(j) = M < \infty$ because the state space is finite. Hence, $\pi = \frac{1}{M}\gamma$ satisfies

$$\pi Q = \frac{1}{M}\gamma Q = 0 \quad \text{and} \quad \sum_j \pi(j) = \frac{1}{M}\sum_j \gamma(j) = 1.$$

Hence, since the continuous time Markov chain is finite and therefore non-explosive, π is a stationary distribution of the continuous time Markov chain. Moreover, for any state j

$$\pi(j) = \frac{1}{M}\gamma(j) = \frac{\mu(j)}{M\lambda(j)} > 0.$$

Stationary measures CTMS vs. embedded DTMC

Uniqueness: Let π and π' be two stationary distributions of the continuous time Markov chain. We want to prove that necessarily $\pi = \pi'$. Let

$$C = \sum_j \pi(j)\lambda(j) \quad \text{and} \quad C' = \sum_j \pi'(j)\lambda(j)$$

Since $\tilde{\pi}$ is unique, we have that for any state j

$$\tilde{\pi}(j) = \frac{\pi(j)\lambda(j)}{C} = \frac{\pi'(j)\lambda(j)}{C'},$$

which implies that for any state j, $\pi(j) = \frac{C'}{C}\pi'(j)$. If are able to show that $C'/C = 1$ we are done. But this follows from

$$1 = \sum_j \pi(j) = \sum_j \frac{C'}{C}\pi'(j) = \frac{C'}{C}\sum_j \pi'(j) = \frac{C'}{C}$$

so the proof is concluded.