Discrete-time Markov chains

We have seen:

- definition;
- Chapman-Kolmogorov equations;
- strong Markov property.

Classification of states, recurrence and transience

Let T_y be the time of the first visit to y, without counting X_0.

$$T_y = \min\{n \geq 1 : X_n = y\}$$

T_y is called **hitting time of** y, and if the chain starts in $X_0 = y$ **return time to** y.
T_y is a random variable expressing how many steps are needed to visit y.
Let

$$\rho_{xy} = P_x(T_y < \infty) = P(\text{we will visit } y \text{ again} | X_0 = x),$$

be the probability of returning to y in a finite time if we start at y.

There are two distinct types of states:

- y is recurrent if $\rho_{yy} = 1$;
- y is transient if $\rho_{yy} < 1$.

Classification of states, recurrence and transience

The names recurrent and transient are better justified by the following properties:

- Recurrent states are visited infinitely many times if the chain starts there.
- on the contrary, the number of visits to any transient state is finite. Therefore you will always find a large enough time such that after that time a transient state is never visited any more.

More on recurrence and communication

Theorem

Assume that x is recurrent and $x \Rightarrow y$. Then, y is recurrent, $P_x(T_y < \infty) = 1$, and with probability 1 y is visited infinitely many times if $X_0 = x$.

Proof.

- If x is recurrent, $x \Rightarrow y$ implies $y \Rightarrow x$. Since $x \Leftrightarrow y$, we have proved that both x, y need to be recurrent.
- Each time x is visited, there is a positive probability $P_x(T_y < T_x)$ of visiting y before returning to x. The number of times we return to x before visiting y is therefore an a.s. finite geometric random variable with parameter $P_x(T_y < T_x)$.
- Once we visit y, and we will do it a.s. with $X_0 = x$, we will come back to it infinitely many times a.s. for what shown in the previous class.

$\square$

Recurrence and expected number of visits

We defined

$$N(y) = \sum_{i=1}^{\infty} \mathbb{1}_{\{X_i = y\}}$$

as the total number of visits to y (without considering X_0). By monotone convergence

$$\mathbb{E}_x[N(y)] = \sum_{n=1}^{\infty} p^{(n)}(x, y)$$

Theorem

y is recurrent if and only if $\mathbb{E}_y[N(y)] = \infty$

Theorem

$$E_x[N(y)] = \begin{cases} 0 & \text{if } \rho_{xy} = 0 \\ \infty & \text{if } \rho_{xy} > 0 \text{ and } \rho_{yy} = 1 \\ \dfrac{\rho_{xy}}{1 - \rho_{yy}} & \text{if } \rho_{xy} > 0 \text{ and } \rho_{yy} < 1 \end{cases}$$

Finite state spaces and recurrence

Theorem

Consider a Markov chain with state space S.

1. *If $A \subset S$ is a finite absorbing communication class, all its states are recurrent*
2. *if S is finite, it can be partitioned as $S = T \cup R_1 \cup \cdots \cup R_k$, where T contains all the transient states and the R_i are absorbing communication classes (and therefore they only contain recurrent states)*
3. *if S is finite and the DTMC is irreducible, it is recurrent (that means that all states are recurrent)*

What about infinite state spaces?

Is it still true that states in an absorbing communication class are recurrent?

1d Random Walk

Consider the discrete time Markov chain on $S = \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ where for some $0 < p < 1$ we have

$$p(i, i+1) = p, \quad p(i, i-1) = q, \quad \text{with} \quad q = 1 - p.$$

It is irreducible, meaning that from any state we can move forward as we like, and backwards as we like. Every state communicates with every other. We claim the following:

1. all states are recurrent if $p = 1/2$;
2. all states are transient if $p \neq 1/2$.

Hence, in this example we see that if S is infinite, then the fact that S is irreducible does not imply recurrence

1d Random Walk, $p = 1/2$

Since it is an irreducible chain, either all state are recurrent or all states are transient. We prove that 0 is recurrent. showing that

$$\mathbb{E}_0[N(0)] = \sum_{n=1}^{\infty} p^{(n)}(0,0) = \infty$$

We have already seen that $p^{(n)}(0,0) \neq 0$ only if n is even. Moreover to be at zero again after $2n$ the process needs to have dome the same amount of steps (n) forward and backward. Hence

$$p^{(2n)}(0,0) = \binom{2n}{n} p^n q^n = \frac{(2n)!}{n!n!}(pq)^n.$$

1d Random Walk, $p = 1/2$

Now, the latter is a bit complicated. Stirling's formula states that

$$\frac{n!}{n^n e^{-n}\sqrt{2\pi n}} \xrightarrow[n\to\infty]{} 1.$$

Hence, for large n

$$p^{(2n)}(0,0) = \frac{(2n)!}{n!n!}(pq)^n \approx \frac{\sqrt{4\pi n}(2n)^{2n}e^{-2n}}{2\,\pi\,n\,n^{2n}\,e^{-2n}}(pq)^n = \frac{1}{\sqrt{\pi n}}(4pq)^n.$$

Now, remember that $p = 1/2$, so $4pq = 4(1/2)(1/2) = 1$. Hence, for n big enough, say for $n \geq M$,

$$p^{(2n)}(0,0) \approx \frac{1}{\sqrt{\pi n}} > \frac{1}{2\sqrt{\pi n}}.$$

Hence,

$$\mathbb{E}_0[N(0)] = \sum_{n=1}^{\infty} p^{(2n)}(0,0) \geq \sum_{n=M}^{\infty} \frac{1}{2\sqrt{\pi n}} = \infty,$$

1d Random Walk, $p \neq 1/2$

We have already seen that

$$E_0[N(0)] = \sum_{n=1}^{\infty} p^{(n)}(0,0) = \sum_{n=1}^{\infty} p^{(2n)}(0,0)$$

and for large n, say for $n \geq M$,

$$p^{(2n)}(0,0) \approx \frac{1}{\sqrt{\pi n}}(4pq)^n \leq (4pq)^n.$$

1d Random Walk, $p \neq 1/2$

If $p \neq 1/2$ the function $4pq = 4p(1-p)$ is strictly less than one (this function is a parabola with maximum 1, obtained at $p = 1/2$). Hence,

$$\sum_{n=1}^{\infty} p^{(2n)}(0,0) \leq \sum_{n=1}^{M-1} p^{(2n)}(0,0) + \sum_{n=M}^{\infty} (4pq)^n \leq M - 1 + \frac{1}{1-4pq} < \infty,$$

where in the second last passage we used that the series starting from $n = M$ is less than the series starting from $n = 0$, and we used the formula for geometric series.

Definition

A random walk in $\mathbb{Z}^n$ is given by the discrete-time process $(X(\ell))_{\ell=1}^{\infty}$ such that for any m

$$X(m) = (X_1(m), X_2(m), \dots, X_n(m))$$

where $(X_1(\ell))_{\ell=1}^{\infty}, (X_2(\ell))_{\ell=1}^{\infty}, \dots, (X_n(\ell))_{\ell=1}^{\infty}$ are i.i.d. 1d random walks.

We have (considering 0 as a vector in $\mathbb{Z}^d$)

$$\mathbb{E}_0[N(0)] = \sum_{n=1}^{\infty} p^{(n)}(0,0) = \sum_{n=1}^{\infty} p^{(2n)}(0,0) = \sum_{n=1}^{\infty} \left[\binom{2n}{n} p^n q^n\right]^d$$

$$\approx \sum_{n=1}^{\infty} \left[\frac{1}{\sqrt{\pi n}}\right]^d = \sum_{n=1}^{\infty} \left[\frac{1}{\pi n}\right]^{d/2}.$$

"A drunk man will always find his way home, but a drunk bird may get lost forever."

— Prof. Shizuo Kakutani, on the same behaviour of the Brownian motion

"A drunk man will always find his way home, but a drunk bird may get lost forever."

— Prof. Shizuo Kakutani, on the same behaviour of the Brownian motion

"That's why birds don't drink."

— Anonymous

Stationary distributions of Markov chains

Stationary distributions, limit distributions and asymptotic frequencies

In the next slides we are going to understand the relations between three distinct notions that we will see are strongly related one to each other

- A stationary distribution π is a distribution on S such that if $X_0 \sim \pi$, that means that the chain is initialized in state i with probability $\pi(i)$, then $X_1 \sim \pi$, and for every n, $X_n \sim \pi$
- A limiting distribution is a distribution π su that

$$\lim_{n\to\infty} p^{(n)}(x,y) = \pi(y)$$

  regardless of x.
- The asymptotic frequency of state y is the long run proportion of visits to y

$$\lim_{n\to\infty} \frac{N_n(y)}{n}$$

Stationary distributions

We have seen that we can find the probability distribution ν of X_1 if we know the initial probability distribution α and the transition matrix P, by calculating $\nu = \alpha \cdot P$. Therefore if we want to find a stationary distribution we need to solve the following algebraic equation

$$\pi = \pi \cdot P$$

and require that $\sum_i \pi(i) = 1$. In coordinates

$$\pi(i) = \sum_j \pi(j) \cdot p(j,i).$$

Hence, π is a left eigenvector of P corresponding to the eigenvalue 1.

We want to investigate conditions that may assure that a stationary distribution of a DTMC exists and is unique.

Example

Consider a DTMC with $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{pmatrix}$$

Find a stationary distribution.

Solution:

$$\pi = \left(\frac{22}{47}, \frac{16}{47}, \frac{9}{47} \right).$$

Example: 2-state DTMCs

Consider a DTMC with state space $S = \{1,2\}$ and transition matrix

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

(Exercise) Prove that if $a+b > 0$

$$\pi = \left(\frac{b}{a+b}, \frac{a}{a+b} \right)$$

is a stationary distribution, and it is the unique one. If $a+b=0$ then any probability vector is a stationary distribution.

Example: doubly stochastic matrices

A transition matrix P is called doubly stochastic, if besides $\sum_j p(i,j) = 1$ for all i, also for every j, $\sum_i p(i,j) = 1$. That means that both rows an columns sums up to 1.

(Exercise) Prove that for a DTMC with a $k \times k$ doubly stochastic transition matrix, the uniform distribution

$$\pi(i) = \frac{1}{k} \text{ for all } i$$

is a stationary distribution.

Stationary distributions and recurrence

Lemma

Let π be a stationary distribution. If $\pi(y) > 0$, then y is recurrent.

Proof.

We have (by monotone convergence)

$$\mathbb{E}_\pi[N(y)] = \sum_{n=1}^{\infty} \mathbb{E}_\pi[\mathbb{1}_{\{X_n=y\}}] = \sum_{n=1}^{\infty} \sum_{x\in S} p^{(n)}(x,y)\pi(x) = \sum_{n=1}^{\infty} \pi(y) = \infty.$$

$\square$

We will state a stronger result later...

We wonder whether the converse holds...

Stationary distributions and stationary measures

Finding a stationary distribution amounts at solving $\pi = \pi \cdot P$ and requiring $\sum_x \pi(x) = 1$. It might be not possible to normalize a solution of $\eta = \eta \cdot p$ if $\sum_x \eta(x) = \infty$.

Any vector η that solves $\eta = \eta \cdot p$ is called a stationary measure. If it is normalizable, it yields a stationary distribution.

Stationary distributions and stationary measures

Theorem

Suppose a Markov chain is irreducible and recurrent. Then there exist a stationary measure η with $\eta(x) > 0$ for all $x \in S$ (not necessarily finite!)

Proof.

For finite chains, it follows from Perron Frobenius theorem. In general, fix $x \in S$ and define η by

$$\eta(y) = \mathbb{E}_x\left[\sum_{n=1}^{\infty} \mathbb{1}_{\{X_n=y, T_x \geq n\}}\right] = \sum_{n=1}^{\infty} P_x(X_n = y, T_x \geq n),$$

the expected number of visits to y by time T_x, starting from $X_0 = x$. We can show that η is a stationary measure. Positivity is implyed by irreducibility (full proof in the course material). □

To guarantee the existence of a stationary distribution we need a stronger condition.

Positive recurrence

Definition

A state x is positive recurrent if $\mathbb{E}_x[T_x] < \infty$. A recurrent state that is not positive recurrent is called null recurrent.

Positive recurrence is a class property, i.e., if $x \leftrightarrow y$ and x is positive recurrent, so is y, because

$$\mathbb{E}_x[T_x] = P_x(T_y < T_x)\left(\mathbb{E}_x[T_y | T_y < T_x] + \mathbb{E}_y[T_x]\right) + P_x(T_y > T_x)\mathbb{E}_x[T_x | T_y > T_x],$$

hence $\mathbb{E}_y[T_x] < \infty$ and $\mathbb{E}_x[T_y | T_y < T_x] < \infty$. Let $G \sim \text{geom}(P_x(T_y < T_x))$, we get

$$\mathbb{E}_x[T_y] = \mathbb{E}[G]\mathbb{E}_x[T_x] + \mathbb{E}_x[T_y | T_y < T_x] < \infty.$$

Hence

$$\mathbb{E}_y[T_y] \leq \mathbb{E}_y[T_x] + \mathbb{E}_x[T_y] < \infty.$$

Stationary distributions and positive recurrence

Theorem

Suppose a Markov chain is irreducible and positive recurrent (i.e. all states are). Then there exist a stationary distribution π.

Proof.

We know

$$\eta(y) = \sum_{n=1}^{\infty} P_x(X_n = y, T_x \geq n)$$

is a stationary measure. Note that (by monotonicity)

$$\sum_{y \in S} \eta(y) = \sum_{y \in S} \sum_{n=1}^{\infty} P_x(X_n = y, T_x \geq n) = \sum_{n=1}^{\infty} \sum_{y \in S} P_x(X_n = y, T_x \geq n)$$
$$= \sum_{n=1}^{\infty} P_x(T_x \geq n) = \mathbb{E}_x[T_x] < \infty.$$

$\square$

1d random walk

Is the 1d random walk with $p = 1/2$ positive recurrent?

Asymptotic frequency and stationary distributions

Theorem (Asymptotic frequency)

Suppose a DTMC is irreducible and recurrent (i.e. all states are recurrent). Then, a.s.

$$\lim_{n\to\infty} \frac{N_n(y)}{n} = \frac{1}{\mathbb{E}_y[T_y]}.$$

where $1/\infty = 0$.

Positive recurrence and stationary distributions

Theorem

Suppose a DTMC is irreducible and a stationary distribution π exists. Then, π is uniquely determined by

$$\pi(y) = \frac{1}{\mathbb{E}_y[T_y]}$$

and all states are positive recurrent.

Positive recurrence and stationary distributions

Proof.

There is y with $\pi(y) > 0$, hence y is recurrent, hence all states are by irreducibility. Note that $N_n(y) \leq n$ so by dominated convergence

$$\lim_{n\to\infty} \mathbb{E}_\pi\left[\frac{N_n(y)}{n}\right] = \frac{1}{\mathbb{E}_y[T_y]}.$$

Now, note that

$$\mathbb{E}_\pi[N_n(y)] = \sum_{i=1}^{n}\sum_{x\in S} p^{(i)}(x,y)\pi(x) = n\pi(y).$$

Hence,

$$\frac{1}{\mathbb{E}_y[T_y]} = \lim_{n\to\infty} \mathbb{E}_\pi\left[\frac{N_n(y)}{n}\right] = \pi(y).$$

There is y with $\pi(y) > 0$, hence y is positive recurrent, hence all states are by irreducibility. $\square$

Comments

- The mean waiting time for the occurrence of an event is the inverse of its probability (cf. the mean of a geometric rv).
- The formula if often used to calculate $\mathbb{E}_y[T_y]$, that is often hard to calculate more directly.
- Combining with previous results, we can conclude

Corollary

Suppose a DTMC is irreducible. Then a stationary distribution π exists if and only if the states are positive recurrent, in which case π is unique and $\pi(y) > 0$ for all states y.

Finite chains

Irreducible finite chains are always positive recurrent, therefore they always allow for a stationary distribution, and this distribution is also unique.

Finite chains that are not irreducible The state space is partitioned in $S = T \cup R_1 \cup \cdots \cup R_k$. For each R_i we can find a stationary distribution $\pi_i(\cdot)$ with support equal to R_i. Therefore different stationary distributions exists and it can be proved that a distribution π is stationary if and only if it is a linear combination of the π_i.

Try to prove the results above yourself!

Ergodic theorem (asymptotic reward)

Theorem

Suppose a DTMC is irreducible and a stationary distribution π exists. Assume that for some function f, $\sum_x |f(x)|\pi(x) < \infty$. Then, a.s.

$$\frac{1}{n}\sum_{m=1}^{n} f(X_m) \to \sum_x f(x)\pi(x).$$

I.e. time averages (left) equal space averages in stationary regime (right).

You can interpret $f(i)$ as the reward (or cost) you get for visiting i.

The theorem is as a **generalization of the law of large numbers** for discrete random variables, when the random variables we are averaging are not necessarily independent.

Limit distributions

Assume that α is a limit distribution, meaning that α is a probability distribution and for any $x \in S$ we have

$$\lim_{n\to\infty} P_y(X_n = x) = \alpha(x).$$

Then, by dominated convergence

$$\alpha(x) = \lim_{n\to\infty} P(X_n = x) = \lim_{n\to\infty} \sum_{y\in S} p(y,x)P(X_{n-1} = y)$$
$$= \sum_{y\in S} p(y,x) \lim_{n\to\infty} P(X_{n-1} = y) = \sum_{y\in S} p(y,x)\alpha(y)$$

That is, we must have $\alpha = \alpha P$ and therefore a limit distribution must be a stationary distribution!

Limit distributions and periodicity

A limit distribution must be a stationary distribution, but the opposite implication does not hold. This is a counterexample. Consider a discrete time Markov chain with $S = \{1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This is a finite irreducible chain, therefore positive recurrent. We can see that $\pi = (\frac{1}{2}, \frac{1}{2})$ is a stationary distribution.

However a limit distribution cannot exist since if we start at state 1, at all even times we will be in 1 with probability one, while in all odd times we will be in state 2 with probability one.

Periodicity

The problem with the previous chain is that the chain is periodic.

Definition

The *period* of a state y is the Greatest Common Divisor

$$\operatorname{per}(y) = \gcd\{k \in \mathbb{N} : p^{(k)}(y,y) > 0\}.$$

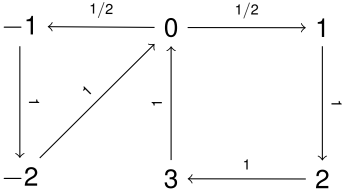
Theorem (Lemma 1.18 in Durrett's "Essentials of Stochastic Processes")

All states in the same communication class have the same period.

Definition

A state is aperiodic if its period is 1.

Periodicity, example



Limit distributions and periodicity

Theorem

Suppose a DTMC is irreducible, aperiodic (i.e. all states have period 1), and there is a stationary distribution π. Then, for all $x, y \in S$,

$$\lim_{n\to\infty} P(X_n = y | X_0 = x) = \lim_{n\to\infty} p^{(n)}(x,y) = \pi(y).$$

Total variation distance

Definition

Given two probabilities μ and ν on the same probability space $(\Omega, \mathcal{A})$, we define their total variation distance as

$$\|\mu - \nu\|_{TV} = 2 \sup_{A \in \mathcal{A}} |\mu(A) - \nu(A)|$$

GOAL: Prove that $\|p^{(n)}(x, \cdot) - \pi(\cdot)\|_{TV} \xrightarrow[n \to \infty]{} 0$.

Couplings

Definition

Given two probabilities μ and ν on the same probability space $(\Omega, \mathcal{A})$, a coupling of μ and ν is a random variable (X, Y) on $\Omega \times \Omega$ with $X \sim \mu$ and $Y \sim \nu$.

Theorem (Proposition 4.7 in Levin, Peres, Wilmer's "Markov Chains and Mixing Times")

Given two probabilities μ and ν on the same probability space $(\Omega, \mathcal{A})$,

$$\|\mu - \nu\|_{TV} = \min\{P(X \neq Y) : (X, Y) \textit{ is a coupling of } \mu \textit{ and } \nu\}.$$

Proof.

$$\mu(A) - \nu(A) = P(X \in A) - P(Y \in A) \leq P(X \in A, Y \notin A) \leq P(X \neq Y).$$

We can construct a coupling that attains the minimum (see the book). □

Coupling of DTMCs

Consider an irreducible, aperiodic DTMC on S with transition matrix P and stationary distribution π. Let $(X_n)_{n=0}^{\infty}$ and $(Y_n)_{n=0}^{\infty}$ be two *independent* DTMCs with state spae S and transition matrix P such that $X_0 = x$ and $Y_0 \sim \pi$.

Let $\tau = \min\{n \geq 1 : X_n = Y_n\}$ and define

$$\hat{X}_n = \begin{cases} X_n & \text{if } n < \tau \\ Y_n & \text{if } n \geq \tau \end{cases}.$$

Then, $(\hat{X}_n)_{n=0}^{\infty}$ is still a DTMCs with state space S and transition matrix P, and $\hat{X}_0 = X_0 = x$.

Hence, for any n the random variable $(\hat{X}_n, Y_n)$ is a coupling of $p^{(n)}(x, \cdot)$ and π, so

$$\|p^{(n)}(x,\cdot) - \pi(\cdot)\|_{TV} \leq P(\hat{X}_n \neq Y_n) = P(\tau > n).$$

Coupling of DTMCs

- We can prove the following lemma (the proof is in the course material):

Lemma

If the original DTMC is irreducible and aperiodic, then $((X_n, Y_n))_{n=0}^{\infty}$ is irreducible.

- $\tilde{\pi}(x, y) = \pi(x)\pi(y)$ is a stationary distribution for $((X_n, Y_n))_{n=0}^{\infty}$ (check it!). Hence, all states of $S \times S$ are positive recurrent.

Then,

$$\|p^{(n)}(x, \cdot) - \pi(\cdot)\|_{TV} \le P(\tau > n) \le P(T_{(x,x)} > n) \xrightarrow[n\to\infty]{} 0$$

because, by continuity of the probability measure,

$$\lim_{n\to\infty} P(T_{(x,x)} \le n) = P(T_{(x,x)} < \infty) = \sum_{y\in S} P_{(x,y)}(T_{(x,x)} < \infty)\pi(y) = \sum_{y\in S} \pi(y) = 1.$$

Detailed balanced distributions

Definition

A distribution π is said to be detailed balanced if for any $x, y \in S$

$$\pi(x)p(x,y) = \pi(y)p(y,x)$$

Summing over y,

$$\sum_{y \in S} \pi(y)p(y,x) = \sum_{y \in S} \pi(x)p(x,y) = \pi(x) \sum_{y \in S} p(x,y) = \pi(x),$$

hence a detailed balanced distribution is a **stationary distribution**. The converse does not hold in general.

Discrete time birth and death processes

Definition

A DTMC is called a (discrete time) birth and death process if the following conditions hold.

- The state space is a sequence of consecutive integers, bounded from below. That is either $S = \{x \in \mathbb{Z} : a \leq x \leq b\}$ for some integers $a < b$, or $S = \{x \in \mathbb{Z} : a \leq x\}$ for some integer a.
- The size of the allowed jumps is at most one, that is, for every $x, y \in S$

$$
\begin{aligned}
p(x,y) &= 0 \qquad \text{if } |x-y| > 1. \\
p(x,x+1) &= p_x \\
p(x,x-1) &= q_x \\
p(x,x) &= r_x \quad (\text{therefore } p_x + q_x + r_x = 1).
\end{aligned}
$$

E.g. the gambler model is a birth and death process.

Theorem

If a discrete time birth and death chain has a stationary distribution π, then π is detailed balanced. Namely $\pi(x)p_x = \pi(x+1)q_{x+1}$

Proof.

We prove it by induction. If π is a stationary distribution, then $\pi(a) = \pi(a)r_a + \pi(a+1)q_{a+1}$ and

$$\pi(a)(1-r_a) = \pi(a+1)q_{a+1}.$$

Since $q_a = 0$ then $p_a = 1 - r_a$ and

$$\pi(a)p_a = \pi(a+1)q_{a+1}.$$

$\square$

Discrete time birth and death processes

Theorem

If a discrete time birth and death chain has a stationary distribution π*, then* π *is detailed balanced. Namely* $\pi(x)p_x = \pi(x+1)q_{x+1}$

Proof.

Assume that $\pi(x)p_x = \pi(x+1)q_{x+1}$. Since π is a stationary distribution,

$$\pi(x+1) = \pi(x)p_x + \pi(x+1)r_{x+1} + \pi(x+2)q_{x+2}.$$

Hence,

$$\pi(x+1)p_{x+1} = \pi(x+1)(1 - r_{x+1} - q_{x+1}) = \pi(x+2)q_{x+2}.$$

□

Discrete time birth and death processes

This gives us an easier way to calculate the stationary distribution of a B&D chain, especially useful when we have infinite states: if a stationary distribution π exists, then

$$\pi(x) = \frac{p_{x-1}}{q_x}\pi(x-1) = \frac{p_{x-1}}{q_x}\frac{p_{x-2}}{q_{x-1}}\pi(x-2) = \cdots = \left(\prod_{i=a}^{x-1}\frac{p_i}{q_{i+1}}\right)\pi(a).$$

It follows that

$$1 = \sum_{x=a}^{b}\pi(x) = \pi(a)\sum_{x=a}^{b}\prod_{i=a}^{x-1}\frac{p_i}{q_{i+1}}$$

Hence we must have

$$\pi(a) = \frac{1}{M} \qquad \text{where } M = \sum_{x=a}^{b}\prod_{i=a}^{x-1}\frac{p_i}{q_{i+1}}$$

A stationary distribution for the B&D chain exists and is unique if and only if M is finite and greater than 0. Note that checking that $M > 0$ is never an issue, but in many cases M could be infinite.

1d partially reflected Random Walk

Consider a birth and death chain on the natural numbers $\mathbb{N}$ including 0, with transition probabilities

$$\begin{aligned}
p(x, x+1) &= p_x = p, \text{ for any } x \geq 0, \\
p(x, x-1) &= q_x = 1-p = q, \text{ for any } x \geq 1, \quad (\text{hence } r_x = 0) \\
p(0,0) &= r_0 = 1-p = q, \quad (\text{hence } q_0 = 0).
\end{aligned}$$

Is there a stationary distribution? Since this is a B&D Markov chain, we know that it has a unique stationary distribution if and only if

$$\sum_{x=0}^{\infty} \prod_{i=0}^{x-1} \frac{p_i}{q_{i+1}} < \infty.$$

1d partially reflected Random Walk

In our case,

$$\sum_{x=0}^{\infty}\prod_{i=0}^{x-1}\frac{p_i}{q_{i+1}}=\sum_{x=0}^{\infty}\prod_{i=0}^{x-1}\frac{p}{q}=\sum_{x=0}^{\infty}\left(\frac{p}{q}\right)^x$$

and the latter is finite if and only if $p<q$, that is if and only if $p<1/2$. In this case, a unique stationary distribution exists and it is given by

$$\pi(0)=\frac{1}{\sum_{x=0}^{\infty}\left(\frac{p}{q}\right)^x}=1-\frac{p}{q}$$

and

$$\pi(x)=\pi(0)\prod_{i=0}^{x-1}\frac{p_i}{q_{i+1}}=\left(1-\frac{p}{q}\right)\left(\frac{p}{q}\right)^x.$$

Hence, if $p<1/2$ then all the states are positive recurrent and we know what the stationary distribution is. Specifically, π is a *shifted geometric distribution* with success parameter p/q.

1d partially reflected Random Walk

What happens if $p \geq \frac{1}{2}$? Is the process transient or null recurrent?

1d partially reflected Random Walk

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