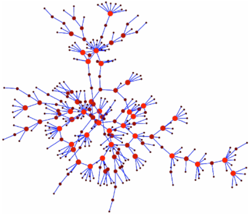
01RMHNG–03RMHPF–01RMING
Network Dynamics
Week 4
Network Flow Optimization



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This week

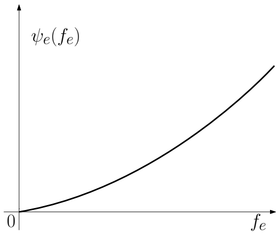
- Network Flow Optimization
- Lagrangian Techniques
- User Equilibrium vs System Optimum in Traffic Networks
- Price of Anarchy and Optimal Tolling

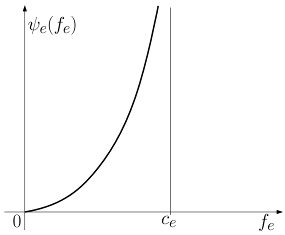
Network flows

- multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, node-link incidence matrix B
- exogenous net flow vector $\nu \in \mathbb{R}^{\mathcal{V}}$ such that $\sum_i \nu_i = 0$
- network flow is a vector $f \in \mathbb{R}_+^{\mathcal{E}}$ such that $Bf = \nu$
- typically several feasible solutions when problem feasible
- network flow optimization: selection of "best" flow

Network flow optimization

- convex nondecreasing cost functions $\psi_e(f_e)$ on every link $e \in \mathcal{E}$

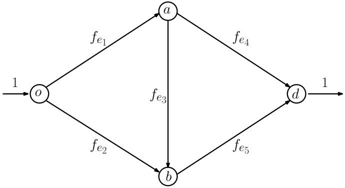


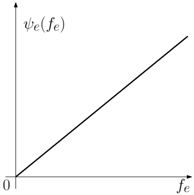


- capacity $c_e = \inf\{f_e \geq 0 : \psi_e(f_e) = +\infty\}$ either finite or infinte
- Network flow optimization problem: given exogenous net-flow ν

$$M(\nu) := \min_{\substack{f \in \mathbb{R}_+^{\mathcal{E}} \\ Bf = \nu}} \sum_{e \in \mathcal{E}} \psi_e(f_e)$$

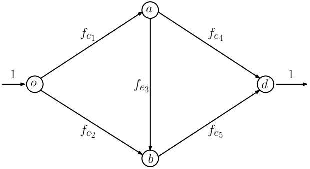
Example 1: Shortest path and optimal transport

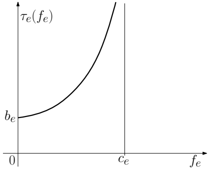


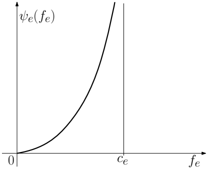


- $b_e =$ length of link $e > 0$
- link cost functions
$$\psi_e(f_e) = b_e f_e$$
- $\nu = \delta^{(o)} - \delta^{(d)}$
- if $\nu = \nu^+ - \nu^-$ with $\nu^+, \nu^- \in \mathbb{R}_+^{\mathcal{V}}$, optimal transport

Ex. 2: System-Optimum Traffic Assignment (SO-TAP)







- convex nondecreasing delay functions $\tau_e(f_e)$. E.g.,

$$\tau_e(f_e) = \frac{b_e}{1 - f_e/c_e}\,, \qquad e \in \mathcal{E}$$

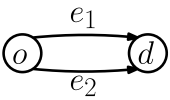
- corresponding convex nondecreasing cost

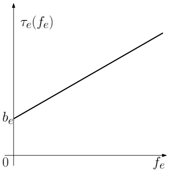
$$\psi_e(f_e) = f_e \cdot \tau_e(f_e)$$

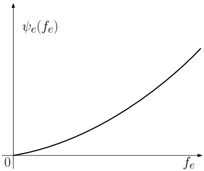
- SO-TAP:

$$\min_{\substack{f \in \mathbb{R}_+^{\mathcal{E}} \\ Bf = \nu}} \sum_{e \in \mathcal{E}} f_e \tau_e(f_e)$$

Ex. 2a: SO-TAP on two parallel links, affine delays







- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{o, d\}$, $\mathcal{E} = \{e_1, e_2\}$, where

$$\theta(e_1) = \theta(e_2) = o \qquad \kappa(e_1) = \kappa(e_2) = d$$

- let $\tau_e(f_e) = a_e f_e + b_e$, where $a_e > 0$ and $b_e > 0$, so that

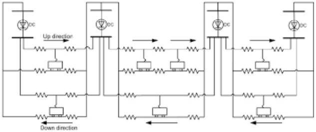
$$\psi_e(f_e) = f_e \cdot \tau_e(f_e) = a_e f_e^2 + b_e f_e$$

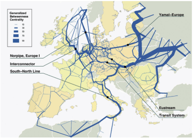
- for $\upsilon > 0$ solution of $\min\limits_{\substack{f_{e_1} \geq 0, f_{e_2} \geq 0 \\ f_{e_1} + f_{e_2} = \upsilon}} \{\psi_{e_1}(f_{e_1}) + \psi_{e_2}(f_{e_2})\}$ is

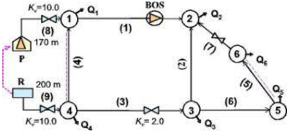
$$f_{e_1} = \upsilon - f_{e_2} = \begin{cases} 0 & \text{if} \quad \frac{b_2 - b_1}{2\upsilon} < -a_2 \\ \frac{2a_2\upsilon + b_2 - b_1}{2(a_1 + a_2)} & \text{if} \quad -a_2 \leq \frac{b_2 - b_1}{2\upsilon} \leq a_1 \\ \upsilon & \text{if} \quad \frac{b_2 - b_1}{2\upsilon} > a_1 \end{cases}$$

Example 3: Power dissipation

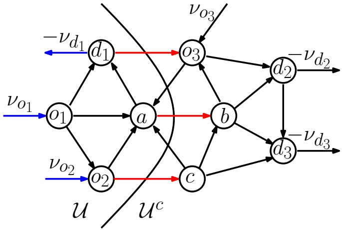
- Undirected $\mathcal{G} = (\mathcal{V}, \mathcal{E}, h)$, link conductances $h_e = h_{\overline{e}} > 0$
- Power dissipation on link e: $\psi_e(f_e) = \dfrac{f_e^{\alpha+1}}{(\alpha+1)h_e^{\alpha}}$
- Total power dissipation: $\displaystyle\sum_{e \in \mathcal{E}} \psi_e(f_e) = \sum_{e \in \mathcal{E}} \frac{f_e^{\alpha+1}}{(\alpha+1)h_e^{\alpha}}$
- $\alpha = 1 \quad \Rightarrow \quad$ direct current (DC) power networks
- $\alpha = 2 \quad \Rightarrow \quad$ gas networks
- $\alpha = 1/1.85 \simeq 0.54 \quad \Rightarrow \quad$ hydraulic networks







Feasibility



Proposition: On $\mathcal{G}$ with capacities c_e, network flow optimization

$$M(\nu) := \min_{\substack{f \in \mathbb{R}_+^{\mathcal{E}} \\ Bf = \nu}} \sum_{e \in \mathcal{E}} \psi_e(f_e)$$

is feasible if and only if

$$\sum_{i \in \mathcal{U}} \nu_i < c_{\mathcal{U}} \qquad \text{for every } \mathcal{U} \subseteq \mathcal{V} \text{ s.t. } \sum_{i \in \mathcal{U}} \nu_i > 0$$

Lagrangian techniques and duality

$$M(\nu) := \min_{\substack{f \in \mathbb{R}_+^{\mathcal{E}} \\ Bf = \nu}} \sum_{e \in \mathcal{E}} \psi_e(f_e)$$

- Lagrange multipliers λ_i for every node $i \in \mathcal{V}$
- Lagrangian function

$$L(f, \lambda, \nu) \;=\; \sum_{e \in \mathcal{E}} \psi_e(f_e) + \sum_{i \in \mathcal{V}} \lambda_i \left(\sum_{e \in \mathcal{E}:\kappa(e)=i} f_e - \sum_{e \in \mathcal{E}:\theta(e)=i} f_e + \nu_i \right)$$

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$$\begin{aligned} L(f, \lambda, \nu) &= \sum_{e \in \mathcal{E}} \psi_e(f_e) + \sum_{i \in \mathcal{V}} \lambda_i \left(\sum_{e \in \mathcal{E} : \kappa(e) = i} f_e - \sum_{e \in \mathcal{E} : \theta(e) = i} f_e + \nu_i \right) \\ &= \sum_{e \in \mathcal{E}} \left(\psi_e(f_e) - f_e(\lambda_{\theta(e)} - \lambda_{\kappa(e)}) \right) + \sum_{i \in \mathcal{V}} \lambda_i \cdot \nu_i \end{aligned}$$

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- Dual function (always concave in λ):

$$\begin{aligned} D(\lambda, \nu) &:= \inf_{f \in \mathbb{R}_+^{\mathcal{E}}} L(f, \lambda, \nu) \\ &= \sum_{e \in \mathcal{E}} \inf_{f_e \geq 0} \left\{ \psi_e(f_e) - f_e(\lambda_{\theta(e)} - \lambda_{\kappa(e)}) \right\} + \sum_{i \in \mathcal{V}} \lambda_i \cdot \nu_i \end{aligned}$$

Complementary slackness conditions

- Lemma: For convex nondecreasing cost $\psi_e : [0, +\infty) \to [0, +\infty]$

and pair of Lagrange multipliers $(\lambda_{\theta(e)}, \lambda_{\kappa(e)})$ in $\mathbb{R}^2$

$$f_e^* \in \operatorname*{argmin}_{f_e \geq 0} \left\{ \psi_e(f_e) - (\lambda_{\theta(e)} - \lambda_{\kappa(e)}) f_e \right\}$$

if and only if the following CS conditions hold true

$$f_e^* \geq 0 \qquad \psi_e'(f_e^*) \geq \lambda_{\theta(e)} - \lambda_{\kappa(e)} \qquad f_e^* \left(\psi_e'(f_e^*) - (\lambda_{\theta(e)} - \lambda_{\kappa(e)}) \right) = 0$$

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- **Remark**: if $\psi_e(x)$ is strictly convex on the interval $[0, c_e)$, and

$$\lambda_{\theta(e)} - \lambda_{\kappa(e)} < \sup\{\psi_e'(x) : x \in [0, c_e)\},$$

then the solution of CS is unique and given by

$$f_e^* = \begin{cases} 0 & \text{if} \quad \lambda_{\theta(e)} - \lambda_{\kappa(e)} \leq \psi_e'(0) \\ (\psi_e')^{-1}(\lambda_{\theta(e)} - \lambda_{\kappa(e)}) & \text{if} \quad \lambda_{\theta(e)} - \lambda_{\kappa(e)} > \psi_e'(0). \end{cases}$$

Lagrangian techniques and duality

Proposition: f^* in $\mathbb{R}_+^{\mathcal{E}}$ s.t. $Bf^* = \nu$ and λ^* in $\mathbb{R}^{\mathcal{V}}$ satisfy CS conditions on every e in $\mathcal{E}$ if and only if

(i) f^* is optimal solution of

$$M(\nu) := \min_{\substack{f \in \mathbb{R}_+^{\mathcal{E}} \\ Bf = \nu}} \sum_{e \in \mathcal{E}} \psi_e(f_e)$$

(ii) λ^* is an optimal solution of

$$M^*(\nu) = \max_{\lambda \in \mathbb{R}^{\mathcal{E}}} D(\lambda, \nu)$$

(iii) $M(\nu) = M^*(\nu)$

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- Application: to find f^*, solve CS as a function of λ, finding $f^*(\lambda)$, then impose conservation constraints

$$Bf^*(\lambda) = \nu\,.$$

Solution λ^* to this nonlinear system is such that $f^*(\lambda^*)$ is an optimal flow and λ^* is an optimal solution of dual problem.

Lagrangian techniques and duality

- note that

$$D(\lambda, \nu) = \sum_i \lambda_i \cdot \nu_i - \sum_e \psi_e^*(\lambda_{\theta(e)} - \lambda_{\kappa(e)})$$

where the dual cost on link $e \in \mathcal{E}$

$$\psi_e^*(y_e) = \sup_{f_e \geq 0} \{y_e f_e - \psi_e(f_e)\}$$

is the Fenchel transform of the cost $\psi_e(f_e)$ and represents the maximum profit that a link operator can make if it charges y_e per unit of flow and pays $\psi_e(f_e)$ to transport f_e units

Lagrangian techniques and duality

Theorem: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, nonincreasing costs $\psi_e(f_e)$, capacities c_e

(i) optimal network flow problem feasible, i.e., $M(\nu) < +\infty$, if and only if $\sum_{i \in \mathcal{U}} \nu_i < c_{\mathcal{U}}$, for every $\mathcal{U} \subset \mathcal{V}$ s.t. $\sum_{i \in \mathcal{U}} \nu_i > 0$.

If $\psi_e(f_e)$ convex differentiable on $[0, +\infty)$, then for all ν as above

(ii) flow vector f^* optimal $\Leftrightarrow$ satisfies CS conditions for Lagrange multipliers $\lambda = \lambda^*$ that solve dual optimization problem

$$M^*(\nu) = \max_{\lambda \in \mathbb{R}^{\mathcal{V}}} D(\lambda, \nu)$$

(iii) if optimal cost $M(\nu)$ is differentiable in ν, then

$$\frac{\partial}{\partial \nu_i} M(\nu) - \frac{\partial}{\partial \nu_j} M(\nu) = \lambda_i^* - \lambda_j^*$$

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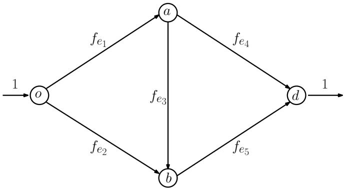
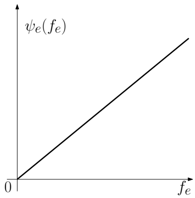
$$\frac{\partial}{\partial \nu_i} M(\nu) - \frac{\partial}{\partial \nu_j} M(\nu) = \lambda_i^* - \lambda_j^*$$

▶ Application (of (ii)): find λ^* solution of dual optimization problem (which is convex and unconstrained), then solve CS to find $f^*(\lambda^*)$. Such $f^*(\lambda^*)$ is an optimal flow.

Example – complementary slackness for shortest path





- cost functions $\psi_e(f_e) = b_e f_e$ where $b_e =$ length of link $e > 0$
- $\nu = \delta^{(o)} - \delta^{(d)}$
- CS conditions yield

$$b_e \begin{cases} = \lambda_{\theta(e)} - \lambda_{\kappa(e)} & \text{if } f_e^* > 0 \\ \geq \lambda_{\theta(e)} - \lambda_{\kappa(e)} & \text{if } f_e^* = 0 \end{cases} \qquad \forall e \in \mathcal{E}$$

- For every $o - d$ path $\gamma = (e_1, \ldots, e_l)$

$$\sum_{j=1}^{l} b_{e_j} \begin{cases} = \lambda_o - \lambda_d & \text{if } f_{e_j}^* > 0 & \text{for all } 1 \leq j \leq l \\ \geq \lambda_o - \lambda_d & \text{if } f_{e_j}^* = 0 & \text{for some } 1 \leq j \leq l \end{cases}$$

Example – complementary slackness for power dissipation

$$\psi_e(f_e) = \frac{f_e^{\alpha+1}}{(\alpha+1)h_e^\alpha} \qquad e \in \mathcal{E}$$

- For $f_e \geq 0$, $\psi'_e(f_e) = f_e^\alpha / h_e^\alpha$
- complementary slackness conditions yield

$$f_e^* = h_e[\lambda_{\theta(e)} - \lambda_{\kappa(e)}]_+$$

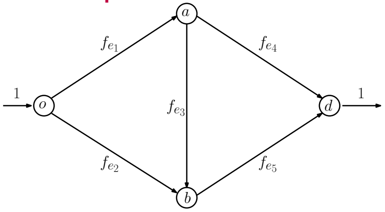
- for DC power networks ($\alpha = 1$), let

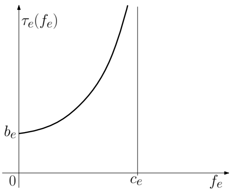
$$W_{ij} = \sum_{e \in \mathcal{E}: \theta(e)=i, \kappa(e)=j} h_e$$

then the net-flow z_{ij} from node i to node j is

$$z_{ij} = W_{ij}(\lambda_i - \lambda_j) \qquad \Longleftrightarrow \qquad \text{Ohm's law}$$

System Optimum Traffic Assignment (SO-TAP)





- convex nondecreasing delay functions $\tau_e(f_e)$. E.g.,

$$\tau_e(f_e) = \frac{b_e}{1 - f_e/c_e}\,, \qquad e \in \mathcal{E}$$

- corresponding convex nondecreasing cost

$$\psi_e(f_e) = f_e \cdot \tau_e(f_e)$$

- SO-TAP:

$$\min_{\substack{f \in \mathbb{R}_+^{\mathcal{E}} \\ Bf = \nu}} \sum_{e \in \mathcal{E}} f_e \cdot \tau_e(f_e)$$

When drivers choose their route

- Drivers, total amount υ, can choose different o-$d$paths
- Γ_{od} the set of all o-d paths
- Link-path incidence matrix $A \in \{0,1\}^{\mathcal{E} \times \Gamma_{od}}$

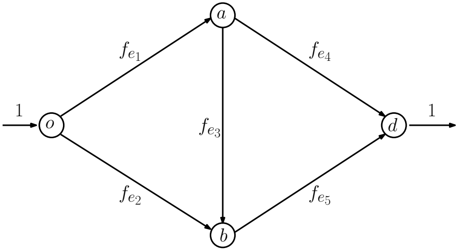
$$A_{e\gamma} = \begin{cases} 1 & \text{if link } e \text{ is along path } \gamma \\ 0 & \text{if link } e \text{ is not along path } \gamma \end{cases}$$

- Path flow $z \in \mathbb{R}^{\Gamma_{od}}$, $\mathbb{1}'z = \upsilon$, $z \geq 0$
- Recall that

$$(BA)_{i\gamma} = \begin{cases} +1 & \text{if } i = o \\ -1 & \text{if } i = d \\ 0 & \text{if } i \neq o, d \end{cases}$$

- Link flow $f = Az$

Example – Link-path incidence matrix



$$A = \begin{array}{c@{}c} \begin{array}{ccc} \gamma_1 & \gamma_2 & \gamma_3 \end{array} & \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} & \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \end{array}$$

$$B = \begin{array}{c@{}c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} & \\ \begin{bmatrix} +1 & +1 & 0 & 0 & 0 \\ -1 & 0 & +1 & +1 & 0 \\ 0 & -1 & -1 & 0 & +1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} & \begin{array}{c} o \\ a \\ b \\ d \end{array} \end{array}$$

Wardrop equilibrium

- Wardrop equilibrium $f^{(0)}$: The flow vector

$$f^{(0)} = Az$$

where $z \in \mathbb{R}^{\Gamma_{od}}$ is such that $z \geq 0$, $\mathbb{1}'z = \upsilon$, and for $\gamma \in \Gamma_{od}$

$$z_\gamma > 0 \quad \Longrightarrow \quad \underbrace{\sum_{e \in \mathcal{E}} A_{e\gamma} \tau_e(f_e^{(0)})}_{\substack{\text{total delay} \\ \text{on path } \gamma}} \quad \leq \quad \underbrace{\sum_{e \in \mathcal{E}} A_{e\tilde{\gamma}} \tau_e(f_e^{(0)})}_{\substack{\text{total delay} \\ \text{on path } \tilde{\gamma}}} \qquad \forall \tilde{\gamma} \in \Gamma_{od}$$

- Interpretation: drivers choose their fastest path

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where $z \in \mathbb{R}^{\Gamma_{od}}$ is such that $z \geq 0$, $\mathbb{1}'z = \upsilon$, and for $\gamma \in \Gamma_{od}$

$$z_\gamma > 0 \quad \Longrightarrow \quad \underbrace{\sum_{e \in \mathcal{E}} A_{e\gamma} \tau_e(f_e^{(0)})}_{\substack{\text{total travel time} \\ \text{on path } \gamma}} \quad \leq \quad \underbrace{\sum_{e \in \mathcal{E}} A_{e\tilde{\gamma}} \tau_e(f_e^{(0)})}_{\substack{\text{total travel time} \\ \text{on path } \tilde{\gamma}}} \qquad \forall \tilde{\gamma} \in \Gamma_{od}$$

- Interpretation: drivers choose their fastest path

- Proposition: Wardrop equilibrium $=$ solution of UO-TAP

$$\min_{\substack{f \in \mathbb{R}_+^{\mathcal{E}} \\ Bf = \upsilon(\delta^{(o)} - \delta^{(d)})}} \quad \sum_{e \in \mathcal{E}} \int_0^{f_e} \tau_e(s) \mathrm{d}s$$

Wardrop equilibrium

- Proposition: Wardrop equilibrium = solution of UO-TAP

$$\begin{array}{cc} \min & \displaystyle\sum_{e\in\mathcal{E}} \int_0^{f_e} \tau_e(s)\mathrm{d}s \\ f \in \mathbb{R}_+^{\mathcal{E}} & \\ Bf = \upsilon\big(\delta^{(o)} - \delta^{(d)}\big) & \end{array}$$

- Proof:

Price of Anarchy

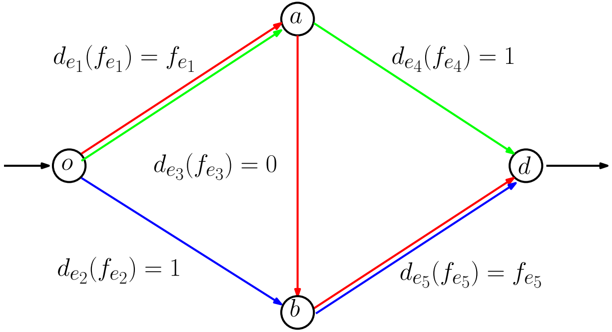
- price of anarchy of Wardrop equilibrium $f^{(0)}$ is

$$\mathrm{PoA}(0) = \frac{\sum_{e\in\mathcal{E}} f_e^{(0)} \tau_e(f_e^{(0)})}{\min\limits_{\substack{f\in\mathbb{R}_+^{\mathcal{E}} \\ Bf=\upsilon(\delta^{(o)}-\delta^{(d)})}} \sum_{e\in\mathcal{E}} f_e \tau_e(f_e)},$$

total delay at the Wardrop equilibrium / total delay at system optimum

- Observe: $\mathrm{PoA}(0) \geq 1$

Example - Braess paradox



- Three paths γ_1, γ_2, γ_3
- Wardrop equilibrium: $f^{(0)} = (1, 0, 1, 0, 1)$
- Social optimum: $f^* == (1/2, 1/2, 0, 1/2, 1/2)$

Toll design

- Vector of tolls: $\omega = (\omega_e)_{e \in \mathcal{E}}$
- Perceived cost $= \omega_e + \tau_e(f_e)$ on link e
- Wardrop equilibrium with tolls

$$f^{(\omega)} = Az$$

where $z \in \mathbb{R}^{\Gamma_{od}}$ is such that $z \geq 0$, $\mathbb{1}'z = \upsilon$, and for $\gamma \in \Gamma_{od}$

$$z_\gamma > 0 \Longrightarrow \underbrace{\sum_{e \in \mathcal{E}} A_{e\gamma} \left(\tau_e(f_e^{(\omega)}) + \omega_e \right)}_{\substack{\text{total perceived} \\ \text{cost on path } \gamma}} \leq \underbrace{\sum_{e \in \mathcal{E}} A_{e\tilde{\gamma}} \left(\tau_e(f_e^{(\omega)}) + \omega_e \right)}_{\substack{\text{total perceived} \\ \text{cost on path } \tilde{\gamma}}} \quad \forall \tilde{\gamma}$$

- Can we find ω s.t. PoA(ω)$= 1$?

Toll design

- Theorem: For nondecreasing $\tau_e(f_e)$, convex $f_e \cdot \tau_e(f_e)$

Wardrop equilibrium with tolls ω satisfies

$$f^{(\omega)} = \underset{\substack{f \in \mathbb{R}_+^{\mathcal{E}} \\ Bf = \upsilon(\delta^{(o)} - \delta^{(d)})}}{\operatorname{argmin}} \sum_{e \in \mathcal{E}} \left(\int_0^{f_e} \tau_e(s) \mathrm{d}s + \omega_e f_e \right)$$

- Corollary: With marginal cost tolls

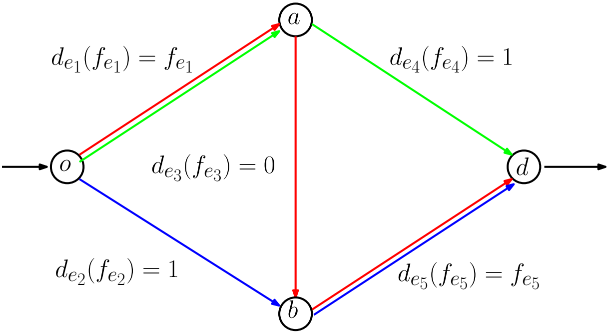
$$\omega_e^* = f_e^* \cdot \tau_e'(f_e^*)$$

computed at social optimum $f^* = \underset{Bf = \upsilon(\delta^{(o)} - \delta^{(d)})}{\operatorname{argmin}} \sum_{e \in \mathcal{E}} f_e \cdot \tau_e(f_e)$

$$f^{(\omega^*)} = f^*$$

- marginal cost tolls $\Leftrightarrow$ "internalize negative externality"

Example – Toll design



- Tolls $\omega_1^* = \omega_5^* = 1/2$, $\omega_2^* = \omega_3^* = \omega_4^* = 0$
- Wardrop equilibrium with tolls: $f^{(\omega^*)} = (0.5, 0.5, 0, 0.5, 0.5)$
- Social optimum: $f^* = (0.5, 0.5, 0, 0.5, 0.5)$
- $\exists$ other optimal toll choices, e,g., $\omega = (0, 0, \beta, 0, 0)$, $\beta \geq 1/2$