Stationary measures CTMS vs. embedded DTMC

Theorem

Consider a CTMC with no absorbing states. Then μ is a stationary measure of the embedded DTMC if and only if γ, defined entry-wise by

$$\gamma(j) = \frac{\mu(j)}{\lambda(j)},$$

(where $\lambda(j)$ is as usual the rate of the holding time in j), satisfies $\gamma Q = 0$.

Corollary

An irreducible CTMC with finitely many states has a ***unique*** *stationary distribution* π. *Moreover, for every state* j *we have* $\pi(j) > 0$.

Existence of a stationary distribution

When does a stationary distribution exist, if the state space is infinite?

For DTMCs, the concept of recurrence and positive recurrence was crucial.

DTMCs: $\quad j$ REC. $\iff P_j(T_j < \infty) = 1$

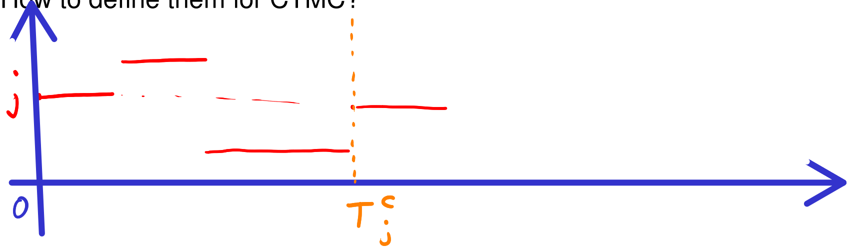
$\quad j$ POS REC. $\iff E_j[T_j] < \infty$

Return times

DTMC: $T_j^D = \inf\{ n \geq 1 : X_n = j \}$

CTMC: $T_j^C \cancel{=} \inf\{ t > 0 : X_t = j \}$

How to define them for CTMC?



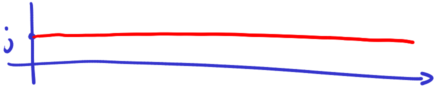
$$\inf\{ t > 0 : X_t = j \} = 0$$

$$P_j\big(\underbrace{\inf\{ t > 0 : X_t = j \}}_{=0} < \infty \big) = 1$$

In continuous time if $X_0 = j$, then X_t will stay in state j for a positive amount of time. To account for this fact, the concept of return time needs to be redefined as follows

$$\begin{aligned} T_j^c &= \min\{t \geq 0 : X_t \text{ enters in } j \text{ from another state}\} \\ &= \min\{t \geq 0 : X_t = j \text{ and } X_s \neq j \text{ for some } 0 \leq s < t\} \end{aligned}$$

IF j ABSORBING



$T_j^c = \infty$

Remark: with the given definition the return time to an absorbing state is infinite.

Return times and recurrence

Definition

A state j of a CTMC is recurrent if either $P_j(T_j^c < \infty) = 1$ or if j is absorbing.

As for DTMCs,

Lemma

If i is recurrent and $p_t(i,j) > 0$, then j is recurrent.

Lemma

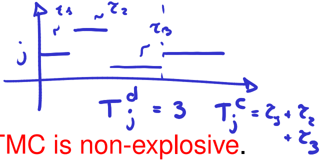
If a stationary distribution π exists and $\pi(j) > 0$, then j is recurrent.

Proofs in the course material.

Return times and recurrence

For non-explosive CTMCs the return time T_j^c is related to the return time T_j^d of the embedded DTMC as follows. Let us define τ_n the $n-th$ waiting time of the CTMC, that is the time elapsed between the $(n-1)$th and the nth jump. We have $\tau_n \sim \mathrm{Exp}(\lambda(Y_{n-1}))$, and hence it is a.s. finite. We have

$$T_j^c = \sum_{n=1}^{T_j^d} \tau_n \ ,$$



therefore T_j^c is finite if and only if T_j^d is finite if the CTMC is non-explosive.

Theorem

Let $(X_t)_{t=0}^{\infty}$ be a non-explosive CTMC. Then, j is a transient (resp. recurrent) state for the CTMC $(X_t)_{t=0}^{\infty}$ if and only if it is a transient (resp. recurrent) state for the embedded DTMC $(Y_n)_{n=0}^{\infty}$.

FINITE STATES : REC. STATES FOR CTMC ARE
(NON-EXPLOSIVE) THOSE IN ABSORBING. C.C.

Positive and null recurrence

Definition

Let $(X_t)_{t=0}^{\infty}$ be a CTMC, and let j be a **recurrent** state. We say that

- j is positive recurrent if $E_j[T_j^c] < \infty$ or if j is absorbing;
- j is null recurrent if it is not positive recurrent: that is, if $E_j[T_j^c] = \infty$ and j is not absorbing;

As for DTMCs, positive recurrence and null recurrence are class properties:

Theorem

If i is positive recurrent and $p_t(i,j) > 0$, then j is positive recurrent.

Proof in the course material.

Asymptotic frequency

$N_t(j) =$ TEMPO IN $[0, t]$ SPESO IN j

$= \int_0^t \mathbb{1}\{X_s = j\}\, ds$

The asymptotic freq. of state j is defined as the fraction of time spent therein:

$$\lim_{t\to\infty} \frac{N_t(j)}{t} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{1}_{\{j\}}(X_s) ds.$$

DTMC: $\lim_{n \to +\infty} \frac{N_n(j)}{n} \longrightarrow \frac{1}{E_j[T_j^D]}$

DIM: RENEWAL PROCESS.

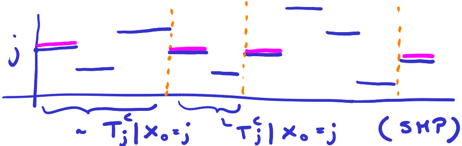
Theorem

Let $(X_t)_{t=0}^{\infty}$ be an irreducible CTMC, with arbitrary initial distribution. To avoid dealing with absorbing states (trivial), assume that the state space has at least 2 states. Then, for any $j \in S$,

$$\lim_{t\to\infty} \frac{N_t(j)}{t} = \frac{1}{E_j[T_j^c]\lambda(j)} \qquad a.s.$$

PROOF (SKETCH):

RENEWAL REWARD.



RENEWAL EVENT : VISITS TO j

$\tau_i \sim \exp(\lambda(j))$

$$\lim_{t\to+\infty} \frac{R(t)}{t} = \lim_{t\to+\infty} \frac{N_t(j)}{t} = \frac{1/\lambda(j)}{E_j[T_j^c]}$$

Positive recurrence and stationary distributions

DTMC : $\pi(j) = \dfrac{1}{E_j[T_j^D]}$

Theorem

Let $(X_t)_{t=0}^{\infty}$ be an irreducible CTMC. We assume that S contains at least two states. If π is a stationary distribution, then π is uniquely determined by

$$\pi(j) = \frac{1}{E_j[T_j^c]\lambda(j)}.$$

and all states are positive recurrent.

Positive recurrence and stationary distributions

VERY SIMILAR TO THAT OF DTMCS

Proof.

There is j with $\pi(j) > 0$, hence j is recurrent, hence all states are by irreducibility. Note that $N_t(j) \leq t$ so by dominated convergence

$$\lim_{t\to\infty} \mathbb{E}_\pi \left[\frac{N_t(j)}{n}\right] = \frac{1}{\mathbb{E}_j[T_j^c]\lambda(j)}.$$

Now, note that

$$\mathbb{E}_\pi[N_t(j)] = \int_0^t \mathbb{E}_\pi[\mathbb{1}_{\{j\}}(X_s)]ds = \int_0^t P_\pi(X_s = j)ds = t\pi(j).$$

Hence,

$$\frac{1}{\mathbb{E}_j[T_j^c]\lambda(j)} = \lim_{t\to\infty} \mathbb{E}_\pi \left[\frac{N_t(j)}{t}\right] = \pi(j).$$

There is j with $\pi(j) > 0$, hence j is positive recurrent, hence all states are by irreducibility. □

Positive recurrence and stationary distributions

As for DTMCs, we have the following.

Theorem

Suppose a CTMC is irreducible. Then a stationary distribution π exists if and only if the states are positive recurrent, in which case $\pi(j) > 0$ for all states j.

The proof is in the course material, and it is almost identical to that for DTMCs.

VALID FOR EXPLOSIVE CHAINS TOO!

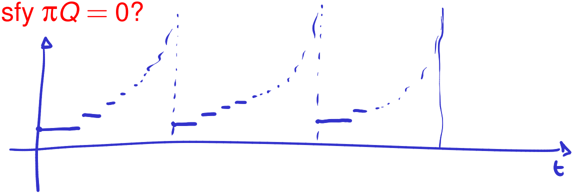
Exotic Example

Consider again the pure birth CTMC with state space $S = \{1, 2, 3, \dots\}$,

$$q(i, i+1) = i^2 \text{ for all } i \in S \text{ and } q(i,j) = 0 \text{ otherwise.}$$

We set $X_{T_\infty} = 1$ at all explosion times.

Are the state positive recurrent? What is the stationary distribution π? Does it satisfy $\pi Q = 0$?



EXPLOSIVE

IRRIDUCIBILE

eDTMC $\quad Y_0 = 1 \quad Y_n = 1 + n \quad$ TRANSIENT.

$P_1(T_1^c < \infty) = P_1(T_\infty < \infty) = 1$ $\rightsquigarrow$ 1 IS RECURRENT

$E_1[T_1^c] \overset{\text{MONOTONE}}{=} \sum_{i=1}^{\infty} E[\exp(i^2)] = \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$ $\rightsquigarrow$ 1 IS POS. REC.

$\rightsquigarrow$ ALL STATES ARE

Exotic Example

$$\pi(j) = \frac{1}{E_j[T_j^c] \cdot \lambda(j)} =$$

$$E_j[T_j^c] \overset{\text{MONOTONE}}{=} \sum_{i=j}^{+\infty} E[\exp(i^2)] + \sum_{i=1}^{j-1} E[\exp(i^2)]$$

$$= \sum_{i=1}^{+\infty} E[\exp(i^2)] = \sum_{i=1}^{+\infty} \frac{1}{i^2} = M < \infty$$

$$\pi(j) = \frac{1}{M} \cdot \frac{1}{j^2}$$

$$\pi \cdot Q \neq 0 \quad : \quad (\pi \cdot Q)(1) = \sum_{i=1}^{+\infty} \pi(i) \cdot Q(i,1) = \pi(1) \cdot Q(1,1)$$

$$= \pi(1) \cdot (-\lambda(1)) = -\pi(1)$$

$$= 1/M \neq 0$$

Another explosion detection technique

CTMC EXPLOSIVE $\Rightarrow$ eDTMC TRANSIENT

$\not\Leftarrow$

PROCESSO DI POISSON : CTMC NON EXPLOSIVE

Corollary

CTMCs with transient embedded DTMC and a distribution γ satisfying $\gamma Q = 0$ are explosive.

Proof.

If the CTMC were non-explosive, then:

- transience of the embedded DTMC would imply transience of the CTMC;
- γ would be a stationary distribution.

However, we know that in an irreducible chain there is a stationary distribution if and only if all the states are positive recurrent. Therefore, we have a contradiction! The CTMC is explosive. $\square$

Asymptotic reward (ERGODIC THEOREM)

DTMC. $\frac{1}{n}\sum_{m=1}^{n} f(X_m) \xrightarrow[n\to+\infty]{} \sum_{j\in S} f(j)\pi(j)$

Theorem

Let $(X_t)_{t=0}^{\infty}$ be an irreducible CTMC, with arbitrary initial distribution. Assume there exists a stationary distribution π. Then, for any function $f: S \to \mathbb{R}$ satisfying

$$\sum_{j\in S} |f(j)|\pi(j) < \infty$$

we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X_s)ds = \sum_{j\in S} f(j)\pi(j) = E_\pi[f(X)] \quad a.s.$$

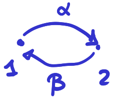
That is the time average converges to the space average, computed over π.

Limit distributions

Limit distribution, example

Consider a 2-state continuous time Markov chain with state space $S = \{1,2\}$ and transition rate matrix

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix},$$



with $\alpha, \beta > 0$. We have seen that the stationary distribution for this model is given by

$$\pi = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta} \right).$$

The transition probabilities are given by

$$P_t = e^{Qt} = \begin{pmatrix} \frac{\beta}{\alpha+\beta} + \overbrace{\boxed{\frac{\alpha}{\alpha+\beta} e^{-(\alpha+\beta)t}}}^{\xrightarrow[t\to+\infty]{} 0} & \frac{\alpha}{\alpha+\beta}\left(1 - \overbrace{\boxed{e^{-(\alpha+\beta)t}}}^{\xrightarrow[t\to+\infty]{} 0}\right) \\ \frac{\beta}{\alpha+\beta}\left(1 - \underbrace{\boxed{e^{-(\alpha+\beta)t}}}_{\xrightarrow[t\to+\infty]{} 0}\right) & \frac{\alpha}{\alpha+\beta} + \underbrace{\boxed{\frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)t}}}_{\xrightarrow[t\to+\infty]{} 0} \end{pmatrix}$$

$P_t(i,j) = P(X_t = j \mid X_0 = i) \qquad i,j = 1,2$

Limit distribution, example

it follows that

$$P_t \xrightarrow[t\to\infty]{} \begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix} = \begin{pmatrix} \pi(1) & \pi(2) \\ \pi(1) & \pi(2) \end{pmatrix}$$

Specifically,

$$\lim_{t\to\infty} p_t(i,1) = \pi(1) \quad \text{for all } i \in \{1,2\}$$

$$\lim_{t\to\infty} p_t(i,2) = \pi(2) \quad \text{for all } i \in \{1,2\}$$

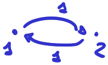
CTMC: $\exists t : p_t(i,j) > 0 \iff p_t(i,j) > 0 \ \forall t$

Hence, the transition probabilities converge to the stationary distribution!

REMARK: the embedded DTMC is periodic, therefore it does not have a limit distribution.

$$p_n(1,1) = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

NOT CONVERGING!!!



Limit distribution

Since periodicity was the only thing preventing an irreducible DTMC from converging, we expect that convergence will occur more often in the continuous time setting. This is true!

Theorem

If a CTMC $(X_t)_{t=0}^{\infty}$ is irreducible and has stationary distribution π, then for any $i, j \in S$, we have $\lim_{t\to\infty} p_t(i,j) = \pi(j)$.

However, to prove the existence of a stationary distribution γ we need to take care of potential explosions (that do not exist in DTMC). It is not enough to check that $\gamma Q = 0$ and $\sum_i \gamma(i) = 1$, but we also need to know that the chain is not explosive!

Limit distribution, proof

It is identical to that for the discrete case. In fact, the lack of periodicity makes it easier. Try to prove it as an exercise!

Continuous time birth and death chains

Continuous time birth and death chains

A continuous time birth and death chain is a continuous time Markov chain whose embedded DTMC is a birth and death chain.

Specifically, the state space is

$$S = \{a, a+1, a+2, \dots, b\} \quad \text{or} \quad S = \{a, a+1, a+2, \dots\}$$

The transition rates (called "birth rate" and "death rate") are given by

$$\begin{aligned} q(j, j+1) &= \lambda_j \qquad \text{for } a \leq j < b \\ q(j, j-1) &= \mu_j \qquad \text{for } a < j \leq b, \end{aligned}$$

with $\lambda_j, \mu_j > 0$ for any state j, with b potentially equal to ∞. We further define $\mu_a = 0$ and $\lambda_b = 0$. The above notation may lead to confusion, as we denote the rate of the holding time in j by $\lambda(j)$. Note that in this case,

$$\lambda(j) = \lambda_j + \mu_j.$$

Continuous time birth and death chains

The embedded DTMC in this case is given by the birth death chain on S, with transition probabilities

$$\begin{aligned} r(j, j+1) &= \frac{\lambda_j}{\lambda_j + \mu_j} && \text{for } a \le j < b \\ r(j, j-1) &= \frac{\mu_j}{\lambda_j + \mu_j} && \text{for } a < j \le b, \end{aligned}$$

We know that μ is a stationary measure for the embedded DTMC **if and only if**

$$\mu(j) = \kappa \prod_{i=a}^{j-1} \frac{r(i, i+1)}{r(i+1, i)},$$

for some constant $\kappa \ge 0$. In this case, we have (most of the factors cancel)

$$\mu(j) = \kappa \prod_{i=a}^{j-1} \frac{\lambda_i}{\lambda_i + \mu_i} \frac{\lambda_{i+1} + \mu_{i+1}}{\mu_{i+1}} = \kappa \frac{\lambda_j + \mu_j}{\lambda_a} \prod_{i=a}^{j-1} \frac{\lambda_i}{\mu_{i+1}}.$$

Hence, $\gamma Q = 0$ if and only if $\gamma(j) = \frac{\mu(j)}{\lambda_j + \mu_j}$ for some stationary measure μ of the embedded chain. Hence, $\gamma Q = 0$ **if and only if**

$$\gamma(j) = \frac{\kappa}{\lambda_a} \prod_{i=a}^{j-1} \frac{\lambda_i}{\mu_{i+1}} = \kappa' \prod_{i=a}^{j-1} \frac{\lambda_i}{\mu_{i+1}}$$

for some constant $\kappa' \geq 0$.

Theorem

A CT B&D chain admits a non-negative γ with $\gamma Q = 0$ and $\sum_{j=a}^{b} \gamma(j) = 1$ if and only if

$$M = \sum_{j=a}^{b} \prod_{i=a}^{j-1} \frac{\lambda_i}{\mu_{i+1}} < \infty,$$

where we set

$$\prod_{i=a}^{a-1} \frac{\lambda_i}{\mu_{i+1}} = 1.$$

DT B&D

STAZIONARIA

$\Updownarrow$

$$M = \sum_{j=a}^{b} \prod_{i=a}^{j-1} \frac{P(i,i+1)}{P(i+1,i)} < \infty$$

If this is the case, then γ is given by

$$\gamma(j) = \frac{1}{M} \prod_{i=a}^{j-1} \frac{\lambda_i}{\mu_{i+1}},$$

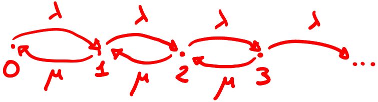
for all $a \leq j \leq b$ (remember that b can be ∞). Finally, if such γ exists and if the CT B&D chain is non-explosive, then γ is the unique stationary distribution.

$\lambda(0) = \lambda \qquad \lambda(i) = \lambda + \mu \qquad i > 0$

Consider a CTMC with state space $S = \{0, 1, 2, \dots\}$ and transition rates

$$q(j, j+1) = \lambda \qquad \text{for } j \geq 0, \quad q(j, j-1) = \mu \qquad \text{for } j > 0,$$

and $q(i, j) = 0$ otherwise. We assume $\lambda < \mu$.



$\lambda < \mu$

$$r(i, i+1) = \frac{\lambda}{\lambda + \mu} < \frac{\mu}{\lambda + \mu} = r(i, i-1)$$

Example 1

Consider a CTMC with state space $S = \{0, 1, 2, \dots\}$ and transition rates

$$q(j, j+1) = \lambda \qquad \text{for } j \geq 0, \quad q(j, j-1) = \mu \qquad \text{for } j > 0,$$

and $q(i,j) = 0$ otherwise. We assume $\lambda < \mu$. We have:

- the CTMC is non-explosive because $\max_j \lambda(j) = \lambda + \mu$ is finite;
- we have

$$M = \sum_{j=0}^{\infty} \prod_{i=0}^{j-1} \frac{\lambda_i}{\mu_{i+1}} = \sum_{j=0}^{\infty} \boxed{\prod_{i=0}^{j-1} \frac{\lambda}{\mu}} = \sum_{j=0}^{\infty} \boxed{\left(\frac{\lambda}{\mu}\right)^j} \overset{\lambda < \mu}{=} \frac{1}{1 - \lambda/\mu} < \infty.$$

$\pi Q = 0$

Hence, there exists a unique stationary distribution and it is given by $\pi \cdot e = 1$

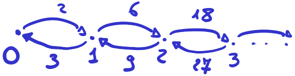
$$\pi(j) = \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^j.$$

$\pi(j) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j$

Example 2

Consider the CTMC with state space $S = \{0, 1, 2, \dots\}$ and transition rates

$$q(j, j+1) = 2 \cdot 3^j \qquad \text{for } j \geq 0 \quad \text{and} \quad q(j, j-1) \quad = 3^j \qquad \text{for } j > 0.$$



We have

$$M = \sum_{j=0}^{\infty} \prod_{i=0}^{j-1} \frac{2 \cdot \cancel{3^i}}{3^{\cancel{i}+1}} = \sum_{j=0}^{\infty} \overset{\left(\frac{2}{3}\right)^j}{\boxed{\prod_{i=0}^{j-1} \frac{2}{3}}} = \sum_{j=0}^{\infty} \left(\frac{2}{3}\right)^j = \frac{1}{1 - 2/3} = 3. < \infty$$

Define γ by

$$\gamma(j) = \frac{1}{M} \boxed{\prod_{i=0}^{j-1} \frac{2 \cdot 3^i}{3^{i+1}}} = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^j$$

for all $j \in \{0, 1, 2, \dots\}$. We know that $\gamma Q = 0$ and $\gamma e = 1$. However, we don't know whether this is a stationary distribution yet, because we don't know whether the chain is explosive.

Example 2

The embedded DTMC has transition probabilities

$\lambda(j) = 2 \cdot 3^j + 3^j = 3 \cdot 3^j = 3^{j+1}$

$j \geq 1$

$$r(j, j+1) = \frac{2 \cdot \cancel{3^j}}{3 \cdot \cancel{3^j}} = \frac{2}{3} \quad \text{for } j \geq 1$$

$p > \frac{1}{2}$

$$r(j, j-1) = \frac{\cancel{3^j}}{3 \cdot \cancel{3^j}} = \frac{1}{3} \quad \text{for } j \geq 1$$

eDTMC TRANSIENTE

$$r(0,1) = \frac{2 \cdot 3^j}{2 \cdot 3^j} = 1$$

Hence, the embedded DTMC is a random walk reflected at zero with $p = 2/3$, and we know that it is transient. If the CTMC were non-explosive, then:

- transience of the embedded DTMC would imply transience of the CTMC;
- γ would be a stationary distribution.

However, we know that in an irreducible chain there is a stationary distribution if and only if all the states are positive recurrent. Therefore, we have a contradiction! The CTMC is explosive.

Detailed balanced distributions

Detailed balanced distributions

DTMC: DET. BAL. $\pi(i)\cdot p(i,j) = \pi(j)\, p(j,i)$

We define detailed balanced distributions in the continuous time framework:

Definition

Let X_t be a CTMC. A distribution π is said to be detailed balanced if for all states $i, j \in S$

$$\pi(i)q(i,j) = \pi(j)q(j,i).$$

$$\begin{tikzcd} i \arrow[r, bend left, "\pi(i)\cdot q(i,j)"] & j \arrow[l, bend left, "\pi(j)q(j,i)"] \end{tikzcd}$$

We have the following,

Theorem

If the CTMC is non-explosive and π is a detailed balanced distribution, then π is a stationary distribution.

DTMC: DET. BAL. $\Rightarrow$ STAT. DISTR.

Detailed balanced distributions

Proof.

We simply need to sum. Take the sum over the states i, different from j:

DET. BAL.

$$\sum_{\substack{i\in S\\ i\neq j}} \pi(i)q(i,j) = \sum_{\substack{i\in S\\ i\neq j}} \pi(j)q(j,i) = \pi(j)\lambda(j) = -\pi(j)Q(j,j),$$

where we used the detailed balanced condition for the first equality. By bringing all the terms to the left-hand side we have

$$\sum_{i\in S} \pi(i)Q(i,j) = 0, \qquad \forall j \quad \rightsquigarrow \quad \pi\cdot Q = 0$$

which is $(\pi Q)(j) = 0$. Since this holds for any j, we have

$$\pi Q = 0.$$

Since π is a distribution by assumption, then $\pi e = 1$. Hence, if the chain is non-explosive we have that π is a stationary distribution. □

Birth and death chains and detailed balanced distributions

DTMC: DT B&D π STAZ $\Rightarrow$ π DET. BAL.

We have the following result, similar to what we have for DTMCs.

Theorem

If a non-explosive continuous time birth and death chain has a stationary distribution π, then π is detailed balanced.

Proof.

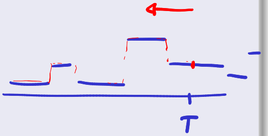
We already know the shape of π, we just need to check it is detailed balanced. $\square$

What happens if we observe a CTMC in reverse time? Do we still have the Markov property? If so, what are the transition rates?

Theorem

Let $(X_t)_{t=0}^{\infty}$ be a non-explosive CTMC with stationary distribution π. If the chain is in stationary regime (that is, $X_0 \sim \pi$), then

$$Z_t = X_{T-t} \qquad \text{for } t \leq T$$



is a CTMC with transition probabilities

$$\hat{p}_t(i,j) = \frac{\pi(j) p_t(j,i)}{\pi(i)} \qquad \text{for } t \leq T$$

Reverse time

Proof: Consider a sequence of times $0 \le t_1 < \cdots < t_n < t < t+h \le T$ and a sequence of states $i_1, i_2, \dots, i_n, i, j$. We have:

$$
\begin{aligned}
P(Z_{t+h} = j | Z_t = i, Z_{t_1} = i_1, \dots, Z_{t_n} = i_n) &= \frac{P(Z_{t+h} = j, Z_t = i, Z_{t_1} = i_1, \dots, Z_{t_n} = i_n)}{P(Z_t = i, Z_{t_1} = i_1, \dots, Z_{t_n} = i_n)} \\
&= \frac{P(X_{T-t-h} = j, X_{T-t} = i, X_{T-t_1} = i_1, \dots, X_{T-t_n} = i_n)}{P(X_{T-t} = i, X_{T-t_1} = i_1, \dots, X_{T-t_n} = i_n)} \\
&= \frac{\boxed{\pi(j)}\boxed{p_h(j,i)}\cancel{P(X_{T-t_1} = i_1, \dots, X_{T-t_n} = i_n | X_{T-t} = i)}}{\boxed{\pi(i)}\cancel{P(X_{T-t_1} = i_1, \dots, X_{T-t_n} = i_n | X_{T-t} = i)}} \\
&= \frac{\pi(j) p_h(j,i)}{\pi(i)} = \frac{P(X_{T-t-h} = j, X_{T-t} = i)}{P(X_{T-t} = i)} \qquad (\star) \\
&= P(X_{T-t-h} = j | X_{T-t} = i) = \underline{P(Z_{t+h} = j | Z_t = i)}
\end{aligned}
$$

Hence, Z_t is a CTMC. Moreover, the transition probabilities are the desired ones.

Reverse time

Corollary

Let $(X_t)_{t=0}^{\infty}$ be a non-explosive *CTMC with stationary distribution π. If the chain is in stationary regime (that is, $X_0 \sim \pi$), then*

$$Z_t = X_{T-t} \qquad \text{for } t \leq T$$

is a CTMC with rates

$$\hat{q}(i,j) = \frac{\pi(j)q(j,i)}{\pi(i)}.$$

IF DET. BAL.

$$= \frac{\cancel{\pi(i)} \cdot q(i,j)}{\cancel{\pi(i)}} = q(i,j)$$

Proof.

Simply take derivatives of $\hat{p}_t(i,j)$. □

If X_t has a detailed balanced stationary distribution, then the reversed time chain $Z_t = X_{T-t}$ has the same distribution as the original X_t.

Historical remark

The nature of particles is naturally symmetric with respect to time: in a very naive approximation, particles can be thought to as “balls” colliding with each other, more or less like balls on a pool table. if you watch a movie of balls on a pool table hitting each other, you cannot say whether the time is reversed or not, because the rules governing the determination of the new angle after the collision do not change.

On the other hand, we know that in physical systems “the entropy always increases”. Nowadays we know how to make sense of this, however when Boltzmann first introduced the concept of entropy in 1872, he got most of the scientific community against him: if we can reverse time and still obtain the same system, how can we possibly say that the entropy increases? If it were so, it would also increase when the time is reversed, which does not make sense. Moreover, by the Poincare ergodicity theorem it was known that every state of a system would be visited again with probability 1. In particular, a state with a smaller entropy level can be visited. Isn’t this in contrast with a theory that states that the entropy always increases?

Historical remark

We know how to make sense of all of this now:

- the time can be reversed only if the system is in a stationary regime, so at equilibrium. Otherwise, we can guess what is the true direction of the time (think of a pool table and imagine to start with all the balls in the triangle: it is very unlikely to obtain exactly this ball configuration again).
- the statement “the entropy always increases” can be simply thought as “the system converges to its stationary distribution”. In physics, systems in stationary regimes are often with particles as much spread as possible, so with maximum entropy.
- the Poincare ergodicity theorem can be thought as “all the system states are recurrent”. This is not in contrast with the fact that we have a stationary distribution. Recurrent states will be visited infinitely many times with probability one, but of course some of them are so unlikely that the time needed to visit them could be greater than the age of the universe.

Historical remark

So, no mystery anymore: with what you have learned in this course, you can understand what goes on. However, note that what seems natural to us now, was cause of a great deal of debate in history. In particular, Boltzmann theories were attacked up to and after his death, and someone argues that they were one of the causes that led him to commit suicide (even though he had other problems as well). If you want to know more details about this discussions, you can for example take a look at “Zermelo, Boltzmann, and the recurrence paradox” by Vincent Steckline.