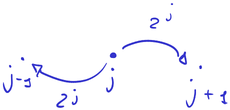
Positive and null recurrence of CTMCs vs embedded DTMCs

We have seen that a non-explosive irreducible CTMC is recurrent if and only if its embedded DTMC is recurrent. What about positive recurrence?

CTMC and DTMC relationship

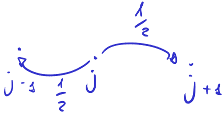
Consider the CTMC with state space $S = \{0, 1, 2, 3, \dots\}$ and transition rates

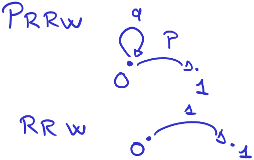
$$q(j, j+1) = 2^j \qquad \text{for } j \geq 0$$
$$q(j, j-1) = 2^j \qquad \text{for } j > 0.$$



Let us take a look at the embedded DTMC! It has transition probabilities

$$r(j, j+1) = \frac{1}{2} \qquad \text{for } j > 0$$
$$r(j, j-1) = \frac{1}{2} \qquad \text{for } j > 0$$
$$r(0, 1) = 1.$$





That is, the embedded DTMC is a ~~Partially~~ Reflected RW, with $p = 1/2$. We know that it is null recurrent, in particular it is not transient. Hence the CTMC is non-explosive.

CTMC and DTMC relationship

Consider again the CTMC with rates

$$q(j, j+1) = 2^j \quad \text{for } j \geq 0 \quad \text{and} \quad q(j, j-1) \quad = 2^j \quad \text{for } j > 0.$$

We have that for any $j \geq 0$

$$M = \sum_{j=0}^{\infty} \prod_{i=0}^{j-1} \frac{\lambda_i}{\mu_{i+1}} = \sum_{j=0}^{\infty} \prod_{i=0}^{j-1} \frac{2^i}{2^{i+1}} = \sum_{j=0}^{\infty} \prod_{i=0}^{j-1} \frac{1}{2} = \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j = 2, \quad < \infty$$

which is clearly finite (we used the geometric series in the last step). Hence, the CTMC admits a non-negative γ with $\gamma Q = 0$ and $\sum_{j=a}^{b} \gamma(j) = 1$, which is

$$\gamma(j) = \frac{1}{M} \prod_{i=0}^{j-1} \frac{2^i}{2^{i+1}} = \left(\frac{1}{2}\right)^{j+1}.$$

We know that this CTMC is non-explosive. Hence, the γ we found is the unique stationary distribution!

CTMC and DTMC relationship

Consider again the CTMC with rates

$$q(j, j+1) = 2^j \qquad \text{for } j \geq 0 \quad \text{and} \quad q(j, j-1) \qquad = 2^j \qquad \text{for } j > 0.$$

The model is non-explosive and the stationary distribution is given by

$$\pi(j) = \left(\frac{1}{2}\right)^{j+1}$$

for any $j \geq 0$. Hence all states are positive recurrent. However, the embedded DTMC is a Partially Reflected Random Walk with $p = 1/2$, which is null recurrent.

What is happening here? The expected number of steps it takes to come back to zero is still infinity, but since the holding times are shorter and shorter for larger states, the time it takes to come back to 0 ends up having finite expectation.

The converse can also happen! EXERCISE!

Markovian queues

Markovian Queues

In several practical problems it is important to model a queue (bank, supermarkets, hospitals, call centers, processing of tasks in a parallel computing architecture...). Real queues may be very complicated, but simple models of queues can help understanding the phenomena.

Queues may well be non Markovian: people may arrive in groups or may be attracted (or discouraged) by the fact that a long queue is present. E.g. if at a restaurant there is people waiting some other people may think it is because food is particularly good, and may want to stop and taste that food. The arrival process of new customers is not Poisson in those cases.

In real queues there might be different service policies, E.g. in a hospital the first served is often not the first come, but the one with highest priority

Queues

There are universal labels for referring to some specific basic queues. The labels are given by the following sequence of 5 objects:

arrival distribution / service distribution / number of servers / capacity / service discipline

1. **arrival distribution.** It can be any of the following symbols (there are more): M (Markovian), D(deterministic), G(general) ...
2. **service distribution.** Same options
3. **number of servers.** A natural number or ∞
4. **capacity.** It is the maximum number allowed in queue. Further arrivals are rejected
5. **service discipline.** It can be FIFO, LIFO, priorities, random ...

M/M/1 M/M/∞
M/M/s

The state space is $S = \{0, 1, 2, 3, \dots\}$ and a state indicates the number of individuals in the queue including those who are being served.

M/M/1

For the M/M/1 queue model, we have $S = \{0, 1, 2, 3, \dots\}$. Since there is only one server, the transition rates are

$$\begin{aligned} q(j, j+1) &= \lambda \qquad \text{for } j \geq 0 \\ q(j, j-1) &= \mu \qquad \text{for } j \geq 1. \end{aligned}$$

The model is a non-explosive continuous time B&D process, for any choice of $\lambda, \mu > 0$. The reason is that the holding rates

$$\lambda(0) = \lambda \qquad \lambda(j) = \lambda + \mu \text{ for } j \geq 1$$

are bounded. A stationary distribution exists if and only if

$$M = \sum_{j=0}^{\infty} \prod_{n=0}^{j-1} \frac{q(n, n+1)}{q(n+1, n)} < \infty.$$

M/M/1

Here,

$$M = \sum_{j=0}^{\infty} \prod_{n=0}^{j-1} \frac{q(n,n+1)}{q(n+1,n)} = \sum_{j=0}^{\infty} \boxed{\prod_{n=0}^{j-1} \frac{\lambda}{\mu}} = \sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^{j},$$

is finite if and only if $\lambda < \mu$.
In this case,

$$M = \frac{1}{1 - \lambda/\mu}$$

and the unique stationary distribution for $(X_t)_{t=0}$ is given by

$$\pi(j) = \frac{\gamma(j)}{M} = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{j} \qquad \text{for all } j \geq 0,$$

which is a shifted geometric distribution.

M/M/1

A non-explosive CTMC is recurrent (or transient) if and only if its embedded DTMC is recurrent (or transient). In this case, the embedded chain is the birth and death chain with transition probabilities

$$r(0,1) = 1$$
$$r(j,j+1) = \frac{\lambda}{\lambda+\mu} \qquad \text{for } j \geq 1$$
$$r(j,j-1) = \frac{\mu}{\lambda+\mu} \qquad \text{for } j \geq 1.$$

By the analogy with the PRRW (the proofs are identical) we can conclude that this DTMC is recurrent if and only if

$$\frac{\lambda}{\cancel{\lambda+\mu}} \leq \frac{\mu}{\cancel{\lambda+\mu}},$$

that is if and only if $\lambda \leq \mu$. Hence, $(X_t)_{t=0}^{\infty}$ is recurrent if and only if $\lambda \leq \mu$ (and transient if and only if $\lambda > \mu$).

Now, we have two cases to consider:

- If $\lambda < \mu$, then there exists a stationary distribution. Together with the fact that the chain is irreducible, this implies that $(X_t)_{t=0}^{\infty}$ is positive recurrent.
- If $\lambda = \mu$, then no stationary distribution exists. Since the chain is recurrent, it can only be null recurrent.

M/M/1

To sum up:

The M/M/1 queue is always non-explosive. Moreover:

- if $\lambda < \mu$, then all states are positive recurrent and the unique stationary distribution is

$$\pi(j) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j \qquad \text{for all } j \geq 0;$$

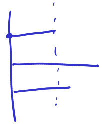
- if $\lambda = \mu$, then all states are null recurrent;
- if $\lambda > \mu$, then all states are transient.

M/M/s

If there are s servers, each of them with service rate μ, then the transition rates of the model are given by

$$q(j, j+1) = \lambda \qquad \text{for } j \geq 0$$

$$q(j, j-1) = \begin{cases} j\mu & \text{for } 1 \leq j \leq s \\ s\mu & \text{for } j > s \end{cases}$$



$\sim exp(\mu)$

$\sim exp(\mu)$

$\sim exp(\mu)$

$\min \sim exp(3\mu)$

The model can't explode, for any choice of $\lambda, \mu > 0$, since the holding rates $\lambda(j) \leq \lambda + s\mu$ are bounded. it has a stationary distribution if and only if

$$M = \sum_{j=0}^{\infty} \prod_{n=0}^{j-1} \frac{q(n, n+1)}{q(n+1, n)} < \infty.$$

$$= \frac{1}{1}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdots\frac{1}{j}$$

$$= \prod_{n=0}^{j-1}\frac{\lambda}{(n+1)\cdot\mu} = \boxed{\prod_{n=0}^{j-1}\cdot\frac{\lambda}{\mu}}\cdot\boxed{\prod_{n=0}^{j-1}\cdot\frac{1}{n+1}}\ \cdots$$

Here,

$$\gamma(j) = \prod_{n=0}^{j-1}\frac{q(n,n+1)}{q(n+1,n)} = \begin{cases} \prod_{n=0}^{j-1}\frac{\lambda}{(n+1)\mu} = \underbrace{\left(\frac{\lambda}{\mu}\right)^j}_{\left(\frac{\lambda}{\mu}\right)^j}\boxed{\frac{1}{j!}} & \text{if } j\le s \\ \boxed{\prod_{n=0}^{s-1}\frac{\lambda}{(n+1)\mu}}\boxed{\prod_{n=s}^{j-1}\frac{\lambda}{s\mu}} = \boxed{\left(\frac{\lambda}{\mu}\right)^s\frac{1}{s!}}\frac{\lambda^{j-s}}{s^{j-s}}\cdot\frac{1}{\mu^{j-s}} & \text{if } j>s \end{cases}$$

$$= \left(\frac{\lambda}{\mu}\right)^j\cdot\frac{1}{s!}\cdot\frac{1}{s^{j-s}}$$

and

$$M = \sum_{j=0}^{\infty}\gamma(j) = \underbrace{\boxed{\sum_{j=0}^{s}\left(\frac{\lambda}{\mu}\right)^j\frac{1}{j!}}}_{<\infty} + \underbrace{\boxed{\left(\frac{\lambda}{\mu}\right)^s\frac{1}{s!}}}_{<\infty}\boxed{\sum_{j=s}^{\infty}\left(\frac{\lambda}{s\mu}\right)^{j-s}} = \sum_{j=0}^{\infty}\left(\frac{\lambda}{s\mu}\right)^j$$

is finite if and only if $\lambda < s\mu$. If $\lambda < s\mu$, then $(X_t)_{t=0}^{\infty}$ is irreducible, non-explosive and there exists a stationary distribution π. Hence, if $\lambda < s\mu$ then $(X_t)_{t=0}^{\infty}$ is positive recurrent.

M/M/s

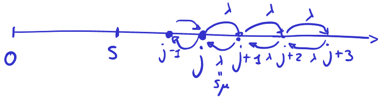
If $\lambda > s\mu$, the $(X_t)_{t=0}^{\infty}$ is transient.

We show this by comparison with the M/M/1 queue where the single server is given "superpowers" and has a service rate of $s\mu$. In this scenario, the M/M/1 queue is more efficient than the M/M/s queue, since the rates $q(j, j-1)$ are always equal to $s\mu$ independently on the number of customers present.

If $\lambda > s\mu$, then the M/M/1 queue with death rate $s\mu$ is transient, and so must be the less efficient M/M/s model.

M/M/s

If $\lambda = s\mu$, then $(X_t)_{t=0}^{\infty}$ is null recurrent.



Since for $\lambda = s\mu$ there is no stationary distribution, to show null recurrence it is enough to show recurrence.

To this aim, we consider a state $j > s$ and show that, if $X_0 = j$, then with probability 1 the chain will visit j again.

Indeed, if the chain moves down (i.e. to $j - 1$), then it will come back to j almost surely since the number of states smaller than or equal to j are finite and all communicating with each other.

On the other hand, if the chain moves up (i.e. to $j + 1$), then the model will come back to j almost surely because the transitions between states greater than or equal to j are the same as in the M/M/1 model with service rate $s\mu$, and the latter is recurrent for $\lambda = s\mu$.

To sum up:

The M/M/s queue is always non-explosive. Moreover:

- if $\lambda < s\mu$, then all states are positive recurrent and the unique stationary distribution is known;
- if $\lambda = s\mu$, then all states are null recurrent;
- if $\lambda > s\mu$, then all states are transient.

In the M/M/∞ model, there are infinitely many servers, that is whenever a new customer arrives, he is served immediately. It follows that the transition rates are given by

$$
\begin{aligned}
q(j, j+1) &= \lambda \qquad \text{for } j \geq 0 \\
q(j, j-1) &= j\mu \qquad \text{for } j \geq 1.
\end{aligned}
$$

Before studying the model, let's see some concrete examples of M/M/∞ queues.

Consider a museum, and assume that people arrive according to a Poisson process with rate λ. Here there is no service, and the people visiting the museum will leave after an exponential time with rate μ. Hence, the transition rates in this case are given by

$$
\begin{aligned}
q(j, j+1) &= \lambda \qquad \text{for } j \geq 0 \\
q(j, j-1) &= j\mu \qquad \text{for } j \geq 1.
\end{aligned}
$$

Another example:
Consider the situation in which proteins of a certain kind are introduced inside a cell with a constant flow (you can assume the protein is transported by the blood). Specifically, we assume that the proteins will enter the cell according to a Poisson process with rate λ. We further assume that the proteins of this kind naturally degrade after an exponential time of rate μ. Hence, if we consider the number of proteins inside the cell we have that the state space is $\{0,1,2,3,\dots\}$ and the transition rates are given by

$$
\begin{aligned}
q(j,j+1) &= \lambda \qquad \text{for } j \geq 0 \\
q(j,j-1) &= j\mu \qquad \text{for } j \geq 1.
\end{aligned}
$$

That is, the model is again a M/M/∞ queue.

M/M/∞

The model can't explode, for any choice of $\lambda, \mu > 0$. Why?

$\lambda(n) = \lambda + n \cdot \mu$ CRESCENTE IN n

The model can't explode, for any choice of $\lambda, \mu > 0$. Why?
The reason is that, if $Y_0 = k$, then we have $Y_n \leq k + n$. It follows that $\lambda(Y_n) \leq \lambda(k+n) = \lambda + (k+n)\mu$. Hence, regardless of the specific path of the embedded discrete time Markov chain Y_n,

$$\sum_{n=0}^{\infty} \frac{1}{\lambda(Y_n)} \geq \sum_{n=0}^{\infty} \frac{1}{\lambda + (k+n)\mu} = \infty.$$

$$\lambda(Y_n) \leq \lambda + (k+n)\mu$$

$$\frac{1}{\lambda(Y_n)} \geq \frac{1}{\lambda + (k+n)\mu}$$

M/M/∞

We have

$$\gamma(j) = \prod_{n=0}^{j-1} \frac{q(n,n+1)}{q(n+1,n)} = \prod_{n=0}^{j-1} \frac{\lambda}{(n+1)\mu} = \left(\frac{\lambda}{\mu}\right)^j \frac{1}{j!},$$

and

$$M = \sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j \frac{1}{j!} = e^{\lambda/\mu} < \infty \; !$$

is always finite for any choice of $\lambda, \mu > 0$. Moreover, the unique stationary distribution is given by

$$\pi(j) = \frac{1}{M} \prod_{n=0}^{j-1} \frac{q(n,n+1)}{q(n+1,n)} = e^{-\lambda/\mu} \left(\frac{\lambda}{\mu}\right)^j \frac{1}{j!} \quad \text{for all } j \geq 0.$$

That is, the stationary distribution is Poisson with parameter λ/μ and the process is positive recurrent for any choice of positive parameters.

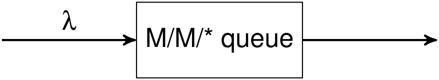
M/M/∞

To sum up:

The M/M/∞ queue is always non-explosive and the states are always positive recurrent. The unique stationary distribution is Poisson with parameter λ/μ.

The output process

Consider a queue



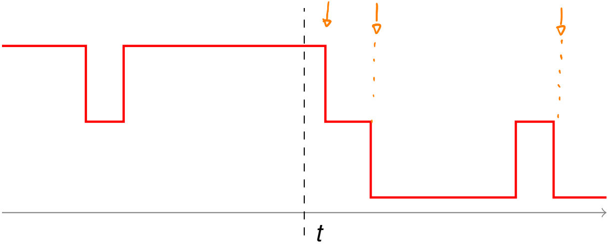
Definition (The output process)

The output process $(U_t)_{t\in[0,\infty)}$ of a queue is a counting process defined by

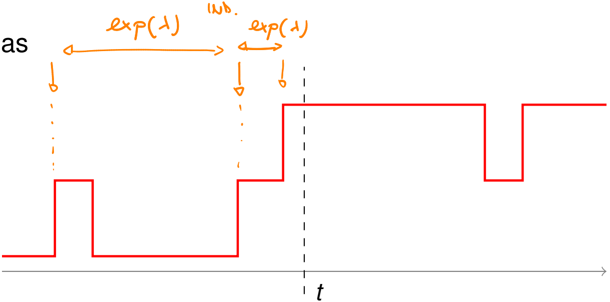
$$U_t = \#\text{individuals that left the queue by time } t$$

By time reversibility...

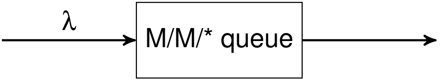
At stationary regime



is distributed as



The output process



$X_0 \sim \pi$

Theorem (The output process is a Poisson process)

Let $(X_t)_{t\in[0,\infty)}$ be a queue of type M/M/ at stationary regime, with arrival rate λ. Then the output process $(U_t)_{t\in[0,\infty)}$ is a Poisson process with rate λ.*

At stationary regime, the distribution of the output process does not depend on μ or s, or on what kind of queue we have! This may be counterintuitive at first. However, the point is that at stationary regime the service rate and the arrival rate should intuitively equilibrate.

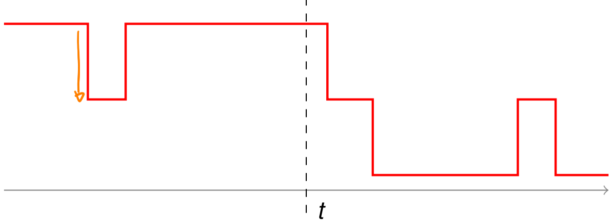
The output process

What is the joint distribution of X_t and U_t at stationary regime?

By time reversibility...

At stationary regime

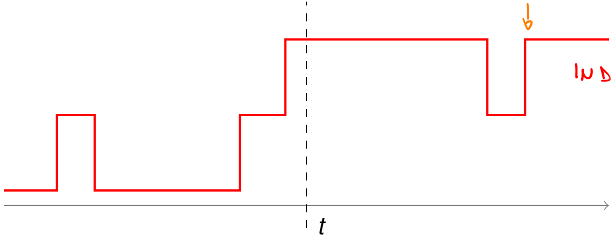
$V_t = $ # PARTENZE IN $[0, t]$



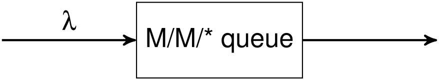
is distributed as

ARRIVI IN $[t, 2t]$

INDIP. DA X_t



The output process



Theorem (The output process is independent of the current state)

Let $(X_t)_{t\in[0,\infty)}$ be a queue of type M/M/ at stationary regime, with arrival rate λ. Then for each $t \in [0,\infty)$ the state of the queue X_t is independent of $(U_s)_{s\in[0,t]}$.*

Burke's theorem

By simply putting the two results together we have

Theorem (Burke, 1956)

Let $(X_t)_{t\in[0,\infty)}$ be a queue of type M/M/ at stationary regime, with arrival rate λ. Then the output process $(U_t)_{t\in[0,\infty)}$ is a Poisson process with rate λ and for each $t \in [0,\infty)$ the state of the queue X_t is independent of $(U_s)_{s\in[0,t]}$.*

Historical remark

The theorem was stated and proved (with an alternative proof that makes use of the properties of exponential random variables) by Paul J. Burke while he was working at Bell Telephone Laboratories.

Example

In a post office there are s servers, and the system can be modeled as a M/M/s queue, with arrival rate of λ/minute. Assume that the system is at stationary regime.

- What is the probability that exactly 4 people leave the post office in the interval of time $[20, 50]$ (in minutes)?

$$P(U_{50} - U_{20} = 4) = e^{-30\cdot\lambda} \cdot \frac{(30\cdot\lambda)^4}{4!}$$

$U_t \sim$ P.P. CON RATE λ

Example

Since the output process is a Poisson process at stationarity, it has all the properties of a Poisson process!

The Thinning, the Superposition, and the Conditioning properties apply...

Example

Now, assume that a customer that leaves the post office (at stationary regime) is satisfied with probability p, independently on the other customers.

- What is the probability that at least one satisfied customer leaves the post office in the interval of time $[20, 50]$ (in minutes)?

THINNING!

V_t = # DEPARTURES OF HAPPY CUSTOMERS

$\sim$ P.P. WITH RATE $\lambda \cdot p$

$$P(V_{50} - V_{20} \geq 1) = 1 - P(V_{50} - V_{20} = 0) = 1 - e^{-30 \cdot \lambda \cdot p}$$

Example of the counterintuitive property of independence

Assume that in the last 10 minutes we have observed exactly 4 people leaving the post office (at stationary regime).

- Given this observation, what is the probability that 10 people are currently inside the office?

$$P(X_t = 10 \mid \cancel{U_t - U_{t-10} = 4}) = P(X_t = 10) = \pi(10)$$

Jackson Networks

First example: tandem queues

Consider a post office with a single server, with arrivals following a Poisson process with rate λ. A portion p_1 of the customers will need to fill out some documents before getting in line, while the other customers get in line to the single server. The time it takes to prepare the documents is an exponential random variable with rate μ_1. After this time, with probability p_2 the customer will enter the queue to the single server, and with probability $(1 - p_2)$ the customer realizes they forgot an important document at home and leaves. The single server takes an exponential amount of time with rate μ_2 to serve the customers. All customers behave independently.

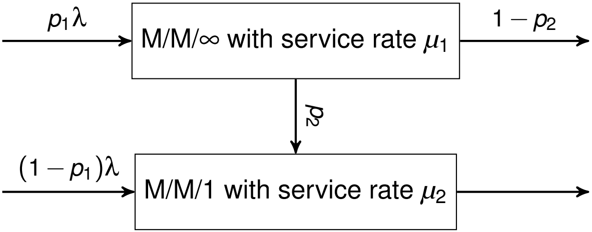
- How to model this situation?

First example: tandem queues

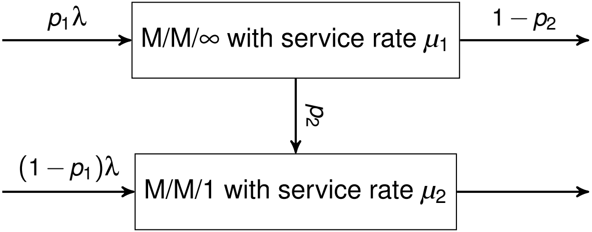
The queue with people filling out documents is a M/M/∞ queue, and the queue with the single server is a M/M/1 queue. The state space is

$$S = \{(n_1, n_2) : n_1, n_2 \in \{0, 1, 2, 3, \dots\}\},$$

where n_1 and n_2 are the states of the two queues.
By **thinning**, the arrivals to the first queue follow a Poisson process with rate λp_1, and the direct arrivals to the second queue follow a Poisson process with rate $\lambda(1 - p_1)$. The two Poisson processes are **independent**.



First example: tandem queues

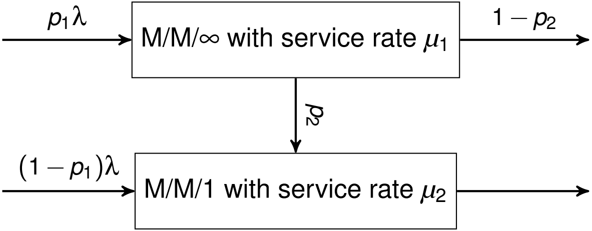


When is there a stationary distribution?

If and only if both queues can reach a stationary regime! In this case the queues are visited with a precise order. Hence, we can study the positive recurrence of the two queues separately, starting from the first one.

- The first queue is an M/M/∞ queue with arrival rate $p_1\lambda$ and service rate μ_1. Hence, it is always positive recurrent!

First example: tandem queues



When is there a stationary distribution?

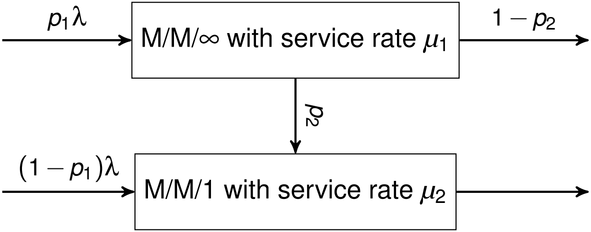
- When the first queue is at stationary regime, its **output process is a Poisson process** with rate $p_1\lambda$. By **thinning** and **superposition**, the arrivals to the second queue follow a Poisson process with rate

$$p_1 p_2 \lambda + (1 - p_1)\lambda$$

Hence, a stationary distribution exists if and only if

$$\mu_2 > p_1 p_2 \lambda + (1 - p_1)\lambda.$$

First example: tandem queues

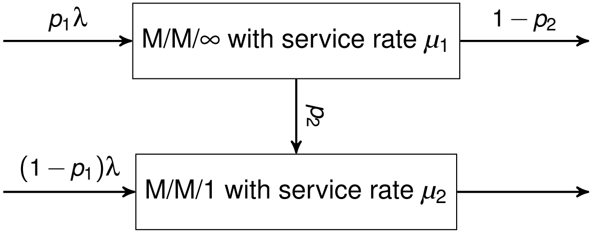


What is the stationary distribution?

The two queues depend on each other only via the output process of the first queue. **but at stationarity the output process is independent of the current state of the queue**! Hence **at stationarity the states of the two queues are independent**, and the stationary distribution is

$$\pi(n_1, n_2) = \pi_1(n_1)\pi_2(n_2)$$

First example: tandem queues

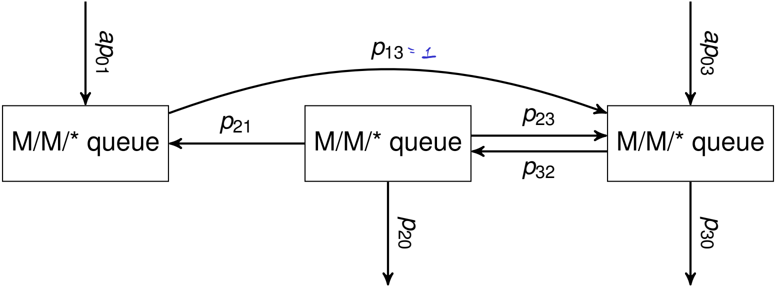


What is the stationary distribution?

By what we have seen so far,

$$\begin{aligned}\pi(n_1,n_2) &= \pi_1(n_1)\pi_2(n_2)\\ &= e^{-\frac{p_1\lambda}{\mu_1}}\left(\frac{p_1\lambda}{\mu_1}\right)^{n_1}\frac{1}{n_1!}\cdot\left(1-\frac{p_1p_2\lambda+(1-p_1)\lambda}{\mu_2}\right)\left(\frac{p_1p_2\lambda+(1-p_1)\lambda}{\mu_2}\right)^{n_2}\end{aligned}$$

A more complex example



Let λ_i be the arrival rate to the ith queue at stationary regime (if it exists). Then,

$$\lambda_1 = ap_{01} + p_{21}\lambda_2, \quad \lambda_2 = p_{32}\lambda_3, \quad \lambda_3 = ap_{03} + p_{13}\lambda_1 + p_{23}\lambda_2.$$

It is still true that at stationary regime (if it exists), the queues are independent.

Jackson networks

Definition (Jackson networks)

A Jackson network is a network of $m < \infty$ queues of type M/M/* such that all present individuals are in line at exactly one queue and behave independently of each other. Moreover,

- the total number of arrivals follow a Poisson process with rate a;
- An arriving customer will enter the ith queue with probability p_{0i};
- A customer leaving the ith queue enters the jth queue with probability p_{ij};
- A customer leaving the ith queue leaves the system with probability p_{i0}.

We require

$$\sum_{j=0}^{m} p_{ij} = 1$$

for all $i = 0, 1, \ldots, m$, where $p_{00} = 0$.

Jackson networks

Theorem

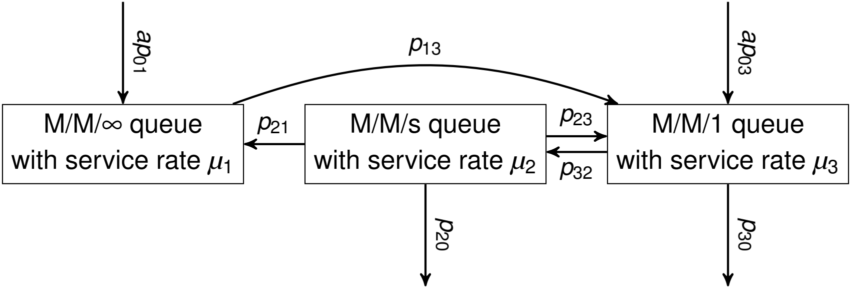
Consider a Jackson network with m queues. Then, the system is positive recurrent if and only if the following conditions are both satisfied:

- *the traffic equation* $\lambda_i = ap_{0i} + \sum_{j=1}^{m} p_{ji}\lambda_j, \quad i = 1, 2, \ldots, m$ *has a solution;*
- *the ith queue with arrival rate λ_i is positive recurrent, and has stationary distribution π_i.*

In this case, the unique stationary distribution is

$$\pi(n_1, n_2, \ldots, n_m) = \prod_{i=1}^{m} \pi_i(n_i).$$

Example



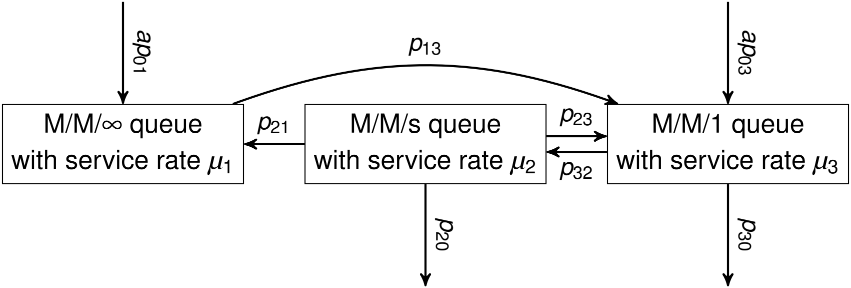
Traffic equation:

$$\lambda_1 = ap_{01} + p_{21}\lambda_2, \quad \lambda_2 = p_{32}\lambda_3, \quad \lambda_3 = ap_{03} + p_{13}\lambda_1 + p_{23}\lambda_2.$$

solved for

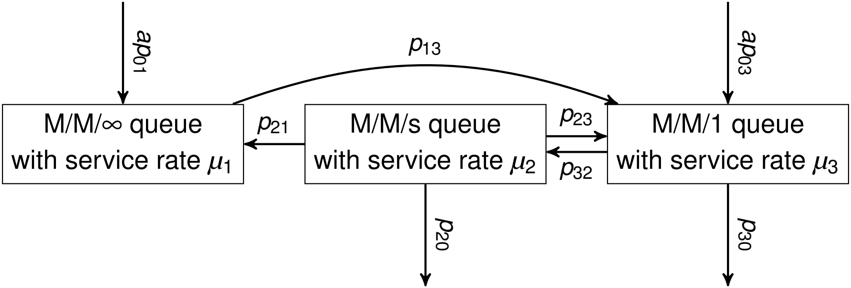
$$\lambda_1 = ap_{01} + p_{21}p_{32}\lambda_3, \quad \lambda_2 = p_{32}\lambda_3, \quad \lambda_3 = a\frac{p_{03} + p_{13}p_{01}}{1 - p_{32}(p_{13}p_{21} + p_{23})}.$$

Example



- The first queue is always positive recurrent.

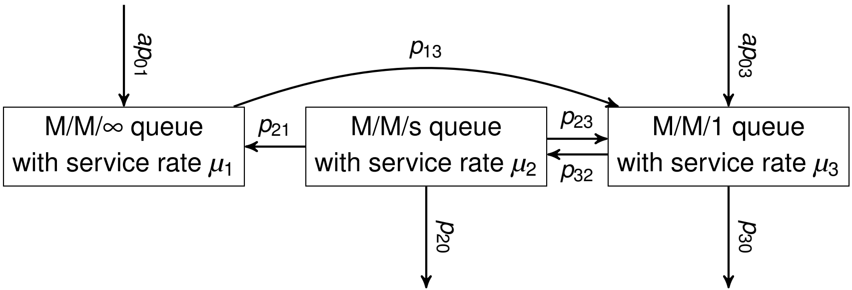
Example



- The second queue is positive recurrent if and only if $\lambda_2 < s\mu_2$, hence if

$$ap_{32}(p_{03} + p_{13}p_{01}) < [1 - p_{32}(p_{13}p_{21} + p_{23})]s\mu_2.$$

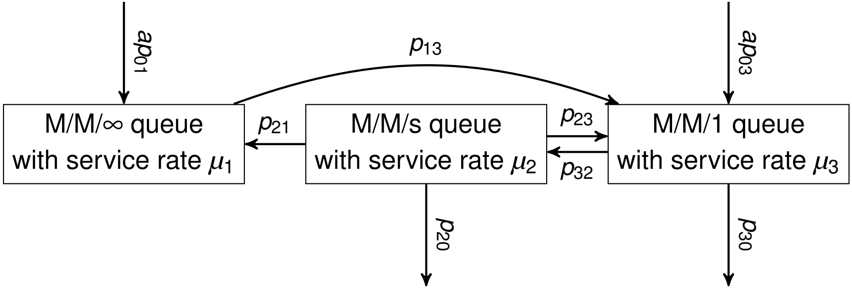
Example



- The third queue is positive recurrent if and only if $\lambda_3 < \mu_3$, hence if

$$a(p_{03} + p_{13}p_{01}) < [1 - p_{32}(p_{13}p_{21} + p_{23})]\mu_3.$$

Example



Hence, the all model is positive recurrent if and only if **both** the following inequalities are satisfied:

$$\begin{cases} ap_{32}(p_{03}+p_{13}p_{01}) < [1-p_{32}(p_{13}p_{21}+p_{23})]s\mu_2 \\ a(p_{03}+p_{13}p_{01}) < [1-p_{32}(p_{13}p_{21}+p_{23})]\mu_3. \end{cases}$$

In this case, the stationary distribution is a product of the stationary distributions of the individual queues.

Martingales

Conditional expectation

Definition (Conditional expectation)

Given a random variable X that is $\mathcal{F}-$measurable and a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, the conditional expectation $\mathbb{E}[X|\mathcal{G}]$ is a *random variable* such that

1. $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}-$measurable;
2. $\mathbb{E}\Big[\mathbb{1}_A \mathbb{E}[X|\mathcal{G}]\Big] = \mathbb{E}[\mathbb{1}_A X]$ for all $A \in \mathcal{G}$. Equivalently,

$$\int_A \mathbb{E}[X|\mathcal{G}](\omega) dP(\omega) = \int_A X(\omega) dP(\omega).$$

Existence and uniqueness (almost everywhere) of $\mathbb{E}[X|\mathcal{G}]$ follow from the Radon-Nikodym Theorem.

Conditional expectation

Intuitively, $\mathbb{E}[X|\mathcal{G}]$ is *the best approximation of X given the information contained in $\mathcal{G}$*, or the *projection of X onto the sub-σ-algebra $\mathcal{G}$*. It can be proven that if $\mathbb{E}[X^2] < \infty$, then $\mathbb{E}[X|\mathcal{G}]$ minimizes

$$\mathbb{E}[(X-\xi)^2]$$

over all random variables ξ such that

1. ξ is $\mathcal{G}-$measurable;
2. $\mathbb{E}[\xi^2] < \infty$.

If $\mathcal{G}$ is the trivial σ-algebra $\{\varnothing, \Omega\}$ then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$.

Properties of conditional expectation

Linearity $\mathbb{E}[aX+bY|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}]$.

Monotonicity if $X \le Y$ a.s. then $\mathbb{E}[X|\mathcal{G}] \le \mathbb{E}[Y|\mathcal{G}]$ a.s. .

Identity if X is $\mathcal{G}-$measurable, then $\mathbb{E}[X|\mathcal{G}] = X$ a.s. .

Jensen's inequality if ϕ is convex and $\mathbb{E}[|\phi(X)|] < \infty$, then

$$\mathbb{E}[\phi(X)|\mathcal{G}] \ge \phi\Big(\mathbb{E}[X|\mathcal{G}]\Big). \text{ a.s.}$$

Pulling out what's known if Y is measurable and $\mathbb{E}[|Y|] < \infty$ then

$$\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}]. \text{ a.s.}$$

Properties of conditional expectation

Tower property if $\mathcal{G}' \subseteq \mathcal{G}$ then $\mathbb{E}[\overbrace{\mathbb{E}[X|\mathcal{G}]}^{\text{v.A.}}|\mathcal{G}'] = \mathbb{E}[X|\mathcal{G}']$.

Irrelevance of independent information if $\mathcal{H}$ is independent of $\sigma(X, \mathcal{G})$ then $\mathbb{E}[X|\sigma(\mathcal{G}, \mathcal{H})] = \mathbb{E}[X|\mathcal{G}]$. a.s. In particular, if $\mathcal{H}$ is independent of X then $\mathbb{E}[X|\mathcal{H}] = \mathbb{E}[X]$. a.s.

Monotone convergence if $X_n \uparrow X$ a.s. and $\mathbb{E}[|X_n|], \mathbb{E}[|X|] < \infty$ then $\mathbb{E}[X_n|\mathcal{G}] \uparrow \mathbb{E}[X|\mathcal{G}]$ a.s. .

Fatou's lemma if $X_n \geq 0$ and $\mathbb{E}[|\liminf_{n\to\infty} X_n|] < \infty$ then $\mathbb{E}[\liminf_{n\to\infty} X_n|\mathcal{G}] \leq \liminf_{n\to\infty} \mathbb{E}[X_n|\mathcal{G}]$ a.s. .

Dominated convergence if $X_n \leq Y$ with $\mathbb{E}[|Y|] < \infty$ and $X_n \to X$ a.s. then $\mathbb{E}[X_n|\mathcal{G}] \to \mathbb{E}[X|\mathcal{G}]$ a.s. .