Homework I

- La soluzione degli esercizi *(ovvero un PDF contenente la risoluzione analitica degli esercizi ed il codice delle funzioni sviluppate nell'Esercizio 3)* deve essere caricata sul portale del corso entro le ore 23:59 di domenica 17 Novembre 2024, sotto il nome di Homework1. Il voto massimo in caso di consegna in ritardo è diminuito di 1 punto. In ogni caso, il PDF finale deve essere consegnato entro 5 giorni dalla data dell'esame orale.
- Le qualità dell'esposizione, la capacità di sintesi e la chiarezza del documento finale rientrano nella valutazione dell'homework. La scrittura del documento finale in Latex o in qualsiasi altro formato elettronico è fortemente incoraggiata. Se il documento finale è scritto a mano deve essere facilmente leggibile.
- La collaborazione in gruppi (fino a un massimo di 5 persone) e lo scambio di idee sono incoraggiati. In ogni caso, ogni studente deve sottomettere una copia del documento finale (in formato PDF) e del codice numerico, e specificare con chi ha collaborato e per quale specifica parte del lavoro.

Esercizio 1. Consider the network in Figure 1 with link capacities

$$c_1 = c_3 = c_5 = 3, \qquad c_6 = c_7 = 1 \qquad c_2 = c_4 = 2.$$

(a) Compute the capacity of all the cuts and find the minimum capacity to be removed for no feasible flow from o to d to exist.

(b) You are given $x > 0$ extra units of capacity ($x \in \mathbf{Z}$). How should you distribute them in order to maximize the throughput that can be sent from o to d? Plot the maximum throughput from o to d as a function of $x \geq 0$.

(c) You are given the possibility of adding to the network a directed link e_8 with capacity $c_8 = 1$ and $x > 0$ extra units of capacity ($x \in \mathbf{Z}$). Where should you add the link and how should you distribute the additional capacity in order to maximize the throughput that can be sent from o to d? Plot the maximum throughput from o to d as a function of $x \geq 0$.

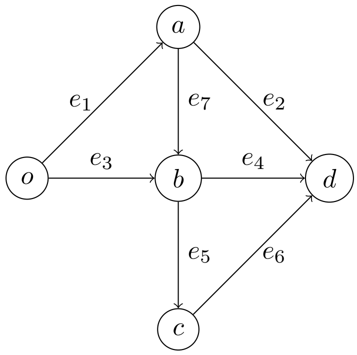

Figure 1

Esercizio 2. Consider o-d network flows on the graph in Figure 2. The links are endowed with delay functions

$$\tau_1(x) = \tau_6(x) = 3x\,, \quad \tau_2(x) = \tau_4(x) = x + 1\,, \quad \tau_3(x) = 3, \quad \tau_5(x) = 2x + 2, \tag{1}$$

and the throughput is 2.

(a) Compute the social optimum flow vector, i.e., the flow vector that minimizes the average delay from o to d.

(b) Compute the user optimum flow vector, i.e., the Wardrop equilibrium, and the price of anarchy.

(c) Consider a new link e_7 with delay function $\tau_7(x) = x$. Find a head and a tail of the link e_7 such that Braess' paradox arises, and compute the price of anarchy on the new graph.

(d) Compute an optimal toll vector ω on the new graph, i.e., a non-negative toll vector that reduces the price of anarchy to 1. If possible, compute a full-support optimal toll vector, i.e., such that $\omega_e > 0$ for every link e. Construct an optimal toll vector with the smallest possible support.

(e) Consider the original network in Figure 2. Assume that

$$\tau_3(x) = 1,$$

and $\tau_e(x)$ are as defined in (1) for all $e = 1, 2, 4, 5, 6$. Consider o-d network flows on the new graph with throughput $\chi > 0$. Find an optimal toll vector independent of χ and α. *Hint*: focus on the optimization problems related to social optimum flows and Wardrop equilibria flows.

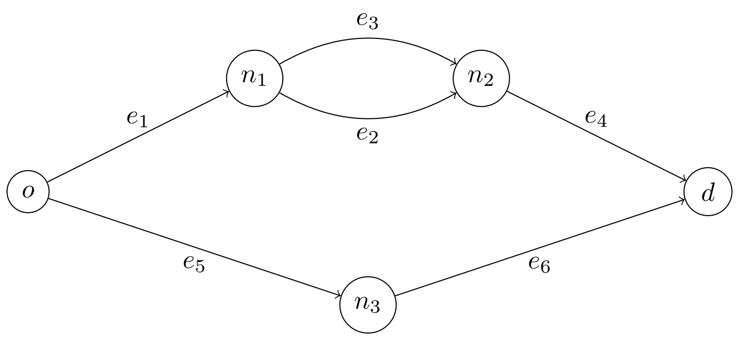

Figure 2

Esercizio 3. Consider the simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in Figure 3.

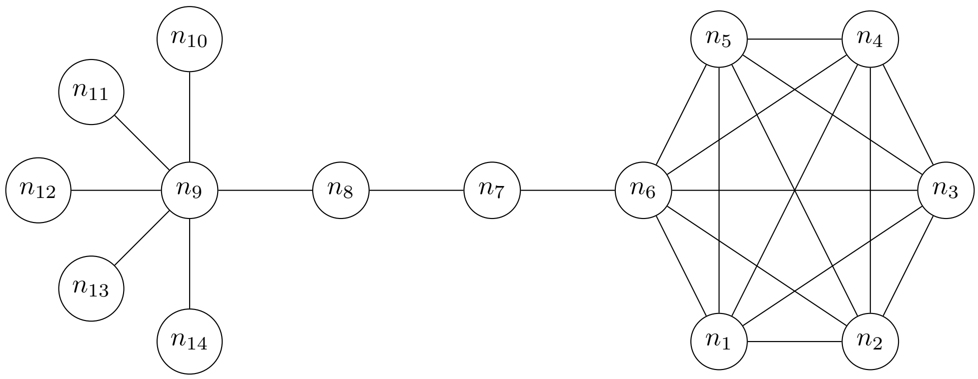

Figure 3

(a) Compute the degree centrality, the eigenvector centrality, the invariant distribution centrality, and comment the results. You can implement the computation in Matlab or Python.

(b) Compute the Katz centrality, with $\beta = 0.15$ and uniform intrinsic centrality μ. You can implement the computation in Matlab or Python.

(c) Write a *distributed* algorithm in Matlab or Python for the computation of Page-rank centrality, with $\beta = 0.15$ and uniform intrinsic centrality μ.

(d) Explain the results of points (b) and (c), focusing on the centralities of nodes n_6 and n_9.