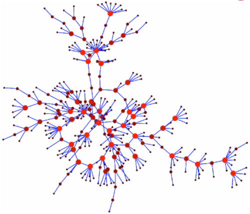
01RMHNG–03RMHPF–01RMING
Network Dynamics
Week 9 – Part I
Introduction to Strategic Games



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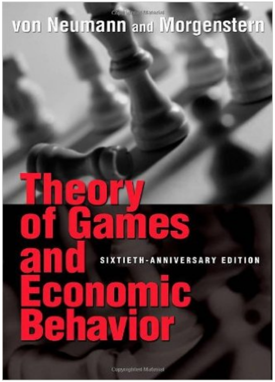
Non-cooperative strategic games

- Historical remarks
- Fundamental examples
- Formal definitions (strategy, best response, Nash equilibrium)
- More discrete and continuous examples

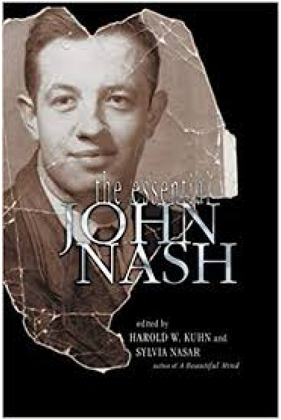
The birth of Game Theory



- 1713: Waldegrave, card games
- 1838: Cournot, theory of duopolies
- 1913: Zermelo, chess optimal strategy
- 1928: Von Neumann, dominant strategy
- 1944: Von Neumann and Mongenstern *Theory of games and economic behavior*
- game theory develops during cold war

Game Theory

John Forbes Nash (1928-2015)



- 1950: *Non-cooperative games*
  PhD Thesis (Princeton)
- 1950: *Equilibrium points in n-person games* PNAS
- 1950: *The bargaining problem*
  Econometrica
- 1951: *Non-cooperative games*
  Annals of Mathematics
- 1953: *Two-person Cooperative Games*
  Econometrica

  ...

- 1994 Nobel Price for Economics: Harsanyi, Nash, and Salten

Impact of Game Theory

in Economics:

- 1994 Nobel Price for Economics: Harsanyi, Nash, and Salten
- 2005 Nobel Price for Economics: Shelling and Aumann
- 2007 Nobel Price for Economics: Hurwicz, Maskin, and Myerson
- 2012 Nobel Price for Economics: Roth and Shapley
- 2014 Nobel Price for Economics: Tirole
- 2021 Nobel Price for Economics: Milgrom and Wilson

and many other fields:

- Political Sciences: Downs (1957), Cuban missile crisis (1962)
- Biology: evolutionary game theory, John Maynard Smith (1970s)
- Computer Science: semantics, algorithmic mechanism design,
- Engineering: multi-agent systems, Internet, networks
- Machine Learning, Artificial Intelligence: GANs, AI planning, ...

The Prisoner Dilemma

Frank and Cora get arrested and are accused of a crime.

They get questioned separately and offered the same deal:

- if they both confess, each of them gets sentenced to 3 years
- if only one confesses (s)he gets free, the other one gets 5 years
- if neither of them confesses, they get 1 year each (a minor crime)

Formalizing the Prisoner Dilemma

Cora's choice

Frank's choice

	CONFESS	SILENT
CONFESS	-3	0
SILENT	-5	-1

	CONFESS	SILENT
CONFESS	-3	-5
SILENT	0	-1

Dominant Strategies in Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	-3, -3	0, -5
SILENT	-5, 0	-1, -1

- whatever Cora does, Frank is better off confessing
- Confess is a dominant strategy for Frank

Dominant Strategies in Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	-3 -3	-5 0
SILENT	0 -5	-1 -1

- whatever Frank does, Cora is better off confessing
- Confess is a dominant strategy for Cora

Nash Equilibrium in Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	-3 -3	-5 0
SILENT	0 -5	-1 -1

- for both confessing is dominant strategy
- both have no incentive to deviate unilaterally
- if both remained silent, both would be better off

Nash Equilibrium in Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	-3 / -3	-5 / 0
SILENT	0 / -5	-1 / -1

Key assumptions:

- game played only once
- Cora and Frank do not communicate
- their interest is just to minimize their time in prison

Modified Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	-3 -3	-5 -2
SILENT	-2 -5	-1 -1

- if Cora confesses, Frank is better off confessing
- if Cora remains silent, Frank is better off remaining silent
- no dominant strategy, best response depends on Cora's choice

Modified Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	-3, -3	-5, -2
SILENT	-2, -5	-1, -1

- if Frank confesses, Cora is better off confessing
- if Frank remains silent, Cora is better off remaining silent
- no dominant strategy, best response depends on Frank's choice

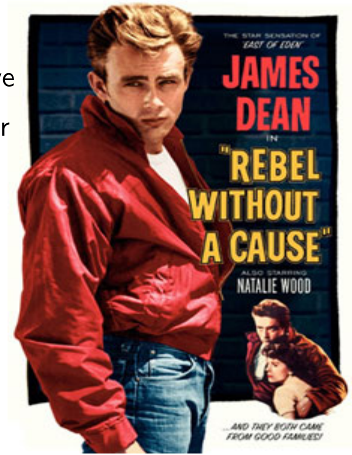
Modified Prisoner Dilemma

F \ C	CONFESS	SILENT
CONFESS	-3 -3	-5 -2
SILENT	-2 -5	-1 -1

- no dominant strategy
- best responses depend on the other player's choice
- 2 Nash equilibria: both have no incentive to deviate unilaterally

The Game of Chicken

- two cars in collision course
- each driver can go straight or swerve
- the one that swerves while the other goes straight is the “chicken”
- the one that goes straight while the other swerves is the brave
- if both go straight they both die
- if both swerve they both lose some appeal but avoid main damage



The Game of Chicken

	SWERVE	STRAIGHT
SWERVE	0 0	1 -1
STRAIGHT	-1 1	-100 -100

- if the other one goes straight better swerve
- if the other one swerves better go straight
- best response depends on the other player's choice
- 2 Nash equilibria: both have no incentive to deviate unilaterally

The Game of Chicken

	SWERVE	STRAIGHT
SWERVE	0, 0	-1, 1
STRAIGHT	1, -1	-100, -100

- 1959 B. Russell "Common Sense and Nuclear Warfare"
- 1973 Maynard Smith and Price, "The logic of animal conflict"
- 1964 S. Kubrik "Dr. Strangelove"

Basic definitions

- $\mathcal{V}$ finite set of players
- $\mathcal{A}_i$ set of actions (a.k.a. strategies) for player i
- $\mathcal{X} = \prod_{i \in \mathcal{V}} \mathcal{A}_i$ set of configurations (a.k.a. strategy profiles)
- $u_i : \mathcal{X} \to \mathbb{R}$ utility function
- $x \in \mathcal{X}$ configuration (a.k.a. action/strategy profile, or outcome)
- x_i action played by player i
- x_{-i} vector of actions played by everyone but i
- utility of player i when each player j plays action x_j:

$$u_i(x_i, x_{-i}) = u_i(x)$$

$\left(\mathcal{V}, \{\mathcal{A}_i\}_{i \in \mathcal{V}}, \{u_i\}_{i \in \mathcal{V}}\right)$ is called a strategic (a.k.a. normal form) game

- game is zero-sum if $\sum_i u_i(x) = 0$ for all $x \in \mathcal{X}$

Two-player games

- when $\mathcal{V} = \{1, 2\}$: two utility functions $u_i(r, s)$, for $i = 1, 2$
(r = action played by i and s = action played by opponent of i)

- Two-player symmetric game: $u_1(r, s) = u_2(r, s) = \phi(r, s)$
(the role of the two players is exchangeable).

- Table representation for finite action spaces:
rows $\leftrightarrow$ actions of player 1, columns $\leftrightarrow$ action of player 2,
(r, s)-th entry displays the pair $u_1(r, s), u_2(s, r)$

- 2×2 (nonsymmetric) game

	-1	+1
-1	a,e	d,h
+1	c,g	b,f

2×2 symmetric game

	-1	+1
-1	a,a	d,c
+1	c,d	b,b

- Ex.matching penny: 2×2 nonsymmetric zero-sum

	-1	+1
-1	+1,-1	-1,+1
+1	-1,+1	+1,-1

Dominant actions

- Action $x_i \in \mathcal{A}_i$ is dominant for player i if

$$u_i(x_i, x_{-i}) \geq u_i(y_i, x_{-i}) \qquad \forall y_i \in \mathcal{A}_i,\ \forall x_{-i} \in \mathcal{X}_{-i}$$

- Def.: A dominant action equilibrium is a configuration x in $\mathcal{X}$ s.t.

$$x_i \text{ is a dominant action for every player } i \in \mathcal{V}$$

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$$x_i \text{ is a dominant action for every player } i \in \mathcal{V}$$

- Example: (Confess,Confess) is dominant action equilibrium in the original prisoner dilemma game

	Confess	Silent
Confess	-3,-3	0,-5
Silent	-5,0	-1,-1

- not so common for a game to admit dominant action equilibrium

Dominance Solvable Games

- Action x_i in $\mathcal{A}_i$ is strictly dominated for player i if $\exists y_i \in \mathcal{A}_i$

$$u_i(x_i, x_{-i}) < u_i(y_i, x_{-i}) \qquad \forall x_{-i} \in \mathcal{X}_{-i}$$

- common knowledge of utilities and rationality result in iterated elimination of strictly dominated actions
- game is dominance solvable if iterated elimination of strictly dominated actions converges to configuration x^*
- Example: extended prisoner dilemma

	Confess	Silent	Suicide
Confess	-3,-3	0,-5	-3,-10
Silent	-5,0	-1,-1	-1,-10
Suicide	-10,-3	-10,-1	-10,-10

	Confess	Silent
Confess	-3,-3	0,-5
Silent	-5,0	-1,-1

Best response and pure strategy Nash equilibrium

- each player i to be interpreted as a rational agent choosing action x_i so as to maximize her utility $u_i(x_i, x_{-i})$
- player i's utility $u_i(x_i, x_{-i})$ depends not only on her action x_i but also on the actions of the rest of the players x_{-i}
- rational choice for a player: best response

$$\mathcal{B}_i(x_{-i}) = \underset{x_i \in \mathcal{A}_i}{\operatorname{argmax}}\, u_i(x_i, x_{-i})$$

Best response and pure strategy Nash equilibrium

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$$\mathcal{B}_i(x_{-i}) = \underset{x_i \in \mathcal{A}_i}{\operatorname{argmax}}\, u_i(x_i, x_{-i})$$

Definition: A pure strategy (P) Nash equilibrium (NE) for the game $(\mathcal{V}, \mathcal{A}, \{u_i\}_{i\in\mathcal{V}})$ is a configuration $x^* \in \mathcal{X}$ such that

$$x_i^* \in \mathcal{B}_i(x_{-i}^*)\,, \qquad \forall i \in \mathcal{V}\,.$$

- PNE x^* is a configuration from which no player has strict incentive to unilaterally change her action
- PNE x^* is said strict if $|\mathcal{B}_i(x_{-i}^*)| = 1$ for every player $i \in \mathcal{V}$

Example 1: Coordination game

- $|\mathcal{V}| = 2$, $\mathcal{A}_1 = \mathcal{A}_2 = \{\pm 1\}$

$$a > c\,, \qquad b > d$$

	-1	+1
-1	a,a	d,c
+1	c,d	b,b

- Best response for both: copy the other player

$$\mathcal{B}_i(-1) = -1\,, \qquad \mathcal{B}_i(+1) = +1\,, \qquad i = 1, 2$$

- Two PNE: $\mathcal{N} = \{(-1,-1), (+1,+1)\}$
- Positive externality
- Two PNE not equally good: $a > b \Rightarrow (-1,-1)$ payoff dominant

Example 2: Prisoner's dilemma

- $|\mathcal{V}| = 2$, $\mathcal{A}_1 = \mathcal{A}_2 = \{\pm 1\}$

$$a > c, \qquad b < d$$

	-1	+1
-1	a,a	d,c
+1	c,d	b,b

- Interpretation: Action $-1 \leftrightarrow$ Betraying, Action $+1 \leftrightarrow$ Silent
- NOT a coordination game!
- Best response for both:

$$\mathcal{B}_i(+1) = \mathcal{B}_i(-1) = -1 \qquad i = 1, 2$$

- Betraying $= -1$ is a dominant action for both
- One PNE: $x^* = (-1, -1)$

Example 3: Anti-coordination game

- $|\mathcal{V}| = 2$, $\mathcal{A}_1 = \mathcal{A}_2 = \{\pm 1\}$

$$a < c\,, \qquad b < d$$

	-1	+1
-1	a,a	d,c
+1	c,d	b,b

- Best response for both: do the opposite of the other player

$$\mathcal{B}_i(-1) = +1\,, \qquad \mathcal{B}_i(+1) = -1\,, \qquad i = 1, 2$$

- Two PNE: $\mathcal{N} = \{(-1, +1), (+1, -1)\}$
- Popular anti-coordination game: Game of Chicken (a.k.a. Hawk-Dove game) where $a > d$. ($-1 =$ Swerve, $+1 =$ Straight)

Example 4: Discoordination game

- $|\mathcal{V}| = 2$, $\mathcal{A}_1 = \mathcal{A}_2 = \{\pm 1\}$

$$a > c\,, \qquad d > b$$

	-1	+1
-1	a,b	c,d
+1	c,d	a,b

- Discoordination game is not symmetric
- Best response

$$\mathcal{B}_1(-1) = \mathcal{B}_2(1) = -1\,, \qquad \mathcal{B}_1(+1) = \mathcal{B}_2(-1) = +1\,.$$

- No PNE!
- Popular example of a discoordination game is the matching penny game where $a = d = 1$ and $b = c = -1$.

Example 5: Rock-Scissor-Paper

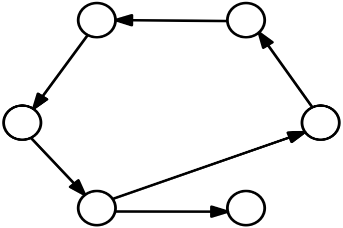
- $|\mathcal{V}| = 2$, $\mathcal{A}_1 = \mathcal{A}_2 = \{R, S, P\}$

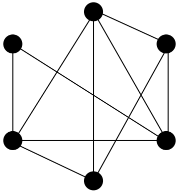
	R	S	P
R	0,0	1,-1	-1,1
S	-1,1	0,0	1,-1
P	1,-1	-1,1	0,0

- Zero-sum game
- No PNE!

Example 6: Majority game

- $|\mathcal{V}| = n$, $\mathcal{A}_i = \{\pm 1\}$ for all i

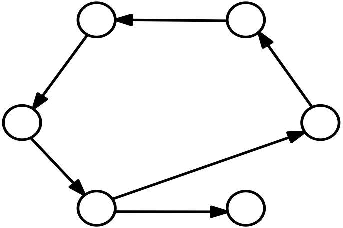


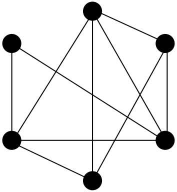


- Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $\{nodes\} = \mathcal{V}$, $\{links\} = \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$
- Adjacency matrix $A \in \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$: $A_{ij} = \begin{cases} 1 & \text{if} \quad (i,j) \in \mathcal{E} \\ 0 & \text{if} \quad (i,j) \notin \mathcal{E} \end{cases}$
- $\mathcal{G}$ undirected if $A' = A$
- Utilities:

$$u_i(x_i, x_{-i}) = \sum_j A_{ij} x_i x_j$$

Example 6: Majority game (cont'd)





- Utilities:

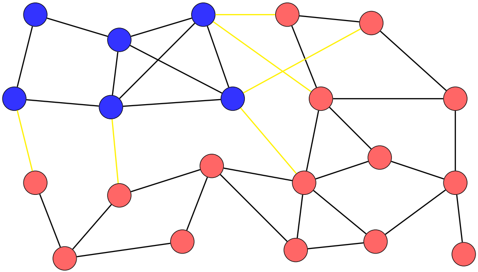
$$u_i(x_i, x_{-i}) = \sum_j A_{ij} x_i x_j$$

- Best response: follow majority of (out-)neighbors

$$\mathcal{B}_i(x_{-i}) = \begin{cases} -1 & \text{if} \quad \sum_j A_{ij} x_j < 0 \\ \{\pm 1\} & \text{if} \quad \sum_j A_{ij} x_j = 0 \\ +1 & \text{if} \quad \sum_j A_{ij} x_j > 0 \end{cases}$$

- Consensus configurations $x^* = \pm \mathbb{1}$ are PNE on every $\mathcal{G}$
- Does the majority game admit any coexistent PNE $x^* \neq \pm \mathbb{1}$?

Example 6: Majority game (cont'd)



- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, adjacency matrix A
- Subset of nodes $\mathcal{U} \subseteq \mathcal{V}$ cohesive if $\sum_{j \in \mathcal{U}} A_{ij} \geq 1/2$, $\forall i \in \mathcal{U}$
- Proposition [Morris,'00]: For majority game on $\mathcal{G}$

$x^* = \mathbb{1}_{\mathcal{U}} - \mathbb{1}_{\mathcal{V} \setminus \mathcal{U}}$ is a PNE $\quad \Longleftrightarrow \quad$ $\mathcal{U}$ and $\mathcal{V} \setminus \mathcal{U}$ both cohesive

Example 7: Minority game

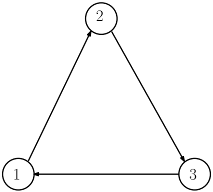
- $|\mathcal{V}| = n$, $\mathcal{A}_i = \{\pm 1\}$ for all i
- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, adjacency matrix A
- Utilities:

$$u_i(x_i, x_{-i}) = -\sum_j A_{ij} x_i x_j$$

- Best response: follow minority of out-neighbors

$$\mathcal{B}_i(x_{-i}) = \begin{cases} +1 & \text{if} \quad \sum_j A_{ij} x_j < 0 \\ \{\pm 1\} & \text{if} \quad \sum_j A_{ij} x_j = 0 \\ -1 & \text{if} \quad \sum_j A_{ij} x_j > 0 \end{cases}$$

- NO PNEs for general directed $\mathcal{G}$



Example 7: Minority game (cont'd)

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$$u_i(x_i, x_{-i}) = -\sum_j A_{ij} x_i x_j$$
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$$\mathcal{B}_i(x_{-i}) = \begin{cases} +1 & \text{if} \quad \sum_j A_{ij} x_j < 0 \\ \{\pm 1\} & \text{if} \quad \sum_j A_{ij} x_j = 0 \\ -1 & \text{if} \quad \sum_j A_{ij} x_j > 0 \end{cases}$$

- NO PNEs for general directed $\mathcal{G}$
- Proposition: $\mathcal{G}$ undirected $\Rightarrow$ Minority game has at least 1 PNE

Example 7: Minority game (cont'd)

- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, adjacency matrix A
- Utilities:
$$u_i(x_i, x_{-i}) = -\sum_j A_{ij} x_i x_j$$
- Best response: follow minority of out-neighbors
$$\mathcal{B}_i(x_{-i}) = \begin{cases} +1 & \text{if} \quad \sum_j A_{ij} x_j < 0 \\ \{\pm 1\} & \text{if} \quad \sum_j A_{ij} x_j = 0 \\ -1 & \text{if} \quad \sum_j A_{ij} x_j > 0 \end{cases}$$
- NO PNEs for general directed $\mathcal{G}$
- Proposition: $\mathcal{G}$ undirected $\Rightarrow$ Minority game has at least 1 PNE
- Proof: Define $\Phi(x) = \frac{1}{2} \sum_{j,k \in \mathcal{V}} A_{jk} x_j x_k$. Observe that $x_{-i} = y_{-i}$

$\Rightarrow \Phi(y) - \Phi(x) = (y_i - x_i) \frac{1}{2} \left(\sum_k A_{jk} x_k + \sum_j A_{ji} x_j \right) = u_i(y) - u_i(x)$

Then, $x^* \in \text{argmax}\{\Phi(x) : x \in \mathcal{X}\}$ is a PNE. (will be generalized)

Example 8: Quadratic Game

- n players with action space $\mathcal{A}_i = \mathbb{R}$ and utilities

$$u_i(x) = h_i x_i - \frac{1}{2}x_i^2 + \beta x_i \sum_{j \neq i} W_{ij} x_j$$

- The best response of player i is linear

$$\mathcal{B}_i(x_{-i}) = h_i + \beta \sum_{j \neq i} W_{ij} x_j$$

- Nash equilibria are the solutions of linear system

$$x = h + \beta W x$$

- If $\beta \rho(W) < 1$, the game has a unique Nash equilibrium given by

$$x^* = (I - \beta W)^{-1} h \qquad x_i^* = \sum_{k \geq 0} \beta^k \sum_j (W^k)_{ij} h_j$$

Example 9: Cournot oligopoly

- A. A. Cournot (1801-1877) philosopher and mathematician
- $n \geq 2$ firms producing a homogeneous good for the same market
- $x_i \in \mathcal{A}_i = [0, +\infty)$ quantity of good produced by firm i
- $c_i(x_i)$= production cost for firm i
- $p\left(\sum_j x_j\right)$ market price of good (a.k.a. inverse demand function)

- profit for firm i

$$u_i(x_1, x_2, \dots, x_n) = x_i \cdot p\left(\sum_j x_j\right) - c_i(x_i)$$

Cournot duopoly with linear costs and affine price

- $n = 2$ firms with same linear costs $c_i(x_i) = cx_i$
- affine inverse demand (market price) function

$$p(q) = [K - q]_+ = \max\{K - q, 0\}$$

- utilities

$$u_i(x_1, x_2) = x_i[K - x_1 - x_2]_+ - cx_i \qquad i = 1, 2$$

- Best response: $\mathcal{B}_i(x_{-i}) = \frac{1}{2}\,[K - c - x_{-i}]_+$

Cournot duopoly with linear costs and affine price

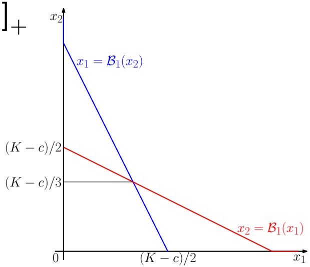
- $n = 2$ firms with same linear costs $c_i(x_i) = cx_i$
- affine inverse demand (market price) function

$$p(q) = [K - q]_+ = \max\{K - q, 0\}$$

- utilities

$$u_i(x_1, x_2) = x_i[K - x_1 - x_2]_+ - cx_i \qquad i = 1, 2$$

- Best response: $\mathcal{B}_i(x_{-i}) = \frac{1}{2}\left[K - c - x_{-i}\right]_+$
- Unique PNE: $x_1^* = x_2^* = \frac{1}{3}[K - c]_+$



- Price: lower than monopoly one
$c < p^c = \frac{1}{3}[K + 2c]_+ < \frac{1}{2}[K + c] = \overline{p}$

- Equilibrium profits: $u_i(x_i, x_{-i}) > 0$

Example 10: Bertrand oligopoly

- J. Bertrand (1822-1900) mathematician (number theory, probability, mechanics...)
- $n \geq 2$ firms producing a homogeneous good for the same market
- $c_i(x_i)$= production cost for firm i
- $x_i \in \mathcal{A}_i = [0, +\infty)$ unit price fixed by firm i
- $q\,(p)$ market demand function (quantity bought at price p)
- Consumers buy at the smallest price $p^*(x) = \min_i x_i$ from the $k(x) = |\operatorname{argmin}_i x_i|$ firms offering such price
- profit for firm i

$$u_i(x_1, x_2, \ldots, x_n) = \begin{cases} 0 & \text{if} \quad x_i > p^*(x) \\ \dfrac{q(p^*(x))}{k(x)} p^*(x) - c_i\left(\dfrac{q(p^*(x))}{k(x)}\right) & \text{if} \quad x_i = p^*(x) \end{cases}$$

Bertrand duopoly with linear costs and affine price

$$u_i(x_i, x_{-i}) = \begin{cases} 0 & \text{if } x_i > x_{-i} \\ (x_i - c)[K - x_i]_+/2 & \text{if } x_i = x_{-i} \\ (x_i - c)[K - x_i]_+ & \text{if } x_i < x_{-i} \end{cases}$$

- There is a unique PNE $x_1^* = x_2^* = c$
- Equilibrium price is $p^b = c$
- Firms make zero utility! (Bertrand paradox)