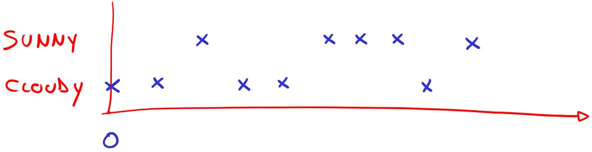
FORUM

Stochastic processes are everywhere! Whenever there is something evolving in time, with a probabilistic outcome, we have a stochastic process.

Stochastic processes and overview of the course

Stochastic processes are everywhere! Whenever there is something evolving in time, with a probabilistic outcome, we have a stochastic process.

weather in different days



$X_t \in \{\text{SUNNY}, \text{CLOUDY}\}$

X_t = WEATHER ON DAY t

$\{X_t,\ t \in \{0, 1, 2, 3, \dots\}\}$

PROCESSO STOCASTICO

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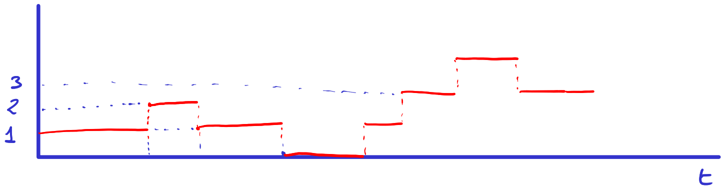
number of job requests at a server



X_t = # REQUESTS IN QUEUE AT TIME t

$\{X_t : t \in [0, +\infty)\}$ PROCESSO STOCASTICO

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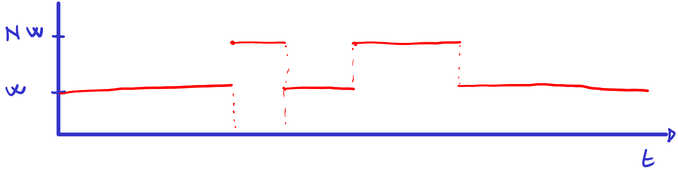
whether a factory machine is working or not



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position of a pollen grain in a glass of water

X_t = POSITION OF THE POLLEN GRAIN AT TIME t $\in \mathbb{R}^2$

$\{X_t, t \in [0,+\infty)\}$ STOCHASTIC PROCESS

Stochastic processes and overview of the course

Definition

A stochastic process is a collection of random variables $\{X_i\}_{i \in I}$ for some set of ordered indices I, with the random variables sharing a common *state space* S.

- If S and I are discrete sets, we have a *discrete stochastic process with discrete time* (ex. weather day by day).
- If S is discrete and I is continuous, we have a *discrete stochastic process with continuous time* (ex. server requests, machine status).
- If S and I are continuous sets, we have a *continuous stochastic process with continuous time* (ex. pollen grain).

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- ESISTE ANCHE IL CASO S CONTINUO E I DISCRETO (MCMC ALGORITHMS)

Quiz time!

Which of the following ones are stochastic processes?

1. $\{X_t\}_{t\in[0,\infty)}$ with X_t being the number of waves hitting a particular rock by time t;
2. $\{X_t\}_{t\in[0,\infty)}$ with X_t being the status of your computer ("on", "off", or "broken") at time t;
3. $\{X_t\}_{t\in[0,\infty)}$ with X_t being the position of a frog jumping around at time t.

TUTTI

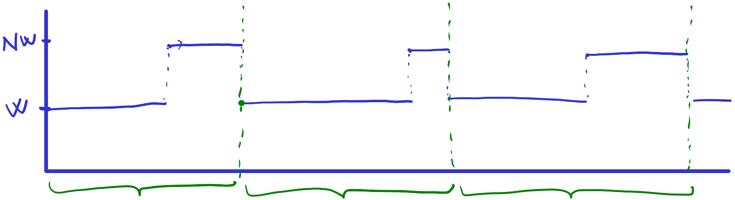
Consider a sequence of i.i.d. random variables $\{X_i\}_{i=1}^{\infty}$. Is it a stochastic process?

Si!

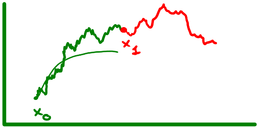
Things become interesting if there is some “structure”

Things become interesting if there is some “structure”

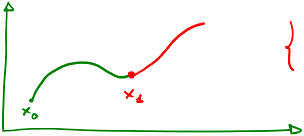
Preview 1 What happens in different interval time intervals is independent. For example the factory machine is as new whenever it is repaired.



Stochastic processes and overview of the course



$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$



$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_1 \end{cases}$$

Things become interesting if there is some “structure”

Preview 2 What happens if the future is independent of the past given the current state.
For example tomorrow’s weather depends only on the current weather situation and not the past weather.

$$P(X_{n+1} = S \mid X_n = S, X_{n-1} = C, X_{n-2} = S, \dots)$$
$$= P(X_{n+1} = S \mid X_n = S) = 0.7$$
$$P(X_{n+1} = C \mid X_n = S) = 1 - 0.7 = 0.3$$
$$P(X_{n+1} = S \mid X_n = C) = 0.4$$
$$P(X_{n+1} = C \mid X_n = C) = 0.6$$

$$P = \begin{array}{c c} & \begin{array}{cc} S & C \end{array} \\ \begin{array}{c} S \\ C \end{array} & \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \end{array}$$

What you should know

Things that you are welcome to ask me about during office hours, but that I will consider as known during classes:

- what a discrete/continuous random variable is;
- how to solve simple integrals, how to multiply two matrices, how to calculate the inverse of a matrix;
- what it means that two random variables are independent;
- what conditional distributions and conditional expectations are;
- the mean (or expected value) of a random variable;
- variance and covariance of random variables.

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Random variables we will be working with

During classes I will assume the law of the following random variables is known:

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- uniform (both discrete and continuous);
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Exam

La verifica dell'apprendimento per questo modulo (processi stocastici) consiste nelle seguenti parti:

1. Due homework (HW1, HW2) che consistono di esercizi teorici ed applicativi ed hanno lo scopo di verificare l'assimilazione da parte dello studente delle teorie sviluppate nel modulo e a valutarne lo sviluppo delle capacità di modellizzazione matematica. Laltro modulo prevede un terzo homework, che prevede un progetto di tipo simulativo/numerico ed ha lo scopo di valutare le capacità progettuali dello studente. I tempi di consegna sono di due settimane per HW1 e HW2.
2. Una prova orale finale che consiste di due parti:

Exam

2.a) Discussione degli homework HW1, HW2 al fine di valutare la capacità critica dello studente rispetto al lavoro svolto.

2.b) Presentazione da parte del candidato di un argomento teorico trattato nel corso e di eventuali relativi esempi di applicazione. L'argomento in questione viene scelto dalla commissione e comunicato al candidato, a cui vengono lasciati 30 minuti per la preparazione della risposta, e durante i quali possono essere consultati libri ed appunti. Ulteriori domande possono poi essere fatte sul resto del programma. Scopo di questa seconda parte è la valutazione delle conoscenze teoriche di base acquisite e la capacità dello studente di presentarle in modo autonomo ed efficace.

Exam

Punteggi: I punteggi massimi per HW1 e HW2, a seguito della discussione di cui a 2a), sono 2 per HW1 e HW2 ciascuno. Nel caso in cui le consegne siano avvenute in ritardo i massimi scendono a 1.5. Il punteggio massimo della prova 2b) è pari a 9. Al termine delle due prove il punteggio viene sommato ed e' al massimo 13. Il resto del voto (al massimo 18) deriva dall'altro modulo del corso, gestito dal prof. Fagnani. Se il voto totale è non superiore a 30, esso costituisce il voto finale (si approssima all'intero piu' vicino e .5 viene approssimato per eccesso), se è superiore il voto finale è 30 e lode.

DOPO ARROTONDAMENTO

Some technical definition

Recall what a σ-algebra is...

$\mathcal{A}$ È σ-ALGEBRA SE

1. $\Omega \in \mathcal{A}$
2. $E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A}$
3. $E_1, E_2, \dots \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$

$P(E)$ SE $E \in \mathcal{A}$

Some technical definition

VITALI

$\mathcal{G}(X)$ = "info su X"

Why do we need σ-algebras?

Some technical definition

Recall what random variables are...

$$X : \Omega \longrightarrow Z$$

$$(\Omega, \mathcal{A}) \qquad (Z, \mathcal{B})$$

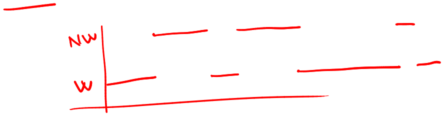
$$\forall\; B \in \mathcal{B} \quad X^{-1}(B) \in \mathcal{A}$$

Some technical definition

A stochastic process is a random variable with infinite dimension: $\{X_i\}_{i \in I}$ is a random variable with image in

$$F = \{f : I \to S\}$$

$\omega \in \Omega$ $\quad X(\omega)$ e' funzione da $I \to S$



$X(\omega)$ e' funzione da $[0, +\infty)$ in $\{w, nw\}$

Some technical definition

Definition

Let $\mathcal{F}$ be a σ-algebra. A *filtration* is a sequence $\{\mathcal{F}_i\}_{i\in I}$ of σ-algebras such that

1. $\mathcal{F}_i \subseteq \mathcal{F}$ for all $i \in I$;
2. $\mathcal{F}_i \subseteq \mathcal{F}_j$ for all $i, j \in I$ with $i < j$.

AL CRESCERE DI j, CRESCE σ-ALGEBRA
CRESCONO INFO A DISPOSIZION

Definition

A stochastic process $\{X_i\}_{i\in I}$ is *adapted* to a filtration $\{\mathcal{F}_i\}_{i\in I}$ if X_i is $\mathcal{F}_i$-measurable for all $i \in I$.

In practice, $\mathcal{F}_i$ contains all the possible evolutions of the process up to time i (it can contain more info).

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$\mathcal{F}$ CONTIENE TUTTI GLI $\mathcal{F}_i$ CONTIENE TUTTE LE INFO SU X_i $\forall i$ $\Rightarrow$ $\omega \longmapsto \{X_i(\omega)\}_{i\in I}$ È MISURABILE RISPETTO A $\mathcal{F}$

Example

For $i \in \{1,2,3,\dots\}$ let $Y_i \sim \text{Uniform}(0,1)$, and

$$X_i = \begin{cases} 0 & \text{if } Y_i < 0.5 \\ 1 & \text{if } Y_i \geq 0.5 \end{cases}$$

Let $\boxed{\mathcal{F}_i = \sigma\{Y_j : j \leq i\}}$. Is $\boxed{\{X_i\}_{i=1}^{\infty} \text{ adapted to } \{\mathcal{F}_i\}_{i=1}^{\infty}?}$

SI!

Y_i È $\mathcal{F}_i$ - MEASURABLE

$X_i = g(Y_i)$ È $\mathcal{F}_i$ - MEASURABLE

$\{\mathcal{F}_i\}_{i=1}^{\infty}$ È FILTRAZIONE:

1) $\mathcal{F}_i \subseteq \mathcal{F}$

$\mathcal{F} = \sigma\{Y_j : j \geq 1\}$

2) $\mathcal{F}_i \subseteq \mathcal{F}_j \quad \forall\, i \leq j$

Some technical definition

Definition

Let $\{X_i\}_{i \in I}$ be a stochastic process. Its *natural filtration* is given by

$$\mathcal{F}_i = \sigma\{X_j : j \leq i\}.$$

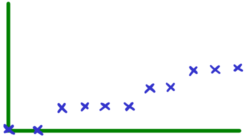
By definition, a stochastic process is always adapted to its natural filtration.

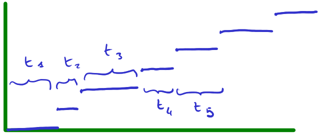
Counting processes

If we count how many times something happened up to a certain time, then we have a counting process!

For example:

- X_t is the number of waves hitting a rock by time t;
- X_t is the number of customers entering a shop by time t;
- X_t is the number of times a machine has been repaired by time t;





Formal definition:

Definition

A counting process is a discrete stochastic process with state space $\{0, 1, 2, \dots\}$ that can only increase by one and can never decrease.

Strong law of large numbers

da $E[X_1] < \infty$ A $E[X_1] = \infty$ CON CONVERGENZA MONOTONA

Theorem (Strong law of large numers)

Let $\{X_i\}_{i=1}^{\infty}$ be a collection of i.i.d. random variables for which $E[X_1]$ exists (it could be infinite). Then, with probability 1 we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n} = E[X_1]$$

Extremely useful! For example in statistics to estimate the proportion of people voting for a given candidate we can take the average of n trials.

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p = proportion people voting for A

$$X_i = \begin{cases} 1 & \text{se VOTA PER A} \\ 0 & \text{ALTRIMENTI} \end{cases} \qquad X_i \sim \text{Bern}(p)$$

$X_1, X_2, \ldots, X_n$ i.i.d.

$$\frac{\sum_{i=1}^{n} X_i}{n} \approx E[X_1] = p$$

Strong law of large numbers and stochastic processes

Independence is important to derive the law of large numbers, but a sequence of independent random variables is a boring stochastic process...

Sweet spot! We can consider independent inter-arrival times in a counting process!

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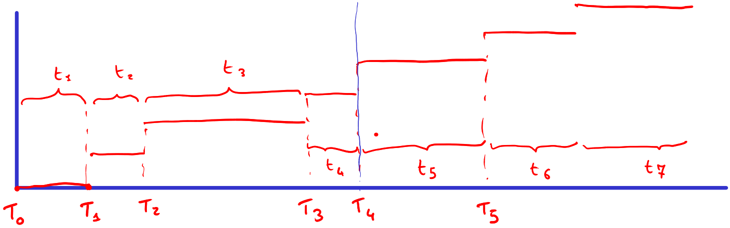

Definition

A *renewal process* is a counting process such that the times in between consecutive arrivals are a sequence of i.i.d. random variables.

Formally, the arrival times are defined by $T_0 = 0$ and:

$$T_\ell = \inf\{i \in I : X_i = \ell\} \quad \text{for } i = 1, 2, 3, \ldots$$

The inter-arrival times are defined by:

$$t_{\ell+1} = T_{\ell+1} - T_\ell$$

for all $\ell \in \{1, 2, 3, \ldots\}$.

$$T_n = \sum_{\ell=1}^{n} t_\ell = \cancel{T_1} - T_0 + \cancel{T_2} - \cancel{T_1} + \cancel{\ldots} + T_n - \cancel{T_{n-1}}$$
$$= T_n - T_0 = T_n$$

$t_1 > 0$

Theorem

Let $\{X_i\}_{i \in I}$ be a renewal process with inter-arrival times $\{t_i\}_{i=1}^{\infty}$, such that $0 < E[t_1]$ and $P(t_1 < \infty) = 1$. Then, with probability 1

$$\lim_{i \to \infty} \frac{X_i}{i} = \frac{1}{E[t_1]}.$$

$\left(= 0 \quad \text{se} \quad E[t_1] = \infty \right)$

Why?

Renewal processes

Theorem

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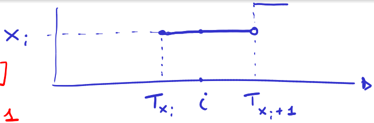


$$T_{X_i} \le i < T_{X_i+1}$$

$$\underbrace{\frac{\sum_{i=1}^{X_i} t_i}{X_i}}_{\xrightarrow{\text{SLLN}} E[t_1]} \le \frac{i}{X_i} < \underbrace{\frac{\sum_{i=1}^{X_i+1} t_i}{X_i+1}}_{\to E[t_1]} \cdot \underbrace{\frac{X_i+1}{X_i}}_{\to 1}$$

$$P(t_i < \infty) = 1$$

$$\Downarrow$$

CON PROB. 1

$$X_i \xrightarrow[i\to+\infty]{} +\infty$$

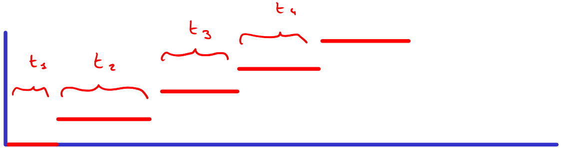
$$\frac{i}{X_i} \xrightarrow[i\to+\infty]{} E[t_1]$$

$$\frac{X_i}{i} \xrightarrow[i\to+\infty]{} \frac{1}{E[t_1]}$$

Application

Assume that the times between requests to a server are i.i.d. with mean $\mu = 5$ seconds. After a day, how many requests per second have been received?

X_t = # REQUESTS BY t. ← COUNTING PROCESS



t_i i.i.d.

$E[t_i] = 5\,s$

$$\frac{X_{24\cdot 60\cdot 60}}{24\cdot 60\cdot 60} \approx \lim_{t\to+\infty} \frac{X_t}{t} = \frac{1}{E[t_i]} = \frac{1}{5}$$

Renewal-reward processes

To extend the possible applications, we assume that in each inter-arrival interval $[T_\ell, T_{\ell+1}]$ a reward $r_{\ell+1}$ is given.

Example: if the factory machine does not break for at least one month, the tecnician receives a bonus of 5 euros.

In the long run, how much does the technician gain in bonuses per week?

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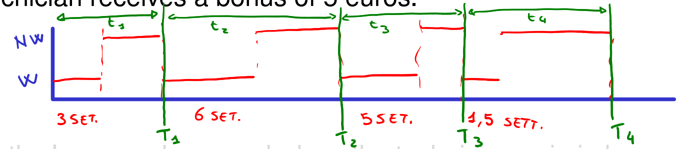

$r_1 = 0$
$r_2 = 5$
$r_3 = 5$
$r_4 = 0$

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$R(t)$ BONUS FINORA PERCEPITI

$$\frac{R(t)}{t}$$

Renewal-reward processes

Rewards can be:

- discrete (ex. fixed amounts of 5 euros);
- continuous (ex. the time the machine works);
- negative (ex. the repair may have a cost of 10 euros).

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Renewal-reward processes

$C_\ell = (t_\ell, r_\ell)$ $\{C_\ell\}_{\ell=1}^{\infty}$ i.i.d.

Definition

Let $\{X_i\}_{i\in I}$ be a renewal process with inter-arrival times $\{t_\ell\}_{\ell=1}^{\infty}$ and rewards $\{r_\ell\}_{\ell=1}^{\infty}$. If the random variables $\{\underbrace{(t_\ell, r_\ell)}_{C_\ell}\}_{\ell=1}^{\infty}$ are i.i.d. then we have a *renewal-reward process*.

Theorem

Let $\{X_i\}_{i\in I}$ be a renewal process with $E[t_1] > 0$, $P(t_1 < \infty) = 1$, and $E[r_1] < \infty$. Then, with probability 1,

$$\lim_{i\to\infty} \frac{R_i}{i} = \frac{E[r_1]}{E[t_1]},$$

where

$$R_i = \sum_{\ell=1}^{X_i} r_i$$

is the cumulative reward up to time i.

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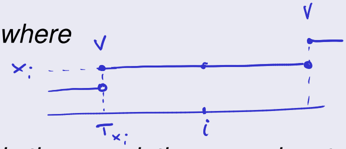


PROOF:

$$\frac{R_i}{i} = \frac{\sum_{\ell=1}^{X_i} r_\ell}{i} = \underbrace{\frac{\sum_{\ell=1}^{X_i} r_\ell}{X_i}}_{\to E[r_1]} \cdot \underbrace{\frac{X_i}{i}}_{\to \frac{1}{E[t_1]}}$$

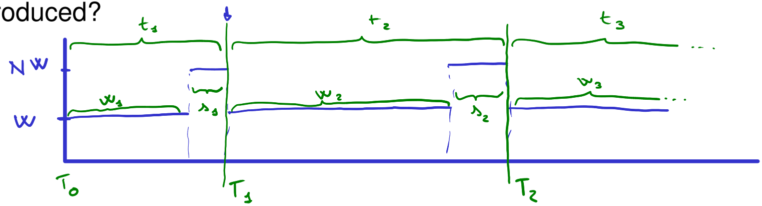
Application

$r_i = 10 \cdot w_i - 50 - 40 \cdot s_i$

$r_i \not\perp\!\!\!\perp t_i$

$\{(r_i, t_i)\}_i$ i.i.d.

Assume a machine create a revenue of 10 euros per day when it works, repairing it costs 50 euros of transportation plus a repairing cost of approximately 40 euros per day. The machine does not break down for times with mean 30 days and repairing it takes on average 5 days. Assume that when the machine is repaired, it can be considered as new.
After 10 years of usage, how much revenue per day has the machine produced?



$E[w_i] = 30$

$E[s_i] = 5$

Application

$$\frac{R_{10\cdot 365}}{10\cdot 365} \approx \lim_{t\to+\infty} \frac{R_t}{t} = \frac{E[r_1]}{E[t_1]} = \frac{10\,E[w_1] - 50 - 40\cdot E[s_1]}{E[w_1] + E[s_1]}$$

$$= \frac{10\cdot 30 - 50 - 40\cdot 5}{30+5}$$

$$= \frac{300 - 250}{35} = \frac{50}{35} \ €/\text{DAY}$$

Application

In the previous question, how does the answer change if repair times are geometric random variables, and times until break-downs are Poisson?

DOES NOT CHANGE!!