Doob's Decomposition Theorem

Doob's decomposition theorem

Theorem (Doob's decomposition theorem, part 1)

Let $(X_n)_{n=0}^{\infty}$ be a discrete-time stochastic process with $\mathbb{E}[|X_n|] < \infty$ for all $n \in \mathbb{N}$, and let $(\mathcal{F}_n)_{n=0}^{\infty}$ be its natural filtration. Then, there exists a unique decomposition

$$X_n = M_n + A_n$$

$M_n = X_n - A_n$

where

- $(M_n)_{n=0}^{\infty}$ *is a martingale w.r.t.* $(\mathcal{F}_n)_{n=0}^{\infty}$;
- $A_0 = 0$ *and* $(A_n)_{n=1}^{\infty}$ *is predictable w.r.t.* $(\mathcal{F}_n)_{n=0}^{\infty}$.

In particular, for all $n \geq 1$

$$A_n = \sum_{i=1}^{n} \overbrace{\mathbb{E}[X_i - X_{i-1} | \mathcal{F}_{i-1}]}^{\mathcal{F}_{i-1}\text{-MEAS} \Rightarrow \mathcal{F}_{n-1}\text{-MEAS.}}.$$

Theorem (Doob's decomposition theorem, part 2)

Moreover,

- *If* $(X_n)_{n=0}^{\infty}$ *is a supermartingale then* $A_{n+1} \leq A_n$ *for all* $n \in \mathbb{N}$ *almost surely;*
- *If* $(X_n)_{n=0}^{\infty}$ *is a submartingale then* $A_{n+1} \geq A_n$ *for all* $n \in \mathbb{N}$ *almost surely.*

Doob's decomposition theorem

The version we state and prove is slightly more general of what typically stated, where only submartingales or supermartingales are considered.

A similar decomposition for continuous time processes exists, but it holds under more technical assumptions (especially the uniqueness). The continuous time version is known as Doob-Meyer decomposition.

The process $(A_n)_{n=0}^{\infty}$ is called the compensator of the process $(X_n)_{n=0}^{\infty}$.

Doob's decomposition theorem

Proof.

First we show existence by considering $A_0 = 0$ and

$$A_n = \sum_{i=1}^{n} \mathbb{E}[X_i - X_{i-1} | \mathcal{F}_{i-1}] = \sum_{i=1}^{n} (\mathbb{E}[X_i | \mathcal{F}_{i-1}] - X_{i-1}) \quad \text{for } n \geq 1.$$

The process $(A_n)_{n=1}^{\infty}$ is predictable w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ by construction. Let $M_n = X_n - A_n$. Then, $\mathbb{E}[|M_0|] = \mathbb{E}[|X_0|] < \infty$ and by triangular inequality and Jensen's inequality for all $n \geq 1$ we have

$$\mathbb{E}[|M_n|] \leq \mathbb{E}[|X_n|] + \mathbb{E}[|A_n|] \leq \overset{<\infty}{\mathbb{E}[|X_n|]} + \sum_{i=1}^{n} \left(\mathbb{E}\underbrace{\left[\mathbb{E}[|X_i| | \mathcal{F}_{i-1}]\right]}_{\overset{\text{T.P.}}{=} \mathbb{E}[|X_i|]} + \mathbb{E}[|X_{i-1}|] \right)$$

$$= \mathbb{E}[|X_n|] + \sum_{i=1}^{n} (\mathbb{E}[|X_i|] + \mathbb{E}[|X_{i-1}|]) < \infty.$$

□

Doob's decomposition theorem

Proof.

Moreover, for all $n \geq 0$, since $M_n = X_n - A_n$,

$$
\begin{aligned}
\mathbb{E}[M_{n+1} | \mathcal{F}_n] &= \mathbb{E}[X_{n+1} - \overbrace{A_{n+1}}^{\mathcal{F}_n\text{-MEAS.}} | \mathcal{F}_n] \\
&= \mathbb{E}[X_{n+1} | \mathcal{F}_n] - (A_{n+1} - A_n) - A_n \\
&= \mathbb{E}[\cancel{X_{n+1}} | \mathcal{F}_n] - \mathbb{E}[\cancel{X_{n+1}} | \mathcal{F}_n] + X_n - A_n \\
&= M_n.
\end{aligned}
$$

□

Doob's decomposition theorem

Proof.

We now prove uniqueness. Assume that $X_n = M_n + A_n$ and $X_n = M'_n + A'_n$ with $(M_n)_{n=0}^{\infty}$ and $(M'_n)_{n=0}^{\infty}$ martingales, $A_0 = A'_0 = 0$, and $(A_n)_{n=1}^{\infty}$ and $(A'_n)_{n=1}^{\infty}$ predictable processes. Then

$$\begin{aligned} A'_{n+1} - A'_n &= X_{n+1} - X_n + M'_{n+1} - M'_n \\ &= (A_{n+1} - A_n) + (M_{n+1} - M_n) + (M'_{n+1} - M'_n). \end{aligned}$$

Taking $\mathbb{E}[\cdot|\mathcal{F}_n]$ leads to (by predictability and martingale properties)

PREDICT. PROC.

$$\begin{aligned} A'_{n+1} - A'_n &\stackrel{\downarrow}{=} \mathbb{E}[A'_{n+1} - A'_n|\mathcal{F}_n] \\ &= \mathbb{E}[A_{n+1} - A_n|\mathcal{F}_n] + \overbrace{\mathbb{E}[M_{n+1} - M_n|\mathcal{F}_n]}^{\overset{0}{''}} + \overbrace{\mathbb{E}[M'_{n+1} - M'_n|\mathcal{F}_n]}^{\overset{0}{''}} \\ &= A_{n+1} - A_n. \end{aligned}$$

Since $A_0 = A'_0 = 0$, $A_n = A'_n$ for all n by induction. $\square$

Doob's decomposition theorem

Proof.

Now assume that $(X_n)_{n=0}^{\infty}$ is a supermartingale. Then, almost surely,

$$A_{n+1} - A_n = \mathbb{E}[X_n - \underbrace{X_{n-1}}_{\mathcal{F}_{n-1}\text{-MEAS.}} | \mathcal{F}_{n-1}] = \mathbb{E}[X_n | \mathcal{F}_{n-1}] - X_{n-1} \leq 0. \quad \text{a.s.}$$

Similarly, if $(X_n)_{n=0}^{\infty}$ is a submartingale then, almost surely,

$$A_{n+1} - A_n = \mathbb{E}[X_n - X_{n-1} | \mathcal{F}_{n-1}] = \mathbb{E}[X_n | \mathcal{F}_{n-1}] - X_{n-1} \geq 0.$$

$\square$

Example 1

Let $(W_n)_{n=0}^{\infty}$ be a random walk with $p = 1/2$. $(W_n)_{n=0}^{\infty}$ is a martingale. Let

$$X_n = |W_n|.$$

By Jensen's inequality, $(X_n)_{n=0}^{\infty}$ is a submartingale.
How can we decompose X_n?

Example 1

Let $(W_n)_{n=0}^{\infty}$ be a random walk with $p = 1/2$. $(W_n)_{n=0}^{\infty}$ is a martingale. Let

$$X_n = |W_n|.$$

Let $(\mathcal{F}_n)_{n=0}^{\infty}$ be the natural filtration of $(X_n)_{n=0}^{\infty}$, which is strictly smaller than the natural filtration of $(W_n)_{n=0}^{\infty}$. We have

$$= \mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n]$$

$$A_{n+1} - A_n = \mathbb{E}[|W_{n+1}| - |W_n| \,|\, \mathcal{F}_n] = \begin{cases} 0 & \text{if } X_n = |W_n| \neq 0 \\ 1 & \text{if } X_n = W_n = 0. \end{cases} = \mathbb{1}_{\{X_n = 0\}}$$

Hence, $A_0 = 0$ and for all $n \geq 1$

$$= \sum_{i=0}^{n-1} (A_{i+1} - A_i)$$

$$A_n = \sum_{i=0}^{n-1} \mathbb{1}_{\{X_n = 0\}}$$

is the number of visits of $(X_n)_{n=0}^{\infty}$ to 0 by time $n-1$ (included). We can write $X_n = M_n + A_n$ with $(M_n)_{n=0}^{\infty}$ martingale and $(A_n)_{n=0}^{\infty}$ predictable w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$.

Example 1

As a byproduct of the Doob's decomposition theorem, we obtained that

$$M_n = X_n - A_n = |W_n| - \sum_{i=0}^{n-1} \mathbb{1}_{\{X_n=0\}}$$

is a martingale with respect to the natural filtration of $(W_n)_{n=0}^{\infty}$, without doing the calculations! Also,

$$\mathbb{E}[|W_n|] = \mathbb{E}[M_n] + \mathbb{E}\left[\overbrace{\sum_{i=0}^{n-1} \mathbb{1}_{\{X_n=0\}}}^{A_n}\right] = \mathbb{E}[M_0] + \mathbb{E}\left[\sum_{i=0}^{n-1} \mathbb{1}_{\{X_n=0\}}\right]$$
$$= \mathbb{E}[|W_0|] + \mathbb{E}\left[\sum_{i=0}^{n-1} \mathbb{1}_{\{X_n=0\}}\right].$$

Example 2

Let $(W_n)_{n=0}^{\infty}$ be a random walk with $p = 1/2$. $(W_n)_{n=0}^{\infty}$ is a martingale. Let

$$X_n = W_n^2.$$

By Jensen's inequality, $(X_n)_{n=0}^{\infty}$ is a submartingale.
How can we decompose X_n?

Example 2

Let $(W_n)_{n=0}^{\infty}$ be a random walk with $p = 1/2$. $(W_n)_{n=0}^{\infty}$ is a martingale. Let

$$X_n = W_n^2.$$

Let $\mathcal{F}_n$ be the natural filtration of $(X_n)_{n=0}^{\infty}$, which is strictly smaller than the natural filtration of $(W_n)_{n=0}^{\infty}$. We have

$$= E[X_{n+1} - X_n | F_n]$$

$$A_{n+1} - A_n = \mathbb{E}[W_{n+1}^2 - W_n^2 | \mathcal{F}_n] \underset{\text{PITAGORA}}{=} \mathbb{E}[(\overbrace{W_{n+1} - W_n}^{1})^2 | \mathcal{F}_n] = 1$$

Hence, $A_0 = 0$ and for all $n \geq 1$

$$= \sum_{i=0}^{n-1} (A_{i+1} - A_i)$$

$$A_n = \sum_{i=0}^{n-1} 1 = n.$$

We can write $X_n = M_n + A_n$ with $(M_n)_{n=0}^{\infty}$ martingale and $(A_n)_{n=0}^{\infty}$ predictable w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$.

$$W_n^2 = M_n + n$$

Example 2

As a byproduct of the Doob's decomposition theorem, we obtained that

$$M_n = X_n - A_n = W_n^2 - n$$

is a martingale with respect to the natural filtration of $(W_n)_{n=0}^{\infty}$, without doing the calculations! Also,

$$\mathbb{E}[W_n^2] = \mathbb{E}[M_n] + n = \mathbb{E}[M_0] + n = \mathbb{E}[W_0^2] + n.$$

Example 3

Let $(X_n)_{n=0}^{\infty}$ be a random walk with $p \neq 1/2$.

How can we decompose X_n?

Example 3

Let $(X_n)_{n=0}^{\infty}$ be a random walk with $p \neq 1/2$.

Let $\mathcal{F}_n$ be the natural filtration of $(X_n)_{n=0}^{\infty}$. Let $q = 1 - p$. We have

$$A_{n+1} - A_n = \mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = p \cdot \underline{1} + q \cdot (\underline{-1}) = \underline{p - q}.$$

Hence, $A_0 = 0$ and for all $n \geq 1$

$$A_n = \sum_{i=0}^{n-1} (p - q) = n(p - q).$$

We can write $X_n = M_n + A_n$ with $(M_n)_{n=0}^{\infty}$ martingale and $(A_n)_{n=0}^{\infty}$ predictable w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$. In particular, we have that

$$M_n = X_n - n(p - q)$$

is a martingale.

Martingales and stopping times

Does the martingale property hold at stopping times?

Let $(X_i)_{i\in I}$ be a martingale w.r.t. $(\mathcal{F}_i)_{i\in I}$. By definition

$$\mathbb{E}[X_j|\mathcal{F}_i] = X_i$$

for all $i < j$. What about stopping times? If T is a stopping time with $T > i$, do we have

$$\mathbb{E}[X_T|\mathcal{F}_i] = X_i?$$

Does the martingale property hold at stopping times?

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for all $i < j$. What about stopping times? If T is a stopping time with $T > i$, do we have

$$\mathbb{E}[X_T|\mathcal{F}_i] = X_i? \qquad \underline{\text{No}}$$

Let $(X_n)_{n=0}^{\infty}$ be the random walk with $p = 1/2$ and let i be any natural number. Let

$$T = \inf\{n \geq i \,:\, X_n = 5\}.$$

Then,

$$\mathbb{E}[X_T|\mathcal{F}_i] = 5 \neq X_i.$$

Does the martingale property hold at stopping times?

Let $(X_i)_{i\in I}$ be a martingale w.r.t. $(\mathcal{F}_i)_{i\in I}$. By definition

$$\mathbb{E}[X_j|\mathcal{F}_i] = X_i$$

for all $i < j$. What about stopping times? If T is a stopping time with $T > i$, do we have

$$\mathbb{E}[X_T|\mathcal{F}_i] = X_i?$$

Let $(X_n)_{n=0}^{\infty}$ be the random walk with $p = 1/2$ and let i be any natural number. Let

$$T = \inf\{n \geq i \,:\, X_n = 5\}.$$

Then,

$$\mathbb{E}[X_T|\mathcal{F}_i] = 5 \neq X_i.$$

However, under certain conditions things work out smoothly, and stopping times and martingales are meant to be together.

Stopped processes

Notation: Given two numbers a, b, we denote

$$a \wedge b = \min\{a, b\}.$$

In particular, if $(X_i)_{i \in I}$ is a stochastic process and τ a random variable with values in I, then $(X_{i \wedge \tau})_{i \in I}$ defines a stochastic process with

$$X_{i \wedge \tau} = \begin{cases} X_i & \text{if } \tau \geq i \\ X_\tau & \text{if } \tau \leq i \end{cases}$$

Stopped σ-algebra

τ STOPPING TIME $\{\tau \leq n\} \in \mathcal{F}_n$

$i \wedge \tau$ STOPPING TIME
$\{i \wedge \tau \leq n\} \in \mathcal{F}_{n \wedge \tau} \subseteq \mathcal{F}_n$

Definition

Consider a filtration $(\mathcal{F}_i)_{i \in I}$ and let τ be a stopping time w.r.t. $(\mathcal{F}_i)_{i \in I}$. Define

$$\mathcal{F}_{i \wedge \tau} = \{A \in \mathcal{F}_i \,:\, A \cap \{\tau \leq j\} \in \mathcal{F}_j \forall j \leq i\} \subseteq \mathcal{F}_i.$$

Theorem

Let $(X_i)_{i \in I}$ be a stochastic process adapted to $(\mathcal{F}_i)_{i \in I}$, and let τ be a stopping time w.r.t. $(\mathcal{F}_i)_{i \in I}$. Then

- $(\mathcal{F}_{i \wedge \tau})_{i \in I}$ *is a filtration;*
- $(X_{i \wedge \tau})_{i \in I}$ *is adapted to* $(\mathcal{F}_{i \wedge \tau})_{i \in I}$*;*
- $(X_{i \wedge \tau})_{i \in I}$ *is adapted to* $(\mathcal{F}_i)_{i \in I}$*.*

Proof.

If $\mathcal{F}$ is such that $\mathcal{F}_i \subseteq \mathcal{F}$ for all $i \in I$, then $\mathcal{F}_{i\wedge\tau} \subseteq \mathcal{F}_i \subseteq \mathcal{F}$ for all $i \in I$. Moreover if $i < k$ then $\mathcal{F}_i \subseteq \mathcal{F}_k$ and

$$\begin{aligned}\mathcal{F}_{i\wedge\tau} &= \{A \in \mathcal{F}_i : A \cap \{\tau \leq j\} \in \mathcal{F}_j \forall j \leq i\} \\ &\subseteq \{A \in \mathcal{F}_k : A \cap \{\tau \leq j\} \in \mathcal{F}_j \forall j \leq k\} = \mathcal{F}_{k\wedge\tau}.\end{aligned}$$

Now note that

$$\underline{X_{i\wedge\tau} = \mathbb{1}_{\{\tau\leq i\}} X_\tau + \mathbb{1}_{\{\tau>i\}} X_i}$$

which is both $\mathcal{F}_i$-measurable and $\mathcal{F}_{i\wedge\tau}$-measurable because τ is a stopping time w.r.t. $(\mathcal{F}_i)_{i\in I}$. □

Discrete time martingales and stopping times

Theorem

Let $(X_n)_{n=0}^{\infty}$ be a martingale, submartingale, or supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$. Let τ be a stopping times w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$. Then,

- *if $(X_n)_{n=0}^{\infty}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ then $(X_{n\wedge\tau})_{n=0}^{\infty}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ and w.r.t. $(\mathcal{F}_{n\wedge\tau})_{n=0}^{\infty}$;*
- *if $(X_n)_{n=0}^{\infty}$ is a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ then $(X_{n\wedge\tau})_{n=0}^{\infty}$ is a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ and w.r.t. $(\mathcal{F}_{n\wedge\tau})_{n=0}^{\infty}$;*
- *if $(X_n)_{n=0}^{\infty}$ is a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ then $(X_{n\wedge\tau})_{n=0}^{\infty}$ is a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ and w.r.t. $(\mathcal{F}_{n\wedge\tau})_{n=0}^{\infty}$.*

Proof.

Since τ is a stopping time, we know that $X_{n\wedge\tau}$ is $\mathcal{F}_n$-measurable and $\mathcal{F}_{n\wedge\tau}$-measurable. Moreover, we have $X_{n\wedge\tau} = X_i$ for some $0 \leq i \leq n$, hence

$|X_{n\wedge\tau}| \leq |X_0| + |X_1| + \dots + |X_n|$

$$\mathbb{E}[|X_{n\wedge\tau}|] \leq \sum_{i=0}^{n} \mathbb{E}[|X_i|] < \infty.$$

Now, note that for any $n \geq 0$

$$X_{(n+1)\wedge\tau} - X_{n\wedge\tau} = \begin{cases} X_{n+1} - X_n & \text{if } \tau \geq n+1 \\ 0 & \text{if } \tau \leq n \end{cases} = \mathbb{1}_{\{\tau \geq n+1\}}(X_{n+1} - X_n).$$

$\square$

Stopped process

Proof.

Assume that $(X_n)_{n=0}^{\infty}$ is a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$. Then

$$\begin{aligned}
\mathbb{E}[X_{(n+1)\wedge\tau}|\mathcal{F}_n] - X_{n\wedge\tau} &= \mathbb{E}[\overset{\mathcal{F}_n\text{-MEAS.}}{\mathbb{1}_{\{\tau\geq n+1\}}}(X_{n+1} - X_n)|\mathcal{F}_n] \\
&= \underbrace{\mathbb{1}_{\{\tau\geq n+1\}}}_{\geq 0}\underbrace{\mathbb{E}[X_{n+1} - X_n|\mathcal{F}_n]}_{\geq 0} \geq 0.
\end{aligned}$$

So, $(X_{n\wedge\tau})_{n=0}^{\infty}$ is a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$. Moreover,

$$\mathbb{E}[X_{(n+1)\wedge\tau}|\mathcal{F}_{n\wedge\tau}] \overset{\text{T.P.}}{=} \mathbb{E}\left[\overbrace{\mathbb{E}[X_{(n+1)\wedge\tau}|\mathcal{F}_n]}^{\geq X_{n\wedge\tau}}\middle|\mathcal{F}_{n\wedge\tau}\right] \geq \mathbb{E}[X_{n\wedge\tau}|\mathcal{F}_{n\wedge\tau}] = \underline{X_{n\wedge\tau}}.$$

So, $(X_{n\wedge\tau})_{n=0}^{\infty}$ is a submartingale w.r.t. $(\mathcal{F}_{n\wedge\tau})_{n=0}^{\infty}$.

We can now either repeat the same arguments for supermartingales and martingales, or apply a common strategy shown in the next slide.

□

Proof.

Assume that $(X_n)_{n=0}^\infty$ is a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$.

- Then $(-X_n)_{n=0}^\infty$ is a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$.
- For what we have already shown, $(-X_{n\wedge\tau})_{n=0}^\infty$ is a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$ and w.r.t. $(\mathcal{F}_{n\wedge\tau})_{n=0}^\infty$.
- Hence, $(X_{n\wedge\tau})_{n=0}^\infty$ is a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$ and w.r.t. $(\mathcal{F}_{n\wedge\tau})_{n=0}^\infty$.

Assume that $(X_n)_{n=0}^\infty$ is a martingale w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$.

- Then $(X_n)_{n=0}^\infty$ is both a submartingale and a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$.
- For what we have already shown, $(X_{n\wedge\tau})_{n=0}^\infty$ is both a submartingale and a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$ and w.r.t. $(\mathcal{F}_{n\wedge\tau})_{n=0}^\infty$.
- Hence, $(X_{n\wedge\tau})_{n=0}^\infty$ is a martingale w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$ and w.r.t. $(\mathcal{F}_{n\wedge\tau})_{n=0}^\infty$.

$\square$

Doob's optional sampling theorem in discrete time

Theorem (Doob's optional sampling theorem in discrete time)

Let $(X_n)_{n=0}^\infty$ be a martingale, submartingale, or supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$. Let τ and σ be two stopping times w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$, such that almost surely,

$$\sigma \leq \tau \leq C$$

for a constant $C \in \mathbb{R}$. Then, $\mathbb{E}[|X_\tau|], \mathbb{E}[|X_\sigma|] < \infty$ and

- *if $(X_n)_{n=0}^\infty$ is a martingale w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$ then $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] = X_\sigma$ a.s.;*
- *if $(X_n)_{n=0}^\infty$ is a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$ then $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] \geq X_\sigma$ a.s.;*
- *if $(X_n)_{n=0}^\infty$ is a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^\infty$ then $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] \leq X_\sigma$ a.s..*

Doob's optional sampling theorem in discrete time

Proof.

$\mathbb{E}[|X_\tau|], \mathbb{E}[|X_\sigma|] < \infty$ because $\sigma, \tau \leq C$ a.s. hence

$|X_\tau| \leq |X_0| + |X_1| + \dots + |X_C|$

$$\mathbb{E}[|X_\tau|] \leq \sum_{n=0}^{C} \mathbb{E}[|X_n|] < \infty \quad \text{and} \quad \mathbb{E}[|X_\sigma|] \leq \sum_{n=0}^{C} \mathbb{E}[|X_n|] < \infty.$$

We now assume that $(X_n)_{n=0}^{\infty}$ is a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$. Note that both X_σ and $\mathbb{E}[X_\tau|\mathcal{F}_\sigma]$ are $\mathcal{F}_\sigma$-measurable. To prove that $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \geq X_\sigma$ a.s. it suffices to show that for all $A \in \mathcal{F}_\tau$ with $P(A) > 0$ we have

$$\int_A E[X_\tau|\mathcal{F}_\sigma](\omega)dP(\omega) = \mathbb{E}\left[\mathbb{1}_A \mathbb{E}[X_\tau|\mathcal{F}_\sigma]\right] \geq \mathbb{E}[\mathbb{1}_A X_\sigma]. = \int_A X_\sigma(\omega)dP(\omega)$$

Indeed, if we had $P(B) > 0$ with $B = \{\omega : \mathbb{E}[X_\tau|\mathcal{F}_\sigma](\omega) < X_\sigma(\omega)\} \in \mathcal{F}_\tau$, then we would have that $\mathbb{1}_B \mathbb{E}[X_\tau|\mathcal{F}_\sigma] - \mathbb{1}_B X_\sigma$ is a non-null, non-positive random variable and as a consequence $\mathbb{E}\left[\mathbb{1}_B \mathbb{E}[X_\tau|\mathcal{F}_\sigma]\right] < \mathbb{E}[\mathbb{1}_B X_\sigma]$. $\square$

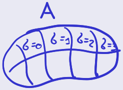
Proof.

By definition of conditional expectation,

$$\mathbb{E}\Big[\mathbb{1}_A \mathbb{E}[X_\tau | \mathcal{F}_\sigma]\Big] = \int_A \mathbb{E}[X_\tau | \mathcal{F}_\sigma](\omega) dP(\omega) \overset{\text{DEF. DI SPER. COND.}}{=} \int_A X_\tau(\omega) dP(\omega) = \mathbb{E}[\mathbb{1}_A X_\tau].$$

$$\geq \mathbb{E}[\mathbb{1}_A X_\sigma]$$

Since $\sigma, \tau < C$ a.s., we further have that a.s.



$$\mathbb{1}_A X_\tau = \sum_{i=0}^{C} \mathbb{1}_{A \cap \{\sigma = i\}} X_\tau = \sum_{i=0}^{C} \mathbb{1}_{A \cap \{\sigma = i\}} X_{C \wedge \tau}.$$

Since $(X_{n\wedge\tau})_{n=0}^\infty$ is a submartingale w.r.t. $\underline{(\mathcal{F}_n)_{n=0}^\infty}$, for all $0 \leq i \leq C$ we have

$$\mathbb{E}[X_{C\wedge\tau} | \mathcal{F}_i] \geq X_{i\wedge\tau} \quad \text{a.s.}$$

□

Doob's optional sampling theorem in discrete time

Proof.

Moreover, since $A \in \mathcal{F}_\sigma$ then $A \cap \{\sigma = i\} \in \mathcal{F}_i$. It follows that

$$E\big[\underbrace{\mathbb{1}_{A\cap\{\sigma=i\}}}_{\mathcal{F}_i\text{-MEAS.}} X_{C\wedge\tau} \,|\, \mathcal{F}_i\big] = \mathbb{1}_{A\cap\{\sigma=i\}}\mathbb{E}[X_{C\wedge\tau}|\mathcal{F}_i] \geq \mathbb{1}_{A\cap\{\sigma=i\}} X_{i\wedge\tau} \quad \text{a.s.}$$

and by taking expectations on both sides (T.P.)

$$\mathbb{E}[\mathbb{1}_{A\cap\{\sigma=i\}} X_{C\wedge\tau}] \geq \mathbb{E}[\mathbb{1}_{A\cap\{\sigma=i\}} X_{i\wedge\tau}]$$

In conclusion, (in the last step we use $\sigma \leq \tau$ a.s.)

$$\mathbb{E}\Big[\mathbb{1}_A \mathbb{E}[X_\tau|\mathcal{F}_\sigma]\Big] = \mathbb{E}[\mathbb{1}_A X_\tau] = \sum_{i=0}^{C} \mathbb{E}[\mathbb{1}_{A\cap\{\sigma=i\}} X_{C\wedge\tau}] \geq \sum_{i=0}^{C} \mathbb{E}[\mathbb{1}_{A\cap\{\sigma=i\}} X_{i\wedge\tau}]$$

$$= \sum_{i=0}^{C} \mathbb{E}[\mathbb{1}_{A\cap\{\sigma=i\}} X_{\sigma\wedge\tau}] = \mathbb{E}[\mathbb{1}_A X_{\sigma\wedge\tau}] \overset{\sigma \leq \tau \text{ a.s.}}{=} \mathbb{E}[\mathbb{1}_A X_\sigma].$$

A

$\sigma=0 \quad \sigma=1$

□

Doob's optional sampling theorem in discrete time

Proof.

We now assume that $(X_n)_{n=0}^{\infty}$ is a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$.

- Then, $(-X_n)_{n=0}^{\infty}$ is a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$.
- It follows that $\mathbb{E}[-X_\tau|\mathcal{F}_\sigma] \geq -X_\sigma$ a.s.
- Hence, $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \leq X_\sigma$ a.s.

We now assume that $(X_n)_{n=0}^{\infty}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$.

- Then, $(X_n)_{n=0}^{\infty}$ is both a submartingale and a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$.
- It follows that $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \geq X_\sigma$ a.s. and $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \leq X_\sigma$ a.s.
- Hence, $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] = X_\sigma$ a.s.

$\square$

Theorem (Doob's optional stopping theorem in discrete time)

Let $(X_n)_{n=0}^{\infty}$ be a martingale, submartingale, or supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$. Let τ and σ be two stopping times w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$, such that

- *almost surely, $\sigma \leq \tau < \infty$;*
- $\mathbb{E}[|X_\sigma|], \mathbb{E}[|X_\tau|] < \infty$;
- $\lim_{n\to\infty} \mathbb{E}[|X_n| \mathbb{1}_{\{\tau > n\}}] = 0$.

Then,

- *if $(X_n)_{n=0}^{\infty}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ then $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] = X_\sigma$ a.s.;*
- *if $(X_n)_{n=0}^{\infty}$ is a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ then $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] \geq X_\sigma$ a.s.;*
- *if $(X_n)_{n=0}^{\infty}$ is a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$ then $\mathbb{E}[X_\tau | \mathcal{F}_\sigma] \leq X_\sigma$ a.s..*

Doob's optional stopping theorem in discrete time

Proof.

We can write

$$X_\tau = X_{n\wedge\tau} + (X_\tau - X_n)\mathbb{1}_{\{\tau>n\}}.$$

Let $(X_n)_{n=0}^{\infty}$ be a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$. By the optional sampling theorem we have

$$\mathbb{E}[X_{n\wedge\tau}|\mathcal{F}_{n\wedge\sigma}] \geq X_{n\wedge\sigma} \quad \text{a.s.}$$

Hence, for every $n \geq 0$,

$$\mathbb{E}[X_\tau|\mathcal{F}_{n\wedge\sigma}] \geq X_{n\wedge\sigma} + \mathbb{E}[X_\tau \mathbb{1}_{\{\tau>n\}}|\mathcal{F}_{n\wedge\sigma}] - \mathbb{E}[X_n \mathbb{1}_{\{\tau>n\}}|\mathcal{F}_{n\wedge\sigma}] \quad \text{a.s.}$$

In order to prove $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \geq X_\sigma$ a.s. it suffices to show that for all $A \in \mathcal{F}_\sigma$ with $P(A) > 0$ we have

DEF. COND. EXP.

$$\int_A \mathbb{E}[X_\tau|\mathcal{F}_\sigma](\omega)dP(\omega) \overset{!}{=} \int_A X_\tau(\omega)dP(\omega) \geq \int_A X_\sigma(\omega)dP(\omega).$$

□

Doob's optional stopping theorem in discrete time

Proof.

Since $\sigma < \infty$ a.s. we have that $\lim_{n\to\infty} X_\tau \mathbb{1}_{A\cap\{\sigma\le n\}} = X_\tau \mathbb{1}_A$ a.s. and using $\mathbb{E}[|X_\tau|] < \infty$ we get by dominated convergence

$$\int_A X_\tau(\omega)dP(\omega) = \mathbb{E}[X_\tau \mathbb{1}_A] = \lim_{n\to\infty} \mathbb{E}[X_\tau \mathbb{1}_{A\cap\{\sigma\le n\}}].$$

By definition of conditional expectation and by the inequality above,

$$\begin{aligned}
\mathbb{E}[X_\tau \mathbb{1}_{A\cap\{\sigma\le n\}}] &= \int_{A\cap\{\sigma\le n\}} X_\tau(\omega)dP(\omega) \overset{\text{DEF. COND. EXP.}}{=} \int_{A\cap\{\sigma\le n\}} \mathbb{E}[X_\tau|\mathcal{F}_{n\wedge\sigma}](\omega)dP(\omega)\\
&\ge \int_{A\cap\{\sigma\le n\}} X_{n\wedge\sigma}(\omega)dP(\omega) + \underbrace{\int_{A\cap\{\sigma\le n\}} \mathbb{E}[X_\tau \mathbb{1}_{\{\tau>n\}}|\mathcal{F}_{n\wedge\sigma}](\omega)dP(\omega)}_{\to 0}\\
&\quad - \underbrace{\int_{A\cap\{\sigma\le n\}} \mathbb{E}[X_n \mathbb{1}_{\{\tau>n\}}|\mathcal{F}_{n\wedge\sigma}](\omega)dP(\omega)}_{\to 0}.
\end{aligned}$$

We need to study the limits of the three terms above. □

Doob's optional stopping theorem in discrete time

Proof.

We first study the first term

$$\int_{A\cap\{\sigma\le n\}} X_{n\wedge\sigma}(\omega)dP(\omega) = \mathbb{E}[X_{n\wedge\sigma}\mathbb{1}_{A\cap\{\sigma\le n\}}] = \mathbb{E}[X_{\sigma}\mathbb{1}_{A\cap\{\sigma\le n\}}]$$

Since $\mathbb{E}[|X_\sigma|] < \infty$ and again by dominated convergence

$$\lim_{n\to\infty} \mathbb{E}[X_\sigma \underbrace{\mathbb{1}_{A\cap\{\sigma\le n\}}}_{\downarrow\ \mathbb{1}_A}] = \mathbb{E}[X_\sigma \mathbb{1}_A].$$

□

Doob's optional stopping theorem in discrete time

Proof.

For the second term, we have by definition of conditional expectation

$$\left|\int_{A\cap\{\sigma\leq n\}} \mathbb{E}[X_\tau \mathbb{1}_{\{\tau>n\}}|\mathcal{F}_{n\wedge\sigma}](\omega)dP(\omega)\right| \overset{\text{DEF. EXP. COND.}}{=} \left|\int_{A\cap\{\sigma\leq n\}} X_\tau(\omega)\mathbb{1}_{\{\tau>n\}}(\omega)dP(\omega)\right|$$

then by Jensen's inequality

$$\leq \int_{A\cap\{\sigma\leq n\}} |X_\tau|(\omega)\mathbb{1}_{\{\tau>n\}}(\omega)dP(\omega) = \mathbb{E}[\underline{|X_\tau|\mathbb{1}_{\{\tau>n\}}\mathbb{1}_{A\cap\{\sigma\leq n\}}}]$$

and finally by positivity of $|X_\tau|$, by $\tau < \infty$ a.s., and by monotone convergence

$$\leq \mathbb{E}[\underline{|X_\tau|\mathbb{1}_{\{\tau>n\}}}] = \sum_{i=n+1}^{\infty} \mathbb{E}[|X_\tau|\,|\,\tau = i]P(\tau = i) \xrightarrow[n\to\infty]{} 0$$

see next slide

□

Proof.

where the last limit holds because, still by $\tau < \infty$ a.s., by monotone convergence and by assumption,

$$\mathbb{E}[|X_\tau|] = \sum_{i=0}^{\infty} \mathbb{E}[|X_\tau| \,|\, \tau = i] P(\tau = i) < \infty.$$

□

Doob's optional stopping theorem in discrete time

Proof.

For the third term we have

$$\left|\int_{A\cap\{\sigma\leq n\}}\mathbb{E}[X_n\mathbb{1}_{\{\tau>n\}}|\mathcal{F}_{n\wedge\sigma}](\omega)dP(\omega)\right| = \left|\int_{A\cap\{\sigma\leq n\}}X_n(\omega)\mathbb{1}_{\{\tau>n\}}(\omega)dP(\omega)\right|$$

then by Jensen's inequality

$$\leq \int_{A\cap\{\sigma\leq n\}}|X_n|(\omega)\mathbb{1}_{\{\tau>n\}}(\omega)dP(\omega) = \mathbb{E}[|X_n|\mathbb{1}_{\{\tau>n\}}\mathbb{1}_{A\cap\{\sigma\leq n\}}]$$

and finally by positivity of $|X_n|$ and by assumption

$$\leq \mathbb{E}[|X_n|\mathbb{1}_{\{\tau>n\}}] \xrightarrow[n\to\infty]{} 0.$$

$\square$

Doob's optional stopping theorem in discrete time

Proof.

Putting everything together,

$$\int_A X_\tau(\omega)dP(\omega) = \lim_{n\to\infty} \mathbb{E}[X_\tau \mathbb{1}_{A\cap\{\sigma\leq n\}}] \geq \mathbb{E}[X_\sigma \mathbb{1}_A] = \int_A X_\sigma(\omega)dP(\omega)$$

which is what we wanted to prove.

□

Doob's optional stopping theorem in discrete time

Proof.

We now assume that $(X_n)_{n=0}^{\infty}$ is a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$.

- Then, $(-X_n)_{n=0}^{\infty}$ is a submartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$.
- It follows that $\mathbb{E}[-X_\tau|\mathcal{F}_\sigma] \geq -X_\sigma$ a.s.
- Hence, $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \leq X_\sigma$ a.s.

We now assume that $(X_n)_{n=0}^{\infty}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$.

- Then, $(X_n)_{n=0}^{\infty}$ is both a submartingale and a supermartingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$.
- It follows that $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \geq X_\sigma$ a.s. and $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \leq X_\sigma$ a.s.
- Hence, $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] = X_\sigma$ a.s.

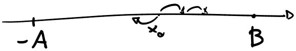
$\square$

A first taste of Martingale power: easy formulas for expected exit times and exit distributions of Markov chains.

Stopped symmetric random walk

Let $(X_n)_{n=0}^{\infty}$ the random walk with $p = 1/2$, and let $X_0 = x_0$. Fix two natural numbers $A, B \geq 0$ with $-A < x_0 < B$. Define

$$\tau = \inf\{n : X_n \leq -A \text{ or } X_n \geq B\}$$



From the point of view of a gambler, this may correspond to the strategy of stop playing when either the winnings reach B or the losses reach A.

We are interested in studying $P(X_\tau = -A)$ and $P(X_\tau = B)$. We simply use Doob's optional stopping theorem with $\sigma = 0$! Indeed, if we have $\mathbb{E}_{x_0}[X_\tau] = \mathbb{E}_{x_0}[X_0] = x_0$ then, by using $P_{x_0}(X_\tau = B) = 1 - P_{x_0}(X_\tau = -A)$

$$x_0 = \mathbb{E}_{x_0}[X_\tau] = -AP_{x_0}(X_\tau = -A) + BP_{x_0}(X_\tau = B) = (-A - B)P_{x_0}(X_\tau = -A) + B$$

hence

$$P(X_\tau = -A) = \frac{B - x_0}{A + B} \quad \text{and} \quad P(X_\tau = B) = \frac{A + x_0}{A + B}.$$

Stopped symmetric random walk

Let's check the assumption of Doob's stopping theorem are satisfied.

- almost surely, $0 \leq \tau < \infty$: we surely exit the set $[-A, B]$ if there are $A+B$ wins or losses in a row, and that will eventually occur with probability 1 (geometric random variables are almost surely finite);
- $\mathbb{E}[|X_\tau|] \leq A+B < \infty$;
- $\mathbb{E}[|X_n|\mathbb{1}_{\{\tau>n\}}] \leq (A+B)P(\tau > n) \xrightarrow[n\to\infty]{} 0$.

What about the average length of the game $\mathbb{E}[\tau]$? We consider the martingale

$(X_n^2 - n)_{n=0}^{\infty}$. If the assumptions of Doob's optional stopping theorem are satisfied, then $\quad \sigma = 0 \quad X_\sigma = X_0 = x_0$

$$x_0^2 = \mathbb{E}_{x_0}[X_0^2 - 0] \overset{\downarrow}{=} \mathbb{E}_{x_0}[X_\tau^2 - \tau] = \mathbb{E}[X_\tau^2] - \mathbb{E}[\tau] = A^2 \frac{B - x_0}{A + B} + B^2 \frac{A + x_0}{A + B} - \mathbb{E}[\tau].$$

Solving for $\mathbb{E}[\tau]$ yields

$$\mathbb{E}[\tau] = \frac{A^2(B - x_0) + B^2(A + x_0)}{A + B} - x_0^2 = (A + x_0)(B - x_0).$$

Stopped symmetric random walk

Let's check the assumption of Doob's stopping theorem are satisfied.

- as before, $0 \leq \tau < \infty$ a.s. : we surely exit the set $[-A, B]$ if there are $A+B$ wins or losses in a row, and that will eventually occur with probability 1 (geometric random variables are almost surely finite). In fact, this tells us that τ is bounded by a geometric random variable and therefore $\mathbb{E}[\tau] < \infty$;
- $\mathbb{E}[|X_\tau^2 - \tau|] \leq \mathbb{E}[|X_\tau^2|] + \mathbb{E}[|\tau|] \leq (A+B)^2 + \mathbb{E}[\tau] < \infty$;
- by triangular inequality

$$\begin{aligned}\mathbb{E}[|X_n^2 - n|\mathbb{1}_{\{\tau>n\}}] &\leq \mathbb{E}[|X_n^2|\mathbb{1}_{\{\tau>n\}}] + \mathbb{E}[n\mathbb{1}_{\{\tau>n\}}] \\ &\leq (A+B)^2 P(\tau > n) + \mathbb{E}[\tau\mathbb{1}_{\{\tau>n\}}] \xrightarrow[n\to\infty]{} 0\end{aligned}$$

because $P(\tau > n)$ goes to 0 as n goes to ∞ and

$$\mathbb{E}[\tau\mathbb{1}_{\{\tau>n\}}] = \sum_{i=n+1}^{\infty} iP(\tau = i)$$

are the tails of the convergent series $\mathbb{E}[\tau] = \sum_{i=0}^{\infty} iP(\tau = i) < \infty$.

Realistic doubling strategy

Consider the realistic scenario where there is an upper bound M to what we can invest in a game. So, in the doubling strategy, if g is the length of the last stretch of consecutive losses at the round n, we bet the amount

$$2^g \wedge (M + X_n).$$

This is still a predictable strategy so what we obtain is still a martingale. If we define

$$\tau = \inf\{n : X_n \leq -M \text{ or } X_n \geq B\}$$

and set the initial gain $X_0 = 0$ we still have

$$0 \overset{\text{DOS THM}}{=} \mathbb{E}[X_\tau] \leq -MP(X_\tau = -M) + BP(X_\tau = B) = (-M - B)P(X_\tau = -M) + B$$

hence

$$P(X_\tau = -M) = \frac{B}{M+B} \quad \text{and} \quad P(X_\tau = B) = \frac{M}{M+B}.$$

The ruin probability is the same as in the constant bet strategy!

Appreciate the tools that martingales gave us

Just a moment to appreciate this:

In the other parte of the course you have studied a linear equation to get the hitting times and absorbing probabilities of discrete time Markov chains. It can always be used (no martingale needed!). However:

- It can be computationally difficult to solve for very large state spaces;
- Nice little formulas formulas could be obtained for the simple case of random walks, but that would be difficult for something more involved such as the realistic doubling strategy model or the Wright-Fisher model below - which we can manage easily.

So, if you see a martingale use it!

Stopped asymmetric random walk

Let $(X_n)_{n=0}^{\infty}$ the random walk with $p \neq 1/2$, let $X_0 = 0$, and let $(\mathcal{F}_n)_{n=0}^{\infty}$ be its natural filtration. Fix two natural numbers $A, B > 0$. Define

$$\tau = \inf\{n : X_n \leq -A \text{ or } X_n \geq B\}$$

To study $P(X_\tau = -A)$ and $P(X_\tau = B)$, we consider the process $(Y_n)_{n=0}^{\infty}$ defined by

$$Y_n = \left(\frac{q}{p}\right)^{X_n}.$$

We prove it is a martingale w.r.t. $(\mathcal{F}_n)_{n=0}^{\infty}$:

- $(Y_n)_{n=0}^{\infty}$ is adapted to $(\mathcal{F}_n)_{n=0}^{\infty}$ because Y_n is a bijective function of X_n;
- $\mathbb{E}[|(q/p)^{X_n}|] \leq (q/p)^n + (q/p)^{-n} < \infty$;
- we have

$$\mathbb{E}\left[\left(\frac{q}{p}\right)^{X_{n+1}} \middle| \mathcal{F}_n\right] = \left(\frac{q}{p}\right)^{X_n}\left(p \cdot \frac{q}{p} + q \cdot \frac{p}{q}\right) = \left(\frac{q}{p}\right)^{X_n} \underbrace{(q+p)}_{=1}.$$

Stopped asymmetric random walk

We now use Doob's optional stopping theorem (check that the conditions are satisfied!). Since $P(X_\tau = B) = 1 - P(X_\tau = -A)$ we get

$$\begin{aligned}\left(\frac{q}{p}\right)^{x_0} &= \mathbb{E}_{x_0}\left[\left(\frac{q}{p}\right)^{X_\tau}\right] = \left(\frac{q}{p}\right)^{-A} P(X_\tau = -A) + \left(\frac{q}{p}\right)^{B} P(X_\tau = B) \\ &= \left(\left(\frac{q}{p}\right)^{-A} - \left(\frac{q}{p}\right)^{B}\right) P(X_\tau = -A) + \left(\frac{q}{p}\right)^{B}.\end{aligned}$$

hence

$$P(X_\tau = -A) = \frac{(q/p)^{x_0} - (q/p)^B}{(q/p)^{-A} - (q/p)^B} \quad \text{and} \quad P(X_\tau = B) = \frac{(q/p)^{-A} - (q/p)^{x_0}}{(q/p)^{-A} - (q/p)^B}.$$

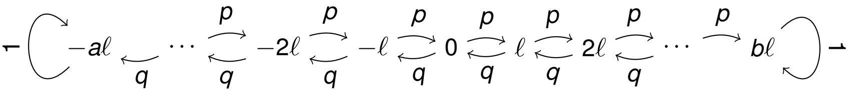
Stopped asymmetric random walk

Assume we bet 1 euro at each round, and start from $X_0 = 0$ euros. We can now easily compute the probability of winning 100 euros before losing 100 euros as a function of p:

p	0.5	0.495	0.49	0.48	0.47
$P(X_\tau = 100)$	0.5	0.119196	0.017977	0.000334	0.000006

Stopped asymmetric random walk

Assume we bet ℓ euros at each round, and start from $X_0 = 0$ euros. If $A = a\ell$ and $B = b\ell$ then the probability of reaching B before $-A$ is the same as reaching b before $-a$ when betting 1 euro each time.



Here is the probability of winning 100 euros before losing 100 euros as a function of p and ℓ:

p	0.5	0.495	0.49	0.48	0.47
bet 1 euro	0.5	0.119196	0.017977	0.000334	0.000006
bet 2 euro	0.5	0.268935	0.119175	0.017949	0.002455
bet 5 euro	0.5	0.401309	0.310003	0.167862	0.082953
bet 10 euro	0.5	0.450164	0.401300	0.309934	0.231219
bet 50 euro	0.5	0.490001	0.480008	0.460064	0.440215
bet 100 euro	0.5	0.495	0.49	0.48	0.47