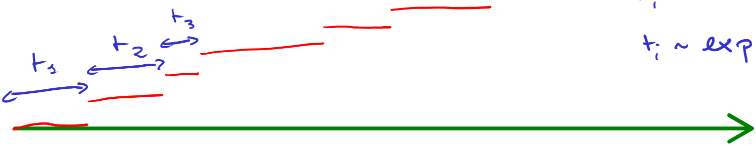
Poisson processes

Definition

A *Poisson process with rate* λ is a counting process $(N(s))_{s\in[0,\infty)}$ with $N(0) = 0$, whose inter-arrival times are i.i.d. exponential random variables with rate λ.



t_i i.i.d.

$t_i \sim exp(\lambda)$

Poisson Process

Theorem

$(N(s))_{s\in[0,\infty)}$ *is a Poisson process with rate* λ *if and only if it is a counting process such that*

1. $N(0) = 0$,
2. *it has independent increments;*
3. $N(t+s) - N(s) \sim \mathit{Poisson}(\lambda t)$. STAZIONARI

Poisson Process

Theorem

$(N(s))_{s\in[0,\infty)}$ is a Poisson process with rate λ if and only if it is a counting process such that:

(i) $N(0) = 0$;

(ii) *it has independent increment;*

(iii) *it has stationary increments;*

(iv) $\lim_{h\to 0} \frac{P(N(h)=1)}{h} = \lambda$ *and* $\lim_{h\to 0} \frac{P(N(h)\geq 2)}{h} = 0.$

Example

Customers arrive at a shipping office at times of a Poisson process with rate 3 per hour. (a) The office was supposed to open at 8AM but the clerk Oscar overslept and came in at 10AM. What is the probability that no customers came in the two-hour period? (b) What is the distribution of the amount of time Oscar has to wait until his first customer arrives?

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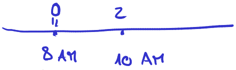
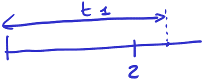
Solution. Let $N(s)$ be the number of customers who have arrived at time s hours after 8am. Then for part (a) we want



$N(2) \sim \text{Pois}(3 \cdot 2)$

$$P(N(2) = 0) = e^{-3 \cdot 2} \frac{(3 \cdot 2)^0}{0!} = e^{-6}.$$

Alternatively, you can realize that we have exponential random variables with parameter 3, and we are asking for $P(t_1 > 2) = \int_2^\infty 3e^{-3t} dt = e^{-6}$.



$t_1 \sim \exp(3) \quad P(t_1 > 2) = e^{-2 \cdot 3}$

Example

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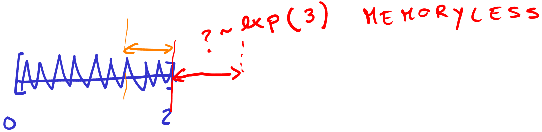
Solution. For part (b), we note that the number of customers arriving after 10am still follows a Poisson process with rate 3 per hour. Hence, the distribution is Exponential with parameter 3 per hour.



The non-homogeneous Poisson Process

A homogeneous Poisson process is often unrealistic because intensities are often variable with time.

Definition

NON-HOM.

We say that $(M(s))_{s\in[0,\infty)}$ is a Poisson process with rate function, or *intensity function*, or *propensity function* $\lambda(u)$ if

1. $M(0)=0$.
2. $M(t)$ has independent increments, and
3. $M(t)-M(s)$ is Poisson with mean $m(t)-m(s)=\int_s^t \lambda(u)du$.

$m(t) = \int_0^t \lambda(u)du$

Note that it is not a Poisson process in the strict sense of the definition! As a matter of fact, here inter-arrival times are not exponentially distributed.

se $\lambda(u)=\lambda$ COSTANTE

ALLORA $M(t)-M(s) \sim \text{Pois}(\overbrace{\int_s^t \lambda(u)du}^{\lambda\cdot t})$

INCREMENTI NON STAZIONARI!

$M(t+h)-M(s+h) \sim \text{Pois}\left(\int_{s+h}^{t+h} \lambda(u)du\right)$

The non-homogeneous Poisson Process

Theorem (Rescaling the time of a unit rate Poisson process)

Let $N(t)$ be Poisson process with rate 1, and let $\lambda(t)$ be a nonnegative function of the time. Then, the process

$$M(t) = N\left(\int_0^t \lambda(u)du\right)$$

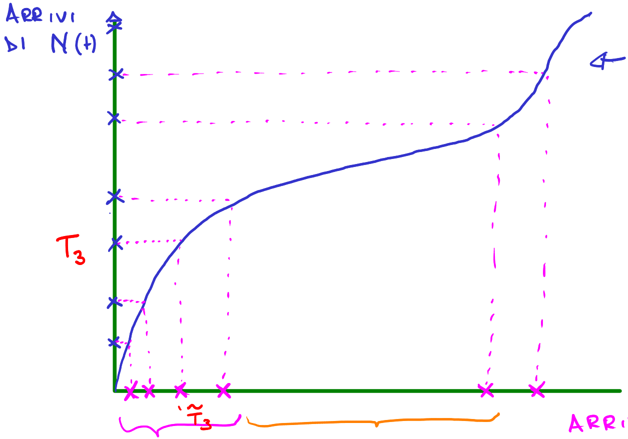
is a (non-homogeneous) Poisson process with rate $\lambda(t)$.

In the special case in which $\lambda(s) = \lambda$ constant, we have that

$$M(t) = N(\lambda t)$$

is a Poisson process with rate λ.

The non-homogeneous Poisson Process



$\leftarrow t \mapsto \int_0^t \lambda(u)\,du$

MONOTONA ($\lambda \geq 0$)

$T_3 = \int_0^{\tilde{T}_3} \lambda(u)\,du$

$m(t) = \int_0^t \lambda(u)\,du$

$T_3 = m(\tilde{T}_3)$

$\tilde{T}_3 = m^{-1}(T_3)$

ARRIVI DI M(t)

$\lambda(u)$ ALTA

$t \mapsto \int_0^t \lambda(u)\,du$

CRESCE RAPIDAMENTE

$\lambda(u)$ BASSA

$t \to \int_0^t \lambda(u)\,du$

CRESCE LENTA

Proof

We check that the definition of (non-homogeneous) Poisson process is met.

i) $M(0) = N\left(\int_0^0 \lambda(u)du\right) = N(0) = 0.$

ii) Let $t_1 < t_2 < \cdots < t_n$. For simplicity, for any $1 \leq i \leq n$ let

$$m(t_i) = \tilde{t}_i = \int_0^{t_i} \lambda(u)du. \qquad \tilde{t}_1 \leq \tilde{t}_2 \leq \dots \leq \tilde{t}_n$$

Then, for any $1 \leq i \leq n-1$, $M(t_{i+1}) - M(t_i) = N(\tilde{t}_{i+1}) - N(\tilde{t}_i)$. Hence, $M(t_{i+1}) - M(t_i)$ are independent because so are $N(\tilde{t}_{i+1}) - N(\tilde{t}_i)$.

iii) For any $s < t$ we have

$$\begin{aligned} M(t) - M(s) &= N\left(\overbrace{\int_0^t \lambda(u)du}^{\tilde{t}}\right) - N\left(\overbrace{\int_0^s \lambda(u)du}^{\tilde{s}}\right) \\ &\sim \text{Pois}\left(\underbrace{\int_0^t \lambda(u)du}_{\tilde{t}} - \underbrace{\int_0^s \lambda(u)du}_{\tilde{s}}\right) \sim \text{Pois}\left(\underline{\int_s^t \lambda(u)du}\right) \end{aligned}$$

Example: frogs 1

Suppose that the arrival times of frogs to a pond can be reasonably modeled by a Poisson process. Frogs are arriving at a rate of 3 per hour. What is the probability that no frogs will arrive in the next hour? What is the probability that 12 or less frogs arrive in the next five hours?

Solution: Let $N(t)$ be the Poisson process of rate 3 determining the number of frogs to arrive in the next t hours. Then,

$\sim \text{Pois}(1 \cdot 3)$

$$P\{N(1) = 0\} = e^{-3*1}\frac{(3*1)^0}{0!} = e^{-3} = 0.04978.$$

Further,

$$P\{N(5) \le 12\} = \sum_{k=0}^{12} P\{\underbrace{N(5)}_{\sim}= k\} = \sum_{k=0}^{12} e^{-3*5}\frac{(3*5)^k}{k!} \approx 0.2676.$$

$\text{Pois}(5 \cdot 3)$

Example: frogs 2

It does not seem plausible that the frog arrivals are uniformly distributed along the day. Instead, suppose the rate of arrival should fluctuate as 
$\lambda(t) = 3 + \sin(\frac{t\pi}{12})$, where $t = 0$ is taken to be 8AM and the unit of time is one hour. Assuming it is 8AM now, what is the probability that no frogs will arrive in the next hour? What is the probability that 12 or less frogs will arrive between 9 and 12?

Solution. We let $N(t)$ be a Poisson process with intensity $\lambda(t) = 3 + \sin(\frac{t\pi}{12})$.
Then, $m(t) = \int_0^t \lambda(u)du = \boxed{3t + \frac{12}{\pi}\left[1 - \cos(\frac{t\pi}{12})\right]}$ and

$$P(N(1) = 0) = e^{-m(1)+m(0)} \approx 0.13.$$

$M(1) \sim \text{Pois}\left(\int_0^1 \lambda(u)\,du\right)$ $\neq$

The incremements are not stationary! Compute $P(\overbrace{N(2) - N(1)}^{\sim \text{Pois}\left(\int_1^2 \lambda(u)\,du\right)} = 0)$. Further

$$P(\underbrace{N(4) - N(1)}_{2} \leq 12) = \sum_{k=0}^{12} e^{-[m(4)-m(1)]} \frac{[m(4) - m(1)]^k}{k!} \approx 0.71.$$

$\text{Pois}\left(\int_1^4 \lambda(u)\,du\right)$

$\longrightarrow m(4) - m(1)$

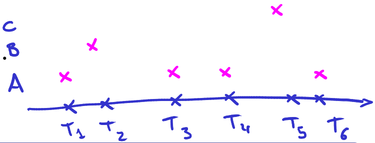
An arrival process, $(N(s))_{s\in[0,\infty)}$ is Poisson with rate λ. Assume that the arrivals can be of k different types, specified by a sequence of iid random variables $\{Y_i\}_{i=1}^{\infty}$, taking values in $\{1,2,3,\dots,k\}$, with probability mass function $P(Y_i = j) = p_j$. Let these random variables be **independent of** $(N(s))_{s\in[0,\infty)}$. Let $N_j(t)$ be the arrivals before time t that are of type j:

$N_C(t)$ = # ARRIVI DI TIPO C IN $[0,t]$

$N_B(t)$ = # ARRIVI DI TIPO B IN $[0,t]$

$N_A(t)$ = # ARRIVI DI TIPO A IN $[0,t]$

$$N_j(t) = \sum_{i=1}^{N(t)} 1_{\{Y_i=j\}}.$$



Theorem

$\{(N_j(t))_{t\in[0,\infty)}\}_j$ are independent Poisson processes with respective rates λp_j.

Thinning, a sketch of the proof

$N_1(0) + N_2(0) = N(0)$

$N_j(0) = 0$ for every j, since $N(0) = 0$. The independence of the increments of each of the N_j follows from that of N and from the independence of each of the Y_i from the others and from the Poisson itself. Let us now consider the case $k = 2$ (events are of two different kind only) and calculate

$$P[N_1(t) - N_1(s) = n, N_2(t) - N_2(s) = m]$$

DOVREI $N_1(t_1) - N_1(s_1) \perp\!\!\!\perp N_2(t_2) - N_2(s_2)$

For such event to occur, we need that $N(t) - N(s) = m + n$. Morover, of the $m + n$ event, n are of the first kind, that happens with a probability of

$P = P_1$

$1 - P = P_2$

$$P[\overbrace{N(t) + N(s)}^{\sim \mathrm{Pois}(\lambda\cdot(t-s))} = m + n, N_1(t) - N_1(s) = n]$$

$$= e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{m+n}}{(m+n)!} \underbrace{\binom{m+n}{n}}_{= \frac{(m+n)!}{n!\,m!}} p^n (1-p)^m =$$

$$= e^{-\lambda p(t-s)} \frac{(\lambda p(t-s))^n}{n!} e^{-\lambda(1-p)(t-s)} \frac{(\lambda(1-p)(t-s))^m}{m!}$$

$= P(\mathrm{Pois}(\lambda p(t-s)) = n)\, P(\mathrm{Pois}(\lambda(1-p)(t-s)) = m)$ □

Thinning, a counter-intuitive example

Assume people arrive at a shop according to a Poisson process with rate 100 per day, and are given coupons independently of each other: there are two kinds of coupons and each is given with probability $1/2$. Knowing that at the end of the day 1000 coupons of type 1 are given, how many coupons of type 2 are expected to be given?

$N_1(t) = \#$ COUPONS GIVEN OF TYPE 1

$N_2(t) = \#$ " " " " 2

$E[N_2(1) \mid N_1(1) = 1000] \neq 1.000$

$\parallel$ THINNING N_1, N_2 P.P. INDIPENDENTI DI RATE $\frac{100}{2}$

$E[\underbrace{N_2(1)}_{\sim \text{Pois}(50)}] = 50$

Non-homogeneous thinning

Thinning can be used to derive a non-homogeneous Poisson process from a homogeneous one:

Theorem

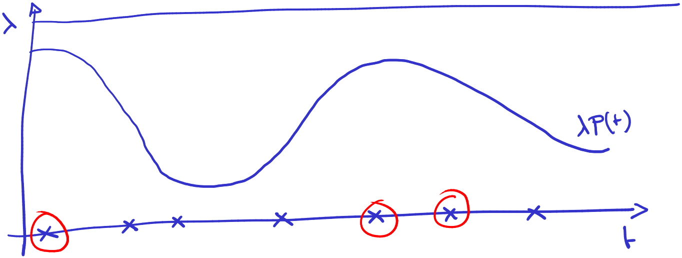
Suppose that in a Poisson process with rate λ, we keep an arrival that takes place at time s with time-dependent probability $p(s)$, independently on the other arrivals. Define

$$M(t) = \#\text{arrivals kept by time } t.$$

Then $(M(t))_{t\in[0,\infty)}$ is a non-homogeneous Poisson process with rate $\lambda p(s)$.

You can take $\lambda = max\{\lambda(s)\}$ and $p(s) = \frac{\lambda(s)}{\lambda}$ and get a Poisson $\lambda(s)$

Non-homogeneous thinning



ARRIVI DI UN P.P. NON OMOGENEO
CON PAR. $\lambda \cdot p(t) = \lambda(t)$

PRO: NON DOVETE INVERTIRE INTEGRALI

CONTRO: SIMULATE ALCUNI ARRIVI PER NIENTE
SPECIE DOVE $p(t)$ E' BASSA $\to$ SFORZO COMP. SPRECATO