Martingales

Conditional expectation

Definition (Conditional expectation)

Given a random variable X that is $\mathcal{F}-$measurable and a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, the conditional expectation $\mathbb{E}[X|\mathcal{G}]$ is a *random variable* such that

1. $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}-$measurable;
2. $\mathbb{E}\Big[\mathbb{1}_A \mathbb{E}[X|\mathcal{G}]\Big] = \mathbb{E}[\mathbb{1}_A X]$ for all $A \in \mathcal{G}$. Equivalently,

$$\int_A \mathbb{E}[X|\mathcal{G}](\omega) dP(\omega) = \int_A X(\omega) dP(\omega).$$

Existence and uniqueness (almost everywhere) of $\mathbb{E}[X|\mathcal{G}]$ follow from the Radon-Nikodym Theorem.

Conditional expectation

Intuitively, $\mathbb{E}[X|\mathcal{G}]$ is *the best approximation of X given the information contained in $\mathcal{G}$*, or the *projection of X onto the sub-σ-algebra $\mathcal{G}$*. It can be proven that if $\mathbb{E}[X^2] < \infty$, then $\mathbb{E}[X|\mathcal{G}]$ minimizes

$$\mathbb{E}[(X-\xi)^2]$$

over all random variables ξ such that

1. ξ is $\mathcal{G}-$measurable;
2. $\mathbb{E}[\xi^2] < \infty$.

If $\mathcal{G}$ is the trivial σ-algebra $\{\varnothing, \Omega\}$ then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$.

Properties of conditional expectation

Linearity $\mathbb{E}[aX + bY|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}]$ a.s. .

Monotonicity if $X \leq Y$ a.s. then $\mathbb{E}[X|\mathcal{G}] \leq \mathbb{E}[Y|\mathcal{G}]$ a.s. .

Identity if X is $\mathcal{G}-$measurable, then $\mathbb{E}[X|\mathcal{G}] = X$ a.s. .

Jensen's inequality if ϕ is convex and $\mathbb{E}[|\phi(X)|] < \infty$, then
$\mathbb{E}[\phi(X)|\mathcal{G}] \geq \phi\Big(\mathbb{E}[X|\mathcal{G}]\Big)$ a.s. .

Pulling out what's known if Y is measurable and $\mathbb{E}[|Y|] < \infty$ then
$\mathbb{E}[XY|\mathcal{G}] = Y\mathbb{E}[X|\mathcal{G}]$ a.s. .

Properties of conditional expectation

Tower property if $\mathcal{G}' \subseteq \mathcal{G}$ then $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{G}'] = \mathbb{E}[X|\mathcal{G}']$ a.s. .

Irrelevance of independent information if $\mathcal{H}$ is independent of $\sigma(X, \mathcal{G})$ then $\mathbb{E}[X|\sigma(\mathcal{G}, \mathcal{H})] = \mathbb{E}[X|\mathcal{G}]$ a.s. . In particular, if $\mathcal{H}$ is independent of X then $\mathbb{E}[X|\mathcal{H}] = \mathbb{E}[X]$ a.s. .

Monotone convergence if $X_n \uparrow X$ a.s. and $\mathbb{E}[|X_n|], \mathbb{E}[X] < \infty$ then $\mathbb{E}[X_n|\mathcal{G}] \uparrow \mathbb{E}[X|\mathcal{G}]$ a.s. .

Fatou's lemma if $X_n \geq 0$ and $\mathbb{E}[|\liminf_{n\to\infty} X_n|] < \infty$ then $\mathbb{E}[\liminf_{n\to\infty} X_n|\mathcal{G}] \leq \liminf_{n\to\infty} \mathbb{E}[X_n|\mathcal{G}]$ a.s. .

Dominated convergence if $X_n \leq Y$ with $\mathbb{E}[|Y|] < \infty$ and $X_n \to X$ a.s. then $\mathbb{E}[X_n|\mathcal{G}] \to \mathbb{E}[X|\mathcal{G}]$ a.s. .

Martingales

Definition (Martingale)

A stochastic process $(X_i)_{i\in I}$ is a martingale with respect to a filtration $(\mathcal{F}_i)_{i\in I}$ if

- X_i is $\mathcal{F}_i$-measurable for each $i \in I$;
- $\mathbb{E}[|X_i|] < \infty$ for each $i \in I$;
- $\mathbb{E}[X_j|\mathcal{F}_i] = X_i$ a.s. for each $\underline{i < j} \in I$.

We simply say that a stochastic process is a martingale if it is so with respect to its natural filtration.

Martingales are *fair games*: if we are playing in a fair game, with $\mathcal{F}_i$ being the information on what happened in the game up to time i, and X_i is our wealth at time i, then the expected value of our wealth in the future, given what has happened so far, is our current wealth.

By the tower property, $\mathbb{E}[X_j] \overset{\text{T.P.}}{=} \mathbb{E}[\overbrace{\mathbb{E}[X_j|\mathcal{F}_i]}^{X_i}] \overset{\text{DEF. MART.}}{=} \mathbb{E}[X_i]$ for all $i < j \in I$.

Discrete time case

Lemma

A discrete time stochastic process $(X_n)_{n=1}^{\infty}$ *adapted to* $(\mathcal{F}_n)_{n=1}^{\infty}$ *satisfies* $\mathbb{E}[X_{n+m}|\mathcal{F}_n] = X_n$ *a.s. for all* $n, m > 0$ *if and only if* $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ *a.s. for all* $n > 0$.

Proof.

The forward direction is trivial.

We prove the reverse direction by induction on $m > 0$. The case $m = 1$ is given by the assumption. If we have $\mathbb{E}[X_{n+m}|\mathcal{F}_n] = X_n$, then by the tower property

$$\mathbb{E}[X_{n+m+1}|\mathcal{F}_n] = \mathbb{E}[\underbrace{\mathbb{E}[X_{n+m+1}|\mathcal{F}_{n+m}]}_{X_{n+m}}|\mathcal{F}_n] = \mathbb{E}[X_{n+m}|\mathcal{F}_n] = X_n.$$

□

Example 1: random walk

$$\begin{tikzcd}
\cdots & -1 \arrow[l, bend left, "1/2"] \arrow[r, bend left, "1/2"] & 0 \arrow[l, bend left, "1/2"] \arrow[r, bend left, "1/2"] & 1 \arrow[l, bend left, "1/2"] \arrow[r, bend left, "1/2"] & 2 \arrow[l, bend left, "1/2"] \arrow[r, bend left, "1/2"] & \cdots \arrow[l, bend left, "1/2"]
\end{tikzcd}$$

The random walk with $p = 1/2$ is a martingale: let $(B_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $P(B_i = -1) = P(B_i = 1) = 1/2$. Then the random walk can be written as

$$X_n = X_0 + \underbrace{\sum_{i=1}^{n} B_i}_{\leq n}.$$

Hence,

$$\mathbb{E}[|X_n|] \leq n + \mathbb{E}[|X_0|] < \infty \qquad \text{if } \mathbb{E}[|X_0|] < \infty$$

and

$$\mathbb{E}[X_{n+m}|\mathcal{F}_n] \overset{X_n + \sum_{i=n+1}^{n+m} B_i}{=} \underbrace{\mathbb{E}[X_n|\mathcal{F}_n]}_{X_n} + \sum_{i=n+1}^{n+m} \mathbb{E}[B_i|\cancel{\mathcal{F}_n}] = X_n + \sum_{i=n+1}^{n+m} \overbrace{\mathbb{E}[B_i]}^{0} = X_n.$$

Example 2: a multiplicative example

Let $(B_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $P(B_i = -1) = P(B_i = 1) = 1/2$. Consider

$$X_n = \prod_{i=1}^{n} \overbrace{(1 + B_i)}^{0 \text{ PROB } \frac{1}{2} \text{ or } 2 \text{ PROB. } \frac{1}{2}} = \begin{cases} 2^n & \text{if } B_1 = B_2 = \cdots = B_n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

We have

$$\mathbb{E}[|X_n|] \leq 2^n < \infty$$

and

$$= E[X_n | F_n] \cdot \prod_{i=n+1}^{n+m} (1 + E[B_i | \cancel{F_n}])$$

$$\mathbb{E}[X_{n+m}|\mathcal{F}_n] = X_n \prod_{i=n+1}^{n+m} \underbrace{(1 + \underbrace{\mathbb{E}[B_i]}_{=0})}_{1} = X_n.$$

$$X_{n+m} = X_n \cdot \prod_{i=n+1}^{n+m} (1 + B_i)$$

Example 3: compensated Poisson process

Let $(M_t)_{t\in[0,\infty)}$ be a non-homogeneous Poisson process with rate $\lambda(t)$, such that

$$\int_0^t \lambda(s)ds < \infty \qquad \text{for all } t \geq 0.$$

Consider

$$W_t = M_t - \int_0^t \lambda(s)ds.$$

The natural filtrations of $(W_t)_{t\in[0,\infty)}$ and $(M_t)_{t\in[0,\infty)}$ coincide (their difference is deterministic at any time point). By the triangular property,

$$\mathbb{E}[|W_t|] \leq \mathbb{E}[M_t] + \int_0^t \lambda(s)ds = 2\int_0^t \lambda(s)ds < \infty.$$

Example 3: compensated Poisson process

Let $(M_t)_{t\in[0,\infty)}$ be a non-homogeneous Poisson process with rate $\lambda(t)$, such that

$$\int_0^t \lambda(s)ds < \infty \qquad \text{for all } t \geq 0.$$

Consider

$$W_t = M_t - \int_0^t \lambda(s)ds.$$

By independence of the increments of $(M_t)_{t\in[0,\infty)}$,

$$\begin{aligned}
\mathbb{E}[W_{t+h}|\mathcal{F}_t] &= \mathbb{E}\left[M_{t+h} - \int_0^{t+h}\lambda(s)ds \,\middle|\, \mathcal{F}_t\right] \boxed{- M_t} + M_t \\
&= \mathbb{E}\left[M_{t+h} - M_t \,|\, \cancel{\mathcal{F}_t}\right] - \int_0^{t+h}\lambda(s)ds + M_t \\
&= \int_t^{t+h}\lambda(s)ds - \int_0^{t+h}\lambda(s)ds + M_t = M_t - \int_0^t \lambda(s)ds = W_t.
\end{aligned}$$

Example 4: a non-Markovian example

Let $(B_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $P(B_i = -1) = P(B_i = 1) = 1/2$. Consider $X_0 \sim \text{Pois}(10)$ and

$$X_n = X_{n-1} + B_n \sum_{i=1}^{n-1} X_i \quad n \geq 1.$$

We have

- $\mathbb{E}[|X_n|] < \infty$.

We prove it by induction: $\mathbb{E}[|X_0|] = 10 < \infty$ and if $\mathbb{E}[|X_i|] < \infty$ for all $i \leq n-1$ then

$$\mathbb{E}[|X_n|] \leq \mathbb{E}[|X_{n-1}|] + \sum_{i=1}^{n-1} \mathbb{E}[|X_i|] < \infty.$$

Example 4: a non-Markovian example

Let $(B_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $P(B_i = -1) = P(B_i = 1) = 1/2$. Consider $X_0 \sim \text{Pois}(10)$ and

$$X_n = X_{n-1} + B_n \sum_{i=1}^{n-1} X_i \quad n \geq 1.$$

We have

- $\mathbb{E}[X_{n+m} | \mathcal{F}_n] = X_n$.

Here it is easier to prove

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n] = \mathbb{E}\left[X_n + B_{n+1} \sum_{i=1}^{n} X_i \,\middle|\, \mathcal{F}_n \right]$$

$$= X_n + \underbrace{\overbrace{\mathbb{E}[B_{n+1}]}^{0} \sum_{i=1}^{n} X_i}_{0} = X_n.$$

Example 4: a non-Markovian example

Let $(B_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $P(B_i = -1) = P(B_i = 1) = 1/2$. Consider $X_0 \sim \text{Pois}(10)$ and

$$X_n = X_{n-1} + B_n \sum_{i=1}^{n-1} X_i \quad n \geq 1.$$

We have

- $(X_n)_{n=0}^{\infty}$ is not Markovian.

$P(X_0 = 5, X_1 = 0, X_2 = 6) = 0$, hence

$$X_2 = X_1 + B_2 \cdot (X_0 + X_1) = 0 + B_2 \cdot 5 \begin{cases} 5 \\ -5 \end{cases} \neq 6$$

$$P(X_2 = 6 | X_1 = 0, X_0 = 5) = 0.$$

However, it is possible that $X_1 = 0$ and $X_2 = 6$ (it can happen if $X_0 = 6$), hence

$$P(X_2 = 6 | X_1 = 0) > 0.$$

$X_0 = 6 \quad B_1 = -1$

$X_1 = 6 - 6 = 0$

$B_2 = 1$

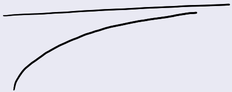
$X_2 = 0 + 6 = 6$

Submartingales

Definition (Submartingale)

A stochastic process $(X_i)_{i\in I}$ is a submartingale with respect to a filtration $(\mathcal{F}_i)_{i\in I}$ if

- X_i is $\mathcal{F}_i$-measurable for each $i \in I$;
- $\mathbb{E}[|X_i|] < \infty$ for each $i \in I$;
- $\mathbb{E}[X_j|\mathcal{F}_i] \geq X_i$ a.s. for each $i < j \in I$.



We simply say that a stochastic process is a submartingale if it is so with respect to its natural filtration.

Submartingales are *favourable games*.

By the tower property, $\mathbb{E}[X_j] = \mathbb{E}[\mathbb{E}[X_j|\mathcal{F}_i]] \geq \mathbb{E}[X_i]$ for all $i < j \in I$.

Supermartingales

Definition (Supermartingale)

A stochastic process $(X_i)_{i \in I}$ is a supermartingale with respect to a filtration $(\mathcal{F}_i)_{i \in I}$ if

- X_i is $\mathcal{F}_i$-measurable for each $i \in I$;
- $\mathbb{E}[|X_i|] < \infty$ for each $i \in I$;
- $\mathbb{E}[X_j | \mathcal{F}_i] \boxed{\leq} X_i$ a.s. for each $i < j \in I$.

We simply say that a stochastic process is a supermartingale if it is so with respect to its natural filtration.

Supermartingales are *unfavourable games*.

By the tower property, $\mathbb{E}[X_j] = \mathbb{E}[\mathbb{E}[X_j | \mathcal{F}_i]] \leq \mathbb{E}[X_i]$ for all $i < j \in I$.

MARTINGALES ARE BOTH SUPER- AND SUB-MARTINGALES

$$\begin{tikzcd} \cdots & -1 \arrow[l, bend left, "q"] \arrow[r, bend left, "p"] & 0 \arrow[l, bend left, "q"] \arrow[r, bend left, "p"] & 1 \arrow[l, bend left, "q"] \arrow[r, bend left, "p"] & 2 \arrow[l, bend left, "q"] \arrow[r, bend left, "p"] & \cdots \arrow[l, bend left, "q"] \end{tikzcd}$$

$p<q$

The random walk with $p < 1/2$ is a supermartingale: let $(B_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $P(B_i = -1) = 1 - p$ and $P(B_i = 1) = p$. Then the random walk can be written as

$$X_n = X_0 + \sum_{i=1}^{n} B_i.$$

Hence,

$$\mathbb{E}[|X_n|] \leq n + E[|X_0|] < \infty \quad \text{if } E[|X_0|] < \infty$$

and

$$\mathbb{E}[X_{n+m}|\mathcal{F}_n] = X_n + \overbrace{\sum_{i=n+1}^{n+m} \mathbb{E}[B_i]}^{<0} < X_n.$$

$$E[B_i] = p + (-1)(1-p) = 2p - 1 < 0$$

Similarly, a random walk with $p > 1/2$ is a submartingale.

Example: Poisson process

A Poisson process with rate λ is a submartingale: by the independence of the increments

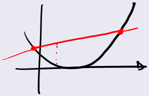
$$\mathbb{E}[X_{t+h}|\mathcal{F}_t] = \mathbb{E}[X_{t+h} - X_t|\cancel{\mathcal{F}_t}] + X_t = \lambda h + X_t > X_t.$$

Jensen's inequality

Theorem

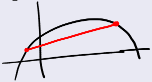
A function $f : \mathbb{R} \to \mathbb{R}$ is convex *if for any $x_1, x_2 \in \mathbb{R}$ and $\alpha \in [0,1]$*

$$f\Big(\alpha x_1 + (1-\alpha)x_2\Big) \leq \alpha f(x_1) + (1-\alpha) f(x_2).$$



We say that $f : \mathbb{R} \to \mathbb{R}$ is concave *if for any $x_1, x_2 \in \mathbb{R}$ and $\alpha \in [0,1]$*

$$f\Big(\alpha x_1 + (1-\alpha)x_2\Big) \geq \alpha f(x_1) + (1-\alpha) f(x_2).$$



Theorem (Jensen's inequality)

If f is convex and X is a real, $\cancel{\mathcal{G}}$ $\mathcal{F}$-measurable random variable with $\mathbb{E}[|X|], \mathbb{E}[|f(X)|] < \infty$, then almost surely

$\mathcal{G} \leq \mathcal{F}$

$$f(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[f(X)|\mathcal{G}].$$

Jensen's inequality

Theorem

Let $(X_i)_{i\in I}$ be a real-valued process, and let $f : \mathbb{R} \to \mathbb{R}$ be such that $\mathbb{E}[f(X_i)] < \infty$ for all $i \in I$.

- *If $(X_i)_{i\in I}$ is a submartingale w.r.t. $(\mathcal{F}_i)_{i\in I}$ and f is convex and non-decreasing, then $(f(X_i))_{i\in I}$ is a submartingale w.r.t. $(\mathcal{F}_i)_{i\in I}$;*
- *If $(X_i)_{i\in I}$ is a supermartingale w.r.t. $(\mathcal{F}_i)_{i\in I}$ and f is concave and non-decreasing, then $(f(X_i))_{i\in I}$ is a supermartingale w.r.t. $(\mathcal{F}_i)_{i\in I}$;*
- *If $(X_i)_{i\in I}$ is a martingale w.r.t. $(\mathcal{F}_i)_{i\in I}$ and f is convex, then $(f(X_i))_{i\in I}$ is a submartingale w.r.t. $(\mathcal{F}_i)_{i\in I}$;*
- *If $(X_i)_{i\in I}$ is a martingale w.r.t. $(\mathcal{F}_i)_{i\in I}$ and f is concave, then $(f(X_i))_{i\in I}$ is a supermartingale w.r.t. $(\mathcal{F}_i)_{i\in I}$.*

Jensen's inequality

Proof.

If $(X_i)_{i\in I}$ is a submartingale w.r.t. $(\mathcal{F}_i)_{i\in I}$ and f is convex and non-decreasing, then by Jensen's inequality

$$\mathbb{E}[f(X_{i+j})|\mathcal{F}_i] \overset{\text{JENSEN}}{\geq} f(\mathbb{E}[X_{i+j}|\mathcal{F}_i]) \overset{\text{NON-DECR.}}{\geq} f(X_i). \quad \to \quad \underbrace{(f(X_i))_i}_{\text{SUB-MART.}}$$

If $(X_i)_{i\in I}$ is a supermartingale w.r.t. $(\mathcal{F}_i)_{i\in I}$ and f is concave and non-decreasing, then $-f$ is convex and non-increasing, so by Jensen's inequality

$$-\mathbb{E}[f(X_{i+j})|\mathcal{F}_i] \geq -f(\mathbb{E}[X_{i+j}|\mathcal{F}_i]) \geq -f(X_i),$$

which implies $\mathbb{E}[f(X_{i+j})|\mathcal{F}_i] \leq f(X_i)$.

$\square$

Jensen's inequality

Proof.

If $(X_i)_{i\in I}$ is a martingale w.r.t. $(\mathcal{F}_i)_{i\in I}$ and f is convex, then by Jensen's inequality

$$\mathbb{E}[f(X_{i+j})|\mathcal{F}_i] \overset{\text{JENSEN}}{\geq} f(\mathbb{E}[X_{i+j}|\mathcal{F}_i]) = f(X_i).$$

$(f(X_i))_{i\in I}$ SUBMARTINGALE

If $(X_i)_{i\in I}$ is a supermartingale w.r.t. $(\mathcal{F}_i)_{i\in I}$ and f is concave, then $-f$ is convex so by Jensen's inequality

$$-\mathbb{E}[f(X_{i+j})|\mathcal{F}_i] \geq -f(\mathbb{E}[X_{i+j}|\mathcal{F}_i]) = -f(X_i),$$

which implies $\mathbb{E}[f(X_{i+j})|\mathcal{F}_i] \leq f(X_i)$.

□

Examples

Let $(X_n)_{n=1}^{\infty}$ be the random walk with $p = 1/2$. Then, $(X_n)_{n=1}^{\infty}$ is a martingale. We have that

- $(X_n^2)_{n=1}^{\infty}$ and $(|X_n|)_{n=1}^{\infty}$ are submartingales;
- $(-X_n^4 + X_n + 7)_{n=1}^{\infty}$ and $(-|X_n|)_{n=1}^{\infty}$ are supermartingales.

Examples

Let $(B_i)_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $P(B_i = -1) = P(B_i = 1) = 1/2$. Consider

$$X_n = \prod_{i=1}^{n}(1 + B_i) = \begin{cases} 2^n & \text{if } B_1 = B_2 = \cdots = B_n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then

- $((X_n + 1)^{-1})_{n=1}^{\infty}$ is a submartingale;
- $(\sqrt{X_n})_{n=1}^{\infty}$ and $(\log(X_n + 1))_{n=1}^{\infty}$ are supermartingales.