01RMHNG–03RMHPF–01RMING
Network Dynamics
Week 10 — Part I
More on Potential Games: Finite Improvement Property, Congestion Games, Network Games

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Outline

- Recap on (Potential) Games
- Finite Improvement Property
- Congestion Games
- Network Games

Strategic Form Games

- $\mathcal{V}$ finite set of players
- $\mathcal{A}_i$ set of actions (a.k.a. strategies) for player i
- $\mathcal{X} = \prod_{i \in \mathcal{V}} \mathcal{A}_i$ set of configurations (a.k.a. strategy profiles)
- $u_i : \mathcal{X} \to \mathbb{R}$ utility function
- $x \in \mathcal{X}$ configuration (a.k.a. action/strategy profile, or outcome)
- x_i action played by player i
- x_{-i} vector of actions played by everyone but i
- utility of player i when each player j plays action x_j:

$$u_i(x_i, x_{-i}) = u_i(x)$$

$\left(\mathcal{V}, \{\mathcal{A}_i\}_{i \in \mathcal{V}}, \{u_i\}_{i \in \mathcal{V}}\right)$ is called a strategic (a.k.a. normal form) game

Exact Potential Games

- Definition: A game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ is an (exact) potential game if there exists $\Phi : \mathcal{X} \to \mathbb{R}$ (called potential function) such that

$$u_i(y_i, x_{-i}) - u_i(x_i, x_{-i}) = \Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}),$$

for every $x \in \mathcal{X}$, $i \in \mathcal{V}$, and $y_i \in \mathcal{A}$, equivalently if

$$x_{-i} = y_{-i} \quad \Longrightarrow \quad u_i(y) - u_i(x) = \Phi(y) - \Phi(x)$$

- In an exact potential game, for any configuration x, the utility variation incurred by player i when changing action unilaterally is the same as the corresponding variation in the potential function

Properties of Exact Potential Games

- Theorem: Game is exact potential if and only if

$$\sum_{i=1}^{4} u_{i_k}(x^{(k)}) - u_{i_k}(x^{(k-1)}) = 0$$

$\forall (x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)} = x^{(0)})$ s.t. $x_{-i_k}^{(k-1)} = x_{-i_k}^{(k)}$, $1 \leq k \leq 4$

- Proposition: A game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$, with $\mathcal{A}_i \subseteq \mathbb{R}$ interval and $u_i \in \mathcal{C}^2$ is an exact potential if and only if

$$\frac{\partial^2}{\partial x_i x_j} u_i(x) = \frac{\partial^2}{\partial x_j x_i} u_j(x)$$

for $i, j \in \mathcal{V}$ and $x \in \mathcal{X}$. In this case a potential function is

$$\Phi(x) = \int_{\Gamma_{\overline{x} \to x}} f(s) \cdot \mathrm{d}s$$

where $\Gamma_{\overline{x} \to x}$ is any simple curve from $\overline{x}$ to x, and

$$f(x) = \left(\frac{\partial u_1}{\partial x_1}(x), \ldots, \frac{\partial u_n}{\partial x_n}(x) \right)$$

Ordinal potential games

- Definition: A game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ is an ordinal potential game if there exists $\Phi : \mathcal{X} \to \mathbb{R}$ (called ordinal potential function) s.t.

$$u_i(y_i, x_{-i}) - u_i(x_i, x_{-i}) > 0 \iff \Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}) > 0$$

for every $x \in \mathcal{X}$, $i \in \mathcal{V}$, and $y_i \in \mathcal{A}$.

- In an ordinal potential game, the sign of the utility variation incurred by player i when changing action unilaterally is the same as the sign of corresponding variation in the potential function:

$$\operatorname{sgn}\left(u_i(y_i, x_{-i}) - u_i(x_i, x_{-i})\right) = \operatorname{sgn}\left(\Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i})\right)$$

Potential games have Pure Strategy Nash Equilibria

Proposition: For an ordinal potential game, every global max point of the ord. potential function $\Phi(x)$ is a pure Nash equilibrium, i.e.,

$$\mathcal{N} \supseteq \mathcal{N}_{max} := \operatorname*{argmax}_{x \in \mathcal{X}} \Phi(x)$$

Proof: Since

$$\mathrm{sgn}(u_i(y_i, x_{-i}) - u_i(x_i, x_{-i})) = \mathrm{sgn}(\Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}))$$

we have that $x^* \in \mathcal{X}$ is PNE if and only if

$$\Phi(y_i, x^*_{-i}) \leq \Phi(x^*_i, x^*_{-i}) \qquad \forall i \in \mathcal{V},\ \forall y_i \in \mathcal{A}_i \tag{1}$$

- Note: There might be pure Nash equilibria outside $\operatorname*{argmax}_{x \in \mathcal{X}} \Phi(x)$

$\mathcal{N}$ = "local maximum points"

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- Corollary 1: Every finite ordinal potential game admits a PNE
- Corollary 2: Every continuous ordinal potential game with compact strategy space admits a PNE

Outline

Finite Improvement Property

- length-l path: sequence of strategy profiles $(x^{(0)}, x^{(1)}, \dots, x^{(l)})$ such that there exist deviating players $i_1, i_2, \dots, i_l$ with

$$x_{i_k}^{(k-1)} \neq x_{i_k}^{(k)} \qquad x_{-i_k}^{(k-1)} = x_{-i_k}^{(k)} \qquad \forall k = 1, \dots, l$$

- improvement path if deviating players have positive utility gain

$$u_{i_k}(x_{i_k}^{(k)}, x_{-i_k}^{(k)}) > u_{i_k}(x_{i_k}^{(k-1)}, x_{-i_k}^{(k)}) \qquad \forall k = 1, \dots, l$$

- useful to model myopic behavior of the players
- Finite Improvement Property (FIP): every improv. path is finite

Finite Improvement Property

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- useful to model myopic behavior of the players
- Finite Improvement Property (FIP): every improv. path is finite
- Lemma: FIP $\Longrightarrow \exists$ PNE x^*

  Proof: every maximal path terminates in a PNE

Finite Improvement Property

- Finite Improvement Property (FIP): every improv. path is finite
- Proposition: every finite ordinal potential game has the FIP

  Proof: Since $\Phi(x^{(0)}) < \Phi(x^{(1)}) < \ldots < \Phi(x^{(l)})$ and $\mathcal{X}$ finite, every improvement path can have length at most $|\mathcal{X}| - 1$

- Converse is NOT true: e.g., the following 2×2 game has the FIP

	-	+
-	1,0	2,0
+	2,0	0,1

but if it existed an every ordinal potential function Φ should satisfy

$$\Phi(-,-) < \Phi(+,-) < \Phi(+,+) < \Phi(-,+) = \Phi(-,-)$$

Finite Improvement Property

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  Proof: Since $\Phi(x^{(0)}) < \Phi(x^{(1)}) < \ldots < \Phi(x^{(l)})$ and $\mathcal{X}$ finite, every improvement path can have length at most $|\mathcal{X}| - 1$

- Converse is NOT true
- Definition: Game is generalized ordinal potential if $\exists \Phi : \mathcal{X} \to \mathbb{R}$

$$u_i(y_i, x_{-i}) - u_i(x_i, x_{-i}) > 0 \implies \Phi(y_i, x_{-i}) - \Phi(x_i, x_{-i}) > 0$$

- Proposition: For finite games

  Generalized Ordinal Potential $\iff$ Finite Improvement Property

- Proposition: Every finite game with FIP and such that

$$u_i(x_i, x_{-i}) \neq u_i(y_i, x_{-i}) \qquad \forall i \in \mathcal{V},\ x_i \neq y_i \in \mathcal{A}_i,\ x_{-i} \in \mathcal{X}_{-i}$$

is an ordinal potential game.

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Congestion games

For player set $\mathcal{V}$, action set $\mathcal{A}$ and $c_a : \mathbb{Z}_+ \to \mathbb{R}$ for $a \in \mathcal{A}$.

$$x \in \mathcal{X},\ a \in \mathcal{A} \quad n_a^x = |\{i \in \mathcal{V} \,|\, x_i = a\}|$$

Utility of unit i: $u_i(x) = -c_{x_i}(n_{x_i}^x)$.

The game $(\mathcal{V}, \mathcal{A}, \{u_i\})$ is called a singleton congestion game.

- utility of a player only depends on total number of players playing the same action.
- Actions $\leftrightarrow$ shared resources. If c_a's are non-decreasing, the more units use the same resource, the worse the performance.

Congestion games (cont'd)

An important extension:

- set of resources $\mathcal{E}$ (e.g., links in a transportation network) and, for $e \in \mathcal{E}$, congestion costs $c_e : \mathbb{Z}_+ \to \mathbb{R}_+$,
- action set $\mathcal{A} \subseteq 2^{\mathcal{E}}$ consists of a family of subsets of $\mathcal{E}$.

$$x \in \mathcal{X},\ e \in \mathcal{E} \quad n_e^x = |\{i \in \mathcal{V} \mid e \in x_i\}|$$

- the game $(\mathcal{V}, \mathcal{A}, \{u_i\})$ with utilities

$$u_i(x) = -\sum_{e \in x_i} c_e(n_e^x) \qquad \forall x \in \mathcal{X}$$

is called a (symmetric) congestion game

Congestion games are exact potential games

- Theorem: A symmetric congestion game with utility functions

$$u_i(x) = -\sum_{e \in x_i} c_e(n_e^x)$$

is an exact potential game with Rosenthal potential function

$$\Phi(x) = -\sum_{e \in \mathcal{E}} \sum_{h=1}^{n_e^x} c_e(h)$$

- Proof: For every $x, y \in \mathcal{X}$ such that $x_{-i} = y_{-i}$ we have

$$\Phi(y) - \Phi(x) = \sum_{e \in x_i} c_e(n_e^x) - \sum_{e \in y_i} c_e(n_e^y) = u_i(y) - u_i(x)$$

■

- result can be extended to homogeneous congestion games where $\mathcal{A}_i$ is player-dependent, while costs c_e remain player-independent

- non-homogeneous congestion games, where costs c_e^i are player-dependent, are not exact potential games in general

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Network games

- Definition: Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a network game on $\mathcal{G}$, or more briefly a $\mathcal{G}$-game, is a game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ such that

$$x_i = y_i \qquad x_{\mathcal{N}_i} = y_{\mathcal{N}_i} \qquad \Longrightarrow \qquad u_i(x) = u_i(y)$$

where $\mathcal{N}_i$ is the (out-)neighborhood of i in $\mathcal{G}$, $\forall\ i \in \mathcal{V}$, $x, y \in \mathcal{X}$

- In other words, a $\mathcal{G}$-game is one where the utility of player i depends only on her own action and on the actions of her neighbors

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- In other words, a $\mathcal{G}$-game is one where the utility of player i depends only on her own action and on the actions of her neighbors

- Example: Best-shot public good game: $\mathcal{A}_i = \{0, 1\}$, $0 < c < 1$

$$u_i(x_i, x_{-i}) = \begin{cases} 1 - c & \text{if } x_i = 1 \\ 1 & \text{if } x_i = 0,\ \sum_{j \in \mathcal{N}_i} x_j \geq 1 \\ 0 & \text{if } x_i = 0,\ \sum_{j \in \mathcal{N}_i} x_j = 0 \end{cases}$$

If player i or anyone in her neighborhood play 1, then i gets a reward 1. Who plays action 1 pays a cost c.

Pairwise separable network games

- Definition: For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ s.t. $W_{ii} = 0 \ \forall i \in \mathcal{V}$, a pairwise separable $\mathcal{G}$-game is $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ where every player i has non-empty action set $\mathcal{A}_i$ and utility function

$$u_i(x_i, x_{-i}) = \rho_i(x_i) + \sum\nolimits_j W_{ij}\lambda_{ij}(x_i, x_j)$$

where $\rho_i : \mathcal{A}_i \to \mathbb{R}$ is the standalone utility of player i, and $\lambda_{ij} : \mathcal{A}_i \times \mathcal{A}_j \to \mathbb{R}$ captures the externality of player j on player i

- Interpretation: every player is simultaneously playing 2-player games with all its out-neighbors, playing the same action in all of them. The utility is the sum of the pairwise utilities $\lambda_{ij}(x_i, x_j)$ and of the standalone $\rho_i(x_i)$ utility not depending on the interactions.

Examples

- the majority game is a pairwise separable game with $W \in \{0,1\}^{\mathcal{V}\times\mathcal{V}}$, $\mathcal{A}_i = \{\pm 1\}$, $\rho_i(x_i) = 0$, and $\lambda_{ij}(x_i, x_j) = +x_i x_j$.
- generalization with $W \in \mathbb{R}_+^{\mathcal{V}\times\mathcal{V}}$ and $\rho_i(x_i) = h_i x_i$ is the network coordination game and accounts for bias towards action $\mathrm{sgn}(h_i)$

- the minority game is a pairwise separable game with $\mathcal{A}_i = \{\pm 1\}$, $\rho_i(x_i) = 0$, and $\lambda_{ij}(x_i, x_j) = -x_i x_j$
- generalization with $W \in \mathbb{R}_+^{\mathcal{V}\times\mathcal{V}}$ and $\rho_i(x_i) = h_i x_i$ is the network anti-coordination game

- the coloring game is pairwise separable game with $\mathcal{A}_i = \mathcal{A} = \{\text{colors}\}$, $\rho_i(x_i) = 0$, and $\lambda_{ij}(x_i, x_j) = -\delta_{x_j}^{x_i}$

- quadratic games are pairwise separable with $\mathcal{A}_i = \mathbb{R}$, $\rho_i(x_i) = h_i x_i - x_i^2/2$, and $\lambda_{ij}(x_i, x_j) = \beta x_i x_j$

- best-shot public good game NOT pairwise separable on general $\mathcal{G}$

Pairwise separable potential games

- Theorem: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be undirected graph ($W = W'$). Consider pairwise separable $\mathcal{G}$-game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ with utilities

$$u_i(x_i, x_{-i}) = \rho_i(x_i) + \sum\nolimits_j W_{ij}\lambda_{ij}(x_i, x_j)$$

where $(\{i, j\}, \{\mathcal{A}_i, \mathcal{A}_j\}, (\lambda_{ij}, \lambda_{ji}))$ is 2-player exact potential game with potential function $\phi_{ij}(x_i, x_j)$ for every i, j s.t. $W_{ij} \neq 0$. Then, $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ is an exact potential game with potential function

$$\Phi(x) = \sum_{i \in \mathcal{V}} \rho_i(x_i) + \frac{1}{2} \sum_{i,j \in \mathcal{V}} W_{ij}\phi_{ij}(x_i, x_j)$$

Pairwise separable potential games

- Theorem: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be undirected graph ($W = W'$). Consider pairwise separable $\mathcal{G}$-game $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ with utilities

$$u_i(x_i, x_{-i}) = \rho_i(x_i) + \sum\nolimits_j W_{ij}\lambda_{ij}(x_i, x_j)$$

where $(\{i,j\}, \{\mathcal{A}_i, \mathcal{A}_j\}, (\lambda_{ij}, \lambda_{ji}))$ is 2-player exact potential game with potential function $\phi_{ij}(x_i, x_j)$ for every i, j s.t. $W_{ij} \neq 0$. Then, $(\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$ is an exact potential game with potential function

$$\Phi(x) = \sum_{i \in \mathcal{V}} \rho_i(x_i) + \frac{1}{2} \sum_{i,j \in \mathcal{V}} W_{ij}\phi_{ij}(x_i, x_j)$$

- Corollary: Network coordination (incl. majority), anti-coordination (incl. minority), coloring, and quadratic games on undirected graphs are exact potential games

Best-Shot public goods game

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ simple graph, $\mathcal{A}_i = \{0, 1\}$, $0 < c < 1$

$$u_i(x_i, x_{-i}) = \begin{cases} 1 - c & \text{if } x_i = 1 \\ 1 & \text{if } x_i = 0,\ \exists j \in N_i \ :\ x_j = 1 \\ 0 & \text{if } x_i = 0,\ \forall j \in N_i \ :\ x_j = 0 \end{cases}$$

Best-Shot public goods games

Utilities $u_i(x_i, x_{-i})$ satisfy the *decreasing difference property*:

$$u_i(b_i, x_{-i}) - u_i(a_i, x_{-i}) \geq u_i(b_i, y_{-i}) - u_i(a_i, y_{-i})$$

if $x_{-i} \leq y_{-i}$ and $a_i \leq b_i$.

In economy, such games model the so called *strategic substitutes effect*: the increase of a player's action, makes less profitable for the others to increase theirs.

Best-Shot public goods games

Theorem

For the Best-Shot public goods game, Nash equilibria always exist:

$x \in \{0, 1\}^n$ *is a Nash equilibrium if and only if* $\{i \in \mathcal{A} : x_i = 1\}$ *forms a maximal independent set of the graph* $\mathcal{G}$

Example: