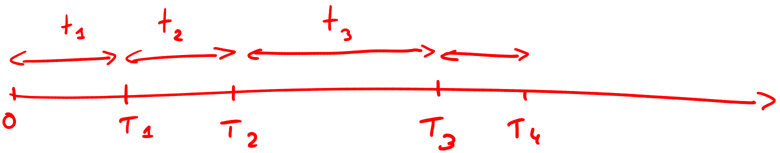
Poisson processes



Definition

A *Poisson process with rate* λ is a counting process $(N(s))_{s\in[0,\infty)}$ with $\boxed{N(0) = 0}$, whose inter-arrival times are i.i.d. exponential random variables with rate λ.

$$t_i \sim \exp(\lambda)$$

A review of the Exponential distribution

We say that $\tau \sim \exp(\lambda)$ if any of the following holds:

$$f_\tau(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \qquad P(\tau \leq t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$\boxed{P(\tau > t) = \begin{cases} e^{-\lambda t} & \text{if } t \geq 0 \\ 1 & \text{if } t < 0 \end{cases}}$$

FUNZIONE DI SOPRAVVIVENZA

$$E[\tau] = 1/\lambda, \quad \mathrm{Var}(\tau) = 1/\lambda^2$$

Memoryless property

For any $t \cancel{\geq}, s \geq 0$ we have $\boxed{P(\tau > t + s | \tau > s) = P(\tau > t)}$

Exponential random variables are the **only** continuous random variables with the memoryless property (geometric random variables are the only discrete random variables with the memoryless property).

Exponential races

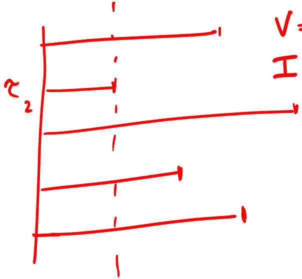
Exponential races: Let $\tau_1, \tau_2, \dots, \tau_n$ be independent random variables, with $\tau_i \sim \exp(\lambda_i)$. Let

$$V = \min\{\tau_1, \tau_2, \dots, \tau_n\} \quad \text{and} \quad I \text{ s.t. } \tau_I = V.$$

$V = \tau_2$
$I = 2$

Then:

1. $V \sim \exp\left(\sum_{i=1}^{n} \lambda_i\right)$
2. $P(I = j) = \frac{\lambda_j}{\sum_{i=1}^{n} \lambda_i}$
3. I and V are independent.



$t < 0 \quad P(V > t) = 1$

$V \sim \exp\left(\sum_{i=1}^n \lambda_i\right)$

$t \geq 0 \quad P(V > t) = P(\min_{i=1,\dots,n} \tau_i > t)$

$= P(\tau_1 > t, \tau_2 > t, \dots, \tau_n > t)$

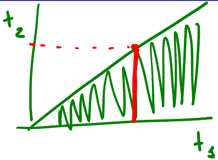
IND. $\overset{}{\rightarrow}$ $= P(\tau_1 > t) P(\tau_2 > t) \cdots P(\tau_n > t)$

$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdot \dots \cdot e^{-\lambda_n t}$

$= e^{-(\lambda_1 + \lambda_2 + \dots + \lambda_n) t}$

c.v.d.

Exponential races



$P(I=j) = \frac{\lambda_j}{\sum_{i=1}^{n} \lambda_i}$ with two exponentials.

IND.

$$P(I=1) = P(\tau_1 < \tau_2) \overset{\text{IND.}}{=} \iint_{t_1 < t_2} \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} dt_1 dt_2$$

$$= \int_0^\infty \left(\int_0^{t_2} \lambda_1 e^{-\lambda_1 t_1} \boxed{\lambda_2 e^{-\lambda_2 t_2}} dt_1 \right) dt_2$$

$$= \int_0^\infty \lambda_2 e^{-\lambda_2 t_2} \left(\int_0^{t_2} \lambda_1 e^{-\lambda_1 t_1} dt_1 \right) dt_2$$

$$= \cdots = \boxed{\frac{\lambda_1}{\lambda_1 + \lambda_2}}.$$

Exponential races

$$P(I=j) = \frac{\lambda_j}{\sum_{i=1}^n \lambda_i}.$$

$$P(I=j) = P\Big(\tau_j < \overbrace{\min_{i\neq j}\{\tau_i\}}^{\textcircled{1}\ \sim\ \exp\left(\sum_{i\neq j}\lambda_i\right)}\Big) \underset{\textcircled{2}}{=} \frac{\lambda_j}{\lambda_j + \sum_{i\neq j}\lambda_i} = \frac{\lambda_j}{\sum_{i=1}^n \lambda_i}$$

I and V are independent.

GOAL: $P(V > t, I = j) = P(V > t)P(I = j)$ for all t, j.

$$P(V > t, I = j) = P\Big(\underbrace{\boxed{\min_{\substack{i=1,\dots,n\\ i\neq j}} \tau_i} > \boxed{\tau_j}}_{I=j}, \underbrace{\tau_j > t}_{V>t}\Big)$$

$\sim exp(\sum_{i\neq j} \lambda_i)$ >INDIPENDENTI!

$\hat{\lambda}_j = \sum_{i \neq j} \lambda_i$

$$= \iint_{\boxed{h > t_j > t}} \hat{\lambda}_j e^{-h\hat{\lambda}_j} \lambda_j e^{-\lambda_j t_j} dh dt_j$$

$$= \int_0^\infty \left(\int_{t_j}^\infty \hat{\lambda}_j e^{-h\hat{\lambda}_j} \lambda_j e^{-\lambda_j t_j} dh \right) dt_j$$

$$= \cdots = \boxed{e^{-(\hat{\lambda}_j + \lambda_j)t} \frac{\lambda_j}{\hat{\lambda}_j + \lambda_j}} = P(V>t)P(I=j)$$

1 + 2

Exponential races

Alice and Bob are doing homework. Alice is done after a time $\tau_A \sim \exp(1)$ and Bob is done after a time $\tau_B \sim \exp(1/4)$. $u =$ hours

What is the probability Alice is done before Bob?

$$P(\tau_A < \tau_B) = P(I = A) = \frac{\lambda_A}{\lambda_A + \lambda_B} = \frac{1}{1 + 1/4} = 4/5$$

Knowing the the first who is done does so after 4 hours, what is the probability Alice is done before Bob? $= P(\tau_A < \tau_B) = 4/5$

$$P(I = A \mid V \geq 4) \cancel{=} P(\tau_A < \tau_B)$$

Knowing that Alice is done before Bob, what is the expected time she finishes homework?

$$E[V \mid V = \tau_A] \cancel{=} E[\tau_A] = 1$$

$$= E[\tau_A \mid V = \tau_A]$$

$$E[V \mid I = A] = E[V] = \frac{1}{1 + \frac{1}{4}} = 4/5$$

Sum of exponential random variables

Let $\tau_1, \tau_2, \ldots, \tau_n$ be independent **and identically distributed** exponential random variables, with common rate λ. Then, $T_n = \tau_1 + \tau_2 + \cdots + \tau_n$ has a gamma distribution with parameters n and λ, that is has density

$$f_{T_n}(t) = \begin{cases} \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Independent increments

Definition

A stochastic process $\{X_i : i \in I\}$ has *independent increments* if for any sequence $i_0 < i_1 < i_2 < \cdots < i_n$ we have that

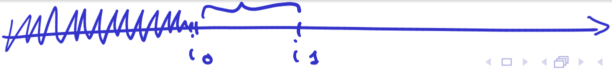
$$(X_{i_1} - X_{i_0}), (X_{i_2} - X_{i_1}), \ldots, (X_{i_n} - X_{i_{n-1}})$$

are independent.

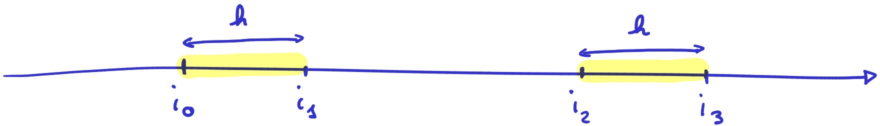
Alternative definition:

Definition

A stochastic process $\{X_i : i \in I\}$ has *independent increments* if for any $\boxed{i_0 < i_1}$ we have that $(X_{i_1} - X_{i_0})$ is independent of $\mathcal{F}_{i_0}$, where $\{\mathcal{F}_i : i \in I\}$ is the natural filtration.



Stationary increments



Definition

A stochastic process $\{X_i : i \in I\}$ has *stationary increments* if for any sequence $i_0 < i_1$ and $i_2 < i_3$ such that $i_1 - i_0 = i_3 - i_2$ we have that

$$(X_{i_1} - X_{i_0}) \quad \text{and} \quad (X_{i_3} - X_{i_2})$$

have the same distribution.

A review of the Poisson distribution

$\lambda > 0$

We say that $N \sim \text{Pois}(\lambda)$ if

$$P\{N = n\} = e^{-\lambda}\frac{\lambda^n}{n!}. \quad \text{IF} \quad n \in \{0, 1, 2, \dots\}$$

Properties:

- $E[X] = \lambda$ and $\text{Var}(X) = \lambda$.
- Let $X_1, X_2, \dots, X_n$ be independent random variables, with $X_i \sim \text{Pois}(\lambda_i)$. Then,

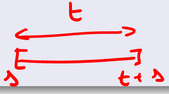
$$X_1 + X_2 + \cdots + X_n \sim \text{Pois}\left(\sum_{i=1}^{n} \lambda_i\right)$$

Poisson Process

Theorem

$(N(s))_{s\in[0,\infty)}$ is a Poisson process with rate λ if and only if it is a counting process such that

1. (i) $N(0) = 0$,
2. (ii) *it has independent increments;*
3. (iii) $N(t+s) - N(s) \sim \text{Poisson}(\lambda t)$.

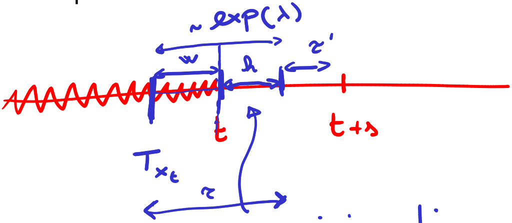


we split the proof into several parts. The first bullet is obvious

Poisson Process

$N(t+s) - N(t)$ IND. DA $\mathcal{F}_t$

$N(t)$ has independent increments. (sketch!)



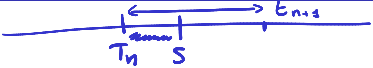
remaining time

$\tau \sim exp(\lambda)$

$$P(h > u \mid \mathcal{F}_t) = P(h > u \mid \tau > w)$$

$$= P(\tau - w > u \mid \tau > w) = P(\tau > w + u \mid \tau > w)$$

$$= P(\tau > u) = e^{-\lambda u}$$



$t_{n+1} > s - T_n$

If $(N(s))_{s\in[0,\infty)}$ is a Poisson process, $N(s) \sim Poisson(\lambda s)$. Indeed $N(s) = n$ if and only if $T_n \leq s < T_{n+1}$. That is, for $n \geq 0$,

$T_n = \sum_{i=1}^{n} t_i \sim$ Gamma

INDIP.

$$P\{N(s) = n\} = P\{T_n \leq s, \boxed{t_{n+1} > s - T_n}\} = \int_0^s \int_{s-t}^{\infty} f_{T_n}(t) f_{t_{n+1}}(r)\, dr\, dt$$

$$= \int_0^s \int_{s-t}^{\infty} \boxed{\lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}} \cdot \lambda e^{-\lambda r}\, dr\; dt$$

$$= \frac{\lambda^n}{(n-1)!} \int_0^s e^{-\lambda t} t^{n-1} \cdot e^{-\lambda(s-t)}\; dt = \frac{\lambda^n}{(n-1)!} \int_0^s t^{n-1} e^{-\lambda s} dt$$

$$= \boxed{e^{-\lambda s} \frac{(\lambda s)^n}{n!}}.$$

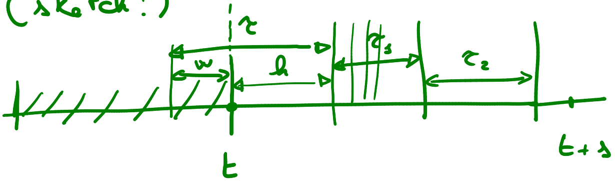
Poisson Process

$$P(t_1 > s) = P(N(s) = 0) = e^{-\lambda s} \cdot \frac{\lambda s^0}{0!} = e^{-\lambda s}$$

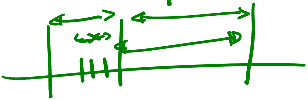
$$\hookrightarrow t_1 \sim \exp(\lambda)$$

Reverse:

(sketch!)



$t_2, t_3, t_4 \ldots$ i.i.d $\sim \exp(\lambda)$



Poisson Process

Theorem

$(N(s))_{s\in[0,\infty)}$ is a Poisson process with rate λ if and only if it is a counting process such that:

1. $N(0) = 0$;
2. *it has independent increment;*
3. *it has stationary increments;*
4. $\lim_{h\to 0} \frac{P(N(h)=1)}{h} = \lambda$ *and* $\lim_{h\to 0} \frac{P(N(h)\geq 2)}{h} = 0$.

Example

Customers arrive at a shipping office at times of a Poisson process with rate 3 per hour. (a) The office was supposed to open at 8AM but the clerk Oscar overslept and came in at 10AM. What is the probability that no customers came in the two-hour period? (b) What is the distribution of the amount of time Oscar has to wait until his first customer arrives?

Example

Customers arrive at a shipping office at times of a Poisson process with rate 3 per hour. (a) The office was supposed to open at 8AM but the clerk Oscar overslept and came in at 10AM. What is the probability that no customers came in the two-hour period? (b) What is the distribution of the amount of time Oscar has to wait until his first customer arrives?
Solution. Let $N(s)$ be the number of customers who have arrived at time s hours after 8am. Then for part (a) we want

$N(s) \sim \mathrm{Pois}(\lambda s)$

$$P(N(2) = 0) = e^{-3\cdot 2}\frac{(3\cdot 2)^0}{0!} = \boxed{e^{-6}}.$$

Alternatively, you can realize that we have exponential random variables with parameter 3, and we are asking for $P(t_1 > 2) = \int_2^\infty 3e^{-3t}dt = e^{-6}$.

Example

Customers arrive at a shipping office at times of a Poisson process with rate 3 per hour. (a) The office was supposed to open at 8AM but the clerk Oscar overslept and came in at 10AM. What is the probability that no customers came in the two-hour period? (b) What is the distribution of the amount of time Oscar has to wait until his first customer arrives?

Solution. For part (b), we note that the number of customers arriving after 10am still follows a Poisson process with rate 3 per hour. Hence, the distribution is Exponential with parameter 3 per hour.