Renewal-reward processes

Definition

Let $\{X_i\}_{i \in I}$ be a renewal process with inter-arrival times $\{t_\ell\}_{\ell=1}^\infty$ and rewards $\{r_\ell\}_{\ell=1}^\infty$. If the random variables $\{(t_\ell, r_\ell)\}_{\ell=1}^\infty$ are i.i.d. then we have a *renewal-reward process*.

Theorem

Let $\{X_i\}_{i \in I}$ be a renewal process with $E[t_1] > 0$, and $P(t_1 < \infty) = 1$. Then, with probability 1,

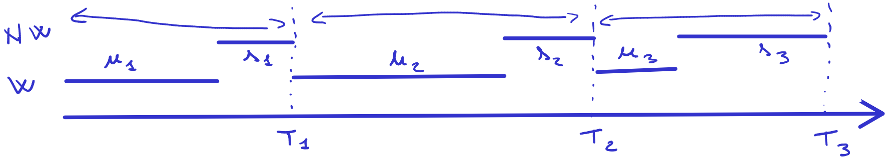
$$\lim_{i \to \infty} \frac{R_i}{i} = \frac{E[r_1]}{E[t_1]},$$

where

$$R_i = \sum_{\ell=1}^{X_i} r_\ell$$

is the cumulative reward up to time i.

Renewal-reward processes



Assume a machine create a revenue of 10 euros per day when it works, repairing it costs 50 euros of transportation plus a repairing cost of approximately 40 euros per day. The machine does not break down for times with mean 30 days and repairing it takes on average 5 days. Assume that when the machine is repaired, it can be considered as new.

After 10 years of usage, how much revenue per day has the machine produced?

EVENTO = MACCHINA RIPARATA

X_t = # EVENTI FINO AL TEMPO t

$r_i = 10 \cdot u_i - 50 - 40 \cdot s_i$

u_i := tempo lavoro i-ESIMO INT.

s_i := tempo riparazione

Renewal-reward processes

$$\frac{R_{10\cdot 365}}{10\cdot 365} \approx \lim_{i\to+\infty} \frac{R_i}{i} = \frac{E[r_1]}{E[t_1]} =$$

$$= \frac{E[10\cdot u_1 - 50 - 40\cdot s_1]}{E[u_1 + s_1]}$$

$$= \frac{10\cdot 30 - 50 - 40\cdot 5}{30+5}$$

$$= \frac{300 - 50 - 200}{35} = \frac{50}{35} \text{ €/giorno}$$

Renewal-reward processes

In the previous question, how does the answer change if repair times are geometric random variables, and times until break-downs are Poisson?

SEMPRE STESSA RISPOSTA!

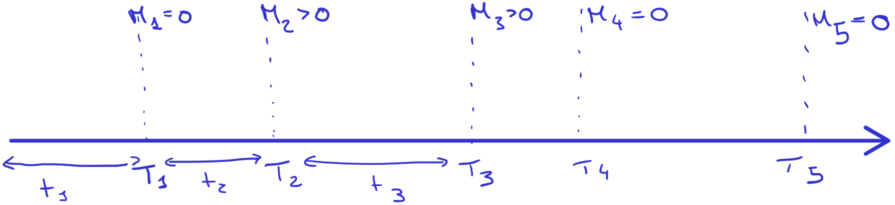
+ IN PIÙ: POTETE STUDIARE
CONFIDENCE INTERVAL...

Renewal-reward processes

Suppose the manager of a restaurant should pay $\$c_1$ per day for health and safety maintenance, enough to guarantee that there will be no health violations. However, the manager is evil and wants to budget less, say $c_2 < c_1$. We want to figure out when this will make him money. Here is what we know:

- The restaurant is inspected, on average, every 45 days, and the number of days between two consecutive inspections are independent and identically distributed random variables (maybe not realistic).
- There is a probability $p = p(c_2)$ that a violation will be found on a given visit. It is a monotonically decreasing function of c_2, with $p(c_1) = 0$.
- The fines have an expected value of $c_3 > 0$ and the fine sizes are all independent and identically distributed, and independent on the inspection times.

Renewal-reward processes



X_i = # ISPEZIONI ENTRO i

$t_1, t_2, t_3, \ldots$ i.i.d.

$$r_i = -c_2 \cdot t_i - M_i$$

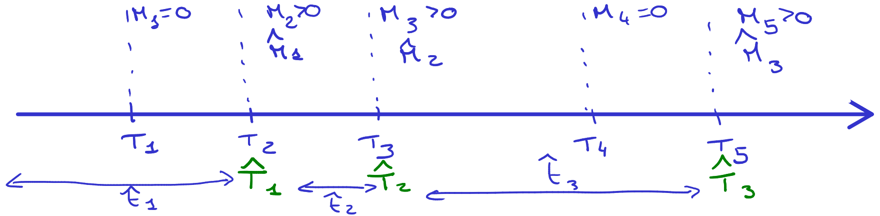
RENEWAL-REWARD PROCESS

$$\lim_{i \to +\infty} \frac{R_i}{i} = \frac{E[r_1]}{E[t_1]}$$

$$E[r_1] = -c_2 \cdot E[t_1] - E[M_1] = -c_2 \cdot 45 - p(c_2) \cdot c_3$$

$$\lim_{i \to +\infty} \frac{R_i}{i} = \frac{-c_2 \cdot 45 - p(c_2) \cdot c_3}{45} = -c_2 - \frac{p(c_2) \cdot c_3}{45}$$

Renewal-reward processes



Y_i = # MULTE RICEVUTE ENTRO i

$$\hat{\tau}_i = -c_2 \cdot \hat{t}_i - \hat{M}_i$$

$$E[\hat{\tau}_1] = -c_2 \cdot E[\hat{t}_1] - c_3$$

$$\hat{t}_1 = \sum_{i=1}^{G} t_i \qquad G \sim Geom(p(c_2))$$

$$E[\hat{t}_1] = E[G] \cdot E[t_1] = \frac{1}{p(c_2)} \cdot 45$$

$$\lim_{i \to +\infty} \frac{\hat{R}_i}{i} = \frac{E[\hat{\tau}_1]}{E[\hat{t}_1]} = -c_2 - \frac{c_3}{\frac{1}{p(c_2)} \cdot 45}$$

Z_i = # GIORNI ENTRO i = i $\qquad \tilde{t}_i = 1$ RENEWAL PROC.

$\tilde{\tau}_i = -c_2 - \boxed{\tilde{M}_i}$ R.R. PROCESS? NO $\qquad \tilde{M}_i$ NON i.i.d.

$p(c_1) = 0$

Assume

$$p(c_2) = \frac{1+c_1}{c_1} \cdot \frac{1}{1+c_2} - \frac{1}{c_1}$$

We want to minimize

$$g(c_2) = c_2 + c_3 \frac{p(c_2)}{45} = c_2 + \frac{c_3}{45} \left(\frac{1+c_1}{c_1} \cdot \frac{1}{1+c_2} - \frac{1}{c_1} \right).$$

We consider

$$g'(c_2) = 1 - \frac{c_3}{45} \cdot \frac{1+c_1}{c_1} \cdot \frac{1}{(1+c_2)^2}$$

and note that

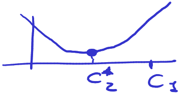
$$g''(c_2) = \frac{c_3}{45} \cdot \frac{1+c_1}{c_1} \cdot \frac{2}{(1+c_2)^3}$$

is positive for any c_2 in $[0, c_1]$.

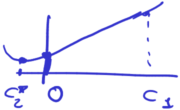
Renewal-reward processes

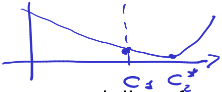
Solving $g'(c_2^*) = 0$ yields

$$c_2^* = \sqrt{\frac{c_3}{45} \cdot \frac{1+c_1}{c_1}} - 1.$$



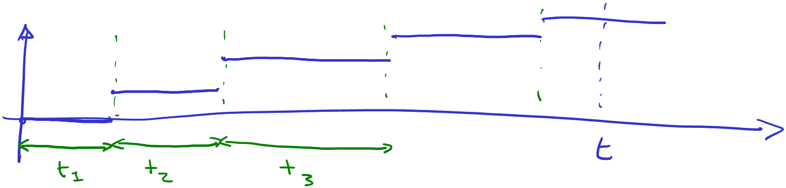
We have three cases:

1. if c_2^* is between 0 and c_1, then it corresponds to the optimal policy the evil manager is looking for;
2. if
$$\frac{c_3}{45} \cdot \frac{1+c_1}{c_1} \leq 1$$


then the fee to pay for the violation is so low that the optimal strategy is to pay nothing for maintenance;
3. if
$$\frac{c_3}{45} \cdot \frac{1+c_1}{c_1} \geq (1+c_1)^2$$


then the fee is adequate, and the optimal policy is to pay c_1 dollars for maintenance.

Poisson processes



Definition

A *Poisson process with rate* λ is a counting process $(N(s))_{s\in[0,\infty)}$ with $N(0) = 0$, whose inter-arrival times are i.i.d. exponential random variables with rate λ.

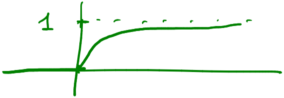
$t_1, t_2, t_3, \dots$ i.i.d

$\sim \exp(\lambda)$

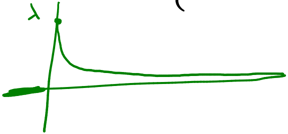
A review of the Exponential distribution

λ RATE

We say that $\tau \sim \exp(\lambda)$ if any of the following holds:



$$f_\tau(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \qquad F(t) = P(\tau \leq t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$



$$P(\tau > t) = \begin{cases} e^{-\lambda t} & \text{if } t \geq 0 \\ 1 & \text{if } t < 0 \end{cases} = 1 - F(t)$$

FUNZIONE SOPRAVVIVENZA

$$E[\tau] = 1/\lambda, \quad \mathrm{Var}(\tau) = 1/\lambda^2$$

Memoryless property

τ DURATA

For any $t, s \geq 0$ we have $P(\tau > t + s | \tau > s) = P(\tau > t)$

$$P(\tau > t+s \mid \tau > s) = \frac{P(\tau > t+s, \tau > s)}{P(\tau > s)} = \frac{P(\tau > t+s)}{P(\tau > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t}$$

Exponential random variables are the **only** continuous random variables with the memoryless property (geometric random variables are the only discrete random variables with the memoryless property).