Risk Management – 2024/25

Risk modeling, measurement, and management

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Part 1

Introductory examples

We will deal with decision making under risk and uncertainty, with a limited scope restricted to problems that are managerial in nature.

We do not consider, for instance; cybersecurity and network intrusion; reliability in construction engineering, mechanical devices, and aircraft; safety issues in chemical plants, etc. These cases are tightly related with an application domain and require deep domain-specific knowledge.

Our aims are more methodological. Hence, we consider cases that allow us to apply more generic mathematical tools from probability, statistics, and optimization, all integrated under the common umbrella of operations research.

A significant part of the examples we will consider are related with finance, but we do not want to consider only financial risk management.

We also consider problem in operations management, supply chain management, etc.

When we deal with uncertain risk factors, it is tempting to base our decisions only on their expected value. Certainly, this makes decision making easier, but does it make sense?

(See also newsvendor model, Hamptonshire Express case)

Consider an asset allocation problem. We have to choose a set of portfolio weights, w_i, $i = 1, \ldots, n$, where n is the number of assets that we are considering for inclusion in the portfolio.

Portfolio weights may be collected into vector $\mathbf{w} \in \mathbb{R}^n$. With this choice of decision variables, a natural constraint is

$$\sum_{i=1}^{n} w_i \equiv \mathbf{1}^{\mathsf{T}}\mathbf{w} = 1,$$

where $\mathbf{1} \in \mathbb{R}^n$ is a column vector with all elements set to 1. If we do not require full investment, we may rewrite the constraint as $\mathbf{1}^{\mathsf{T}}\mathbf{w} \leq 1$.

If we rule out short-selling, we also require a non-negativity condition,

$$w_i \geq 0, \quad i = 1, \ldots, n.$$

We may also represent this condition in the compact form $\mathbf{w} \geq \mathbf{0}$. These two constraints imply $w_i \leq 1$.

In practice, additional constraints may be enforced in order to limit exposure to individual assets or sets of assets, defined on the basis of geography or industrial sectors. These simple constraints may be expressed as linear inequalities.

Let us denote the random return of asset i, over the holding period, by R_i.

Then, the portfolio return, denoted by R_p, can be easily related to our decision variables w_i,

$$R_p = \sum_{i=1}^{n} w_i R_i.$$

Let us denote the expected return of individual assets as

$$\mu_i \doteq \mathrm{E}[R_i], \quad i = 1, \ldots, n.$$

The expected value of a linear combination of random variables is just the corresponding linear combination of expected values. Hence, we find

$$\mu_p \doteq \mathrm{E}[R_p] = \sum_{i=1}^{n} w_i \mu_i = \boldsymbol{\mu}^\mathsf{T} \mathbf{w},$$

where we collect expected returns into vector $\boldsymbol{\mu} \in \mathbb{R}^n$.

Maximizing expected return leads to a nonsensical solution (why?).

In order to account for risk, we may consider standard deviation or, equivalently, the variance or return.

The variance of portfolio return as a function of portfolio weights is

$$\sigma_p^2 \doteq \mathsf{Var}(R_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \sigma_{ij} w_j = \mathbf{w}^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{w},$$

where $\sigma_{ij} \doteq \mathsf{Cov}(R_i, R_j)$ is the covariance between returns of assets i and j, and covariances are collected into the square covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$. Note that the diagonal of this matrix collects the return variances, $\mathsf{Var}(R_i) \doteq \sigma_i^2 \equiv \sigma_{ii}$.

There is a good reason to consider standard deviation as the risk measure, rather than variance: Standard deviation is measured in the same units as return. However, variance may be mathematically more convenient, as it leads to convex quadratic programming (QP) problems, which are easy to solve.

Actually, the kind of optimization problem we end up with depends on how we represent the tradeoff between risk and reward.

If we minimize risk subject to a lower bound μ_{min} on expected return, we obtain the QP problem:

$$\begin{aligned} \min \quad & \mathbf{w}^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{w} \\ \text{s.t.} \quad & \mathbf{1}^{\mathsf{T}}\mathbf{w} = 1 \\ & \boldsymbol{\mu}^{\mathsf{T}}\mathbf{w} \geq \mu_{\mathsf{min}} \\ & \mathbf{w} \geq \mathbf{0}. \end{aligned}$$

By changing the lower bound on expected return, we generate a set of portfolios, which is called the mean–variance efficient frontier.

From a computational viewpoint, this approach is quite convenient. This simplicity should not mask the limitations and the difficulties of such a model:

- Variance and standard deviation are symmetric risk measures, since they penalize both under- and above-average returns in the same way, even though the former may correspond to a huge loss and the latter to a welcome extra-profit. Hence, they may not be quite adequate to deal with possibly asymmetric distributions of returns. A quantile-based measure could better capture risk.

- Estimating the covariance matrix Σ is no easy task.

  [See `PortfolioNoise.mlx`]

- What about *forecasting* expected returns $\boldsymbol{\mu}$ for the future, rather than collecting past averages that might not be relevant anymore?

Despite all of its pitfalls, the mean–variance model plays a pivotal role in financial theory, leading to a body of knowledge known as Modern Portfolio Theory, and it is associated with the name of its inventor, Harry Markowitz.

From a theoretical viewpoint, mean–variance efficiency is also related to an important, albeit controversial, equilibrium model, the capital asset pricing model (CAPM).

In standard asset allocation problems, a portfolio of financial investments must be selected, where wealth must be allocated to a set of financial assets like stocks and bonds, featuring an uncertain return.

The uncertain return is a random variable, and we say that an equity portfolio is subject to **market risk**. However, if we invest in foreign stock markets, another risk factor comes into play, **foreign exchange (forex) risk** or **currency risk**.
Question: Are these two sources of risk independent?

We may consider stock returns as risk factors, but there may be a hierarchy of risk factors. For instance, we may decompose the stock return into a set of common risk factors (systematic risk factors like market indexes or oil price) and a specific (idiosyncratic) risk factor.

If we consider bonds, an obvious concern is **interest rate risk**, which is difficult to model, since there is a whole term structure of interest rates (i.e., there is an array of related interest rates with different maturities).

In the case of bonds, other issues are related with **credit risk**. If we consider a long term problem, **inflation risk** may be a concern too.

Things get more complicated when we consider alternative assets like commodities and real estate, or derivatives like futures, swaps, and options. In these cases, we may also want to control, e.g., **liquidity risk** and **volatility risk**.

Are all of the above risk factors observable? How do they relate with asset prices?

We may get a clue about risk management in banking by looking at a stylized balance sheet.*

Assets Investments of the firm		Liabilities Obligations from fundraising	
Cash	£10M	Customer deposits	£80M
(and central bank balance)			
Securities	£50M	Bonds issued	
- bonds, stocks, derivatives		- senior bond issues	£25M
Loans and mortgages	£100M	- subordinated bond issues	£15M
- corporates		Short-term borrowing	£30M
- retail and smaller clients		Reserves (for losses on loans)	£20M
- government			
Other assets	£20M	Debt (sum of above)	£170M
- property			
- investments in companies		Equity	£30M
Short-term lending	£20M		
Total	£200M	Total	£200M

*Borrowed from McNeil et al., chapter 2.

Risk management in banking cannot be limited to the asset side. We need an integrated Asset–Liability Management (**ALM**) perspective.

Equity is the difference between the asset side and the liability side of the balance sheet. If a bank loses on the asset side, it may not be a problem if there is a corresponding drop on the liability side.

Both can be subject to financial risk factors, but the impact may be different. A typical reason of concern is the difference in maturities, which means that there is a mismatch in the exposure to interest rate risk.

Which bond is riskier? A short term or a long term bond?

Some liabilities may be non-maturing (bank accounts), but subject to liquidity risk. An otherwise solvent bank may go bankrupt if there is a mismatch between long-term assets and short-term liabilities (think of bank runs).

Banks are also subject to regulatory risk, operational risk, and retail credit risk (which is of a different nature with respect to the credit risk in corporate or sovereign bonds).

Another source of peculiar risks are mortgages. Of course, there is a credit risk issue, as the debtors may default. Risk pooling should help, but painful experience (see the subprime mortgage crisis) has shown the impact of correlation risk.

Another issue with mortgages is early repayment risk, as this exposes the bank to reinvestment risk. On the other hand, holders of callable bonds issued by banks suffer from the same problem.

To understand some peculiarities of the insurance risk landscape, we consider again a stylized balance sheet.†

Assets		Liabilities	
Investments		Reserves for policies written (technical provisions)	£80M
- bonds	£50M		
- stocks	£5M	Bonds issued	£10M
- property	£5M		
Investments for unit-linked contracts	£30M	Debt (sum of above)	£90M
Other assets	£10M	Equity	£10M
- property			
Total	£100M	Total	£100M

In insurance, we must distinguish life vs. non-life insurance. Different branches of actuarial mathematics deal with mathematical models in the two subsectors.

A general issue, is solvency, which is carefully regulated.

†Borrowed again from McNeil et al., chapter 2.

In the life case financial risk factors, like interest rates and inflation, play a definite role. A more specific factor is longevity risk.

Consider the two standard kinds of pension fund: defined benefits and defined contributions? Who is bearing what risk?

From the viewpoint of the employee, accumulation and decumulation policies must be selected. From the insurer viewpoint, annuity pricing is another issue.

Some products offer a minimum guaranteed return, which entails an optionality component.

Pricing a policy is a typical problem in non-life insurance, as is fraud detection. Apart from traditional tools, models from statistical/machine learning are a hot topic (e.g., generalized linear models, classifiers, etc.).

Modern trends also worth mentioning are explainable AI and roboadvising.

Securitization is a way to engineer new assets by converting illiquid assets, such as a pool of mortgages, into a tradable security. **Asset-backed securities** (ABS) collect the cash flows from a pool of assets and are sold as bonds.

In the specific case of a **mortgage-backed security** (MBS), there are two risks that the investor is subject to:

- The prepayment risk: If interest rates drop, the homeowner may find it convenient to terminate the old mortgage and open a new one at a lower rate. Hence, there is a reinvestment risk (see callable bonds).

- The default risk, as homeowners may fail to comply with periodic payments.

Default risk can be diversified away by pooling mortgages, unless risks are strongly correlated (which may be the case under an economic stress). Also the value of the collateral is risky.

It is worth emphasizing that ABSs are sometimes improperly referred to as *derivatives*.

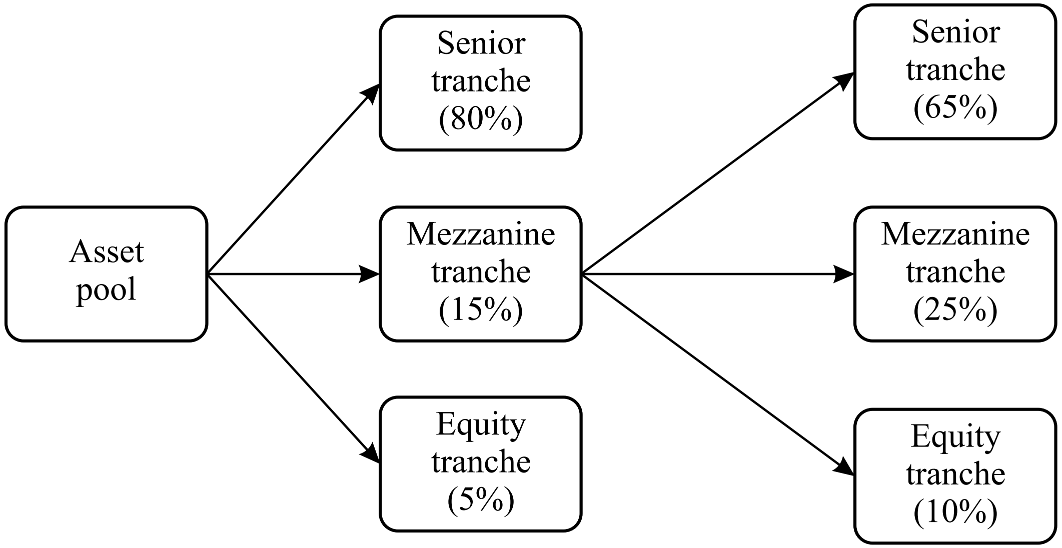
An important concept related to securitization is **tranching**, a mechanism by which different securities, with different risks, are issued. The idea is that losses are sustained by different tranches in a well-defined sequence.

To keep it simple, we may imagine that an ABS is tranched into the following three levels (example taken from Hull):

- The **equity tranche**, which is the first one to face any loss. For instance, the equity tranche may have to cover the first, say, 5% of loss. The equity tranche consists of cheap, but speculative-level securities.

- The **mezzanine tranche**, which has to sustain, say, the next 15% of loss.

- The **senior tranche**, which is supposed to be fairly safe, as it has to sustain only the loss in excess of the previous levels.



Since the assets in the mezzanine tranche were fairly risky, they were not easily sold to investors. This suggested the possibility of yet another level of securitization, whereby assets in the mezzanine tranche were pooled, repackaged, tranched again, and sold.

The 15% of the original mezzanine tranche is split into three new tranches amounting to 10%, 25%, and 65%, respectively. One might assume that the senior tranche of the second-level security was fairly safe, but let us analyze a few scenarios.

1. If the loss on the original ABS is 10%, the first 5% is sustained by the equity tranche, and the second 5% is sustained by the mezzanine tranche. This 5% actually amounts to 33.3% (one third, 5% out of 15%) of the total potential loss of the original mezzanine tranche. This means that the equity tranche at the second level sustains a 10% of loss and is wiped out, and that the mezzanine tranche in the second-level CDO has to sustain $33.3\% - 10\% = 23.3\%$ of the loss deriving from the original mezzanine tranche, which amounts to $23.3/25 = 93.2\%$ of the second-level potential loss. The senior tranche of the CDO is safe in this scenario.

2. A slight increase in loss on the original assets, say, from 10% to 13%, has a significant impact. Now the original mezzanine tranche loses

$$\frac{13-5}{15} = 53.3\%$$

of its value, which means that the mezzanine tranche of the CDO is wiped out, too, while the senior tranche loses

$$\frac{53.3-(10+25)}{65} = 28.2\%$$

of its value.

3. If the loss on the first-level ABS is 15%, the original mezzanine tranche loses $10/15 = 66.7\%$ of its value. Hence, the senior tranche of the second-level CDO loses

$$\frac{66.7-(10+25)}{65} = 48.8\%$$

of its value.

Thus, we realize that senior tranches at the second level of securitization were actually quite risky. Despite this fact, these second-level CDOs received quite good ratings from specialized agencies, possibly due to a conflict of interest, since the agencies in charge of rating these securities were paid for this service by the investment banks originating them.

- **A risk model.** Definition of the primary risk factors and characterization of their uncertainty (not necessarily stochastic), as well as their mapping to the relevant outcome (P/L, service level, etc.). Statistical estimation issues must be taken into account (see the trouble with mean–variance). We may need advanced tools like extreme value and copula theory. There is a cross-sectional, but also a longitudinal side of the coin.

- **A risk measure.** Standard deviation is a symmetric risk measure, but we are not likely to complain about windfalls and unexpected profits. Thus, we should look for alternative definitions leading to asymmetric measures. Furthermore, we must characterize the properties of a *coherent* risk measure. As an alternative, we may try to characterize *subjective* risk aversion by a utility function.

- **A risk management strategy.** Risk measurement is useful in monitoring the consequence of decisions, but we need a proper way to *make* those decisions. We may use simple immunization or hedging strategies, or build a sophisticated stochastic or robust optimization model.

Part 2

The risk model

Risk models are needed:

- to describe an environment in which we have to make decisions;
- to forecast the future evolution of the environment;
- to predict the consequences of our decisions within the environment;
- to drive scenario generation to support decision models and to assess risk measures.

A wide array of tools may be used, from several disciplines.

- Probabilistic models: extreme values to model tail behavior; multivariate distributions and copula theory; time series models; stochastic differential equations; Bayesian networks (subjective risk assessment).
- Statistical models (supervised learning): forecasting and prediction models; robust estimation.
- Statistical models (unsupervised): dimensionality reduction (PCA etc.); anomaly detection.

Not all models have to be stochastic in nature:

- Scenarios may be stress scenarios, to which we do not associate any probability measure.
- In classical worst-case robust optimization, we use non stochastic uncertainty sets (hybrid approach: DRO – Distributionally Robust Optimization).

Sometimes, a distinction is made in the literature between risk and uncertain (in the Knightian sense). We face risk when we have an exposure to uncertain outcomes, but the rules of the game are clear and the probabilities are known (e.g., dice throwing). In true uncertainty, we do not know probabilities, and we possibly don't even know all of the possible outcomes (black swans; unknown unknowns).

Risk models may be static or dynamic. In the latter case we may have any combination of discrete/continuous time and discrete/continuous states. Also note that risk factors may be observable or not (volatility risk in option pricing), as well as purely exogenous or at least partially endogenous.

The primary risk factors are transformed into a relevant outcome (service level, P/L), like a function of a random variable (e.g., by a pricing model).

Let us consider a classical linear factor model for equity portfolio management.

A factor model may be needed to shed light on the structure of risk factors and to ease estimation issues.

In a mean-variance portfolio optimization model, we could estimate the covariance σ_{ij} by its sample counterpart

$$S_{ij} = \frac{1}{T-1} \sum_{t=1}^{T} (R_{it} - \overline{R}_i)(R_{jt} - \overline{R}_j),$$

based on a sample of returns R_{it} and R_{jt}, $t = 1, \dots, T$, where $\overline{R}_i$ and $\overline{R}_j$ are sample means.

Clearly, this is not feasible in practice. For $n = 500$ assets, we should estimate $n(n-1)/2$ covariances and n variances, i.e,

$$n + \frac{n \cdot (n-1)}{2} = \frac{n \cdot (n+1)}{2} = 125{,}250$$

parameters.

A commonly suggested alternative is a linear factor model like

$$R_i = \alpha_i + \sum_{k=1}^{m} \beta_{ik} F_k + \epsilon_i, \quad i = 1, \ldots, n, \tag{1}$$

where the random return of each asset i is related with a set of m common (systematic) risk factors F_k, $k = 1, \ldots, m$, where $m \ll n$, and a specific (idiosyncratic) factor ϵ_i.

Sometimes, excess returns with respect to a risk free rate are used, rather than plain returns.

The parameter α_i is a constant, related with a specific risk premium, and β_{ik} is the sensitivity (exposure) of asset i to risk factor k.

Note that diversification may protect against specific risk only.

From the theory of linear regression we know that the specific risk is an error term, uncorrelated with the common factors (the regressors). If we further assume uncorrelated risk factors (diagonal model), the covariance is

$$\sigma_{ij} = \sum_{k=1}^{m} \beta_{ik} \beta_{jk} \sigma_k^2, \qquad i \neq j,$$

where σ_k^2 is the variance of factor F_k.

For variance we have

$$\sigma_i^2 \equiv \sigma_{ii} = \sum_{k=1}^{m} \beta_{ik}^2 \sigma_k^2 + \sigma_{\epsilon i}^2,$$

where $\sigma_{\epsilon i}^2$ is the variance of the specific factor.

Now, in order to estimate the covariance matrix, we need $m \times n$ betas, m volatilities of common risk factors, and n specific volatilities. If $n = 500$ and $m = 3$, we need 2003 parameters, a couple of order of magnitues less than a full covariance matrix.

Apart from statistical issues, a factor model may also shed light on the structure of risk (see the CAPM equilibrium model).

The simplest factor model is the single-index model, where the only common factor is a broad market index (say, S&P 500).

In general, risk factors may be:

- macroeconomic factors (e.g., inflation, oil price, etc.);
- fundamental factors (firm leverage and other indicators used in corporate finance);
- statistical factors (PCA is one possibility, which also give orthogonal factors; a difficulty may be the lack of interpretability; minimum torsion has been proposed as an alternative);
- factors obtained from alternative data (and possibly use of machine learning/AI).

An example of a nonlinear risk model is the bond pricing formula

$$P_T(t, r(t, \cdot)) = \sum_{i=1}^{m} \frac{cF}{2} e^{-r(t,t_i)\cdot(t_i - t)} + F e^{-r(t,t_m)\cdot(t_m - t)},$$

where c is the (semiannual) coupon rate and F is the face value, cash flows occur at time instants t_i, $i = 1, \ldots, m$, maturity is $T \equiv t_m$, e $r(t, \tau)$ gives the term structure, i.e., the annual interest rate for time period (t, τ), with continuous-time compounding.

Let us consider the impact of an annually compounded interest rate on a zero:

$$P_z(0; R_{0,T}, T) = \frac{F}{(1 + r_1(0, T))^T}. \tag{2}$$

Here, the interest rate gives the required yield for the bond.

Equation (2) gives a nonlinear mapping between price and yield.

To get a feeling, consider three zeros with face value $F = 100$, maturing in 3, 10, e 30 years, respectively. Assume a flat structure (yield is the same for every maturity).

If $r_1(0, \cdot) = 4\%$, the price of the first zero is

$$P_z(0; 0.04, 3) = \frac{100}{(1 + 0.04)^3} = 88.90.$$

If yield goes up by 100 basis points to 5%, we find

$$P_z(0; 0.05, 3) = \frac{100}{(1 + 0.05)^3} = 86.38,$$

with a percentage loss of

$$\frac{86.38 - 88.90}{88.90} = -2.83\%.$$

The table shows the risk in long-term zeros.

T (years)	3	10	30
$P_z(0; 0.04, T)$	88.90	67.56	30.83
$P_z(0; 0.05, T)$	86.38	61.39	23.14
% loss	−2.83	−9.13	−24.96

Coupon-bearing bonds depend on a range of rates within the term structure. One way to simplify the problem is to resort to data reduction approaches like PCA or factor models.

[See bond durations and option deltas]

Is it true that increasing correlation increases risk?

It is well known that

$$\sigma_{1+2}^2 \doteq \mathrm{Var}(X_1 + X_2) = \sigma_1^2 + 2\rho_{1,2}\sigma_1\sigma_2 + \sigma_2^2.$$

If we measure risk by variance or standard deviation, increasing correlation does increase risk.

The same applies if we measure risk by a quantile for the sum of two jointly normal variables, as the quantile does not really provide any additional information.

But what if we consider the *maximum* between the two variables?

[See `PlayRho.mlx`]

Common wisdom suggests that increasing correlation between risk sources has an adverse effect on investors. Actually, there may be somewhat paradoxical results. Imagine an ABS (or a CDO), where loss is related to default on the part of 100 debtors, and all potential losses are the same in monetary terms. The probability of default is 2% for each debtor. Consider an equity tranche liable to pay the first 5% of defaults. What is the probability of losing the whole value of the asset?

If default events are both equiprobable and independent, the probability distribution of the number X of defaults is a binomial random variable, with probability 0.02 and size 100. The probability of a total loss for the equity tranche is

$$\begin{aligned} \mathsf{P}\{X \geq 5\} &= 1 - \mathsf{P}\{X \leq 4\} \\ &= 1 - \sum_{k=0}^{4} \binom{100}{k} \times 0.02^k \times 0.98^{100-k} \\ &= 1 - (0.133 + 0.271 + 0.273 + 0.182 + 0.090) \\ &\approx 5.08\%. \end{aligned}$$

If correlation is increased to 1, i.e., in the case of perfect correlation, there are only two scenarios:

- A name defaults, hence, all of them do the same, with probability 2%, which is the probability of total loss.

- No one defaults, with probability 98%.

Therefore, we notice that increasing the correlation reduces the risk of losing the whole value of an equity tranche asset, contrary to common wisdom.

Remark. We are using the term "correlation" in a somewhat improper way, as the correlation coefficient refers to values of numerical random variables, whereas we are talking about correlation of events. We may refer to the related indicator variables, anyway. Moreover, we consider extreme settings (no correlation, perfect correlation). Intermediate cases require copula theory.

On the contrary, a senior tranche covering the last 80% of defaults is fairly safe with independent defaults, as the probability of observing more than 20 defaults is negligible. However, total loss has probability 2% in the case of perfect correlation. Hence, an increasing correlation increases risk in this case, as one would expect.

Clearly, this example is a limit case, but it shows that the effect of correlation risk may be counterintuitive.

Part 3

Risk measures

In mean–variance portfolio optimization we use standard deviation as a way to measure risk. However, a significant inconvenience is that this is a symmetric risk measure. A commonly used alternative is value-at-risk, which is the $1-\alpha$ quantile of the distribution of loss L_H on time horizon of length H:

$$\mathsf{P}\{L_H \leq \mathsf{V@R}_{1-\alpha,H}\} = 1 - \alpha. \tag{3}$$

This definition makes sense for a continuous random variable. Some care is needed in the case of a discrete distribution.

V@R is an asymmetric risk measure and it may be interpreted as the capital requirement to make up for possible losses.

However, it suffers from some limitations:

- It does not say anything about losses on the nasty tail (see stockout probability vs. fill rate in SCM).
- It features diversification anomalies, as diversification may increase risk, when measured by V@R.
- It is not a convex risk measure (what happens in the case of two assets when uncertainty is modeled by a discrete set of scenarios?).

A possibly better risk measure is conditional value-at-risk (CV@R), defined as the tail expectation

$$\mathsf{CV@R}_{1-\alpha,H} \doteq \mathsf{E}\big[L_H \mid L_H > \mathsf{V@R}_{1-\alpha,H}\big].$$

Generally speaking, we need a risk measure that features some coherence properties, can be reliably estimated, and lends itself to optimization.

Alternative risk functionals are the probability of nasty events (leading to chance constraints). Generally speaking, risk functionals may be used in defining objectives (e.g., mean–risk models) or constraints.

A risk measure is an *objective* assessment of risk. Sometimes we want to measure the *subjective* aversion (or perception) of risk. To this aim, we may introduce utility functions.

A utility function $u(x)$ may be used to define an expected utility $U(X) \doteq \mathsf{E}[u(X)]$, for a random prospect X.

The idea is that a concave utility function models risk aversion, as a certain outcome is preferred to an uncertain prospect with the same expected value. This may account for objective risk, or for the subjective assessment of risk.

Expected utility induces a complete ordering of random alternatives. However, eliciting a utility function is difficult, and the theory leads to some paradoxes (and it is heavily criticized by behavioral finance theory).

An alternative is based on stochastic dominance, which only provides us with a partial ordering of random prospects.

Further complications arise if we consider *dynamic* risk measures.

Part 4

Decision models and strategies for risk management

Risk **management** is not only about stochastic modeling o measuring risk. Probability and statistics are fundamental tools but we have to *make* decisions, and this is an **Operations Research** problem.

Many tools can be used to build a decision model or a simple policy based on decision rules, depending on the specific problem. The list includes:

- Subjective preference modeling: utility functions, stochastic dominance.
- Mean-risk models (based on objective risk measures).
- Stochastic and robust optimization.
- Real options.
- Immunization models (first and second-order).

Monte Carlo simulation may play a key role.

When facing a risky situations, two basic strategies are:

- Risk avoidance. For instance, in a retail credit application (loan or mortgage) we may build a classifier (SVM, decision tree, logistic regression, etc.) to decide whether a request should be accepted or not.

  Note: issues with explainable AI.

- Risk mitigation, which may be accomplished by a mix of the following:
  - Diversification: build a diversified portfolio of financial assets or products/services; risk pooling in insurance and SCM (by suitable product design strategies).
  - Hedging: use forward/futures contracts to offset forex, or energy, or commodity price risk; safety stocks in logistics; system redundancy.
  - Insurance and risk transfer: insurance against death, disability and longevity risk; credit risk derivatives; risk sharing contracts in SCM; reinsurance (e.g., cat bonds).

Note that boundaries are not sharp and in the literature the terminology may be ambiguous. In principle, hedging by linear contracts (forward) eliminates risk completely in finance, whereas non linear options maintain some upside potential. This may be considered as an insurance (see the protective put strategy).

Risk mitigation is a mix of the the three building blocks. We will use the term insurance when risk transfer is involved.

A hedging policy may use linear and nonlinear contracts (see the AIFS case).

In principle, hedging is perfect when the underlying asset and the maturity of a forward contract match our exposure.

Say that we hold N units of an asset, with current spot price S_0, at time instant $t = 0$. If we plan to sell the asset at time instant $t = T$, we are exposed to risk due to the uncertaintinity in the future spot price S_T.

If we sell N forward contracts (short position) with price $F_0 \equiv F(0, T)$ and maturity T, the cash flow will be (let us assume cash settlement):

$$\underbrace{N \cdot S_T}_{\text{spot sale}} + \underbrace{N \cdot (F_0 - S_T)}_{\text{payoff fwd}} = N \cdot F_0, \tag{4}$$

which is equivalent to selling the underlying asset at F_0 to the holder of the long position.

If we use standard deviation (or variance) as a risk measure, we can claim that risk has been hedged away. However, we also have wiped out any upside potential. To retain some upside potential, we may resort to hedging with N put options, with strike K. Then, the cash flow at maturity will be

$$N \cdot [S_T + \max\{K - S_T, 0\}] = N \cdot \max\{K, S_T\}$$

This looks bright, but the upside potential is reduced by the option premium.

In the case (4) we have a perfect hedge, but in real life we may have to use standardized and liquid instruments like traded futures, rather than over-the-counter OTC forwards.

Let us ignore some further complications of futures (like marking-to-market, daily settlement, and margin calls). We only consider misalignments in the underlying asset and the maturity of the contract.

We apply a cross-hedge strategy, with contracts written on a risk factor which is correlated with our exposure. Furthermore, the contract maturity is T_2, whereas we have to close our position in the asset at time $T_1 < T_2$.

We have to find a hedging ratio h, i.e., the number of contracts per unit of the asset, in order to minimize variance (which we assume as a risk measure).

The cash flow at time T_1 is

$$C_1 = N \cdot [S_{T_1} + h \cdot (F_0 - F_{T_1})] \,. \tag{5}$$

A possible approach is to minimize its variance. Since S_0 is given, we may equivalently define the random variations $\delta S \doteq S_{T_1} - S_0$ and $\delta F \doteq F_{T_1} - F_0$, and minimize the variance $\delta S - h \cdot \delta F$:

$$\min_h \quad \mathrm{Var}[\delta S - h \cdot \delta F] = \mathrm{Var}(\delta S) + h^2 \mathrm{Var}(\delta F) - 2h\mathrm{Cov}(\delta S, \delta F).$$

The first-order optimality condition (note that the problem is convex) is

$$2h\mathrm{Var}(\delta F) - 2\mathrm{Cov}(\delta S, \delta F) = 0.$$

Therefore, we find

$$h^* = \frac{\mathrm{Cov}(\delta S, \delta F)}{\mathrm{Var}(\delta F)} = \rho \cdot \frac{\sigma_S}{\sigma_F}, \tag{6}$$

where σ_F is the standard deviation (volatility) of δF, σ_S the standard deviation of δS, and ρ is the correlation coefficient.

Note the clear link with linear regression. There, we want to minimize unexplained variability; here, we want to minimize residual risk.

In practice, we have to deal with additional complications like:

- Standardized volumes of traded contracts.
- Volume risk.
- Estimation risk.
- Impact of marking to market (see the Metallgesellschaft debacle).
- The limits of symmetric risk measures like variance or volatility.

A rather traditional method to manage nonlinear risk (in fixed income portfolios and asset–liability management) is first-order immunization, which is based on Taylor expansions.

Assume that the value V of a financial portfolio depends on m risk factors R_i, $i = 1, \ldots, m$ (they could be interest rates with different maturities, within a term structure of interest rates). If the risk factors undergo a shock δR_i, there will be a corresponding impact (profit or loss; P&L) on the portfolio value. Let us denote by δV the variation in the portfolio value.

We may apply a first-order approximation of the variation:

$$\delta V \approx \sum_{i=1}^{m} \frac{\partial V}{\partial R_i} \cdot \delta R_i.$$

To compensate for this variation, we may include m hedging instruments in the portfolio. If the risk factors are interest rates, we could use bond options, interest rate futures, interest rate swaps, etc.

The value of H_j, $j = 1, \ldots, m$, of the hedging instruments must depend on the same risk factors to which V is exposed. Note that the number of hedging instruments matches the number of risk factors. Now we have to decide how many instruments to include (let us disregard their cost; swaps have zero value at their inception, but options are different).

We may apply the same first-order approximation to the hedging instruments:

$$\delta H_j \approx \sum_{i=1}^{m} \frac{\partial H_j}{\partial R_i} \cdot \delta R_i, \qquad j = 1, \ldots, m.$$

If we include ϕ_j units of each instruments in the portfolio, we obtain a hedged portfolio with value:

$$V^H = V + \sum_{j=1}^{m} \phi_j H_j.$$

Given the shocks on the risk factors, the variation in the hedged portfolio value is approximated as follows:

$$\begin{aligned}
\delta V^H = \delta V + \sum_{j=1}^{m} \phi_j \cdot \delta H_j &\approx \sum_{i=1}^{m} \frac{\partial V}{\partial R_i} \cdot \delta R_i + \sum_{j=1}^{m} \left(\phi_j \sum_{i=1}^{m} \frac{\partial H_j}{\partial R_i} \cdot \delta R_i \right) \\
&= \sum_{i=1}^{m} \left(\frac{\partial V}{\partial R_i} + \sum_{j=1}^{m} \phi_j \frac{\partial H_j}{\partial R_i} \right) \cdot \delta R_i. \qquad (7)
\end{aligned}$$

Note that we need to obtain the sensitivities in some way, possibly through a pricing model.

In order to set $\delta V^H = 0$ for (small) shocks δR_i, we have to find hedging positions such that the risk exposures in Eq. (7) are zero.

$$\frac{\partial V}{\partial R_i} + \sum_{j=1}^{m} \phi_j \frac{\partial H_j}{\partial R_i} = 0, \qquad i = 1, \ldots, m.$$

Since we have matched the number of hedging instruments and the number of risk factors, this is just a system of m linear equations in the m unknown variables ϕ_j.

This simple approach suffers from definite limitations:

- We have to dynamically rebalance the portfolio, since first-order sensitivities depend on the current value of risk factors.

- We may somewhat improve the hedge by considering second-order sensitivities (bond convexities, option gammas). However, we are still perfectly hedged against small shocks. It may be preferable to be approximately hedged against large shocks.

- The evaluation of sensitivities relies on a model, so we are introducing model risk. If we estimate sensitivities statistically, we introduce estimation risk.