Poisson processes

Definition

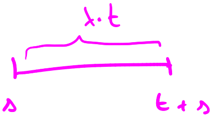
A *Poisson process with rate* λ is a counting process $(N(s))_{s\in[0,\infty)}$ with $N(0) = 0$, whose inter-arrival times are i.i.d. exponential random variables with rate $\boxed{\lambda.}$

Poisson Process

Theorem

$(N(s))_{s\in[0,\infty)}$ is a Poisson process with rate λ if and only if it is a counting process such that

1. $N(0) = 0$,
2. *it has independent increments;*
3. $N(t+s) - N(s) \sim \mathit{Poisson}(\lambda t)$.



Poisson Process

Theorem

$(N(s))_{s\in[0,\infty)}$ is a Poisson process with rate λ if and only if it is a counting process such that:

(i) $N(0) = 0$;

(ii) *it has independent increment;*

(iii) *it has stationary increments;*

(iv) $\lim_{h\to 0} \frac{P(N(h)=1)}{h} = \lambda$ *and* $\lim_{h\to 0} \frac{P(N(h)\geq 2)}{h} = 0$.

The non-homogeneous Poisson Process

A homogeneous Poisson process is often unrealistic because intensities may vary with time.

Definition

We say that $(M(s))_{s\in[0,\infty)}$ is a Poisson process with rate function, or *intensity function*, or *propensity function* $\underline{\underline{\lambda(u)}}$ if

1. $M(0) = 0$.
2. $M(t)$ has independent increments, and
3. $M(t) - M(s)$ is Poisson with mean $m(t) - m(s) = \int_s^t \lambda(u)du$.

$t > s$

NOT STATIONARY

Note that it is not a Poisson process in the strict sense of the definition! As a matter of fact, here inter-arrival times are not exponentially distributed.

s t

$M(t) - M(s) \sim \mathrm{Pois}\left(\int_s^t \lambda(u)du\right)$

$\lambda(u) = \lambda$ $\quad N(t) - N(s) \sim \mathrm{Pois}\left(\overbrace{\int_s^t \lambda(u)du}^{\lambda\cdot(t-s)}\right)$

The non-homogeneous Poisson Process

Theorem (Rescaling the time of a unit rate Poisson process)

Let $N(t)$ be Poisson process with rate 1, and let $\lambda(t)$ be a nonnegative function of the time. Then, the process

$$M(t) = N\left(\boxed{\int_0^t \lambda(u)du}\right)$$

is a (non-homogeneous) Poisson process with rate $\lambda(t)$.

In the special case in which $\lambda(s) = \lambda$ constant, we have that

$$M(t) = N(\lambda t)$$

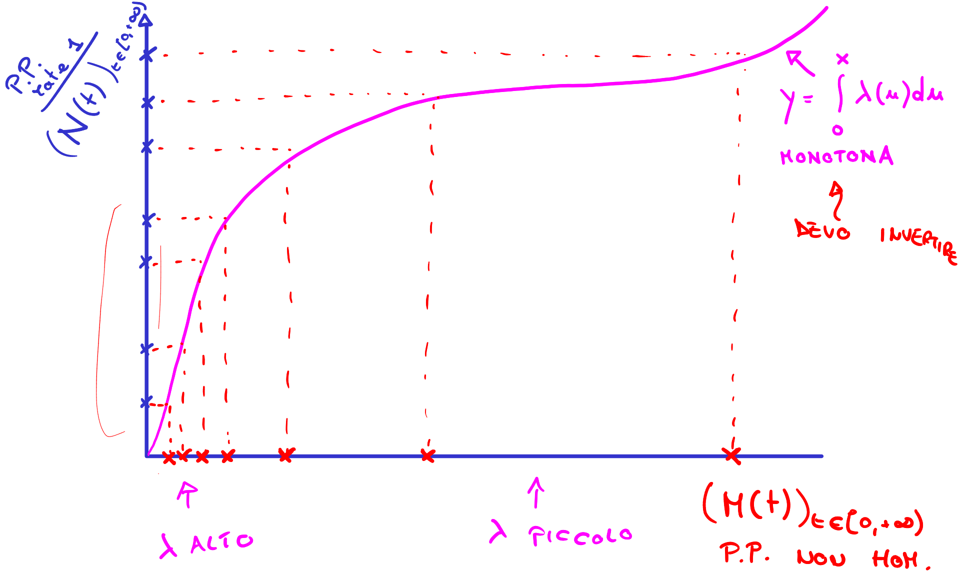
$\tilde{t}_1$ tempo 1° arrivo di M

is a Poisson process with rate λ.

$t_1 = \int_0^{\tilde{t}_1} \lambda(u)du$

t_1 tempo 1° arrivo di N

The non-homogeneous Poisson Process



Proof

We check that the definition of (non-homogeneous) Poisson process is met.

i) $M(0) = N\left(\int_0^0 \lambda(u)du\right) = N(0) = 0.$

ii) Let $t_1 < t_2 < \cdots < t_n$. For simplicity, for any $1 \le i \le n$ let

$$\tilde{t}_i = \int_0^{t_i} \lambda(u)du. = m(t_i) \quad \tilde{t}_1 \le \tilde{t}_2 \le \ldots \le \tilde{t}_n$$

Then, for any $1 \le i \le n-1$, $M(t_{i+1}) - M(t_i) = N(\tilde{t}_{i+1}) - N(\tilde{t}_i)$. Hence, $M(t_{i+1}) - M(t_i)$ are independent because so are $N(\tilde{t}_{i+1}) - N(\tilde{t}_i)$.

iii) For any $s < t$ we have

$$\begin{aligned} M(t) - M(s) &= N\left(\int_0^t \lambda(u)du\right) - N\left(\int_0^s \lambda(u)du\right) \\ &\sim \text{Pois}\left(\int_0^t \lambda(u)du - \int_0^s \lambda(u)du\right) \sim \text{Pois}\left(\int_s^t \lambda(u)du\right) \end{aligned}$$

Example: frogs 1

Suppose that the arrival times of frogs to a pond can be reasonably modeled by a Poisson process. Frogs are arriving at a rate of 2 per hour. What is the probability that at most one frog will arrive in the next three hours? And between 9 and 12?

$N(t) = \#\text{ARRIVALS IN } [0, t]$ $\qquad \{N(t), t \in [0, +\infty)\}$ P.P. RATE 2/h

$$P(\underbrace{N(3)}_{\sim \text{Pois}(3 \cdot 2)} \leq 1) = e^{-6} \cdot \frac{6^0}{0!} + e^{-6} \cdot \frac{6^1}{1!} = 7 \cdot e^{-6}$$

$$P(N(12) - N(9) \leq 1) = P(N(3) \leq 1) = 7 \cdot e^{-6}$$

INCREMENTI STAZIONARI

Example: frogs 2

It does not seem plausible that the frog arrivals are uniformly distributed along the day. Instead, suppose the rate of arrival should fluctuate as $\lambda(t) = 2 + \sin(\frac{t\pi}{12})$, where $t = 0$ is taken to be 8AM and the unit of time is one hour. Assuming it is 8AM now, what is the probability that at most one frog will arrive in the next three hours? And between 9 and 12?

$M(t) = \#$ ARRIVALS IN $[0, t]$ P.P. N. HOM.

$$P(\underbrace{M(3)}_{\sim \text{Pois}\left(\underbrace{\boxed{\int_0^3 \lambda(u)\,du}}_{m(3)}\right)} \le 1) = e^{-m(3)} \cdot \frac{m(3)^0}{0!} + e^{-m(3)} \cdot \frac{m(3)^1}{1!}$$

$$P(\underbrace{M(12) - M(9)}_{\sim \text{Pois}\left(\boxed{\int_9^{12} \lambda(u)\,du}\right)} \le 1) \not= P(M(3) \le 1)$$

Example: frogs 2

It does not seem plausible that the frog arrivals are uniformly distributed along the day. Instead, suppose the rate of arrival should fluctuate as $\lambda(t) = 2 + \sin(\frac{t\pi}{12})$, where $t = 0$ is taken to be 8AM and the unit of time is one hour. Assuming it is 8AM now, what is the probability that at most one frog will arrive in the next three hours? And between 9 and 12?

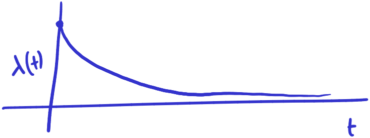
Solution. We let $\{M(t), t \in [0, \infty)\}$ be a Poisson process with intensity $\lambda(t) = 2 + \sin(\frac{t\pi}{12})$. Then, $m(t) = \int_0^t \lambda(u)du = \boxed{2t + \frac{12}{\pi}\left[1 - \cos(\frac{t\pi}{12})\right]}$ and

$$P(N(3) \leq 1) = e^{-(m(3)-m(0))}\frac{(m(3)-m(0))^0}{0!} + e^{-(m(3)-m(0))}\frac{(m(3)-m(0))^1}{1!}$$
$$\approx 0.0066.$$

The incremements are not stationary!

$$P(N(12) - N(9) \leq 1) = e^{-(m(12)-m(9))}\frac{(m(12)-m(9))^0}{0!}$$
$$+ e^{-(m(12)-m(9))}\frac{(m(12)-m(9))^1}{1!}$$
$$\approx 0.0463.$$

Simulation of non-homogeneous Poisson process



Let $\{M(t)\}_{t\in[0,+\infty)}$ be a non-homogeneous Poisson process with rate function $\lambda(t) = 50/(t+1)$ per minute. How can we simulate the first 150 minutes of the process?

ARRIVI P.P. RATE 1

$$\hookrightarrow \quad T_i = \int_0^{\tilde{T}_i} \lambda(u)\,du$$

$\tilde{T}_i$ ARRIVI DI M

$$t \longmapsto \int_0^t \lambda(u)\,du = \int_0^t \frac{50}{u+1}\,du = 50 \cdot \log(t+1)$$

$$t = 50 \cdot \log(\tilde{t}+1) \rightsquigarrow \tilde{t}+1 = e^{t/50}$$

$$\tilde{t} = e^{t/50} - 1$$

Let $\{M(t)\}_{t\in[0,+\infty)}$ be a non-homogeneous Poisson process with rate function $\lambda(t) = 2 + \sin(\frac{t\pi}{12})$ per minute. How can we simulate the first 150 minutes of the process?

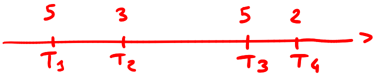
$$m(t) = \int_0^t \left(2 + \sin\left(\frac{u\pi}{12}\right)\right) du = 2t + \frac{12}{\pi}\left[1 - \cos\left(\frac{t\pi}{12}\right)\right]$$

$$t = 2\tilde{t} + \frac{12}{\pi}\left[1 - \cos\left(\frac{\tilde{t}\pi}{12}\right)\right]$$

COME LA INVERTO ???

Thinning

$Y_1 = 5 \quad Y_2 = 3 \quad Y_3 = 5 \quad Y_4 = 2$



An arrival process, $(N(s))_{s\in[0,\infty)}$ is Poisson with rate λ. Assume that the arrivals can be of k different types, specified by a sequence of iid random variables $\{Y_i\}_{i=1}^{\infty}$, taking values in $\{1,2,3,\dots,k\}$, with probability mass function $P(Y_i = j) = p_j$. Let these random variables be **independent of** $(N(s))_{s\in[0,\infty)}$. Let $N_j(t)$ be the arrivals before time t that are of type j:

$$N_j(t_2) - N_j(t_1) = \sum_{i=N(t_1)+1}^{N(t_2)} \mathbb{1}\{Y_i = j\}$$

$t_1 < t_2 < t_3 < t_4$

IND.

$$N_j(t_4) - N_j(t_3) = \sum_{i=N(t_3)+1}^{N(t_4)} \mathbb{1}\{Y_i = j\}$$

$$\boxed{N_j(t) = \sum_{i=1}^{N(t)} 1_{\{Y_i=j\}}.} = \#\text{ ARRIVALS TYPE } j \text{ IN } [0,t]$$

Theorem

$\{(N_j(t))_{t\in[0,\infty)}\}_j$ are independent Poisson processes with respective rates λp_j.

Thinning, a sketch of the proof

$N_j(0) = 0$ for every j, since $N(0) = 0$. The independence of the increments of each of the N_j follows from that of N and from the independence of each of the Y_i from the others and from the Poisson itself. Let us now consider the case $k = 2$ (events are of two different kind only) and calculate

PER IND. DOVREI CONSIDERARE $N_1(t_1)-N_1(s_1) \perp\!\!\!\perp N_2(t_2)-N_2(s_2)$

$$P(N_1(t)-N_1(s))\cdot P(N_2(t)-N_2(t)) \overset{\text{GOAL}}{=} P[N_1(t) - N_1(s) = n, N_2(t) - N_2(s) = m]$$

For such event to occur, we need that $N(t) - N(s) = m + n$. Morover, of the $m + n$ event, n are of the first kind, that happens with a probability of

$$\begin{aligned}
P[N(t) - N(s) = m+n, N_1(t) - N_1(s) = n] \\
&= e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{m+n}}{(m+n)!} \binom{m+n}{n} p^n (1-p)^m = \\
&= \underbrace{e^{-\lambda p(t-s)} \frac{(\lambda p(t-s))^n}{n!}}_{P(\mathrm{Pois}(\lambda p(t-s))=n)} \underbrace{e^{-\lambda(1-p)(t-s)} \frac{(\lambda(1-p)(t-s))^m}{m!}}_{P(\mathrm{Pois}(\lambda(1-p)(t-s))=m)}
\end{aligned}$$

$N_1(t) = \#$ COUPON TYPE 1 IN $[0,t]$

$N_2(t) = \#$ COUPON TYPE 2 IN $[0,t]$

THINNING

P. POISSON

$\boxed{\text{INDIPENDENTI}}$!!!

CON PAR. 50

Assume people arrive at a shop according to a Poisson process with rate 100 per day, and are given coupons independently of each other: there are two kinds of coupons and each is given with probability $1/2$. Knowing that at the end of the day 1000 coupons of type 1 are given, how many coupons of type 2 are expected to be given?

$$E[N_2(1) \mid \cancel{N_1(1) = 1000}]$$

$$= E[N_2(1)] = 50$$

Non-homogeneous thinning

T_i TENUTO CON PROB. $p(T_i)$

Thinning can be used to derive a non-homogeneous Poisson process from a homogeneous one:

Theorem

Suppose that in a Poisson process with rate λ, we keep an arrival that takes place at time $\boxed{s}$ with time-dependent probability $\boxed{p(s)}$, independently on the other arrivals. Define

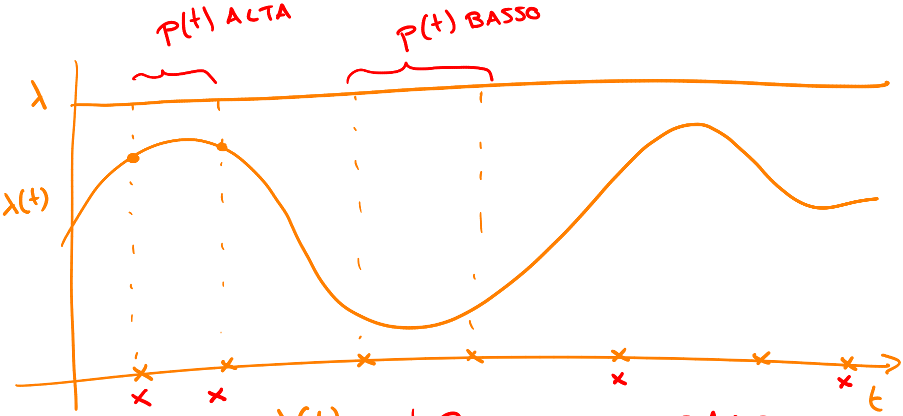
$$M(t) = \#\text{arrivals kept by time } t.$$

Then $(M(t))_{t\in[0,\infty)}$ is a non-homogeneous Poisson process with rate $\boxed{\lambda p(s).}$

You can take $\underbrace{\lambda = max\{\lambda(s)\}}$ and $\underbrace{p(s) = \frac{\lambda(s)}{\lambda}}$ and get a Poisson $\lambda(s)$ $= \lambda \cdot \underbrace{\frac{\lambda(s)}{\lambda}}_{p(s)}$

NON HOM.

Non-homogeneous thinning



$$p(t) = \frac{\lambda(t)}{\lambda}$$

PRO: NON CALCOLATE INTEGRALI E INVERSE

CONTRO: POTREI SPRECARE TEMPO COMPUTAZIONALE A SIMULARE TEMPI ARRIVO RIFIUTATI

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