Cardan Polynomials and the Reduction of Radicals

Article *in* Mathematics Magazine · February 2001

DOI: 10.1080/0025570X.2001.11953029

1 author:

Thomas Osler
Rowan University

Cardan Polynomials and the Reduction of Radicals

THOMAS J. OSLER
Rowan University
Glassboro, NJ 08028

1. Introduction

Expressions of the form $\sqrt[n]{a+\sqrt{b}}$ occur frequently in the literature. To begin with, Cardan's solution of the cubic equation

$$x^3 - 3cx = 2a \tag{1}$$

is given by the sum of two radicals

$$x = \sqrt[3]{a+\sqrt{a^2-c^3}} + \sqrt[3]{a-\sqrt{a^2-c^3}}. \tag{2}$$

The author was teaching a course in the history of mathematics and designing simple problems of this type for solution by his students when he encountered an awkward algebraic result. With $c = -1$ and $a = 2$, equation 1 becomes

$$x^3 + 3x = 4$$

which has the solution $x = 1$. However, the solution given by (2) is

$$x = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}. \tag{3}$$

A quick check with a calculator shows that this expression is really 1, but how can one algebraically manipulate it into that result? All attempts to do so led the author back to the original cubic from which it started. (The reader might try now to reduce (3) to the value 1.) That such a combination of square and cube roots should produce a rational number, let alone an integer, is startling. Later we will see that the individual radicals are related to the golden section and its reciprocal:

$$\sqrt[3]{2\pm\sqrt{5}} = \frac{1\pm\sqrt{5}}{2}. \tag{4}$$

This last relation is especially interesting because it shows that these cube roots are constructible with straightedge and compass. In general, we can construct any number defined by an expression composed of a finite number of additions, subtractions, multiplications, divisions and square roots of rational numbers. It is known that the number $\sqrt[3]{2}$ is not constructible. Yet (4) shows that $\sqrt[3]{2+\sqrt{5}}$, which looks more complicated than $\sqrt[3]{2}$, is constructible because it can be expressed by the golden section. This is indeed an unexpected result. It is surprises of this type that make it a joy to be a mathematician.

In general, expressions like $\sqrt[n]{a\pm\sqrt{b}}$ with rational a and b are not constructible and their sum

$$x = \sqrt[n]{a+\sqrt{b}} + \sqrt[n]{a-\sqrt{b}}$$

is not a rational number. However for special rational values of a and b the individual radicals are constructible and their sum is rational. We will investigate how to determine these values.

Following are several interesting radicals with simple values:

$$\sqrt[3]{2 \pm \sqrt{5}} = \frac{1 \pm \sqrt{5}}{2}, \quad \sqrt[6]{9 \pm 4\sqrt{5}} = \frac{1 \pm \sqrt{5}}{2}, \quad \sqrt[9]{38 \pm 17\sqrt{5}} = \frac{1 \pm \sqrt{5}}{2},$$

$$\sqrt[3]{7 \pm 5\sqrt{2}} = 1 \pm \sqrt{2}, \quad \sqrt[5]{41 \pm 29\sqrt{2}} = 1 \pm \sqrt{2}, \quad \sqrt[6]{99 \pm 70\sqrt{2}} = 1 \pm \sqrt{2},$$

$$\sqrt[3]{-5 \pm i\sqrt{2}} = 1 \pm i\sqrt{2}, \quad \sqrt[5]{1 \pm 11i\sqrt{2}} = 1 \pm i\sqrt{2}, \quad \sqrt[9]{17 \pm 56i\sqrt{2}} = 1 \pm i\sqrt{2}.$$

In each case, the sum of pairs of conjugate radicals is an integer because the radicals cancel. For example $\sqrt[5]{41 + 29\sqrt{2}} + \sqrt[5]{41 - 29\sqrt{2}} = 2$. (Many radical expressions, some like those mentioned here, appear in the work of Ramanujan. Our method for simplifying them has been referred to in [**2**] and [**3**].)

Our main tools for examining a given radical expression for possible simplification will be what we will call the *Cardan polynomials* which we denote by $C_n(c, x)$. (We use the name "Cardan polynomials" because the expression for their roots closely resembles Cardan's solution of the cubic equation.) The Cardan polynomials are related to the familiar Chebyshev polynomials by

$$C_n(c, x) = 2c^{n/2} T_n\left(\frac{x}{2\sqrt{c}}\right).$$

Cardan polynomials seem to have first appeared in a note by J. E. Woko [**12**] in a different context. The use of Cardan polynomials in simplifying radicals may be new. For other approaches to the simplification of radicals see [**6**], [**8**] and [**9**]. (In [**2**], Chebyshev polynomials are used to reduce a radical expression.)

A note on radical values: A radical expression of the form $\sqrt[n]{a + \sqrt{b}}$ has, in general, n complex values. When such radicals appear in an equation, it may be that the equation is true for only one of the n possible values. When two such radicals appear in a single equation, then appropriate pairs of values must be selected. In specific cases, the meaning is usually clear in context. For instance, in (3), we take the real cube roots.

2. Cardan polynomials

Let us start with the expression

$$x = \sqrt[n]{a + \sqrt{b}} + \sqrt[n]{a - \sqrt{b}} \tag{5}$$

We will define c as

$$c = \sqrt[n]{a + \sqrt{b}}\,\sqrt[n]{a - \sqrt{b}} = \sqrt[n]{a^2 - b}, \tag{6}$$

so $c^n = a^2 - b$.

If we raise both sides of (5) to the power n, and make appropriate substitutions, we will generate a polynomial equation in x, of degree n, with integer coefficients. We write the equation in the form $C_n(c, x) = 2a$; this defines the *Cardan polynomial of* *degree n*. In the case $n = 3$, for instance, cubing (5) gives

$$\begin{aligned}
x^3 &= \left(\sqrt[3]{a+\sqrt{b}} + \sqrt[3]{a-\sqrt{b}}\right)^3 \\
&= \left(\sqrt[3]{a+\sqrt{b}}\right)^3 + 3\left(\sqrt[3]{a+\sqrt{b}}\right)^2\left(\sqrt[3]{a-\sqrt{b}}\right) \\
&\quad + 3\left(\sqrt[3]{a+\sqrt{b}}\right)\left(\sqrt[3]{a-\sqrt{b}}\right)^2 + \left(\sqrt[3]{a-\sqrt{b}}\right)^3 \\
&= \left(\sqrt[3]{a+\sqrt{b}}\right)^3 + 3\sqrt[3]{a+\sqrt{b}}\sqrt[3]{a-\sqrt{b}}\left(\sqrt[3]{a+\sqrt{b}} + \sqrt[3]{a-\sqrt{b}}\right) \\
&\quad + \left(\sqrt[3]{a-\sqrt{b}}\right)^3.
\end{aligned}$$

Using (5) and (6) to simplify the middle summand we get

$$x^3 = a + \sqrt{b} + 3cx + a - \sqrt{b} = 3cx + 2a.$$

We now have $x^3 - 3cx = 2a$. If we write $C_3(c, x) = x^3 - 3cx$, then $C_3(c, x) = 2a$. The reader can continue in this way and derive algebraically the Cardan polynomials $C_n(c, x)$ of higher degree, and find that

$$C_n(c, x) = 2a. \tag{7}$$

Another approach to our Cardan polynomials is simpler, and reveals their connection to the Chebyshev polynomials. Writing

$$\sqrt[n]{a \pm \sqrt{b}} = r\exp(\pm i\theta) \tag{8}$$

(where r and θ need not be real) gives

$$a \pm \sqrt{b} = r^n \exp(\pm in\theta). \tag{9}$$

Using (8) to rewrite (5) we get

$$x = \sqrt[n]{a+\sqrt{b}} + \sqrt[n]{a-\sqrt{b}} = re^{i\theta} + re^{-i\theta} = 2r\cos\theta.$$

From (9) we have $r^n e^{in\theta} + r^n e^{-in\theta} = 2a$, so

$$\cos n\theta = \frac{a}{r^n}. \tag{10}$$

The Chebyshev polynomials $T_n(\cos\theta)$ are defined by $T_n(\cos\theta) = \cos n\theta$. Now (10) gives

$$T_n(x/2r) = \frac{a}{r^n} \tag{11}$$

Comparing (11) with (7) we have

$$C_n(c, x) = 2r^n T_n(x/2r). \tag{12}$$

To determine r, use (6) and (8) to get

$$c = \sqrt[n]{a+\sqrt{b}}\sqrt[n]{a-\sqrt{b}} = re^{i\theta}re^{-i\theta} = r^2.$$

Now (12) becomes

$$C_n(c,x) = 2c^{n/2}T_n\left(\frac{x}{2\sqrt{c}}\right). \tag{13}$$

We see that the exact nature of the n^{th} Cardan polynomial can be determined from the n^{th} Chebyshev polynomial.

3. Constructing a table of Cardan polynomials

Starting with a table of the familiar Chebyshev polynomials $T_n(x)$ and using (13) we can tabulate the $C_n(c,x)$:

TABLE 1: Cardan polynomials

$C_1(c,x)$	$=$	x
$C_2(c,x)$	$=$	x^2-2c
$C_3(c,x)$	$=$	x^3-3cx
$C_4(c,x)$	$=$	$x^4-4cx^2+2c^2$
$C_5(c,x)$	$=$	$x^5-5cx^3+5c^2x$
$C_6(c,x)$	$=$	$x^6-6cx^4+9c^2x^2-2c^3$
$C_7(c,x)$	$=$	$x^7-7cx^5+14c^2x^3-7c^3x$
$C_8(c,x)$	$=$	$x^8-8cx^6+20c^2x^4-16c^3x^2+2c^4$
$C_9(c,x)$	$=$	$x^9-9cx^7+27c^2x^5-30c^3x^3+9c^4x$
$C_{10}(c,x)$	$=$	$x^{10}-10cx^8+35c^2x^6-50c^3x^4+25c^4x^2-2c^5$
$C_{11}(c,x)$	$=$	$x^{11}-11cx^9+44c^2x^7-77c^3x^5+55c^4x^3-11c^5x$

If we denote the coefficients by $d(n,k)$, where

$$C_n(c,x) = x^n - d(n,1)cx^{n-2} + d(n,2)c^2x^{n-4} - d(n,3)c^3x^{n-6} + \cdots,$$

the table above suggests that

$$d(n,k) = d(n-1,k) + d(n-2,k-1). \tag{14}$$

This means that any coefficient in the above table is the sum of the one directly above it and the one two rows above and to the left. Using this simple recursion relation, we can extend the table indefinitely. This is simpler than transforming the Chebyshev polynomials with (13).

Relation (14) follows from $C_n(c,x) = xC_{n-1}(c,x) - cC_{n-2}(c,x)$, which in turn follows from the known relation [**10**] $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$.

4. The quadratic connection

Another important relation is

$$\sqrt[n]{a \pm \sqrt{b}} = \frac{x \pm \sqrt{x^2 - 4c}}{2}, \tag{15}$$

where, as before, $x = \sqrt[n]{a + \sqrt{b}} + \sqrt[n]{a - \sqrt{b}}$ and $c = \sqrt[n]{a + \sqrt{b}}\sqrt[n]{a - \sqrt{b}} = \sqrt[n]{a^2 - b}$. To derive (15), we start with

$$y = \sqrt[n]{a + \sqrt{b}}. \tag{16}$$

Multiply (16) by $\sqrt[n]{a - \sqrt{b}}$ and use (6) to get

$$y\sqrt[n]{a - \sqrt{b}} = c. \tag{17}$$

It follows from (16) and (17) that $x = \sqrt[n]{a + \sqrt{b}} + \sqrt[n]{a - \sqrt{b}} = y + c/y$, which implies the quadratic equation $y^2 - xy + c = 0$, with the solutions

$$y = \frac{x \pm \sqrt{x^2 - 4c}}{2} \tag{18}$$

Comparing (16) and (18) we see that (15) is true.

5. A technique for resolving radicals

To decide whether a radical $\sqrt[n]{a \pm \sqrt{b}}$ can be reduced to some nice form, we first solve

$$b = a^2 - c^n \tag{19}$$

for c. If c is an integer or rational number, there is hope that the radical can be simplified, however, even when this is not the case do not give up. Next, examine the equation

$$C_n(c, x) = 2a \tag{20}$$

using the table of Cardan polynomials, and search for an integer or rational root x, where

$$\sqrt[n]{a + \sqrt{b}} + \sqrt[n]{a - \sqrt{b}} = x. \tag{21}$$

If such a root is found, then our radical can be written as

$$\sqrt[n]{a \pm \sqrt{b}} = \frac{x \pm \sqrt{x^2 - 4c}}{2}. \tag{22}$$

If the above procedure fails to simplify the radical then try the same steps with n replaced by any divisor d of n, where $n = d\,e$. In this case, we examine $\sqrt[e]{\sqrt[d]{a \pm \sqrt{b}}}$

for simplification. Start with the largest divisor $d < n$ and continue trying successively smaller divisors until all divisors are exhausted or a nice simplified expression emerges. We illustrate with several examples.

Example 1: A calculator shows that $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} = 1.000000000$ out to nine zeroes. Show that the expression is exactly 1.

Solution: Comparing our expression with (21) we see that we need $n = 3$, $a = 2$, and $b = 5$. Using (19) we get $c = -1$. From the table given in Section 3 and (20) we get $C_3(c, x) = x^3 - 3cx = 2a$, which reduces to $x^3 + 3x = 4$, of which $x = 1$ is an exact root. Notice also that (22) yields $\sqrt[3]{2 \pm \sqrt{5}} = \frac{1\pm\sqrt{5}}{2}$, from which it follows that $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} = 1$. (This problem is also considered in [**11**].)

Example 2: Show that $\sqrt[7]{\frac{29}{2} + \frac{1}{2}\sqrt{845}} + \sqrt[7]{\frac{29}{2} - \frac{1}{2}\sqrt{845}} = 1$.

Solution: In this case $n = 7$, $a = 29/2$, $b = 845/4$, and $c = -1$. Relation (20) becomes now $C_7(-1, x) = 29$. From the table in section 3 we get $x^7 + 7x^5 + 14x^3 + 7x = 29$. Since $x = 1$ is a root, we are finished. We see, also, that the quadratic relation (22) implies

$$\sqrt[7]{\frac{29}{2} \pm \frac{1}{2}\sqrt{845}} = \frac{1 \pm \sqrt{5}}{2}.$$

Example 3: (See also [**5**] and [**7**].) For what positive numbers b is $\sqrt[3]{2+\sqrt{b}} + \sqrt[3]{2-\sqrt{b}}$ an integer?

Solution: The desired values of b are 5 and 100/27. To see why, let $a = 2$ and $n = 3$ in (19) to get $c^3 = 4 - b$. Also (20) becomes $x^3 - 3cx - 4 = 0$, and solving for c we get

$$c = \frac{x^3 - 4}{3x}. \tag{23}$$

We can now look at the problem another way: Let x be an integer, and use (23) to find c. Now $b = 4 - c^3$; that this last expression must be positive limits the number of possible solutions. If we try $x = 1$, (23) gives us $c = -1$, so $b = 4 - c^3 = 5$. If we try $x = 2$, then $c = 2/3$ and $b = 100/27$. All other values of x yield values of c that give $b < 0$.

Example 4: The dedication for the paper [**1**] reads: "In memory of Ramanujan on the

$$\left(32\left(\frac{146410001}{48400}\right)^3 - 6\left(\frac{146410001}{48400}\right) + \sqrt{\left(32\left(\frac{146410001}{48400}\right)^3 - 6\left(\frac{146410001}{48400}\right)\right)^2 - 1}\right)^{1/6} \text{th}$$

anniversary of his birth." What is this number?

Solution: We see at once that with $a = 32(\frac{146410001}{48400})^3 - 6(\frac{146410001}{48400})$, and $b = a^2 - 1$, the complicated number above can be written as $y = (a + \sqrt{b})^{1/6}$. Since $b = a^2 - c^6$, we see that $c = 1$ or -1. Also, $146410001 = 110^4 + 1$ and $48400 = 4(110)^2$, so $2a = (\frac{110^4+1}{110^2})^3 - 3(\frac{110^4+1}{110^2})$. We recognize this last expression as $2a = C_3(1, x)$, with

$x = \frac{110^4+1}{110^2}$. Hence, using (22) we get $\sqrt[3]{a+\sqrt{b}} = \frac{x+\sqrt{x^2-4c}}{2} = 110^2$, and our original number is $y = 110$.

Notice that the original problem involved a sixth root and therefore we would expect to look at the sixth Cardan polynomial. The sixth Cardan polynomial would have failed to help. When this occurs, we should use the fact that $\sqrt[mn]{z} = \sqrt[m]{\sqrt[n]{z}}$ to see if Cardan polynomials of lower order can reduce the radical.

Example 5: (This example is motivated by [**12**], where the author approaches the Cardan polynomials from an entirely different direction.) Let α and β be the roots of the quadratic equation $y^2 - (\alpha+\beta)y + \alpha\beta = 0$. Show that $\alpha^n + \beta^n = C_n(\alpha\beta, \alpha+\beta)$. (This is an expansion of the symmetric function of the roots $\alpha^n + \beta^n$, in powers of the coefficients $\alpha\beta$ and $\alpha+\beta$ using Cardan polynomials.)

Solution: Consider the quadratic connection (18) which is derived from $y^2 - xy + c = 0$. If α and β are the two roots of this quadratic in y, then $c = \alpha\beta$ and $x = \alpha+\beta$. From (15) we see that $\alpha = \sqrt[n]{a+\sqrt{b}}$ and $\beta = \sqrt[n]{a-\sqrt{b}}$, so $\alpha^n + \beta^n = 2a$. The result now follows immediately from the fact that $C_n(c, x) = 2a$.

Acknowledgment. The author wishes to thank Richard Askey, Bruce Berndt, Heng Huat Chan, Theodore Rivlin, and Marcus Wright for their help and encouragement. The author also thanks the referee for several important references and other improvements.

REFERENCES

1. B. C. Berndt, H. H. Chan, L.-C. Zhang, Ramanujan's association with radicals in India, *Amer. Math. Monthly* **104** (1997), 905–911.
2. B. C. Berndt, H. H. Chan, and L.-C. Zhang, Radicals and units in Ramanujan's work, *Acta Arith.* **87** (1998), 145–158.
3. B. C. Berndt, Y.-S. Choi, and S.-Y. Kang, Problems submitted by Ramanujan to the Journal of the Indian Mathematical Society. (In *Continued Fractions: From Analytic Number Theory to Constructive Approximation*, B. C. Berndt and F. Gesztesy, eds., Amer. Math. Soc., Providence, RI, pp. 15–56.)
4. G. Cardano, *Ars Magna or the Rules of Algebra*, Dover, New York, NY, 1993.
5. G. H. Hardy, *A Course in Pure Mathematics*, Cambridge Univ. Press, New York, NY, 1952.
6. M. Honsbeek, Denesting certain nested radicals of depth two, to appear in *Journal de Théorie de Nombres de Bordeaux*.
7. D. C. Kay, Problem 1529, this MAGAZINE **70** (1997), 298.
8. S. Landau, How to tangle with a nested radical, *Math. Intelligencer*, **16** (1994), 49–55.
9. S. Landau, Simplification of nested radicals, *SIAM Journal of Computing* **21** (1992), 85–110.
10. T. J. Rivlin, *Chebyshev Polynomials*, John Wiley & Sons, New York, NY, 1990.
11. A. Sofo, Generalization of a radical identity, *Math. Gazette* **83** (1999), 274–276.
12. J. E. Woko, A Pascal-like triangle for $\alpha^n + \beta^n$, *Math. Gazette* **81** (1997), 75–79.

The Editor owes a debt of gratitude to computer consultants Eric V. Level of 47 Software and Scott A. Dick of Sun Vista Technologies for their expert help in designing a database to track articles, authors, and referees for the MAGAZINE.