Automatic differentiation for general relativity's calculations

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1 Introduction :

In the course of the general relativity lectures given during the first year of Master, I couldn't help to think that something was amiss : a brilliant theory, revolutionizing the idea of space and time, of the origin of physics, using a powerful and quite unique approach to resolve problems sometimes even not observed, and yet, we would not calculate values. The subject studied was purely theoretical, and even if it could be understood given the nature of general relativity-we were not going to put a black hole in the center of the class to study it-, a interrogation arose : what could be done to better visualize this theory, to have a better grasp at it ?

I came up to J.Donini, my teacher of general relativity about his issue and about the idea of doing a internship on it. He then proposed to me this internship ; doing a research on general relativity calculation's with automatic differentiation. This last part, "with automatic differentiation" would be interesting for me, as I've never heard of before, for itself as it was not something done for now in the community.

The focus of the internship would be to use automatic differentiation to calculate the Riemann's tensor, write down a supervised work for manually calculating the terms in general relativity and have some modelization. The question we could ourselves is :

How could we better understand general relativity ? And how does automatic differentiation can help us in this mean?

2 General relativity

The general relativity theory was developed during the early twentieth century. Albert Einstein, in 1905, had his magical year : he wrote three majors contribution to physics ; light quanta, Brownian motion and the premises of General Relativity as we know it, *the theory of Special Relativity.*
At first being a correction of others theory such as Maxwell's laws, Einstein decided to dig further and spent the next ten years trying to find an explanation to gravity and special relativity. That what he has done in 1915 by presenting the *Theory of General Relativity.*
By doing that, he contributed the most to the physics community, opening a field of research and a brand new way to see physics and even gravity, not as objects and forces, but also as geometry itself.

Even if general relativity has little use in common day life ; albeit Global Positioning System (also known as **GPS**) which without general relativity would be pointlessly inaccurate, it tells us a great deal about symmetries, astrophysics and cosmology.
Purely derived from thought and intuition and which has always managed to describe every aspect of the gravitational physic observed and succeeded in predicting the existence of objects like black hole a century before observing it.

2.1 Principle of equivalence, the basis of general relativity

The keystone of General relativity is **"Einstein Equivalence Principle"**, or more simply put, the local equivalence of gravitation and inertia. By using this postulate, he managed to describe and understand relativity as purely the manifestation of the space-time architecture : curvature and geometry are at the source of every gravitational force we have experienced and existing. We can postulate it as :

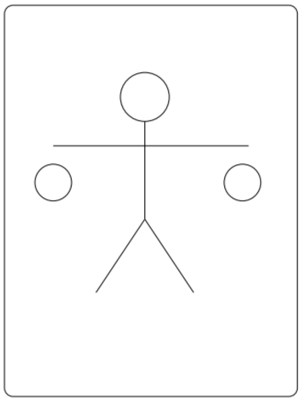
" At every space-time point in an arbitrary gravitational field it is possible to choose a locally inertial (or freely falling) coordinate system such that, within a sufficiently small region of this point, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation."[1]

We can formulate it also as :

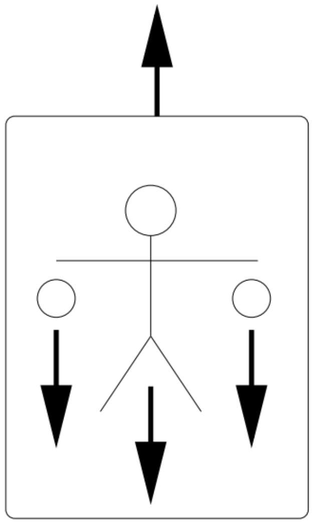
"Experiments in a sufficiently small freely falling laboratory, over a sufficiently short time, give results that are indistinguishable from those of the same experiments in an inertial frame in empty space." [2]

To explain it more clearly, this "Gedankenexperiment", which is the German for "thought experiment", will help us a lot.

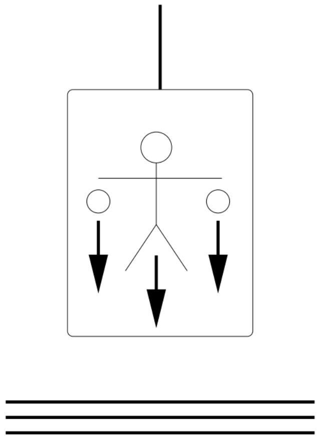
Let it be a elevator, closed from the outside, so anyone inside can just deduce what is happening from their sensations. There is one person in this rocket with two stones. We'll call him "Keyzar".



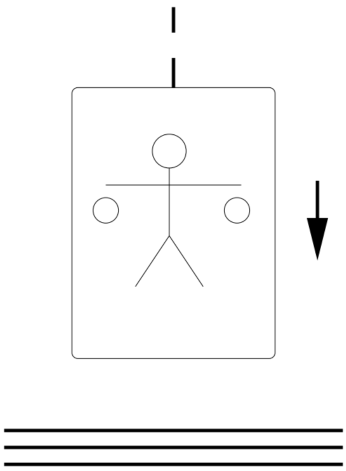
(a) Keyzar and his two stones freely floating in outer space, with the absence of forces [3].



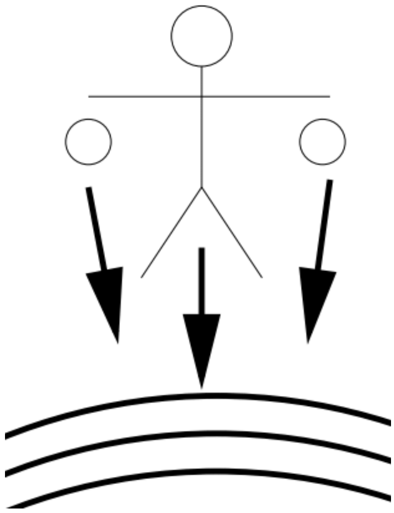
(b) Keyzar in a upward constant accelerated elevator. This acceleration replicates a gravitational field downward, and Keyzar and his stone will drop to the bottom of the elevator [3].



(c) If put in a gravitational field, Keyzar and his stones would be drawn to the ground by gravity [3].



(d) Keyzar and his two stones floating in the elevator. The elevator is in a free fall, but Keyzar is not able to see the outside [3].



(e) Keyzar and his two stones are falling toward the center of the Earth, each with their own shortest line to reach it, which make them slightly approach each other in the end [3].

Figure 2: If this elevator us put in outer space, Keyzar would not feel any gravitational pull, he'll be floating and likewise the stones he brought with him, as shown in Figure 2a.
If is suddenly applied a constant acceleration ”upward” on the elevator, Keyzar and the stones would fall to the floor of said elevator, as they're subject to a constant force opposed to the one pushing the elevator upward. This acceleration replicates a gravitational field downward, as shown in Figure 2b.
Assume that this elevator is now put in a gravitational field and held back by a rope as in Figure 2c. Keyzar is still drawn to the floor. Without seeing the outside, he has now way of determining if the cause is the constant acceleration or just a gravitational field.
Finally, if the rope is cut, the elevator would be in free fall. Keyzar and his stones would float again, and yet without seeing the outside, he would have no idea whatsoever if he is in outer space again or currently falling.
The local notion of the formulation means that for Keyzar and his stone, if they are falling toward the Earth, and so are subjects to a non-uniform gravitational field, here a spherical one, as they are not strictly at the same location, they'll approach each others slightly as seen in Figure 2e.

By this simple Gedankenexperiment and what we know of Special relativity (that something moving relatively to another has a different space-time referential), one can easily understand that space and time are bent by matter, and that gravity is just an expression of this deformation, curvature of space and time.
The trajectory of anything is bent by space time curvature, and hence the explanation of the deviation of the trajectory of light : the light passing near the sun is bent by the deformation the sun provokes in the neighbouring space-time fabric such as seen in Figure 3. It was observed in 1919 by Arthur Eddington, Frank Watson Dyson, and their collaborators during the total solar eclipse on May 29, and has greatly contributed to prove at the time the credibility of the theory of General Relativity.

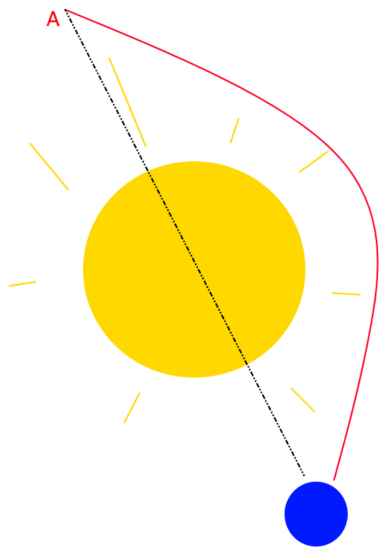


Figure 3: An object behind the sun should not be visible from Earth, as is object A. But since the trajectory of the light it emits is curved by the Sun, it is possible to see it. Newton's law don't predict that exactly, while general relativity predict it perfectly. It's a proof of general relativity well observed and an example of a deviation of the geodesic trajectory of light.

What happened exactly here ? The light follows the path with the minimum distance between two points. In a common euclidean space-time, it is a straight line. But for a curved space, it changes, hence the deviation of light for the Earth referential.

2.2 Einstein summation convention

Einstein has introduced a notational convention that implies summation over a number of indexed terms in a formula. For example :

$$\sum_{i=1}^{3} a_i x^i = a_1 x^1 + a_2 x^2 + a_3 x^3 \rightarrow a_i x^i = a_1 x^1 + a_2 x^2 + a_3 x^3 \tag{1}$$

This convention will be used for the rest of the report.

2.3 Geodesic equations and Christoffel symbol, and the notion of curvature

To express the trajectory of our system in space and time, we shall manipulate quantities such as Four-vector such as the four-vector of space-time :

$$\bar{x} = \begin{bmatrix} x^0 = ct \\ x^1 = x \\ x^2 = y \\ x^3 = z \end{bmatrix} = [ct, \vec{x}] \tag{2}$$

By changing coordinate from an inertial referential, such as one falling Keyzar, to a accelerated referential, such as Keyzar seen from the surface of the Earth.

One can find the trajectory of lesser distance above all the others ; **the geodesic equation :**

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\nu}{d\tau} \frac{dx^\mu}{d\tau} = 0 \tag{3}$$

with $\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial X^{\lambda}} \frac{\partial^2 X^{\lambda}}{\partial x^{\mu} \partial x^{\nu}}$ being the affine connection, or also the Christoffel's symbol.
If we would to try to understand it, let's see several cases :

- $\frac{d^2 x^{\lambda}}{d\tau^2} = 0 \rightarrow$ Inertial referential, the acceleration is null.
- $\frac{d^2 x^{\lambda}}{d\tau^2} = -\Gamma^{\lambda}_{\mu\nu} \frac{dx^{\nu}}{d\tau} \frac{dx^{\mu}}{d\tau} \neq 0 \rightarrow$ Earth's referential, Keyzar is falling toward us, his acceleration is non null.

Here, $\Gamma^{\lambda}_{\mu\nu}$ is the term of inertial gravitational force. To express our falling Keyzar in different referential, it would be wise to introduce a new object giving the changes needed to pass from one flat space-time to a curved one, $X^{\alpha} \rightarrow x^{\alpha}$, **the metric tensor :**

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial X^{\alpha}}{\partial x^{\mu}} \frac{\partial X^{\beta}}{\partial x^{\nu}} \tag{4}$$

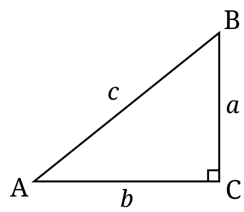
where $\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is the Minkowski space-time, the one used for euclidean space, and stating that while all the individual components in Euclidean space and time may differ due to general relativity, in this spacetime, all the frame of reference will agree on the total distance in spacetime between events with spacetime being : $dS^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$.
One can use this equation to do a change of a referential to an other by just putting the ancient metric tensor What is the relation between the metric tensor and the geodesic equation one could ask ? Those two quantities are linked to the derivate of $\frac{\partial X^{\alpha}}{\partial x^{\mu}}$ and thus are not independent. One could write the Christoffel's symbols as :

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha} \right) \tag{5}$$

where $g_{\alpha\mu,\nu} = \frac{\partial g_{\alpha\mu}}{\partial x^{\nu}}$ and one could interpret $\Gamma^{\beta}_{\mu\nu}$ as the inertial gravitational forces and $\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}}$ is the geometry of space and time. Thus, Geometry is directly the source of the gravitational force.

To better understand what is this curvature, one could say that curvature is a geometric notion, characterizing the distance on a surface.

For example, the distance in classical geometry, in an euclidean space is given by Pythagoras theorem : $c^2 = a^2 + b^2$ and the total angle is 180° for a rectangle triangle. For a spherical surface, our distance would be different, as it's a curve space : the angle would be more than 180° and the very Pythagoras theorem would be false : $c^2 \neq a^2 + b^2$. To make it work, we need to express the distance in a different coordinate system : the spherical one where : $L^2 = R^2\theta^2 + R^2 sin^2(\theta)\phi^2$.



2.4 Riemann tensor, Ricci tensor, and scalar curvature

There is a lot of information in the Christoffel symbol, but one would like to manipulate it more efficiently with having a tensor-like object.
Alas, the Christoffel symbol is not a tensor. That's why the curvature tensor $R^{\lambda}_{\mu\nu\alpha}$ is built with the Christoffel's symbols to contain every information on the curvature of referential, it's **the Riemann curvature tensor :**

$$R^{\lambda}_{\mu\nu\alpha} = \Gamma^{\lambda}_{\mu\alpha,\nu} - \Gamma^{\lambda}_{\mu\nu,\alpha} - \Gamma^{\lambda}_{\nu\eta}\Gamma^{\eta}_{\mu\alpha} - \Gamma^{\lambda}_{\alpha\eta}\Gamma^{\eta}_{\mu\nu} \tag{6}$$

- As $\Gamma^{\lambda}_{\mu\nu}$ is built from $g_{\mu\nu}$, $R^{\lambda}_{\mu\nu\alpha}$ contain all the information on the curvature.
- All the components of the tensor annihilate each others if and only if the space-time is flat.
- In four dimension, the Riemann tensor has 256 components but only twenty components are independents, thanks to the symmetries properties of the tensor.

This expression wholly describe the curvature of space-time, but it's heavy to manipulate, that's why we will contract this tensor. We want also to have it used in for expressing a physical phenomenon and not only a purely geometric one. To do that, we use the Einstein convention to sum two indices together.

Several contraction of this tensor exists such as the **Ricci tensor** :

$$R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda} = \sum_{\alpha=0}^{3} R^{\lambda}_{\mu\nu\lambda} \tag{7}$$

From this Ricci tensor, we can calculate the **scalar curvature :**

$$R = R^{\lambda}_{\lambda} = g^{\mu\nu} R_{\mu\nu} \tag{8}$$

The scalar curvature is really interesting because it indicates us if the space-time is curved or not. If the space-time is flat, then $R = 0$ and if not, it will take a certain value relative to our metric tensor.
Having those, one can write the **Einstein equation :**

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = kT_{\mu\nu} \tag{9}$$

where $G_{\mu\nu}$ represent the curvature of space and time, and $kT_{\mu\nu}$ is the propagation of the impulse and energy of the matter in the direction ν times a constant. In a empty space as a void, $kT_{\mu\nu} = 0$.
The Riemann tensor is therefore essential to calculating physical quantities in general relativity.

2.5 Schwarschild metric :

To express the curvature done by a single spherical body, we can use the Schwarschild metric :
Coordinates $x^{\alpha} = (ct, R, \theta, \phi)$.

$$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{R_s}{R} & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{R_s}{R}\right)^{-1} & 0 & 0 \\ 0 & 0 & -R^2 & 0 \\ 0 & 0 & 0 & -R^2 \sin^2\theta \end{pmatrix} \tag{10}$$

With $R_s = \frac{2GM}{c^2}$ is the Schwarzschild radius, G the gravitation constant, M the mass of the spherical body, and c the celerity of light.
For high values of R, which mean a high distance from the center of mass, we can see that this metric becomes similar to the surface of a spherical sphere :
Coordinates $x^{\alpha} = (ct, R, \theta, \phi)$.

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -R^2 & 0 \\ 0 & 0 & 0 & -R^2 \sin^2\theta \end{pmatrix} \tag{11}$$

2.6 The arduous task of finding a solution in general relativity

To find a solution of the geodesic equations, except in a flat space-time, which can be done very easily with Newton's equations, it becomes rapidly arduous.
If we take for example a star, and try to see what are the geodesic equations around, we need to take in account the star, but also how it curves space and time. But to determine that, we need also the coordinates of the star, themselves altered by how it curves space and time. To do that, thankfully, we can relay on symmetries. But for the Riemann's tensor in four dimension, we got 256 terms to calculate, and only 20 if we use the symmetrical properties of the Riemann tensor.

Let's for example calculate the Riemann's tensor for this metric tensor which is the two dimensionnal surface of a sphere with a radius of one.
We then got for our distance : $ds^2 = d\theta^2 + sin^2(\theta) d\phi^2$. From that, one can determine the metric tensor :
$(x^0, x^1) = (\theta, \phi)$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & sin^2(\theta) \end{pmatrix} \tag{12}$$

First, one needs to calculate the inverse metric tensor, which is $g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{sin^2(\theta)} \end{pmatrix}$.
Then, one calculates the Christoffel's symbol : $\Gamma^{\beta}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha} \right)$

$$\Gamma^{\theta}_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & -cos(\theta)sin(\theta) \end{pmatrix}$$

$$\Gamma^{\phi}_{\mu\nu} = \begin{pmatrix} 0 & cot(\theta) \\ cot(\theta) & 0 \end{pmatrix}$$

To then calculate the Riemann tensor, one needs the Christoffel's derivatives :
$\Gamma^{\theta}_{\mu\nu,\theta} = \begin{pmatrix} 0 & 0 \\ 0 & -cos(2\theta) \end{pmatrix}$, $\Gamma^{\theta}_{\mu\nu,\phi} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\Gamma^{\phi}_{\mu\nu,\theta} = \begin{pmatrix} 0 & -\frac{1}{sin^2(\theta)} \\ -\frac{1}{sin^2(\theta)} & 0 \end{pmatrix}$, $\Gamma^{\phi}_{\mu\nu,\theta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Finally, one can calculate the Riemann tensor :

$$R^{\theta}_{\mu\nu\theta} = \begin{pmatrix} 0 & 0 \\ 0 & sin^2(\theta) \end{pmatrix},\ R^{\phi}_{\mu\nu\theta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ R^{\theta}_{\mu\nu\phi} = \begin{pmatrix} 0 & -sin^2(\theta) \\ 0 & 0 \end{pmatrix},\ R^{\phi}_{\mu\nu\phi} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

And at the end, the Ricci tensor and the scalar curvature :

$$R_{\mu\nu} = R^{\theta}_{\mu\nu\theta} + R^{\phi}_{\mu\nu\phi} = R^{\theta}_{\mu\nu\theta} = \begin{pmatrix} 0 & 0 \\ 0 & sin^2(\theta) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & sin^2(\theta) \end{pmatrix}$$

$$R = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = 1 + \frac{1}{sin^2\theta} sin^2(\theta) = 2$$

Our result of $R = 2$ indicates that our space is not flat, and indeed it's curved. This calculation is one done for the supervised work written during my intership with the goal of making those calculations more intuitive for the M1 students. You can found it in the **Appendix A**.
This calculation when done by hand can be tedious. That's when automatic differentiation becomes relevant to our problem.

3 Automatic Differentiation:

What is automatic differentiation ? It's a kind of differentiable programming : *" Writing software composed of differentiable and parameterized building blocks that are executed via automatic differentiation and optimized in order to perform a specified task"* [4]. We'll try to understand what automatic differentiation is.

3.1 What automatic differentiation is not

To better understand automatic differentiation(which we will abbreviate now with **AD**), it's easier to understand how it's different of symbolic and numerical differentiation.

3.1.1 Automatic differentiation is not symbolic differentiation:

It's different of **symbolic differentiation** where the symbolic value of the function, its "shape" matters to us at the end, while in AD, the shape of our function doesn't matter since we only manipulate values going through elementary operation such as *logarithmic function, exponential function, addition, multiplication, fraction* which are contained and well documented in Python's library such as NumPy [5]. We can then manipulate blocs with loops, and other purely algorithmic operations that would be prohibited by symbolic differentiation. We don't need a closed block such as shown in Figure 4. Plus, we don't have the problem of "expression swell" as we can see at the center right : the derived expression is expending more and more, it' letting us with something not really convenient to use.

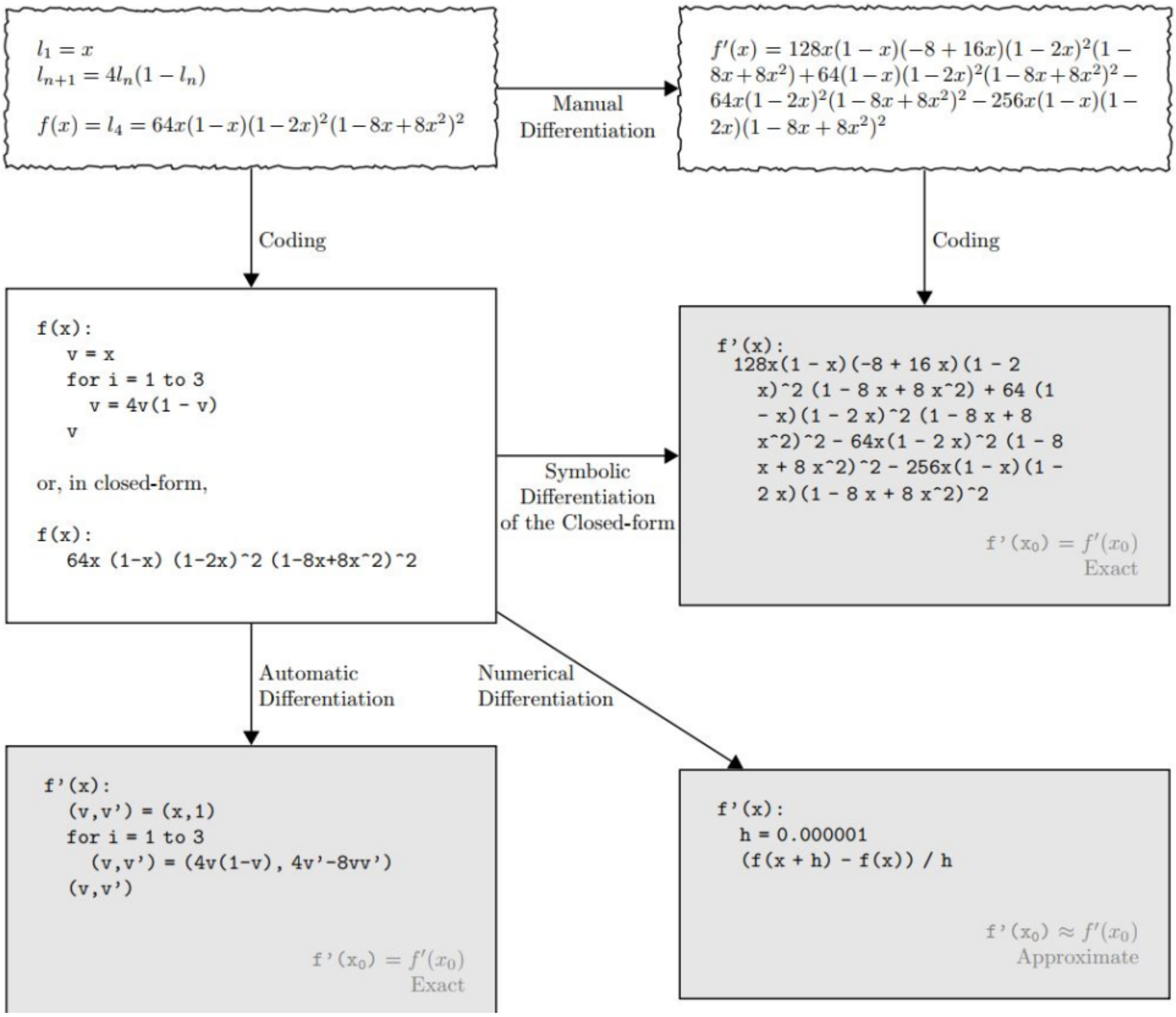


Figure 4: Different types of differentiation. Symbolic differentiation is of the Closed-Form of our function, but it cannot always be the case. Here, in the left center bloc, is used a iteration of the "for i to n" form, that allows more flexibility, but cannot be used in symbolic differentiation. [4].

3.1.2 AD is not numerical differentiation :

Numerical differentiation is the finite difference approximation of derivatives using the values calculated at some defined points(Figure 4) It is really simple to implement, but it has the problem of having to carefully choose to size of the step "h" taken between each evaluation, for if too big, it will create a not accurate simulation, and too small, it will take too much calculating power. The numerical differentiation has the problem that it's only a approximation and so we will always face approximations errors.

3.2 How does AD works ?

There are two main modes of AD, both complementary to each others, backwards and forward. Both consist in taking a function that we break down in numerous fundamental operations such as those numPy can do. By doing this, we can do a computational graph as seen in Figure 5. Then, by using the chain rule, we can easily deduce the relative derivatives to the precedent part of the graph.

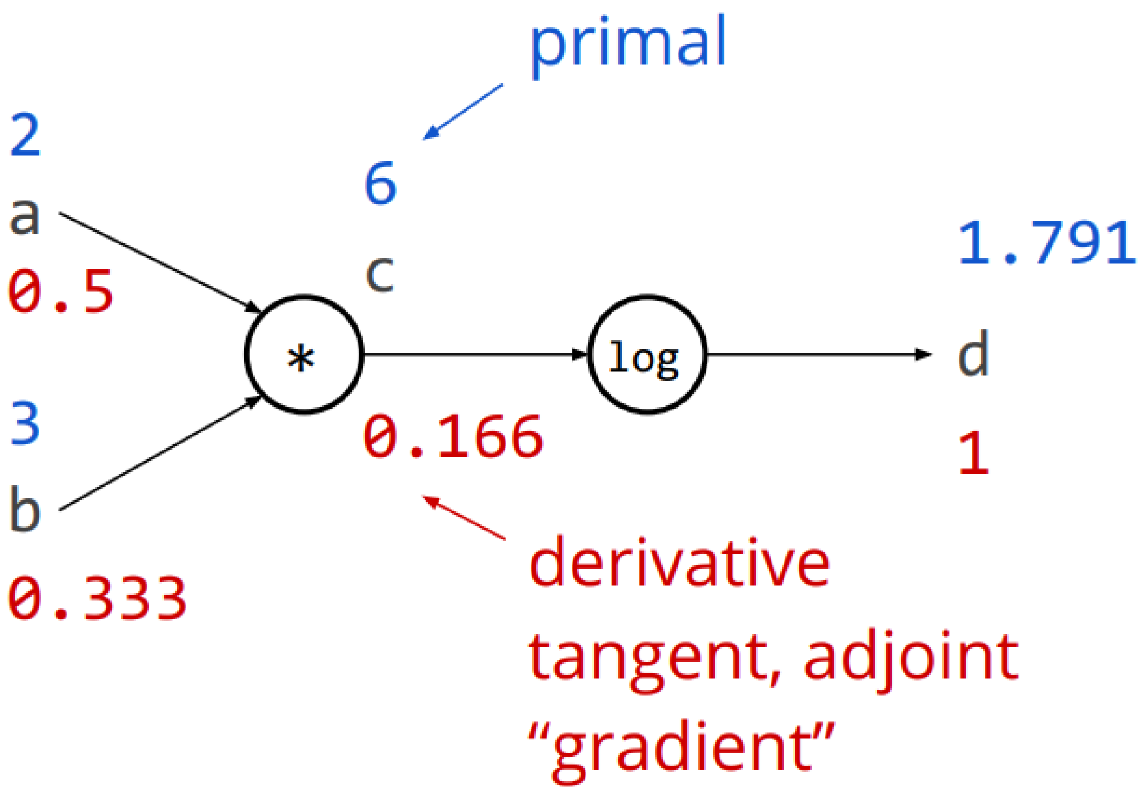


Figure 5: Computational graph of $y = log(ab)$. Each of the calculating steps are decomposed to see the advancement of the calculation : first comes **a** times **b**, then this value **c** is calculated as an input in the log function to give **d**. The primals are values deduced forward from the initial input, and the derivative are the red numbers. If the derivatives are calculated in the same time as the primals, the operation of AD is called forward and if the derivatives are calculated afterward, beginning from the output(here d), it's called backward propagation, or reverse mode.

3.3 What are the strength of AD

Besides the advantages of precision we don't have with numerical differentiation and of flexibility that we don't have with symbolic differentiation, it depends if we are in forward mode or in reverse mode :
In forward mode, the best use is for : $R^n \to R^m$ where $n << m$ or ideally $R \to R^m$ because as all the derivatives can be computed with just one forward pass.

In reverse mode, the best use is for $R^n \to R^m$ where $n >> m$ or ideally $R^n \to R$ so we can calculate all our gradient in one reverse pass.
That's the one we will be using with PyTorch [6], which is an open source machine learning library for Python, that allows us to manipulate automatic differentiation operations.

3.4 PyTorch

We will work with PyTorch, which gives us two tools to differentiate our tensors ; the jacobian and hessian function, that for a given function, and a set of coordinate will calculate the jacobian or hessian tensor of this function at those last ones.

For the jacobian;

$$(r = 1, \theta = 0,5) \to g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \xrightarrow{pytorch.jacobian} g_{\mu\nu,r} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \tag{13}$$

The problem here is that we cannot apply this function two times on the same tensors directly, for it will try to differentiate numerical values, so it will give us a null tensor.

4 The Riemann tensor calculator

4.1 Calculating the Riemann tensor:

Our goal in this internship was to calculate the Riemann tensor (Eq. 6) using AD.

We needed the Christoffel tensor (Eq. 5) but we couldn't just derive it since we obtained it from the jacobian function. All we could have with the jacobian was the first and second derivative of $g_{\mu\nu}$.
So we expressed the Riemann tensor as a function of $g_{\mu\nu}$ and its first and second derivatives.

$$\begin{aligned} R^{\beta}_{\mu\nu\epsilon} = \frac{1}{2}[g^{\alpha\beta,\nu}(g_{\alpha\mu,\epsilon} + g_{\alpha\epsilon,\mu} - g_{\mu\epsilon,\alpha}) + g^{\alpha\beta}(g_{\alpha\mu,\epsilon,\nu} + g_{\alpha\epsilon,\mu,\nu} - g_{\mu\epsilon,\alpha,\nu})] \\ - \frac{1}{2}[g^{\alpha\beta,\epsilon}(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}) + g^{\alpha\beta}(g_{\alpha\mu,\nu,\epsilon} + g_{\alpha\nu,\mu,\epsilon} - g_{\mu\nu,\alpha,\epsilon})] \\ + [\frac{1}{2}g^{\alpha\beta}(g_{\alpha\nu,\eta} + g_{\alpha\eta,\nu} - g_{\nu\eta,\alpha}][\frac{1}{2}g^{\alpha\eta}(g_{\alpha\mu,\epsilon} + g_{\alpha\epsilon,\mu} - g_{\mu\epsilon,\alpha}] \\ - [\frac{1}{2}g^{\alpha\beta}(g_{\alpha\epsilon,\eta} + g_{\alpha\eta,\epsilon} - g_{\epsilon\eta,\alpha}][\frac{1}{2}g^{\alpha\eta}(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}] \end{aligned} \tag{14}$$

Since the expression for the Riemann tensor was redundant with the derivatives of the Christoffel appearing two times with different indices and the product between two Christoffel, we could create a function to call those parts of the formula :

$$R^{\beta}_{\mu\nu\alpha} = \Gamma^{\beta}_{\mu\alpha,\nu} - \Gamma^{\beta}_{\mu\nu,\alpha} - \Gamma^{\beta}_{\nu\eta}\Gamma^{\eta}_{\mu\alpha} - \Gamma^{\beta}_{\alpha\eta}\Gamma^{\eta}_{\mu\nu} \tag{15}$$

$$\Gamma^{\beta}_{\mu\epsilon,\nu} = \frac{1}{2}[g^{\alpha\beta,\nu}(g_{\alpha\mu,\epsilon} + g_{\alpha\epsilon,\mu} - g_{\mu\epsilon,\alpha}) + g^{\alpha\beta}(g_{\alpha\mu,\epsilon,\nu} + g_{\alpha\epsilon,\mu,\nu} - g_{\mu\epsilon,\alpha,\nu})] \tag{16}$$

$$\Gamma^{\beta}_{\nu\eta}\Gamma^{\eta}_{\mu\epsilon} = [\frac{1}{2}g^{\alpha\beta}(g_{\alpha\nu,\eta} + g_{\alpha\eta,\nu} - g_{\nu\eta,\alpha}][\frac{1}{2}g^{\alpha\eta}(g_{\alpha\mu,\epsilon} + g_{\alpha\epsilon,\mu} - g_{\mu\epsilon,\alpha}] \tag{17}$$

4.2 Riemann calculator

In this section, we'll discuss the jupyter notebook :

```
Riemann_Calculations.ipynb
```

which can be found on GitHub :
https://github.com/AndreaAntoniali/Riemann-tensor-calculator

Our program is able to :

- Given a coordinate, and a metric tensor, calculate the Christoffel's symbols values.
- Calculate the Riemann tensor, the Ricci tensor and the scalar curvature values.
- display them with a user-friendly interface.

The final values are correct, and we can put any metric tensor, of any dimension.

4.3 Treating the data

Our Riemann tensor calculator on itself is interesting, but what would be nice is to have something visual to interpret its, such as a map for example.
To do this, we need to be able to calculate our Riemann tensor not just on a point, but on a set of points, to map a n- dimensional space. So we've created a program able to create a map of Riemann tensors, write data to reuse it later, and to read it, such as found on the repository on GitHub. It's the Jupyter notebook :

```
Riemann_Files.ipynb
```

Our program is able to :

- Given a coordinate, make a linear progress of each of the coordinate in a linear direction and write it down in a file. For example, if given as starting coordinates (0, 1) and told to go two steps of 1 further, it would generate a set of coordinates as : (0,1),(1,2),(2,3)
- Given a metric tensor, calculate the Christoffel's symbols values for each of the coordinates points.
- Calculate the Riemann tensor, the Ricci tensor and the scalar curvature values and write it down in a file, for each coordinate of the position.
- read the given coordinates and currently existing files.

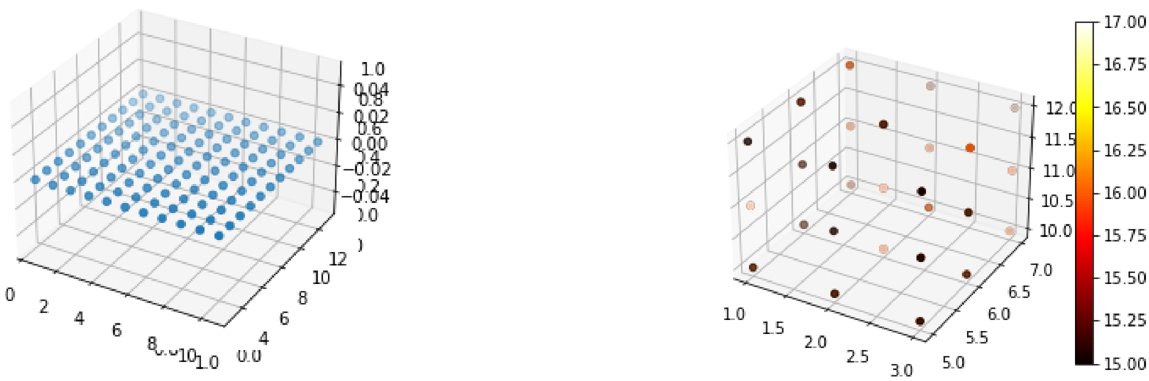
4.4 Visualizing the data

As we have now a way of generating data, yet it's imperfect, we try to go further and create a way of visualising physical values with meaning. The choice was opted to look at the scalar curvature for each point of coordinate and thus a program as been written :

```
Riemann_Map.ipynb
```

Our program is able to :

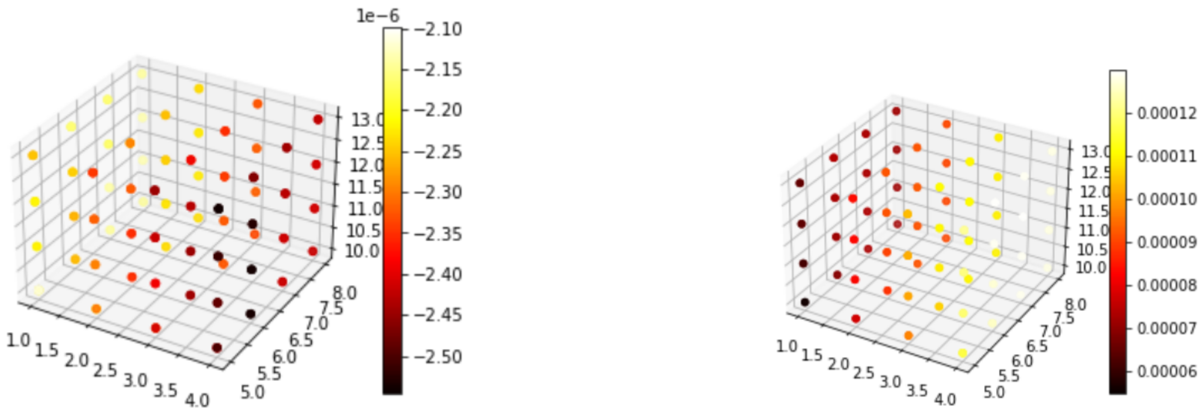
- Given a coordinate, create a map with a grid of points representing those coordinates such as shown in Figure 6a.



(a) A set of coordinate generated for a step of one, beginning at (0.5, 3)

(b) A set of 4 dimension coordinate generated for a step of one, beginning at (1, 5, 10, 15). The color represent the value of time.

Figure 6: Multiple representation generated by the program.

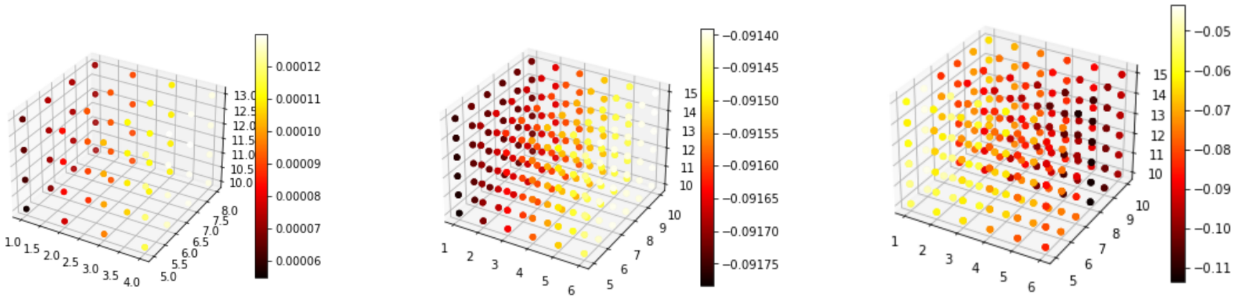


(a) A set of 3 spatial dimension coordinate generated for a step of one, beginning at (1, 5, 10) meters, for a flat space-time metric. As expected, we found a result for scalar curvature near 0. The color represent the value of the curvature of space time for those spatial coordinate.

(b) A set of 3 spatial dimension coordinate generated for a step of one, beginning at (1, 5, 10)meters, for a Schwarschild space-time metric, with a $R_S = 1$ meter. The color represent the value of the curvature of space time for those spatial coordinate.

Figure 7: Multiple representation of the scalar curvature generated by the program. As expected, we found a result for scalar curvature near 0 in Figure 9b, but not the exact same as the flat space-time in Figure 9a.

- Given this metric, represent the scalar curvature and show it in color as shown in Figure 9.



(a) A set of 3 spatial dimension coordinate generated for a step of one, beginning at (1, 5, 10) meters, for a Schwarzschild space-time metric, with a $R_S = 1$ meter.

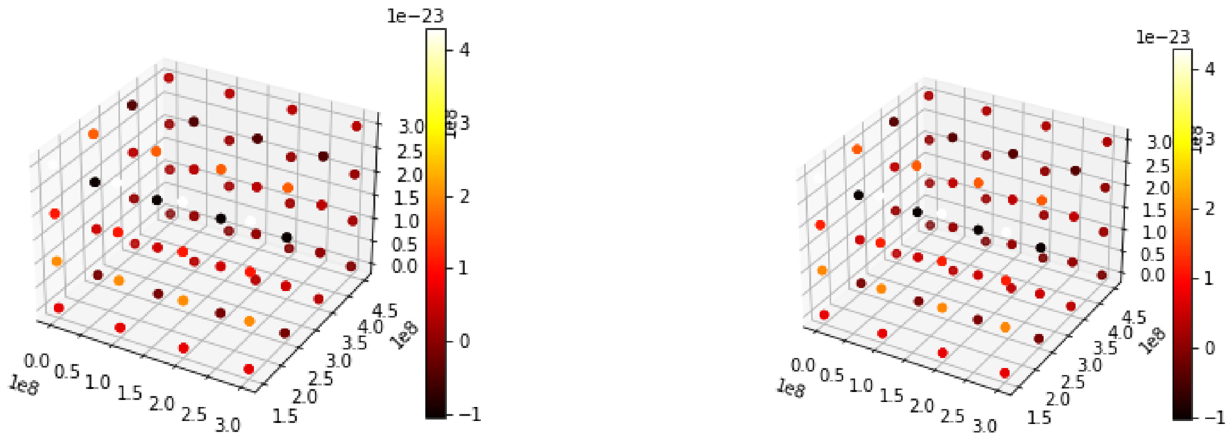
(b) A set of 3 spatial dimension coordinate generated for a step of one, beginning at (1, 5, 10)meters , for a Schwarzschild space-time metric, with a $R_S = 1000$ meters.

(c) A set of 3 spatial dimension coordinate generated for a step of one, beginning at (1, 5, 10), for a Schwarzschild space-time metric, with a $R_S = 1000000$ meters.

Figure 8: Multiple representation of the scalar curvature for the Schwarzschild metric but with different value of R_S As expected, we found a result for scalar curvature near 0 in Figure 8a, but not the exact same as the scalar curvature value in Figures 8b and Figure 8c. It's definitely a proof of the effect of the factor R_S in our metric.

We can try to find a physical variation of our scalar value, and so we plot the same representation, but changing the value of R_S from 1 meter to 1000000 meters.

By doing so, we have seen that if we had more calculating power, we could surely visualize an increase for high value of R_S, which represent the space-time becoming more and curved near the center of mass as the coordinates are more not more distant. We also observe that as soon our coordinates are inside the Schwarzschild radius, our scalar curvature becomes negative. Let's try observing it from very far away, one astronomical unit away, so 150 millions kilometers, using several Schwarzschild radius : The length unit will be in kilometer.



(a) A set of 3 spatial dimension coordinate generated for a step of 100000000 kms, beginning at (150000000, 1, 1) kms, for a Schwarzschild space-time metric, with a $R_S = 2.95$ corresponding to that of the Sun.

(b) A set of 3 spatial dimension coordinate generated for a step of 100000000 kms, beginning at (150000000, 1, 1) kms, for a Schwarzschild space-time metric, with a $R_S = 2.95$ corresponding to that of Sagittarius A*, the Supermassive black hole at the center of the Milky Way.

Figure 9: Multiple representation of the scalar curvature for the Schwarzschild metric but with different value of R_S for a distance of one astronomical unit in kilometers. We observe that the scalar curvature is almost null for both case which indicates a quasi-flat spacetime.

5 Conclusion

We first have discussed of the basis of general relativity : space and time are bent by the matter they contain, and thus we need to change the geometry and our way to calculate distances. This implies that to understand those phenomenons and to calculate the geodesics, the straight lines in a non flat space-time, we need to calculate the space-time curvature. All the information of it is contained in the Riemann tensor, and contractions of this last one.
We used automatic differentiation, a differentiation based on the evaluations of a succession of elementary mathematical operations and their derivatives, and using the chain rules to determinate the final derivatives values, to create a Riemann tensor calculator.
We also created others programs using this Riemann calculator : One generating a trajectory and stocking it down in files, and an other showing a visual representation of the scalar curvature.

References

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A TD Relativité Générale : Approche numérique.

M1 PFA UP - TD Relativité Générale

En posant un modèle géométrique de l'espace temps, la relativité générale nous permet d'expliquer l'origine de la gravitation. Ce modèle, bien que simple dans ses prémices se complexifie rapidement avec des fondements mathématiques nécessaires, comme la manipulation des tenseurs et des sommes implicites d'Einstein. Ce TD a pour but de vous familiariser avec les équations vues dans les cours et d'acquérir un compréhension plus intuitive de la relativité générale, plus proche de la simulation que de la théorie pure.

A.1 Un peu de calcul à la main

Vous avez vu plusieurs tenseurs, quantités et équations durant vos cours. Ils ont pu vous sembler intimidants, de par la profusion d'indices montrés. Cette partie va vous montrer qu'il n'en est rien, et vous permettre de comprendre la logique de calcul derrière chaque formule. [1] Pour ce TD, vous aurez le droit à trois difficultés :

- Chercheur : Vous n'avez que votre connaissance et votre intuition pour avancer. Voyez cela comme un défi, un puzzle. Si vos résultats sont bons, vous devriez trouver **2** à la fin. Bon courage!
- Master : Des indications vous mettront sur la bonne voie, c'est la meilleure façon d'avancer sans y consacrer trop de temps.
- Lecteur : Toute la correction est rédigée pour vous. Un bon moyen de vérifier si l'on a bien réussi les deux versions précédentes.

[1] remerciement à M. Vincent pour ses retours.

Tenseur métrique : Le fondement de la relativité générale est de définir une métrique nouvelle; ici, le tenseur métrique nous aide à passer d'une métrique à une autre.

- Tenseur métrique pour un espace euclidien en coordonnées cartésiennes (qui est celui dans lequel nous réfléchissons le plus facilement ; nous vivons dedans... :
  $(x^1, x^2, x^3) = (x, y, z)$
$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{18}$$

- Tenseur métrique pour un espace euclidien, en coordonnées polaires : $(x^1, x^2) = (r, \theta)$
$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{19}$$

- Tenseur métrique pour un espace-temps de Minkowski plat :
  $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$
$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{20}$$

  Si l'on souhaitait calculer tout les termes du tenseur de Riemann de ce tenseur à la main, à savoir 256, cela nous prendrait un certain temps que nous n'avons pas dans ce TD. Voyons une métrique plus simple :

- Tenseur métrique pour un espace euclidien sphérique à deux dimensions : **la surface d'une sphère :**
  $(x^0, x^1) = (\theta, \phi)$
$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & sin^2(\theta) \end{pmatrix} \tag{21}$$

C'est une métrique familière, typiquement celle dans laquelle nous évoluons sur Terre avec les latitudes et les longitudes.

A.2 Calcul : Cas de la surface d'une sphère en 2D.

A.2.1 Tenseur métrique inverse

Quelle est le tenseur métrique inverse pour la surface d'une sphère ?
Indication : Vous devez trouver une expression telle que :

$$g_{\mu\nu}g^{\mu\nu} = g^{\mu\nu}g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A.2.2 Symboles de Christoffel :

Nous rappelons ici l'expression des symboles de Christoffel :

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right)$$

Calculez les symboles de Christoffel.
Indication : Pour calculer les symboles, vous pouvez commencer par fixer β. De plus, vous remarquerez que votre tenseur métrique inverse est diagonal, et donc si $\alpha \neq \beta$ alors $g^{\alpha\beta} = 0$ et donc $\alpha = \beta$.
Vous aurez alors deux expressions :

$$\Gamma^{\theta}_{\mu\nu} = \frac{1}{2}g^{\theta\theta}\left(g_{\theta\mu,\nu} + g_{\theta\nu,\mu} - g_{\mu\nu,\theta}\right)$$

$$\Gamma^{\phi}_{\mu\nu} = \frac{1}{2}g^{\phi\phi}\left(g_{\phi\mu,\nu} + g_{\phi\nu,\mu} - g_{\mu\nu,\phi}\right)$$

Attention, n'oubliez pas la sommation implicite d'Einstein !

A.2.3 Tenseur de Riemann

Nous rappelons l'expression du tenseur de Riemann :

$$R^{\lambda}_{\mu\nu\alpha} = \Gamma^{\lambda}_{\mu\alpha,\nu} - \Gamma^{\lambda}_{\mu\nu,\alpha} - \Gamma^{\lambda}_{\nu\eta}\Gamma^{\eta}_{\mu\alpha} - \Gamma^{\lambda}_{\alpha\eta}\Gamma^{\eta}_{\mu\nu}$$

Calculez le tenseur de Riemann.
Indication : Ici aussi, il est conseillé de calculer le tenseur de Riemann en plusieurs étapes : D'abord, vous pouvez calculer la dérivée des tenseurs des symboles de Christoffel. Puis, vous pouvez fixer votre λ tel que :

$$R^{\theta}_{\mu\nu\alpha} = \Gamma^{\theta}_{\mu\alpha,\nu} - \Gamma^{\theta}_{\mu\nu,\alpha} - \Gamma^{\theta}_{\nu\eta}\Gamma^{\eta}_{\mu\alpha} - \Gamma^{\theta}_{\alpha\eta}\Gamma^{\eta}_{\mu\nu}$$

Il est vraiment conseillé d'y aller étape par étape :
$R^{\theta}_{\mu\nu\theta}$, $R^{\phi}_{\mu\nu\theta}$, $R^{\theta}_{\mu\nu\phi}$ puis $R^{\phi}_{\mu\nu\phi}$.

A.2.4 Tenseur de Ricci

Nous rappelons l'expression du Tenseur de Ricci, qui est une contraction du tenseur de Riemann :

$$R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda}$$

Calculez le tenseur de Ricci.
Indication :
Ici encore, toute la difficulté est de savoir comment commencer. Souvenez-vous ce que deux indices dans le même tenseur implique une sommation implicite.

A.2.5 Courbure scalaire :

Nous rappelons l'expression de la courbure scalaire, qui est une contraction du tenseur de Ricci :

$$R = R^{\lambda}_{\lambda} = g^{\mu\nu}R_{\mu\nu}$$

Calculez la courbure scalaire.
Indication : Il ne faut pas vous laisser désemparer par les indices et par la supposée complexité. Posez vos deux tenseurs et essayez de faire le produit des deux.

A.3 Correction :

A.3.1 Tenseur métrique inverse

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{sin^2(\theta)} \end{pmatrix}$$

A.3.2 Symboles de Christoffel :

$$\Gamma^{\theta}_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & -cos(\theta)sin(\theta) \end{pmatrix}$$

$$\Gamma^{\phi}_{\mu\nu} = \begin{pmatrix} 0 & cot(\theta) \\ cot(\theta) & 0 \end{pmatrix}$$

Ici, remarquer que les dérivées par rapport à ϕ sont vouées à être nulles nous permet de gagner du temps. De plus, toutes les coordonnées μ et ν pour les tenseurs métriques qui ne sont pas identiques nous dirigent sur une coordonnée nulle.

A.3.3 Tenseur de Riemann

Calculons d'abord nos tenseurs dérivés des symboles de Christoffel :
$\Gamma^{\theta}_{\mu\nu,\theta} = \begin{pmatrix} 0 & 0 \\ 0 & -cos(2\theta) \end{pmatrix}$, $\Gamma^{\theta}_{\mu\nu,\phi} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\Gamma^{\phi}_{\mu\nu,\theta} = \begin{pmatrix} 0 & -\frac{1}{sin^2(\theta)} \\ -\frac{1}{sin^2(\theta)} & 0 \end{pmatrix}$, $\Gamma^{\phi}_{\mu\nu,\theta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Il suffit pour le reste d'injecter tout cela dans notre expression.
Nous obtenons :
$R^{\theta}_{\mu\nu\theta} = \begin{pmatrix} 0 & 0 \\ 0 & sin^2(\theta) \end{pmatrix}$, $R^{\phi}_{\mu\nu\theta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $R^{\theta}_{\mu\nu\phi} = \begin{pmatrix} 0 & -sin^2(\theta) \\ 0 & 0 \end{pmatrix}$,
$R^{\phi}_{\mu\nu\phi} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Nous pouvons remarquer que nous avons du coup sur seize valeurs, seules quatre sont non nulles :

$$R^{\theta}_{\phi\theta\phi} = +sin^2(\theta)$$

$$R^{\theta}_{\phi\phi\theta} = -sin^2(\theta)$$

$$R^{\phi}_{\theta\theta\phi} = 1$$

$$R^{\phi}_{\theta\phi\theta} = 1$$

Dès que les deux indices de débuts ou de fin sont les mêmes entre eux, on se retrouve avec une valeur nulle. C'est dû à l'antisymétrie intrinsèque du tenseur de Riemann.

A.3.4 Tenseur de Ricci

$$R_{\mu\nu} = R^{\theta}_{\mu\nu\theta} + R^{\phi}_{\mu\nu\phi} = R^{\theta}_{\mu\nu\theta} = \begin{pmatrix} 0 & 0 \\ 0 & sin^2(\theta) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & sin^2(\theta) \end{pmatrix}$$

A.3.5 Courbure scalaire :

Il y a une sommation implicite d'Einstein ici :

$$R = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = 1 + \frac{1}{sin^2\theta}sin^2(\theta) = 2$$

$$R = 2$$