Algebraic Brill-Noether Theory

Andrea Barbon

Radboud University Nijmegen
Vrije Universiteit Amsterdam
Universiteit Van Amsterdam

Master Degree in

Mathematics

Academic Year 2013-2014

Supervisor: B.J.J. Moonen

Second reader: R.M.H. de Jeu

Discussion Date: March 2014

Supervisor sign

1

Preliminaries and geometrical intuition

In this Chapter we introduce some of the ideas which form the foundations of the classical Brill-Noether Theory. We define the concepts of divisors and line bundles on a curve and, further, the Abel-Jacobi map which associates a line bundle to any effective divisor. Thanks to the Abel's Theorem, this function can be seen as a quotient map and the corresponding equivalence classes are the so called complete linear series, whose behaviour is one of the central themes of the Theory. Another key ingredient is the famous Riemann-Roch Theorem which, as we will see, describes the duality between a linear series and its residual one.
All of these concepts are rather abstract, nevertheless we will try to give some insight on their geometrical meaning, by proposing informal interpretations and by taking advantage of some pictures. We remark that such drawings only represent the real skeleton of the curve and, therefore, they should not be intended – in any way – as precise representations, but just as sketches which may help the reader to build some geometrical intuition.

1.1 Divisors and the Abel-Jacobi map

Let X be a smooth projective curve over an algebraically closed field k. Consider the set Div_X of divisors over X – i.e. the free abelian group on the points of X – and denote by EDiv_X the subset of effective divisors – i.e. the free monoid on the points of X. One can build an intuitive idea of effective divisors by thinking about the finite formal sum $D = \sum_i m_i P_i$ as a *book-keeping device* containing the points of intersection between X and another variety, where the (positive) coefficient m_i of each point P_i measures the multiplicity of the intersection, as Figure 1.1 shows.

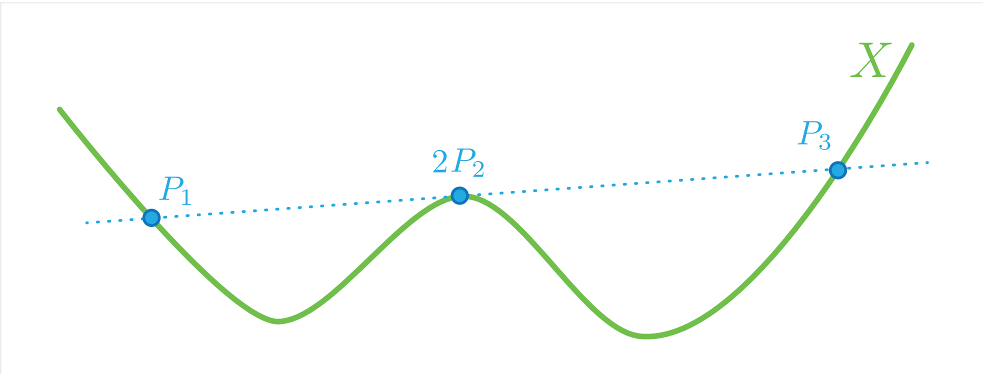

Figure 1.1: The effective divisor $P_1 + 2P_2 + P_3$ obtained as the intersection between the curve X and a line

Another way to obtain (not necessarily effective) divisors is to start from any non-zero rational map g defined locally on the curve and build a divisor by the recipe

$$\mathrm{Div}(g) = (g) = \sum_{P \in X} v_P(g) \cdot P$$

where $v_P(g)$ is the valuation of g at the point P given by the choice of a local uniformizer.

Remark 1.1. If the divisor of g is given by $(g) = \sum_i m_i P_i$ then, motivated by the definition of *local uniformizer*, we say that on the point P_i the map g presents a **zero** of order m_i if $m_i > 0$, and a **pole** of order m_i if $m_i < 0$.

Let $f \in k(X)^*$ be globally defined, then the associated divisor (f) is called **principal divisor** and the well known fact that $\deg(f) = 0$ can be explained, informally, by saying that a global rational map on a complete curve has the same number of zeros and poles counted with multiplicity.

The **degree** of a divisor is defined as the sum of its coefficient, so that it gives a group homomorphism from the set of divisors to $\mathbb{Z}$ and takes non negative values when restricted to effective divisors. We write X_d for the set of effective divisors of degree d.

1.3 The canonical map

The recipe we used to obtain divisor (g) from a locally defined rational map g can be extended to the define divisor associated to sections of arbitrary line bundles. In particular, given any global section ω of the cotangent bundle of X, we can pick an open cover $\bigcup_i U_i$ of X and a local uniformizer π_i for every U_i, then ω can be written locally on every U_i as $\omega = g_i \, d\pi_i$ and the corresponding divisor $(\omega) := \sum_i (g_i)$ is called **canonical divisor**.

Remark 1.4. All the canonical divisors on a curve of genus g are linearly equivalent and, as it follows from the Riemann-Roch Theorem, have degree $2g-2$. Notice that, if $g > 0$, a canonical divisor is not principal and its degree is non-zero.

Notation 1. *From now on we will abuse notation and write K both for any canonical divisor and for the cotangent bundle Ω^1_X, while the corresponding cohomology groups will be denoted by $H^i(K)$.*

Next, recall the definition of a base-point-free linear series which we will need for the following construction.

Definition 1.5. A g^r_d is said to be **base-point-free** if there is no point which is contained in the supports of all the divisors belonging to the linear series.

Any base-point-free g^r_d with linear series $\mathbb{P}V$ can be used to obtain a map of the curve X to a projective space, by considering the assignment

$$\phi_V : X \to \mathbb{P}V^* \qquad P \mapsto \{ \, s \in V \mid s(P) = 0 \, \} \ .$$

Notice that, since the g^r_d is base-point-free, the requirement $s(P) = 0$ gives a non trivial linear condition hence it defines an hyperplane of V. Therefore $\phi_V(P)$ can be seen as a point of the dual projective space $\mathbb{P}V^*$ parametrizing hyperplanes and, thus, ϕ_V is a well-defined function.

Definition 1.6. We extend the map ϕ_V to any effective divisor $D = \sum_{i=1}^d P_i$ with distinct points, by declaring $\phi_V(D) := \operatorname{span}\{\phi_V(P_1), \ldots, \phi_V(P_d)\}$.

We now assume that X has genus $g \geq 2$. Choose any canonical divisor K and recall that the **genus** of X is defined as $g := h^0(X, K)$, so that the complete linear series $|K|$ gives rise to a map $\phi_K : X \to \mathbb{P}^{g-1}$ which is called the **canonical map**.

It is easy to show that, if X is not hyperelliptic, this map is in fact an embedding and gives a canonical, preferred realization of our curve in a $(g-1)$-dimensional projective space. If the curve is hyperelliptic, instead, $\phi_K : X \to \mathbb{P}^{g-1}$ is not an embedding, but a 2 to 1 map exhibiting X as a double cover of a rational normal curve in $\mathbb{P}^{g-1}$.

$$\begin{tikzcd} X \arrow[r, hook, "\phi_K"] & \mathbb{P}^{g-1} \end{tikzcd}$$

(a) X not hyperelliptic

$$\begin{tikzcd} X \arrow[rr, "\phi_K"] \arrow[dr, "h"'] & & \mathbb{P}^{g-1} \\ & \mathbb{P}^1 \arrow[ur, hook, "v_{g-1}"'] & \end{tikzcd}$$

(b) X hyperelliptic

Assumption 1. *For the rest of the Chapter we will assume that X is not hyperelliptic and, for simplicity, we identify X with its isomorphic image $\phi_K(X)$.*

The most interesting feature of the canonical embedding is that, by construction, any global section of the cotangent bundle $\omega \in H^0(X, K)$ corresponds to a hyperplane $H_\omega \subset \mathbb{P}^{g-1}$ which intersects X precisely in the points forming the support of the divisor (ω), counted with the right multiplicity.

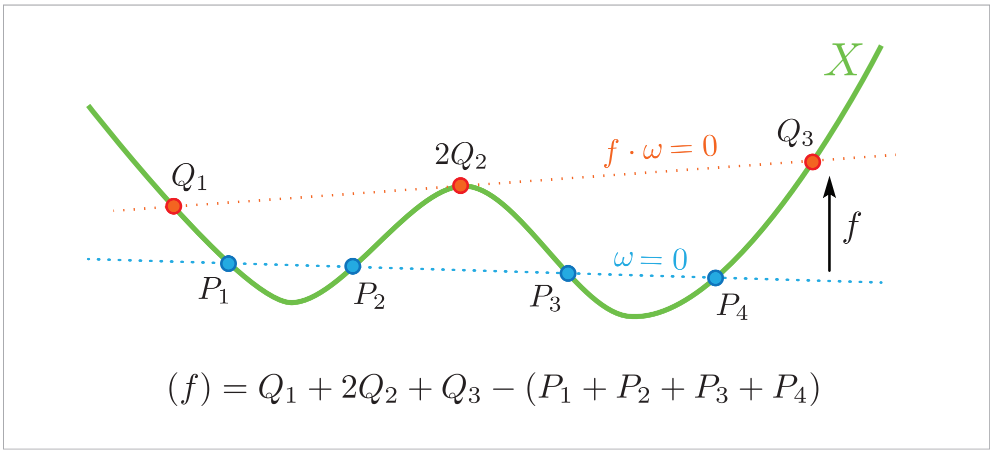

Figure 1.2: The canonical embedding of a non-hyperelliptic curve of genus $g = 3$ in the plane. The picture shows how two canonical divisors are linearly equivalent, being connected by a moving hyperplane

Given any rational map $f \in k(X)$, the product $f \cdot \omega$ is still an element of $H^0(X, K)$ and, therefore, it corresponds to another hyperplane $H_{f\omega}$ via the canonical embedding. Since the support of $(f \cdot \omega) = (f) + (\omega)$ is in general different from the one of (ω), we can interpret f geometrically as a transformation which *moves* the hyperplane H_ω to a different hyperplane $H_{f\omega}$. Hence we could informally say that, under the canonical embedding, two divisors are linearly equivalent if and only if *they are connected by a moving hyperplane.*

This intuitive idea is pictured in Figure 1.2, where an non-hyperelliptic curve of genus $g = 3$ is considered, so that ϕ_K embeds it into $\mathbb{P}^2$. Notice that, in this example, the linearly equivalent divisors $P_1 + P_2 + P_3 + P_4$ and $Q_1 + 2Q_2 + Q_3$ are both canonical and have degree $2g - 2 = 4$.

1.4 The Riemann-Roch Theorem

Given any divisor D on X, define its **residual divisor** or **dual divisor** as $D' := K - D$. Looking at the canonical embedding of our non hyperelliptic curve and considering a divisor D of degree $d < g$ consisting distinct points, we claim that we can always find a canonical divisor $K = (\omega)$ such that $D \leq K$. Indeed, we can always find a hyperplane of $\mathbb{P}^{g-1}$ passing through the $\phi_K(D)$, whose dimension is at most $d - 1$. In this setting we can think of the support of the residual D' as those points of intersection between the hyperplane and the curve which do not belong to the support of D – see Figure 1.3.

The famous Riemann-Roch Theorem 6.4 can be interpreted, thanks to Serre duality, as a statement on the relationship between a divisor and its residual. In fact, given a divisor $D \in \mathrm{Div}_X$ of degree d, the Riemann-Roch formula

$$h^0(D) - h^0(D') = d - g + 1$$

implies that the knowledge of the dimension of $H^0(D)$ is equivalent to that of the dimension of $H^0(D')$ and viceversa.

We highlight, moreover, that the statement of the Riemann-Roch is completely symmetric with respect to residual duality. Indeed, since the degree of the canonical divisor equals $2g - 2$ and the degree of D' is given by $d' = \deg(K) - d$, one can easily see that a completely equivalent formulation of the Theorem is

$$h^0(D') - h^0(D) = d' - g + 1$$

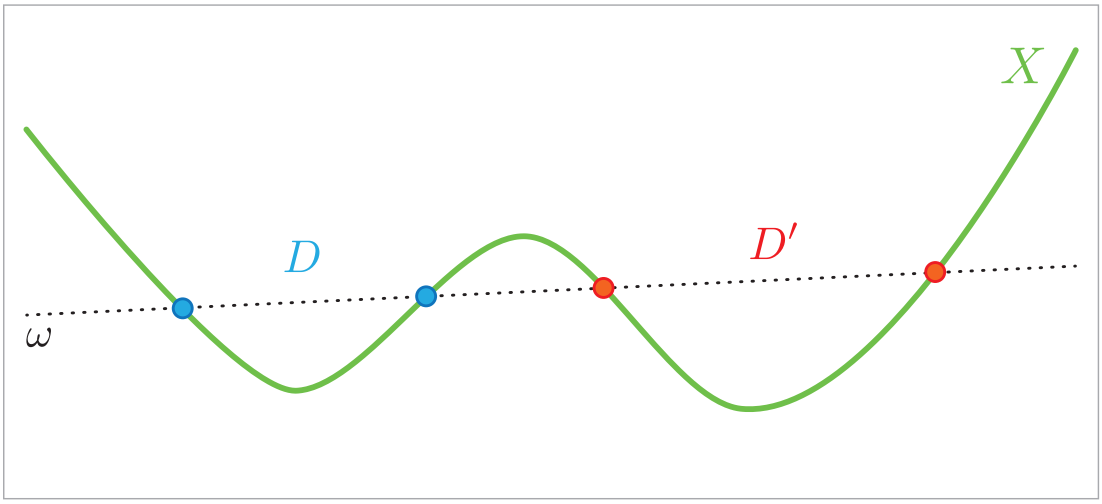

Figure 1.3: A divisor D of degree $d = 2$ and its residual, pictured in the canonical embedding of a genus 3 curve. In this particular case, since $\deg(\omega) = 2g - 2 = 4$, the degree d' of the residual divisor is 2, as well

thus showing that the Riemann-Roch does not discriminate between a divisor and its residual. This implies that the information we can get from the behaviour of a given divisor can be equivalently obtained by looking at its residual and viceversa, a fact which will be exploited to draw Figure 1.4.

The relationship between the Theorem and linear series is therefore well understood by means of the above mentioned canonical isomorphism $\mathbb{P}H^0(D) \cong |D|$ which identifies the complete linear series of D with the projectification of the space of global sections of $\mathcal{O}_X(D)$. Therefore, using the standard notation $r(D) := \dim |D|$ for the dimension of $|D|$, we can rewrite the Riemann-Roch formula as

$$r(D') - r(D) = d' - g + 1 \tag{1.1}$$

thus making it clear that it can be interpreted as a statement on the *dimension spread* between a linear series and its residual.

The Riemann-Roch Theorem is an extremely useful result, with applications ranging from pure mathematics to applied graph theory and even communication engineering. But what is the geometrical meaning of the Riemann-Roch formula? We will try to answer this fascinating question in the following Sections.

1.5 Special exceptional divisors

Some divisors on the curve X – and the corresponding linear series – are *more important* than others, in the sense that they contain more information about the curve itself.
A good reason to study the so called **special exceptional divisors** of a given curve, as we will exemplify in Section 1.7, is that the behaviour of the corresponding linear series may help distinguish the curve among other curves.
Put $r = r(D)$. For a curve of genus g the region of special exceptional divisors is given, as a subset of the (d, r)-plane, by the inequalities

$$r > 0 \qquad \text{and} \qquad 2r < d < g$$

as we picture in Figure 1.4. The reasons for these constraints are the following:

- If $r = 0$ then the linear series $|D|$ is trivial;
- Due to the duality involved in the Riemann-Roch formula, we can restrict our attention to linear series of degree $d < g$;
- For non trivial linear series of degree $d < g$, Clifford's Theorem 5.5 gives the upper bound $r < d/2$.

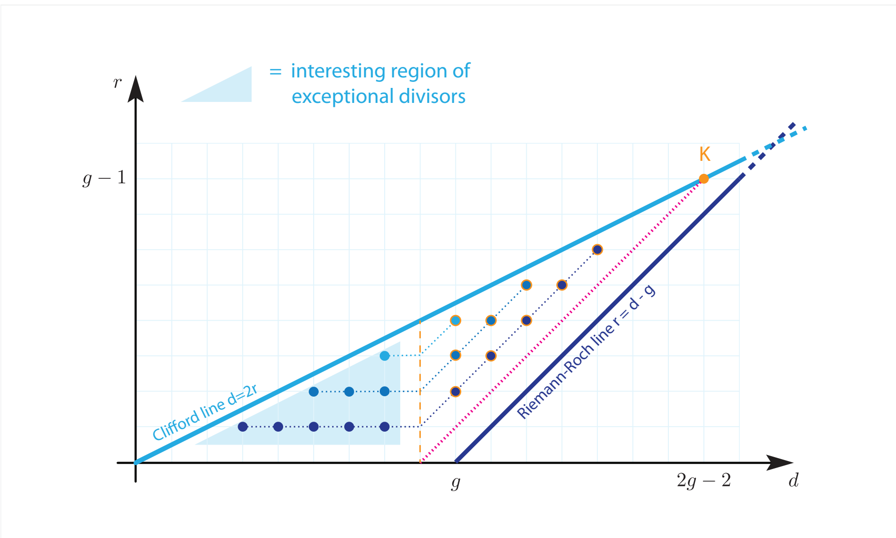

Figure 1.4: The region of Special Exceptional divisors for a curve of genus $g = 9$

1.6 Geometrical interpretation

In the following we will give some insights into the geometrical meaning of the Riemann-Roch formula. In order to do so, we start by introducing the cup-product homomorphism

$$\mu_0 : H^0(D) \otimes H^0(K-D) \longrightarrow H^0(K)$$

which is also know as the **Petri's map** and, as we will see in Chapter 3, plays a fundamental role in the linear approximation of the varieties X^r_d and W^r_d parametrizing linear series.

Consider an effective divisor D of degree $d < g$ consisting of distinct points and notice that the vector space $H^0(K-D)$ can be interpreted as the linear subspace consisting of those $\omega \in H^0(K)$ such that $(\omega) \geq K$ or, in other words, $\mathbb{P}H^0(K-D)$ parametrizes the hyperplanes of $\mathbb{P}^{g-1}$ which cut X in a set of points containing D. Among these hyperplanes there is a unique one passing through the support of the residual D', which can be identified with the unique generator of $H^0(K-D-D') \cong H^0(\mathcal{O}_X)$.
Now, by restricting the Petri's map to $H^0(D) \otimes H^0(K-D-D')$, we get a map

$$\mu_0 : H^0(D) \otimes H^0(K-D-D') \longrightarrow H^0(K-D')$$

whose target space corresponds (up to scalar multiplication) to the hyperplanes cutting the curve in a set of points containing the support of D'. From the natural isomorphism $H^0(K-D') \cong H^0(D)$ we deduce that the number of such hyperplanes is given by the integer $r(D) = \dim \mathbb{P}H^0(D)$, a fact which can be used to shade light on the geometrical interpretation of the Riemann-Roch . Indeed, one can equivalently rewrite the formula (1.1) as

$$r(D) = [\ g-1\ ]\ -\ [\ d' - r(D')\ ]$$

and observe that $g-1$ is nothing but the dimension of the projective space $\mathbb{P}^{g-1}$ in which X is canonically embedded. Therefore we see that $d' - r(D')$ equals the number of linearly independent points of the support of D' and, consequently, that the integer $r(D')$ counts the number of independent linear relations among these points. Hence the Riemann-Roch is *simply* telling us that the complete linear series $|D|$ can be geometrically visualized as a family of hyperplanes passing through $\phi_K(D')$, each one of them cutting X in a set of points consisting of D' plus a divisor in $|D|$.

Therefore we understand how the fact that the *dimension spread* $r(D) - r(D')$ is a constant depending on d and g has a clear geometrical meaning: a higher number $r(D')$ of independent linear relations among the support of D' corresponds to a larger family of hyperplanes passing through $\phi_K(D')$, the dimension of this family being precisely the dimension $r(D)$ of the linear series $|D|$.

In the next page, we show two pictures that might help the reader visualizing the geometrical meaning of $r(D)$, in the context of the canonical embedding of a curve of genus 4. Figure 1.6 presents an example of a g^3_1, with one linear relation among the 3 points of D, while Figure 1.7 shows a trivial linear series, where no linear relations among the points of D is present.

Moreover, notice that the above geometrical interpretation allows to exclude some otherwise possible scenarios. For instance, the canonical embedding of a genus 4 curve cannot appear as pictured in Figure 1.5, because this would correspond to the values $d = 2$, $d' = 4$ and $r(D) = 0$, $r(D') = 2$ which do not satisfy the Riemann-Roch formula.

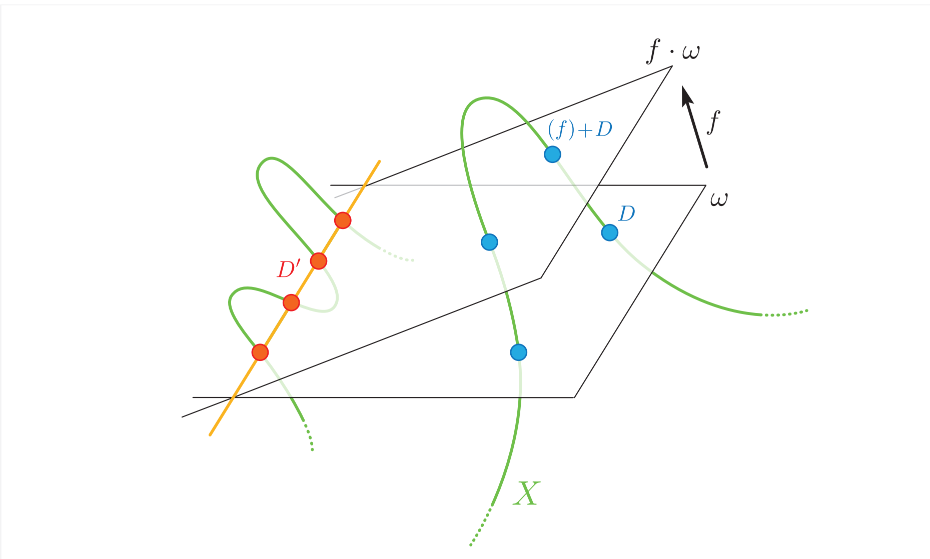

Figure 1.5: An hypothetical curve of genus 4 embedded in $\mathbb{P}^3$, showing a g^0_2 whose residual series is a g^2_4. This situation is actually impossible, as one can deduce form the Riemann-Roch Theorem

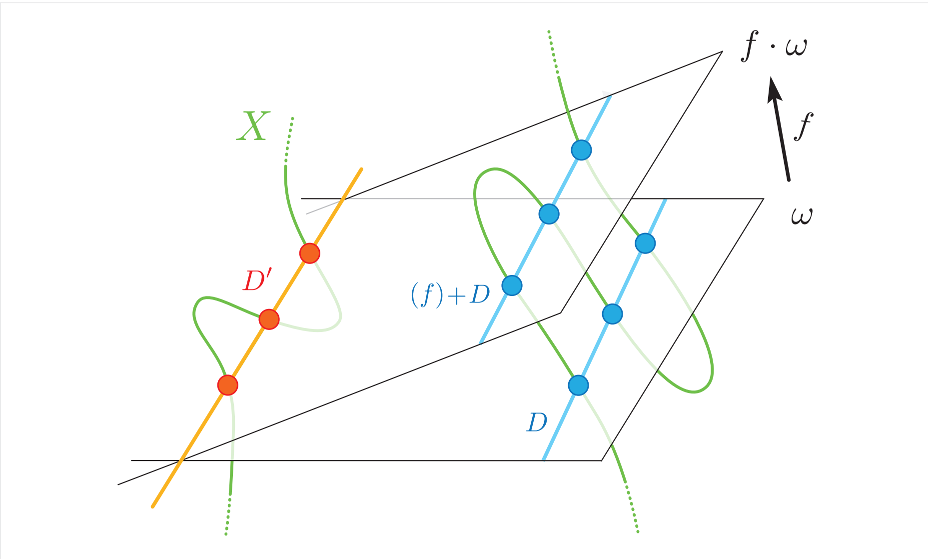

Figure 1.6: An example of a complete linear series on a curve of genus 4 canonically embedded into $\mathbb{P}^3$. The global section $f \in H^0(D)$ has poles only on the points of D, hence the hyperplane H_ω associated to $\omega \in H^0(K - D - D')$ is moved away from D by f, but stays on the points of the residual divisor D'. Notice that, for this particular choice of D, there is one linear relation among the points of both D and D', so that $r(D) = r(D') = 1$ and each divisor gives rise to a g^1_3.

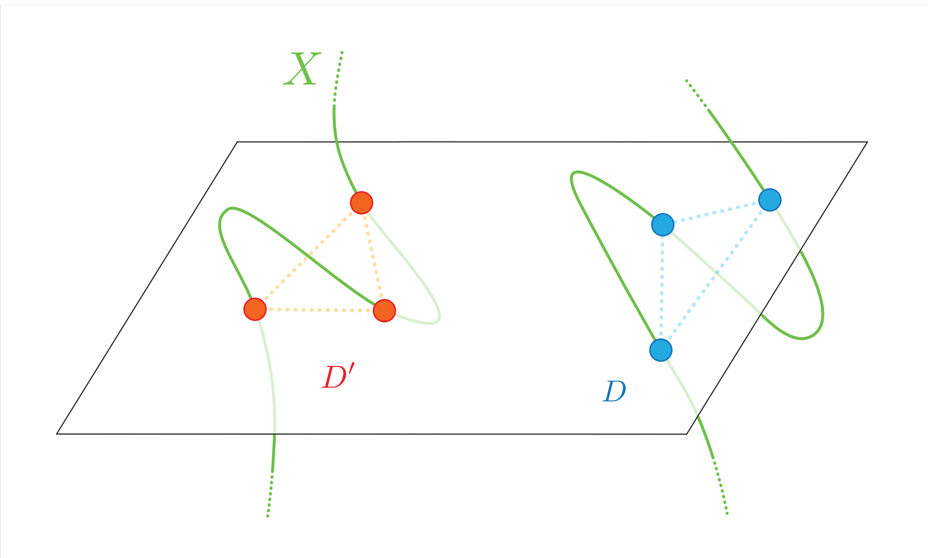

Figure 1.7: In this example a divisor D of degree 3 gives rise to a trivial linear series. The reason is that the points of D' are linearly independent, therefore there is only one hyperplane passing through $\phi_K(D + D')$.

1.7 Examples

In this Section we will analyse the case of a non-hyperelliptic curve X of genus 4 in which, as it follows from the discussion of Section 1.5, the only special exceptional linear series – if any exists – are g_1^3. Actually, the Existence Theorem 4.3 states that whenever the **Brill-Noether number**

$$\rho(g, d, r) := g - (r + 1)(g - d + r)$$

is non-negative, then there exists at least one g_d^r on the curve. In our case we find $\rho(4, 3, 1) = 0$ and, as a consequence we deduce that our curve admits a g_3^1.
It is a well known fact in algebraic geometry that any smooth projective curve of genus 4 comes as the complete intersection of a quadric surface with a cubic surface inside $\mathbb{P}^3$ and, moreover, that any quadric of $\mathbb{P}^3$ is (up to projective equivalence) a ruled surface. Hence we see that there are two distinct possibilities:

i) If the quadric is smooth, then it is a saddle surface, doubly-ruled by two families of perpendicular lines;

ii) If the quadric is singular, then it is a conic surface and there is only one family of ruling lines, all passing through the singular point.

Let us start by looking at the first case of a curve on a smooth quadric surface.

Example 1. Up to projective equivalence, the smooth quadratic surface Q corresponds to the equation

$$X_0X_3 = X_1X_2$$

and it is naturally isomorphic to the product $\mathbb{P}^1 \times \mathbb{P}^1$ of two projective lines. The double ruling of Q is given by two $\mathbb{P}^1$ families of lines

$$A = \begin{cases} X_0 = aX_1 \\ X_2 = aX_3 \end{cases} \qquad \text{and} \qquad B = \begin{cases} X_0 = bX_2 \\ X_1 = bX_3 \end{cases}$$

where the parameters a and b vary in $\mathbb{P}^1$. It is easy to check that any two lines $L_\alpha \in A$ and $L_\beta \in B$ intersect in the unique point $[\alpha\beta, \beta, \alpha, 1]$ and, hence, their span is a plane $H_{\alpha\beta} = \mathrm{Span}(L_\alpha, L_\beta)$. Suppose that such a plane cuts X on the effective divisor

$$H_{\alpha\beta} \cdot X = D_\alpha + D_\beta, \qquad D_\alpha \in L_\alpha,\ D_\beta \in L_\beta$$

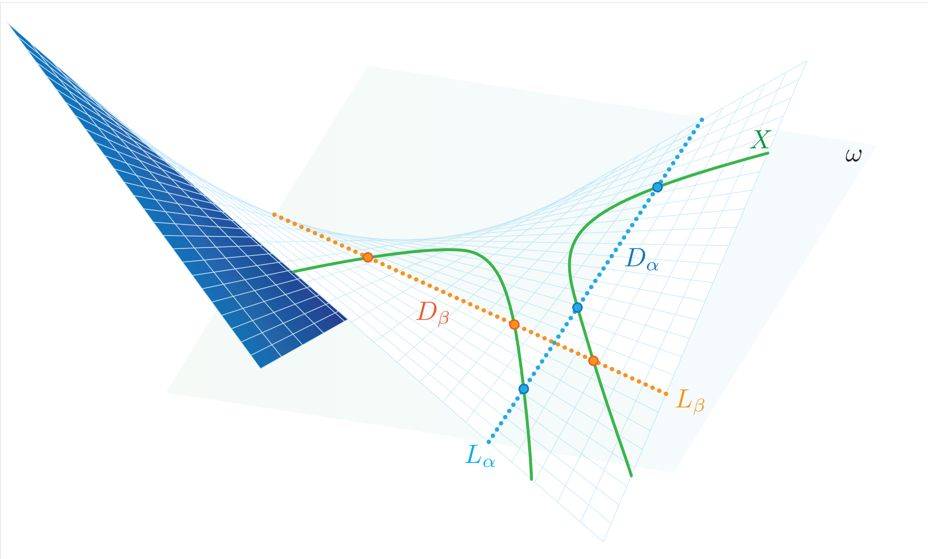

Figure 1.8: A genus 4 curve contained in a doubly-ruled smooth quadratic surface – a hyperbolic paraboloid. The blue dots form the support of a divisor D_α contained in a g^1_3, while the orange ones form the support of the residual D_β belonging to the other g^1_3

and recall that, since $H_{\alpha\beta}$ is a hyperplane of $\mathbb{P}^3$, the divisor $D_\alpha + D_\beta$ is canonical of degree $2g-2=6$. Each line L_α and L_β moves in a $\mathbb{P}^1$-family, so it follows that

$$r(D_\alpha) \geq 1 \qquad \text{and} \qquad r(D_\beta) \geq 1$$

and, as a consequence, Clifford's Theorem 5.5 implies that both $\deg(D_\alpha) \geq 2$ and $\deg(D_\beta) \geq 2$. But, since X is not hyperelliptic, these inequalities are actually strict and from $\deg(D_\alpha + D_\beta) = 6$ we deduce $\deg(D_\alpha) = \deg(D_\beta) = 3$, thus another application of the Clifford's Theorem ensures that

$$r(D_\alpha) = r(D_\beta) = 1\,.$$

Hence we conclude that the linear series $|D_\alpha|$ and $|D_\beta|$ are both g^3_1 or, in other words, X admits two triple covers of $\mathbb{P}^1$ obtained by projecting in the directions of L_α and L_β. This situation is pictured in Figure 1.6, where the orange and the blue lines belong to distinct families of lines and cut a pair of residual divisors.

Example 2. Up to projective equivalence, the singular quadratic surface Q corresponds to the equation

$$X_0^2 = X_1 X_2$$

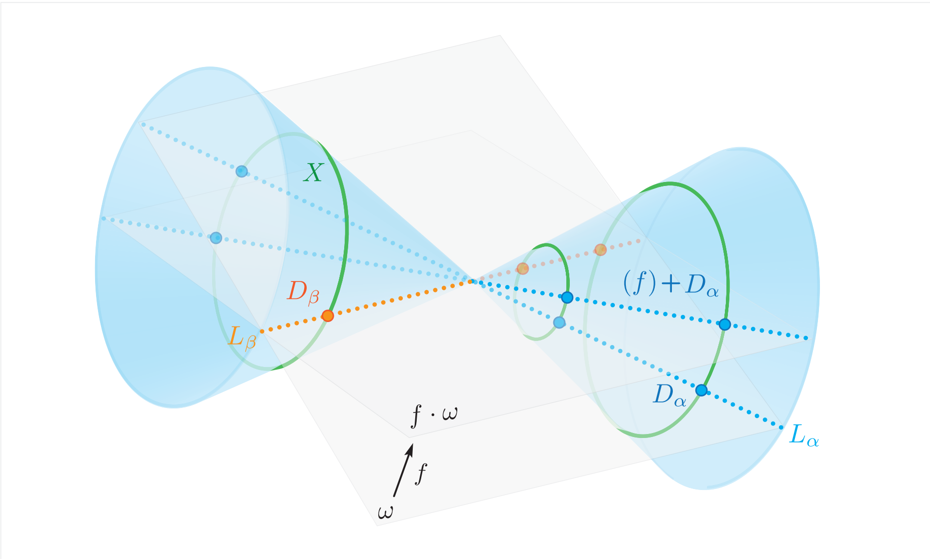

Figure 1.9: A genus 4 curve contained in a singular ruled quadratic surface – namely a cone. There is a $\mathbb{P}^1$-family of hyperplanes passing through $\phi_k(D')$, each one cutting the curve in D' plus a divisor belonging to the unique g_3^1. Notice that, in contrast with Figure 1.8, every divisor of the g_3^1 can be obtained by rotating a plane on the orange axis

and can be viewed as the union of a $\mathbb{P}^1$ family of lines, parametrized by a plane conic, which can be described as

$$A = \begin{cases} X_0 = aX_1 \\ X_2 = a^2 X_1 \end{cases}$$

where the parameter a varies in $\mathbb{P}^1$. It is easy to check that any two lines L_α, $L_\beta \in A$ intersect in the singular point $[0,0,0,1]$ of Q and, hence, that their span is a plane $H_{\alpha\beta} = \mathrm{Span}(L_\alpha, L_\beta)$. Again, such a plane cuts X on the canonical divisor

$$H_{\alpha\beta} \cdot X = D_\alpha + D_\beta, \qquad D_\alpha \in L_\alpha,\ D_\beta \in L_\beta$$

and we realize that the situation is closely related to the one of Example 1. Reasoning in a similar way one can check that both D_α and D_β give rise to a g_3^1, but the crucial difference from the previous Example is that this time there is a unique family of lines and, as a consequence, the two linear series coincide: $|D_\alpha| = |D_\beta|$.
The situation is pictured in Figure 1.7, with a $\mathbb{P}^1$ family of planes *rotating around* $\phi_K(D')$, where each plane intersects the curve in a set of points consisting of D' and a divisor in $|D|$. The reader should notice that a rotation of H_ω by 90 degrees gives a plane which is tangent to the cone Q and which cuts X in the divisor $2D'$, thus showing that also D' belongs to the linear series $|D|$.

We therefore see that, in the case of a genus 4 curve, it is sufficient to count the number of g^1_3's to be able to distinguish between the two possible scenarios described above. More precisely, the variety G^1_3 parametrizing linear series of degree 3 and dimension 1 is zero dimensional in both cases, but in Example 1 it consists of 2 distinct points, while it *degenerates* to a unique point in the case of Example 2.
It is important to remark that the cup-product homomorphism μ_0 is injective in the situation of Example 1, while it presents a 1-dimensional kernel in the *degenerate* situation of Example 2. As we will explain in Chapter 3, this is an instance of the general fact that a non trivial kernel of μ_0 indicates the presence of singularities in the moduli varieties.

associated to the universal divisor, which is in some sense the global analogue of the second sequence appearing in Remark 2.11. Since D is 0-dimensional and $h^1(D_D) = 0$, Proposition 5.3 of Appendix A implies that $R^1\pi_*\mathcal{O}_\Delta(\Delta) = 0$, therefore the last part of the direct image sequence of 2.6 is given by

$$\pi_*\mathcal{O}_\Delta(\Delta) \xrightarrow{\delta} R^1\pi_*\mathcal{O}_Z \to R^1\pi_*\mathcal{O}_Z(\Delta) \to 0.$$

As we showed in Section 2.5, for every $D \in X_d$ the cohomology groups

$$H^0(X, D_D) \cong T_D X_d \qquad \text{and} \qquad H^1(X, \mathcal{O}_X) \cong T_{\mathcal{O}_X(D)} \operatorname{Pic}^d_X$$

are fiberwise vector spaces of dimensions d and g. Hence Proposition 5.3 implies that $\pi_*\mathcal{O}_\Delta(\Delta)$ is locally free of rank d and that $R^1\pi_*\mathcal{O}_Z$ is locally free of rank g, thus showing that (2.4) is in fact a **free** presentation.

Moving towards the second sequence, notice that we have the natural short exact sequence

$$0 \to \mathscr{L} \to \mathscr{L}(\Gamma) \to \mathscr{L}(\Gamma)/\mathscr{L} \to 0 \tag{2.7}$$

and, since by Lemma 5.5 of Appendix A we have $R^1\nu_*\mathscr{L}(\Gamma) = 0$, the corresponding direct image sequence is given by

$$0 \to \nu_*\mathscr{L} \to \nu_*\mathscr{L}(\Gamma) \xrightarrow{\gamma} \nu_*\mathscr{L}(\Gamma)/\mathscr{L} \to R^1\nu_*\mathscr{L} \to 0\,. \tag{2.8}$$

Moreover, since for every $L \in \operatorname{Pic}^d_X$ the Riemann-Roch implies $h^0(X, L(M)) = d + m - g + 1$, another application of Proposition 5.3 implies that $\nu_*\mathscr{L}(\Gamma)$ is locally free of rank $d + m - g + 1$. Finally, we notice that $\mathscr{L}(\Gamma)/\mathscr{L}$ can be seen as a line bundle on Γ, so it follows that $\nu_*\mathscr{L}(\Gamma)/\mathscr{L}$ is locally free of rank m, as ν restricts to a finite locally free morphism of degree m on $\Gamma \subset X \times \operatorname{Pic}^d_X$. □

Remark 2.21. During the proof of the above Lemma we showed that the ranks of the first two bundles appearing in the presentation (2.5) are $d + m - g + 1$ and m. Keep it in mind, because this fact will be exploited during the proof of the Connectedness Theorem.

Collecting the results of this Chapter, we are now able to write down a commutative diagram in the category of coherent sheaves which is in some sense the *global version* of (2.3) and, thus, gives an identification of the coboundary map $\delta : \pi_*\mathcal{O}_\Delta(\Delta) \to R^1\pi_*\mathcal{O}_Z$ with the morphism of locally free sheaves $Tu : TX_d \to u^*T \operatorname{Pic}^d_X$, representing the tangent map of the Abel-Jacobi map u restricted to divisors of degree d.

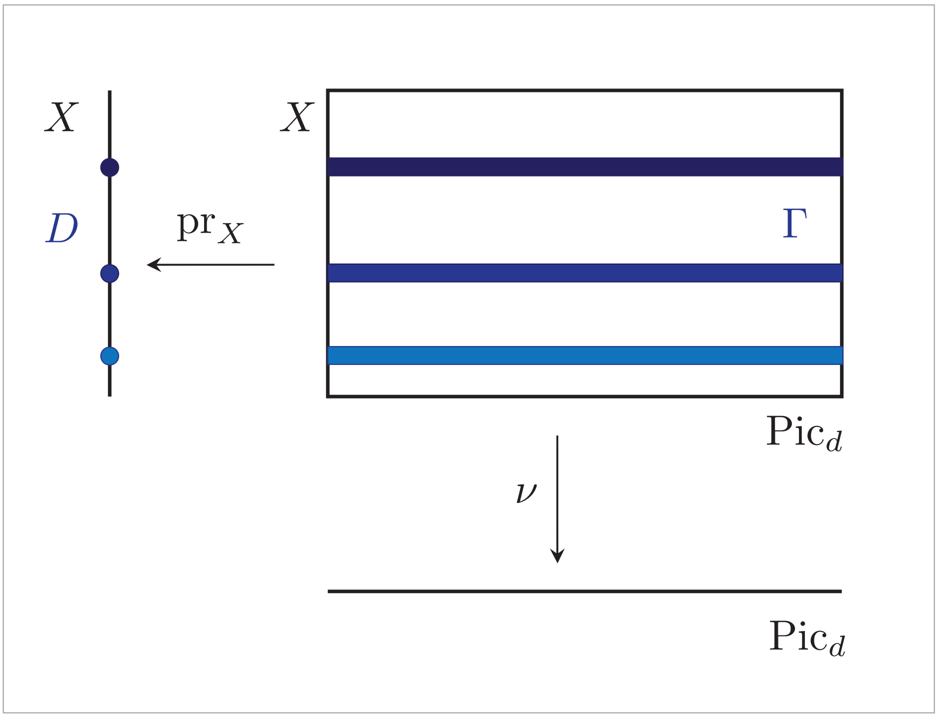

Figure 2.1: The projection ν restricts to a finite locally free morphism of degree m on the product $\Gamma \subset X \times \operatorname{Pic}_X^d$

$$\begin{tikzcd}
\pi_*\mathcal{O}_Z(\Delta) \arrow[r] & \pi_*\mathcal{O}_\Delta(\Delta) \arrow[r, "\delta"] \arrow[d, "\cong"'] & R^1\pi_*\mathcal{O}_Z \arrow[r] \arrow[d, "\cong"] & R^1\pi_*\mathcal{O}_Z(\Delta) \\
 & TX_d \arrow[r, "Tu"] & u^*T\operatorname{Pic}_X^d &
\end{tikzcd}$$

The commutativity of this *global* diagram follows from the commutativity of its fiberwise counterparts – achieved in Proposition 2.18 – together with the fact that the sheaves appearing in the central square are locally free, as we observed in Remark 2.15.

and we can therefore apply Lemma 5.9 of Appendix A to get a line bundle F over X_d such that

$$u^*\mathscr{L} \;\cong\; \mathcal{O}_Z(\Delta) \otimes \pi^* F\,,$$

where $\pi : X \times X_d \to X_d$ is the natural projection map. Finally, from Corollary 3.10 we get a natural isomorphism

$$u^* R^1\nu_*\mathscr{L} \;\cong\; R^1\pi_*\mathcal{O}_Z(\Delta) \otimes F$$

and, since $\square \otimes F$ is a right-exact functor and does not affect Fitting ideals, we therefore conclude that

$$\mathrm{Fitt}(u^* R^1\nu_*\mathscr{L}) \;\cong\; \mathrm{Fitt}(R^1\pi_*\mathcal{O}_Z(\Delta))\,,$$

as desired. □

We will now show that the support of X_d^r consists of divisors with rank at least r. To start, recall from Proposition 2.20 that the terms appearing in (2.4) are locally free sheaves over X_d, and the first two have rank respectively d and g, so that with respect to the notation used in the identities (3.1) we have

$$e = d, \qquad f = g \qquad \text{and} \qquad s = g - d + r. \tag{3.3}$$

Moreover from Proposition 2.18 we know that we can identify δ with Tu and thus, exploiting the above mentioned identities , we find

$$\mathrm{Supp}(X_d^r) = \mathrm{Fiber}_{(r)}(Tu) = \{\; D \in X_d \mid r(D) \geq r \;\}\,.$$

Since from Proposition 3.11 it follows in particular that u maps X_d^r onto W_d^r, we therefore immediately see that the support of W_d^r is given by

$$\mathrm{Supp}(W_d^r) = \left\{\; L \in \mathrm{Pic}_X^d \mid r(L) \geq r \;\right\}$$

3.3 Dimensional lower bounds

First of all let us introduce to the reader the so called Brill-Noether number, which will turn to be a crucial ingredient of Brill-Noether theory.

Definition 3.12. Let d, g and r be natural numbers. The **Brill-Noether number** is defined as

$$\rho = \rho(d, g, r) = g - (r+1)(g - d + r)$$

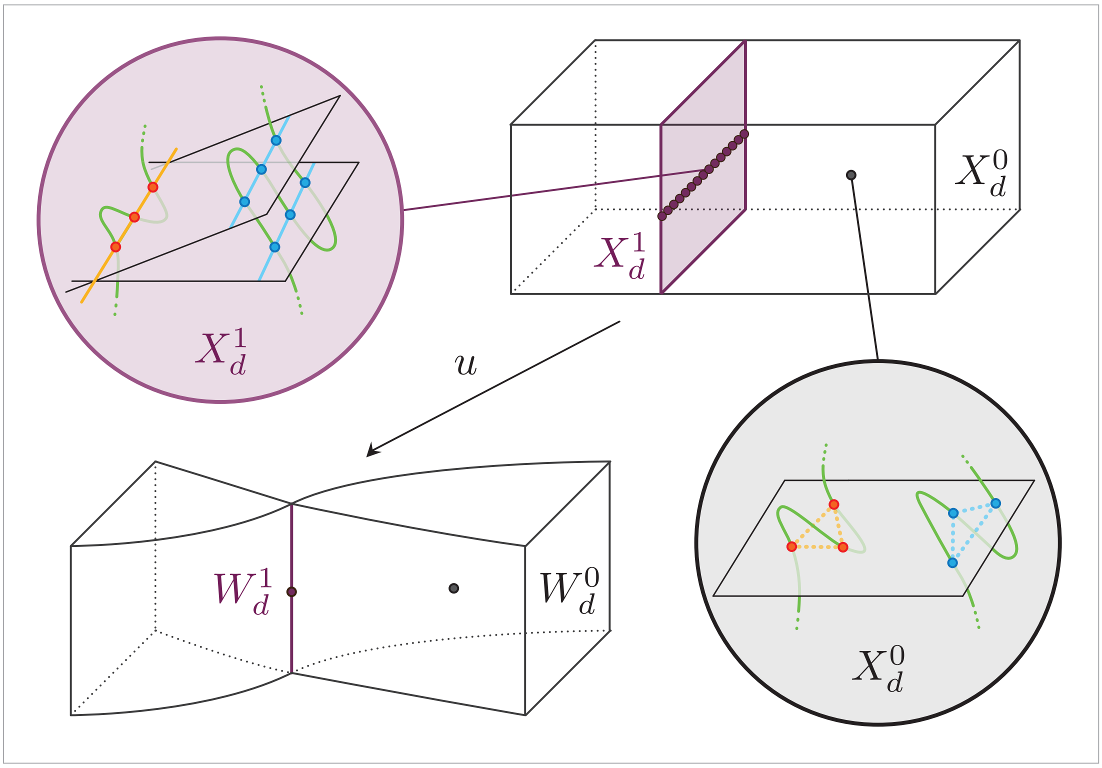

Figure 3.1: An intuitive picture of the Degeneracy loci of the Abel-Jacobi map u.

Let now G be a finitely presented sheaf which admits the free presentation

$$E \xrightarrow{\varphi} F \to G \to 0$$

with E of rank e and F of rank f. Applying Theorem 5.1 of Appendix A to G we get

$$\text{height}(\text{Fitt}_t(G)) \leq (e - f + t + 1)(t + 1).$$

In the case of X_d^r, the sheaves appearing in the presentation (2.4) have rank d and g as in (3.3). Hence, denoting by I the (Fitting) ideal sheaf of X_d^r we have

$$\text{height}(I) \leq r(g - d + r).$$

Since X_d is a variety over a field, it is a catenary scheme and we have the equality $\operatorname{codim} X_d^r + \dim X_d^r = \dim X_d$. Thus, recalling that $\dim X_d = d$ we get the lower bound

$$\dim X_d^r \geq d - r(g - d + r) = g - (r + 1)(g - d + r) + r = \rho + r$$

Dimension Theorem. *Let X be a smooth projective curve of genus g over the complex numbers and fix integers $d \geq 1$ and $r \geq 0$. Then the variety G^r_d is empty if $\rho < 0$, while it is reduced of pure dimension ρ if $\rho \geq 0$.*

The Dimension Theorem was out of the scope of this thesis, nevertheless our intuition suggests that such a result should remain valid over a more general algebraically closed field k. In any case we would like to highlight the fact that the Dimension Theorem allows to further restrict the region of special exceptional divisors, thus obtaining a refined version of Figure 1.4, as showed below

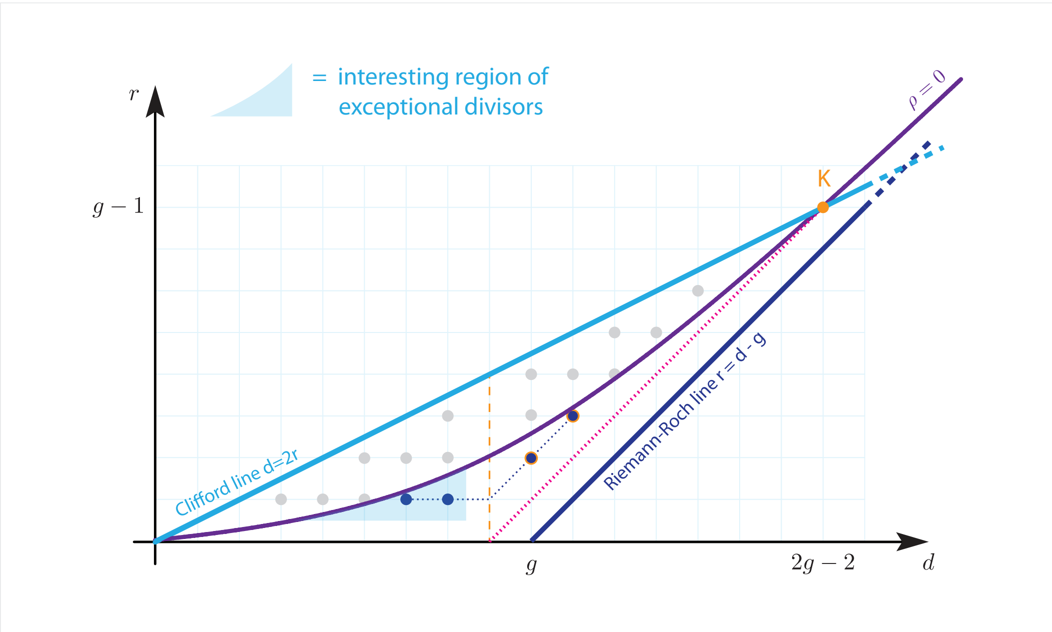

Figure 4.1: The interesting region of exceptional special divisors in the case of a curve of genus $g = 9$, further refined exploiting the Dimension Theorem

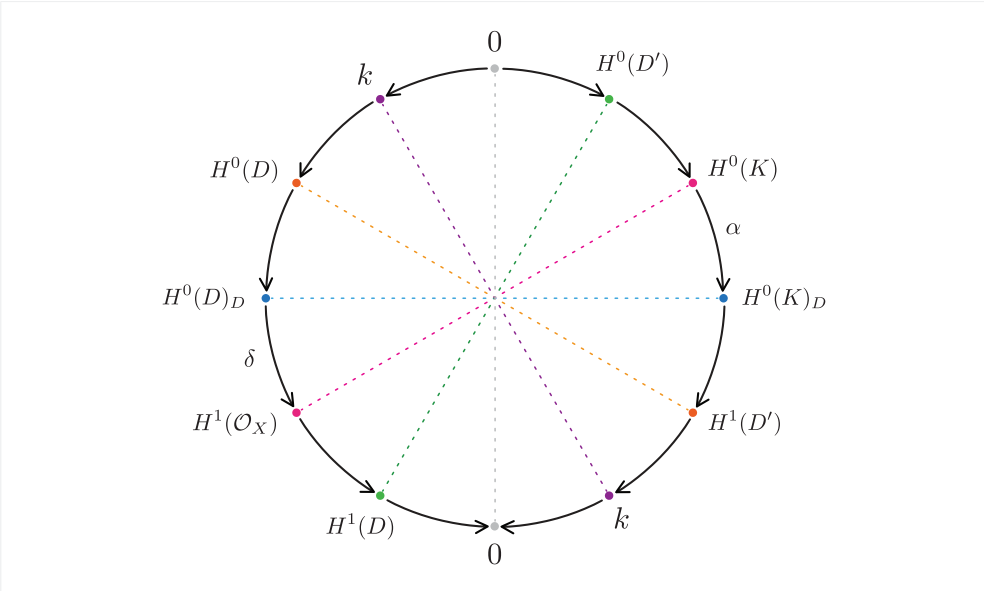

Figure 6.1: A more *artistic* representation of the duality involved in diagram (6.2)

6.4 Riemann-Roch Theorem

Using Serre duality we immediately get the final version of the theorem.

Riemann-Roch Theorem. *Let X be a complete curve of genus g, K a canonical divisor and $D \in X_d$. Then*

$$h^0(D) - h^0(K - D) = d - g + 1 \tag{6.4}$$

Notice that, plugging-in the canonical class K in the Riemann-Roch formula and recalling that by definition $h^0(K) = g$, we get

$$\deg(K) = 2g - 2$$

which, since K is the dual of the tangent bundle, is an analogue of the Hopf index theorem for Riemann surfaces.

6. APPENDIX B: SERRE DUALITY AND RIEMANN-ROCH