Exercise 42

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1 Part A

Let $E, F \in \mathcal{F}$. We can find some linear basis e_j, f_j s.t.

$$E_j = \text{span}(e_j) \qquad \text{and} \qquad F_j = \text{span}(f_j)$$

2 Part B

Let $m = (m_1, m_2) \in M$. Since M does not contain the origin, we can assume $m_1 \neq 0$ (if $m_1 = 0$ than $m_2 \neq 0$, and we can just interchange the names of m_1 and m_2). We will prove that there exists an A-invariant neighborhood U of m such that the restriction of the map

$$\varphi : M \times A \to M \times M, \qquad ((m_1, m_2), t) \mapsto ((m_1, m_2), (e^t m_1, e^{-t} m_2))$$

to $U \times A$ is proper. We can define U as following

$$U := \begin{cases}]0, +\infty[\times] - \infty, +\infty[& \text{if} \quad m_1 > 0 \\ ] - \infty, 0[\times] - \infty, +\infty[& \text{if} \quad m_1 < 0 \end{cases}$$

and it is clear that such set is an open neighborhood of m and it is invariant under the action of A. We can now define the map

$$\psi : U \times U \to U \times A, \qquad ((m_1, m_2), (x, y)) \mapsto ((m_1, m_2), \log(|x|) - \log(|m_1|))$$

which is well-defined, because $m_1 \neq 0$ and $x \neq 0$. We easily see that this is a local inverse of $\varphi_{|U \times A}$, infact we have

$$\begin{aligned} \psi \circ \varphi((m_1, m_2), t) &= \psi((m_1, m_2), (e^t m_1, e^{-t} m_2)) \\ &= ((m_1, m_2), \log(|e^t m_1|) - \log(|m_1|)) \qquad \text{since} \quad e^t > 0 \\ &= ((m_1, m_2), \log(e^t |m_1|) - \log(|m_1|)) \\ &= ((m_1, m_2), \log(e^t) + \log(|m_1|) - \log(|m_1|)) \\ &= ((m_1, m_2), \log(e^t)) \\ &= ((m_1, m_2), t) \end{aligned}$$

and, restricing to the image $\varphi(U \times A)$,

$$\begin{aligned}
\varphi \circ \psi((m_1, m_2), (e^t m_1, e^{-t} m_2)) &= \varphi((m_1, m_2), \log(|e^t m_1|) - \log(|m_1|)) \\
&= \varphi((m_1, m_2), t) \\
&= ((m_1, m_2), (e^t m_1, e^{-t} m_2)).
\end{aligned}$$

Hence we see that $\varphi_{|U \times A}$ has a continuous inverse, which maps compact sets to compact sets. We therefore conclude that $\varphi_{|U \times A}$ is proper. Moreover from *Part A* we know that the action is free, so by *Theorem 13.5* we conclude that the restriction of A to U is of principal fiber bundle type.

3 Part C

Consider the compact set $\{(m_1, 0)\} \subset M$, where $m_1 \neq 0$. It's easy to see that the set

$$A \cdot C = \{(e^t m_1, 0) \mid t \in A\} = \{(x, 0) \mid x \in \mathbb{R}_{>0}\}$$

is not closed. M is locally compact, so we can apply *lemma 11.8* and conclude that the quotient topology A/M is not Hausdorff.

4 Part D

In view of *Lemma 11.8*, the condition (b) of *Definition 12.2* is equivalent to the quotient A/M being Hausdorff. In item (c) we showed that this is not the case, hence the action of A on M is not of principal fiber bundle type.