

Exercise 42

Andrea Barbon - VU 2206157

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1 Part A

Let $E, F \in \mathcal{F}$. We can find some linear basis e_j, f_j s.t.

$$E_j = \text{span}(e_j) \quad \text{and} \quad F_j = \text{span}(f_j)$$

and the map g defined by $e_j \mapsto f_j$ is an element of G , and has the property that $g \cdot E = F$. We therefore see that the action of α is transitive.

2 Part B

From what we saw in Part A it is clear that the stabilizer is given by

$$P = \{\text{diag}(M_j) \mid M_j \in \text{GL}(d_j, \mathbb{K}), j = 1, \dots, k\}.$$

Using the trivial global chart $\chi : \mathbb{R}^{n^2} \rightarrow G$ of G one can see (after a rearranging of the basis of \mathbb{R}^{n^2}) that P can be described by

$$P = \chi(\mathbb{R}^{n^2} \cap \{x_1 = \dots = x_l = 0\})$$

where the coordinates set to zero are the ones outside the diagonal blocks. It follows by definition that P is a submanifold of G , and applying Theorem 9.1 we conclude that P is a closed subset of G . Obviously P is a subgroup of G , hence a closed subgroup as required.

3 Part C

We can give \mathcal{F} a group structure by declaring $\phi(g) \star \phi(h) = \phi(gh)$. The product \star is well defined on the whole domain since from Part A it follows that ϕ is surjective, and it is defined in the right way to make ϕ a homomorphism of groups. We can then apply the isomorphism theorem for groups to conclude that such a bijection $\bar{\phi}$ is induced.

4 Part D

To see tht $\phi(K) = \mathcal{F}$ we look at what we did in Part A, and we observe that using the Grant-Schmidt process we can make all the e_j and the f_j orthonormal, thus making g an orthogonal tranformation. Hence $\forall F \in \mathcal{F}$ we can find a map $g \in K$ s.t. $\phi(g) = F$, or in other word $\phi(K) = \mathcal{F}$.

From above it follows that the action of K on \mathcal{F} is transitive, and it is clear that $H = \ker(\phi|_K) = K_E$. Hence we can apply Proposition 15.5 the get the desired diffeomorphism.

Finally we know that K is compact and so it is its quotient K/H . Therefore \mathcal{F} is compact.

5 Part E

6 Part F

Consider the set $D = \{d \in \mathcal{F} \mid d = (k, n - k)\}$, and define the map

$$\psi : D \rightarrow G_{n,k}(\mathbb{K}), \quad d \mapsto F_1$$

. ψ is surjective because F_1 can be any linear subspace of dimension k , and it is aswell injective because $F_2 = \mathbb{K}^n, F_0 = 0$ and $F_1 = F'_1 \implies d = d'$. We covered the general case of $G_{n,k}(\mathbb{K})$, and the projective space is just the special case with $k = 1$.