Exercise 42

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1 Part A

Let $E, F \in \mathcal{F}$. We can find some linear basis e_j, f_j s.t.

$$E_j = \text{span}(e_j) \qquad \text{and} \qquad F_j = \text{span}(f_j)$$

and the map g defined by $e_j \mapsto f_j$ is an element of G, and has the property that $g \cdot E = F$. We therefore see that the action of α is transitive.

2 Part B

From what we saw in Part A it is clear that the stabilizer is given by

$$P = \{\text{diag}(M_j) \mid M_j \in \text{GL}(d_j,\ \mathbb{K}), j = 1, \ldots, k\}.$$

Using the trivial global chart $\chi : \mathbb{R}^{n^2} \to G$ of G one can see (after a rearranging of the basis of $\mathbb{R}^{n^2}$) that P can be described by

$$P = \chi(\mathbb{R}^{n^2} \cap \{x_1 = \cdots = x_l = 0\})$$

where the coordinates set to zero are the ones outside the diagonal blocks. It follows by definition that P is a submanifold of G, and applying Theorem 9.1 we conclude that P is a closed subset of G. Obviously P is a subgroup of G, hence a closed subgroup as required.

3 Part C

We can give $\mathcal{F}$ a group structure by declaring $\phi(g) \star \phi(h) = \phi(gh)$. The product $\star$ is well defined on the whole domain since from Part A it follows that ϕ is surjective, and it is defined in the right way to make ϕ a homomorphism of groups. We can then apply the isomorphism theorem for groups to conclude that such a bijection $\bar{\phi}$ is induced.

4 Part D

To see tht $\phi(K) = \mathcal{F}$ we look at what we did in Part A, and we observe that using the Grant-Schmidt process we can make all the e_j and the f_j orthonormal, thus making g an orthogonal tranformation. Hence $\forall F \in \mathcal{F}$ we can find a map $g \in K$ s.t. $\phi(g) = F$, or in other word $\phi(K) = \mathcal{F}$.
From above it follows that the action of K on $\mathcal{F}$ is transitive, and it is clear that $H = \ker(\phi_{|K}) = K_E$. Hence we can apply Proposition 15.5 the get the desired diffeomorphism.
Finally we know that K is compact and so it is its quotient K/H. Therefore $\mathcal{F}$ is compact.

5 Part E

We can decompose G as $G = P \cup P^c$, and by looking at $k = e$ we see that $P \subset \mathrm{Im}(m)$. Moreover if $g \in P^c$ then $g \cdot E = F$, and from $\phi(K) = \mathcal{F}$ we know that $\exists k \in K \text{s.t.} k \cdot E = F$. By the isomorphism fro Part D we deduce that g and k are in the same class, i.e. exists a $p \in P$ s.t. $g = kp$. The surjectivity of m follows.

6 Part F

Consider the set $D = \{d \in \mathcal{F} \mid d = (k, n-k)\}$, and define the map

$$\psi : D \to G_{n,k}(\mathbb{K}), \qquad d \mapsto F_1.$$

ψ is surjective because F_1 can be any linear subspace of dimension k, and it is aswell injective because $F_2 = \mathbb{K}^n, F_0 = 0$ and $F_1 = F_1' \implies d = d'$.We covered the general case of $G_{n,k}(\mathbb{K})$, and the projective space is just the special case with $k = 1$.