Exercise 42

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1 Part A

Let $E, F \in \mathcal{F}$. We can find some linear basis e_j, f_j s.t.

$$E_j = \operatorname{span}(e_j) \qquad \text{and} \qquad F_j = \operatorname{span}(f_j)$$

and the map g defined by $e_j \mapsto f_j$ is an element of G, and has the property that $g \cdot E = F$. We therefore see that the action of α is transitive.

2 Part B

From what we saw in Part A it is clear that the stabilizer is given by

$$P = \{\operatorname{diag}(M_j) \mid M_j \in \operatorname{GL}(d_j,\ \mathbb{K}), j = 1, \dots, k\}.$$

Using the trivial global chart $\chi : \mathbb{R}^{n^2} \to G$ of G one can see (after a rearranging of the basis of $\mathbb{R}^{n^2}$) that P can be described by

$$P = \chi(\mathbb{R}^{n^2} \cap \{x_1 = \cdots = x_l = 0\})$$

where the coordinates set to zero are the ones outside the diagonal blocks. It follows by definition that P is a submanifold of G, and applying Theorem 9.1 we conclude that P is a closed subset of G. Obviously P is a subgroup of G, hence a closed subgroup as required.

3 Part C

Consider the compact set $\{(m_1, 0)\} \subset M$, where $m_1 \neq 0$. It's easy to see that the set

$$A \cdot C = \{(e^t m_1, 0) \mid t \in A\} = \{(x, 0) \mid x \in \mathbb{R}_{>0}\}$$

is not closed. M is locally compact, so we can apply *lemma 11.8* and conclude that the quotient topology A/M is not Hausdorff.

4 Part D

In view of *Lemma 11.8*, the condition (b) of *Definition 12.2* is equivalent to the quotient A/M being Hausdorff. In item (c) we showed that this is not the case, hence the action of A on M is not of principal fiber bundle type.