Final Exam - MACQM

Andrea Barbon - VU 2206157

2012-3-9

1 Problem 1

...

2 Problem 2

2.1

First of all we will show that b, c and d can all be determined by a.

Proof. Using the self-adjointness of the operators we have

$$a_s = \langle\, S_x e_s \,,\, e_{s-1} \,\rangle = \langle\, e_s \,,\, S_x e_{s-1} \,\rangle = \overline{b_{s-1}}$$
$$c_s = \langle\, S_y e_s \,,\, e_{s-1} \,\rangle = \langle\, e_s \,,\, S_y e_{s-1} \,\rangle = \overline{d_{s-1}}$$

and from the second commutator relation $[S_y,\ S_z] = iS_x$ we also get

$$\begin{aligned} i(a_{s-1} + b_{s+1})e_s &= iS_x e_s = [S_y,\ S_z]e_s = \\ &= sc_s e_{s-1} + sd_s e_{s+1} - ((s-1)c_s e_{s-1} + (s+1)d_s e_{s+1}) = \\ &= c_s e_{s-1} - d_s e_{s+1}. \end{aligned}$$

Putting all together we conclude that

$$b_s = \overline{a_{s+1}}, \qquad c_s = ia_s, \qquad d_s = -i\overline{a_{s+1}}$$

□

2.2

We have $|a_s|^2 - |a_{s+1}|^2 = s/2$, and in particular $|a_{L/2}|^2 = L/4$

Proof. From an easy computation it follows that

$$\begin{aligned} ise_s &= iS_z e_s = [S_x,\ S_y]e_s = (S_x S_y - S_y S_x)e_s = \\ &= S_x(ia_s e_{s-1} - i\overline{a_{s+1}}e_{s+1}) - S_y(a_s e_{s-1} + \overline{a_{s+1}}e_{s+1}) = \\ &= 2i(|a_s|^2 - |a_{s+1}|^2)e_s \end{aligned}$$

and using the fact that $a_{L/2+1} = 0$ we get also $|a_{L/2}|^2 = L/4$. □

2.3

The following formula holds

$$|a_s|^2 = \frac{1}{4}(\frac{L}{2} - s + 1)(\frac{L}{2} + s)$$

Proof. ... □

Hence we only know the complex norm of a_s, b_S, c_s and d_s. If we assume that a_s is real and positive we can pick

$$a_s = \frac{1}{2}\sqrt{(\frac{L}{2} - s + 1)(\frac{L}{2} + s)}$$

and using relations from 2.1 we get also the values for b_s, c_s, d_s.

2.4

We introduce the new couple of operators

$$S_+ = S_x + iS_y \qquad \text{and} \qquad S_- = S_x - iS_y$$

and we claim that the following commutator relations hold

$$[S_z,\ S_\pm] = \pm S_\pm$$

Proof. ... □

2.5

The operators S_- and S_+ are respectively lowering and rising operators. More precisely

$$S_\pm e_s = c_\pm^s$$

where the $c_\pm^s$ are complex numbers dependent on the sign and s.

Proof. ... □