Final Exam - MACQM

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2012-3-9

1 Problem 1

...

2 Problem 2

2.1

First of all we will show that b, c and d can all be determined by a.

Proof. Using the self-adjointness of the operators we have

$$a_s = \langle\, S_x e_s \ ,\ e_{s-1} \,\rangle = \langle\, e_s \ ,\ S_x e_{s-1} \,\rangle = \overline{b_{s-1}}$$
$$c_s = \langle\, S_y e_s \ ,\ e_{s-1} \,\rangle = \langle\, e_s \ ,\ S_y e_{s-1} \,\rangle = \overline{d_{s-1}}$$

and from the second commutator relation $[S_y,\ S_z] = iS_x$ we also get

$$\begin{aligned} i(a_{s-1} + b_{s+1})e_s &= iS_x e_s = [S_y,\ S_z]e_s = \\ &= sc_s e_{s-1} + sd_s e_{s+1} - ((s-1)c_s e_{s-1} + (s+1)d_s e_{s+1}) = \\ &= c_s e_{s-1} - d_s e_{s+1}. \end{aligned}$$

Putting all together we conclude that

$$b_s = \overline{a_{s+1}}, \qquad c_s = ia_s, \qquad d_s = -i\overline{a_{s+1}}$$

□

2.2

We have $|a_s|^2 - |a_{s+1}|^2 = s/2$, and in particular $|a_{L/2}|^2 = L/4$

Proof. From an easy computation it follows that

$$\begin{aligned} ise_s &= iS_z e_s = [S_x,\ S_y]e_s = (S_x S_y - S_y S_x)e_s = \\ &= S_x(ia_s e_{s-1} - i\overline{a_{s+1}}e_{s+1}) - S_y(a_s e_{s-1} + \overline{a_{s+1}}e_{s+1}) = \\ &= 2i(|a_s|^2 - |a_{s+1}|^2)e_s \end{aligned}$$

and using the fact that $a_{L/2+1} = 0$ we get also $|a_{L/2}|^2 = L/4$. □

2.3

The following formula holds

$$|a_s|^2 = \frac{1}{4}(\frac{L}{2} - s + 1)(\frac{L}{2} + s)$$

Proof. ... □

Hence we only know the complex norm of a_s, b_S, c_s and d_s. If we assume that a_s is real and positive we can pick

$$a_s = \frac{1}{2}\sqrt{(\frac{L}{2} - s + 1)(\frac{L}{2} + s)}$$

and using relations from 2.1 we get also the values for b_s, c_s, d_s.

2.4

We introduce three new operators

$$S_\pm = S_x \pm iS_y \qquad \text{and} \qquad S^2 = S_x^2 + S_y^2 + S_z^2$$

and we claim that the following commutator relations hold

$$[S_z,\ S_\pm] = \pm S_\pm \qquad \text{and} \qquad [S^2,\ S_\pm] = 0$$

Proof. ... □

Let now $V = \{-\frac{L}{2}, -\frac{L}{2}+1, \ldots, \frac{L}{2}-1, \frac{L}{2}\}$. The last relation says that if for some $t \in V$ we have $S^2 e_t = \lambda e_t$, then we know that λ works as an eigenvalue for S^s for all the values of $s \in V$, i.e. more formally

$$\exists t \in V \quad \text{s.t.} \quad S^2 e_t = \lambda e_t \implies S^2 e_s = \lambda e_s, \quad \forall s \in V. \tag{1}$$

2.5

The operators S_- and S_+ are respectively lowering and rising operators. More precisely

$$S_\pm e_s = c_\pm^s e_{s\pm 1}$$

where the $c_\pm^s$ are complex numbers dependent on the sign and s.

Proof. ... □

Now from the identity $S_x = \frac{1}{2}(S_+ + S_-)$ we see that

$$c_+^s = b_s = \overline{a_{s+1}} \qquad \text{and} \qquad c_-^s = a_s \tag{2}$$

2.6

We are now ready to show the eigenvalue of S^2. We have

$$S^2 e_s = \frac{L}{2}(\frac{L}{2} + 1)e_s, \quad \forall s \in V$$

Proof. Using the identity $S_x^2 + S_y^2 = S_-S_+ - i[S_x,\ S_y]$ we see that $S^2 e_s = (S_-S_+ + S_z + S_z^2)e_s$. Now look at this equality for $s = \frac{L}{2}$ and recall that, since S_+ is a rising operator, $S_+ e_{\frac{L}{2}} = 0$. We conclude that $S^2 e_{\frac{L}{2}} = (S_z + S_z^2)e_{\frac{L}{2}} = \frac{L}{2}(\frac{L}{2}+1)e_{\frac{L}{2}}$. Therefore the claim follows from what we saw in (1). □

This also tell us that $|Se_s| = \sqrt{\frac{L}{2}(\frac{L}{2}+1)}, \quad \forall s \in V$, or in other words all the possible observed values of the spin lies in a sphere of radious $\sqrt{\frac{L}{2}(\frac{L}{2}+1)}$.

2.7 Answers

(a) The possible values of S_z are its eigenvalues, i.e. every $s \in \{-\frac{L}{2}, -\frac{L}{2}+1, \ldots, \frac{L}{2}-1, \frac{L}{2}\}$.

(b) The commutators of S_z with S_x and S_y are not trivial, hence we cannot measure the values of two or more such operators at the same time. Together with the last remark of the previous section, this means that if we first measure S_z, then we cannot have a precise knowledge of the values of S_x and S_y, but we know that Se_s lies in a sphere with fixed radious. Hence at first it seems that S_x and S_y can take all the values in a latitude circle of height equal tho the measured value of S_z. However the simmetry of the problem implies that we could have chosen another prefferred axis instead of the z one, finding a basis of eigenstates for S_x or S_y, and all the results we showed here would have been the same (modulo a cyclic permutation of the indexes). Therefore we conclude that the possible values of S_x and S_y are the same ones as S_z.

(c) Look at paragraph 2.6.

3 Problem 3

The probability is given by the inner product between ψ and $e_{-L/2}$.

4 Problem 4

First we notice that

$$S_{t+s}(q'', q) = S_t(q'(q, q''), q) + S_s(q'', q'(q, q'')) \tag{3}$$

from which, since S_{t+s} does not depend on q', we deduce

$$\frac{\partial(S_t + S_s)}{\partial q'} = 0 \tag{4}$$

and taking the derivatices of such identity with respect to q and q'' we also get

$$\frac{\partial^2(S_t + S_s)}{\partial q' \partial q'} \cdot \frac{\partial q'}{\partial q} + \frac{\partial^2 S_t}{\partial q \partial q'} = \frac{\partial^2(S_t + S_s)}{\partial q' \partial q'} \cdot \frac{\partial q'}{\partial q''} + \frac{\partial^2 S_s}{\partial q' \partial q''} = 0 \tag{5}$$

Now we can start our computation

$$\begin{aligned}
\frac{\partial^2 S_{t+s}}{\partial q \partial q''} &= \frac{\partial^2 S_t}{\partial q \partial q'} \cdot \frac{\partial q'}{\partial q''} + \frac{\partial^2 S_s}{\partial q' \partial q''} \cdot \frac{\partial q'}{\partial q} + \\
&+ \frac{\partial^2(S_t + S_s)}{\partial q' \partial q'} \cdot \frac{\partial q'}{\partial q''} \cdot \frac{\partial q'}{\partial q} + \frac{\partial(S_t + S_s)}{\partial q'} \cdot \frac{\partial^2 q'}{\partial q \partial q''} = \qquad \text{using (4)} \\
&= \frac{\partial^2 S_t}{\partial q \partial q'} \cdot \frac{\partial q'}{\partial q''} + \frac{\partial^2 S_s}{\partial q' \partial q''} \cdot \frac{\partial q'}{\partial q} = \qquad \text{using (5)} \\
&= -\left(\frac{\partial^2 S_t}{\partial q' \partial q} \cdot \frac{\partial^2 S_s}{\partial q' \partial q''} + \frac{\partial^2 S_s}{\partial q' \partial q''} \cdot \frac{\partial^2 S_t}{\partial q' \partial q} \right) \cdot \left(\frac{\partial^2(S_t + S_s)}{\partial q' \partial q'} \right)^{-1} = \\
&= -2 \frac{\partial^2 S_t}{\partial q' \partial q} \cdot \frac{\partial^2 S_s}{\partial q' \partial q''} \cdot \left(\frac{\partial^2(S_t + S_s)}{\partial q' \partial q'} \right)^{-1}
\end{aligned}$$

which shows that

$$\frac{\partial^2 S_{t+s}}{\partial q \partial q''} \cdot \frac{\partial^2(S_t + S_s)}{\partial q' \partial q'} = -2 \left(\frac{\partial^2 S_t}{\partial q' \partial q} \cdot \frac{\partial^2 S_s}{\partial q' \partial q''} \right)$$

and the required equation follows from the multiplicativity of the determinant.

5 Problem 5

Let $\pi : L \to \{p_i = 0\}$ be the projection. By hypothesis π is a bijection, hence it admits an inverse g such that

$$(q^i, p_i) = (q^i, g(q^i)).$$

Therefore we see that

$$0 = \omega_{|L} = \sum_i dg(q^i) \wedge dq^i = d(\sum_i g(q^i)dq^i)$$

or in other words the one form $\alpha = \sum_i g(q^i)dq^i$ is closed in $\mathbb{R}^n$. But the latter has trivial cohomology, hence it follows that α is exact aswell. We conclude that $\alpha = df = \sum_i \frac{\partial f}{\partial q^i}$ for some function $f \in C^\infty(\mathbb{R}^n)$, and so we have

$$p_i = g(q^i) = \frac{\partial f}{\partial q^i}.$$