Regression Analysis

- Regression problem
- Analytical solution
- Regression results
- Application to capital markets

Introduction

Imagine you bought a car and the seller told you that it can run for 35 kilometer per liter of gasoline.

This seems to good to be true, so you decide to investigate the matter.

You start using your new car and every time you drive from a place to another you keep track of

$$x_i = \text{ liters of gasoline consumed during trip } i$$

$$y_i = \text{ length in kilometers of trip } i$$

After some months you made $N = 50$ trips and you have a dataset with N observations in the form

$$(x_i, y_i) \qquad \text{for} \qquad i = 1, \ldots N$$

How can you use this data to check if you can *actually* drive for 35 km with one liter of gasoline?

Use linear regression!

It makes sense to conjecture a linear relation between liters and kilometers

Hence you assume a linear model of the form

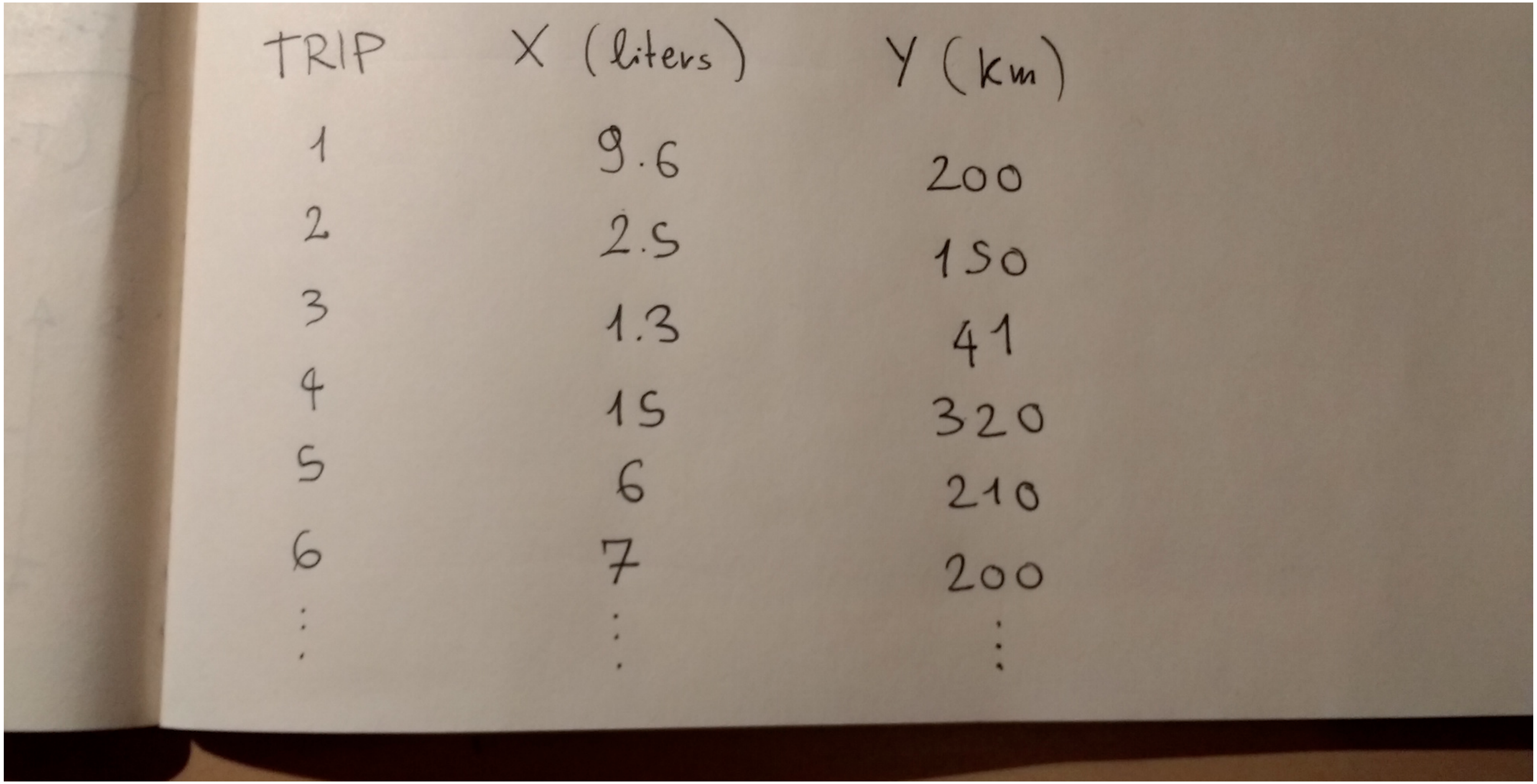
$$y_i = \alpha + \beta x_i + \varepsilon_i$$

Of course, you expect $\alpha = 0$ and $\beta > 0$ (why?)

But the precise value of β is an *empirical question.*

Will it be close to 35, as promised, or not?

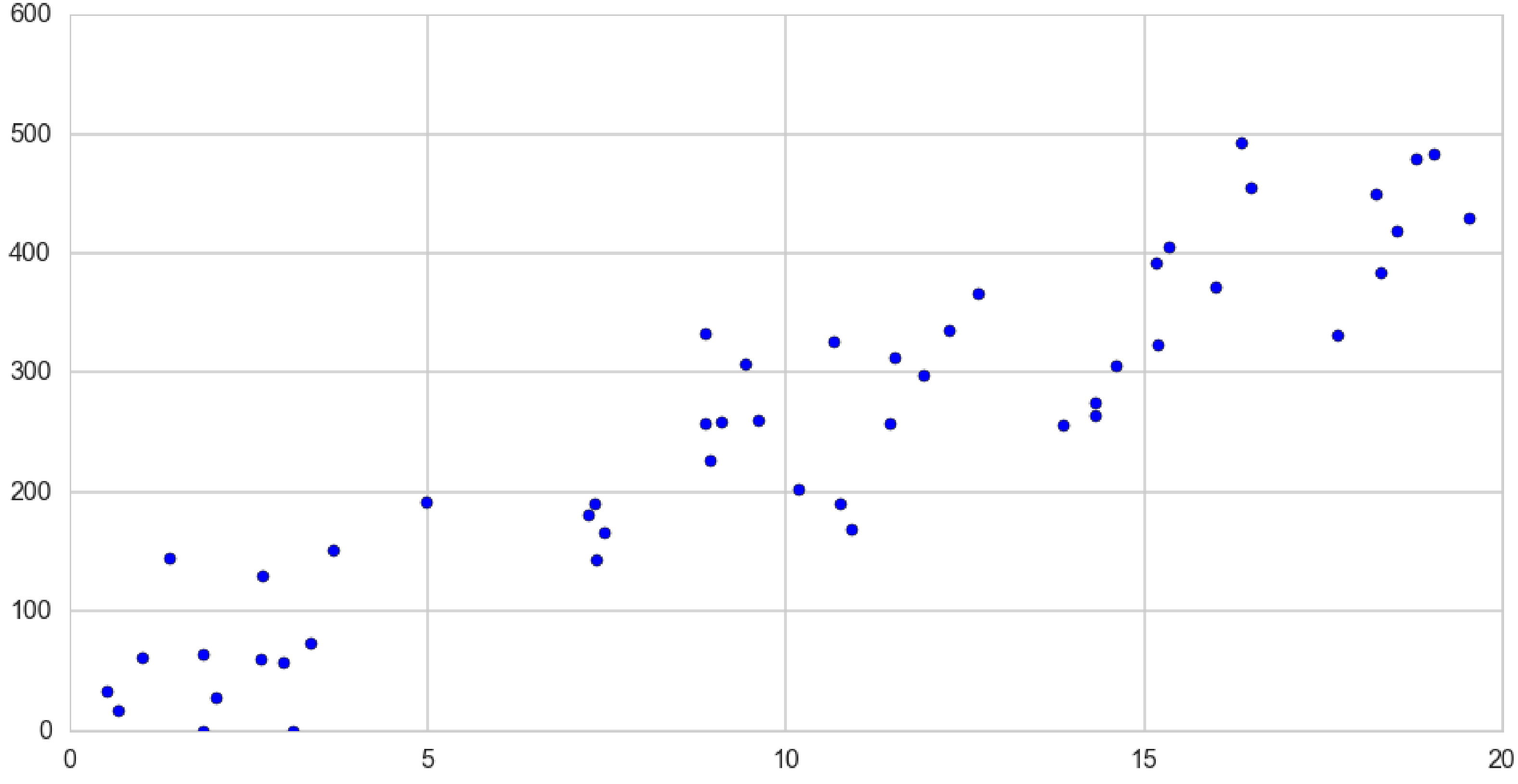
This is how your dataset looks like on your notebook:



TRIP	X (liters)	Y (km)
1	9.6	200
2	2.5	150
3	1.3	41
4	15	320
5	6	210
6	7	200
⋮	⋮	⋮

You can also plot it on the (x,y) plane:

In [213]:
```
car_data = show_dataset()
```

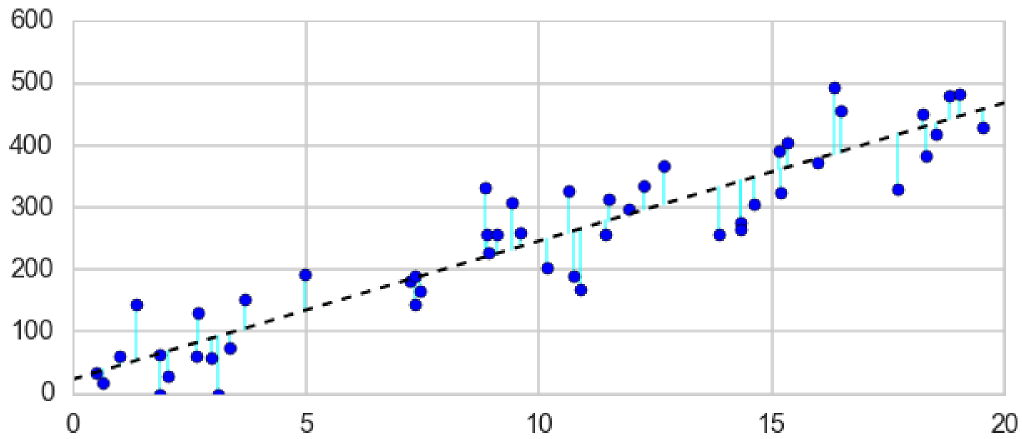


You fit a linear regression to estimate the coefficients

In [255]:
```
reg = fit_dataset(car_data, prt=False); reg.summary().tables[1]
```

Out[255]:

	coef	std err	t	P>|t|	[95.0% Conf. Int.]
const	23.2423	14.999	1.550	0.128	-6.914 53.399
X	22.2526	1.295	17.187	0.000	19.649 24.856



According to the estimate, the car runs for less than 25 km per liter of gasoline. The seller was lying!

Regression Problem

We are given N points (x_i, y_i) with $i = 1, \ldots N$.

We can think of those as sample draws from the the joint distribution of two random variables X and Y.

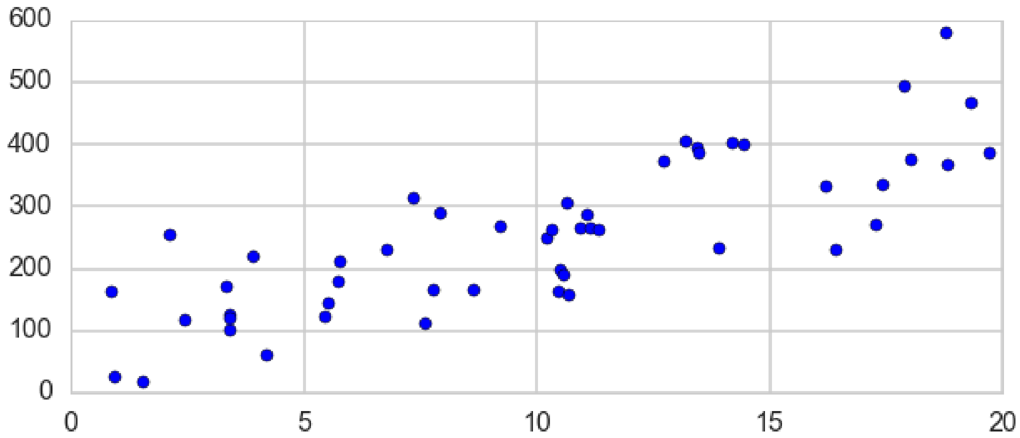
We want to check if there is a linear relation between X and Y.

If that is the case, we should find a non-zero β when estimating the linear model

$$y_i = \alpha + \beta x_i + \varepsilon_i$$

Geometric Interpretation

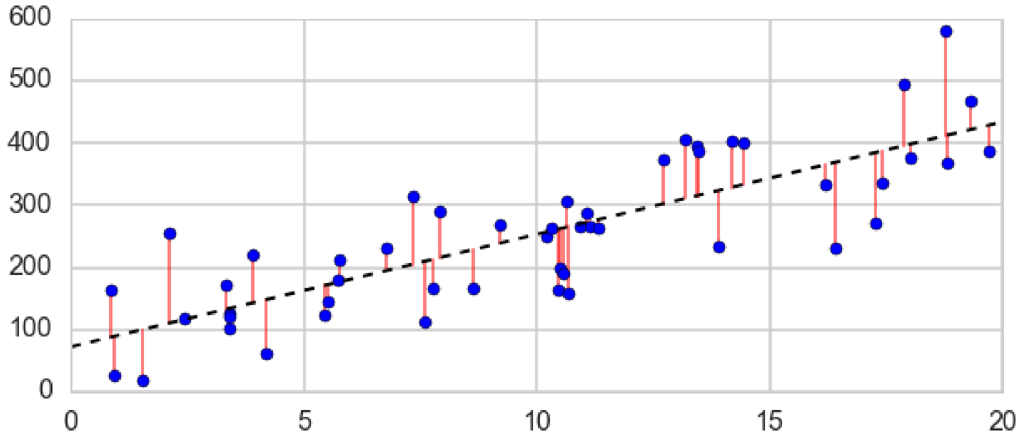
We can scatter-plot the datapoints on the (x,y) plane



We want the line that fits the points in the best possible way.

Then the **slope** of the line will be the estimated β, while its **intercept** will be the estimated α.

The best fitting line is the one that is closer to most of the points.

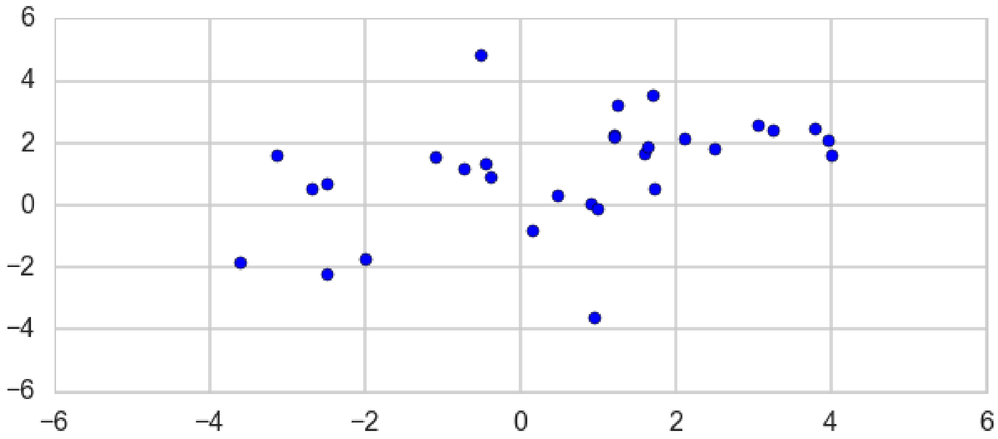


More precisely, it is the line with intercept α and slope β which minimizes the **sum of squared residuals**

$$SS_{res} = \sum_{i=1}^{N} \varepsilon_i^2 = \sum_{i=1}^{N} (y_i - \alpha - \beta x_i)^2$$

Example

In [281]:
```
df  = gen_regression_problem(); plot_reg(df)
```

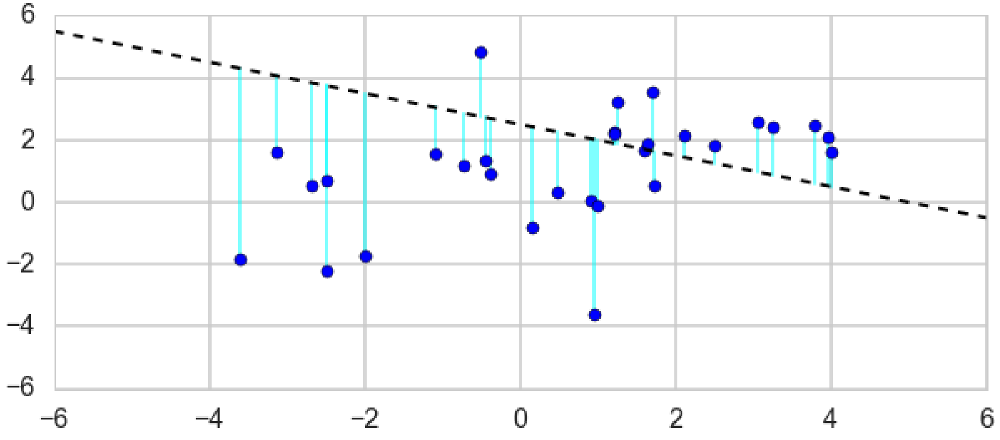


First guess

In [284]:
```
alpha = 2.5
beta  = -0.5
```

In [285]:
```
plot_reg(df, alpha, beta)
```

```
alpha = 2.5, beta = -0.5
Sum of Squared Residuals : 214.54
```



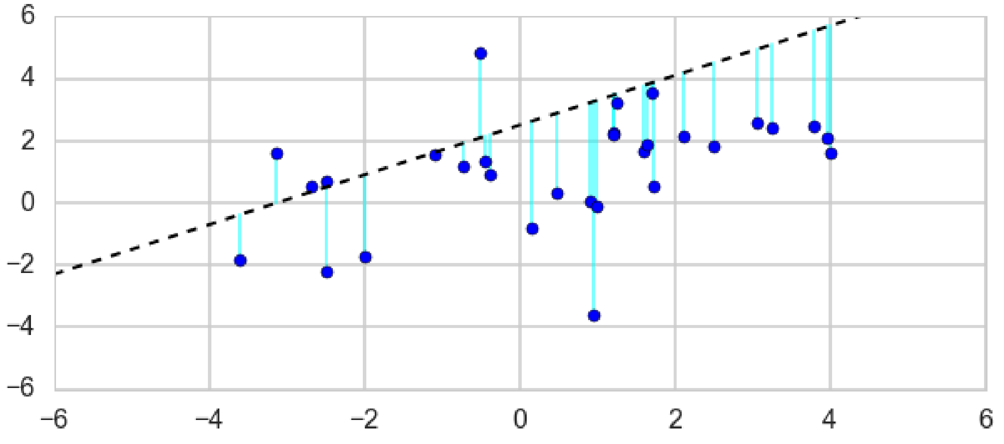
Not good at all! The slope of the line β should be positive

Second guess

In [287]:
```
alpha = 2.5
beta  = 0.8
```

In [288]:
```
plot_reg(df, alpha, beta)
```

```
alpha = 2.5, beta = 0.8
Sum of Squared Residuals : 202.30
```



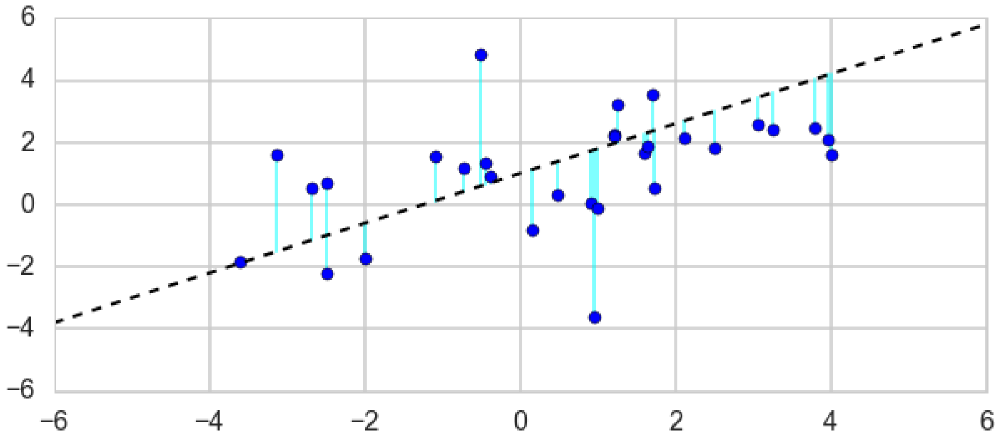
Better, but the α is too high

Third guess

In [289]:
```
alpha =  1
beta  =  0.8
```

In [290]:
```
plot_reg(df, alpha, beta)
```

```
alpha = 1.0, beta = 0.8
Sum of Squared Residuals : 104.19
```



Not bad! But can we do better?

Analytical Solution

We want to minimize the sum of squared resuduals, SS_{res} , as a funcion of α and β. How?

Let's take derivatives!

First, we impose the FOC with respect to α and we get

$$0 = \frac{\partial}{\partial \alpha} SS_{res} \implies \alpha = \frac{1}{N}\sum_{i=1}^{N} y_i - \frac{\beta}{N}\sum_{i=1}^{N} x_i = \bar{y} - \beta\bar{x}$$

Thus we can re-write SS_{res} as

$$SS_{res} = \sum_{i=1}^{N} (y_i - \bar{y} - \beta(x_i - \bar{x}))^2$$

and imposing the FOC wrt β we find

$$\beta = \frac{\frac{1}{N} \sum_i (y_i - \bar{y})(x_i - \bar{x})}{\frac{1}{N} \sum_i (x_i - \bar{x})^2} = \frac{\hat{Cov}(X, Y)}{\hat{V}(X)}$$

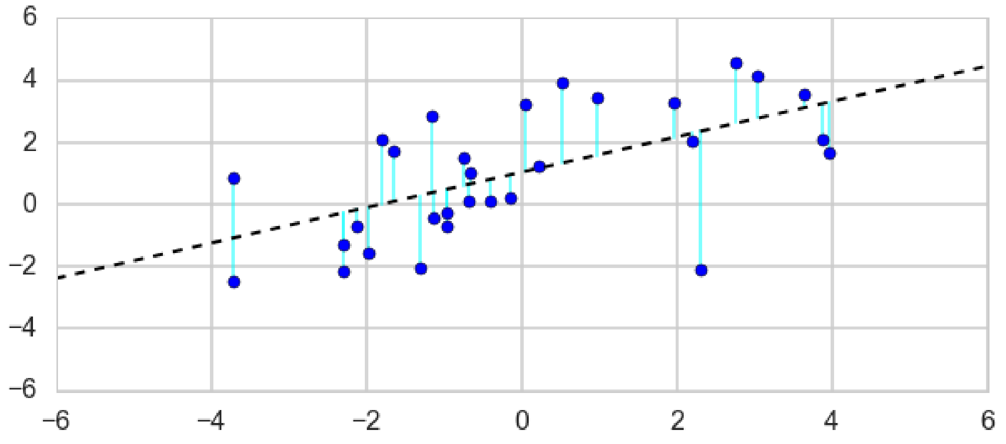
Hence we see that the optimal β is the sample covariance of X and Y divided by the sample variance of the explanatory variable X.

Optimal α and β

In [278]:

```
X1    = sm.add_constant(df[['X']])  # Define the explanatory variable
reg   = sm.OLS( df['Y'], X1 ).fit() # Fit the regression
alpha = reg.params['const']         # Estimated alpha
beta  = reg.params['X']             # Estimated beta
plot_reg(df, alpha, beta)
```

```
alpha = 1.0, beta = 0.6
Sum of Squared Residuals : 81.68
```



R^2

The R^2 or **coefficient of determination** is the proportion of the variance of Y that is predictable from X.

In other words it measures how well X can explain Y.

To define the R^2 we first need to introduce two objects:

1. The total sum of squares SS_{tot}
2. The regression sum of squares SS_{reg}

Total Sum of Squares

The total sum of squares SS_{tot} is proportional to the sample variance of Y

$$SS_{tot} = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{y})^2 = \hat{V}(Y)$$

This is the variation we would like to explain using the explanatory variable X.

Regression Sum of Squares

The *regression sum of squares* or *explained sum of squares* SS_{tot} is proportional to the sample variance of βX

$$SS_{reg} = \frac{1}{N-1} \sum_{i=1}^{N} (\beta x_i - \beta \bar{x})^2 = \hat{V}(\beta X)$$

This is the variation we can explain using the explanatory variable X.

R^2 Definition

The coefficient of determination or R^2 is defined as

$$R^2 = \frac{SS_{tot} - SS_{reg}}{SS_{tot}}$$

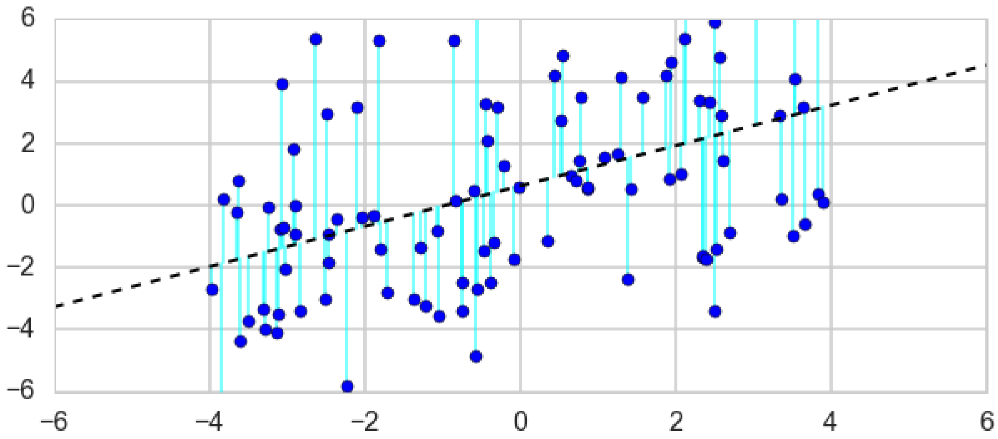
It measures the fraction of variance that can be explained by X

Example : Low R^2

In [279]:
```
df = gen_regression_problem(100, 4, 3, 0.2, 0.5 ); reg_plot_table(df)
```

Model:	OLS	Adj. R-squared:	0.202
Dependent Variable:	Y	AIC:	504.1063
No. Observations:	100	Log-Likelihood:	-250.05
R-squared:	0.210	Scale:	8.8763

	coef	std err	t	P>|t|	[95.0% Conf. Int.]
const	0.6221	0.298	2.085	0.040	0.030 1.214
X	0.6493	0.127	5.101	0.000	0.397 0.902

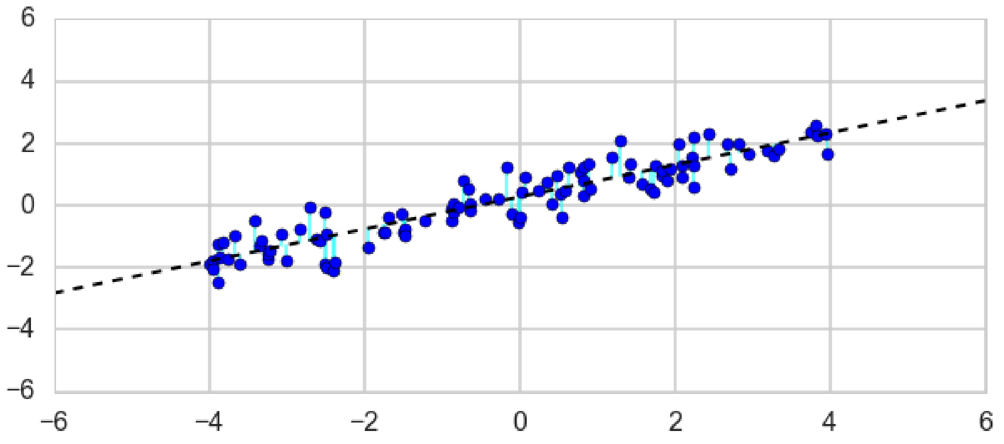


Example : High R^2

In [280]:
```
df = gen_regression_problem(100, 4, 0.5, 0.2, 0.5 ); reg_plot_table(df)
```

Model:	OLS	Adj. R-squared:	0.854
Dependent Variable:	Y	AIC:	145.2269
No. Observations:	100	Log-Likelihood:	-70.613
R-squared:	0.856	Scale:	0.24527

	coef	std err	t	P>|t|	[95.0% Conf. Int.]
const	0.2722	0.050	5.464	0.000	0.173 0.371
X	0.5162	0.021	24.090	0.000	0.474 0.559



Significance of β

When we estimate a model on the data, we want to measure the statistical significance of the estimated parameters.

To asses the significance of the estimated β, for instance, we first need to compute the *volatility* of the estimated parameter.

Then we compare the magnitude of the estimated β with its volatility.

If β is large enough wrt its volatility, we conclude that it is **significantly** different from zero.

Standard Error

The *standard error* of β is the volatility of the estimated β, given X.

Since $V(y_i \mid X) = V(\alpha + \beta x_i + \varepsilon_i \mid X) = V(\varepsilon)$

$$V(\beta \mid X) = V\left(\frac{\sum_i (y_i - \bar{y})(x_i - \bar{x})}{\sum_i (x_i - \bar{x})^2} \mid X\right) = \frac{V(\varepsilon)}{\sum_i (x_i - \bar{x})^2}$$

$$= \frac{1}{(N-1)} \frac{V(\varepsilon)}{\hat{V}(X)}$$

Therefore an empirical estimate of the standard error of β is

$$SE(\beta) = \sqrt{\frac{\hat{V}(\varepsilon)}{(N-1)\hat{V}(X)}}$$

The intuition is the following:

- The standard error of β is the ratio between the sample volatility of ε and the sample volatility of X
- When this ratio is large, it means that *points are stacked together* (see example later)
- Thus, if the $SE(\beta)$ is too large, our estimate of β is *unstable*

t-statistics

The *t-statistics* is defined as the ratio

$$\frac{\beta}{SE(\beta)}$$

If the t-stat is far enough from zero, then we say that the estimated β is significantly different from zero.

The threshold for significance depends on the number of explanatory variables in our linear model.

p-value

A more practical way to check if an estimated coefficient is significantly different from zero is to look at the p-value.

It is computed from the t-stat and it (roughly) gives the probability that the estimated coefficient is non-zero even if the real coefficient is zero (false positive).

The thresholds are fixed by convention as:

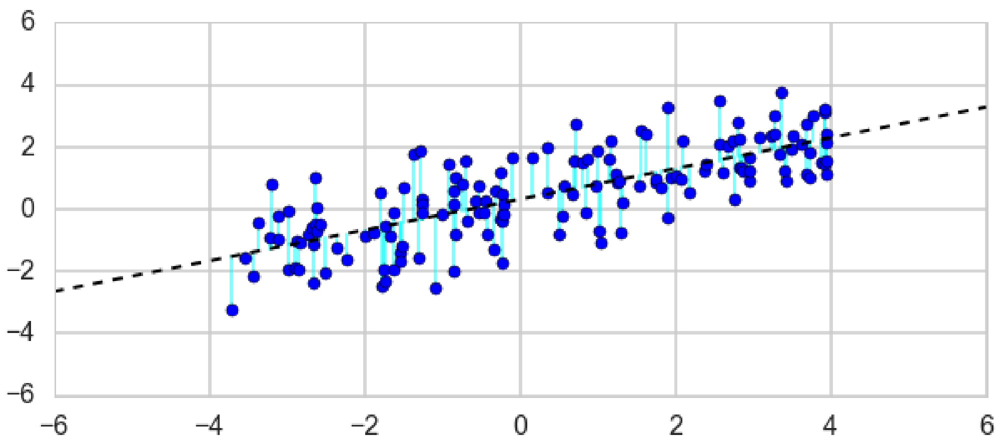
- p-value $\geq 5\%$ $\Longrightarrow$ not significant
- p-value $< 5\%$ $\Longrightarrow$ weakly significant
- p-value $< 1\%$ $\Longrightarrow$ significant
- p-value $< 0.1\%$ $\Longrightarrow$ highly significant

Example : Highly Significant β

```
In [291]: df  = gen_regression_problem(150, 4, 1, 0.2, 0.5 ); reg_plot_table(df)
```

Model:	OLS	Adj. R-squared:	0.564
Dependent Variable:	Y	AIC:	422.1055
No. Observations:	150	Log-Likelihood:	-209.05
R-squared:	0.567	Scale:	0.96360

	coef	**std err**	**t**	**P>|t|**	**[95.0% Conf. Int.]**
const	0.3161	0.081	3.896	0.000	0.156 0.476
X	0.4947	0.036	13.928	0.000	0.424 0.565

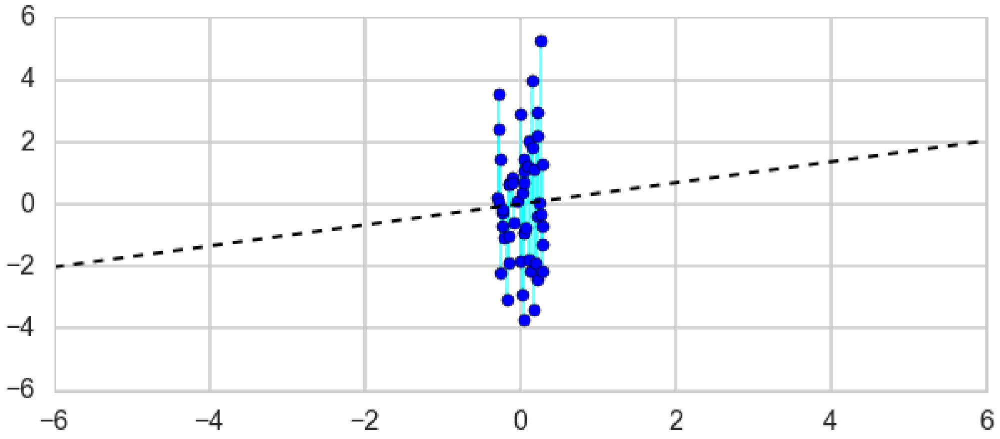


Example : Not Significant β

In [292]:
```
df  = gen_regression_problem(50, 0.3, 2, 0.2, 0.5 ); reg_plot_table(df)
```

Model:	OLS	Adj. R-squared:	-0.020
Dependent Variable:	Y	AIC:	212.0924
No. Observations:	50	Log-Likelihood:	-104.05
R-squared:	0.001	Scale:	3.9149

	coef	std err	t	P>|t|	[95.0% Conf. Int.]	
const	0.0074	0.282	0.026	0.979	-0.559	0.574
X	0.3385	1.521	0.223	0.825	-2.719	3.396



Capital Markets

We have daily return data on a bunch of stocks

In [332]:
```
data.groupby(['permno']).head(3)
```

Out[332]:

	permno	asset	prc	vol	ret	shrout	market
date							
2007-01-03	10107	MICROSOFT CORP	29.8600	77630458	0.0000	9777000	-0.0012
2007-01-04	10107	MICROSOFT CORP	29.8100	46650955	-0.0017	9777000	0.0012
2007-01-05	10107	MICROSOFT CORP	29.6400	44681937	-0.0057	9777000	-0.0061
2007-01-03	11308	COCA COLA CO	48.5800	7877300	0.0068	2343796	-0.0012
2007-01-04	11308	COCA COLA CO	48.6000	5908400	0.0004	2343796	0.0012
2007-01-05	11308	COCA COLA CO	48.2600	5803900	-0.0070	2343796	-0.0061
2007-01-03	14593	APPLE COMPUTER INC	83.8000	44545733	-0.0123	860220	-0.0012
2007-01-04	14593	APPLE COMPUTER INC	85.6600	34097551	0.0222	860220	0.0012
2007-01-05	14593	APPLE COMPUTER INC	85.0500	29842335	-0.0071	860220	-0.0061
2007-01-03	84788	AMAZON COM INC	38.7000	12450246	-0.0193	414000	-0.0012
2007-01-04	84788	AMAZON COM INC	38.9000	6329815	0.0052	414000	0.0012
2007-01-05	84788	AMAZON COM INC	38.3700	6624080	-0.0136	414000	-0.0061
2007-01-03	90319	GOOGLE INC	467.5900	7725290	0.0154	227670	-0.0012
2007-01-04	90319	GOOGLE INC	483.2600	7944390	0.0335	227670	0.0012
2007-01-05	90319	GOOGLE INC	487.1900	6880018	0.0081	227670	-0.0061

Market Model

To estimate the market β^i of company i, we regress its returns on the market returns

$$R_t^i = \alpha^i + \beta^i R_t^M + \varepsilon_t^i$$

Notice that here i identifies the company (so it is fixed), while the sample points are indexed by t

The higher the estimated β^i, the more company i is exposed to the market risk factor

For example, let's estimate the β of Coca-Cola

In [318]:
```
CAPM_reg(CocaCola)
```

Model:	OLS	Adj. R-squared:	0.403
Dependent Variable:	ret	AIC:	-14719.1211
No. Observations:	2266	Log-Likelihood:	7361.6
R-squared:	0.403	Scale:	8.8333e-05

	coef	**std err**	**t**	**P>|t|**	**[95.0% Conf. Int.]**
const	0.0003	0.000	1.524	0.128	-8.63e-05 0.001
market	0.5675	0.015	39.133	0.000	0.539 0.596

- The estimated β is 0.57 and highly significant
- The estimated α is not significantly different from zero
- The R^2 is about 40%

Now let's try with Amazon

In [322]:
```
CAPM_reg(Amazon)
```

Model:	OLS	Adj. R-squared:	0.318
Dependent Variable:	ret	AIC:	-10836.9389
No. Observations:	2266	Log-Likelihood:	5420.5
R-squared:	0.318	Scale:	0.00048997

	coef	std err	t	P>|t|	[95.0% Conf. Int.]
const	0.0013	0.000	2.844	0.004	0.000 0.002
market	1.1092	0.034	32.480	0.000	1.042 1.176

- The beta is higher, larger than one
- The alpha is positive and highly significant

Neutralize the market risk

Amazon has a significant alpha over the sample.

Imagine we invested in research and we were able to predict it.

Can we pocket the alpha, without being exposed to the market risk?

We can form a portfolio which is long Amazon and short the market in the right proportions

Use the fact that the market beta of Amazon is 1.11

Arbitrage Portfolio

In [333]:
```
mu_A = 1;  mu_M = -1.11
pf = form_portfolio(Amazon, 'market', mu_A, mu_M)
```

In [334]:
```
reg_table( pf['ret'], sm.add_constant(pf['market']) )
```

Model:	OLS	Adj. R-squared:	-0.000
Dependent Variable:	ret	AIC:	-10836.9389
No. Observations:	2266	Log-Likelihood:	5420.5
R-squared:	0.000	Scale:	0.00048997

	coef	std err	t	P>|t|	[95.0% Conf. Int.]
const	0.0013	0.000	2.844	0.004	0.000 0.002
market	-0.0008	0.034	-0.022	0.982	-0.068 0.066

The alpha is positive and significant, while the market beta is not significantly different from zero.

With this strategy we are able to pocket the alpha, without being exposed to the market risk.

More realistic : estimate market beta ex-ante

Suppose we want to implement the strategy starting from 2009.

We estimate the market beta in a backward-looking window, say during the previous quarter.

In [337]:

```
est_win = pd.date_range( pd.datetime(2008,10,1), pd.datetime(2008,12,31)
inv_win = pd.date_range( pd.datetime(2009,1 ,1), pd.datetime(2015,12,31)
```

In [338]:

```
reg = CAPM_reg(Amazon, est_win)
```

Model:	OLS	Adj. R-squared:	0.657
Dependent Variable:	ret	AIC:	-251.8932
No. Observations:	64	Log-Likelihood:	127.95
R-squared:	0.663	Scale:	0.0011088

	coef	**std err**	**t**	**P>|t|**	**[95.0% Conf. Int.]**
const	-0.0005	0.004	-0.115	0.909	-0.009 0.008
market	1.0891	0.099	11.034	0.000	0.892 1.286

Realistic Arbitrage Portfolio

In [335]:
```
mu_A = 1;  mu_M = -1.0891
pf = form_portfolio(Amazon, 'market', mu_A, mu_M)
```

In [336]:
```
reg_table( pf['ret'], sm.add_constant(pf['market']) )
```

Model:	OLS	Adj. R-squared:	-0.000
Dependent Variable:	ret	AIC:	-10836.9389
No. Observations:	2266	Log-Likelihood:	5420.5
R-squared:	0.000	Scale:	0.00048997

	coef	**std err**	**t**	**P>|t|**	**[95.0% Conf. Int.]**
const	0.0013	0.000	2.844	0.004	0.000 0.002
market	0.0201	0.034	0.590	0.555	-0.047 0.087

Great! We still get the alpha and the market exposure is not significant.

Multivariate Regression : Boston house prices

In [341]:
```
reg_table(bost['price'],bost_x)
```

Model:	OLS	Adj. R-squared:	0.567
Dependent Variable:	price	AIC:	3263.4529
No. Observations:	506	Log-Likelihood:	-1626.7
R-squared:	0.570	Scale:	36.664

	coef	std err	t	P>|t|	[95.0% Conf. Int.]
const	-21.0523	2.887	-7.292	0.000	-26.725 -15.380
RM	7.6751	0.418	18.359	0.000	6.854 8.496
CRIM	-0.1860	0.035	-5.365	0.000	-0.254 -0.118
AGE	-0.0317	0.013	-2.506	0.013	-0.056 -0.007
INDUS	-0.1626	0.056	-2.929	0.004	-0.272 -0.054