

$$\begin{aligned}
 1) \quad a. \quad x^{(k+1)} &= x^{(k)} + \alpha(b - Ax^{(k)}) \\
 &= \alpha\left(\frac{1}{\alpha}I x^{(k)}\right) + \alpha(b - Ax^{(k)}) \\
 &= \alpha\left(b - \left(A - \frac{1}{\alpha}I\right)x^{(k)}\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{Then } A_2^{-1} &= \alpha I, \text{ so } A_2 = \frac{1}{\alpha}I \\
 &\Rightarrow A_1 = A - \frac{1}{\alpha}I
 \end{aligned}$$

$$b. \text{ Let } x^{(k+1)} = G(x^{(k)}) = x^{(k)} + \alpha(b - Ax^{(k)})$$

We need G to be a contraction mapping.

$$\begin{aligned}
 \|G(x) - G(y)\|_2 &= \|x + \alpha(b - Ax) - y - \alpha(b - Ay)\|_2 \\
 &= \|x - y + \alpha A(y - x)\|_2 \\
 &= \|x - y - \alpha A(x - y)\|_2 \\
 &= \|(I - \alpha A)(x - y)\|_2 \\
 &\leq \|I - \alpha A\|_2 \|x - y\|_2
 \end{aligned}$$

Let (λ, x) be an eigenvalue - eigenvector pair of A . Then

$$(I - \alpha A)(x) = x - \alpha \lambda x = (1 - \alpha \lambda)x$$

So $1 - \alpha \lambda$ is an eigenvalue of $I - \alpha A$. Since A is symmetric, so is $I - \alpha A$. Then $\|I - \alpha A\|_2 = \rho(I - \alpha A)$, which will be either $|1 - \alpha \lambda_1|$ or $|1 - \alpha \lambda_n|$. Then

$$\begin{aligned}
 |1 - \alpha \lambda_1| &< 1 & |1 - \alpha \lambda_n| &< 1 \\
 -2 < -\alpha \lambda_1 < 0 & & -2 < -\alpha \lambda_n < 0 \\
 \frac{2}{\lambda_1} > \alpha > 0 & & \frac{2}{\lambda_2} > \alpha > 0
 \end{aligned}$$

Then $\alpha \in (0, \frac{2}{\lambda_1}) \cap (0, \frac{2}{\lambda_n}) = (0, \frac{2}{\lambda_1})$ since $\frac{2}{\lambda_1} < \frac{2}{\lambda_2}$. Then we need $\alpha \in (0, \frac{2}{\lambda_1})$

$$\begin{aligned}
 c. \text{ From (c), } x^{(k+1)} &= \alpha(b - (A - \frac{1}{\alpha}I)x^{(k)}) \\
 &= \alpha b + [-(\alpha A - I)]x^{(k)} \\
 &= \alpha b + \underbrace{[I - \alpha A]}_{\text{iteration matrix}} x^{(k)}
 \end{aligned}$$

From earlier, the eigenvalues of $I - \alpha A$ are $1 - \alpha \lambda_1, \dots, 1 - \alpha \lambda_n$. Then we seek to minimize $\rho(I - \alpha A) = \max\{|1 - \alpha \lambda_1|, \dots, |1 - \alpha \lambda_n|\} = \max\{|1 - \alpha \lambda_1|, |1 - \alpha \lambda_n|\}$. We can do so by centering the two at 0, such that $1 - \alpha \lambda_1 < 0$, so

we want to impose that

$$\| \alpha \lambda_1 \| = \| 1 - \alpha \lambda_n \|$$

$$\lambda_1 \alpha - 1 = 1 - \alpha \lambda_n$$

$$(\lambda_1 + \lambda_n) \alpha = 2$$

$$\alpha = \frac{2}{\lambda_1 + \lambda_n}$$

As desired.

d. Let (λ, x) be an eigenvalue-eigenvector pair of A . Then

$$\lambda x = Ax$$

$$\lambda A^{-1}x = x \Rightarrow A^{-1}x = \frac{1}{\lambda}x$$

Then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} , so the eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$. Further, since A is symmetric, it is orthogonally diagonalizable, so there exists a unitary W , diagonal D such that

$$A = W D W^*$$

$$A^{-1} = W^{*-1} D^{-1} W^{-1} = W D^{-1} W^*$$

Then A^{-1} is a real, orthogonally diagonalizable matrix, so it is symmetric. Then $\|A^{-1}\|_2 = \rho(A^{-1}) = \frac{1}{\lambda_n}$.

Then $k_2(A) = \|A\|_2 \|A^{-1}\|_2 = \lambda_1 \frac{1}{\lambda_n}$. Further,

$$M = -A_2^{-1} A_1$$

$$= -(\alpha I)(A - \frac{1}{2}I)$$

$$= -(\alpha A - I) = I - \alpha A$$

From (c), for $\alpha = \frac{2}{\lambda_1 + \lambda_n}$,

$$\rho(M) = \rho(I - \alpha A) = \frac{\alpha \lambda_1 - 1}{1} = \frac{\frac{2\lambda_1}{\lambda_1 + \lambda_n} - 1}{1}$$

$$= \frac{\frac{2k_2(A)\lambda_n}{k_2(A)\lambda_n + \lambda_n} - 1}{1}$$

$$= \frac{\frac{2k_2(A)}{k_2(A) + 1} - 1}{1} = \frac{k_2(A) - 1}{k_2(A) + 1}$$

$$= \frac{k_2(A) - 1}{k_2(A) + 1}$$

$$2) a. G(x_1, x_2) = \begin{bmatrix} \gamma(x_2 - x_1) \cos(\frac{\alpha}{10}) - \gamma(x_2 - 1) \sin(\frac{\alpha}{10}) + 1 \\ \gamma(x_1 - 1) \sin(\frac{\alpha}{10}) + \gamma(x_2 - 1) \cos(\frac{\alpha}{10}) + 5(x_1 - 1)^3 + 1 \end{bmatrix}$$

$$G(x^*) = G(1, 1) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = x^*$$

$$b. \frac{\partial G_1}{\partial x_1} = \gamma \cos(\frac{\alpha}{10}) \quad \frac{\partial G_1}{\partial x_2} = -\gamma \sin(\frac{\alpha}{10})$$

$$\frac{\partial G_2}{\partial x_1} = \gamma \sin(\frac{\alpha}{10}) + 15(x_1 - 1)^2 \quad \frac{\partial G_2}{\partial x_2} = \gamma \cos(\frac{\alpha}{10})$$

Then

$$\nabla G(x^*) = \begin{bmatrix} \gamma \cos(\frac{\alpha}{10}) & -\gamma \sin(\frac{\alpha}{10}) \\ \gamma \sin(\frac{\alpha}{10}) & \gamma \cos(\frac{\alpha}{10}) \end{bmatrix}$$

$$c. \|\nabla G(x^*)\|_2 = |\lambda| \left\| \begin{bmatrix} \cos \frac{\alpha}{10} & -\sin \frac{\alpha}{10} \\ \sin \frac{\alpha}{10} & \cos \frac{\alpha}{10} \end{bmatrix} \right\|_2$$

$$= |\lambda| \sup_{\substack{x \in \mathbb{R}^2 \\ x \neq 0}} \frac{\left\| \begin{bmatrix} \cos \frac{\alpha}{10} & -\sin \frac{\alpha}{10} \\ \sin \frac{\alpha}{10} & \cos \frac{\alpha}{10} \end{bmatrix} x \right\|}{\|x\|} = |\lambda| \sup_{\substack{x \in \mathbb{R}^2 \\ x \neq 0}} \frac{\|x\|}{\|x\|} = |\lambda|$$

Rotation matrix, orthogonal

d. Then fixed point iteration converges for $|\lambda| < 1$.

$$3) a. F(x_1, x_2) = \begin{bmatrix} x_1^2 + x_2^2 + 5x_1 \\ 2x_1x_2 + 3x_2^2 + x_2 \end{bmatrix}$$

$$F(x^*) = F(0, 0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = x^*$$

$$\frac{\partial F_1}{\partial x_1} = 2x_1 + 5$$

$$\frac{\partial F_2}{\partial x_2} = 2x_2$$

$$\frac{\partial F_2}{\partial x_1} = 2x_2$$

$$\frac{\partial F_2}{\partial x_2} = 2x_1 + 6x_2 + 1$$

Then

$$\nabla F(x_1, x_2) = \begin{bmatrix} 2x_1 + 5 & 2x_2 \\ 2x_2 & 2x_1 + 6x_2 + 1 \end{bmatrix}$$

$$\begin{aligned} b. \|\nabla b(x^*)\|_2 &= \|I - \alpha \nabla F(x^*)\|_2 \\ &= \|I - \alpha \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\|_2 \\ &= \left\| \begin{bmatrix} 1-5\alpha & 0 \\ 0 & 1-\alpha \end{bmatrix} \right\|_2 = \rho \left(\begin{bmatrix} 1-5\alpha & 0 \\ 0 & 1-\alpha \end{bmatrix} \right) \end{aligned}$$

By symmetry. Then we need

$$|1-5\alpha| < 1$$

$$|1-\alpha| < 1$$

$$-2 < -5\alpha < 0$$

$$-2 < -\alpha < 0$$

$$\frac{2}{5} > \alpha > 0$$

$$2 > \alpha > 0$$

$$\text{So we need } 0 < \alpha < \frac{2}{5}$$

4) Sherman-Morrison - $(A + xy^T)^{-1} = A^{-1} - \frac{A^{-1}xy^TA^{-1}}{1 + y^TA^{-1}x}$

from Broyden's,

$$A_k = A_{k-1} + \frac{1}{\|s^{(k)}\|_2^2} [y^{(k)} - A_{k-1}s^{(k)}] s^{(k)T}$$

$$\begin{aligned} A_k^{-1} &= (A_{k-1} + \frac{1}{\|s^{(k)}\|_2^2} [y^{(k)} - A_{k-1}s^{(k)}] s^{(k)T})^{-1} \\ &= A_{k-1}^{-1} - \frac{A_{k-1}^{-1} \frac{1}{\|s^{(k)}\|_2^2} (y^{(k)} - A_{k-1}s^{(k)}) s^{(k)T} A_{k-1}^{-1}}{1 + \frac{1}{\|s^{(k)}\|_2^2} (y^{(k)} - A_{k-1}s^{(k)})^T A_{k-1}^{-1} s^{(k)}} \end{aligned}$$

$$= A_{k-1}^{-1} - \frac{A_{k-1}^{-1} (y^{(k)} - A_{k-1}s^{(k)}) s^{(k)T} A_{k-1}^{-1}}{\|s^{(k)}\|_2^2 + (y^{(k)T} - s^{(k)T} A_{k-1}^{-1}) A_{k-1}^{-1} s^{(k)}}$$

$$= A_{k-1}^{-1} - \frac{A_{k-1}^{-1} y^{(k)} s^{(k)T} A_{k-1}^{-1} - s^{(k)} s^{(k)T} A_{k-1}^{-1}}{\|s^{(k)}\|_2^2 + y^{(k)T} A_{k-1}^{-1} s^{(k)} - \|s^{(k)}\|_2^2}$$

$$= A_{k-1}^{-1} + \left(\frac{s^{(k)} - A_{k-1}^{-1} y^{(k)}}{y^{(k)T} A_{k-1}^{-1} s^{(k)}} \right) s^{(k)T} A_{k-1}^{-1}$$

Homework3

February 25, 2024

```
[64]: import numpy as np
import scipy
import matplotlib.pyplot as plt
```

```
[72]: def F(x):
    return (np.array([x[0]**2 + x[1]**2 + 5*x[0],
                      2*x[0]*x[1] + 3*x[1]**2 + x[1]]),
            np.array([[2*x[0]+5, 2*x[1]],
                      [2*x[1], 2*x[0]+6*x[1]+1]]))
```

```
[86]: def newton(x0, F, it=50):
    points = np.zeros((it, len(x0)))
    points[0] = x0
    x = x0
    for i in range(1, it):
        f, grad = F(x)
        h = scipy.linalg.solve(grad, -f)
        x += h
        points[i] = x
    return points

def fpi(x0, F, alpha, it=50):
    points = np.zeros((it, len(x0)))
    points[0] = x0
    x = x0
    for i in range(1, it):
        f, _ = F(x)
        x = x - alpha*f
        points[i] = x
    return points
```

```
[89]: newton_pts = newton(np.array([0.1,0.1]), F, 50)
fpi_pts = fpi(np.array([0.1,0.1]), F, 0.1, 50)
```

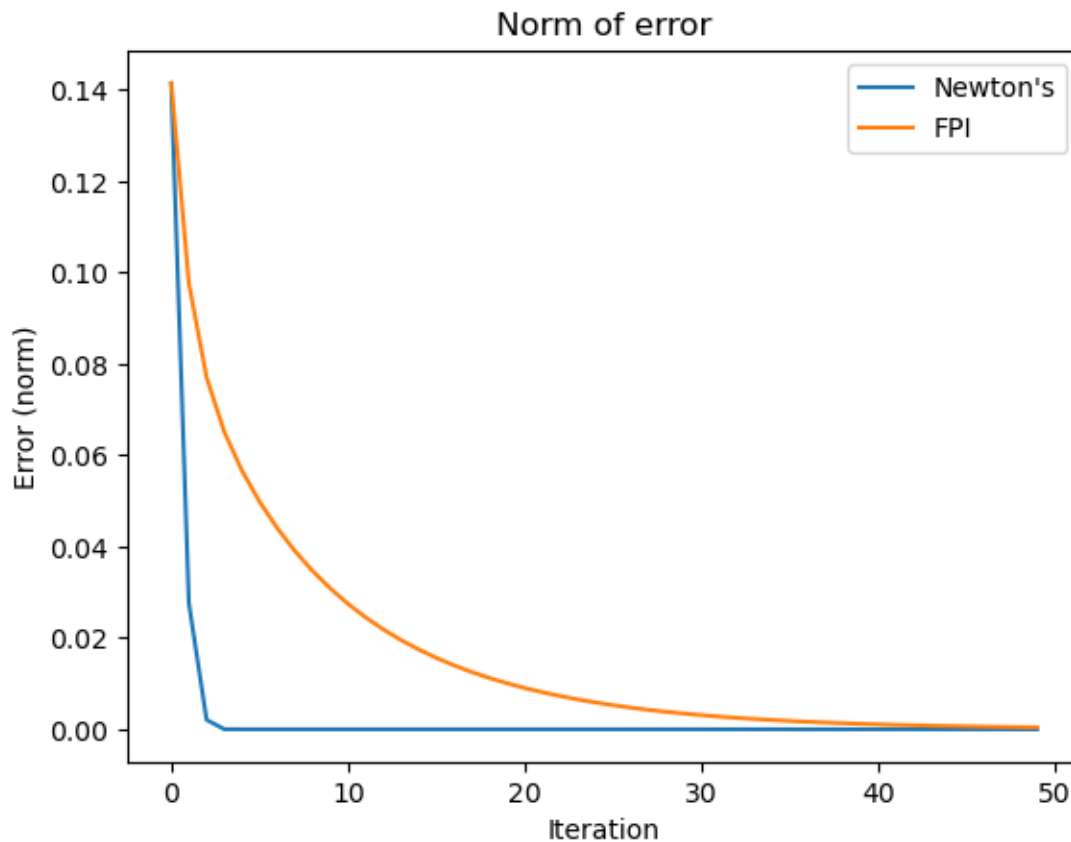
```
[90]: fig, ax = plt.subplots()
ax.set_title('Norm of error')
ax.set_xlabel("Iteration")
```

```

ax.set_ylabel("Error (norm)")
ax.plot(np.arange(0,len(newton_pts)), np.linalg.norm(newton_pts,axis=1),
        label="Newton's")
ax.plot(np.arange(0,len(fpi_pts)), np.linalg.norm(fpi_pts,axis=1), label="FPI")
plt.legend()

```

[90]: <matplotlib.legend.Legend at 0x7ff468be0410>



```

[93]: fig, ax = plt.subplots()
ax.set_title('Estimate of solution')
ax.set_xlabel("x1")
ax.set_ylabel("x2")
ax.plot(newton_pts[:,0], newton_pts[:,1], label="Newton's")
ax.plot(fpi_pts[:,0], fpi_pts[:,1], label="FPI")
plt.legend()

```

[93]: <matplotlib.legend.Legend at 0x7ff468c42120>

