Math 151AH HW 4 Andrea Boretto

1) From lecture,

$$\|f - P_n\|_\infty \le \frac{1}{(n+1)!} 2^{-n} \|f^{(n+1)}\|_\infty$$
$$= \frac{1}{(n+1)!} 2^{-n} \|e^{-x}\|_\infty$$
$$= \frac{1}{(n+1)!} 2^{-n} e$$

We want this to be bounded by 10^{-10}, so

$$\|f - P_n\|_\infty \le \frac{1}{(n+1)!} 2^{-n} e \le 10^{-10}$$
$$\frac{1}{(n+1)!} 2^{-n} \le \frac{1}{e} 10^{-10}$$

Solving this numerically, $n = 10$ is the lowest number of points guaranteeing $\|f - P_n\|_\infty \le 10^{-10}$.

2) a. $T_{n+1}(x) = \cos(n\cos^{-1}(x) + \cos^{-1}(x))$
$= \cos(n\cos^{-1}(x))\cos(\cos^{-1}(x)) - \sin(n\cos^{-1}(x))\sin(\cos^{-1}(x))$

$T_{n-1}(x) = \cos(n\cos^{-1}(x) - \cos^{-1}(x))$
$= \cos(n\cos^{-1}(x))\cos(-\cos^{-1}(x)) - \sin(n\cos^{-1}(x))\sin(-\cos^{-1}(x))$
$= \cos(n\cos^{-1}(x))\cos(\cos^{-1}(x)) + \sin(n\cos^{-1}(x))\sin(\cos^{-1}(x))$

Then

$$T_{n+1}(x) + T_{n-1}(x) = 2\cos(n\cos^{-1}(x))\,x = 2x\,T_n(x)$$

So

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Then

$$T_2(x) = 2x(x) - 1 = 2x^2 - 1$$
$$T_3(x) = 2x(2x^2-1) - x = 4x^3 - 2x - x$$
$$= 4x^3 - 3x$$
$$T_4(x) = 2x(4x^3 - 3x) - (2x^2-1) = 8x^4 - 6x^2 - (2x^2-1)$$
$$= 8x^4 - 8x^2 + 1$$

b. By induction: Base case: for $T_1(x) = x$, the leading coefficient is clearly $2^{n-1} = 2^{1-1} = 1$. Also $T_1 \in P_1$, $T_0 \in P_0$.
Induction step: Assume the leading coefficient of $T_n(x)$ is 2^{n-1}, and that $T_n \in P_n$, $T_{n-1} \in P_{n-1}$.

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$
$$= 2x(2^{n-1}x^n + q(x)) - T_{n-1}(x)$$

where $q(x) \in P^{n-1}$, $T_{n-1} \in P^{n-1}$. Then
$$T_{n+1}(x) = 2^n x^{n+1} + 2x\,q(x) - T_{n-1}(x)$$
where $2x\,q(x) \in P^n$. Then the leading term of $T_{n+1}(x)$ is $2^n x^{n+1}$, and so the leading coefficient is 2^n, as desired.

c. $\{s_k\}_{k=0}^{n}$ $\quad s_k = \cos\left(\frac{2k-1}{2n}\pi\right)$

$$\begin{aligned} T_n(s_k) &= \cos\left(n \cos^{-1}\left(\cos\left(\frac{2k-1}{2n}\pi\right)\right)\right) \\ &= \cos\left(n \frac{2k-1}{2n}\pi\right) \\ &= \cos\left((2k-1)\frac{\pi}{2}\right) \end{aligned}$$

Since $2k-1$ is always odd, $(2k-1)\frac{\pi}{2} \in \{-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots\}$, so
$$T_n(s_k) = \cos\left((2k-1)\frac{\pi}{2}\right) = 0$$

d. $T_n(x) = \cos(n \cos^{-1}(x))$
$$\frac{d}{dx} T_n(x) = -\sin(n\cos^{-1}(x))\, n \frac{-1}{\sqrt{1-x^2}} \qquad x \neq 1, -1$$
$$= \sin(n\cos^{-1}(x)) \frac{n}{\sqrt{1-x^2}} = 0$$
$$n\cos^{-1}(x) = \sin^{-1}(0) = k\pi \quad \text{for any } k \in \mathbb{N}$$
$$x = \cos\left(\frac{k\pi}{n}\right)$$

Since $x \in (-1, 1)$, we have to bound k by $(0, n) \cap \mathbb{N}$. Then, for $t_k = \cos\left(\frac{k\pi}{n}\right)$, $0 < k < n$, each t_k is a local extrema of $T_n(x)$, and every local extrema to $T_n(x)$ is one of the t_k's. Further
$$\begin{aligned} T_n(t_k) &= \cos\left(n \cos^{-1}\left(\frac{k\pi}{n}\right)\right) \\ &= \cos(k\pi) = (-1)^k \end{aligned}$$

For $k = 0$: $t_0 = \cos(0) = 1$, and
$$\begin{aligned} T_n(t_0) &= \cos\left(n\cos^{-1}\left(\cos\left(\tfrac{0}{n}\right)\right)\right) \\ &= \cos(0) = 1 = (-1)^0 \end{aligned}$$

For $k = n$: $t_n = \cos(\pi) = -1$, and
$$\begin{aligned} T_n(t_n) &= \cos(n\cos^{-1}(\cos(\pi))) \\ &= \cos(n\pi) = (-1)^n \end{aligned}$$

Then the $\{t_k\}_{k=0}^{n}$ is the set of absolute extrema of $T_n(x)$.

1. Suppose for contradiction there exists $P \in \hat{\Pi}_n$ such that $\|P\|_\infty < \|T_n\|_\infty$ on $x \in [-1, 1]$. Since $P, \tilde{T}_n \in P_n$ are both monic polynomials, their leading term is the same, so $Q = T_n - P \in P_{n-1}$, so Q has $n-2$ roots. However, since $\|T_n\|_\infty > \|P\|_\infty$, and T_n achieves its absolute extrema at each $\{t_k\}_{k=0}^n$, $Q = T_n - P$ has sign $(-1)^k$ at each $k = 0, \dots, n$. Then, by the intermediate value theorem, Q has $n-1$ roots, a contradiction.