A bound on the number of lines through a point in the Wang-Shi split

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Abstract

This contribution answers the question that Tom Lyche addressed to the public of the 2022-INDAM meeting in Cortona "Approximation Theory and Numerical Analysis meet Algebra, Geometry, Topology" while presenting his work with Carla Manni and Hendrik Speelers.

The question is if the number of lines through a point in the Wang-Shi split of degree d is always less than or equal to $d+1$. While expressed in simple terms it has application to the construction of piecewise polynomials spaces with maximal degree of continuity on general triangulation by *splitting* each triangle in sub-polygons.

1 Introduction

Let $T \subset \mathbb{R}^2$ be a triangle with vertices w_0, w_1, w_2. The Wang-Shi-split of degree d of T is obtained by defining the set of knots $V(d,T) = \{v_0, \ldots, v_{3d-1}\}$ with

$$v_i = d^{-1} \begin{cases} iw_1 + (d-i)w_0 & i = 0, \ldots, d \\ (i-d)w_2 + (2d-i)w_1 & i = d+1, \ldots, 2d \\ (i-2d)w_0 + (3d-i)w_2 & i = 2d+1, \ldots, 3d-1. \end{cases}$$

For ease of discussion let $\ell_{i,j} = \overline{v_i v_j}$ be the segment connecting v_i and v_j,

$$\begin{aligned} L(d,T) &= \{\ell_{i,j} : \lfloor i/d \rfloor \neq \lfloor j/d \rfloor, \lceil i/d \rceil \neq \lceil j/d \rceil\}, \\ P(d,T) &= \{\ell_1 \cap \ell_2 : \ell_1 \neq \ell_2 \in L(d,T)\}. \end{aligned}$$

Then the Wang–Shi split has for subdomains the connected components of

$$T \setminus \bigcup L(d,T),$$

which we denote by $\mathcal{T}^d_{\mathrm{WS}}$. The subdomains $\tau \in \mathcal{T}^d_{\mathrm{WS}}$ are convex, but not necessarily triangles.

Tom Lyche presented the following open problem to the public:

Is the maximum number of graph edges
intersecting at the same point less or equal to $d+1$?

and, since the question is independent of T it is about

$$Q(d) := \max_{p \in P(d,T)} \#\{\ell \in L(d,T) : p \in \ell\}.$$

The answer is related to the dimension formula for the space of piecewise polynomials of degree d on the Wang–Shi split; the formula simplifies significantly if the answer is affirmative, see [3, 1] and Section 5 for details.

Partial answers were already known at the meeting: Tom Lyche with Carla Manni and Hendrik Speleers had checked the result computationally for $d \leq 8$; other participants like Francesco Patrizi and Frank Sottile extended the checked cases using their own code at least till $d \leq 18$. Our main theoretical result is

Theorem 1. *For all $d \geq 17$,*

$$Q(d) \leq d+1.$$

It is clear from the definition that $Q(d)$ can become arbitrarily large: take T with rational vertices and $p \in T^\circ$ with rational coordinates. Then take k lines through p with rational slope. The intersections of the lines with ∂T have rational coordinates and spit each edge in segments whose length is a fraction of the edge length. If all the denominators of these fractions divide d then all lines are in $L(d,T)$ so that $Q(d) \geq k$.

Section 2 is the proof of Theorem 1. Section 3 explains how to numerically verify the remaining cases. In fact, asymptotically $Q(d)$ is much smaller, as is shown in Section 4 using a number theoretical argument. Section 5 shows the application to the spline dimension formula.

To ease notation $X(d,T)$ will be shortened to X, and the indexing of vertices and edges of T is done in $\mathbb{Z}/3\mathbb{Z}$, i.e. $w_3 = w_0$. With this convention $e_i = \overline{w_i w_{i+1}}$, $i = 0, 1, 2$, are the edges of T.

2 A proof for large degrees: $d \geq 17$

Let $p \in P$ be any intersection point, and L_p the set of segments containing p

$$L_p := \{\ell \in L : \ell \ni p\}.$$

The segments in L_p connect two knots in two adjacent edges e_i and e_{i+1} so it is covered by

$$L_{p,i} := \{\ell \in L_p : e_i \cap \ell \neq \emptyset,\, e_{i+1} \cap \ell \neq \emptyset,\, w_{i+1} \notin \ell\}.$$

The $L_{p,i}$, $i = 0, 1, 2$ are not necessarily disjoint as both $L_{p,i}$, $L_{p,i+1}$ can contain a segment connecting the vertex $w_i = v_{id} = w_{i+3}$. This gives that $k_i(p) := \#L_{p,i}$ and $k(p) := \#L_p$ are related by

$$k_0(p) + k_1(p) + k_2(p) - 3 \leq k(p) \leq k_0(p) + k_1(p) + k_2(p). \tag{1}$$

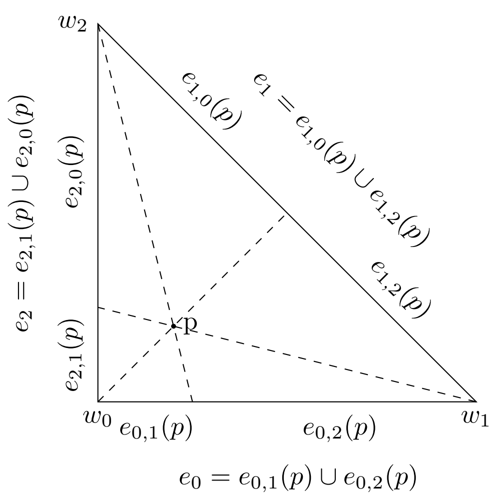

Figure 1: Picture of a triangle T with a chosen point p and the parts of the edges $e_{i,j}$ that correspond through the map S_p. Each line through p intersects the boundary of T in two corresponding segments $e_{i,j}$ and $e_{j,i}$.

To $p \in P$ corresponds an homeomorphism $S_p : \partial T \to \partial T$ that maps a point g to the other intersection of the line through p and g with ∂T. Of interest for the following discussion are the sets

$$e_{i,j}(p) := S_p(e_j) \cap e_i = S_p(e_{j,i}).$$

The extrema of the segments in $L_{p,i}$ are contained in $e_{i,i+1}(p) \cup e_{i+1,i}(p)$. See Figure 1.

Let μ_i be the distance between two consecutive knots in e_i. Let $\delta_{i,j}(p)$ be the maximum distance between two knots in $e_{i,j}(p) \cap \bigcup L$ and for $i \in \mathbb{Z}/3\mathbb{Z}$

$$u_i(p) = \frac{\delta_{i,i+1}(p)}{\mu_i} + \frac{\delta_{i+1,i}(p)}{\mu_{i+1}}.$$

By construction

$$u_i(p) \leq \frac{|e_{i,i+1}|}{\mu_i} + \frac{|e_{i+1,i}|}{\mu_{i+1}} \tag{2}$$

so that

$$\begin{aligned} u(p) :=& u_0(p) + u_1(p) + u_2(p) \\ \leq& \frac{|e_{0,1}|}{\mu_0} + \frac{|e_{0,2}|}{\mu_0} + \frac{|e_{1,2}|}{\mu_1} + \frac{|e_{1,0}|}{\mu_1} + \frac{|e_{2,0}|}{\mu_2} + \frac{|e_{2,1}|}{\mu_2} \\ =& 3d. \end{aligned}$$

On the other hand, $u(p)$ is bound from below as a function of the $k_i = \#L_{p,i}$ and this will imply a maximum for k as a function of d.

This can be analyzed on the reference triangle for segments connecting the horizontal with the vertical edge, i.e. studying $L_{p,2}$ in the case $w_0 =$

$(0,0)$, $w_1 = (d,0)$ and $w_2 = (0,d)$. For this case let the segments in $L_{p,2}$ be

$$\overline{(x_i,0)(0,y_i)}, \qquad i = 1,\dots,k_2,$$

ordered by increasing x_i so that

$$u_2(p) = (x_{k_2} - x_1) + (y_1 - y_{k_2}) = \sum_{i=1}^{k_2-1} (x_{i+1} - x_i + y_i - y_{i+1}).$$

The relation between x_i and y_i is given by $y_i = \pi_2 S_p(0,x_i)$, where π_2 is the projection on the ordinate, and extends to rational bijection between $\mathbb{R}_{>p_1}$ and $\mathbb{R}_{>p_2}$ (where $p = (p_1,p_2)$)

$$y(x) = \frac{p_1 p_2}{x - p_1} + p_2. \tag{3}$$

Studying the integer solutions of (3) leads to number theoretical questions, see Section 4. However, in the following, we remain in the context of real numbers.

Lemma 1. *Given $s,t > 0$ there exists a unique $x > p_1$ such that*

$$y(x+s) = y(x) - t. \tag{4}$$

Proof. By (3)

$$y(x) - y(x+s) = \frac{p_1 p_2 s}{(x-p_1)(x+s-p_1)}.$$

The right hand side is a monotone decreasing function of x on $\{x > p_1\}$ with limits

$$\lim_{x\to +p_1} \frac{p_1 p_2 s}{(x-p_1)(x+s-p_1)} = +\infty, \qquad \lim_{x\to +\infty} \frac{p_1 p_2 s}{(x-p_1)(x+s-p_1)} = 0.$$

This implies that for all s and t, (4) has a unique solution. □

Corollary 1. *The map $\{1,\dots,k_2-1\} \to \mathbb{N}^2_{>0}$ defined by*

$$i \to (x_{i+1} - x_i, y_i - y_{i+1}).$$

is injective.

Lemma 2. *Let π_i, $i = 1,2$ be the coordinate projection, and*

$$m(t) := \min_{I\subset \mathbb{N}^2_{>0}, \#I = t-1} \Big\{ \sum_{K\in I} (\pi_1 + \pi_2)K \Big\}$$

then

$$m(k_i) \le u_i(p).$$

Proof. It is enough to prove the result for $i = 2$. Since the map $i \to (x_{i+1} - x_i, y_i - y_{i+1})$ is injective by Corollary 1 its range R is a subset of $\mathbb{N}^2_{>0}$ having cardinality $k_2 - 1$. Then $\sum_{K\in R} (\pi_1 + \pi_2)K = u_2(p)$ is greater than the minimum over all subsets. □

Lemma 3. *The following estimate holds*

$$\underline{m}(k) := (k-1)\frac{\sqrt{8k-7}+3}{3} \leq m(k)$$

Proof. For $r \in \mathbb{N}_{>0}$ there are r elements whose sum-of-components is $r+1$ in $\mathbb{N}^2_{>0}$, namely

$$(1, r), (2, r-1), \ldots (r, 1).$$

This means that m is the restriction to $\mathbb{N}_{>0}$ of a piecewise linear function that for $k = 1 + \sum_{r=1}^{t} r$ is

$$m(k) = m\Big(1 + \sum_{r=1}^{t} r\Big) = \sum_{r=1}^{t} r(r+1) = \frac{t(t+1)(t+2)}{3}.$$

Now from

$$k = \frac{t(t+1)}{2} + 1$$

we get

$$t + 2 = \frac{\sqrt{8k-7}+3}{2}$$

and, for the selected k, m agrees with $\underline{m}$

$$m(k) = t(t+1)\frac{t+2}{3} = 2(k-1)\frac{\sqrt{8k-7}+3}{6} = \underline{m}(k).$$

Since m is piecewise linear and $\underline{m}$ is convex,

$$\underline{m}''(k) = \frac{8}{3}\frac{6k-5}{(8k-7)^{3/2}} > 0, \tag{5}$$

it follows that $\underline{m}$ is a global lower bound for m. □

Lemma 4. *The number k of segments in $\ell \in L$ passing through a point $p \in P$ satisfies*

$$\underline{\underline{m}}(k) := \frac{1}{9}(k-3)(\sqrt{3}\sqrt{8k-21}+9) \leq 3d. \tag{6}$$

Proof. Indeed we have

$$\underline{m}(k_0) + \underline{m}(k_1) + \underline{m}(k_2) \leq u(p) \leq 3d.$$

The Hessian of $\underline{m}(k_0) + \underline{m}(k_1) + \underline{m}(k_2)$ is diagonal with positive entries, see (5). Consequently the left hand side is a convex function, and for fixed $k_0 + k_1 + k_2$ the stationary point $k_0 = k_1 = k_2$ is a minimum. Using (1), we replace k_i with $k/3$ leading to

$$3\Big(\frac{k}{3} - 1\Big)\frac{3 + \sqrt{8/3k - 7}}{3} \leq 3d,$$

which is the thesis up to algebraic manipulations. □

Proof of Theorem 1. For $Q(d)$ to be $d+2$ the inequality

$$\underline{\underline{m}}(d+2) \leq 3d$$

must be satisfied. Solving for equality gives

$$d = \frac{43 + \left(96011 - 72\sqrt{19653}\right)^{1/3} + \left(96011 + 72\sqrt{19653}\right)^{1/3}}{8} \approx 16.8\ldots$$

To conclude it is sufficient to notice that $\underline{\underline{m}}$ is super-linear and convex so that for all d grater than the value above the inequality cannot be satisfied and the maximum number of lines is $\leq d+1$. $\square$

3 Checking small degrees: $d \leq 109$

Two programs that given d compute $Q(d)$ have been written independently by the authors and tested for agreement. They are available in the public repository github.

The programs work as follows:

- The three vertices w_0, w_1 and w_2 of the triangle are initialized.
- The homogeneous coordinates $\tilde{v}_i$ of the points $v_i \in V(d,T)$ are computed.
- The homogeneous coordinates $\tilde{\ell}_{i,j} = \tilde{v}_i \times \tilde{v}_j$ are computed for all lines $\ell_{i,j} \in L(d,T)$. Here the indices are taken such that each line is computed uniquely.
- The homogeneous coordinates $\tilde{p}_{\mu,\nu} = \tilde{\ell}_\mu \times \tilde{\ell}_\nu$ are computed for all intersections of pairs of lines $\{\ell_\mu, \ell_\nu\} \subset L(d,T)$, with $\mu \neq \nu$, and then transformed to Cartesian coordinates $p_{\mu,\nu}$.
- All intersection points outside the triangle are dropped.
- All intersection points inside the triangle are tallied, i.e., for each intersection point $p \in T^\circ$ we count the number $n(p)$ of pairs $\{\ell_\mu, \ell_\nu\}$ that yield $p_{\mu,\nu} = p$.
- Let $k(p)$ be the number of lines through p, then $k(p)(k(p)-1)/2$ is the number of pairs of line that intersect at p. Thus we compute $k(p) = \frac{1}{2}(1 + \sqrt{1 + 8n(p)})$. We then compute $Q(d)$ as the maximum of $k(p)$ for all $p \in P(d,T)$.

The computation of all lines can be done in $\#L(d,T) = 3d(d-1) = O(d^2)$ steps, whereas the computation, handling and tallying of all intersection points can be performed in $O(d^4)$ steps, which is the overall complexity of the algorithm.

A version of the code exploits the symmetries of T, i.e., the invariance by permutation of the vertices to decrease the constant by analyzing only intersections in a sixth of the triangle.

The first eight cases of $d = 1, \ldots, 8$ for

$$w_0 = (0,0), \qquad w_1 = (0,d), \qquad w_2 = (d,0)$$

are depicted in Figures 2–4. Table 1 contains the computed values of $Q(d)$ for $d \leq 109$. Next to the table is a plot of $Q(d), \underline{\underline{m}}(k)$ and the line $k = d+2$ that gives a graphical representation of the proof.

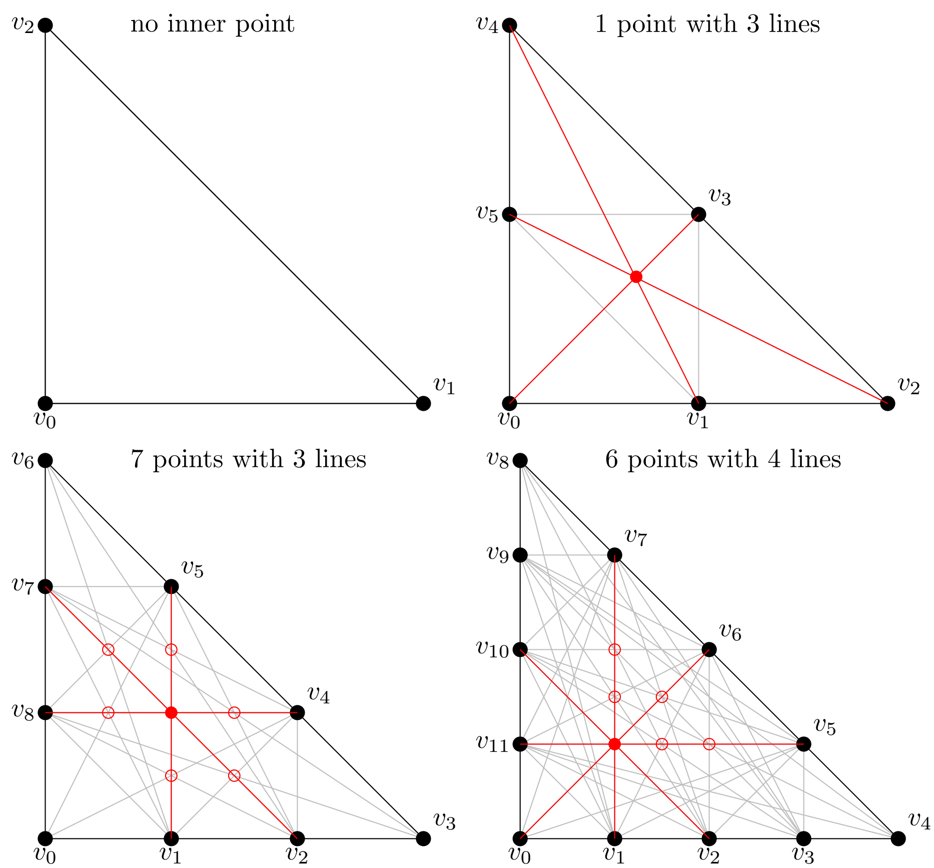

Figure 2: The Wang-Shi split of degree 1–4 with highlighted the intersection points with maximum degrees and the lines through one of them.

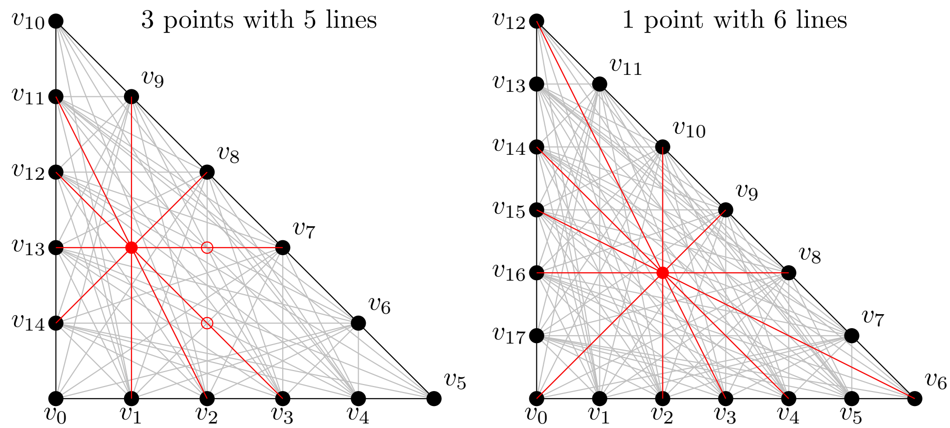

Figure 3: The Wang-Shi split of degree 5, 6 with highlighted the intersection points with maximum degrees and the lines through one of them.

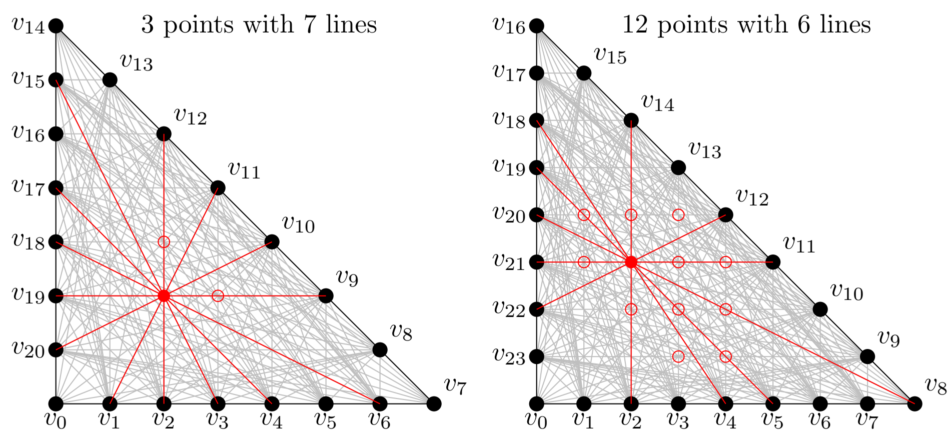

Figure 4: The Wang-Shi split of degree 7, 8 with highlighted the intersection points with maximum degrees and the lines through one of them.

d	0	1	2	3	4	5	6	7	8	9
_		0	3	3	4	5	6	7	6	8
1_	8	9	9	10	10	10	12	11	12	12
2_	12	13	14	13	14	14	14	15	16	17
3_	16	17	16	17	18	17	18	18	19	19
4_	19	18	18	20	20	18	22	20	19	20
5_	21	21	21	22	22	21	23	22	22	21
6_	21	21	25	25	24	25	24	25	24	23
7_	26	24	26	24	25	26	26	27	26	25
8_	27	25	28	27	28	27	28	26	28	29
9_	27	27	28	27	30	28	29	28	30	28
10_	29	29	31	29	31	31	31	31	31	30

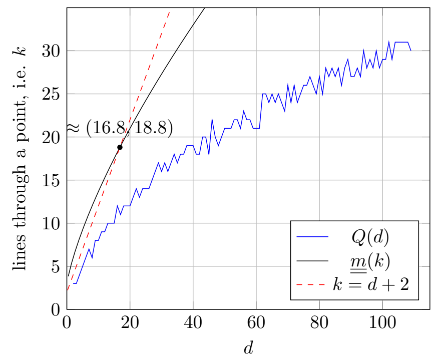

Table 1: Computed values of $Q(d)$ for $d = 1, \ldots, 109$. The entries until $d = 18$ coincide with those published by Frank Sottile in [4].

4 Asymptotic behavior

Let us now follow up on the considerations that lead us to the relation (3), i.e., all line segments $\ell = \overline{(x, 0)(0, y)}$ going through a point $p = (p_1, p_2)$ in $P(d, T)$ satisfy

$$y = \frac{p_1 p_2}{x - p_1} + p_2. \tag{7}$$

Let ℓ_1 and ℓ_2 be line segments, with $\ell_i = \overline{(x_i, 0)(0, y_i)}$, that intersect at p. Then we have

$$p_1 = \frac{x_1 x_2 (y_1 - y_2)}{x_2 y_1 - x_1 y_2} = \frac{\hat{p}_1}{q} \quad \text{and} \quad p_2 = \frac{y_1 y_2 (x_2 - x_1)}{x_2 y_1 - x_1 y_2} = \frac{\hat{p}_2}{q},$$

where $\hat{p}_1$, $\hat{p}_2$ and q are integers. Hence, (7) is equivalent to

$$(y - p_2)(x - p_1) = p_1 p_2,$$

which can be rewritten as

$$(y\, q - \hat{p}_2)(x\, q - \hat{p}_1) = \hat{p}_1 \hat{p}_2. \tag{8}$$

Thus, for a fixed point p the number of pairs (x, y) that satisfy the equation is bounded by the number of divisors of $\hat{p}_1 \hat{p}_2$, for which asymptotic growth rates are known. This leads to the following.

Corollary 2. *We have*

$$Q(d) = o(d^{\varepsilon})$$

for all $\varepsilon > 0$.

Proof. We follow the notation in Section 2. We use (1) and obtain

$$Q(d) \leq \max_{p \in P(d,T)} |L_p| \leq \max_{p \in P(d,T)} (k_0(p) + k_1(p) + k_2(p)) \leq 3 \max_{p \in P(d,T)} k_2(p).$$

Here $k_2(p)$ is the number of line segments of the form $\ell = \overline{(x, 0)(0, y)}$, with integers $1 \leq x, y \leq d$, going through the point p. Following from (8), this number is bounded by the number of divisors of $\hat{p}_1 \hat{p}_2$, i.e.,

$$k_2(p) \leq \delta(\hat{p}_1 \hat{p}_2),$$

since the mapping

$$\begin{aligned} L_{p,2} &\mapsto \{\rho \in \mathbb{Z}^+ : \rho \mid \hat{p}_1\hat{p}_2\} \\ \overline{(x,0)(0,y)} &\to x\,q - \hat{p}_1 \end{aligned}$$

is an injection. Every integer pair (x, y) yields one divisor pair ρ and $\hat{p}_1\hat{p}_2/\rho$, but not vice versa, since, for an arbitrary divisor ρ the terms $x = (\rho + \hat{p}_1)/q$ and $y = (\hat{p}_1\hat{p}_2/\rho + \hat{p}_2)/q$ are not necessarily integers between 1 and d.

Let $\mathfrak{d}(n)$ be the number of (positive) divisors of the integer n (unfortunately, in number theory the standard notation for this function is d, to avoid confusion, we use $\mathfrak{d}$). It is a well-known result that $\mathfrak{d}(n) = o(n^\delta)$ for all $\delta > 0$, cf. [2, Sec. 18.1, Thm. 315]. We have by definition $\hat{p}_1\hat{p}_2 \le d^6$, therefore

$$Q(d) \le 3 \max_{p\in P(d,T)} k_2(p) \le 3 \max_{N\le d^6} \mathfrak{d}(N) = o((d^6)^\delta).$$

Selecting $\delta = \varepsilon/6$ yields the thesis. □

Thus, the growth rate of $Q(d)$, as d goes to infinity, is very small. However, the argument that is used in the proof is not yet strong enough to yield any meaningful estimates for reasonably sized d. Further studies would require deeper insights into the integer solutions of equations of the form (8).

5 Splines on the Wang–Shi split

Knowing that $Q(d) \le d + 1$, the dimension formula for splines defined on the Wang–Shi split simplifies. We denote by $S_d^{d-1}(\mathcal{T}_{\mathrm{WS}}^d)$ the spline space of degree d and smoothness C^{d-1} over the partition $\mathcal{T}_{\mathrm{WS}}^d$ generated by the Wang–Shi split of degree d, i.e.,

$$S_d^{d-1}(\mathcal{T}_{\mathrm{WS}}^d) = \{s \in C^{d-1}(T) : s|_\tau \in \mathbb{P}_d,\ \forall\tau \in \mathcal{T}_{\mathrm{WS}}^d\}.$$

It was shown by Lyche, Manni and Speleers in [3, Theorem 1] that the dimension of this space satisfies

$$\dim S_d^{d-1}(\mathcal{T}_{\mathrm{WS}}^d) = \dim \mathbb{P}_d + m + \sum_{p\in P(d,T)} \zeta(k(p)),$$

where $m = \#L(d, T)$ is the number of lines through T and $\zeta(k(p))$ is a function that depends non-trivially on the number of lines $k(p)$ through p. This result is based on a general dimension formula for splines over cross-cut partitions developed in [1, Theorem 3.1]. The function $\zeta(k(p))$ is such that it vanishes if $k(p) \le d + 1$, thus we have the following.

Corollary 3. *We have*

$$\dim S_d^{d-1}(\mathcal{T}_{\mathrm{WS}}^d) = \binom{d+2}{2} + 3d(d-1),$$

for all $d \ge 1$.

Proof. This result follows from [1, Theorem 3.1], together with Theorem 1 and the results in Table 1, cf. [4] and [3, Theorems 1 and 2]. □

References

[1] Charles K. Chui and Ren Hong Wang. Multivariate spline spaces. *J. Math. Anal. Appl.*, 94(1):197–221, 1983.

[2] Godfrey Harold Hardy and Edward Maitland Wright. *An introduction to the theory of numbers.* Oxford university press, 2008.

[3] Tom Lyche, Carla Manni, and Hendrik Speleers. Construction of C^2 cubic splines on arbitrary triangulations. *Found. Comput. Math.*, 22(5):1309–1350, 2022.

[4] Frank Sottile. Some problems at the interface of approximation theory and algebraic geometry. *arXiv preprint arXiv:2308.03211*, 2023.