Answers to questions in
Lab 1: Filtering operations

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Instructions: Complete the lab according to the instructions in the notes and respond to the questions stated below. Keep the answers short and focus on what is essential. Illustrate with figures only when explicitly requested.

Good luck!

Question 1: Repeat this exercise with the coordinates p and q set to (5, 9), (9, 5), (17, 9), (17, 121), (5, 1) and (125, 1) respectively. What do you observe?

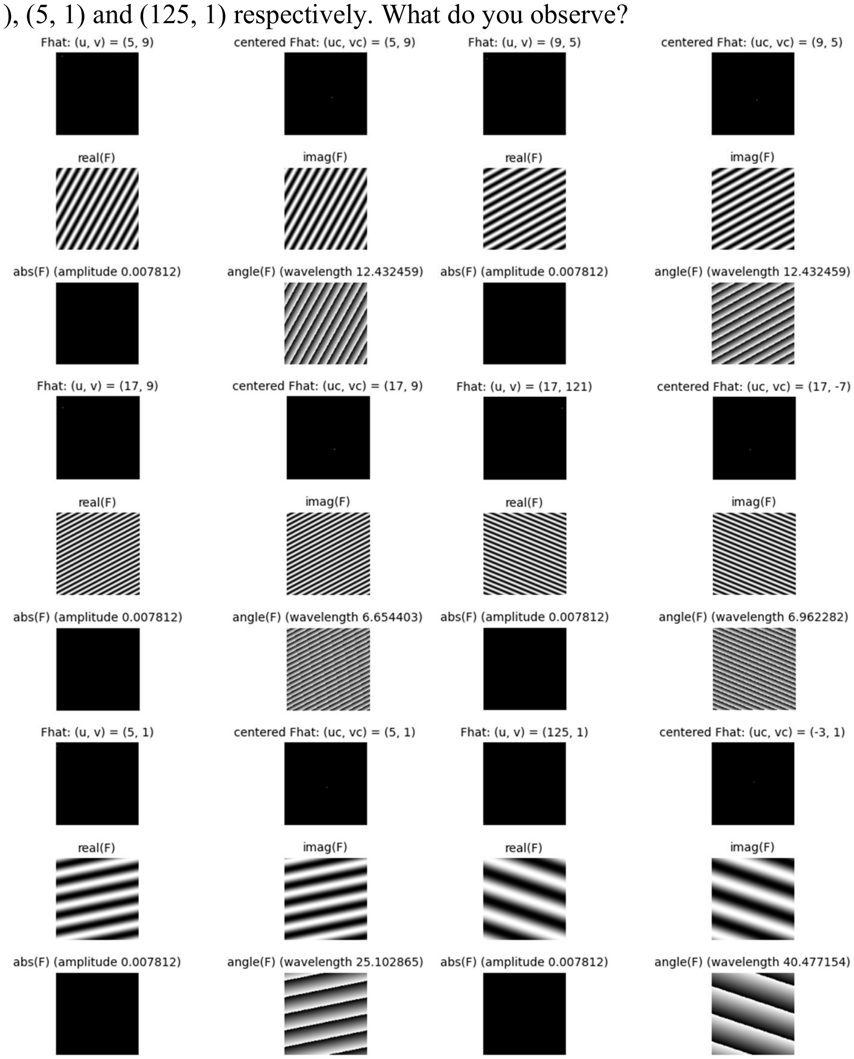

Fig. 1: fftwave() computed wrt the provided coordinates

Answers:

- Lower frequencies on the edges (e.g. zero-frequency top-left corner)
- Higher frequencies near the center (e.g highest frequency on the center)
- The position of the white dot determines the waves direction
- Moving along axis u (Vertical axis) yields horizontal waves (e.g. (63,0), (125,1), (5,1))
- Moving along axis v (Horizontal axis) yields vertical waves (e.g. 0,63)
- Moving along both axes determines diagonal waves
- The amplitude doesn't vary (same amplitude)
- Real and Imagine parts are shifted with a phase of π2π2
- The waves have the same direction of the line from the origin (0,0) to (p,q), and the more the distance from the origin the higher the frequency
- The wavelength depends on the frequency (The higher the frequency the lower the wavelength)

Question 2: Explain how a position (p, q) in the Fourier domain will be projected as a sine wave in the spatial domain. Illustrate with a Matlab figure.

Answers:

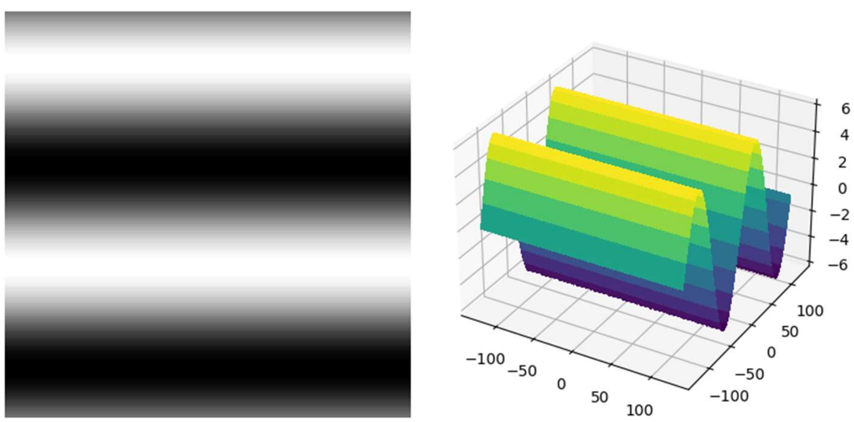

Fig. 2: Left: imaginary part of F in 2D. Right: imaginary part of F in 3D

Question 3: How large is the amplitude? Write down the expression derived from Equation (4) in the notes. Complement the code (variable amplitude) accordingly.

Answers:

From (4), $F(m,n) = \frac{1}{\sqrt{MN}}\sum_{m:0}^{M}\sum_{n:0}^{N}\left[\hat{F}(u,v)e^{\frac{2\pi inv}{N}}\right]e^{\frac{2\pi imu}{M}}$,

and from euler's equation $e^{iwx} = cos(wx) + isin(wx)$

$$F(m,n) = \frac{1}{\sqrt{MN}}\sum_{m:0}^{M}\sum_{n:0}^{N}\hat{F}(u,v)cos\left(\frac{2\pi inv}{N} + \frac{2\pi imu}{M}\right) + isin\left(\frac{2\pi inv}{N} + \frac{2\pi imu}{M}\right)$$

Spectrum of Fourier transform = $|F(u,v)| = \sqrt{Re(F)^2 + Im(F)^2}$,

and amplitude = $max|F(u,v)| = \frac{1}{\sqrt{MN}}max\left[\sum_{m:0}^{M}\sum_{n:0}^{N}\hat{F}(u,v)\right]$

M,N = sz, sz ➔ amplitude = $\frac{\widehat{F_{max}}}{sz}$

Question 4: How does the direction and length of the sine wave depend on p and q? Write down the explicit expression that can be found in the lecture notes. Complement the code (variable wavelength) accordingly.

Answers:

$$\lambda = \frac{2\pi}{||w||} = \frac{2\pi}{\sqrt{w_1^2 + w_2^2}}; w_D = \frac{2\pi u}{N}$$

$$\Rightarrow \lambda = \frac{2\pi}{\sqrt{\left(\frac{2\pi u}{N}\right)^2 + \left(\frac{2\pi v}{M}\right)^2}} = \frac{sz}{\sqrt{uc^2 + vc^2}}$$

Question 5: What happens when we pass the point in the center and either p or q exceeds half the image size? Explain and illustrate graphically with Matlab!

Answers:

- (p,q) becomes (uc,vc) after the re-centering
- if either p or q exceeds half the image size, the centered point will be computed as p-sz, q-sz respectively.

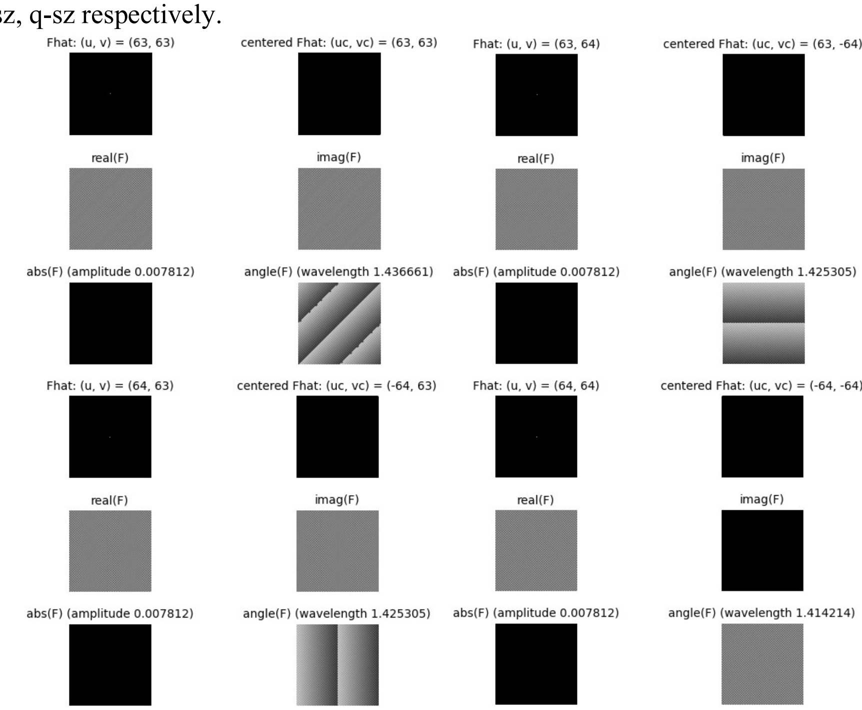

Fig. 3: fftwave() computed for coordinates in the center

Question 6: What is the purpose of the instructions following the question *What is done by these instructions?* in the code?

Answers: The purpose of those instructions is to perferm the re-centering operation. This consists in a change of variables (u,v) into (uc,vc). The new domain for uc and vc is $\left[\frac{-N}{2}, \ldots, \frac{N}{2} - 1\right]$

Question 7: Why are these Fourier spectra concentrated to the borders of the images? Can you give a mathematical interpretation? Hint: think of the frequencies in the source image and consider the resulting image as a Fourier transform applied to a 2D function. It might be easier to analyze each dimension separately!

Fig. 4: F,G and H in the spatial domain.

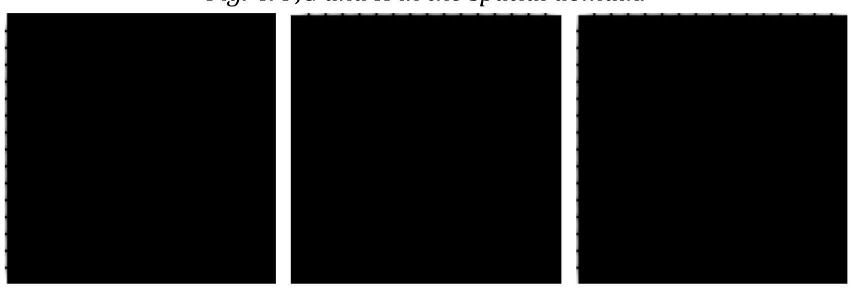

Fig. 5: log(1+|Fourier(x)|), Fourier() is computed with respect to F, G and H.

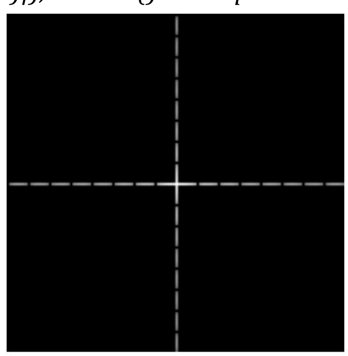

Fig. 6: log(1+|Fourier(H)_shifted|)

Answers:

$$\hat{F}(u,v) = \frac{1}{\sqrt{M\,\mathrm{N}}} \sum_{m:0}^{M-1} \sum_{n:0}^{N-1} \quad \mathrm{f}(m,n) \quad {}^{-2\pi\,\mathrm{i}\left(\frac{m\,\mathrm{u}}{M}+\frac{n\,\mathrm{v}}{N}\right)}$$

$$f(m,n) = \begin{cases} 1 & m_1 \leq m \leq m_2 \\ 0 & otherwise \end{cases}$$

$$\hat{F}(u,v) = \frac{1}{\sqrt{MN}} \sum_{m:m_1}^{m_2} e^{-2\pi i\left(\frac{mu}{M}\right)} \sum_{n:0}^{N-1} e^{-2\pi i\left(\frac{nu}{N}\right)} = \frac{1}{\sqrt{MN}} \sum_{m:m_1}^{m_2} e^{-2\pi i\left(\frac{mu}{M}\right)} * \delta(n)$$

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$

Since $\delta(n) = 1$ only for n=0, we will have Fourier spectra for F concentrated to the left border. And similarly, for G, will be concentrated on the upper border.

Question 8: Why is the logarithm function applied?

Answers:
By substituting each pixel value with its logarithm, we are enhancing low intensity pixel values (grey-level transformation).
1 is added to have non-negative values => $log(1) = 0$ and $log(1 + |\hat{F}|) \geq 0$

Pixel having zero values will be always mapped with zero values since $log(1 + 0) = 0$

Question 9: What conclusions can be drawn regarding linearity? From your observations can you derive a mathematical expression in the general case?

Answers:
H is a linear combination of F and G: $H = F + 2G$, and also its Fourier spectra $\hat{H} = \hat{F} + 2\hat{G}$.
This can be derived by the following properties of the Fourier transform and its inverse:

$$\mathcal{F}[\alpha f(m,n) + \beta g(m,n)] = \alpha \hat{f}(u,v) + \beta \hat{g}(u,v)$$

$$\mathcal{F}^{-1}\left[\alpha \hat{f}(u,v) + \beta \hat{g}(u,v)\right] = \alpha f(m,n) + \beta g(m,n)$$

Question 10: Are there any other ways to compute the last image? Remember what multiplication in Fourier domain equals to in the spatial domain! Perform these alternative computations in practice.

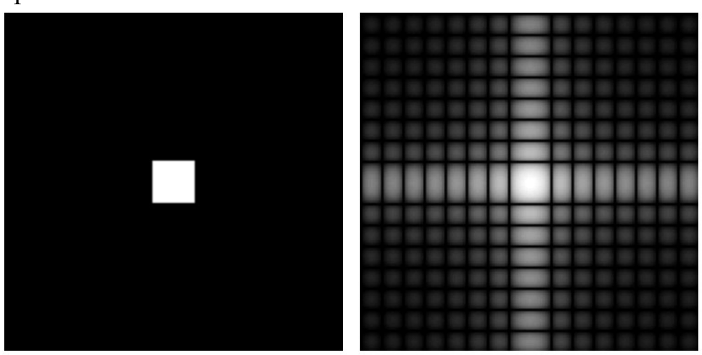

Fig. 7: Left: pointwise multiplication between F and G in the spatial domain. Right: Fourier transform of the previous result

Answers:
A multiplication in the spatial domain is the same as a convolution in the Fourier domain, and vicevera.
$\mathcal{F}(f) * \mathcal{F}(g) = \mathcal{F}(f \cdot g)$

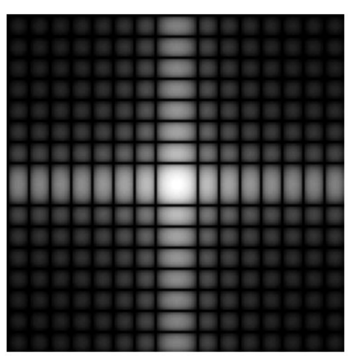

Fig. 8: Convolution between the fourier transform of F and G

Question 11: What conclusions can be drawn from comparing the results with those in the previous exercise? See how the source images have changed and analyze the effects of scaling.

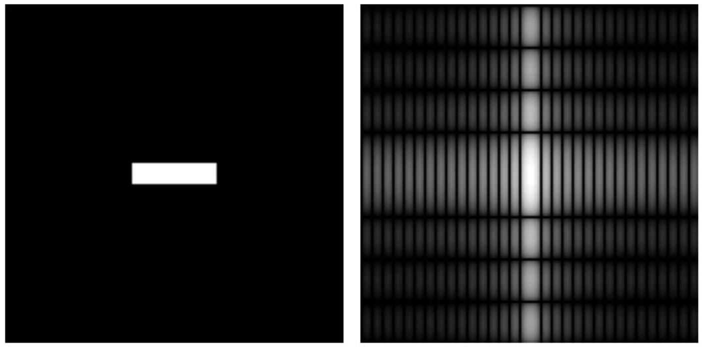

Fig. 9: Left: F in spatial domain. Right: Fourier transform of F

Answers:
Compression (scale down) in the spatial domain is same as expansion (scale up) in Fourier domain, and viceversa.

$$g(x,y) = f(ax,by)$$
$$\hat{g}(u,v) = \frac{1}{ab}\hat{f}\left(\frac{u}{a},\frac{v}{b}\right)$$

First image has 16x16 white square, while second image 8x32. Basically, we are scaling down the height and up the width ($a = 1/2, b = 2$)

In the fourier domain, the compression of the height in the spatial domain, corresponds to an expansion in the vertical direction, and the expansion of the width corresponds to an horizontal compression.

Question 12: What can be said about possible similarities and differences? Hint: think of the frequencies and how they are affected by the rotation.

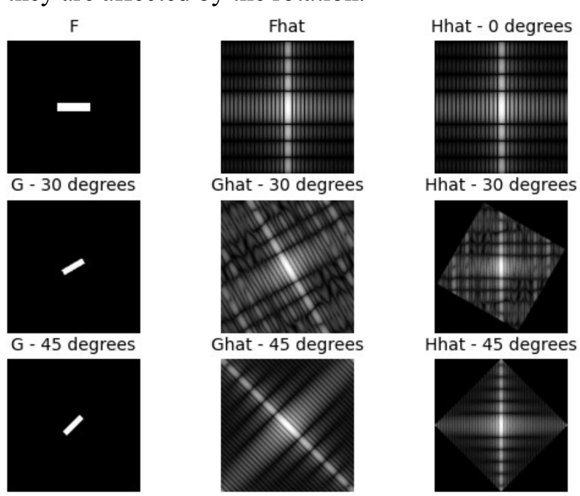

Fig. 10: F original image in spatial domain, G image rotated of α degrees. $\widehat{H}$ fourier transform centered and rotated back by -α

Answers:
Rotation of the original image determines a rotation by the same angle in the fourier domain. Magnitude and phase don't change.

Both in the spatial and fourier domain the rotation determines some distortions, introducing noise (especially for 30 degrees).

Question 13: What information is contained in the phase and in the magnitude of the Fourier transform?

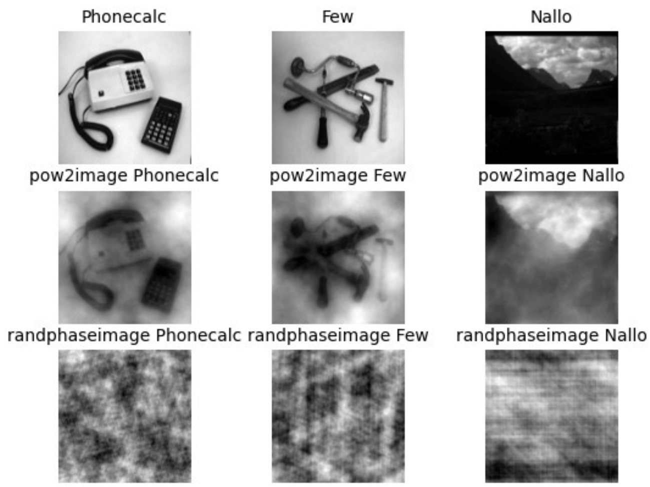

Fig. 11: Original images, images after computing pow2image() and randphaseimage()

Answers:

The function pow2image() replace the original power spectrum and we can observe that this affect the magnitude (because of non linear transformation) but not the phase.

The function randphaseimage() keeps the magnitude but not the phase, which is replaced with a random distribution.

What we can conclude is that, the phase is far more important than the magnitude for images, in fact,

- the phase determines the edges and contains most of the information,
- the magnitude determines the intensity of the image.

Question 14: Show the impulse response and variance for the above-mentioned t-values. What are the variances of your discretized Gaussian kernel for t = 0.1, 0.3, 1.0, 10.0 and 100.0?

****See the notebook for the variance results.**

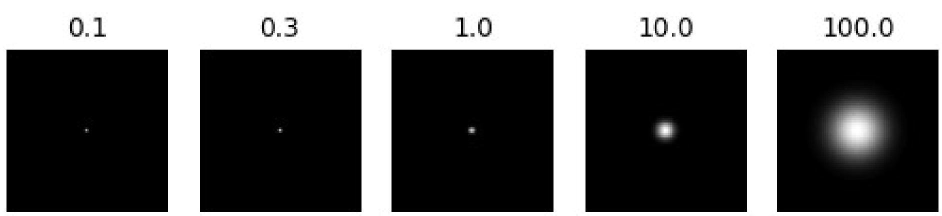

Fig. 12: Gaussian filters

Question 15: Are the results different from or similar to the estimated variance? How does the result correspond to the ideal continuous case? Lead: think of the relation between spatial and Fourier domains for different values of t.

Answers:

For higher t ($t \geq 1$) the results are closer to the estimated variance. For $t < 1$ is difficult to model the distribution in the discrete case since we only have a fixed set of pixels. As a result, this discretization will introduce Fourier domain errors, which will get smaller as the variance grows.

Question 16: Convolve a couple of images with Gaussian functions of different variances (like t = 1.0, 4.0, 16.0, 64.0 and 256.0) and present your results. What effects can you observe?

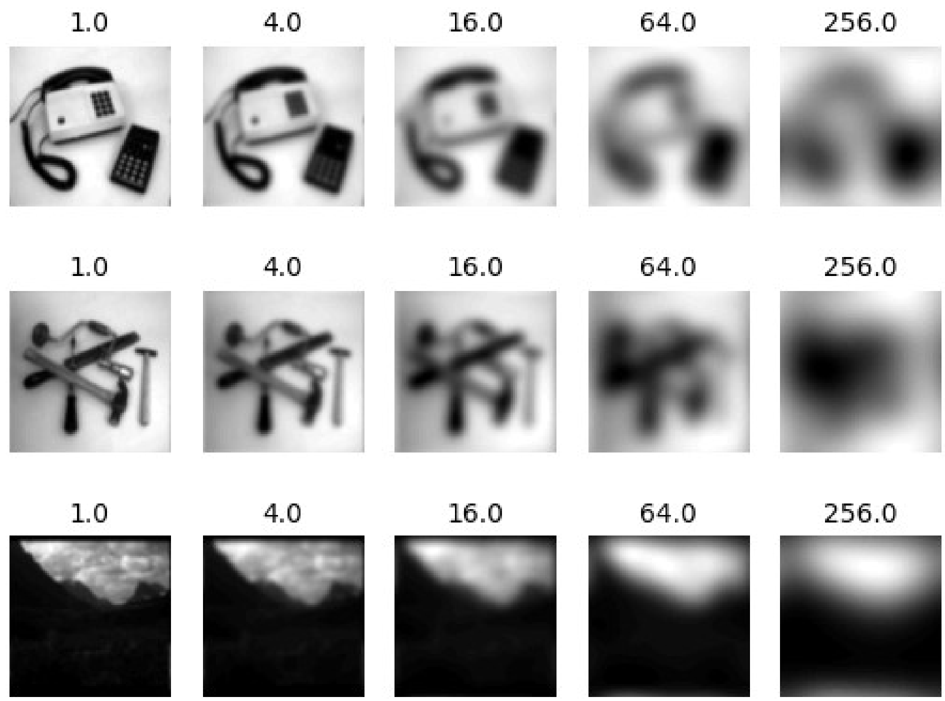

Fig. 13: Images after convolution with gaussian filters

Answers:
The pictures become increasingly blurry as the variance rises. The higher the variance, the higher are the frequencies retained, i.e the more the noise.

Question 17: What are the positive and negative effects for each type of filter? Describe what you observe and name the effects that you recognize. How do the results depend on the filter parameters?

Fig. 14: Left: original image. Middle: image + gaussian noise. Right: image + salt and pepper noise

Answers:

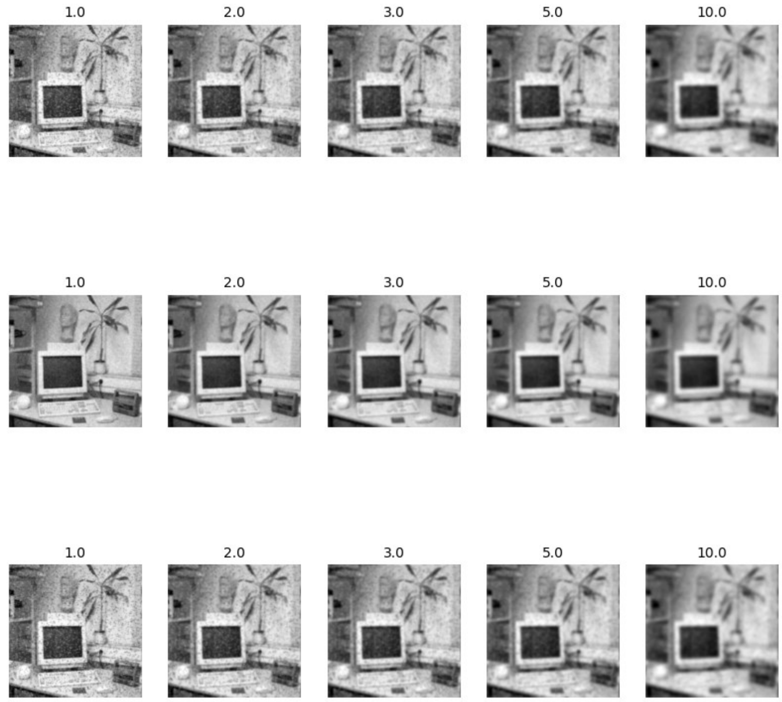

Fig. 15: Gaussian smoothing applied to the 3 images in figure

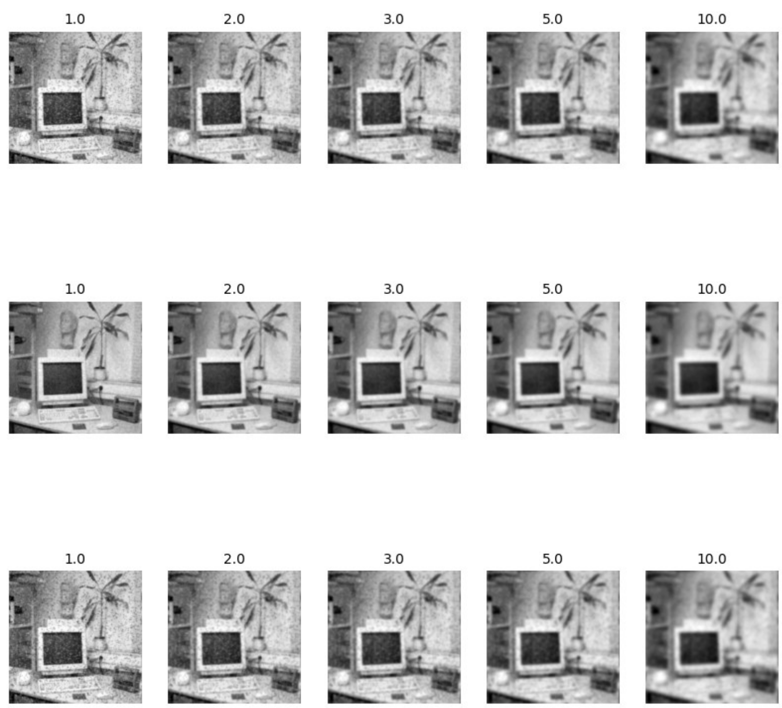

Fig. 16: Median filter applied to the 3 images in figure

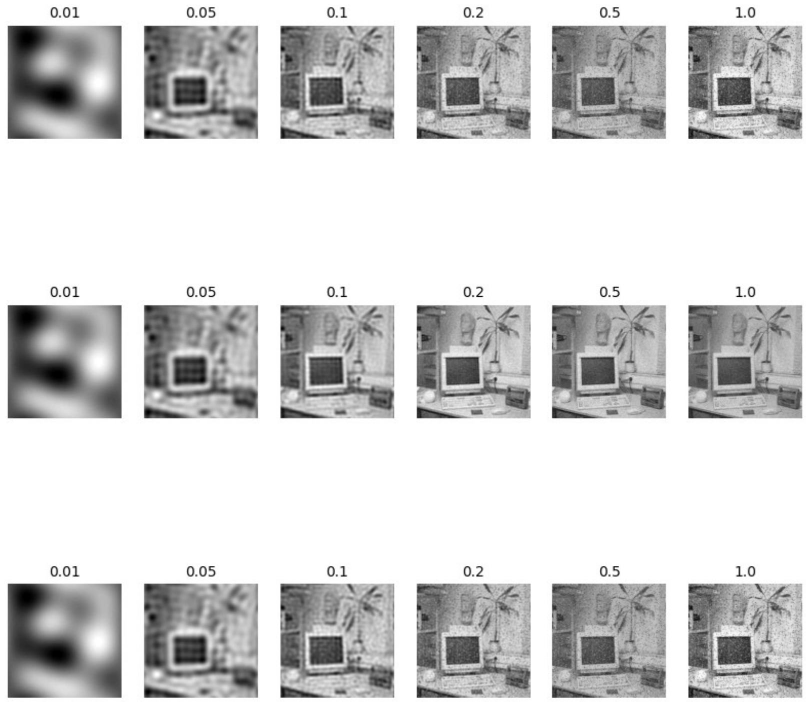

Fig. 17: Ideal low-pass filter applied to the 3 images in figure

From the above figures we can observe:

- Gaussian smoothing
    - image smoothing,
    - image blurring with higher variance, does not preserve edge information,
    - good in removing the gaussian white noise (with variance in [1,4]),
    - while not good in removing the salt and pepper noise.
- Median filter,
    - it preserves the edges with lower values of w.
    - keep high frequency
    - good in reducing sap, especially shown with w=2 and 4 (a value in between them could be optimal, e.g 3) .
    - image becomes illegible with w wider (image looks like paintings).
- Ideal low-pass filter
    - images distorted and blurred with cut-off frequency small (0.01),
    - with cut-off frequency high, the noise is kept,
    - keep low frequencies,
    - not smooth enough,
    - not good in reducing both the noises.

Question 18: What conclusions can you draw from comparing the results of the respective methods?

Answers:

As stated above, the gaussian filter is good in removing the gaussian noise, by increasing the blurring of the images, but value of the variance too high will determine images totally blurred.

The median filter is a non linear filter in which each output sample is computed as the median value of the input samples under the window, and works better in removing the salt and pepper noise, while preserving the edges.

The ideal low-pass filter cuts-off the higher frequency, but is not good in reducing the gaussian and the sap noises.

Question 19: What effects do you observe when subsampling the original image and the smoothed variants? Illustrate both filters with the best results found for iteration i = 4.

By subsampling we are losing information, the images are still smoother than the original one.

Question 19: What effects do you observe when subsampling the original image and the smoothed variants? Illustrate both filters with the best results found for iteration i = 4.

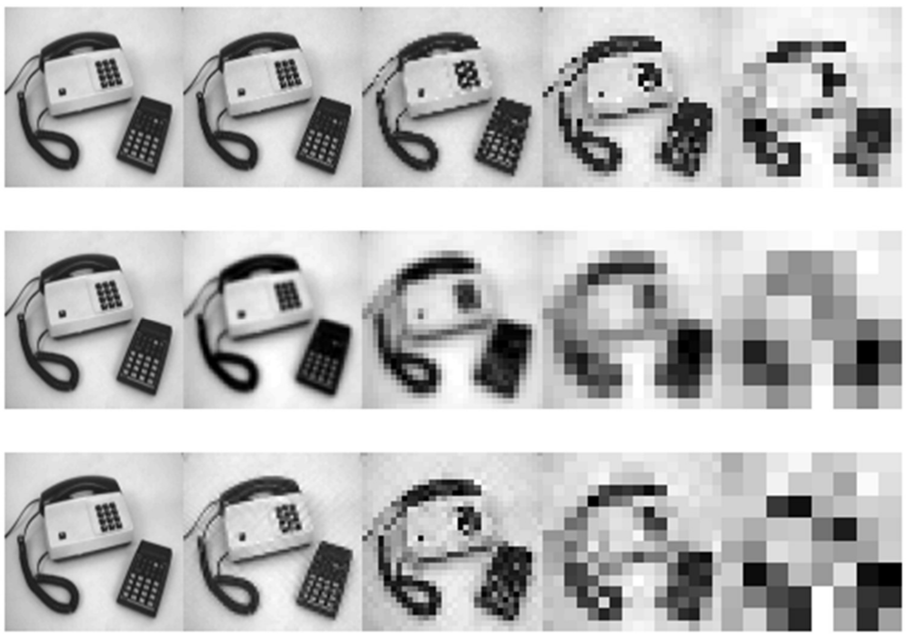

Fig. 18: Subsampling applied to 1. Original image, 2. Original image filtered with gaussian smoothing, 3. Original image filtered with low-pass filter

Answers:
By subsampling we are losing information, the images are still smoother than the original one.

Question 20: What conclusions can you draw regarding the effects of smoothing when combined with subsampling? Hint: think in terms of frequencies and side effects.

Answers:
Subsampling decreases the information content, thus resulting in losing details. Details are found in the higher frequencies, and thus together with a filter will be even more discarded. As mentioned above, aliasing is a common error when subsampling a signal.
