Decision Support Systems

Coalitional Game Theory

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Introduction to Coalitional Game Theory

So far, we've talked about non-cooperative game theory: selfish agents...

Sometimes, however, we are interested in settings in which we explicitly consider collaboration among agents: this is the aim of **Coalitional Game Theory** (also called **Cooperative**)

Here, we're interested in understanding the behaviour of *coalitions* (i.e., groups) of agents: Which coalitions will form? How they should divide their utility?

Introduction to Coalitional Game Theory (cont.)

Basically, groups of agents work together to complete a task and receive some utility: the group utility must then be partitioned among the participating agents

Formally, let $N = \{o_1, \ldots, o_n\}$ be our agents. A **transferable utility game** is a pair (N, v), where $v : 2^N \to \mathbb{R}$ s.t. $v(\emptyset) = 0$

Given a group of agents S, $v(S)$ represents the utility received (collectively) by the agents in S, when the coalition that completed the task is precisely S

Introduction to Coalitional Game Theory: Examples

A game is **super-additive** iff $v(S \cup T) \geq v(S) + v(T)$

A game is **additive** iff $v(S \cup T) = v(S) + v(T)$

A game is **convex** iff $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$

A game is **constant-sum** iff $v(S) + v(N \setminus S) = v(N)$

A game is **simple** iff $\forall S, v(S) \in \{0, 1\}$

Introduction to Coalitional Game Theory: Examples (cont.)

Voting Game (simple, super-additive and constant-sum)

We have N parties that may want to pass a law: each party i has a number of seats equal to p_i, denote with $P = \sum_i p_i$ the total number of seats

A coalition of parties S can pass the law iff $\sum_{i \in S} p_i > \frac{P}{2}$

Therefore, we can set $v(S) = 1$ iff $\sum_{i \in S} p_i > \frac{P}{2}$ holds, otherwise $v(S) = 0$

Airport Game (convex)

A number of cities need airport capacity. If a new regional airport is built the cities will have to share its cost, which will depend on the largest aircraft that the runway can accommodate. Otherwise each city will have to build its own airport.

Therefore, N is the set of cities and $v(S) = -\max_{i \in S} c_i - \sum_{j \notin S} c_j$, where c_i is the cost for building an airport in city i

Solution Concepts

Given a transferable utility game, we have two basic questions:

1. Which coalitions will form? Specifically: under which conditions will the *grand coalition* (i.e., the coalition of all agents) form?
2. Given that a coalition has formed, how do the agents in it should share the utility?

We will start from question 2, assuming that the grand coalition forms.

Solution Concepts: Shapley Values

A division of utility among agents will be modeled as a **payoff vector** $\phi \in \mathbb{R}^{|N|}$: ϕ_i represents the the utility assigned to agent o_i

A payoff is **feasible** iff $\sum_i \phi \leq v(N)$: we do not *magically create* utility

The **pre-imputation set** is defined as $\mathcal{P} = \{\phi : \sum_i \phi_i = v(N)\}$: the assignment of utility is *efficient* (we do not lose utility)

The **imputation set** is defined as $\mathcal{I} = \{\phi : \forall i, \phi_i \geq v(\{o_i\})\}$: the agents are *rational* (if this does not hold, the agent would prefer working alone)

Solution Concepts: Shapley Values (cont.)

We are interested in a division of utility that is **fair**: we define this axiomatically!

1. **Symmetry**: if i, j are *interchangeable* (that is, $\forall S \cap \{i,j\} = \emptyset$, it holds $v(S \cup \{o_i\}) = v(S \cup \{o_j\})$), then $\phi_i = \phi_j$;
2. **Dummy Player**: if i is *dummy* (that is, $\forall S$ with $i \notin S$, it holds that $v(S \cup \{o_i\}) - v(S) = v(\{o_i\})$), then $\phi_i = v(\{o_i\})$
3. **Additivity**: if v_1, v_2 are two games, define $v_1 + v_2(S) = v_1(S) + v_2(S)$. Let ϕ^1, ϕ^2, ϕ^+ be the respective payoff vectors. Then $\forall i$, $\phi_i^+ = \phi_i^1 + \phi_i^2$

Solution Concepts: Shapley Values (cont.)

Shapley Theorem

There exists a unique pre-imputation that satisfies Symmetry, Dummy Player and Additivity. Moreover, this pre-imputation is given by:

$$\phi_i = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{o_i\}} |S|!(|N| - |S| - 1)! \left[v(S \cup \{o_i\}) - v(S)\right]$$

Intuitively, the **Shapley value** corresponds to the *average marginal contribution* of an agent.

Assume that a coalition is gradually built by adding agents. If we want to add o_i to coalition S its contribution is $v(S \cup \{o_i\}) - v(S)$: there are $|S|!$ ways in which S could be built, and $(|N| - |S| - 1)!$ to complete the sequence (after having added o_i). Then, we sum over all possible coalitions, and normalize.

Shapley Values: Example

Consider the voting game, with four parties s.t. $p_1 = 45$, $p_2 = 25$ and $p_3 = p_4 = 15$: we compute the Shapley values!

$$\phi_1 = \frac{0!*3!*(0-0)+3*1!*2!*(1-0)+3*2!*1!*(1-0)+3!*0!*(1-1)}{24} = 0.5$$

$$\phi_2 = \frac{0!*3!*0+1!*2!*1+2*1!*2!*0+2*2!*1!*(1-1)+1*2!*1!*1+3!*0!*(1-1)}{24} = \frac{1}{6}$$

Similarly, we can show that $\phi_3 = \phi_4 = \frac{1}{6}$

Assume that if the law passes the parties (who supported it) will receive 1 million € to split: then, the Shapley values describe the proportion of money to each party (if the law is supported by the grand coalition of all parties).

Solution Concepts: Core

We now get to the first question: which coalition will form?

The question is relevant, because in certain contexts agents might have some incentive to not form the grand coalition

In the voting game, parties o_1, o_2 may benefit from forming a coalition excluding the two other parties: e.g., they might split 0.75 to 0.25 and have a benefit!

We then ask: which payoffs would ensure that the agents form the grand coalition (but can be extended to other coalitions of interest)?

Solution Concepts: Core (cont.)

A payoff ϕ is in the **core** if $\forall S$, $\sum_i \phi_i \geq v(S)$

The definition implies that no sub-coalition has any incentive to break from the grand coalition: note that this implies that a payoff in the core *must be* an imputation!

Two questions: Is the core always non-empty? Does it contain a unique imputation?

Solution Concepts: Core (cont.)

Two questions: Is the core always non-empty? Does it contain a unique imputation?

Unfortunately the answer is no: consider the voting game...

The set of minimal winning coalitions is $\{o_1, o_2\}$, $\{o_1, o_3\}$, $\{o_1, o_4\}$ and $\{o_2, o_3, o_4\}$: if the sum of payoffs to o_2, o_3, o_4 is less than 1, they will have incentive to cooperate and exclude o_1...

But in this case $\phi_1 = 0$, and o_1 could have a strictly greater payoff by cooperating with the one among o_2, o_3, o_4 taking the smallest payoff: hence, the core is empty!

Solution Concepts: Core (cont.)

Two questions: Is the core always non-empty? Does it contain a unique imputation?

Unfortunately the answer is no: consider the voting game...

Assume instead we change the voting game by requiring a 80% majority to win: in this case, parties o_1 and o_2 must be included in any winning coalition...

Thus, the core contains two coalitions: $\{o_1, o_2, o_3\}$ and $\{o_1, o_2, o_4\}$

Can we guarantee that the core is non-empty?

A **weight vector** $\lambda \in \mathbb{R}^{2^{|N|}}$ is **balanced** if $\forall i, \sum_{S: o_i \in S} \lambda(S) = 1$

Bondereva-Shapley Theorem

A game has non-empty core, iff for all balanced weights λ it holds that

$$v(N) \geq \sum_{S} \lambda(S) v(S)$$

While this theorem completely characterizes the games with non-empty core, it is not practical from a computational point of view!

Solution Concepts: Core (cont.)

Some other (more practical) conditions can be used to characterize the non-emptiness of the core

A constant-sum game that is not additive has a non-empty core

In a simple game, the core is non-empty iff it exists a **veto agent** (that is, $\exists o_i$ s.t. $v(N \setminus \{o_i\}) = 0$). If there are veto agents, the core consists of all payoffs in which non-veto agents get 0

Every convex game has a non-empty core. Furthermore, the payoff of Shapley values is in the core

Solution Concepts: Nucleolus

Still, even when non-empty, the core may contain multiple payoff vectors... we try to solve this issue by introducing a more refined solution concept!

A payoff ϕ is in the ϵ-**core** if $\forall S$, $\sum_{o_i \in S} \phi_i \geq v(S) - \epsilon$: the interpretation is that there is an additional cost ϵ for not forming the grand coalition (note, ϵ can also be negative!)

There is always an ϵ for which the ϵ-core is not empty: we say that ϕ is in the **least core** iff it is a solution to

$$\text{minimize } \epsilon \tag{1}$$

$$\text{subject to } \forall S, \sum_{o_i \in S} \phi_i \geq v(S) - \epsilon \tag{2}$$

Solution Concepts: Nucleolus (cont.)

The least core does not contain a unique payoff if the constraints for some coalitions are slack (not met with equality)... it seems we can arrive to a unique solution by *iteratively pruning away coalitions*, removing their slack. We say that a payoff is in the **nucleolus** if it solves the optimization problems $O_1, \ldots, O_{|N|}$ where O_i is:

$$\text{minimize } \epsilon \tag{3}$$

$$\text{subject to } \forall S \in S_1, \sum_{o_i \in S} \phi_i = v(S) - \epsilon_0 \tag{4}$$

$$\forall S \in S_{i-1} \setminus S_{i-2}, \sum_{o_i \in S} \phi_i = v(S) - \epsilon_{i-1} \tag{5}$$

$$\forall S \in 2^N \setminus S_{i-1}, \sum_{o_i \in S} \phi_i \geq v(S) - \epsilon \tag{6}$$

with ϵ_i being the solution of O_i and S_i the set of coalitions for which the inequality constraints are met with equality.

Solution Concepts: Nucleolus (cont.)

Uniqueness of the Nucleolus

The nucleolus of a game contains a unique payoff vector

We can also give a simpler interpretation of the nucleolus: the **excess** of coalition S, for payoff *phi*, is $e(S) = v(S) - \sum_{o_i \in S} \phi_i$

We say that the excesses due to ϕ^1 are *lexicographicall smaller* than those due to ϕ^2 if $\exists j$ s.t. $\forall i < j, \phi^1_i = \phi^2_i$ and $\phi^1_j < \phi^2_j$

The nucleolus is the unique payoff ϕ that lexicographically minimizes the excess for all coalitions except $\emptyset$ and N