Decision Support Systems

Introduction

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Introduction to Game Theory

In Game Theory, we consider decision-making settings in which we have multiple agents $o_1, \ldots, o_m$

Basically we refine the **environment**: instead of having an abstract environment, we have a collection of **self-interested** and **rational** agents.

Introduction to Game Theory (cont.)

A **game** is defined by a set of agents $O = \{o_1, \dots, o_m\}$ with:

- For each agent o_i, a set of actions $A_i = \{a^i_1, \dots, a^i_{n(i)}\}$: actions are also called **pure strategies**;
- A utility function $U : \Pi_{i=1}^m A_i \to \mathbb{R}^m$, that associates to each combination of pure strategies the utility of that configuration for each of the agents.

$U(\bar{a})_i$ is the utility for agent o_i when the vector of actions is $\bar{a}$

Introduction to Game Theory (cont.)

We will be interested also in strategies that are not pure

- A **strategy** s_i for agent o_i is a probability distribution over A_i: o_i can **randomize** its actions;
- A **strategy profile** is $\overline{s} = (s_1, \ldots, s_m)$: one strategy for each agent.

We can extend the utility function to strategies as:

$$U(\overline{s})_i = \sum_{\overline{a} \in \Pi_{j=1}^m A_j} (\Pi_{j=1}^m s_j(\overline{a}_j)) U(\overline{a})_i$$

Normal-Form Representation

We can represent games using decision matrices: this representation is called *normal form*

Agent 2 / Agent 1	a^2_1	$\dots$	$a^2_{n(2)}$
a^1_1	$o(a^1_1, a^2_1)_1, o(a^1_1, a^2_1)_2$	$\dots$	$o(a^1_1, a^2_{n(2)})_1, o(a^1_1, a^2_{n(2)})_2$
$\vdots$		$\ddots$	
$a^1_{n(1)}$	$o(a^1_{n(1)}, a^2_1)_1, o(a^1_{n(1)}, a^2_1)_2$	$\dots$	$o(a^1_{n(1)}, a^2_{n(2)})_1, o(a^1_{n(1)}, a^2_{n(2)})_2$

The normal form hides away information about the structure of the game (e.g., in the game of chess each action actually corresponds to full sequence of moves): in some cases one may want to consider more *extensive* representations...

Normal-Form Representation: Some Examples

The **Driving** game

Agent 1 \ Agent 2	Left	Right
Left	1, 1	-1, -1
Right	-1, -1	1, 1

Pure coordination game: for all action profiles $\overline{a} \in \Pi_{i=1}^{m} A_i$ and all $i, j \in \{1, \ldots, m\}$, it holds that $U(\overline{a})_i = U(\overline{a})_j$... in a pure coordination game, the agents have no interest in competing, coordination is maximally beneficial to all!

Normal-Form Representation: Some Examples

The **Matching Pennies** game

Agent 1 \ Agent 2	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

Zero-sum game: **two agents** and for all action profiles $\overline{a} \in \Pi_{i=1}^{m} A_i$, it holds that $\sum_{i=1}^{m} U(\overline{a})_i = 0$... purely competitive game!

Normal-Form Representation: Some Examples (cont.)

The **Prisoner's Dilemma**

Agent 1 \ Agent 2	Cooperate	Defect
Cooperate	-1, -1	-4, 0
Defect	0, -4	-3, -3

This game (and most games, indeed) exhibits both elements of cooperation and of competition

Analyzing Games

As in decision theory, having a game, we are interested in understanding how to reason about (i.e., how we can draw conclusions) about it...

While in decision theory we were interested in *optimal strategies*, in game theory the situation is more complex, because the optimal strategy of an agent depends on what the other agents do!

We will study different **solution concepts**, that correspond to sets of outcomes that are *in some sense* interesting...

Pareto Optimality

If we take the role of an outside observer, can we say whether an outcome is better than another?

Strategy profile $\bar{s}$ **Pareto dominates** $\bar{t}$ if $\forall i \in \{1, \ldots, m\}, U(\bar{s})_i \geq U(\bar{t})_i$ and $\exists j \in \{1, \ldots, m\}$ with $U(\bar{s})_j > U(\bar{t})_j$

Strategy profile $\bar{s}$ if **Pareto optimal** if $\nexists \bar{t}$ that Pareto dominates it: at least one exists, in general there may be many Pareto optimal strategy profile...

In a zero-sum game all strategy profile are Pareto optimal... so it is not a very informative solution concepts (similar to dominance in decision theory)

Pareto Optimality: an Example

Agent 1 \ Agent 2	Left	Right
Left	**1, 1**	-1, -1
Right	-1, -1	**1, 1**

Nash Equilibria: Best Response

Aside from being only weakly informative, Pareto optimality requires the advantage point of an external observer... sometimes we only know (and generally we only care) about **our** utilities!

So, let's take the point of view of an agent... how we should play?

Define $\overline{s}_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m)$, then $\overline{s} = (s_i, \overline{s}_{-i})$

If $\overline{s}_{-i}$ is fixed, it is easy to play, simply select $s_i^* \in \arg\max_{s_i} U(s_i, \overline{s}_{-i})_i$

s_i^* is called a **best response** strategy (w.r.t. $\overline{s}_{-i}$)

Nash Equilibria

Best response strategies lead to the most important solution concept:

Strategy profile $\overline{s}$ is a **Nash equilibrium** if, for all agents o_i, s_i is a best response to $\overline{s}_{-i}$

A Nash equilibrium is **strict** if, for all agents o_i, s_i is a **strict** best response to $\overline{s}_{-i}$, i.e., $\forall t_i, U(s_i, \overline{s}_{-i})_i > U(t_i, \overline{s}_{-i})$, otherwise is **weak**

Strict Nash equilibria represent a situation of **stability**: no agent has a desire to unilaterally deviate from its strategy... with weak equilibria, some agents can select among different strategies (only non-pure ones)!

Nash Equilibria: Examples

Agent 2 / Agent 1	Left	Right
Left	**1, 1**	-1, -1
Right	-1, -1	**1, 1**

In this case the Nash equilibria coincide with the Pareto optimal strategies

Nash Equilibria: Examples (cont.)

Agent 1 \ Agent 2	Cooperate	Defect
Cooperate	-1, -1	-4, 0
Defect	0, -4	**-3, -3**

Here we have a single Nash equilibrium... and it is strange: notice that it is dominated by strategy (Cooperate, Cooperate)!

Nash Equilibria: Properties

As for Pareto optimality, we want to reason about Nash Equilibria:

1. Do they always exist?
2. How many of them?
3. Why Nash equilibria are interesting?
4. How can we compute them?

Nash Equilibria: Existence

Nash Theorem

In any game with a finite number of agents, each of which with a finite number of actions, it exists at least one Nash equilibrium.

The proof of Nash Theorem is quite complex and relies on some advanced results from convex analysis and topology (cfr. Brouwer's fixed point theorem)... if you're interested, you can look at the book by Shoam and Leyton-Brown.

Nash Equilibria: Existence (cont.)

In Driving and Prisoner's dilemma we have shown (some of) the Nash equilibria... what about Matching Pennies?

Agent 1 \ Agent 2	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

It seems there are no Nash equilibria... indeed, no equilibrium of *pure actions*! But the strategy:

$$\bar{s} = ((1/2 : \mathit{Heads}, 1/2 : \mathit{Tails}), (1/2 : \mathit{Heads}, 1/2 : \mathit{Tails}))$$

is a (in fact, the unique) Nash equilibrium!

Nash Equilibria: Uniqueness

In general, Nash Equilibria are not unique...

If there are at least two Nash equilibria $\bar{s}, \bar{t}$, then there are infinite equilibria. Indeed:

$$p\bar{s} + (1 - p)\bar{t}$$

is an equilibrium for every $p \in [0, 1]$.

Nash Equilibria: Interpretation

A natural question when we study a solution concept is: why this concept is interesting?

Equivalently, we are asking why we would expect this kind of solutions to arise in practice, when observing sets of agents...

A natural interpretation for Nash equilibria (other than stability) arises in two-player zero-sum games

Nash Equilibria: Interpretation (cont.)

Recall maximin decision rule from decision theory: it encodes the behavior of a pessimistic agent...

In the game theory settings it means that agent o_i selects the strategy that would give the maximum benefit in the worst case (optimal strategy of agent all the other agent o_{-i}).

So, we say that s_i^{maxmin} is a **maximin strategy** for agent o_i if:

$$s_1^{maxmin} = \arg\max_{s_i} \min_{s_{-i}} U(s_i, s_{-i})_i$$

Nash Equilibria: Interpretation (cont.)

Dually, we can define a **minimax strategy** s_i^{minmax} for agent o_i (against agent o_j)

$$s_i^{minmax} = (\arg\min_{s_{-j}} \min_{s_j} U(s_j, s_{-j})_j)_i$$

Intuitively, it is i-th component in a strategy profile that guarantees o_j the worst possible outcome

Nash Equilibria: Intepretation (cont.)

In a two-player, zero-sum game, minimax and maximin can be see to be equivalent (because the gain for one agent is a loss for the other)... most interestingly:

Minimax Theorem

In any finite, two-player, zero-sum game, Nash equilibria coincide with minimax strategies (which coincide with maximin strategies)!

This results gives a nice interpretation of Nash equilibria:

1. Each player maximin value is equal to its minimax value: the value for agent o_1 is called the *value* of the game
2. Nash equilibria can be understood to arise as the result of a minimax (or maximin) process: agents make decisions as if they were pessimistic about their opponent's behavior!

Nash Equilibria: Computation

The minimax theorem also gives a way to efficiently compute Nash equilibria for zero-sum games

minimize U_1^*
subject to

$$\forall a_j^1 \in A_1, \sum_{a_k^2 \in A_2} U(a_j^1, a_k^2)_1 s_k^2 \leq U_1^*$$

$$\sum_{a_k^2 \in A_2} s_k^2 = 1$$

$$\forall a_k^2 \in A_2, s_k^2 \geq 0$$

U_1^* represents the utility of agent o_1 under a minimax strategy by player o_2, while s_k^2 is the probability that agent o_2 assigns to action a_k^2.

Nash Equilibria: Computation (cont.)

The minimax theorem also gives a way to efficiently compute Nash equilibria for zero-sum games

maximize U_1^*
subject to

$$\forall a_k^2 \in A_2, \sum_{a_j^1 \in A_1} U(a_j^1, a_k^2)_1 s_j^1 \geq U_1^*$$

$$\sum_{a_j^1 \in A_1} s_j^1 = 1$$

$$\forall a_j^1 \in A_1, s_j^1 \geq 0$$

U_1^* represents the maximin value for agent o_1, while s_j^1 is the probability that agent o_1 assigns to action a_j^1.

Nash Equilibria: Computation (cont.)

Solving the two previous problems computes the strategies for (one of) the Nash equilibria for the two players: note that the objective optimal value for the two problems is the same (and is the value of the game)!

Thus, finding Nash equilibria in zero-sum games can performed efficiently by reduction to linear programming, which is a problem in `P`.

Unfortunately, in general games, it is believed that Nash equilibria cannot be computed efficiently: they belong to a class of problems known as `PPAD` (and actually, they are *complete* for the class)