Decision Support Systems

Social Choice Theory

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Table of Contents

Introduction to Social Choice Theory

Similarly to Game Theory, here we study settings in which we have multiple agents...

However, our interest is in understanding how to aggregate the preferences of the agents so as to make a decision (imagine a central planner or a government aggregating the preferences of the voters)

Introduction to Social Choice Theory (cont.)

Let $O = o_1, \ldots, o_n$ be our agents, and $A = a_1, \ldots, a_m$ be some alternatives among which we want to decide.

For each agent o_i, its preference is expressed in terms of a linear order $<_i$ on A: $a_l <_i a_k$ means that agent o_i *strictly prefers* alternative a_k to a_l.

Remark

We won't consider a numeric utility function: the reason is that in SCT we assume that the utility scales of different agents are **incomparable**

We denote with $\mathcal{L}(A)^n$ the set of possible linear orders generated by our agents, which are called *profiles*.

Introduction to Social Choice Theory (cont.)

The intuitive idea that we want to **aggregate the preferences** is formalized by means of a *social welfare function* (SWF)

$$f : \mathcal{L}(A)^n \rightarrow \mathcal{R}(A),$$

where $\mathcal{R}(A)$ is the set of *weak orders* over A.

Basically, f takes as input a vector of preferences $(<_1, \dots, <_n)$, called a *profile*, and gives a new preference (possibly with ties).

Social Welfare Functions: An Example

Assume we have 303 agents expressing their preference among three alternatives a, b, c. Assume that 102 agents said $a > b > c$, 101 agents said $b > c > a$ and the remaining 100 agents have said $c > b > a$. Their preferences can be summarized as:

102	101	100
a	b	c
b	c	b
c	a	a

Plurality Voting

Plurality voting is the SWF that orders the alternatives by how many times they were the top-ranked alternative.

102	101	100
a	b	c
b	c	b
c	a	a

Formally, let $s : O \times A \to [m]$ be defined as $s(o_i, a_j) = \{a_k : a_j <_i a_k\} + 1$. Then $Plurality(<_1, \ldots, <_n) = (a^{(1)}, \ldots, a^{(m)})$, where $a^{(i)} \geq a^{(j)}$ iff $|\{o_k : s(o_k, a^{(i)}) = m\}| \geq |\{o_k : s(o_k, a^{(j)}) = m\}|$.

In the example $Plurality = (a, b, c)$.

Plurality Voting (cont.)

Social Choice Theory emerges from a critique of plurality voting: the winner can be unpopular!

102	101	100
a	b	c
b	c	b
c	a	a

In our example, even if *a* was top-ranked by a plurality (note: **not a majority!**) of agents, it was the worst alternative according to the majority of them: 201 agents out of 303 consider *a* the worst alternative!

Net Preference and tournaments

Social Choice Theory studies SWFs that make better use of the full preferences, instead of only using the top-ranked alternatives.

Given a_i, a_j the *net preference* $Net_P(a_i, a_j)$ is defined as

$$Net_P(a_i, a_j) = |\{o_k | a_i >_k a_j\}| - |\{o_k | a_j >_k a_i\}|$$

We write $a_i \geq^\mu a_j$ if $Net_P(a_i, a_j) \geq 0$: intuitively, a_i was preferred to a_j by a majority of agents!

Copeland Rule

Copeland rule refines plurality voting by using the win-loss record defined by the net preference and $\geq^\mu$

The *Copeland score* of alternative a is defined as $Copeland(a) = |\{a' \in A | a >^\mu a'\}| - |\{a' \in A | a' >^\mu a\}|$. *Copeland rule* ranks alternatives by their Copeland score.

102	101	100
a	b	c
b	c	b
c	a	a

In the example $Net_P(a, b) = -101$, $Net(a, c) = -101$, $Net(b, c) = 103$, hence $b >^\mu c >^\mu a$. Therefore, $Copeland(b) = 2$, $Copeland(c) = 0$, $Copeland(a) = -2$.

Borda Rule

Borda rule refines plurality voting by explicitly using the net preference scores

The *Borda score* of alternative a is defined as $Borda(a) = \sum_{a' \in A} Net_P(a, a')$. *Borda rule* ranks alternatives by their Borda score.

102	101	100
a	b	c
b	c	b
c	a	a

In the example $Net_P(a, b) = -101$, $Net(a, c) = -101$, $Net(b, c) = 103$. Therefore, $Borda(a) = -202$, $Borda(b) = 204$, $Borda(c) = -2$.

Basic Theoretical Results

Motivated by limitations of plurality voting, we have seen that multiple SWFs can be considered: however, these rules can provide different results!

We will adopt our usual axiomatic approach to define desirable characteristics of a SWF: as in Coalitional Game Theory, we will be interested in SWFs that are **fair** and **rational**!

So we need to define what fairness and rationality mean... however we will focus not on SWFs but on *social choice functions* (SCF) $c : \mathcal{L}(A)^n \to 2^A$: c gets as input a vector of preferences and gives a set of top-ranked alternatives

Given a SWF f we can construct a SCF c by

$$c(<_1, \ldots, <_n) = \arg\max_{a \in A} f(<_1, \ldots, <_n)$$

Basic Theoretical Results (cont.)

Some intuitive properties that a *fair* SCF should have:

- **Anonymity**: c is anonymous if, for all o_i, o_j s.t. $<_i = <_j$, it holds that $c(<_1, \ldots, <_i, \ldots, <_j, \ldots, <_n) = c(<_1, \ldots, <_j, \ldots, <_i, \ldots, <_n)$: intuitively, c does not care about the identities of the agents!
- **Non-Dictatorship**: c is non-dictatorial if $\nexists o_i$ s.t. $c(<_1, \ldots, <_n)$ is always o_i top-ranked alternative: namely, o_i is a *dictator* that always determines the outcome!
- **Neutrality**: c is neutral if whenever given two profiles P_1, P_2 that are equivalent except the votes for two alternatives a_i, a_j have been swapped, $c(P_1), c(P_2)$ are obtained by a similar swap: intuitively, neutrality means that voting is not *rigged* in favor of some alternatives!

Basic Theoretical Results (cont.)

Some intuitive properties that a *rational* SCF should have:

- We say that a **Pareto dominates** a' if all agents rank a above a'. An SCF c is **Pareto** if it never contains a Pareto dominated alternative: intuitively, it means that c is rational in terms of the preferences!
- **Independence of Irrelevant Alternatives** (IIA): if $c(<_1, \ldots, <_n) = S$ and we obtain a new profile by only changing the rankings of alternatives not in S, then $c(<_1, \ldots, <_n) = S$. Intuitively, the ranking between two alternatives is not influenced by the ranking of other alternatives!
- **Monotonicity**: c is **monotonic** if $c(<_1, \ldots, <_n) = a$ and only one agent changes its top-ranked alternative $a' \neq a$ into a, then a is still the result of c.
- **Resoluteness**: c is resolute if for all profile P, $|c(P)| = 1$. Intuitively means that it is easy to make decisions using c because it never gives ties!

Basic Theoretical Results (cont.)

If we restrict ourselves to only two alternatives, however, there is a single best SCF... unsurprisingly it is plurality voting (which in this case coincides with *majority voting*!

May's Theorem

If $m = 2$ and n is odd, then plurality voting is the unique neutral, anonymous, non-dictatorial, Pareto, IIA, monotonic and resolute SCF.

The setting of the Theorem, however, is quite limited: in many cases we will more than two alternatives... is there a single best (fair, rational) SCF in general settings? The answer seems to be no, and we will look at the three most famous **negative results**!

Basic Theoretical Results (cont.)

Moulin's Theorem

If $m \geq 3$ and n is divisible by any integer $1 < r \leq m$, then no neutral, anonymous, Pareto SCF is *resolute*.

The theorem implies that if we are given an SCF that is fair (neutral, anonymous and Pareto) then there exists at least one profile that will result in a tie!

If you imagine that the SCF is used to make a decision, then ties are complicated, because they imply we are undecided among several alternatives and we have no way to decide among them... however, it seems hard to renounce the other criteria, so we must accept the presence of ties!

Basic Theoretical Results (cont.)

Arrow's Impossibility Theorem

If $m \geq 3$ then every SWF that is Pareto and IIA is also dictatorial.

The theorem is also sometimes called the *Democracy is not rational* theorem as it implies that democracy (non-dictatorship) is not rational (as it cannot be both Pareto and IIA)...

However, Arrow's theorem is much weaker than Moulin's, as IIA is a very strong requirement: sometimes it makes sense to assume that the ranking among two alternatives can change w.r.t. a third, unrelated one (e.g., when we have considerations about *rarity*)!

Basic Theoretical Results (cont.)

The third theorem requires an additional rationality requirement.

Strategy-proofness: An SCF c is *single-voter manipulable* if for some profiles P, P' and agent o_i, with $\forall o_j \neq o_i, <_j^P = <_j^{P'}$, it holds that $c(P') >_i c(P)$. Otherwise c is *strategyproof*. Intuitively, if we interpret P as o_i's true preference, strategyproofness means that o_i cannot improve its utility by expressing a *fake preference*.

Gibbard-Satterthwaite Theorem

If $m \geq 3$, then any resolute, neutral and strategyproof SCF must be a dictatorship.

As Arrow's theorem, also this is weaker than it seems: if we accept the consequence of Moulin's theorem (i.e., we must accept ties), then we can avoid Gibbard-Satterthwaite by considering SCFs that can be non-resolute!