

# QFT for Cosmology, Achim Kempf, Lecture 8

Note Title

## The Unruh effect

(W.B. Unruh, 1976)



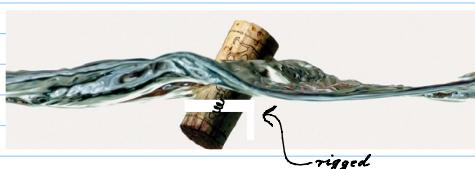
An accelerated ice cube will melt, even in vacuum.

The Unruh effect is the observation, by accelerated observers, of particles, even when the field is in the vacuum state in Minkowski space, i.e., even if inertial observers don't see particles.

### Intuition:

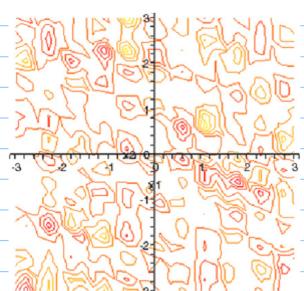


If rigged cork can act as sender and detector.



When accelerated, it gets excited - as if detecting.

### Similarly:



If simplified to a 2-level system, we say that we have an Unruh DeWitt detector system.

An atom or molecule can also both emit and detect particles. This can serve as the definition of particles.



Unruh effect

When accelerated, expect particle emission and detection.

We'll consider detectors at rest and in motion:

\* A detector at rest has:  $x^\mu(\tau) = (\tau, 0, 0, 0)$

\* Case of constant velocity:

$$x^\mu(\tau) = (a\tau, \vec{b}\tau)$$

$$\text{with } a^2 - \vec{b}^2 = 1. \quad \text{Exercise: verify}$$

\* Case of constant acceleration in the x-direction:

$$x^0(\tau) = \omega \sinh(\tau/\omega)$$

$$x^1(\tau) = \omega (1 + \sinh^2(\tau/\omega))^{1/2}$$

$$x^2(\tau) = x^3(\tau) = 0$$

Exercise:  $\square$  verify that  $\ddot{x}^\mu \dot{x}_\mu = \text{const}$   
(i.e. for still small velocities)  
 $\square$  show that for  $\tau \ll 1$ :  $x(\tau) \approx (\tau, a + b\tau^2)$

## The quantum field

$\square$  We assume that, for an inertial observer, the field  $\hat{\phi}$  is in the Minkowski vacuum. Recall:

$$\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{k} \cdot \vec{x}} \hat{\phi}_k(x^0) d^3k \text{ with } \hat{\phi}_k(x^0) = \frac{1}{\sqrt{2}} (v_k^+(x^0) a_k + v_k^-(x^0) a_k^\dagger)$$

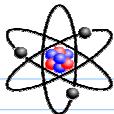
$$\text{and } v_k(x^0) = \frac{1}{\sqrt{\omega_k}} e^{i\omega_k x^0} \text{ with } \omega_k = \sqrt{k^2 + m^2}.$$

$$\square \text{ Thus: } \hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \left( \frac{1}{\sqrt{\omega_k}} e^{i\omega_k x^0 + i\vec{k} \cdot \vec{x}} a_k + \frac{1}{\sqrt{\omega_k}} e^{-i\omega_k x^0 + i\vec{k} \cdot \vec{x}} a_k^\dagger \right) d^3k$$

$\square$  Note:  $\hat{\phi}(x)$  acts on Hilbert space  $\mathcal{H}^{\text{field}}$ .

Next: Consider a system (e.g. an atom) that can detect particles of the Klein Gordon field  $\hat{\phi}$ .

## The detector system



Inertial observer's  
cartesian coordinates.

□ Small, localized system with path  $\vec{x}^*(\tau)$

E.g.: \* An atom

\* An oscillator, such as the diatomic molecule  $H_2$ .

↓  
detector's eigen time

□ First quantized.

□ Hamiltonian  $\hat{H}^{\text{detector}}$  acts on Hilbert space  $\mathcal{H}^{\text{detector}}$ .

□ Assume  $\text{spec}(\hat{H}^{\text{detector}}) = \{E_0, E_1, E_2, \dots\}$  is discrete.

⇒ The total quantum system thus consists of two subsystems, with:

$$\hat{H}^{\text{total}} = \hat{H}_0^{\text{detector}} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}_0^{\text{field}} + \hat{H}^{\text{interaction}}$$

□ On the total Hilbert space:

$$\mathcal{H}^{\text{total}} = \mathcal{H}^{\text{detector}} \otimes \mathcal{H}^{\text{field}}$$

□ The interaction Hamiltonian  $\hat{H}^{\text{interaction}}$  consists of operators of both subsystems, usually:

$$\hat{H}^{\text{interaction}} (\tau) = \varepsilon(\tau) \hat{Q}(\tau) \hat{\phi}(x^*(\tau), \vec{x}(\tau))$$

↓  
 Detector efficiency      ↓  
 (can also be used  
as on/off switch)      An observable  
 of the detector's  
quantum system      ↓  
 The field  $\hat{\phi}$   
 at the current  
detector location

□ Examples:  $\hat{H}^{\text{int}} = \hat{S}_3(\tau) \otimes \hat{B}_3(x(\tau))$

↑  
detector is a spin.

↑  
field is magnetic field.

or:  $\hat{H}^{\text{int}} = -\frac{e}{mc} \hat{p}_i \otimes \hat{A}^i(x(\tau))$  (use Axial gauge:  $\partial_i A^i = 0$ )

## Time evolution

- If we (realistically) assume that the detector efficiency  $\epsilon(\tau)$  is small, we can use perturbation theory.
- In this case, the Dirac picture of time evolution is convenient:

\* Operators evolve according to

$$\hat{H}^{\text{tree}} = \hat{H}_{\text{detector}} \otimes 1 + 1 \otimes \hat{H}_{\text{field}} \quad (\star)$$

For example:

$$\begin{aligned}\hat{Q}(\tau) &= e^{i\hat{H}^{\text{tree}}\tau} (\hat{Q}_0 \otimes 1) e^{-i\hat{H}^{\text{tree}}\tau} \\ &= e^{i\hat{H}_{\text{detector}}\tau} \hat{Q}_0 e^{-i\hat{H}_{\text{detector}}\tau} \otimes 1\end{aligned}$$

\* States evolve according to  $\hat{H}^{\text{int}}(\tau)$ , i.e., according to the time evolution operator:

$$\hat{U}(\tau) = T \exp \left( i \int_{\tau_i}^{\tau} \hat{H}^{\text{interaction}}(\tau') d\tau' \right)$$

↑  
time-ordering symbol      ↑  
In  $\hat{H}^{\text{interaction}}$  the operators are time  
dependent, evolving according to  $(\star)$

## Perturbative ansatz

- For small detector efficiency  $\epsilon(\tau)$  we can expand:

$$\hat{U}(\tau) = 1 + i \int_{-\infty}^{\tau} \epsilon(\tau') \hat{Q}(\tau') \hat{\phi}(x'(\tau'), \vec{x}(\tau')) d\tau' + \mathcal{O}(\epsilon^2)$$

- Note: We can set  $\tau_i = -\infty$  since we can always switch  $\epsilon(\tau)$  on or off.

## Initial conditions

- We assume that the quantum field  $\phi$  starts out in a state  $|\Omega\rangle$  with  $|0\rangle = \text{Minkowski vacuum}$ ,  $|d\rangle = |0\rangle$ , or a 1-particle state  $|d\rangle = |\downarrow\rangle$ .
- We assume that the detector starts out in its ground state  $|E_0\rangle$ .
- Thus, the combined system starts out in the state:

$$|\Psi_{in}\rangle = |E_0\rangle \otimes |\Omega\rangle$$

## Time evolution:

At time  $\tau$  the total system is in the state

$$|\Psi(\tau)\rangle = \hat{U}(\tau) |\Psi_{in}\rangle$$

## Particle creation

### The problem:

What is the probability amplitude that, if we measure at time  $\tau$  the detector system will be found to have detected something, i.e., to be in an excited state  $|E_n\rangle$ ?

### □ To this end, calculate:

$$p(\tau) := \langle E_n | \otimes \langle \Omega | \hat{U}(\tau) | \Psi(\tau) \rangle$$

for an arbitrary end state  $|\Omega\rangle$  of the quantum field  $\phi$ .

### □ Note: We will see that not all states $|\Omega\rangle$ yield a nonzero $p(\tau)$ .

## Total detection probability:

II The probability for detection eventually is:

$$p(\infty) \approx \langle E_n | \otimes \langle \Omega | \left( 1 + i \int_{-\infty}^{+\infty} \varepsilon(\tau) \hat{Q}(\tau) \otimes \hat{\phi}(x(\tau)) d\tau \right) | E_o \rangle \otimes |\omega \rangle$$

(we may choose  $\varepsilon(\tau)$  so as to make it finite)

Note:  $\langle E_n | E_o \rangle = 0 \Rightarrow 1^{\text{st}} \text{ term vanishes} \Rightarrow$

$$= i \int_{-\infty}^{+\infty} \varepsilon(\tau) \langle E_n | \hat{Q}(\tau) | E_o \rangle \langle \Omega | \hat{\phi}(x(\tau)) | \omega \rangle d\tau$$

↑

Recall:  $\hat{Q}(\tau) = e^{i H_0 \text{dilator} \tau} Q_o e^{-i H_0 \text{dilator} \tau}$

$$p(\infty) = i \int_{-\infty}^{+\infty} \varepsilon(\tau) e^{i(E_n - E_o)\tau} \langle E_n | \hat{Q}_o | E_o \rangle \langle \Omega | \hat{\phi}(x(\tau)) | \omega \rangle d\tau$$

↑

Recall:  $\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \left( \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k x^0 - ik^2 \vec{x}} \hat{a}_k^\dagger + \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k x^0 + ik^2 \vec{x}} \hat{a}_k \right) d^3 k$

Note: We can now calculate all absorption and emission processes.

Here: Let's focus on particle detection in the vacuum,  $|\omega\rangle := |0\rangle$ :

\* In  $\hat{\phi}(x)$ , only the terms  $\sim a_k^\dagger$  can contribute,  
because  $a_n |0\rangle = 0$

\* Thus, in  $|\Omega\rangle$  only the one-particle components contribute:

$$|\Omega\rangle = \Omega_o |0\rangle + \int \Omega_\omega a_\omega^\dagger |0\rangle d^3 k + \int \Omega_{\omega'} a_{\omega'}^\dagger a_{\omega'}^\dagger |0\rangle d^3 k d^3 k' + \dots$$

\* Thus, let us consider a  $|\Omega\rangle = a_\omega^\dagger |0\rangle$ :

$$\Rightarrow p(\infty) = i \frac{\langle E_n | \hat{a}_k | E_0 \rangle}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \epsilon(\tau) e^{i(E_n - E_0)\tau} \langle 0 | a_k^+ \int \frac{e^{i\omega_k x_k(\tau) - \tilde{k}\vec{x}(\tau)}}{\sqrt{2\omega_k}} a_k^+ d^3 k | 0 \rangle d\tau$$

↑   ↑  
leads to:

$$\langle 0 | a_k^+ a_k^- | 0 \rangle = \langle 0 | a_k^+ a_k^- + \delta^3(\tilde{k}-k) | 0 \rangle = \delta^3(\tilde{k}-k)$$

⇒

$$p(\infty) = i \underbrace{\frac{\langle E_n | \hat{a}_k | E_0 \rangle}{(2\pi)^{3/2}}}_{\text{some constant}} \int_{-\infty}^{+\infty} \frac{e^{i(E_n - E_0)\tau}}{\sqrt{2\omega_k}} e^{i(\omega_k x_k(\tau) - \tilde{k}\vec{x}(\tau))} \epsilon(\tau) d\tau$$

Special case:  $|d\rangle = |0\rangle$  and detector inertial:

\* Choose the detector's rest frame:  $x^\mu(\tau) = (\tau, 0, 0, 0)$

\* Thus:

$$p(\infty) = i \frac{\langle E_n | \hat{a}_k | E_0 \rangle}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \frac{e^{i(E_n - E_0)\tau}}{\sqrt{2\omega_k}} e^{i(\omega_k x_k(\tau) - \tilde{k}\vec{x}(\tau))} \epsilon(\tau) d\tau$$

assume  $\epsilon(\tau) = 1$ , i.e., "always on".

$$= i \frac{\langle E_n | \hat{a}_k | E_0 \rangle}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \frac{i(E_n - E_0)\tau}{\sqrt{2\omega_k}} e^{i\omega_k \tau} d\tau$$

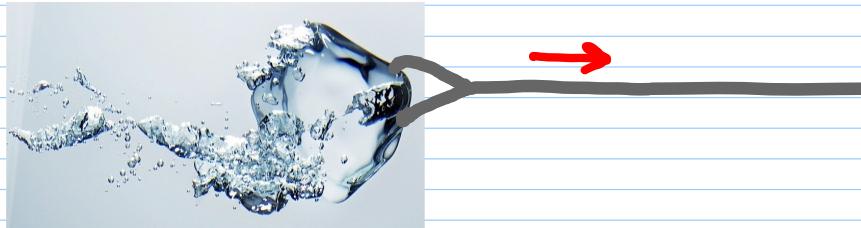
$$= i \frac{\langle E_n | \hat{a}_k | E_0 \rangle}{(2\pi)^{3/2}} (2\pi)^{1/2} \delta(\underbrace{E_n - E_0 + \omega_k}_{>0}) \frac{1}{\sqrt{2\omega_k}}$$

$\sqrt{k^2 + m^2} > 0$   
 $\parallel$   
 this cannot be 0

= 0

⇒ No excitation of the detector, as expected.

Special case:  $|\alpha\rangle = |\alpha\rangle$  and detector non-inertial:



□ The probability amplitude for the detector to become excited will depend on the excitation energy  $E_{\text{ex}} := E_n - E_0$ :

$$p(\infty) = i \frac{\langle E_n | \hat{Q}_0 | E_0 \rangle}{(2\pi)^{3/2} \sqrt{2\omega_i}} \int_{-\infty}^{+\infty} e^{i(E_n - E_0)\tau} e^{i(\omega_i x^*(\tau) - \tilde{k} \vec{x}(\tau))} \epsilon(\tau) d\tau$$

↓ a constant      ↓ Fourier factor  
 i.e.  $\tau$  and  $E_{\text{ex}}$       ↓ function that is being  
 ↙ are a Fourier pair      ↓ Fourier transformed  
 (if neglecting the "constant")

□ Clearly: For generic, accelerated detectors the function

$$f(\tau) := e^{i(\omega_i x^*(\tau) - \tilde{k} \vec{x}(\tau))} \epsilon(\tau)$$

possesses a Fourier transform

$$\mathcal{F}(E_{\text{ex}}) = \int_{-\infty}^{+\infty} e^{iE_{\text{ex}}\tau} f(\tau) d\tau, \quad E_{\text{ex}} = E_n - E_0$$

which is generally non-zero for positive  $E_{\text{ex}}$ .

$\Rightarrow p(\infty) \sim \mathcal{F}(E_{\text{ex}}) \neq 0 \Rightarrow$  detector does get excited.  
 ↑ proportional to  
 (European notation) (while also the field gets excited)

Unruh effect

Example: The constantly accelerated detector.

\* One finds that the prob. of excitation is identical to the case in which the detector is in a heat bath of temperature  $T \approx \frac{1}{2}$  where  $\omega$  is the acceleration.

Assigned reading: Birrell & Davies: p 52 - 58.

Remark: \* Note that both the detector and the quantum field become excited. Is energy conservation violated?

\* One can show that the energy stems from the accelerating agent.

E.g.: Think of a regular antenna.  
If the accelerated  $e^-$  were excitable little systems, they would get excited.

\* It's the case of an system with charge in time-dependent interaction with the field: An antenna where field & system get excited.

Special case:  $|d\rangle = |1_k\rangle$ :

Recall:

$$P = i \int_{-\infty}^{+\infty} \frac{\epsilon(\tau)}{\sqrt{2\omega_k}} e^{i(E_n - E_0)\tau} \langle E_n | \hat{Q}_0 | E_0 \rangle \langle \Omega | \hat{\phi}(x(\tau)) | d \rangle d\tau$$

Prob. amplitude for detector to get excited

Recall:

$$\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \left( \frac{1}{\sqrt{\omega_k}} e^{i\omega_k x^0 - ik^2} a_k^+ + \frac{1}{\sqrt{\omega_k}} e^{-i\omega_k x^0 + ik^2} a_k^- \right) d^3 k$$

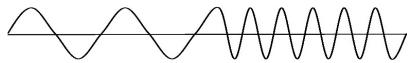
$\Rightarrow$  For  $|d\rangle = |1_k\rangle = a_k^+ |0\rangle$ , we can have:

a.)  $|\Omega\rangle = |2_k\rangle$ , or  $|1_k, 1_k\rangle$  Would mean detector excites the field further

i.e., not only "detects" a particle.

b.)  $|\Omega\rangle = |0\rangle$ : Means detector absorbs a particle.

## Alternative intuition



- ◻ A monochromatic wave in an inertial frame is not monochromatic for an accelerated observer.
- ◻ Thus, the accelerated observer's modes are coupled oscillators; he sees wavelengths change.
- ◻ There oscillator's ground state is different.

→ Calculation strategy:

- ◻ Use accelerated observers' mode decomposition.
- ◻ Relate it to inertial observer's mode decomposition.
- ◻ Choose vacuum for the inertial observer
- ◻ Calculate particle production for accelerated observer analogous to  $|in_{inertial}\rangle \rightarrow |out_{inertial}\rangle$  transform for driven harmonic oscillators' evolution above.