

QFT for Cosmology, Achim Kempf, Lecture 5

Note Title

Particles in QFT

Back in the Heisenberg picture,
to solve the QFT is to solve:

□ The hermiticity conditions:

$$\hat{\phi}^+(x, t) = \hat{\phi}(x, t), \quad \hat{\pi}^+(x, t) = \hat{\pi}(x, t)$$

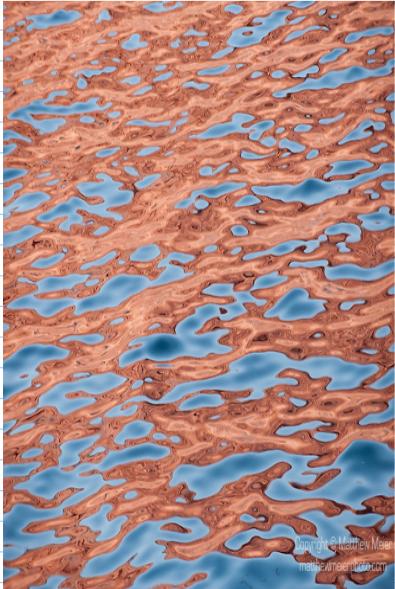
□ The canonical commutation relations:

$$[\hat{\phi}(x, t), \hat{\pi}(x', t)] = i \cdot \delta(x - x')$$

□ The equations of motion:

$$\dot{\hat{\phi}}(x, t) - \Delta \hat{\phi}(x, t) + m^2 \hat{\phi}(x, t) = 0$$

$$\dot{\hat{\pi}}(x, t) = \dot{\hat{\phi}}(x, t)$$



To simplify:

□ Infrared regularization:

Box size $L \times L \times L$ with

periodic boundary conditions.

↑ Project: uses Dirichlet boundary conditions.

□ Then Fourier series expansion:

$$\hat{\phi}(x, t) = L^{-3/2} \sum_{\mathbf{k}} \hat{\phi}_{\mathbf{k}}(t) e^{i \mathbf{k} \cdot \mathbf{x}}$$
$$\sum_{\mathbf{k}} \mathbf{k} = \frac{2\pi}{L} (n_1, n_2, n_3), n_i \in \mathbb{Z}$$

Obtain:

$$\ddot{\hat{\phi}}_{\mathbf{k}}(t) = -(\mathbf{k}^2 + m^2) \hat{\phi}_{\mathbf{k}}(t) \quad \text{and} \quad [\hat{\phi}_{\mathbf{k}}, \hat{\phi}_{\mathbf{k}'}] = i \delta_{\mathbf{k}, -\mathbf{k}'}$$

$$\hat{H} = \sum_{\mathbf{k}} \hat{H}_{\mathbf{k}} \quad \text{with} \quad \hat{H}_{\mathbf{k}} = \frac{1}{2} \hat{\pi}_{\mathbf{k}}^+ \hat{\pi}_{\mathbf{k}} + \frac{1}{2} \hat{\phi}_{\mathbf{k}}^+ (\mathbf{k}^2 + m^2) \hat{\phi}_{\mathbf{k}}$$

$$\text{i.e.:} \quad \hat{H} = \sum_{\mathbf{k}} \left(\frac{1}{2} \hat{\pi}_{\mathbf{k}}^+ \hat{\pi}_{\mathbf{k}} + \frac{1}{2} \hat{\phi}_{\mathbf{k}}^+ (\mathbf{k}^2 + m^2) \hat{\phi}_{\mathbf{k}} \right)$$

□ Crucial observations:

* For each wave vector $\mathbf{k} = (k_x, k_y, k_z)$ there is an independent harmonic oscillator with frequency $\omega_k = \sqrt{k^2 + m^2}$ and spectrum $\text{spec}(H_k) = \hbar \omega_k \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \right\}$.

\Rightarrow The excitation levels of H_k differ by the energy

$$E = \omega_k = \sqrt{k^2 + m^2} \quad (\hbar = 1)$$

* This is also the energy of a particle of momentum \mathbf{k} !

\rightsquigarrow Hypothesis: Mode excitation = particle creation

Does this interpretation work?

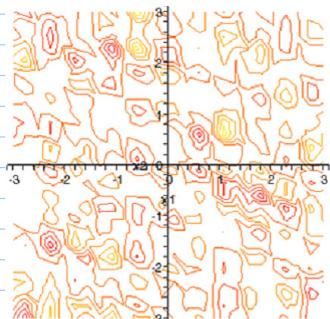
Water:

$$\phi(x, t)$$



Quantum field:

$$\hat{\phi}(x, t)$$



Probe amplitudes,
e.g., with a cork:



Probe amplitudes, e.g.,
with atoms.



Use as a
detector for
the field's particles
(e.g. photons for EM field)

One finds:

- Interpretation works but is acceleration and curvature dependent.
- Interpretation simple only in Minkowski space for inertial detectors.

Note: Conventional particle physics is based on that special case.

Then: Which is, e.g., the state $|4\rangle$ in which we have

3 particles of momentum k_a and 7 particles of momentum k_b ?

$$|4\rangle = |n_{k_a}=3, n_{k_b}=7, \text{all other } n_k=0\rangle \\ = |n_{k_a}=3\rangle \otimes |n_{k_b}=7\rangle \left(\bigotimes_{\substack{\text{all other} \\ k_c}} |n_{k_c}=0\rangle \right)$$

Energy: $\hat{H}_k |4\rangle = \begin{cases} \hbar \omega_a (\frac{1}{2} + 3) & \text{if } k=k_a \\ \hbar \omega_b (\frac{1}{2} + 7) & \text{if } k=k_b \\ \hbar \omega_a \frac{1}{2} & \text{if } k \neq k_a, k_b \end{cases} |4\rangle$

$$\Rightarrow \hat{H} |4\rangle = \left(3\omega_{k_a} + 7\omega_{k_b} + \sum_{\text{all } k} \frac{1}{2} \omega_k \right) |4\rangle$$

And one can have linear combinations:

Which is, e.g., the state $|xc\rangle$ in which we have

3 particles of momentum k_a or 7 particles of momentum k_b ,

with probability amplitudes $\alpha, \beta = \sqrt{1-\alpha^2}$?

$$|xc\rangle = \alpha |n_{k_a}=3, \text{other } n_k=0\rangle + \beta |n_{k_b}=7, \text{other } n_k=0\rangle$$

Notice: This is not a state of fixed particle number!

{ Remark: Some particle species have a number conservation

law, e.g., leptons, i.e. e^- , μ^- , τ^- , ν_e , ν_μ , ν_τ ,

(where the antiparticles count negatively).

"Superslection rule" then says we can't have such linear combinations: only number eigenstates allowed.

Mechanisms for mode excitation/particle creation?

J. e.: What are mechanisms for exciting harmonic oscillators?

□ 2 types of mechanism: (here, $\hat{q}(t)$ stands for $\hat{\phi}_k(t)$)

we'll begin → **a.) A "driving force" shakes the oscillator:**
with this effect

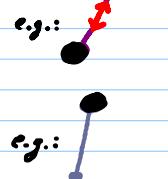
$$\ddot{\hat{q}}(t) = -\omega^2 \hat{q}(t) + \hat{j}(t)$$



b.) A time dependence of ω affects the oscillator:

$$\ddot{\hat{q}}(t) = -\omega^2(t) \hat{q}(t)$$

And, $\omega^2(t)$ could even go negative!



All occur in QFT:

a.) Multiple fields enter into the Hamiltonian and into the eqns of motion. Thus, fields provide each other with \hat{J} terms, e.g.:

$$H(\hat{\phi}, \hat{\psi}) = \hat{H}_1(\hat{\phi}) + \hat{H}_2(\hat{\psi}) + \int_{\mathbb{R}^3} \lambda \hat{\phi}(x,t) \hat{\psi}(x,t) d^3x$$

□ Wave interpretation: Nontrivial interaction of waves of different types of fields

□ Particle interpretation: The collision of particles happens when their mode oscillators drive another.

→ Collisions can create and annihilate particles.

□ Strongest effects?

When oscillator "resonates" with driving force.

E. g.: It takes high energy particles to make high energy particles

b) The presence of gravity can effectively influence the $\omega_r(t)$.

□ Wave interpretation: * E.g., cosmic expansion stretches the wavelength

\Rightarrow expect $\omega = \omega(t)$ decreases. True, and also:

* if wavelength > horizon then $\omega^2(t) < 0$!

\Rightarrow runaway harmonic mode "oscillators"

(then: field amplification but no particle interpretation)



□ Particle interpretation:

Gravity can excite mode oscillators, i.e. it can create particles from the vacuum.

□ Strongest effects? When oscillator resonates with $\omega(t)$. This effect is called parametric resonance.

Case a: Particle creation through external driving of mode oscillators.

Example: Production of photons by an antenna:

□ We model the electromagnetic field as a Klein-Gordon field.

(The fact that EM fields have polarization and have $m=0$ is not important here)

□ Consider an arbitrary mode of the electromagnetic field:

$\phi_e(t)$

should really
be quantized too

□ We model the electric current as a given classical field $j(x, t)$ whose modes are $j_n(t)$.

j should really be vector-valued

□ In a rough simplification, the EM k mode obeys:

$$\hat{H}_k = \frac{1}{2} \hat{\pi}_k^*(t) \hat{\pi}_k(t) + \frac{1}{2} \omega_k^2 \hat{\phi}_k^*(t) \hat{\phi}_k(t) + \hat{\phi}_k(t) j_k(t)$$

⇒ If the current $j(t)$ varies in time it can excite the mode oscillators, thus creating photons.

⇒ Need to study the quantized driven harmonic oscillator!

$$\hat{H}(t) = \frac{1}{2} \hat{p}(t)^2 + \frac{\omega^2}{2} \hat{q}(t)^2 - J(t) \hat{q}(t)$$

[for $\hat{H}_k(t)$] [for $\hat{\pi}_k(t)$] [stands for a field mode $\hat{\phi}_k(t)$]
 [stands for a mode $j_k(t)$ of another classical (or better quantum) field].

I Preparation:

□ Recall that for all observables \hat{f} :

$$\hat{f}(t) = \langle \psi_0 | \hat{U}^*(t) \hat{f}_0 \hat{U}(t) | \psi_0 \rangle$$

↓ state at initial time
 ↓ operator at initial time

with the time-evolution operator obeying:

$$\hat{U}(t_0) = 1, \quad i \frac{d}{dt} \hat{U}(t) = \hat{U}(t) \hat{H}(t)$$

↴ the original Hamiltonian
 ↑ "Heisenberg Hamiltonian"

□ Schrödinger picture? We write, equivalently:

$$\begin{aligned} \hat{f}(t) &= (\langle \psi_0 | \hat{U}^*(t)) \hat{f}_0 \underbrace{(\hat{U}(t) | \psi_0 \rangle)}_{= |\psi(t)\rangle} \\ &= \langle \psi(t) | \hat{f}_0 | \psi(t) \rangle \end{aligned}$$

Recall: $\hat{A}_S(t) = \hat{H}(t)$ only if $\frac{d}{dt} \hat{H}(t) = 0$

The dynamics is $i \frac{d}{dt} |\psi(t)\rangle = \hat{H}_S(t) |\psi(t)\rangle$

with Schrödinger Hamiltonian: $\hat{H}_S(t) = \hat{U}(t) \hat{H}(t) \hat{U}^\dagger(t)$

□ We will use, equivalently, the Heisenberg picture:

$$\begin{aligned}\hat{f}(t) &= \langle \psi_0 | \underbrace{\left(\hat{U}^\dagger(t) f_0 \hat{U}(t) \right)}_{\text{"}} | \psi_0 \rangle \\ &= \langle \psi_0 | \hat{f}(t) | \psi_0 \rangle\end{aligned}$$

with dynamics:

$$i \frac{d}{dt} \hat{f}(t) = [\hat{f}(t), \hat{H}(t)]$$

II Aspects of the Heisenberg picture:

□ The state of the quantum system stays the same Hilbert space vector, say $|\psi\rangle \in \mathcal{H}$ (from measurement to measurement).

□ The observables, say $\hat{H}(t)$, $\hat{f}(t)$, etc., are time-dependent operators in Hilbert space.

□ Important implication:

The eigenbases and the eigenvalues of observables, such as $\hat{H}(t)$ and any $\hat{f}(t)$ depend on time!

$$\hat{f}(t) |f_n(t)\rangle = f_n(t) |f_n(t)\rangle$$

$$\hat{H}(t) |E_m(t)\rangle = E_m(t) |E_m(t)\rangle$$

Example: * Assume the driven harmonic oscillator starts out at time t_1 in n 'th energy state, say $|x\rangle = |E_n(t_1)\rangle$:

$$\hat{H}(t_1)|E_n(t_1)\rangle = E_n(t_1)|E_n(t_1)\rangle$$

* State vector of the system stays $|x\rangle$ for $t > t_1$.

* But at later times, say $t > t_1$, the Hamiltonian and its eigenvectors and eigenvalues are

$$\hat{H}(t)|E_m(t)\rangle = E_m(t)|E_m(t)\rangle$$

and we generally have

$$E_m(t) \neq E_n(t_1), |E_m(t)\rangle \neq |E_n(t_1)\rangle$$

\Rightarrow At time t_2 system is still in state $|x\rangle$ and still $|x\rangle = |E_n(t_1)\rangle$

but $|x\rangle$ is generally no longer with (or any other) energy eigenstate!

In particular:

* Assume system starts out at t_1 in lowest energy state (i.e. in vacuum): $|x\rangle = |E_0(t_1)\rangle$

* Then if $|x\rangle = |E_0(t_1)\rangle \neq |E_0(t_2)\rangle$

\Rightarrow At t_2 the system's state $|x\rangle$ is not the ground state i.e. not the vacuum state, i.e. particles (e.g. photons) exist at time t_2 .

III Strategy for solving quantized driven harmonic oscillator

□ Problem: * CCR: $[\hat{q}(t), \hat{p}(t)] = i\mathbb{1}$

* Hermiticity: $\hat{q}^+(t) = \hat{q}(t)$, $\hat{p}^+(t) = \hat{p}(t)$

* Hamiltonian: $\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\hat{q}(t)$

* Heisenberg eqns $i\dot{\hat{f}}(t) = [\hat{f}(t), \hat{H}(t)]$ yield:

This is a good strategy
with and without a
driving force

$$\dot{\hat{q}}(t) = \hat{p}(t)$$

$$\dot{\hat{p}}(t) = -\omega^2 \hat{q}(t) + J(t)$$

↓
□ Strategy: * Combine $\hat{a}(t) := \underline{\lambda} \hat{q}(t) + i\underline{\beta} \hat{p}(t)$ (operator even though no "hat".)
T. Mukhan calls it $a^-(t)$ (analogous to "real" &
"imaginary" parts)

* Choose λ, β so that $\hat{H}(t)$ and eqn of motion simplify.

IV Determine λ and β :

□ Notice first that once we have $a(t)$ we immediately obtain $\hat{q}(t), \hat{p}(t)$: Use of $\hat{a}^+(t) = \lambda \hat{q}(t) - i\beta \hat{p}(t)$ yields:

$$\hat{q}(t) = \frac{1}{2\lambda} (\hat{a}^+(t) + a(t))$$

$$\hat{p}(t) = \frac{i}{2\beta} (\hat{a}^+(t) - a(t))$$

□ Use this to express $[\hat{q}, \hat{p}] = i$ in terms of new variable $a(t)$:

$$\Rightarrow [a(t), \hat{a}^+(t)] = 2\lambda\beta$$

For simplicity, we choose $\beta = \frac{1}{2\lambda}$ so that:

$$[a(t), \hat{a}^+(t)] = 1$$

□ Now express $\hat{H}(t)$ in terms of new variable $a(t)$:

$$\hat{H}(t) = -\frac{1}{2}\omega^2(a^+(t)-a(t))^2 + \frac{\omega^2}{2}\frac{1}{4\omega^2}(a^+(t)+a(t))^2$$

$$- J(t) \frac{1}{2\omega} (a^+(t) + a(t))$$

We notice that the terms $\sim a^+(t)^2$ and $\sim a(t)^2$ drop out if we choose:

$$-\frac{1}{2}\omega^2 + \frac{\omega^2}{2}\frac{1}{4\omega^2} = 0$$

Thus, we choose: $\omega = \sqrt{\frac{\omega}{2}}$ and therefore $\beta = \frac{1}{\sqrt{2\omega}}$

□ Thus, by definition:

$$a(t) := \sqrt{\frac{\omega}{2}} \hat{q}(t) + i \frac{1}{\sqrt{2\omega}} \hat{p}(t)$$

□ The Hamiltonian simplifies to become:

$$\hat{H}(t) = \omega(a^+(t)a(t) + \frac{1}{2}) - J(t)\frac{1}{\sqrt{2\omega}}(a^+(t) + a(t))$$

IV Solve for $a(t)$:

□ The Heisenberg equation $i\dot{f}(t) = [f(t), \hat{H}(t)]$ reads for $a(t)$:

$$i\dot{a}(t) = \omega a(t) - \frac{i}{\sqrt{2\omega}} J(t)$$

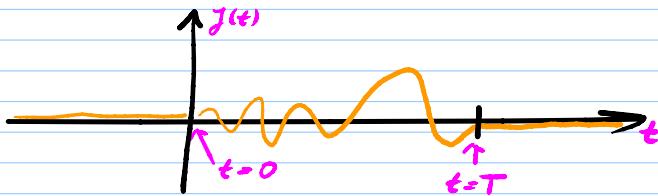
□ Let us give $a(t=0)$ a name: $a_{in} = a(0)$. Then:

Exercise:
verify.

$$a(t) = a_{in} e^{-i\omega t} + 1 \frac{i}{\sqrt{2\omega}} \int_0^t J(t') e^{i\omega(t'-t)} dt'$$

VII Case of force of finite duration

□ Assume $J(t) = 0$ for all $t \notin [0, T]$



□ Define $J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'} dt'$

□ Then: $a(t) = \begin{cases} a_{in} e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } t \in [0, T] \\ (a_{in} + J_0) e^{-i\omega t} & \text{for } t > T \end{cases}$

Implications in terms of particle (e.g. photon) production?